Potential Fraction Concept Images Afforded in Textbooks: A Comparison of Northern Ireland and Singapore

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Fractions are among the most problematic concepts that children encounter in their primary school years because of the many different conceptions of fractions. Textbook analyses have tried to provide insights into how fractions are introduced, focusing on the different concepts and representations of fractions. In this paper, we contribute to these efforts by investigating the way fractions are first introduced using the notion of potential concept images as afforded by the textbooks. Analyses of two textbooks, one from Northern Ireland and the other from Singapore, will be presented to highlight these potential concept images and their implications for practice.

It is widely argued that fractions are among the most problematic concepts that students encounter in the primary school years (Charalambous & Pitta-Pantazi, 2006; Gabriel et al., 2013; Siegler & Lortie-Forgues, 2017; Smith, 2002). Despite numerous efforts by researchers, governments, mathematics educators, and textbook writers to reform the learning of fractions over the past few decades, students' performance on fraction arithmetic tasks has shown little improvement (Siegler & Lortie-Forgues, 2017). Fractions, as a "powerful mathematical idea", are difficult to learn because they can be used to "express so many different kinds of relationships" (Smith, 2002, p. 4). As Kieren (1976) had suggested, fractions are best conceptualised not as a single conception, but as a set of inter-related conceptions: part-whole, ratio, operator, quotient, and measure. However, fractions are often introduced only as a part-whole model in many countries. This is limiting and may have contributed to the difficulties faced by students when learning fractions (Simon et al., 2018). Furthermore, textbooks, which make up the main instructional or curriculum materials, do not introduce fractions using a *concept definition*; instead, they often present real-world scenarios, such as sharing pizzas, and other representations to support students in forming their own *concept images* (Tall & Vinner, 1981). Given that textbooks offer examples and representations that may facilitate or hinder learners' formation of useful concept images (Tall & Vinner, 1981; Zhang et al., 2014), it is important for mathematics educators to analyse how fraction concepts are presented in textbooks. Although there have been textbook analyses on the concept of fractions (e.g., see Lee et al., 2021), these analyses focused on the kind of fraction conceptions, tasks, and representations presented in the textbooks. In this paper, we contribute to this conversation by examining the *potential concept images* afforded by how fraction concepts are first introduced in mathematics textbooks from two relatively different education systems—Northern Ireland and Singapore. According to PISA 2022, Singapore remains the world leader in mathematics education, whereas pupils in Northern Ireland achieved a score of 475, not significantly different to the OECD average score of 472, but significantly lower than the score of 492 achieved in 2018. Besides international assessment data, the two countries have different education systems, which offer opportunities for interesting comparisons. The paper is framed by the following question:

> What are the potential concept images of fractions afforded by the textbooks from Northern Ireland and Singapore in the way they first introduce fractions?

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Review of Literature

We begin by reviewing the different conceptions of fractions and highlighting the challenges faced by students as they grapple with these various concepts. Furthermore, the way fractions are presented to students (manipulative, pictorial, verbal, real-world, and symbolic) may privilege certain conceptions over others. Hence, students' difficulties when learning fractions may be partially explained by the limited concept images they might have (Tall & Vinner, 1981; Zhang et al., 2014). As most students develop these mental images through their interactions with textbooks, we argue that it is important for us to investigate how fractions are introduced and represented in the textbooks. Finally, we suggest the notion of potential concept image to better understand how students may perceive fractions from textbooks.

Conceptualisations and Representations of Fractions

According to Smith (2002, p. 3), "no area of elementary mathematics is as mathematically rich, cognitively complicated, and difficult to teach as fractions, ratios and proportionality". Whilst several factors have been identified as contributing to this difficulty, the multifaceted conception of fraction is a key factor associated with the complexities of teaching and learning fractions (Kieren, 1976; Lamon, 2012; Simon et al., 2018). Fractions can be conceptualised in at least five different ways: part-whole, measure, operator, quotient, and ratio (Kieren, 1976; Lamon, 2012). For example, the fraction 2/3 is usually interpreted as two out of three equal parts (a part-whole conception). As a measure, 2/3 is perceived as the distance of two 1/3 units from zero on the number line, which presents fractions as numbers (Simon et al., 2018). Seeing fractions as numbers is also related to the idea that 2/3 is the answer to $2 \div 3$ (a quotient). As an operator construct, 2/3 of a quantity is found by multiplying the unit by two and dividing the result by 3 or dividing the quantity by 3 and multiplying the result by 2. Finally, 2/3 can also be seen as a ratio, which represents a multiplicative relationship between two quantities in a particular order. These different conceptions make fractions a "powerful mathematical idea" because they are used to "express so many different kinds of relationships" (Smith, 2002, p. 4).

However, these different conceptions are not all well understood by students. For instance, Simon et al. (2018) highlight that students often do not see fractions as numbers (measure concept); instead, they often perceive fractions as two numbers: the numerator denotes the number of shaded parts and the denominator denotes the total number of equal parts (a limited understanding of fraction as part-whole). This partial understanding of the various fraction conceptions may hinder students' ability to work with different fraction constructs (Lee & Hackenberg, 2014). As argued by Simon et al. (2018), introducing fractions as "part of a whole" is limiting and may hinder students in developing more "powerful conceptions of fractions" (p. 123). Hence, it is critical for students to encounter these different fraction ideas as part of their learning experiences.

To this end, Cramer et al. (2002) advocated providing opportunities for students to use multiple physical models of fractions and translate between and within the different modes of representation: pictorial, manipulative, verbal, real-world, and symbolic. Similarly, van de Walle et al. (2013) recommended the use of all three types of pedagogical models (area models (e.g., fraction discs), linear models (e.g., fraction strips, number lines, etc.), and set models (e.g., collection of discrete objects) to support students in developing a more robust understanding of fractions. Despite these recommendations, textbooks often favour the use of area models over the other representations (Lee et al., 2021; Simon et al., 2018; Watanabe, 2002), leading to a limited mental image of fractions, which could lead to an erroneous understanding of operations involving fractions (see Watanabe, 2002, p. 462 for an example).

Concept Images

Therefore, it is the networking of fraction conceptions and representations that enable students to perceive fractions in different ways that are useful to the tasks at hand. Specifically, it is crucial for students to have a rich network of mental images associated with the different fraction conceptions. These mental images, which could take the form of pictures, diagrams, graphs, symbols, or other representations, are called concept images. According to Tall and Vinner (1981), a concept image describes “all the cognitive structure in the individual’s mind that is associated with a given concept”, including “all the mental pictures and associated properties and processes” (pp. 151–152). The complication arises because learners’ concept images may not always be aligned to the corresponding formal concept definitions. Hershkowitz and Vinner (1980) suggested that students may develop concept images of a concept before they can make sense and use an appropriate concept definition. These images are developed through the students’ interactions with examples, diagrams, and other representations presented in the classrooms.

But for fractions, there seems to be an over-emphasis on the use of area models, and this results in a limited part-whole fraction conception. This limited conception “seemed to generate concept images of unit fractions which resulted in students learning much about drawing and shading circles, rectangles, and squares, but without developing sound conceptual understandings” of unit fractions (Zhang et al., 2014, p. 228). This issue is further exacerbated by the fact that most primary school textbooks do not introduce the concept of rational numbers, which has a clearly defined concept definition. These findings suggest the importance of textbook writers moving away from “area-model representations of unit fractions toward multiple embodiments” (p. 228) when writing textbooks.

We wonder, therefore, how textbooks can facilitate or hinder the development of concept images that can be productively evoked for solving fraction tasks. Studies that investigated students’ concept images often examined their evoked images when responding to carefully designed tasks (Hershkowitz & Vinner, 1980; Tall & Vinner, 1981; Zhang et al., 2014). But, if textbooks are seen as “objectively given structure” without considering how the textbooks might be used with students (Herbel-Eisenmann, 2007, p. 346), then there is a need to examine how textbooks might play a role in enriching or hindering students’ concept images. Most textbook analyses focus on characterising the concepts, representations, and tasks presented in the textbooks (Charalambous et al., 2010; Choy et al., 2020; Lee et al., 2021). In this paper, we want to analyse textbooks through the lens of concept images by using the notion of *potential* concept images. To be clear, we are not looking at evoked concept images; instead, we are trying to determine if various combinations of conceptions and representations might afford certain concept images, which give rise to opportunities for students to understand fractions.

Methods

To derive the potential concept images as afforded by textbooks, we first examined the textbooks with respect to the contextual variables of the two countries (Huntley, 2008) before we analysed the content and instructional variables (Charalambous et al., 2010; Lee et al., 2021). We focused on the five fraction conceptions (Kieren, 1976; Lamon, 2012) for the content variable and the fraction representations (Cramer et al., 2002; Watanabe, 2002) for the instructional variable.

Contexts of the Two Countries

In Northern Ireland (NI), it is compulsory for all children to have at least 12 years of education (seven years of primary education and five years of post-primary education) and the opportunity to obtain school leaving qualifications. The 12 years of compulsory education are divided into five Key Stages. Primary education comprises a Foundation Stage (Years 1 and 2), Key Stage 1 (Years 3 and 4) and Key Stage 2 (Years 5, 6 and 7). The NI Primary Curriculum (Council for the Curriculum Examinations & Assessment (CCEA), 2007) sets out the minimum requirement that should be taught at each of the three stages of primary education: Foundation Stage, Key Stage 1 and Key Stage 2. Teachers, however, have considerable flexibility to make decisions about how best to interpret and combine the requirements in order to prepare young people for the future. In the NI Curriculum for Mathematics and Numeracy, fractions are included within the 'Number' strand and are first introduced in Key Stage 1. In Key Stage 1, pupils "should be enabled to … recognise and use simple everyday fractions' (p. 62), and in Key Stage 2 they 'should be enabled to … understand and use vulgar fractions, decimal fractions and percentages and explore the relationships between them" (p. 65).

In Singapore, there is compulsory education for primary school education, which comprises levels from Primary One (students aged 7) to Primary Six (students aged 12). As a centralised education system, there is a standardised curriculum for all primary schools that specify the content learning outcomes, learning experiences, pedagogical approaches, and assessment requirements for each level (Ministry of Education-Singapore, 2019). Unlike in Northern Ireland, Singapore teachers generally follow the standards set out in the curriculum documents very closely. However, they have flexibility to decide on the teaching approaches to cater to the needs of their learners. Like NI, fractions are introduced as part of the 'Number and Algebra' content strand and are first introduced in Primary Two. Fraction concepts are progressively developed from Primary Two through Primary Six, covering key fraction concepts such as fractions as part-whole, equivalent factions, mixed numbers, and improper fractions, as well as fraction operations (see Figure1).

Figure 1

Development of Fraction Concepts in Singapore (Ministry of Education-Singapore, 2012, p. 36).

P2	P3	P4	P5	P6
1. **Fraction of a whole** 1.1 fraction as part of a whole 1.2 notation and representations of fractions 1.3 comparing and ordering fractions with denominators of given fractions not exceeding 12 • unit fractions • like fractions 2. **Addition and subtraction** 2.1 adding and subtracting like fractions within one whole with denominators of given fractions not exceeding 12	1. **Equivalent fractions** 1.1 equivalent fractions 1.2 expressing a fraction in its simplest form 1.3 comparing and ordering unlike fractions with denominators of given fractions not exceeding 12 1.4 writing the equivalent fraction of a fraction given the denominator or the numerator 2. **Addition and subtraction** 2.1 adding and subtracting two related fractions within one whole with denominators of given fractions not exceeding 12	1. **Mixed numbers and improper fractions** 1.1 mixed numbers, improper fractions and their relationships 2. **Fraction of a set of objects** 2.1 fraction as part of a set of objects 3. **Addition and subtraction** 3.1 adding and subtracting fractions with denominators of given fractions not exceeding 12 and not more than two different denominators 3.2 solving up to 2-step word problems involving addition and subtraction	1. **Fraction and division** 1.1 dividing a whole number by a whole number with quotient as a fraction 1.2 converting fractions to decimals 2. **Four operations** 2.1 adding and subtracting mixed numbers 2.2 multiplying a proper/improper fraction and a whole number **without calculator** 2.3 multiplying a proper fraction and a proper/improper fractions **without calculator** 2.4 multiplying two improper fractions 2.5 multiplying a mixed number and a whole number 2.6 solving word problems involving addition, subtraction and multiplication	1. **Four operations** 1.1 dividing a proper fraction by a whole number **without calculator** 1.2 dividing a whole number/ proper fraction by a proper fraction **without calculator** 1.3 solving word problems involving the 4 operations

Selection of Textbooks

Schools in NI tend to rely on materials published in the United Kingdom to support their mathematics teaching. Anecdotal evidence suggests that, as in England, there is a vast array of mathematics curriculum resources in use, including textbook-schemes, online-schemes, and teacher-curated resources. For this study, we decided to focus on the New Heinemann Maths (NHM) series, a complete mathematics programme for children aged 4 to 11. Produced in England, NHM was first developed over 20 years ago and has been one of the more popular textbook-schemes in NI. In our analyses, we looked at NHM 3 (Scottish Primary Mathematics Group, 2000), where fractions were introduced for the first time in textbooks.

In contrast, there is only one textbook series, written by the Ministry of Education, to be used in Singapore primary schools for the current syllabus (Ministry of Education-Singapore, 2019), which was implemented in 2021. The former series of textbooks used for the 2013 syllabus will be progressively retired as the current syllabus progressively replaces the 2013 version from 2021 to 2026. As we are looking at how fractions are introduced, we will only analyse the current textbook for Primary Two, *Primary Mathematics Textbook 2B* (PMT 2B), published by Star Publications (Curriculum Planning & Development Division, 2022).

Data Analysis

With the aim of discerning the potential concept images as afforded by the textbooks, we followed Herbel-Eisenmann (2007) in seeing textbooks as "objectively given structure" (p. 346) without considering how the textbooks might be used with students. We wanted to examine how the examples, diagrams, and explanations might come together in presenting certain concept images to the readers. As we were interested in how fractions are first introduced in primary schools, we only analysed the relevant chapters that first present fractions as a mathematical concept (e.g., we did not analyse the chapter on time where terms such as "half past nine in the morning" might suggest some concepts of fractions). Although this may appear to be limited, it is important to consider how the foundations are established as this will impact how the learner's knowledge and understanding develops over time and how this knowledge and understanding will be used and applied in other contexts.

Analyses occurred at the item level where we saw each example, question, representation, paragraph of explanatory text, or task as an item. We first coded the items using the five conceptions of fractions—part-whole, measure, quotient, operator, and ratio (Kieren, 1976; Lamon, 2012)—depending on what was explained or needed to complete the item. We then coded the representations used (i.e., concrete, pictorial (area, length and set models), verbal, and symbolic) using the categorisation used by Lee et al. (2021). Coding was done independently in which the first author coded the Singapore textbook while the second author coded the NI textbook, before we came together to share and justify our analyses for our respective textbooks. Consensus was reached through discussion. We then proposed possible concept images based on our analyses of the conceptions and representations by connecting these two aspects of textbooks.

Findings and Discussions

In this section, we describe, compare, and contrast two key potential concept images afforded by the chosen textbooks from the two countries. They are "fractions as equal parts of a whole" and "fractions can have different sizes". As we describe these potential images, we also attempt to relate them to students' learning difficulties in extant literature before we conclude with some implications for practice.

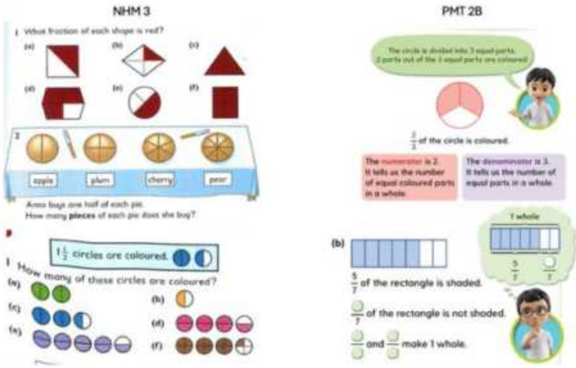
Fractions are Equal Parts of a Whole

We expected that "fractions are equal parts of a whole [shape]" to be one of the potential concept images afforded by the two textbooks. However, we made a nuanced distinction from the usual part-whole conception of fractions (Lamon, 2012) by highlighting what students might perceive about the equality of parts from the fraction representations. As seen in Figure 2, fractions are often represented in area models, comprising circles or rectangles, with equal parts that are *congruent* to each other. This image is also reinforced by explanations such as "The circle is divided into 3 equal parts. 2 parts out of three equal parts are coloured. So, 2/3 of the circle is coloured" [PMT 2B, p. 55].

This conception-pictorial-textual link might be wrongly perceived by students (and even teachers) to mean that there is no fraction when the parts are not equal (Choy, 2013). Seeing that the correct answers always involve equal parts with congruent parts, it might seem logical for students that fractions must always be equal and *congruent* parts of a whole (Simon et al., 2018), resulting in this flawed concept image. This flawed concept image may be potentially evoked by students when solving tasks which require students to identify pictures that show a given fraction or to identify the fraction in each picture [PMT 2B, p. 54; NHM 3, p. 70]. Therefore, our textbooks present at least two potential concept images associated with the part-whole conception. First, the mental image that a shaded part is $1/n$ of a given shape (area model) because its area is $1/n$ of the area of the given shape. Second, the flawed mental image that a shaded part is $1/n$ of a given shape because the shape is divided into n congruent parts.

Figure 2

Part-Whole Representations of Fractions (Curriculum Planning & Development Division, 2022, p. 55; Scottish Primary Mathematics Group, 2000, p. 70)

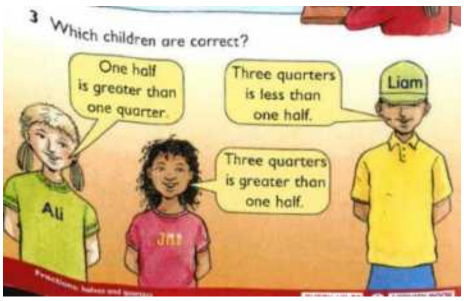


Fractions can Have Different "Sizes"

Several items in both the NHM 3 and PMT 2B afford the potential concept image of fractions having different sizes. Clearly, as a measure, fractions can be ordered *on a number line*. Here, the NHM 3 included a task that aims to encourage students to count in halves on a number line from 0 to 7. This task is followed by another one which provides students opportunities to determine which statement is true (see Figure 3). There are two possible answers to this task. First, if we take these fractions as a number (measure), then there is a fixed answer in that ¼ < ½ < ¾. Second, if we think of each fraction as part of a whole, then we will need to be clear about the referent before we can decide. Taken together, these tasks afford the potential image of fractions have different sizes because they can be *ordered* on a number line.

Figure 3

Task in NHM 3 on Comparing Fractions (Scottish Primary Mathematics Group, 2000, p. 71)

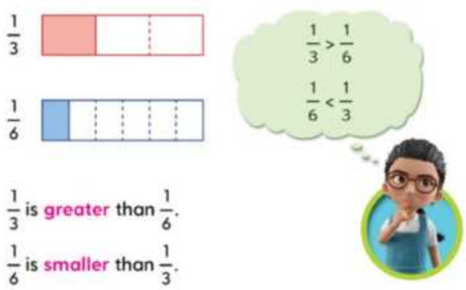


However, for the PMT 2B, there is no number line to indicate that fractions can be ordered independent of context. Instead, the potential concept image with respect to comparing and ordering fractions is associated with the *size of the shaded area* of the figure, which is consistent with the focus on the part-whole conception of fraction. Referring to Figure 4, we see that the basis for deciding which fraction is greater is the area of the shaded region—the bigger the area, the greater the fraction.

Although both textbooks offer potential concept images about the possibility of comparing and ordering fractions, they differ qualitatively in terms of what counts as size. The use of the number line in NHM 3 to illustrate the measure conception of fraction affords students a concept image that potentially relates the magnitude of a fraction to its location along the number line. Whereas in PMT 2B, the area model provides the basis for comparing and ordering fractions using the size (area) of the fraction pieces. This potential concept image may not be robust enough to deal with cases where the whole is of different sizes. The NHM 3, on the other hand, provides another potential image through the task in Figure 3.

Figure 4

Task in PMT 2B on Comparing Fractions (Curriculum Planning & Development Division, 2022, p. 57)



Concluding Remarks

Our analyses of the two textbooks through the lens of potential concept images is by no means comprehensive as there may be other potential concept images. However, they suggest a layer of complexity beyond coding for conceptions and representations in textbook analyses: the same representation in a textbook can potentially evoke different concept images in students. For instance, the heavy focus on the "part-whole" fraction concepts using the area models in PMT 2B can give rise to both an acceptable concept image and a flawed one. However, the surfacing of flawed potential concept images in a textbook does not mean that the textbook is inferior in any way. Nor would the number of potential concept images mean anything about the quality of a textbook. What we offer through the notion of potential concept images is to provide a way to connect conceptions, representations, and tasks in the textbooks for the purpose of highlighting different ways of understanding the same concept.

These potential concept images, as developed through our content and instructional analyses of the textbooks, can be seen as students' possible understanding of fraction ideas. As suggested by Smith (2002), it is useful to "work with students to understand fractions and ratios in their own terms first" for the purpose of making sense of the formal concepts of fractions later (p. 5). Hence, these potential concept images can serve as teachers' anticipation of students' ideas about fractions. They also sensitise teachers to the possibilities that students may perceive what was written in the textbooks differently. In addition, in the hands of a skilled teacher, an awareness of these flawed potential concept images can provide the locus of action for teachers to take when planning their instruction. Finally, given the powerful role that textbooks play in mathematics teaching and learning, it is vital that textbook writers carefully consider the potential concept images that may be evoked from the pedagogical approaches recommended.

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