# Learning to Share Fairly: The Importance of Spatial Reasoning in Early Partitioning Experiences

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Young children often explore partitioning as the idea of fair sharing in contexts where equal parts are created and distributed based on spatial constructs of the objects, rather than enumerating parts or collections. However, the presence of spatial reasoning in children's early fraction experiences is implicit within much of the literature and has not been explored pedagogically in a range of early schooling contexts. This study reports on a selection of data from a larger Design Based Research study that demonstrates the power spatial reasoning plays in developing early partitioning. Implications for teaching and learning are discussed.

Partitioning is considered the foundational concept that enables children to develop the multiplicative foundations needed to work with an extended range of fractions (Confrey et al., 2014; Kieren, 1993; Lamon, 2007). Young children typically explore partitioning by creating fair shares and equal parts in everyday play experiences (such as sharing a set of counters equally or dividing lumps of playdough into equal shares). Siemon (2003) argued that partitioning is the missing link between young children's experiences with fractions in the early years of schooling and their performance and capabilities for multiplicative and proportional reasoning in later years. While Siemon's (2003) research reports on data from over 20 years ago, these issues still exist in primary and middle school children's understandings today in Australia and internationally (Callingham & Siemon, 2021; Vanluydt et al., 2022). This implies that there is still a great need to examine the ways in which young children develop flexible understandings about partitioning, and what pedagogical approaches may better support their learning.

## Theoretical Framing

Regardless of the theoretical orientation of young children's development of fractions, partitioning is considered a critical foundation for fraction understanding because it is derived from division and reassembly of division (i.e., multiplication). This understanding is required to work with the various interpretations of fractions and the broader network of rational number concepts (Confrey et al., 2014; Kieren, 1993). The extensive body of research on fractions illustrates that it is a complex area of mathematics (Kieren, 1993) however, this paper does not have the scope to expand on these theoretical foundations and will focus specifically on young children's development of partitioning as the basis of this mathematical domain.

To establish a conceptual understanding of partitioning, it needs to be understood beyond the physical act of creating equal parts or shares from discrete or continuous objects. One of the critical ideas of this concept is understanding the relationship between how parts are formed and named; that is, the inverse relationship between the number of parts and their size (Lamon, 2007). Partitioning, therefore, develops closely in relation to the concepts of unitising (i.e., assigning a unit of measure to name a fraction in relation to how the quantity has been partitioned), and quantitative equivalence (i.e., fractions can be renamed infinitely) (Kieren, 1993; Lamon, 2007). Exploring 'half' is the typical introduction to fractions young children experience when fair sharing in everyday contexts (e.g., cutting a sandwich in half or filling a bucket half full of water). Halving is considered a critical entry point for exploring the inverse property of partitioning more broadly, as it introduces children to 'fraction families' (e.g.,

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halves, quarters, eighths, etc.), which integrates unitising and quantitative equivalence understandings (Confrey, 2012; Siemon, 2003). Even in the early years, teachers need to possess a deep understanding of how these concepts are connected so that the choice of pedagogical approach—including models and representations utilised—assists children in developing such connections. Examining current research on young children's early partitioning capabilities, in addition to the typical models, materials, and tasks teachers employ, provides an opportunity to re-examine early partitioning instruction.

## Examining Children's Partitioning Experiences

Children are often provided with learning experiences that explore partitioning in continuous contexts, such as using 2D shapes and pattern blocks, and paper folding (Siemon, 2003; Clarke et al., 2011). However, research suggests that children will often be prompted to count the parts generated from partitioning, rather than examining their magnitude in relation to the parts generated, which masks the inverse property of fair sharing (Clarke, 2011; Confrey, 2012). While such literature emphasises the need for teachers to utilise different continuous models to explore the relationship between partitioning and fraction magnitude, the focus largely remains on the types of models used for a particular fraction task. Conversely, we must consider the underlying foundations of how the models support children in constructing the relationship between partitioning, unitising, and equivalence.

English (1997) advocates that by reframing the representation as the vehicle for mathematical reasoning rather than a tool to represent an end product, the opportunity to consider how we utilise various models is revealed. For example, taking a Cuisenaire rod or pattern block (2D shape) and asking, '*If this is ... [1-half; 1-quarter; 1 and a half], what is 1?*' is a common task teachers currently employ in fraction instruction (Clarke et al., 2011). However, the power in such models is when they are used as a vehicle to discuss *how* and *why* different combinations of rods or blocks may represent equivalent fractions. From this perspective, the spatial attributes of the materials, such as how the orientation, symmetrical and proportional properties of the parts can be used to represent fraction magnitude, are of focus. When we consider using the materials in this way, it suggests spatial proportional reasoning is critical to how these materials can be used as vehicles to develop early partitioning understanding.

Spatial proportional reasoning is described as the ability to reason about non-symbolic, relative quantities (Möhring et al., 2015). Several studies have demonstrated children's capabilities to reason proportionally in activities that involve non-symbolic partitioning contexts-such as number line estimation tasks (see Gunderson & Hildebrand, 2021; Möhring et al., 2015). Spinillo and Bryant (1991) conducted an experiment where 6–7-year-old children were asked to compare a referent rectangle that was partly blue and white (typically half and half; three-quarters and quarter combinations of the colours) to two other larger rectangles, one of which had the same proportion of blue and white. The results revealed children's sensitivity to the 'half boundary', which children consistently used as a visual benchmark to reason proportionally about the size of the different coloured parts. Moreover, some of the rectangles were presented in different orientations to the referent image (e.g., horizontal versus vertical partitioning of the colours). This suggests that spatial visualisation was also a useful skill for children to draw on in this context. Spatial visualisation can be defined as the ability to mentally transform or manipulate spatial properties of an object or image (Lowrie et al., 2016). In this study, the results imply that the children also utilised spatial visualisation to imagine 'rearranging' the coloured parts to make comparisons between the rectangles. It is important to note that the children in that study demonstrated little understanding of symbolic notation relating to fractions. Thus, these findings suggest a strong connection between children's conceptualisation of what it means to partition and name a fraction as half through utilising spatial proportional reasoning and spatial visualisation in continuous contexts.

Discrete materials are also critical for the development of partitioning (Siemon, 2003) and can support children's early fraction understandings by emphasising their spatial attributes. Typical discrete materials include counters and small blocks to explore how sets can be shared fairly. Several studies illustrate that children create different structures and arrangements of objects to determine the equality of shares in a discrete context (Confrey, 2012; Wilson et al., 2012). Such arrangements included arrays or visual patterns (including symmetry and congruence of the arrangement of the objects) to justify the fair shares. Therefore, even in contexts that may imply a counting strategy is required, children will use a range of 'non-numerical' based strategies that help them reason about the magnitude and relationship of the shares created. Using arrays or patterns to create and distribute fair shares from a spatial reasoning lens implies that children use spatial structures in early partitioning experiences. Mulligan and Mitchelmore (2009) describe spatial structure as an awareness of mathematical relationships that are supported by spatial patterns and arrangements. Literature on spatial structure and early fraction understanding is limited. However, several studies have found that structure and pattern promote a multiplicative understanding of unit structures in measurement and whole number contexts (see Battista & Clements, 1996; Mulligan & Mitchelmore, 2009). Given the multiplicative relationship that exists between unitising and partitioning in the domain of fractions, spatial structuring appears to be an important construct to leverage in the teaching and learning of fractions—particularly in discrete contexts.

In summary, several spatial reasoning constructs have been identified as important in the development of children's early partitioning experiences. Yet, no studies to date have explored this connection through a pedagogical approach in the early years of school. The present study investigated this issue, and the following question frames this paper:

- How does spatial reasoning help young children develop a flexible understanding of partitioning?

## Research Design

This doctoral study employed a Design-Based Research (DBR) methodology to iteratively explore a teaching intervention that employed a spatial reasoning approach to teaching and learning fractions in the early years of primary school. DBR is an important methodology to the mathematics education research community because it enables a dual focus on (a) designing innovative forms of instruction to explore children's processes of learning and (b) refining local instruction theories through iterative teaching experiments for wider theoretical and practical application (Prediger et al., 2015). This paper will report on a subset of data from two teaching experiments conducted as part of this study. Data from two junior primary classes from two public schools in regional South Australia (referred to as Class B and Class C). Forty-four children aged 6–7 years participated in this component of the study. Furthermore, the classroom teacher of each class acted as an additional researcher in this project.

### Instruments

The intervention program comprised 13, 60-minute lessons taught by the researcher. The lessons were designed for children to explore continuous and discrete contexts, focusing on developing an understanding of the inverse property of partitioning through providing contexts that emphasised spatial reasoning strategies. Examples of activities from this intervention will be provided in the results and discussion section.

## Data Collection and Analysis

The researcher collected and thematically analysed work samples, anecdotal notes, and observational data from each lesson during the two teaching experiments. Consistent with DBR, the classroom teacher acted as an additional researcher by collating their observations and anecdotal notes throughout each lesson for comparison with the researcher. Thematic analysis was employed to analyse children's thinking and behaviour throughout the intervention program. Thematic analysis provides researchers with a method for identifying patterns of meaning within the data that the researcher deems important concerning the research questions (Braun & Clarke, 2013).

# Results and Discussion

The results of this paper are organised into three themes, which represent how spatial reasoning was integral to the development of the various partitioning ideas in discrete and continuous contexts.

## Creating Fair Shares

The first theme that illustrates children's partitioning behaviours in this study is their ability to create fair shares. The first activity for discussion asked the children to collect 12 plastic counters that represented 'cookies' and explore how they could share them fairly between different groups of friends (as discrete collections). In the introduction of the activity, the researcher used six counters to model a 3 x 2 array to the class and asked how many counters there were and how they knew. Specifically, the children were asked to look at the way the counters were arranged and if that helped them describe how many counters were there. This was intended to elicit a discussion about how they might describe these parts and the share of cookies each person receives based on the structure and arrangement of the counters. Several children could articulate that they could see that '*six is three equal columns*; *three as a row is half of six*; *six can be shared fairly between three people*, *there are two equal rows (shares) of three cookies'.* The children were asked to think about how they might arrange their counters to help them describe and 'see' how many ways they could share 12 'cookies' fairly.

During this task, one group of children made three groups, distributing one counter to each group in a square-like arrangement (see Figure 1).

**Figure 1**

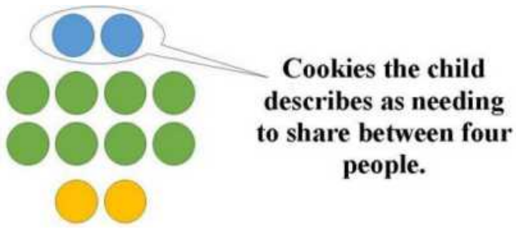
*Children's Representation of Sharing 12 Cookies Between Three People*

The children discussed how many counters there were in each group when sharing with three friends. Child 45 explained, "There are four, four and four because I can see a square in a group of four counters; my [12] counters make three squares!" This representation and description demonstrated how they utilised the spatial structure of the shares within the whole set to justify equal parts. That is, their creation of squares as a unit was created through the processes of partitioning, in this context, dealing one cookie to each 'square' at a time to exhaust the collection. However, their description of the 12 counters representing three shares of four indicates they unitised twelve as three–fours through the geometrical structure the child imposed on the shares. After this group experimented with sharing 12 cookies between four, five and six friends, Child 41 also stated that "when you make patterns with the shares, it's easier to tell if they are all the same or not, even when they get smaller because there's more people". This indicates that the spatial structure the children utilised within their created models was connected to their understanding of the inverse relationship generated through fair sharing.

In an extension of the cookie-sharing activity, the children were asked if they could share 12 cookies between 8 people fairly and how much each person would receive. A range of materials (paper circles, plastic counters) were provided for the children to work with. Although this was challenging for many children, three children worked together to make the following representation using the counters and presented it to the class (see Figure 2).

**Figure 2**

*Child 57's Representation*



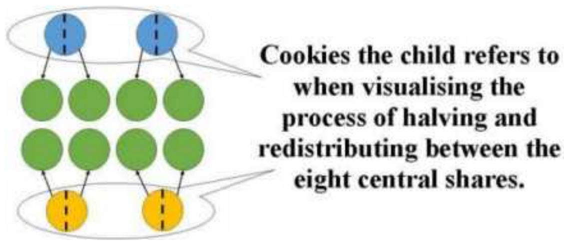

Child 57 explained their representation:

Child 57: We knew that there would be one cookie for each person, so we decided to line them up like this so you can see the four and four rows [gesturing horizontal movements with hand to indicate the central two rows of four]. This left four over, but to work it out it's easy because these cookies [referring to the two counters at the top of Figure 2] have to be shared between four people (Figure 3).

Child 57: This one cookie [referring to one of the two cookies at the bottom of Figure 3] also gets broken in half and shared to these two people and same with this one [gestures halving the two top cookies].

**Figure 3**

*Child 57's Interpretation of Creating Fair Shares*




This explanation highlights how the children imposed a spatial structure again to help them conceptualise the shares. However, they also indicated they used spatial visualisation to imagine partitioning the remaining four cookies in half to create equal shares of the collection, as the plastic counters were not able to be physically partitioned. This is a complex problem for children at this age, as mixed fractions are not typically considered appropriate for teaching in the early years and are not reflected in the current curriculum standards, at least in Australia.

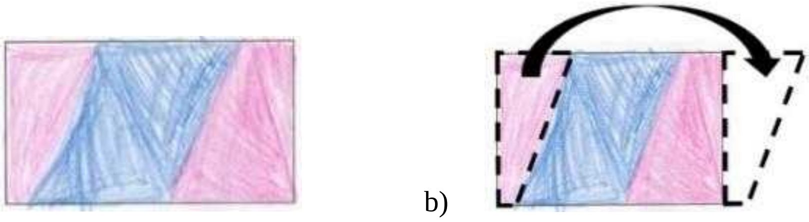
## Justifying Equal Shares

The second theme emerging from this study that highlights the connection between spatial reasoning and partitioning was children's ability to justify that they had created equal shares. The following activity was based on the context of designing and comparing tablecloths. The children were provided with blank paper rectangles and asked if they could design a tablecloth that met a specific brief (e.g., a design that had equal parts of blue and green, a design that was more purple than orange etc.) Some children chose to cut and fold their rectangles while others drew and coloured regions to represent their tablecloth designs.

Child 67 created the following design (Figure 4a) and stated that they had created equal parts of pink and purple on their tablecloth, that is, their tablecloth would be half pink and half purple. When asked how they knew the parts were equal, they stated that if they moved the left-hand pink section to join the pink region on the right-hand side, it would "look the same size as the purple part" (see Figure 4b).

**Figure 4**

*Child 67's Tablecloth and Interpretation of Their Visualisation Process*



a) b)

This child's explanation demonstrated a complex visualisation process to justify that their parts were equal. The interpretation of this child engaging in spatial visualisation was supported by their use of spatial transformation vocabulary; that is, words indicating the child was mentally changing parts of the image in some way—such as move and slide—were captured in their descriptions. This child also indicated a sensitivity to the 'half boundary', consistent with Spinillo and Bryant's (1991) study, by noticing the symmetrical relationship of the parts after (mentally) manipulating the tablecloth. Spatial visualisation was also evident in Child 30's discussion about justifying their equal parts "You can have something that has lines all over it, and all different shapes, but it's still a whole, and you can still make a half or a fourth if you look inside these patterns and move them in your head."

These examples represent how many of the children were engaging in spatial visualisation, which allowed them to mentally move and transform parts of the tablecloth to make a judgement on the relative proportions of different colours. These findings represent early understandings of how fractions can be generated (partitioning), named (unitising), and renamed (quantitative equivalence), utilising spatial visualisation and spatial proportional reasoning.

## Justifying Proportionally Equal Shares

The final theme exemplifying children's partitioning development in this study was their use of spatial proportional reasoning when comparing and justifying fractions between unlike wholes. An activity that exemplified this theme was based on the context of fictional town maps (approximately 1.5m x 2m). The children were provided with a set of clues for where members of the public had seen dinosaurs (that had escaped their enclosure). The children had to use the clues to work out where the dinosaurs were hiding. The clues involved statements like, "The T-Rex was spotted halfway between the fountain and the bike path; …two-thirds along the train track, etc.). The children were not allowed to use any formal measurement tools as such, as the focus was on them considering the spatial elements of the map's images.

The second part of the task required children to draw scaled representations of their pathways in their workbooks. Child 32 explained to the classroom teacher that the representation of their paths were different lengths, "so half is going to look different on each, but still in the middle [of each path]". Teacher B also recorded that the child suggested that for the curved path, they had to take into account the impact of the curves in relation to the length of the whole path. In other words, the child could not just consider halfway between the destinations 'as the crow flies' but had to visualise the length of the entire pathway in relation to its orientation to determine the equal parts.

**Figure 5**

*Child 45's Work Sample*

Child 45 demonstrated a spatial proportional awareness of half in their explanation of "my lines [pathways] are all crazy, but there's still two same-size parts," referring to each half as proportionally equal to its relevant whole (Figure 5). With this explanation, the child used a gesture that suggested they were visualising straightening out the 2-halves of the curvy path at the top of their picture like it was a piece of string they were flattening out with the palms of their hands. This example also illustrates the close connection between spatial visualisation and spatial proportional reasoning in the creation and justification of equal parts. Finally, Child 32 concluded this activity with the following point in a whole class discussion:

> I could imagine that I can walk from here to far, far away, and its only half to where I'm going—like Adelaide or something. But I could walk from here to that table, and it's halfway of this room. You have to think about what the end is to know how big you've walked.

This is evidence that spatial visualisation and spatial proportional reasoning assisted the children in developing flexible understandings of partitioning. The ideas and understanding the children expressed in these statements are also consistent with Spinillo and Bryant's (1991) description of the 'half boundary' idea, which they argue young children utilise to judge the magnitude of parts within an object. Through emphasising spatial visualisation and spatial proportional reasoning, the children demonstrated the ability to justify the equality of the parts created, rather than seeing 'half' (or other fractions) as a quantity that is fixed to a particular object. This evidence demonstrates that utilising spatial reasoning is a powerful vehicle for developing early partitioning generalisations, that is, the inverse property of fractions. This is a critical finding as, according to the current literature, anticipating the outcome of partitioning and recognising the multiplicative foundations of fractions are ideas that many older children fail to comprehend (Callingham & Siemon, 2021).

## Conclusion and Implications

This study demonstrates that the young children in this study were very capable of exploring early multiplicative and proportional foundations of partitioning through spatial contexts. That is, spatial reasoning enabled children to visualise the outcome of performing different partitions and determine the relationships between the part size and number of parts generated of like and unlike quantities. Furthermore, children's understanding of partitioning and the tightly connected concepts of unitising and equivalence was evident through focusing on structural and proportional comparisons of the objects and sets of objects they were partitioning. This paper is intended to be used as an opportunity for teachers and researchers to consider young children's capabilities for exploring the foundations for fractions, as the tasks presented in this study are not unlike activities and materials that are commonly used in primary classrooms for the teaching and learning of early fraction understanding. However, an implication of this study suggests that it is the way spatial reasoning was employed pedagogically that enabled children to make these connections about the foundations of partitioning, meaning teachers need to be supported on how to employ such an approach to provide rich opportunities for their students, whilst still meeting curriculum requirements. The evidence presented in this study is based on a modest sample, and therefore, more research is needed to explore the impact such an approach may have in the early years of schooling and beyond. Nevertheless, this study provides strong support for how spatial reasoning can be used as a powerful teaching and learning tool for developing flexible and sophisticated understandings of partitioning.

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