Social Mathematical Practices in Multi-Digit Multiplication

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Despite substantial research exploring multiplicative thinking and students' difficulty in the domain, the topic of multi-digit multiplication is under-researched. In this paper, I share a learning trajectory for multi-digit multiplication that combined social and cognitive perspectives of learning. Using Design Research methods and involving 45 Year 5 (9–11-year-olds) students from two different schools, an instructional sequence based on the trajectory was implemented. Findings led to the refinement of a trajectory that has implications for teaching practice.

Multiplicative thinking is widely recognised as an important understanding in students' mathematical development (Clark & Kamii, 1996; Park & Nunes, 2001). Multiplicative reasoning is the ability to think and reason using a deep conceptual understanding of the multiplicative structure, which underlies important concepts including fractions, decimals, and proportional reasoning (Siemon, 2013). Concerningly, evidence suggests that students' 10-15 years of age struggle to think multiplicatively (Siemon, 2013). Students' fluency with multi-digit multiplication develops from their ability to think multiplicatively. Although multiplicative thinking is well-researched, there is limited work in the field of multi-digit multiplication (Hickendorff et al., 2019).

In the last two decades, there has been an increased research focus on using learning trajectories to study the development of learning in mathematical domains. In this paper, a learning trajectory is presented for multi-digit multiplication that fosters the development of multiplicative thinking. Unlike other trajectories in the domain of multiplication, this one combines cognitive and social perspectives of learning. The taken-as-shared learning route that a class community may follow is described and a means by which an individual student's learning might be supported as s/he participates in and contributes to the collective learning of the class.

Literature

Developing understanding in multiplication is complex and acquired over multiple years (Clark & Kamii, 1996). Multiplication is a binary operation (Barmby et al., 2009) that requires the coordination of composite units. Multiple theories have been presented through relevant literature into early multiplicative understanding. In his study of the counting scheme, Steffe (1994) explained that understanding of multiplication is based on the construction of a composite, iterable unit. Other researchers have argued that this repeated addition model of multiplication is incomplete (Clark & Kamii, 1996; Park & Nunes, 2001) and that repeated addition is a procedure for solving multiplicative problems, not a conceptual basis. Park and Nunes (2001) present an alternate theory, stating that multiplicative understanding is defined by an invariant relationship between two quantities.

Much less is known about the development of understanding in multi-digit multiplication compared to single-digit multiplication (Hickendorff et al., 2019). In one study, Larsson (2016) described understanding in multi-digit multiplication as the result of connections between three elements: the arithmetic properties of commutativity, distributivity and associativity; models of multiplication; and strategies for solving multiplicative calculation. Although there is not a developmental hierarchy evident in multiplicative properties or models, Larsson (2016) reports an observable progression in students' solution strategies for multi-digit multiplication,

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developing from addition-based strategies such as repeated doubling through to strategies that draw on multiplicative thinking, including distribution and decomposition. Similar strategy progressions relating to multi-digit multiplication are reported in other studies (Ambrose et al., 2003; Barmby et al., 2009; Izsak, 2004).

Student difficulties in multi-digit multiplication result in many relying on additive strategies for prolonged periods of time (Ambrose et al., 2003; Barmby et al., 2009; Izsak, 2004). Evidence suggests that there is scope to extend additive strategies to more efficient ways of calculating. For example, students' intuitive use of repeated doubling to solve multi-digit multiplication problems has supported students to develop more complex multiplication strategies that draw on the associative property (Ambrose et al., 2003; Tripet, 2019). On the flip side, there is a danger that overgeneralising addition strategies can impede students' conceptualisation of the binary nature of multiplication (Larsson, 2016).

The use of effective multiplicative strategies demonstrates a shift in students' understanding of multiplication (Larsson, 2016) and their efficiency in performing calculations (Hickendorf et al., 2019). These strategies for multi-digit multiplication draw on the associative and distributive properties. Students recognise the practicality of partitioning and grouping numbers to solve more complex multiplication problems (Barmby et al., 2009). Ambrose et al. (2003) found that students in Years 3 to 5 instinctively used partitioning strategies to solve multiplication and division problems, concluding that many students hold an intuitive understanding of the distributive property. Izsak (2004) reported similar findings, noting that the coordinated rows and columns of the array supported students' calculations.

Although the associative property is an important multiplicative understanding, there is very limited research based on this property (Ding et al., 2013). In one notable study, Ding et al. (2013) evaluated primary pre-service teachers' understanding of the associative property of multiplication. Their work showed general misunderstandings around the associative property, with many pre-service teachers confusing the associative and commutative properties. They concluded that students' conceptual understanding of the associative property would be impeded by teachers' misconceptions.

Learning Trajectories

Learning trajectories were introduced by Simon (1995) as a description of "what teaching might be like if it were built on a constructivist view of knowledge development" (p. 115). Simon's (1995) original trajectory was based on a single instructional episode. Others have since applied the construct to sequences over longer periods of time.

As "a vehicle for planning learning of a particular concept" (Simon & Tzur, 2004, p. 93), trajectories refocus teaching from content transmission to students' cognitive constructions (Gravemeijer, 2004) and help provide focus and direction for instruction (Wright et al., 2006). Although most trajectories are concerned with an individual's cognitive development, few acknowledge the complementary nature of social and individual aspects of learning (Stephan & Rasmussen, 2002). Recognising that student learning is rarely uniform, Cobb et al. (2011a) adapted the construct of a learning trajectory as a sequence of "mathematical practices" (p. 80). Mathematical practices are descriptions of the collective learning pathway of the class community, and individual learning is understood in terms of individual students' reasoning as they participate in and contribute to the collective mathematical practices of the class (Cobb et al., 2011b). Describing social and individual learning provides a more comprehensive description of the learning that takes place in the classroom (Stephan & Rasmussen, 2002).

The intent of this study was to better understand how students construct understanding in multi-digit multiplication in the social setting of the classroom and how this process of understanding might be reflected through a learning trajectory. My work was guided by the following question: How can the social and cognitive aspects of learning be accounted for in a learning trajectory for multi-digit multiplication?

Methodology

Theoretical Perspective

The theoretical perspective for this study draws on Cobb and Yackel's (1996) emergent perspective, in which learning is recognised as both a social and individual endeavour. From this perspective, a reflexive relationship exists between the constructions of the individual and the social culture of learning in the classroom. Individuals construct new knowledge and understandings through mathematical activity while participating in the social role of learning in the classroom, and, in turn, students' interactions and contributions influence the evolving learning culture in the classroom. As such, the social and cognitive aspects of learning cannot exist in isolation. Concern is not about which perspective is more dominant, rather how the two aspects work together to support the development of students' mathematical learning.

Methods

Design Research methods were employed that allowed observation of students' thinking and reasoning of multi-digit multiplication first-hand (Cobb & Gravemeijer, 2008). In the preparatory phase, a domain-specific instructional theory was developed based on a detailed analysis of literature (Bobis & Tripet, 2023; Tripet, 2019). The teaching experiment phase involved a cyclic process of designing, testing, and refining the implementation of the instructional theory in the classroom setting. The experiment was conducted in two different Year 5 (9–11-year-olds) classes in Sydney, Australia: 23 students in Class 1 and 22 in Class 2, creating a sample size of 45 students. The same instructional theory was used as the basis for teaching in both classes and was implemented over a two-week period. The author was the primary researcher and adopted the role of teacher in each experiment, with the regular class teacher present to help facilitate student activity.

Data Collection and Analysis

The experimental phase involved the collection and ongoing analysis of data. Data collected included student work samples, classroom video recordings and field notes taken by the primary researcher. Following each lesson, the new data were compiled with existing data and reviewed. To coordinate the differing cognitive and social aspects, the dataset was iteratively analysed from three perspectives: the social learning of the class, the cognitive constructions of individual students, and then the relationship between the two.

Analysis of data from the social perspective identified the emerging mathematical practices (Cobb & Yackel, 1995). Cobb et al. (2011b) explain that mathematical practices comprise three interrelated and interdependent mathematical norms: mathematical activity, argumentation, and ways of reasoning with tools and symbols. Based on this, analysis of data focused on mathematical activity, reasoning and argumentation that became taken-as-shared in the classroom. A practice was considered taken-as-shared when most students were observed acknowledging acceptance and/or employing the practice. Along with analysis of student work samples after each lesson, video footage of class discussions was viewed to identify regularities and patterns in the way students acted, reasoned, or spoke mathematically.

Each mathematical practice identified was attributed to conceptual events. Events were considered conceptual when a shift in the collective reasoning of the class was observed. Conceptual events were considered significant when they were observed over multiple instances and influenced the collective knowledge of the class.

The cognitive perspective was informed by individual students' reasoning and their participation in, and contribution to, the collective ways of acting, reasoning and arguing mathematically that emerged in the class. Students' participation and contributions were documented on three levels: students' emerging use of representation, the pathways of strategy reinvention and the associated key understandings for these strategies, and through their discourse as they explained and justified their thinking.

The final stage of analysis in the teaching experiment was to consider the relationship between the social and cognitive aspects of learning, that is, how students' cognitive constructions contributed to the emerging social mathematical practices in the class, and then how students' participation in these practices led them to more sophisticated mathematical thinking. Analysis of video footage of each instructional episode examined how the actions of the teacher, the classroom culture and the role of the context supported student cognition and the development of social mathematical practices.

Results

Four socially constructed mathematical practices were identified in the study (Table 1). Each mathematical practice was linked to two conceptual events. The following section presents a detailed description of one of the conceptual events that led to the negotiation of the first mathematical practice (MP1). The conceptual event describes individual students' mathematical reasoning and argumentation, which serve as illustrations from both classes.

Table 1

The Four Mathematical Practices and Associated Conceptual Events

Mathematical practices (MP)	Conceptual events (CE)
MP1—The array as a sense-making tool: Partitioning the array	CE1—Using complete rows and columns to partition the array CE2—The use of place value across strategies
MP2—The array as a sense-making tool: Rearranging the array	CE1—Noticing a relationship between the numbers CE2—Using factors to manipulate the array
MP3—Working mathematically: Thinking multiplicatively	CE1—Recognising and adding all partial products CE2—Differing between additive and multiplicative compensation
MP4—Working mathematically: Using friendly numbers	CE1—Looking for efficiency CE2—Use of multiple strategies

MP1—The Array as a Tool for Sense-Making: Partitioning the Array

The instructional sequence followed the narrative of a cupcake bakery. The first teaching episode presented the narrative: *A baker makes and sells eight different flavours of cupcakes. The cakes are baked in a tray that has four rows with six cakes in each row. He bakes one tray of each flavour. How many cupcakes does he bake each day?* Students were asked to answer the question and explain how they obtained their answer. To assist in this process, students had access to different manipulatives, including a graphic of the cakes in an array.

A whole class discussion formed the inciting incident for the first conceptual events central to the negotiation and acceptance of the first mathematical practice. The intent of the discussion was to compare different student strategies, and for individuals to use these observations as a stimulus for modifying and refining their own strategies.

MP1 Conceptual Event 1—Using Complete Rows and Columns to Partition the Array

Three different strategies that focused on partitioning the array were presented during the class discussion. Zoe and Lucille (Figure 1) demonstrated that the structure of the array allowed for a straight partition to create a group of 20 and a group of 4.

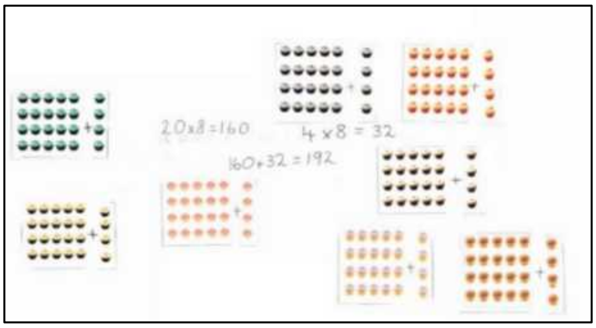
Zoe: We cut it down here (pointing to an array on their poster) to make a group of 20 and a group of 4.

Teacher: Why did you use 20 and 4?

Zoe: This one is 5 times 4 and this one is 1 times 4. You can do it just by cutting down here (showing the cut made on one of the trays of cakes) … also we thought that 20 and 4 would be easy.

Figure 1

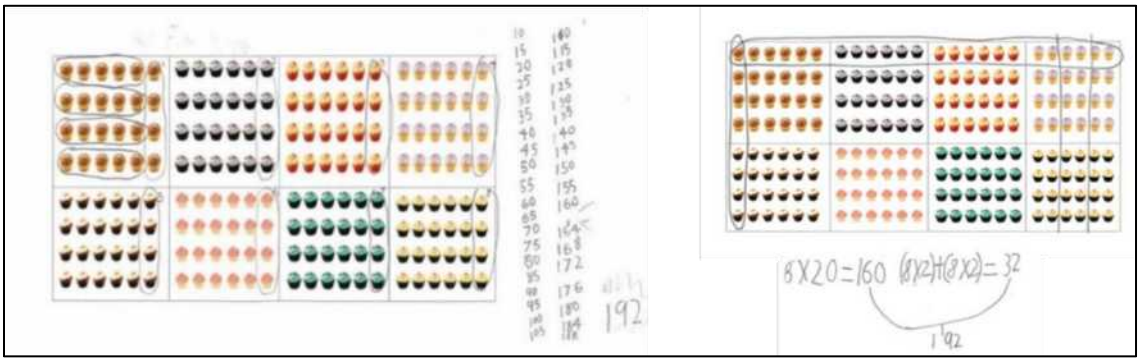
Strategy Used by Zoe and Lucille



Jake (Figure 2) explained his strategy of skip counting fives and then adding on the remaining fours. Jasper (Figure 3) explained how he used the larger array to partition 20 × 8 and two groups of 2 × 8, as these were multiplication calculations that he could perform mentally.

Figure 2 and 3

Strategies Used by Jake (1) and Jasper (2)



Students examined the similarities and differences between the three strategies. The partitioning methods were identified as a key difference between strategies: Jake circled groups of five, Zoe and Lucille had physically cut the array, and Jasper had drawn lines. One student identified the similar way in which Jasper, Zoe and Lucille used columns to partition multiple rows. Jake commented that he could have used columns to partition rows as well:

Teacher: Could you have split your array in a similar way to Zoe, Lucille and Jasper?

Jake: Yes … I think … I could have just cut down there and … just cut it like Zoe and Lucille's. I think our ways are sort of the same … more than Jasper's anyway.

Teacher: Why is your way more like Zoe and Lucille's?

Jake: Well, where I circled is sort of like… it's just like where they cut. They are just the same really.

Teacher: Do people agree with Jake? Do you think his strategy is like Zoe and Lucille's? … Frederique?

Frederique: Well, I think it is sort of. You can see that they both used the 4s at the end …

Lucy: Jake could use 20 too because he has 20 with his 5s.

Jake's contribution was an important element in the first mathematical practice becoming taken-as-shared in the class. There was verbal consensus in the class that Jake could partition multiple rows using columns, rather than focusing on just one row at a time. Similarly, there was verbal consensus that Jake's multiple rows of fives were like Zoe and Lucille's partitioning of the smaller arrays into 4 × 5 and 4 × 1. The similarity between Jake's strategy and Zoe and

Lucille's strategy was accepted and reinforced by other class members, who indicated how the partitioning used by Zoe and Lucille was evident in Jake's skip counting.

Students were asked to compare Zoe and Lucille's work with Jasper's work. One student observed that both used 8×20 and 8×4, although Jasper had partitioned his 8×4 into two groups of 8×2, which became a focus for discussion.

Teacher: How could they both use the same calculations?

Luke: If you look at Jasper's there is 24 across the top and there is 8 [motioning down the column]. You can just make 20 by putting in the line and then there is 4 on this side. And then in that one [pointing at Zoe and Lucille's] they have 20 and then they have 4 and … if you count … there are 8 of them.

The class indicated their agreement with Luke's explanation and that Jasper's method of partitioning was more efficient. Significantly, the students' reasoning centred on the array which indicated that the representation was a useful sense-making tool. The array provided a means for students to reason conceptually, not just procedurally, about multiplication. Students' argumentation, even their gesturing, centred on the array. In subsequent activity, a shift was observed in students' strategy use. For example, when asked to calculate 8 trays of 16 cakes, most students partitioned 16 into 10 and 6. This shift in students' strategy use demonstrated an acceptance of the power of the distributive property to aid efficient computation.

Discussion

The purpose of this research was to articulate a learning trajectory for multi-digit multiplication that accounted for the social and cognitive aspects of learning. The final trajectory is presented in Table 2. The trajectory is structured on Cobb's et al. (2011b) description of mathematical practices of the interrelated and interdependent elements of mathematical activity, argumentation, and ways of reasoning with tools and symbols. The first section in the trajectory identifies the four social mathematical practices that emerged through the experiment. Initially, students used the array as a tool for sense-making as they explored partitioning the array and using factors to manipulate the array. The two subsequent practices related to ways that students worked mathematically, as they transitioned to thinking multiplicatively and identifying 'friendly' numbers within more complex problems.

Students' cognitive advances were primarily seen through use of more sophisticated strategies, changing interactions with the array, and their mathematical argumentation. These aspects form the other three categories listed in the trajectory. The computational strategies observed in this study were comparable to those reported in previous studies (Ambrose et al., 2003; Barmby et al., 2009; Izsak, 2004; Larsson, 2016). The array proved integral to students' reasoning. As Gravemeijer explains (2004), mathematical models bridge informal and formal mathematics. In this instance, the array bridged students' experimental strategies to the properties of distributivity and associativity. It also formed a bridge across students' strategies and their argumentation. Students were able to make sense of others' thinking and argumentation using the array. This was particularly evident in the class discussion that formed the inciting incident for the first conceptual events central to the negotiation and acceptance of MP1. As students participated in this discourse, they were reorganising their own thinking, resulting in cognitive shifts.

This trajectory offers significant implications to teaching practice. Typically, learning trajectories focus on the mathematical learning of individual students, suggesting (whether intentional or not) that students will engage in tasks in similar ways and develop mathematical insights at similar times. The uniform descriptions of learning can mean that the diversity of students' reasoning fade into the background (Cobb et al., 2011a). This presents a challenge to teachers: how can one use a trajectory to inform instruction while still accounting for and responding to the diversity of student thinking?

The power of this trajectory and its potential impact on classroom practice is realised in the reflexive relationship between the social mathematical practices and the development of individual cognitive learning; social and individual aspects of learning are interrelated and interdependent, one cannot exist without the other (Cobb et al., 2011b). In this study, students' participation in and contribution to classroom activity shaped the mathematical practices, and through their participation and contributions, students reorganised their own thinking resulting in cognitive advances. The mathematical practices in the trajectory provide teachers with instructional directionality (Wright et al., 2006). By guiding the establishment of mathematical practices, teachers are simultaneously supporting individual students' cognitive advances.

Table 2

A Learning Trajectory for Multi-Digit Multiplication

Mathematical practices (MP)	Shared purpose	Reasoning with tools and symbols	Mathematical discourse
MP1 The array as a sense-making tool: Partitioning the array	Calculate the total number of cakes on 8 trays of 24	Partitioning the array by physically cutting or drawing lines to make smaller, more friendly parts to aid computation Informal recording with symbols	The distributive property—Noticing the similarity between the different partitioning strategies used by students
MP2 The array as a sense-making tool: Rearranging the array	Calculate the total number of cakes—16 boxes with 12 in each box	Physically cutting the array and then rearranging all parts, ensuring the rectangular structure is maintained Informal recording with symbols	The associative property—The array can be rearranged using factors and multiples
MP3 Working mathematically: Thinking multiplicatively	Calculate the area of two trays	Using the array to see: • all partial products formed need to be added together. • only factors and multiples can be used to rearrange the array More formalised symbolic recording	Additive v multiplicative thinking—Noticing the 2D structure of the array and the way it impacts the working of strategies
MP4 Working mathematically: Looking for friendly numbers	Calculate the money raised through cake orders	Reduced use of the array Formalising methods of symbolic recording	Identifying friendly numbers in calculations and determining which strategy to use based on the numbers to be multiplied

Conclusion

Students' fluency with multi-digit multiplication develops from their ability to think multiplicatively. In this paper, a learning trajectory for multi-digit multiplication is presented. Uniquely, the trajectory coordinates social and cognitive aspects of learning and, as such, provides a rich description of learning over the course of the instructional sequence. Practically, this is significant. The trajectory provides a viable theory that teachers can use to provide directionality for their teaching while also acknowledging student diversity.

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