Secondary Mathematics Teachers' Mathematical Competence

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Whilst educational goals in recent years for mathematics education are foregrounded the development of mathematical competencies, little is known about mathematics teachers' competencies. In this study, a group of practising teachers were asked to solve an algebra problem, and their solutions were analysed to determine the competencies apparent (or, equally importantly, absent) in these solutions. The most demonstrated competencies were *devising strategies* and *mathematising*, whilst *communication* and *reasoning* were mostly missed. The research provides a detailed methodology on how mathematical competencies can be assessed in the context of problem solving.

In recent decades, efforts in mathematics education reform and development have been directed to mathematical competencies. Many countries have introduced new standards influenced by mathematical competencies. Denmark, for example, has competency descriptions of its mathematics programmes all the way from primary school to tertiary level, and in teacher training (Jankvist et al., 2022). In Australia, the notion of mathematical competencies has become more explicit and established in the mathematics curriculum, by implementing a mathematical proficiencies framework (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2022). A shared understanding within the mathematics education community is that students' mathematical competencies are affected by their teachers' competencies (Jankvist et al., 2022). To be able to shed some light on this potential impact, we need to know what mathematical competencies teachers have. Extending on our earlier works with secondary mathematics teachers, in this paper, we present a study that examines mathematical competencies evident (or not) in a sample of secondary mathematics teachers' solutions to an open-ended algebra problem that is based on the idea of consecutive numbers. We aim to answer the research questions: (1) *What can be said about the teachers' mathematical competence for the given problem in terms of their ability to activate the mathematical competencies and arrive at a desirable answer?* and (2) *How (if at all) is the teachers' performance in the problem impacted by activation of the mathematical competencies?*

Mathematical Competencies

Broadly, mathematical competence is "someone's insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to a given situation" (Niss & Højgaard, 2019, p. 12). Mathematical competence consists of six distinct but mutually related mathematical competencies: *Communication*; *Devising strategies*; *Representation*; *Mathematising*; *Symbols and formalism*; *Reasoning and argument* (see Turner et al., 2015). The *communication* competency involves two aspects: *receptive* and *expressive*. The *receptive* aspect includes reading, deciphering, and understanding mathematical statements and data. The *expressive* aspect involves explaining, presenting, and debating mathematical concepts. *Devising strategies* is creating and executing a mathematical strategy to solve problems that arise from the task or context. *Representation* is creating or utilising representations of mathematical entities or relationships. These representations can be equations, formulas, graphs, tables, diagrams, or textual descriptions. *Mathematising* is converting real-world

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problems into mathematical ones. It also involves interpreting mathematical entities or data in the context of the modelled situation. *Symbols and formalism* is understanding, manipulating, and utilising symbolic expressions. It also involves using constructs based on definitions, rules, conventions and formal systems. Finally, *reasoning and argument* is logical thinking processes that explore and connect problem elements to draw conclusions from them. It also involves checking a given justification or providing a justification (Niss & Højgaard, 2019).

Previous Research on Mathematical Competencies

Most relevant to the current study, Büchele and Feudel (2023) carried out a study on data from a non-standardised mathematics entry test taken by 3076 economics students divided into different cohorts from 2012 to 2019. The authors used regression analyses to examine the changes in students' mathematical competencies. The students' ability to carry out symbolic calculations decreased over the years. However, their performance increased in some test questions focusing on other process competencies like *reasoning*, *mathematising*, or *using different representations*. The authors concluded that a curriculum reform emphasising these competencies was likely to have had the desired effect of improving students' abilities in the mathematical process. Albano and Pierri (2014) introduced a role-play activity designed to foster conceptual understanding of mathematics for first-year engineering students. The findings showed that the quality of the questions posed by the students highlighted a shift from the instrumental approach they were used to towards a relational one. This suggests that the role-play activity was successful in fostering a deeper, more conceptual understanding of mathematics among the students. Wess and Greefrath (2019) studied the professionalisation of preservice teachers through reflective practice when they were training for the mathematical modelling competency. The authors designed and implemented a teaching laboratory, which was a learning environment that simulated authentic classroom situations and allowed the participants to practice and reflect on their teaching skills. The teaching laboratory had a positive impact on the participants' task competency, as well as their self-efficacy and enthusiasm for teaching mathematical modelling. In none of these studies have the competencies been investigated in the context of practising schoolteachers, nor has it been shown what relationship exists between the various competencies and success in problem solving.

The Study

The data for this study comes from a workshop delivered by Hatisaru, in 2022, at an annual mathematics teacher conference in Western Australia that aimed to support lower secondary mathematics teachers' pedagogical content knowledge for teaching algebra. The teachers were teaching mathematics to students in years 7 to 12 (aged 12 to 18), and most were qualified mathematics teachers, with 16 out of 22 having at least a mathematics minor, and all others having completed some tertiary mathematics units. Each participant was provided with three Reflection Forms, each with an algebra word problem presented on it and the prompt: *Think of and explain as many different possible solutions to the problem as you can. Name the solutions as Solution A, Solution B, Solution C and so on*.

The participants completed this task in 20–25 minutes. Out of the 42 teachers, 22 of them gave consent for their responses to contribute to this research. They were assigned a code from P1 to P22 to protect their anonymity. In this paper, we focus on participants' solutions to one of these problems, referred to as *three consecutive numbers*: *Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two. Now do it with another three consecutive numbers. Can you explain it with numbers? Can you use algebra to explain it?* (Kieran, 1992). Here, we describe the mathematical content of this problem for each competency presented earlier.

Communication: The text is fairly straightforward to interpret, involving some reasonably low-level technical language (e.g., 'consecutive', 'square', 'subtract', 'product'). One must interpret the text (receptive component) as instructions to do various things—starting with deciding what numbers to work with, but also interpreting and dealing with the two somewhat ambiguous, or at least tentative 'can you…?' questions. The expressive or constructive component consists of explaining the process of their investigation, presenting a hypothesis, and arguing the validity of their findings. *Devising strategies*: Several steps are involved, starting with negotiating a string of instructions to be applied to three numbers chosen, applying the instructions again to another set of three numbers, before hopefully arriving at an 'explanation' of the observed result. The instructions are then to be applied again in a generalised setting using algebra, together with a generalised solution, and an explanation drawn/devised. One needs to keep the goal in mind, work out what meeting the goal might look like, monitor their progress, and move on from stage to stage. *Representation*: The heart of the problem lies in finding a way to express the context and text instructions in a mathematical form. The need to represent the task elements with numbers, arithmetic operations, and letters (algebraic entities) is fairly obvious, so only a very low-level representation demand is present. There is no given mathematical representation to interpret; the 'devise' aspect of representation fits better under mathematisation.

Mathematising: When one moves beyond the numeric exploration that falls directly out of the interpretation of the problem statement, to an algebraic representation, one needs to mathematise the phrase 'three consecutive numbers'. This can be done in a number of possible mathematical formulations, for example $a - 1, a, a + 1$; or $n, n + 1, n + 2$; or as $n - 2, n - 1, n$. Each of these leads to slightly different algebraic processing steps once the phrases 'square the middle number', 'subtract from it' and 'the product of the other two' have been mathematised. We decided to characterise these demands under mathematisation, because one must come up with a mathematical formulation of essentially real-world (extra-mathematical) context elements. Finally, interpreting the mathematical result ('Square − Product = 1') in the real-world context (i.e., in relation to any three consecutive numbers) is a 'de-mathematisation' step that is rather important. *Symbols and formalism*: One needs to apply their formal mathematical knowledge to carry out the arithmetic calculations (multiplication and subtraction), and then perform algebraic manipulation once that has been arranged (square an expression, multiply two expressions, and find the difference, in the specified direction), leading to the mathematical result 'Square − Product = 1'. *Reasoning and argument*: Reasoning and argumentation are needed to reflect on the numerical results found, link these to the generalised algebraic result, and to conceptualise an argument that expresses the finding that the difference found at the subtract step is always a number 1 less than the square.

Data Analysis Approach

Among 22 teachers, one teacher did not respond to the problem; 18 of the remaining teachers generated one solution and 3 of them gave two solutions. For teachers with more than one solution, we decided to only analyse their best solution. We first analysed these 21 solutions according to the marking scheme presented in Table 1. To find out what mathematical competencies were manifested in these solutions, next, we analysed the solutions according to the scheme presented in Table 2 (for examples, see Figure 1). The codes 'evident' and 'absent' were used where the relevant competency was apparent or not in the respective solution. As none of the participants attempted to explain the findings, *the constructive aspect of communication* was not coded. Finally, we compared differences in the observed mathematical competencies by teachers showing stronger performance in the problem and those showing weaker performance. A strong performance is defined as a solution receiving a score of 4, while a weaker performance is defined as a solution receiving a score of 3 or lower.

The data were analysed by two of the authors of this paper (Hatisaru and Richardson). To establish interrater reliability, they independently coded all the participant solutions. Their agreement for the coding of participants' solutions to the problem was approximately 86% and for the coding of competencies evident in these solutions was approximately 81%. Any differences or disagreements were resolved through discussion.

Table 1

Marking Scheme Used to Assess Participant Solutions to Three Consecutive Numbers

Score	Observed solution
5	Solution demonstrates correct and complete interpretation; correct mathematical formulation, result 'Square − Product = 1' is found together with explanation connecting the algebra to the original number context
4	Solution demonstrates correct and complete interpretation; correct mathematical formulation, result 'Square − Product = 1' is found but without a clear statement linking the result to the original number context
3	Solution demonstrates mostly correct interpretation; formulation correct, but with sense of difference reversed ('Product − Square = −1'); there is not a clear statement linking the result to the original number context
2	Solution demonstrates correct interpretation applied to numeric examples with the result of 'Square − Product = 1' without algebraic formulation being correctly found and/or correctly applied
1	Solution demonstrates mostly correct interpretation applied to numeric examples with the result of 'Product − Square= −1' without algebraic formulation being correctly found and/or correctly applied
0	Solution demonstrates no progress with mathematical formulation of problem situation

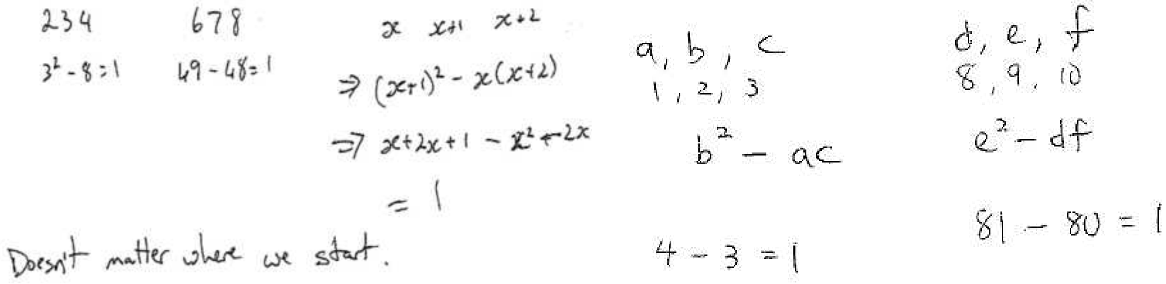
Table 2

Competencies for Solving Three Consecutive Numbers

Competency	Observed behaviour
Communication (COM)	The text is interpreted correctly; the terms 'consecutive', 'square', 'subtract' and 'product' are understood
Devising strategies (STR)	There is a strategy involving adhering to the steps outlined in the question statement. At least two steps are involved: applying the situation to numbers and to a generalised setting
Representation (REP)	The numerical and algebraic expressions are correct representations of the intent expressed in the problem statement
Mathematising (MAT)	The phrases are mathematised and the formulation of the problem situation 'Square–Product' is found
Symbols and formalism (SYF)	The formal knowledge needed to process both the numerical and the algebraic steps is evident. Algebraic manipulations are carried out correctly
Reasoning and argument (RES)	Numerical results are linked to the generalised algebraic result, a conclusion is made, and the mathematical result (i.e., 'Square − Product = 1') is connected to the 'real-word' result (i.e., any three consecutive numbers)

Figure 1

Example Solutions for P5 (Left) and P10 (Right) Coded Against the Marking Schema and Competencies



P5's response was scored as '4' with the codes COM, STR, REP, MAT, SYF, and RES.

P10's response was scored as '2' with the codes COM, STR, and REP.

Findings

Out of the 21 solutions to three consecutive numbers, none of the solutions were rated at score '5' because none of them included an explanation connecting the algebra to the original number context that was given. Two solutions were incorrect and were scored as '0' (P11 and P15). Two solutions were scored as '1' or '2' because, although these participants were able to interpret and correctly explore the problem with specific numerical examples, they did (or could) not formulate the problem algebraically (P10 and P16) or represent three consecutive numbers algebraically. Four were scored as '3' since they were able to formulate and devise the desired result algebraically, other than getting the order of Square − Product the wrong way around (P4, P6, P8, and P22). The remaining 14 solutions were scored as '4'; these responses presented the correct mathematical formulation (i.e., Square − Product = 1), but without an explanation. These findings suggest that many of the participants got to the essence of the problem, achieving differing levels of performance from scores '4' to '0'. Consideration of possible alternative approaches to the problem was happened in only three cases.

Figure 2 captures the frequency of mathematical competencies, apparent or not, in the solutions. Although participants varied in their use or activation of these competencies, patterns seemed to emerge. The mostly missed competency was *reasoning and argument*, followed by the *communication* and *symbols and formalism* competencies. In three solutions, all *devising strategies*, *representation*, and *mathematising* were absent. For example, in P4's solution in Figure 3 (left), a lack in the *communication* and *symbols and formalism* competencies is evident because P4 did not follow the instruction of implementing the procedure on numerical examples. Doing this might have given them a hint as to what they were looking to see from the algebraic consideration. Also, they did not order the 'Square–Product' correctly. Moreover, P4 lacked the algebra skills to simplify their final expression to -1.

The mathematical competencies in the participants' solutions who showed stronger (n = 13) and who showed weaker (n = 8) performance was compared (Figure 4). This comparison has revealed that all 13 participants who performed at 'score 4' level activated almost all of the first five competencies in solving the given problem, with one not activating the *symbols and formalism* competency. Conversely, the 8 weaker participants, i.e., those who performed at 'score 3' or lower, did not activate one (P6, P8, P22), three (P4, P15), five (P10), or all six (P11, P16) of the relevant competencies. This finding shows that activation of these competencies impacts on one's performance in solving a mathematics problem.

Figure 2

Distribution of Mathematical Competencies by the two Codes

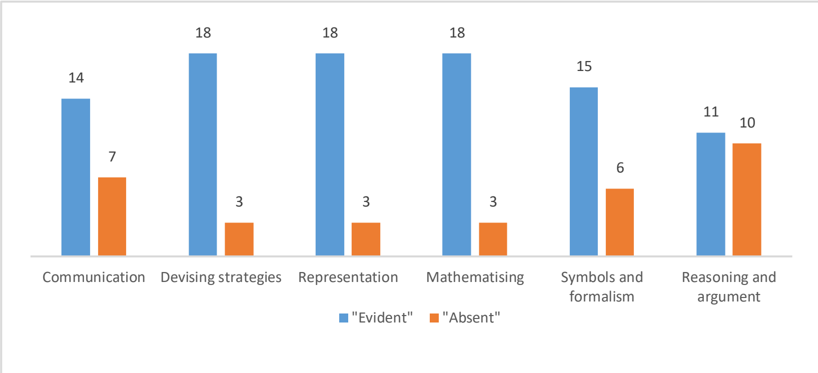
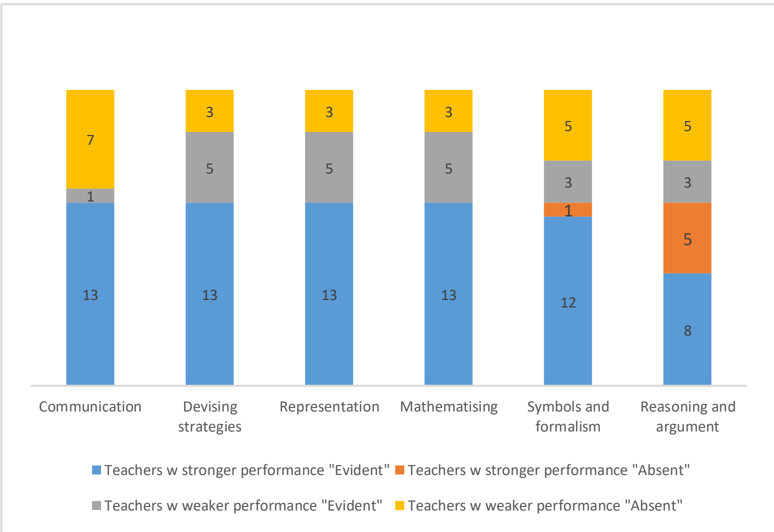


Figure 3

Solutions of P4 (Left) and P11 (Right) to Three Consecutive Numbers

Figure 4

Distribution of Participants with Stronger (n = *13) or Weaker* (n = *8) Performance by the two Codes*



Although the mathematical competencies are distinct, they are utilised in an interconnected way when solving problems. As a result, it is not always possible to accurately assess a specific mathematical competency using a presented solution, since a deficiency in one competency may eliminate the opportunity for another competency to be demonstrated. For example, P10 whose solution is presented in Figure 1 (right), only understood the question up to the numeric example level. The fact that they could not mathematise the question in algebraic form meant that we do not know if they could have simplified the expression (i.e., if they could have demonstrated *symbols and formalism*). In this solution, *mathematising* and *devising strategies* seem linked, as one cannot devise a strategy without being able to mathematise the problem. P11 (Figure 3) knew to square the middle number and multiply the others but was not able to interpret the subtraction part. They were unable to represent three consecutive numbers mathematically (i.e., *mathematising*) which meant that the remaining competencies could not really be assessed. That said, they did correctly expand $(a–1) \times (a+1)$ which is an element of *symbols and formalism*, although not part of a coherent solution.

Discussion and Suggestions for Future Research

In this study, we have provided a perspective for understanding and assessing mathematical competencies in the school education context. By using the case of *three consecutive numbers*, we have developed the process of applying the six mathematical competencies to an algebraic problem and assessing individual solutions to that problem, deconstructing the skills which need to be demonstrated for each competency to be verified. Our associated marking schemes (Tables 1 and 2) can be adapted to different problems, providing the basis for initial teacher education and professional development initiatives for preservice and practising teachers to assess and develop their mathematical competencies. They can also be used for teachers to assess the competencies of their students in the classroom, or for individuals to self-reflect on their work and consider whether they have demonstrated all six competencies in presenting their solutions. The conceptualisation presented here can be used in future investigations to understand the extent to which teacher competencies impact on student competencies, and thereby to gain a deeper understanding of any skills gaps being shown in national tests.

We analysed the mathematical competencies apparent and absent in a selection of practising teacher responses to an algebra problem. The most conspicuous competency absent in the solutions of our participants was that of *expressive communication*, which none of the solutions was deemed to contain. The weaker solutions also almost exclusively lacked *receptive communication* as well—that is, the ability to successfully interpret the instructions in the question. We consider teachers to be expert communicators: their main role is to explain complex concepts to beginners. The participants' solutions, however, showed almost no explanation, no commentary, conclusions, or interpretations. It is possible that the reason for this was contextual; that in the setting of a workshop teachers did not feel it was important to communicate their solutions as thoroughly as they would do in a classroom with students. Further investigation is required here to determine if mathematical *communication* skills are lacking in practising teachers, or if they are underused except in particular contexts. Connected to the competency of *expressive communication* is *reasoning and argument*, a skill that was the main competency lacking among both strong and weak solutions. This shows an inability to reflect on the results that were found, to generalise a result to a broader context, and to justify a result that was found. One could argue that the skill of *reasoning* is the most important competency, defining the very essence of mathematics: the ability to notice patterns and to question and justify the contexts in which they hold true. However, we see that even some successful solutions were missing this skill. As with the lack of *communication* skills, further investigation is required to determine the reasons for this within the context of the workshop. Mathematical competencies are not necessarily being learnt implicitly in initial teacher education. The research of Albano and Pierri (2014) and Wess and Greefrath (2019) show that these mathematical competencies can be developed in professional development sessions, both in terms of making the competencies explicit and emphasising their importance, and also in terms of improving teachers' skills and self-efficacy in these areas. We believe teachers of mathematics need to be supported to develop, use, and teach these competencies.

Like Büchele and Feudel's (2023) German participants, our Australian participants struggled more with *symbolic manipulation and formalism* than they did with *strategy* and *mathematising*. These latter skills were evident in all but three solutions, while *symbolic manipulation* was problematic in six. Further data on participants' educational backgrounds and current teaching focus would be useful here to determine the reasons for this weakness. For example, a teacher may not have taught algebra for some time or may have done their teacher training a long time ago. However, it could also be argued that *three consecutive numbers* required only low-level algebraic skills which we would expect that all secondary school mathematics teachers would have, regardless of their teaching level or educational background. As Australian students' algebra abilities are sometimes weaker than in other areas, our study is likely to representative of a wider problem. While it is good to see more of the higher-level competencies present in our participants (*devising strategies*, *mathematising*), we showed that it is important to focus teacher training initiatives on accurately applying formal mathematical knowledge to close this skills gap. This is likely to aid in the competency of *reasoning and argument*, as arguments cannot progress and conclude unless the intermediate steps are correct.

We acknowledge that our study had a relatively small sample size, with only 22 teachers participating. While all were teaching at the secondary level, we do not know more specifically what teaching experience the participants had. For example, someone teaching Specialist Mathematics at Year 12 would be expected to have stronger mathematical competencies than someone teaching mainly Years 7–9. It would also be interesting to investigate whether the degree program studied had an effect on the strength of the participants' solutions; for example, whether having a mathematics major as compared with a minor resulted in stronger competencies. Mathematical competencies should be investigated in a wider variety of problems and contexts, investigating the extent to which the wording of the problem influences the competencies being demonstrated and if factors such as the time given to solve a problem also have an impact. Finally, *three consecutive numbers* explicitly asked participants to first investigate with numbers and then to explain with algebra. A future investigation might omit these specific instructions to find out if teachers would naturally start from specific cases and generalise to all numbers and investigate how best we can cultivate and motivate this skill among teachers.

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