# Development of Items to Assess Big Ideas of Equivalence and Proportionality

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Big Ideas can be seen as overarching concepts that occur in various mathematical topics and strands within a syllabus. Within our project on Big Ideas in School Mathematics, we developed instruments to measure two Big Ideas: Equivalence and Proportionality. The instruments we developed seek to assess students' ability to see these Big Ideas as common ideas connecting within and across topics, and their ability to apply these Big Ideas in solving problems. In this paper, we discuss the development of items for these instruments.

Big Ideas can be described as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole (Charles, 2005, p. 10)". It would be reasonable to describe Big Ideas as overarching ideas that cut across topics. This follows from NCTM (2000) which highlighted the importance for teachers to "understand the Big Ideas of mathematics and be able to represent topics as a coherent and connected enterprise" (p. 17). In the most recent mathematics syllabus in Singapore (MOE, 2018a & 2018b), a significant addition is the explicit introduction of the need to teach towards Big Ideas. It defines Big Ideas as key ideas that "bring coherence and show connections across different topics, strands and levels" (MOE, 2018a). While the concept of Big Ideas in Mathematics has received increasing attention over recent years (NCTM, 2000; Charles, 2005; MOE, 2018a & 2018b), there is little research so far on the assessment of Big Ideas in Mathematics. One possible reason may be the varied definitions of what a Big Idea in Mathematics is. For our study, we adopt Charles' (2005) definition of a Big Idea as stated earlier. Knowing whether students can make connections between numerous understandings can help teachers improve their pedagogies and teaching approaches. However, there is currently little research done in developing instruments that can assess the ability to see connections using Big Ideas. Hence, the ability of students in Singapore to see the connectedness across and within topics using Big Ideas is a guiding principle in our study. This paper presents the development of an instrument to measure the Big Ideas of Equivalence and Proportionality.

While there are studies on the assessment of Big Ideas like *Equivalence* (Warren & Cooper, 2009; Fyfe et al., 2018; Niemi et al., 2006), and *Proportionality* (Carpenter et al., 1999; Izzatin, 2020), assessing students' ability to connect across or within topics was not evident. To fill this gap, our instrument is developed with the focus on assessing students' ability to apply Equivalence and Proportionality thinking to solve mathematical tasks and to elicit students' ability to 'see' connections across these tasks using Big Ideas.

## Principles in Item Design

Our design of the items for the instruments is guided by two of the three characteristics of Big Ideas as detailed by Hsu et al. (2007): (a) connect different parts of the curriculum under the umbrella of Big Idea; and (b) be a basis for understanding other topics. To achieve a measurement of (a), a multi-part assessment item was developed. This multi-part structure is

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partly motivated by Niemi's (2006) *PowerSource* instrument which assesses students' ability to see Equivalence in algebra.

In this paper we focus on the assessment items measuring two Big Ideas: Equivalence and Proportionality. There are studies that explore and unpack Big Ideas in greater detail. For example, Fyfe et al. (2018), focused their study on the symbolic understanding of equivalence, i.e., the understanding of the equal sign, while Cook et al. (2022) provided three interpretations of equivalence, viz., Common Characteristics, Descriptive and Transformational. For Proportionality, Carpenter et al. (1999) identified four levels of proportional reasoning development, Özgün-Koca and Altay (2009) identified three types of proportional reasoning problems, and Izzatin (2020) identified five levels in the proportional reasoning process. Some studies went in depth into fine-grain details of Big Ideas such as proportional reasoning down to the detail of cross multiplication (Chin et al., 2022).

However, in the development of our items, we focus on tasks that require students to apply any forms or levels of Equivalence or Proportionality thinking while simultaneously teasing out their ability to see connections across tasks (Jahangeer et al., 2023). Thus, we define Equivalence as *a relationship that expresses the equality of two mathematical entities and realises the potential of an easier solution/understanding of one entity by converting it to the other entity*. We operationalise this definition to our Equivalence items by the statement: In this item, Entity 1 is equivalent to Entity 2, the equivalence of which usually enables the problem to be solved in an easier manner. (For example, 29 + 13 is equivalent to 30 + 12; the latter is easier to solve.)

We define Proportionality as "a relationship that expresses the direct variation of two mathematical entities and realises the potential of an easier calculation of one entity by knowing the value of the other entity". We operationalise this definition to our Proportionality items by the statement: In this item, Entity 1 is proportional to Entity 2, the proportionality of which usually enables one entity to be calculated from the other by multiplicative reasoning. (for example, if $y = 3x + 7$, then when $x$ increases by 7, $y$ will increase by $3 \times 7 = 21$).

We developed two separate instruments, one for each Big Idea. They are to be administered at two different levels: Primary (Grades 5 and 6); and Secondary (Grades 7 and 8). The items that we created to assess each Big Idea satisfy the operational definition of the Big Idea. Each item serves to assess students' ability to see the connections across the parts of the item through the Big Idea. The tasks of each item cover concepts either within or across topics in the Singapore school syllabus.

Example 1 shows a mathematics task that requires students to use Equivalence to solve:

> Example 1: Ali had $220, and Colin had $310. After each of them bought an identical pair of sneakers, Colin had thrice as much money as Ali had left. How much did the pair of sneakers cost?

Equivalence is established using the operational definition by stating that Entity 1 is "the difference in the money Ali and Colin had at the beginning" and Entity 2 is "the difference in the money Ali and Colin had at the end", and that Entity 1 is equivalent to Entity 2:

> Example 2 (Figure 1) is another task which shows how the operational definition ensures that it is valid for Equivalence. In this task, the shaded region (Entity 1) is equivalent to the numerical expression 1 + 2 + 3 + 4 + 5 + 6 + 7 (Entity 2). The shaded region is half of the rectangle, which can be easily calculated to be 7 × 8. Thus, the numerical expression is equivalently (7 × 8) ÷ 2.

For Proportionality, we consider Example 3 and Example 4 (Figure 2). In Example 3, the line shown is of the form $y = mx$, in which $y$ varies directly with $x$, i.e., $y$ is proportional to $x$. In Example 4, the line is of the general form $y = mx + c$. While $y$ is not proportional to $x$, the **change in $y$** is proportional to the **change in $x$**. Hence, Example 4 still satisfies the operational definition of Proportionality and is included as one of the mathematical tasks for the instrument on Proportionality.

**Figure 1**

*Example 2*

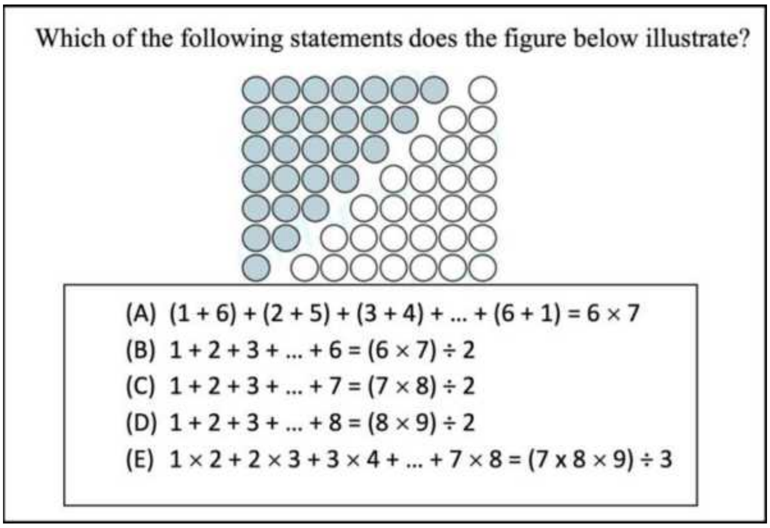
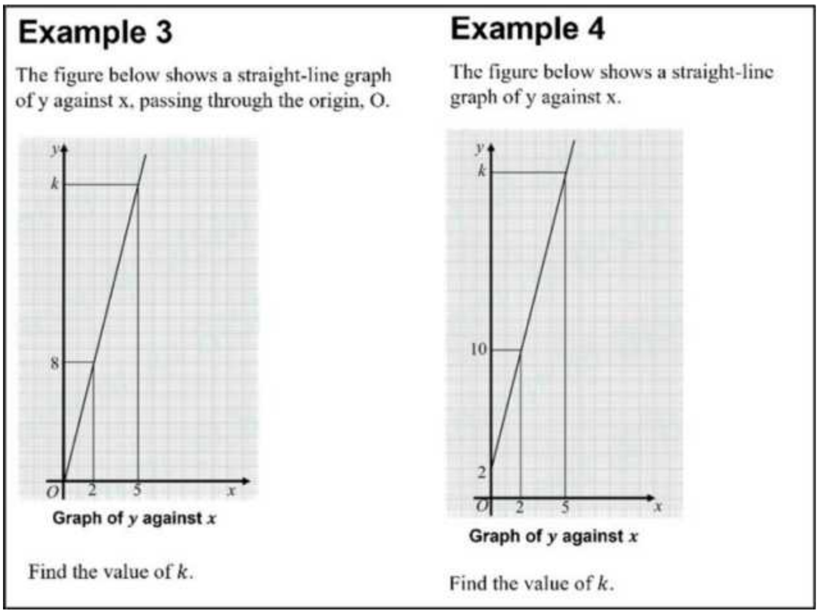



**Figure 2**

*Example 3 and Example 4*




## Item Design

A typical item (Figure 3 is one such example measuring the Big Idea of Equivalence) consists of four tasks and two sets of reflection questions. The item parts are presented in the following order: Task 1, Task 2, Task 3, Reflection A-1, Reflection A-2, Task 4, and Reflection B.

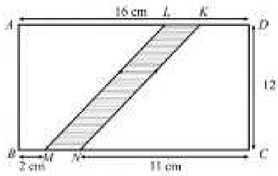
Recall that we define a Big Idea as an idea that is central to the learning of mathematics, one that brings both coherence and connections 'within' and 'across' topics. The task shown in Figure 3 is an example which seeks to elicit students' ability to 'see' connections 'across' topics. The functionality of the task to assess Big Idea thinking across topics is as described in the proceeding paragraphs.

In Task 1 shown in Figure 3, students are supposed to find the shaded area, which is the area of the parallelogram. The shaded area can be found by finding the difference between the two unshaded areas and the larger rectangle. Note that the two unshaded areas can be put together to form a smaller rectangle. The equivalence here is the area of the two trapeziums (Entity 1) and the area of the smaller rectangle formed by joining the two together (Entity 2). Students at this stage have not been introduced to finding the area of trapeziums and parallelograms as it is not within the syllabus requirements for Grades 5 and 6 in Singapore.

**Figure 3**

*A Complete Item for the 'Big Idea of Equivalence'*

**Task 1**
The below shows a rectangle ABCD. The lines KN and LM are parallel.




Find the value of the shaded area.

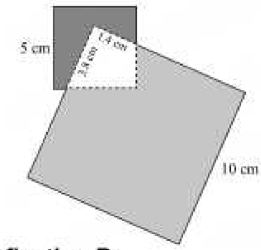
**Task 2**
Charles has 248 Pokemon cards and David has 88 Pokemon cards. They each gave away ¼ of their original total number of cards. At the end, Charles has ____ Pokemon cards more than David.

**Task 3**
Alvin has $2023 and Betty has $1973. They were each given three times as much as their original total amount.
At the end, Alvin has $____ more than Betty.

**Reflection A-1**
What is common across Task 1, Task 2 and Task 3?

**Reflection A-2**
What is common across Task 1, Task 2 and Task 3?
- I used Diagrams for the tasks.
- I used Equivalence for the tasks.
- I used Guess and Check for the tasks.
- I used Proportionality for the tasks.
- Others (Please elaborate).

**Task 4**
The diagram below shows a lighter square with side 10cm placed on top of part of a darker square of length 5 cm. Their common region is unshaded. The difference between the area of the lighter region and the area of the darker region is _____cm2.




**Reflection B**
What is common across Task 1, Task 2, Task 3 and Task 4?
- In all these tasks, I used Diagrams.
- In all these tasks, I used Equivalence.
- In all these tasks, I used Guess and Check.
- In all these tasks, I used Proportionality.
- Others (Please elaborate)

In Task 2, students are asked to find the difference in the number of cards left between the two persons. It will be easy if the actual number of cards given away by each person is known. Since the amount at the start is known, the quantity at the end can be easily calculated and the difference found. However, the actual quantity given away is not mentioned. Since both persons give away an equal number of cards, the difference they have in the end (Entity 1) is equivalent to the difference they have at the beginning (Entity 2). It is easy to find the difference at the start with a known number of cards each person has.

Similar to Task 2, in Task 3, the difference in money they have in the end (Entity 1) is equivalent to the difference they have at the beginning (Entity 2). In this instance, both persons are now given the same amount of money.

Before attempting Task 4, students are asked Reflection A-1 and Reflection A-2. The two reflection questions require students to look for the common idea used to solve the first three tasks. In the first task, students could realise that it is easier to solve when they can find two entities that are equivalent. Thinking along the same idea of equivalence, the student could again look to see if the next two tasks can also be solved using equivalence. As Tasks 2 and 3 are in topics chosen from the syllabus that we know are familiar to students at that level, the students are expected to be able to see that the tasks could also be solved by comparing two entities that are equivalent. Once the connection (the common idea used for all three tasks) is established, students are then given Task 4 to solve.

Task 4 is generally unfamiliar to most students. Students who have made the connections carry over the same idea to try to find two entities that are equivalent. Here, the Big Idea of Equivalence is the bridge that connects the solution process to solve the previous three tasks with Task 4, i.e., looking for two entities that are equivalent. The more able (in that Big Idea) students will realise that the difference between the lighter and shaded regions (Entity 1) is equivalent to the difference between the larger and smaller squares (Entity 2), as the shaded regions are obtained by removing the same unshaded regions from both squares. It is easier to find the difference between two squares than the difference of two irregular shaded regions.

In explaining the example above, we make the assumption that students are encountering the item, consisting of the tasks and reflection questions, for the first time. When doing a second item, students may already start to look for connections of ideas between the first three tasks.

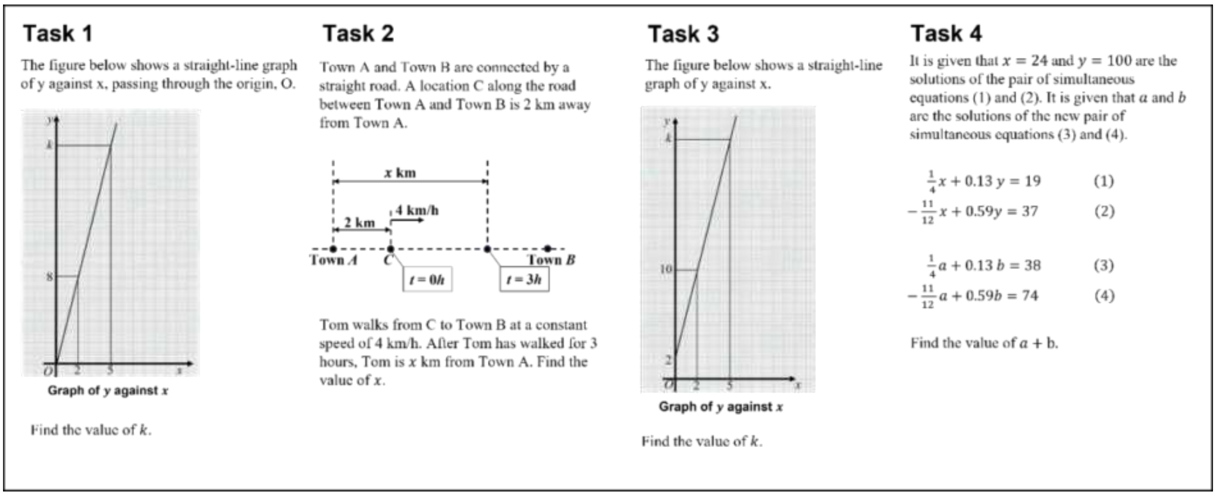
As a second example, we shall look at an item developed for assessing the Big Idea of Proportionality as shown in Figure 4. In Task 1, students are required to find the value of $k$ by analysing the straight-line graph given. As the line cuts through the origin, students could conclude that the equation of the line is of the form $y = mx$, in which $y$ (Entity 1) is proportional to $x$ (Entity 2). Analysing the graph, students could notice that an $x$ value of 2 results in a value of 8 for $y$, indicating that $y$ is four times the value of $x$, i.e., $y = 4x$. With this observation, the value of $k$ can be easily obtained by multiplying 5 by 4 to obtain the answer 20.

We shall proceed to elaborate how students are likely to answer Task 2. A secondary school student would be familiar with tasks of this nature as the topic of speed is part of the primary mathematics syllabus in Singapore. Recalling their prior knowledge, students will focus their attention on the speed given and know that the distance covered by Tom from Town C to Town B (Entity 1) is directly proportional to the time taken (Entity 2). The student could then obtain the distance of 12 km. However, to answer the problem posed, students will need to add the distance, 2 km, of Town C from Town A. We postulate that the more able (in the Big Idea of Proportionality) students would see the similarity between the two parts, in which the solution is based on the equation of a straight line, $y = mx + c$. For Task 1, $c = 0$ and for Task 2, $c = 2$.

Continuing to Task 3, we postulate that the more able (in the Big Idea of Proportionality) students would see that the value of $y$ is related to the value of $x$ by the equation $y = 4x + 2$, and that Task 3 is similar to both earlier tasks. While $y$ is not directly proportional to $x$, the **change in $y$** (Entity 1) is directly proportional to the **change in $x$** (Entity 2). Arriving at Reflection A-1 and A-2, where they are prompted to look for the common ideas across all three parts, able students would be able to see that connecting across all these parts is the Big Idea of Proportionality. We postulate that the ability to see this connection will positively prime them with an efficient approach to solve the task in Task 4 later.

**Figure 4**

*An Example of an Item (Without the Reflection Questions) to Assess the 'Big Idea of Proportionality' That Shows Connections Across Topics*




When presented with Task 4, students could try to see if the Big Idea of Proportionality would be helpful in their solving process. Upon analysing the information given, a properly primed and able student should be able to see that the right-hand-side of equation 3 and the right-hand-side of equation 4 are twice those of equations 1 and 2 respectively. Students will use Proportionality to deduce that the values of $a$ and $b$ (Entity 1) will also be twice the solutions of equations 1 and 2 (Entity 2), resulting in the value of 48 for $a$ and 200 for $b$.

Figure 5 is an example of an item meant for secondary school students under the Big Idea of Proportionality which elicits students' ability to see connections 'within' topics, in this example, the topic of rate. Tasks 1 and 2 involve direct proportion. Students use simple straightforward multiplicative reasoning to solve these tasks. Task 3 seems to require knowledge of inverse proportion. At these grade levels, the students are not yet introduced to inverse proportion. However, the task can be solved using direct proportion if the students can break down the task into parts. If we fix the number of days, then the number of workers (Entity 1) is proportional to the number of houses (Entity 2). We can obtain the fact that 36 workers will build $\frac{36}{60}$ of a house in 180 days. If we now fix the number of workers, then the number of days (Entity 1) is proportional to the number of houses (Entity 2). We can finally obtain the fact that 36 workers will build one house in $180 \times \frac{60}{36}$ days. Task 4 will be administered after the students have answered Reflections A-1 and A-2. Able students will see the common idea of proportionality and attempt to solve Task 4 by using proportion to find the number of chairs the five carpenters can make, and the number of chairs the other two carpenters can make separately and then combine them to obtain the answer.

## Discussion

At this point in the research, we have created four sets of instruments, each containing eight items: two instruments for the primary level (Grades 5 and 6) and two instruments for the secondary levels (Grades 7 and 8). The two instruments for each level are for the Big Ideas of Equivalence and Proportionality separately. All tasks for each item have been carefully designed such that one has to typically use either Equivalence or Proportionality to solve them. Task 4 requires a higher cognitive demand as the task is typically unfamiliar to the students.

**Figure 5**

*An Item (Without the Reflection Questions) for the 'Big Idea of Proportionality' That Shows Connections 'Within' Topics*

**Task 1**

A car takes 120 min to travel 180 km.
At the same speed, how much time does the car take to travel 120 km?

| (A) | (B) | (C) | (D) | (E) |
|---|---|---|---|---|
| 20 min | 40 min | 60 min | 80 min | 180 min |

**Task 2**

60 workers can make 180 chairs in one day. Each worker works at the same rate. How many chairs can 36 workers make in one day?

| (A) | (B) | (C) | (D) | (E) |
|---|---|---|---|---|
| 72 | 90 | 108 | 156 | 300 |

**Task 3**

A team of 60 workers takes 180 days to build one house. All workers work at the same rate. How many days does a team of 36 workers need to build the same house.

| (A) | (B) | (C) | (D) | (E) |
|---|---|---|---|---|
| 72 | 90 | 108 | 252 | 300 |

**Task 4**

10 carpenters take 5 hours to make 40 wooden chairs. All carpenters work at the same rate. In a particular week, 5 carpenters work for 40 hours, and 2 carpenters work for 35 hours. How many wooden chairs did the 7 carpenters make altogether in that week?

| (A) | (B) | (C) | (D) | (E) |
|---|---|---|---|---|
| 28 | 60 | 196 | 216 | 600 |

It may be possible for students to have answered correctly the first three tasks of each item based on their strong topical knowledge. However, we believe the first set of reflection questions will help students seek out the Big Idea among the first three tasks. We believe that the first set of reflection questions, Reflections A-1 and A-2, will trigger the students to 'see' that Task 4 can be solved using the same Big Idea. However, we acknowledge that (i) students may be able to obtain the correct answer by applying the correct Big Idea for the unfamiliar Task 4 but not see the connections between tasks (seen from their 'wrong' answers in the Reflection); and that (ii) students may be able to solve Task 4 using a different approach than that which we anticipated. Firstly, the proportion of students who can solve the task correctly but are not able to make the connections can give us valuable feedback on the current state of mathematics understanding. Such students have a good grasp of mathematical concepts but do not see mathematics as a connected whole. The findings can provide valuable feedback to teachers on how to modify their pedagogical approaches to help students see the connections. For the second case, we believe that such responses will be but a few and more of an exception rather than the norm. We shall explain this by referring to a typical mathematics assessment as an analogy. An item in a mathematics assessment is included to test a certain mathematics concept or process. The marking scheme for the item would be created based on typical responses expected. However, there will be responses that are atypical and the student's performance is scored using a different marking scheme without questioning the validity of the item. This is acceptable as the number of such atypical solutions are usually very small compared to the population. Similarly, while we acknowledge that there will be correct responses obtained by applying a different strategy or concept, the low occurrence of such cases can allow us to ignore its impact on the overall item analysis. In addition, our Task 4s have been designed such that a brute force computational approach would be time-consuming and inefficient.

The instrument, being 8-item long, requires a long duration for the students to complete. When we piloted two of the items to about 230 primary and secondary students, the mean time taken by the students was about 30 minutes. Thus, an eight-item instrument will require a duration of at least 120 mins. The time taken to administer the instruments developed will be a challenge as an assessment this long will take away precious curriculum time required for teaching and learning. At the same time, we risk reducing the validity of the data collected as students may experience fatigue completing the assessment or may decide not to perform to their ability due to the perceived fatigue of completing the lengthy assessment (Ackerman & Kanfer, 2009). A proposed solution will be to consider splitting the instrument into testlets and adopting a matrix design instead. We will be able to elaborate on this in a longer paper.

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