Comprehending and Applying the First Isomorphism Theorem

Marios Ioannou
Larnaca College
maxioannou@elarnaca.com

This qualitative study aims to investigate novice undergraduate mathematics students' first encounter with the First Isomorphism Theorem, which is, more often than not, the pinnacle of a typical introductory course in Group Theory. Several studies have reported on the challenges that this mathematical result poses to inexperienced mathematicians, mostly due to the numerous prerequisite abstract concepts. For the analysis of student responses, there has been used the Commognitive Theoretical Framework. This study suggests that the major challenges are due to the comprehension of the notions of kernel, image and isomorphism, and the application of FIT as a routine in the context of proofs.

First Isomorphism Theorem (FIT) is usually the culminating point of an introductory course in Group Theory (Ioannou, 2012). FIT is a particularly important mathematical result in such educational context, since it is essential in Group Theory, and it is rather complex in relating numerous concepts in Abstract Algebra in general (Melhuish et al., 2023). This complexity is due to the difficulty novice undergraduate mathematics students have in comprehending concepts such as the quotient group, which is reportedly the most challenging concept to grasp in an introductory course in Group Theory (Ioannou, 2016), the cosets, which novice students are challenged due to their inability to visualise them (Ioannou & Iannone, 2011), and the group isomorphisms, together with the kernel and image (Ioannou, 2019). Therefore, this study aims to investigate undergraduate mathematics students' first encounter with FIT, and report on the major pedagogical challenges they face. For this purpose, there will be used the Commognitive Theoretical Framework (CTF), proposed by Sfard (2008). Presmeg (2016, p. 423) suggests it is a theoretical framework of unrealised potential, designed to consider not only issues of teaching and learning of mathematics per se, but to investigate "the entire fabric of human development and what it means to be human." It proves to be an astute tool for the comprehension of diverse aspects of mathematical learning, which although grounded on discrete foundational assumptions, can be integrated to give a more holistic view of the students' learning experience (Sfard, 2012).

Literature Review

Abstract Algebra in general, and Group Theory in particular, is one of the most challenging mathematical fields for novice undergraduate mathematics students. Nardi (2000) reports that the grasp of the newly introduced notion of group is often an arduous task for novice students and causes serious difficulties in the transition from the informal secondary education mathematics to the formalism of undergraduate mathematics. Ioannou (2018) reports that the notion of subgroup, with the application of the Subgroup Test is considered another laborious mission to accomplish for novice students. In fact, novice students face numerous difficulties regarding both the comprehension and use of the notion of subgroup in the context of proofs.

The reported challenges students face in comprehending and applying group theoretic concepts are partly grounded on historical and epistemological factors, "The problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (expected to understand) the concepts today" (Robert and Schwarzenberger, 1991). Nowadays, the presentation of the fundamental group theoretic concepts including cosets, quotient groups, is "historically decontextualized" (Nardi, 2000, p169), since historically the fundamental concepts of Group Theory were permutation and symmetry.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 295–302). Gold Coast: MERGA.

to students who are beginning to study (expected to understand) the concepts today" (Robert and Schwarzenberger, 1991). Nowadays, the presentation of the fundamental group theoretic concepts including cosets, quotient groups, is "historically decontextualized" (Nardi, 2000, p169), since historically the fundamental concepts of Group Theory were permutation and symmetry.

Furthermore, quotient groups and FIT are two of the most challenging topics in an introductory course in Group Theory (Melhuish, 2019). Mena-Lorca and Parraguez (2016) reported that students who used a more generalised theorem had more sophisticated understanding than students who relied exclusively on group theoretic notions. Rupnow (2021) suggests that mathematicians who have managed to link the meaning of homomorphism and quotient groups are much more effective in the application of FIT and the comprehension of proofs. Brenton and Edwards (2003) discovered that understanding quotient groups requires novice students to realise that a coset is both an element in a group, as well as a set by itself. In addition, Siebert and Williams (2003) further the significance of the notion of coset, by suggesting that there are three distinct interpretations, namely coset as a set, coset as element set combinations, and cosets as representative elements, that need to be embraced.

Finally, a significant aspect of mathematical learning and practice at university level is the production of rigorous, explicit and elegant proofs, especially in the context of pure mathematics. Weber (2001) associates student difficulty with Group Theory with the difficulty to construct proofs, "when left to their own devices, students usually fail to acquire optimal strategies for completing mathematical tasks and often acquire deficient ones" (p. 116). Alcock (2010) similarly points out that learning Group Theory is challenging because of the abstract nature of its concepts and because it involves reading and writing proofs involving various learning practices and beliefs.

Theoretical Framework

CTF is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis. It involves a number of different notions such as *metaphor*, *thinking*, *communication*, and *cognition* (Sfard, 2008). In mathematical discourse, objects are discursive constructs and form part of the discourse. Mathematics is an *autopoietic* system of discourse, namely "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p. 129). Moreover, CTF defines discursive characteristics of mathematics as the *word use*, *visual mediators*, *narratives*, and *routines*.

Mathematical discourse involves certain objects of different categories and characteristics. *Primary object* (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p. 169). *Simple discursive objects* (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization". *Compound discursive objects* (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary" (Sfard, 2008, p. 166). In this context, group, coset and quotient group are examples of compound d-objects.

Other notions within the CTF, important for this study, are the rules of discourse, namely the *object-level* and the *metalevel rules*. Object-level rules are defined as "narratives about the regularities in the behaviour of the objects of the discourse" (Sfard, 2008, p. 201). In other words, these are rules that are directly related to the definition of the various objects, e.g., group, subgroup, coset, etc. Metalevel rules "define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (Sfard, 2008, p. 201). In other words, metarules govern the process of proof of new (to novice students) mathematical results.

Sfard (2008) describes two distinct categories of learning, namely the *object-level* and the *metalevel learning*. "Object-level learning … expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse" (Sfard, 2008, p. 253). In addition, "metalevel learning, which involves changes in metarules of the discourse … is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008, p. 254).

Finally, an important concept for the purposes of this study is the *commognitive conflict*, which is defined as a "situation that arises when communication occurs across incommensurable discourses" (Sfard, 2008, p. 296). Commognitive conflict is considered "a gate to the new discourse rather than a barrier to communication, both the newcomer and the old-timers must be genuinely committed to overcoming the hurdle" (Sfard, 2008, p. 282).

Methodology

The present study is a ramification of a larger research project (Ioannou, 2012), which conducted a close examination of Year 2 undergraduate mathematics students' learning experience in their first encounter with Abstract Algebra. The course was taught in a research-intensive mathematics department in the United Kingdom. It was a mandatory course, and a total of 78 students attended it. The course was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a doctorate student in the mathematics department. All members of the teaching team were pure mathematicians. The course assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of the semester.

The gathered data included the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 members of staff's interviews (5 members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave 3 interviews each), student coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, which uses only the student solution artifacts, there have been analysed the data gathered from the thirteen volunteers. The interviews, which covered a wide spectrum of themes, were fully transcribed, and analysed with comments regarding the mood, voice tone, emotions and attitudes, or incidents of laughter, long pauses etc., following the principles of Grounded Theory, and leading to the "Annotated Interview Transcriptions", where the researcher highlighted certain phrases or even parts of the dialogues that were related to a particular theme. Furthermore, coursework and examination solutions were analysed in detail, after the data collection period, using the CTF, and mostly focusing on issues such as students' engagement with certain mathematical concepts, the use of mathematical vocabulary and symbolization, and the application of discursive rules.

Finally, all emerging ethical issues during the data collection and analysis, namely, issues of power, equal opportunities for participation, right to withdraw, procedures of complaint, confidentiality, anonymity, participant consent, sensitive issues in interviews, etc., were addressed accordingly.

Data Analysis

The First Isomorphism Theorem was introduced by the lecturer as follows: *Suppose* G, H *are groups, and* $\varphi: G \longrightarrow H$ *is a homomorphism. Then* ${}^{G}/_{ker\varphi} \cong im\varphi$. There were two mathematical tasks that required the use of the FIT, one in the coursework and one in the final examination, as shown below.

Figure 1

Coursework Exercise on the FIT

6. (i) Suppose H is a non-trivial subgroup of $\mathbb{Z}$, the group of integers under addition, and let d be the smallest natural number in H (- why is there such a thing?). Prove that $H = d\mathbb{Z}$.
(ii) Suppose $G = \langle g \rangle$ is a cyclic group (written multiplicatively). Define $\phi : \mathbb{Z} \to G$ by $\phi(n) = g^n$ (for $n \in \mathbb{Z}$). Prove that this is a homomorphism.
(iii) Using (i) and (ii) and the First Isomorphism Theorem, deduce that if G is a cyclic group then there exists an integer d such that $G \cong \mathbb{Z}/d\mathbb{Z}$.

Figure 2

Examination Exercise on the FIT

5. (i) Suppose G is a group.

(a) What does it mean to say that a subgroup N of G is a *normal* subgroup? If N is a normal subgroup of G, explain how to make the set G/N of left cosets of N in G into a group. **[3 marks]**

(b) State the First Isomorphism Theorem for groups, defining the terms *kernel* and *image* in your statement. **[4 marks]**

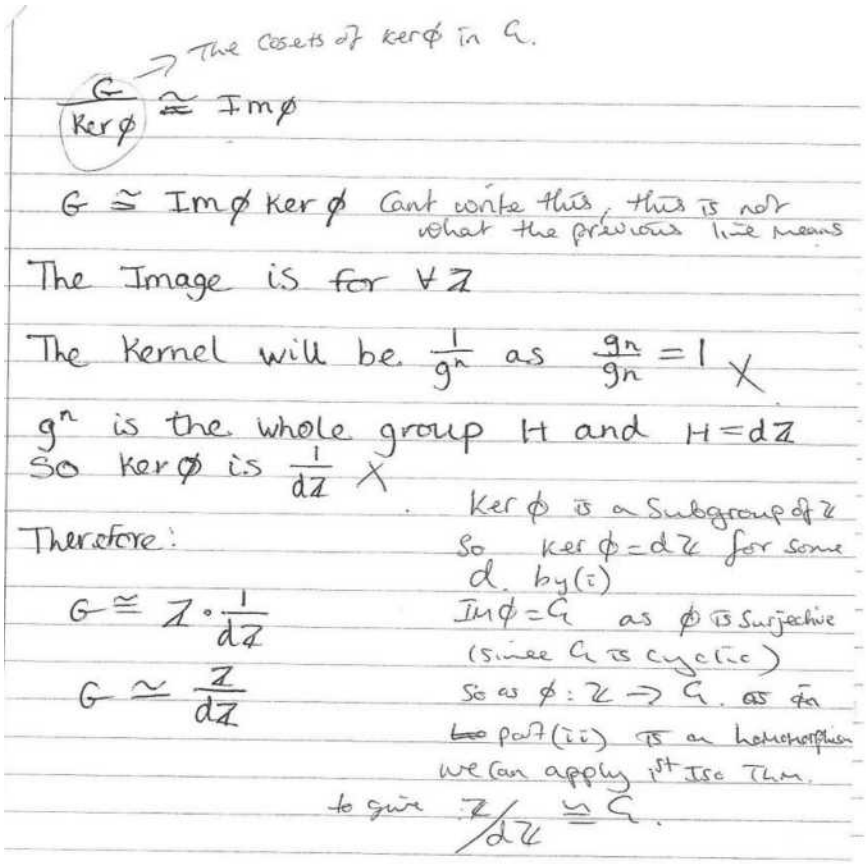
(c) Suppose H is a cyclic group. By defining a suitable homomorphism $\phi : (\mathbb{Z}, +) \to H$, or otherwise, prove that $H \cong \mathbb{Z}/m\mathbb{Z}$ for some $m \in \mathbb{Z}$. **[3 marks]**

Twelve out of thirteen (12/13) students' solutions indicated problematic engagement with FIT. There were strong indications of problematic application of the governing metarules of this routine by the majority of students. As mentioned before FIT is the pinnacle of an introductory course in Group Theory and students are required to overcome any object-level learning issues in order to be able to successfully apply the governing metarules that are required in the application of FIT as a routine. In what follows, we analyse a number of representative examples of the three types of errors that occurred in the student solutions.

Complete object-level learning of the involved d-objects, such as kernel, image and isomorphism, is of vital importance for successfully applying the FIT. For instance, as Figure 3 indicates, Student A's object-level learning of the notion of isomorphism appears to be incomplete and therefore he is still unable, to apply FIT successfully.

Figure 3

Student A's Coursework Solution


→ The cosets of $\ker\phi$ in G.

$\frac{G}{\ker\phi} \cong Im\phi$

$G \cong Im\phi \ker\phi$ Cant write this, this is not what the previous line means

The Image is for $\forall \mathbb{Z}$

The Kernel will be $\frac{1}{g^n}$ as $\frac{g^n}{g^n} = 1$ X

g^n is the whole group H and $H = d\mathbb{Z}$

So $\ker\phi$ is $\frac{1}{d\mathbb{Z}}$ X

Therefore:

$G \cong \mathbb{Z} \cdot \frac{1}{d\mathbb{Z}}$

$G \simeq \frac{\mathbb{Z}}{d\mathbb{Z}}$

$\ker\phi$ is a Subgroup of $\mathbb{Z}$

So $\ker\phi = d\mathbb{Z}$ for some d. by (i)

$Im\phi = G$ as ϕ is Surjective (since G is cyclic)

So as $\phi : \mathbb{Z} \to G$. as from part (ii) is an homomorphism we can apply 1st Iso Thm.

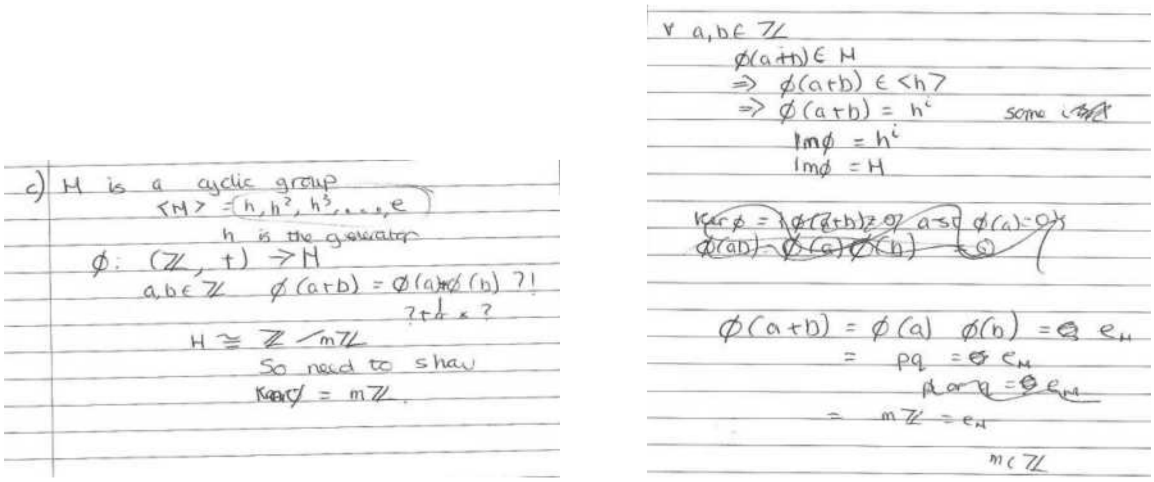
to give $\mathbb{Z}/d\mathbb{Z} \cong G$.

Here appears a typical example of a commognitive conflict. The notation $\frac{G}{ker\varphi} \cong Im\varphi$ is treated as an algebraic equation in which Student A has applied cross-multiplication, i.e., $G \cong Im\varphi Ker\varphi$. This is an erroneous metaphor from elementary algebra that indicates an incomplete object-level learning of the concepts of isomorphism, kernel and image, and therefore it reveals a typical situation which arises when communication occurs across incommensurable discourses, in this case elementary to abstract algebra. Student A has not realised that $G/ker\varphi$ is a mathematical structure and the symbol $\cong$ does not refer to equality relation but to bijective relation. His object-level and metalevel learning of the d-objects of kernel and image is also incomplete and consequently he is still not able to use FIT effectively. He has not realised that $im\varphi$ is a subgroup of H and that $ker\varphi$ is a normal subgroup of G.

Albeit asked to, Student B failed to properly state FIT in the final examination, since she did not provide the definitions for image and kernel. Possibly, failing to define kernel, image and stating FIT suggests that her object-level learning of kernel and image of a group is incomplete when applied in a more advanced context such as the FIT. The third part of her solution, involving the definition of a suitable homomorphism, reinforces the claim that she is still not able to effectively use the FIT. Her attempt to solve the last bit is fragmental and lacks linear reasoning and structure.

Figure 4

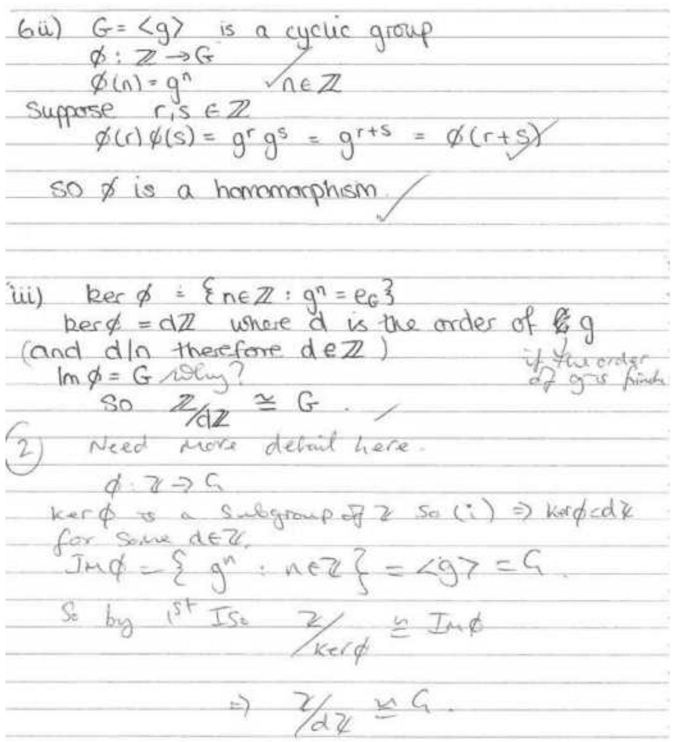
Student B's Examination Solution



A third type of errors regarding the application of FIT occurred Student C's coursework solution. Her attempt to solve this exercise has several problems, especially in the third part regarding the FIT. She correctly states that the image of the homomorphism is the group G itself, and therefore it is an isomorphism, but she does not mention anything about the kernel. This possibly suggests an incomplete object-level learning of the d-object of isomorphism, and in particular relating to the fact that one has to prove that a homomorphism needs to be both injective and surjective in order to be an isomorphism. Moreover, her solution indicates that she is not aware of the importance of the fact that the order of g is finite. Her narratives are not explicit, possibly indicating an incomplete metalevel learning of the involved routine as well as problematic application of the governing metarules. She does not seem to realise that the kernel in this case is $d\boldsymbol{Z}$, and the reasoning behind that.

Figure 5

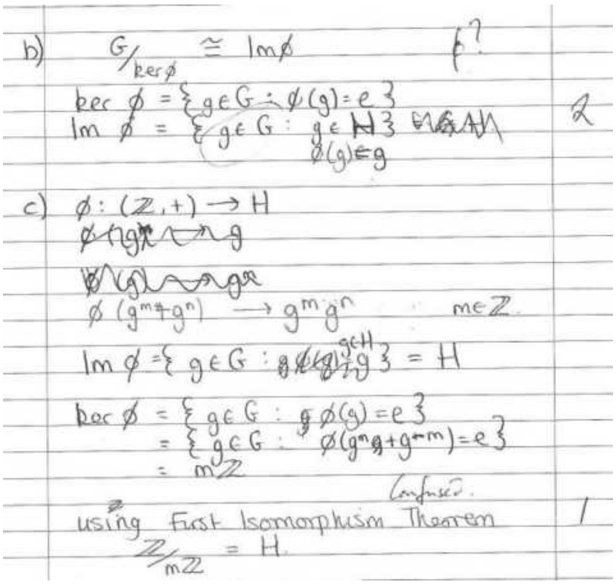
Student C's Coursework Solution



In Student C's examination solution, we can identify several problems. First of all, she has not properly stated FIT, but instead she wrote the mathematical expression $\frac{G}{Ker\varphi} \cong Im\varphi$ without any further explanation. The definition of image is problematic, since instead of writing $Im\varphi = \{\varphi(g): g \in G\}$ she stated $Im\varphi = \{g \in G: g \in H\}$. This probably indicates that she has an incomplete object-level learning of the definition of image and its elements. In addition, her narratives suggest that she is not fully aware yet of what she is writing mathematically. In part (c), her solution lacks explicitness, reflecting her problematic engagement with FIT, indicating incomplete metalevel learning of the particular routine, as well as imprecise application of 'norms' required for proving in this advanced mathematics course. The marker wrote the comment "Confused", which he rarely does in the examination solutions. Student C's performance in the exams indicates a decline regarding the application of FIT.

Figure 6

Student C's Examination Solution



Conclusion

First Isomorphism Theorem is, more often than not, the pinnacle of an introductory course in Group Theory. Its difficulty is due to several reasons. First, it is due to the fact that FIT relates numerous abstract theoretical concepts (Melhuish et al., 2023). These concepts include the quotient group (Author, 2016), which is the most challenging of all fundamental group theoretic concepts in an introductory level, the coset, which is often an obstacle due to students' difficulty to visualise it (Author & Iannone, 2011), and the group isomorphisms, together with the kernel and image (Author, 2019). The current study has analysed students' engagement with FIT.

The analysis has shown that the great majority of students have faced significant challenges with its application, with these challenges being of both object-level and metalevel nature. These challenges have been exemplified by using three representative examples which have repeatedly occurred in the solutions of many students. The first challenge was associated with the incomplete object-level learning of relevant mathematical concepts such as the kernel, image and group isomorphism. This challenge is relevant to the second one, which, according to the analysis, leads to metadiscursive level issues and the successful application of FIT, indicating that without full object-level learning of these notions it is almost impossible to apply the governing metarules successfully. Finally, the third challenge indicates a deeper difficulty with the notion of isomorphism, which emerges from the incomplete object-level learning of the notion of homomorphism, and the special characteristics that make homomorphism to be an isomorphism, namely being injective and surjective. These interesting results require further investigation in order to comprehend more deeply these challenges, and hopefully make Group Theory, and Abstract Algebra in general, more approachable to our students.

Acknowledgments

This study is part of a larger research project which was funded by the School of Education and Lifelong Learning of the University of East Anglia. The participation in the conference was possible due to a research fund provided by Larnaca College.

Ethics approval was granted by the University of East Anglia in 2008, and the participants gave informed consent.

References

Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. *Issues in Mathematics Education*, *16*, 63–91. https://doi.org/10.1090/cbmath/016/03

Brenton, L., & Edwards, T. G. (2003). Sets of sets: A cognitive obstacle. *The College Mathematics Journal*, *34*(1), 31–38. https://doi.org/10.1080/07468342.2003.11921982

Ioannou, M., & Iannone, P. (2011). Students' affective responses to the inability to visualise cosets. *Research in Mathematics Education*, *13*(1), 81–82. https://doi.org/10.1080/14794802.2011.550737

Ioannou, M. (2012). *Conceptual and learning issues in mathematics undergraduates' first encounter with group theory: A commognitive analysis* [Unpublished doctoral dissertation]. University of East Anglia.

Ioannou, M. (2016). Investigating undergraduate mathematics students' first encounter with quotient groups. *Mediterranean Journal for Research in Mathematics Education, 15*, 51–67.

Ioannou, M. (2018). Commognitive analysis of undergraduate mathematics students' first encounter with the subgroup test. *Mathematics Education Research Journal*, *30*(2), 117–142. https://doi.org/10.1007/s13394-017-0222-6

Ioannou, M. (2019, July). On the learning of group isomorphisms. In M. Graven, H. Venkat, A. A. Essien, & P. Vale (Eds), *Proceedings of the 43rd annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 400–407). PME.

Melhuish, K., Guajardo, L., Dawkins, P. C., Zolt, H., & Lew, K. (2023). The role of the partitioning and coset algorithm quotient group partial meanings in comprehending the First Isomorphism Theorem and its proof. *Educational Studies in Mathematics, 113*(3), 499–517. https://doi.org/10.1007/s10649-023-10207-2

Mena-Lorca, A., & Parraguez, A. M. M. (2016). Mental constructions for the group isomorphism theorem. *International Electronic Journal of Mathematics Education, 11*(2), 377–393. https://doi.org/10.29333/iejme/340

Nardi, E. (2000). Mathematics undergraduates' responses to semantic abbreviations, 'geometric' images and multi-level abstractions in Group Theory. *Educational Studies in Mathematics, 43*, 169–189.

Presmeg, N. (2016). Commognition as a lens for research. *Educational Studies in Mathematics, 91*(3), 423–430. https://doi.org/10.1007/s10649-015-9676-1

Robert, A., & Schwarzenberger, R. (1991). Research in teaching and learning mathematics at an advanced lever. D. Tall. (Ed.), *Advanced mathematical thinking* (pp. 127–139). Kluwer Academic Publishers.

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *The Journal of Mathematical Behavior, 62*, Article 100867. https://doi.org/10.1016/j.jmathb.2021.100867

Sfard, A. (2008). *Thinking as communicating: human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sfard, A. (2012). Introduction: Developing mathematical discourse—Some insights from communicational research. *International Journal of Educational Research*, *51–52*(3), 1–9. https://doi.org/10.1016/j.ijer.2011.12.013

Siebert, D., & Williams, S. R. (2003). Students' understanding of ZN. In N. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th International Conference for the Psychology of Mathematics Education* (Vol. 4, pp. 167–173). PME.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119.