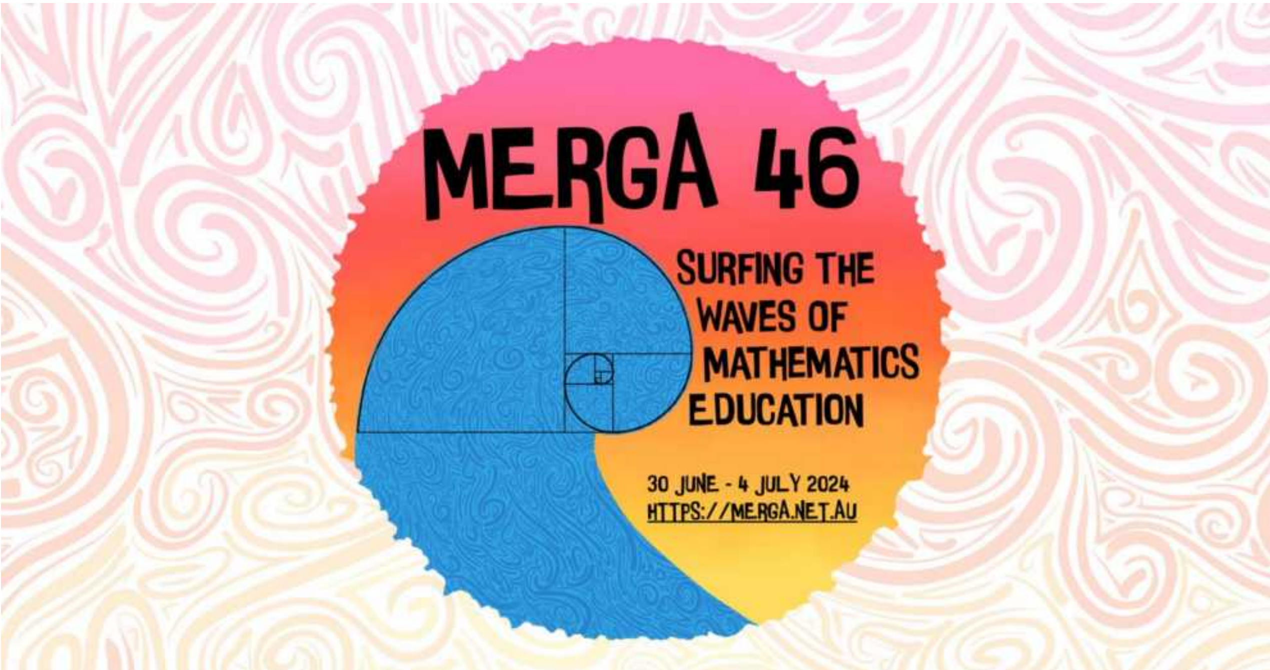
Mathematics Education Research Group of Australasia

Surfing the Waves of Mathematics Education



Proceedings of the 46th Annual Conference of the Mathematics Education Research Group of Australasia 30 June–4 July 2024, Gold Coast

Jana Višňovská, Emily Ross, & Seyum Getenet (Editors)

Surfing the Waves of Mathematics Education

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Edited by Jana Višňovská, Emily Ross, & Seyum Getenet

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Preface

This is a record of the Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia (MERGA). The conference was hosted by Griffith University and organised by colleagues from all the universities across South-East Queensland. The Proceedings were published online at the MERGA website.

Griffith University, the Gold Coast campus, is situated on the land of the Kombumerri people of the Yugambeh Language Region. The Kombumerri, or 'Salt Water' people, lived off the land and the water as the first people to the Gold Coast region. Established in 1971, Griffith is a relatively young university and in four decades has grown from one campus and 451 students to become a comprehensive, multi-campus institution, with 50,000 students and over 200,000 graduates.

The Gold Coast is world famous for its beaches and surfing, and strongly influences the theme of the conference, *Surfing the Waves of Mathematics Education*. This theme is of special importance as it reflects the strength of MERGA as a modern organisation committed to living and learning. Surfing the Waves of Mathematics Education also captures the dynamic and ever-changing nature of the field of mathematics education research: Surfing is an activity that requires navigating constantly changing waves, much like the field of mathematics education, which is continuously evolving with new research, pedagogical approaches, and technological advancements. The theme suggests a readiness to adapt and thrive amidst these changes.

By highlighting the dynamic nature of mathematics education, the local organising committee encouraged participants to be open to new ideas, methods, and technologies. This openness was bound to foster a culture of innovation where attendees could feel inspired to share their latest research, teaching strategies, and technological tools. The conference theme acknowledged the ever-changing landscape of mathematics education through recognising and valuing diverse perspectives and approaches. Featuring a wide range of topics and speakers, including those from different educational backgrounds, cultures, and disciplines, the conference provided a rich learning experience.

Delegates from Australia, Canada, Cyprus, Fiji, Germany, India, Ireland, Italy, Japan, Malawi, New Zealand, Philippines, Singapore, South Africa, and the United States of America participated in the conference.

Keynotes

Two keynote speakers set the pace for this theme. The opening keynote to the conference, *Towards embodied validity in mathematics education research,* was presented by Dr Nathalie Sinclair, Distinguished Simon Fraser University Professor. The address challenged researchers to consider embodiment as a method for accounting for experience in mathematical activity and as a method of research communication.

The keynote by Professor Mercy Kazima, *Interventions and development of mathematics education in primary schools,* highlighted research and development projects from Malawi, conducted over a decade. Two of these projects highlighted improved quality and capacity of mathematics teacher education in Malawi; and professional development projects to strengthen numeracy in the early years of primary school.

The *Clements/Foyster* lecture was delivered by Professor Janette Bobis. In this presentation, Professor Bobis explored what it means for research to be deemed significant. Her reflections drew upon her own experiences and those of others, as part of a 'deep dive' into the search for significance in mathematics education research. This exploration aimed to provide insights and perspectives to help researchers better articulate and elevate the impact of their work in the field.

Simplification of the awards nomination processes resulted in a record number of entries received for the Beth Southwell Practical Implications Award, aimed at highlighting opportunities for embedding research in practice. Eighteen candidates nominated their papers to be considered, and four (two without a self-nomination) were nominated by the review panels. Ten shortlisted papers were considered by a panel of four expert external assessors who arrived at a worthy winner in 2024. For the first time, commendations were also awarded to the two runners-up.

Six papers were shortlisted for the Early Career Award. The shortlisted submissions were judged by a panel of judges for the quality of both the written paper and the presentation during the conference.

Among further notable events of MERGA 46 was the workshop by the Aboriginal and Torres Strait Islander Mathematics Alliance (ATSIMA) CEO Professor Chris Matthews, Quandamooka man, on the Sunday preceding the MERGA Welcome. This pre-conference workshop was well received, connecting mathematics and culture through the Goompi Model and emphasising building relationships with Community.

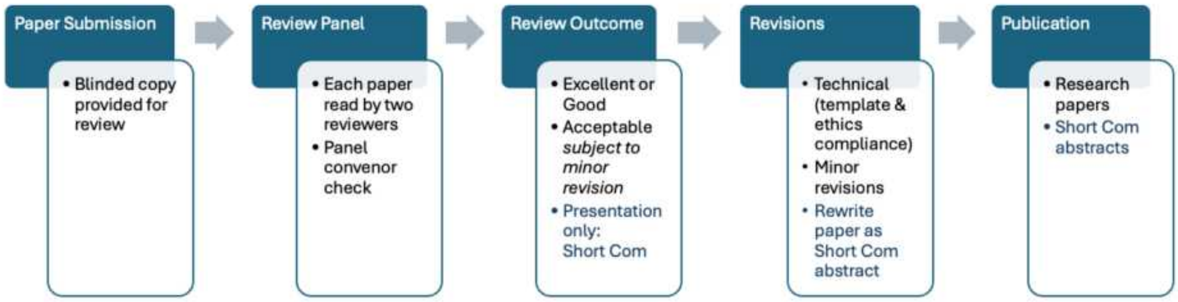
Of significant mention this year was the crossover between MERGA 46 and the Queensland Association of Mathematics Teachers (QAMT) State Conference *Ideas into Action*. Teacher's Day welcomed approximately 75 teachers from around Queensland, and commenced with the Beth Southwell Practical Implications Award keynote and a parallel program of presentations. Eight exhibitors joined the event. A welcome response was felt by the QAMT community as teachers and researchers were supported to meet and connect.

Proceedings

The published Proceedings include the plenary papers, symposia papers, research papers, and abstracts of short communications and round tables. All submitted papers (i.e., in both symposia and research papers sections) were double-blind reviewed in the Main round of submissions by panels of mathematics educators with expertise in the field. Figure 1 captured phases of the review process.

Figure 1

MERGA Review and Revision Process for Research Papers



Authors had an opportunity to receive formative feedback if they first submitted their papers via the Early Bird submission round. This opportunity was taken on by authors of unprecedented 28 papers and three research symposia.

The Proceedings illustrate the breadth of mathematics education research undertaken in the region and beyond by the MERGA research community. The Editorial Team would like to thank the authors for sharing their research. Gratitude also goes to the Review Panel Convenors and all the reviewers for their professionalism and effort in reviewing the papers and providing constructive feedback. The review process ensured that the high academic standards of the MERGA community were upheld.

Thank you

We would like to thank all the many colleagues who have helped bring MERGA 46 together, the Local Organising Committee, the MERGA Executive, the School of Education and Professional Studies at Griffith University, AAMT CEO Allan Dougan, the Queensland Association of Mathematics Teachers, our administrative support officer Megan Quentin-Baxter; and our sponsors:

- Gold Level: Griffith University (griffith.edu.au), reSolve—Australian Academy of Sciences (resolve.edu.au);

- Silver Level: School of Education, The University of Queensland (https://www.uq.edu.au), the Queensland Department of Education (education.qld.gov.au), Springer (www.springer.com); and
- Bronze Level: the Australian Association of Mathematics Teachers (AAMT, aamt.edu.au).

A special word of thanks goes to Dr Candace Kruger, Yugambeh Elder and Songwoman, Kombumerri (Gold Coast) and Ngugi (Moreton Island) Aboriginal woman, and her son Aric Kruger, up-and-coming mathematics education researcher, for conducting the *Welcome to Country*.

Kym Fry & Peter Grootenboer (Conference Convenors)

Jana Višňovská, Emily Ross, & Seyum Getenet (Proceedings Editors)

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Clements-Foyster Lecture

The Clements-Foyster Lecture acknowledges an eminent mathematics education researcher from Australia, New Zealand, or a South East Asian rim country, who is invited to present a keynote address at the annual MERGA conference. This annual keynote address is named in honour of Ken Clements and John Foyster who initiated and organised the first Mathematics Education Research Group of Australia Conference at Monash University in 1977. This led to the establishment of the organisation now known as MERGA.

A Deep Dive Into Mathematics Education Research in Search of Significance

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Doing significant research is critical to building the quality of mathematics education research. But doing substantively significant research is inherently difficult because we are studying the unknown. The ability to clearly articulate or 'sell' the significance of our research often poses even a greater challenge to researchers. Nevertheless, without such statements of significance we are unlikely to win grants or have papers accepted for publication. In this presentation, I consider what it means when we say that research is significant. My reflections draw upon my own experiences and those of others as part of a 'deep dive' in search of significance in mathematics education research.

The annual Clements/Foyster Lecture at MERGA conferences provide an important reminder to us of the foresight John Foyster and McKenzie (Ken) Clements showed in founding an organisation that has played such a major role in the lives and careers of so many mathematics educators since its inception in 1977. I recall being in the audience when the inaugural Clements/Foyster lecture was given by David Clarke in 2005 and although I never had the opportunity to meet John Foyster, Ken Clements was one of the first people to greet me at my maiden MERGA conference in 1991 when I was still a doctoral student. The following year, I had the honour of receiving the MERGA Early Career award and being given a hand-written certificate signed by Ken Clements. The certificate is now a valued memento of not only the award but of Ken Clements as co-founder of MERGA.

In my letter of invitation to present this lecture, it was suggested that I "take an empirically rich and theoretically robust forward-thinking approach that celebrates and thinks about the future potential of our organisation and its members". With this request in mind, I felt that a presentation going beyond just my own research was needed. I wanted to focus on an aspect of research that impacts us all and is critical to strengthening the ongoing credibility and quality of mathematics education research not just in Australasia, but more widely. After assessing high stakes grant applications, research proposals for higher degree research students, and reviewing and writing manuscripts for publication over many years, I often find myself asking "So what?" and wondered why it challenges so many of us to clearly articulate the significance of our research. According to Cai et al. (2019b) about one-third of manuscripts rejected for publication in the *Journal of Research in Mathematics Education* (JRME) received feedback from reviewers that "So what?" had not been adequately addressed despite being a fundamental aspect of doing and reporting research. In keeping with the conference theme, *Surfing the waves of mathematics education*, I decided to take a 'deep dive' into the *significance* of mathematics education research. My aim in this presentation is to raise questions and start conversations about aspects of research that we should be discussing to help build some shared understandings about doing *significant* mathematics education research.

Speaking the Same Research Language

I started my dive into the significance of research by examining how I and other MERGA researchers describe the significance of our work. First, I searched for occurrences of the term and its derivatives in papers from the MERGA 2023 conference proceedings and scrutinised the context in which the terms were used. Excluding occurrences in references and short communications, the term significance or significant occurred 84 times but only 73 of those

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 3–10). Gold Coast: MERGA.

were in research papers. I was only interested in references to the substantive significance of research so mentions of statistically significant results, and use of the term to indicate an issue was noteworthy or sizeable were discounted. Just five instances remained where the term was used to refer to the substantive significance of the research and only three of those (all written by early career researchers) offered elaborations as to why their studies were significant. Of course, the significance of a study can be conveyed in ways other than citing this exact term. Drawing upon papers from the MERGA 2023 proceedings, I created a list of terms that were most frequently used to potentially convey the significance of research. However, I now consider many of those terms to be distinctly different to the meaning intended by 'significance' of research.

Armed with this set of 'synonyms' for describing the significance of our research, I extended my inquiry to a relatively recent issue of the *Mathematics Education Research Journal* (MERJ). An examination of eleven articles from the same issue revealed that the term significance was frequently used in subheadings toward the end of papers (e.g., Significance of findings) but rather than elaborate on the significance in the ensuing paragraphs, authors spoke in terms of other characteristics that typically drew upon terms from the set of so-called synonyms. At the same time, I made a point of asking educational researcher colleagues from a range of fields such as philosophy, linguistics, educational psychology, and mathematics education what the significance of research meant to them during incidental conversations. Almost without exception, even the most experienced researchers used a range of terms and phrases to explain their thinking and all acknowledged that a succinct statement was challenging—especially when not having had time to prepare a considered response. The terms and phrases they most typically used were added to my list of synonyms. From this initial shallow dive, I realised that a fundamental issue surrounding how we convey the significance of our research exists—we are not all speaking the same research language.

To guide a deeper dive into this topic, I did what all researchers do—I developed research questions. The first guiding question was: *What do we mean when we say research is significant?* Before moving deeper into my explorations of research significance, I urge you to pause reading and reflect on your own response and perhaps the responses of those around you to what we mean when we say that our research is significant.

Of interest, is the extent to which the terms and phrases you have just considered match those of other researchers. The most used terms and phrases in my list of synonyms for research significance included: importance, contribution to the field, benefits, impact, innovative solutions to a problem, implications, and extending or building new knowledge. There were others, but these were the terms and phrases most frequently mentioned by colleagues, authors of texts recommended for our higher degree research students, in papers from the MERGA 2023 conference proceedings, and in the *MERJ* issue examined in my initial investigations. From these exploratory beginnings, further questions were formulated to help me delve deeper into issues surrounding the identification and articulation of significance in mathematics education research. Questions included:

- Why is it so challenging for researchers to agree upon a definition of significant research?
- Is the significance of research different to its contribution or its implications? If so, how do they differ?
- How are significant research questions developed?
- How can we judge if a research question is significant?
- Why is it important that we *do* significant research?

The remainder of this paper is structured around responses to each of these guiding questions and draws upon examples from my own research experiences and those of others as part of a 'deep dive' in search of significance in mathematics education research.

Clarifying Key Terms

Why is it so challenging for researchers to agree upon a definition of significant research? Is the significance of research different to its contribution or its implications? If so, how do they differ? Before we can dive more deeply into what we mean by significant research, it is necessary to clarify a few key terms. The lack of consensus surrounding the meaning of key research terms is, according to Hiebert and colleagues (2023), a result of such terms being overused which has led to definitions gradually shifting. Meanwhile, Evans et al. (2014, p. 69) argue that confusion is "due to a looseness of expression" used among researchers. Conversations with established researchers from a range of disciplines, and readings of texts about doing educational research confirmed early in my dive that researchers often use a variety of terms interchangeably when talking about the significance of research. However, one term stood out from the rest—important. All colleagues and reference texts used the term 'important' at some point when discussing the significance of research. In their discussion of research agendas, Ertmer and Glazewski (2014) argue for the importance of research and do not mention significance at all. Although, when defining 'importance' they use almost the exact phrases and examples that Evans et al. (2014) use to define significance. Hiebert et al. (2023) suggest that the importance of our research is judged by its "significance, contributions, and implications" (p. 106). This statement implies that while they are connected, each of these terms refers to a different aspect of research that taken together, argue for the importance of our research.

Significance

From the literature and conversations, there seems to be consensus that significance is mostly associated with the research questions we pose and for them to be potentially significant they need to address important issues or problems in education. Arguing significance purely because there is a 'gap' in the literature is not sufficient. Perhaps the gap exists because it is just not important enough to know. Cai et al. (2019a) suggest that the significance of our research is drawn from the importance of the mathematics content or context in which the problem we are addressing is situated. They agree with Simon (2004), arguing that it is unlikely significant research will result from opportunistic research—studies undertaken purely because participants are conveniently available rather than those planned to address a recognisable problem of importance. This means that the likelihood our mathematics education research will be considered significant will depend on the degree to which it addresses a problem or issue of interest and perceived value by the community including other researchers and practitioners.

However, what is valued and considered to be an important problem in need of solving might not be the case in another place or time. Research significance is bound by context. What mathematics education research and the broader educational community perceive to be significant changes over time. Moreover, the significance of research might be viewed differently in different cultures. The implication of research significance being bound by context is that the significance of a study might not be realised for many years after it is published and might be more (or less) highly valued in another country. Consider the current research interest surrounding artificial intelligence (AI). The term AI was originally coined in the 1950's and made popular by Alan Turing when he posed the question "can machines think?" (Turing, 1950, p. 434). Interestingly, after posing this question, he questioned its significance, "Is this new question a worthy one to investigate?" (p. 434). He also made the prediction that by the end of the 20th century, the idea of machines thinking and learning will be without contradiction. Given the current pervasiveness of AI, the significance of his initial question is perhaps more important today than it was in 1950 when so many dismissed his claims as having little consequence for society. However, the contribution of Turing's paper to the field of AI was acknowledged from its release as being the first published paper to introduce the idea of testing machines for their capacity to exhibit acts of intelligence. From this example, we can surmise that unlike significance, the contribution of research remains the same regardless of time and place. This means that there is a clear distinction between the *significance* of research and its *contribution* to education research and education practice.

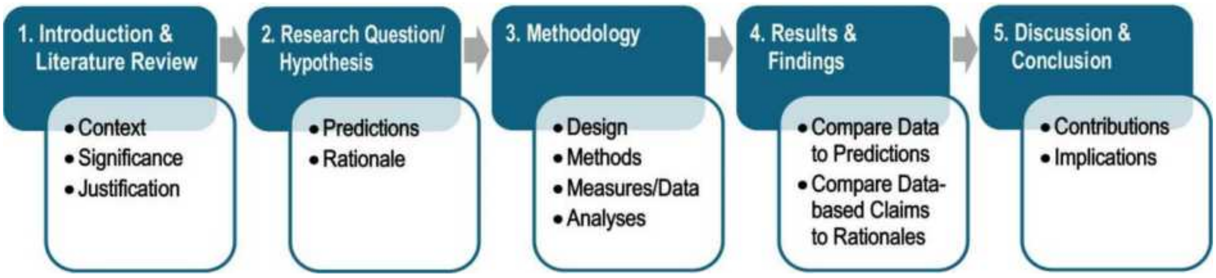
Although nowhere near the level of importance as Turing's seminal work on AI, the realisation that the significance of research is bound by context, but its contribution is not, caused me to reflect on my early experiences as a doctoral student and why my initial ideas for research were rejected by my supervisor but eventually gained traction in mathematics curricula. Emerging directly from a part-time research master's degree and full-time primary teaching to my PhD in the late 1980's I was prepared with a specific mathematics-related problem from the classroom that I wanted to explore. My proposal involved investigating the impact of a teaching intervention designed to facilitate the development of visualisation strategies in young children that would assist mental computation strategies. I had already conducted a pilot study with my kindergarten class from the previous year to refine my instructional activities. To my dismay, my supervisor quickly rejected my proposal during our first meeting on the grounds that he could not see any potential for it to be significant research. I revised my proposal to align with my supervisor's expectations and completed my PhD. As a very early career researcher, I did not question the rejection of my initial proposal. When I finished my PhD, I returned to my pilot data and decided to present it at the 1993 MERGA conference—the year after I had won the Early Career Researcher award with a paper based on my PhD research. The new paper was titled, 'Visualisation and the development of mental computation' (Bobis, 1993) and was one of two papers concerned with visualisation strategies at the conference. I recall a very large audience and feeling intimidated by some influential researchers, such as Professor Bob Wright and Alistair McIntosh sitting in the front row. As far as I was aware, no one in the audience seemed familiar with ten frames or the potential benefits of using subitising strategies to assist the noticing of structure in dot pattern arrangements or the potential for students' early number knowledge. Immediately following my presentation, Bob Wright congratulated me on such an important piece of research. Less than a year later, he had incorporated subitising into his *Learning Framework of Number*, eventually including subitising and ten frames into the New South Wales Department of Education's *Count Me In Too* (CMIT) numeracy program learning and assessment materials (Wright, 1998). Although ten frames and subitising were familiar to primary teachers involved in CMIT since 1996, they were not popularised until they were included in the 2002 *NSW Mathematics Syllabus K–6* (NSW Board of Studies, 2002). Following the conference, I was also invited to expand on my paper for a chapter in a book edited by Joanne Mulligan and Mike Mitchelmore (see Bobis, 1996). This chapter is still one of my top ten cited publications.

Reflecting on this experience caused me to question: Why was my initial proposal not judged as significant research if only a few years later it could be acknowledged as such and impact curriculum documents? I consider that there are at least two reasons for this delayed acknowledgement of significance. First, is the context in which my initial proposal was delivered. In the late 1980's, research surrounding number sense and mental computation was surging, but that of visualisation and visual imagery was still mostly confined to geometry. Its potential for improving young students' number knowledge was relatively undervalued. Additionally, my supervisor was not an expert in early years mathematics learning issues, so was not familiar with this area of research. The second major reason for its rejection was probably my own inexperience as an early career researcher to articulate a convincing argument for the potential significance of the proposed research to my supervisor—something that I was much better at doing in publications post-PhD (Bobis, 1993, 1996). Reflecting on this experience, the importance of establishing a robust case for significance early in the evolution of a research project and of clearly articulating it as such were reinforced. Without the ability to clearly articulate a case for significance, our research may never come to fruition.

If the significance of research is mostly associated with the research problems and questions we pose early in the life of a study, this means that significance can be determined before data collection. Authors of researcher guides and thesis-writing handbooks (e.g., Evans et al., 2014; Hiebert, et al., 2023; Paltridge & Starfield, 2020) invariably discuss research significance when describing the contents of the introductory chapter or opening paragraphs of research work. A colleague described research significance and contributions as the 'bookends' in her writing—a case for significance is built at the start of her work and the contributions of findings are presented at the end. The important aspect of the metaphor for my colleague was the connecting thread between the bookends. In her view, this connection ensures consistency between the aims of the study and the conclusions, providing a robust argument for the importance of the research which starts with the case for significance and ends with a discussion emphasising the value of its contributions and implications. This connecting thread is akin to Simon's (2004, p. 160) "line of reasoning" and to the "coherent chain of reasoning" that Hiebert and colleagues (2023, p. 116) identify as running through all parts of a research study. The visual representation (Figure 1), adapted from Hiebert et al.'s chain, conveys the point that there must be a clear flow between the initial phases of research and the contributions and implications discussed at the end. Methods are viewed as an enabling bridge—selecting the appropriate methods is needed to ensure a strong connection between significance and contributions. The authors warn that contributions are also constrained by the original research questions and hypotheses or revisions to these in the light of findings. A takeaway message of this chain of coherence is that if the significance of the study is established early, and if the methods are appropriate, the likelihood that the contributions will be important is increased.

Figure 1

The Chain of Coherence That Runs Through a Research Study (Adapted from Hiebert et al., 2023)



Contributions

Contributions are determined by the degree to which the research has moved the field forward, what we now understand better, and that can only be determined at the end of the study. The greater the perceived importance or value of the problem or issue investigated, the greater the perceived contribution of the research. An important characteristic of research contribution is that it is most often cumulative. We are regularly led to believe every study we conduct must include the Holy Grail as its contribution! Yet, one of the major criticisms made by reviewers of *JRME* is that authors' claims of contribution are often not justified by the data (Hiebert et al., 2023). The reality is that the merits of the contributions to a field of research by most researchers are judged on their contributions over a series of studies. Consider the contributions of prominent mathematics education researcher Professor Emeritus Peter Sullivan. A driving force behind his work throughout his career has been the identification of and desire to address barriers to learning mathematics particularly for students from disadvantaged backgrounds. Throughout his career, Sullivan has addressed this theme from multiple perspectives including good questioning (Sullivan & Clarke, 1991); open-ended tasks (Sullivan et al., 2001); enabling and extending prompts (Sullivan et al., 2006); lesson structures (Sullivan et al., 2015); challenging tasks (Sullivan et al., 2016); and sequences of challenging lessons (Sullivan et al., 2023). Examining even this narrow representation of his work reveals how Sullivan's underlying research agenda to identify and address barriers to learning mathematics provided direction to his research. More importantly, the linked work yielded a far greater contribution over time to advance our understanding of how to design tasks and lessons that increase the likelihood of challenging and engaging all students in learning mathematics than just one study could possibly achieve.

The critical point derived from the notion of cumulative contribution is that important advancements of knowledge in mathematics education rarely come from a single research study but from the linked contributions made by many. This is a major reason why researchers develop research agendas and base their studies on prior research conducted by themselves and by colleagues. Look at any chapter in *Research in mathematics education Australasia 2020–2023* (Mesiti et al., 2024) for the evidence of this cumulative contribution!

Implications

In the mathematics education research field, implications are most popularly presented in terms of suggestions that are *reasonably* derived from the findings for improving educational practices. Implications for educational policy, curricula development, theory, and research methods are also common. Additionally, implications expressed as recommendations for future research are usually expected in reports of research. Contrary to these popular views regarding the nature of implications, Hiebert et al. (2023) argue that most implications can be considered as research contributions. They propose that only conclusions about the efficacy of certain methods for generating a study's data align with their definition of research implications. Although I can understand this perspective given that there is often a fine line between contributions and implications, I prefer a targeted discussion of implications, especially for educational practice and future research. A thorough discussion of practical implications is valued by MERGA and is a reason why we offer the Beth Southwell Practical Implications Award. Unfortunately, some researchers encounter difficulties writing about implications of their studies. One common difficulty relates to the definition that implications should be *reasonably* derived from data. In their analysis of *JRME* reviewer comments, Cai et al. (2019b) report that nearly 30% of reviews included concerns about implication claims that were unsupported by the findings.

A second potential reason why some authors experience difficulties making a strong case for implications is linked to the positioning of implications in the concluding sections of research text. Word limitations along with an afterthought that implications should be included can result in the generation of meaningless sentences, such as:

> The results have *important implications* for theory and practice.
>
> Findings from this study *contribute significantly* to the development of recommendations for teacher professional learning.
>
> The study's findings will *impact* student learning and have *important implications* for pedagogy.

Regularly these types of sentences comprising sweeping statements appear with little or no details as to what the significance, contributions and implications are.

Inevitably there will be some variation in the way different researchers interpret the meaning of key terms used to argue the significance of their research. Nonetheless, it is still important that we have a clear understanding of what we mean when we use them in our work and that we can differentiate between them. Without this clarity or differentiation, we risk using these terms interchangeably, which can cause confusion for readers. The important point is that although we are aware of the necessity of claiming the significance, contribution, and implications of our work to argue its importance, without clarifying what these terms mean to ourselves, it is almost impossible to present a convincing case to others.

Developing Significant Research Questions

Earlier I stated that significance is mostly associated with the research questions we pose. This statement is in accordance with the National Research Council's (NRC, 2002) six guiding principles for scientific inquiry. The first of these principles is to "pose significant questions that can be investigated empirically" (p. 52). The NRC maintains that such questions in education pertain to "pressing problems of practice and policy" (p. 8) and that the formulation of significant questions can be more important than the solutions. This advice is still not very helpful for developing significant research questions because it is not always obvious what pressing problems of practice or policy exist in education. In fact, Evans et al. (2014) consider that using the term 'problem' may be too strong and suggest that "motivation for the study" (p. 63) might be more appropriate in some contexts. Either way, a major challenge for researchers is often how to develop significant research questions. Moreover, as reviewers of research manuscripts and proposals, how can we judge if a research question is significant? Or at least, has the potential to be significant?

Helpful advice for developing potentially significant research questions in mathematics education is provided by Cai et al. (2019a). They assert that research questions are more likely to be significant if they arise from teachers' problems of practice and should "aim to directly impact practice" (p. 115). They advocate increased interaction between teachers and researchers so that authentic problems of practice can guide research question development. Although Cai et al. focus on teachers' instructional problems as inspiration for research questions, it is important to also consider the problems of learners as a source. Perhaps the most significant questions in mathematics education are derived from our aspirations to improve students' learning of mathematics. This improvement should occur by addressing problems of instruction *and* problems of learning.

Once a question is selected, it will usually go through refinements to ensure it is unambiguous and meaningful. I use the Goldilocks principle as a guide for judging the meaningfulness of a research question. Namely, the question should not be too broad, not too narrow, but just right. Being 'just right' means that it will be focused, address an important problem or issue, and not require only a yes/no answer but seek to understand how and why.

Finally, simply because we consider a research question to be significant does not ensure that readers will also view it as such. It is unlikely anyone can judge the significance of a research proposal simply by reading the research questions. Presenting a strong case or 'selling' the significance of a research question starts early by establishing the research context. The research question(s) and the problem it addresses must be contextualised in terms of existing research, providing a justification as to how addressing the research question will deepen our understanding of an important phenomenon in mathematics education and extend our knowledge in the field. As so eloquently argued by Hiebert et al. (2023) and represented in Figure 1, contextualising the research question based on prior research is only the first stage in a chain of justification that should be coherently woven through each subsequent section of writing; thus, continuously strengthening the case for significance.

Conclusion

An aim of this paper was to raise questions and spark conversations about aspects of doing and reporting significant mathematics education research. These are conversations that we should regularly revisit not only with our research colleagues but also with potential end-users whether they be practitioners or policymakers. I hope that my comments have made a positive contribution to individual researchers and to the MERGA community by raising awareness of how critical it is for researchers to not only *do* significant research, but to think more deeply about how to communicate a coherent case for its importance via the significance, contributions, and implications. Not doing so, runs the risk that the quality of mathematics education research will be questioned. Refining our skills for arguing the case for the importance of research impacts us all—only together can we strengthen the ongoing credibility and quality of mathematics education research and have a truly significant impact on students' learning internationally.

References

Bobis, J. (1996). Visualisation and the development of number sense with kindergarten children. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning* (p. 17–33). Australian Association of Mathematics Teachers (AAMT) and Mathematics Education Research Group of Australasia (MERGA).

Bobis, J. (1993). Visualisation skills and the development of mental computation. In B. Atweh, C. Kanes, M. Carss, & G. Booker, (Eds.), *Contexts in mathematics education. Proceedings of the 16th annual conference of the Mathematics Education Research Group of Australasia* (pp. 117–122), Brisbane: MERGA.

Cai, J., Morris, A., Hohensee, C., Hwang, S., Robison, V., Cirillo, M., Kramer, S. L., & Hiebert, J. (2019a). Posing significant research questions. *Journal for Research in Mathematics Education*, *50*(2), 114–120. https://doi.org/10.5951/jresematheduc.50.2.0114

Cai, J., Morris, A., Hohensee, C., Hwang, S., Robison, V., Cirillo, M., Kramer, S. L., & Hiebert, J. (2019b). So what? Justifying conclusions and interpretations of data. *Journal for Research in Mathematics Education*, *50*(5), 470–477. https://doi.org/10.5951/jresematheduc.50.5.0470

Evans, D., Gruba, P., & Zobel, J. (2014). *How to write a better thesis*. Springer.

Hiebert, J., Cai, J., Hwang, S., Morris, A., & Hohensee, C. (2023). *Doing research: A new researcher's guide*. Springer.

Mesiti, C., Seah, W-T., Kaur, B., Pearn, C., Jones, A., Cameron, S., Every, E., & Copping, K. (2024). *Research in mathematics education in Australasia 2020–2023*. Springer.

National Research Council (NRC). (2002). *Scientific research in education*. National Academy Press.

New South Wales Board of Studies. (2002). *Mathematics K–6*. NSW Board of Studies.

Paltridge, B., & Starfield, S. (2020). *Thesis and dissertation writing in a second language: A handbook for students and their supervisors* (2nd ed.). Routledge.

Simon, M. (2004). Raising issues of quality in mathematics education research. *Journal for Research in Mathematics Education*, *35*(3), 154–163. https://doi.org/10.2307/30034910

Sullivan, P., Askew, M., Cheeseman, J., Clarke, D. M., Mornane, A., Roche, A., & Walker, J. (2015). Supporting teachers in structuring mathematics lessons involving challenging tasks. *Journal of Mathematics Teacher Education*, *18*(2), 123–140. https://doi.org/10.1007/s10857-014-9279-2

Sullivan, P., Bobis, J., Downton, A., Livy, S., McCormack, M., & Russo, J. (2023). *Maths sequences for the early years: Challenging children to reason mathematically*. Oxford University Press.

Sullivan, P., Borcek, C., Walker, N., & Rennie, M. (2016). Exploring a structure for mathematics lessons that initiate learning by activating cognition on challenging tasks. *Journal of Mathematical Behavior*, *41*, 159–170. https://doi.org/10.1016/j.jmathb.2015.12.002

Sullivan. P. & Clarke, D. J. (1991). *Communication in the classroom: The importance of good questioning*. Deakin University.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2006). Teacher actions to maximize mathematics learning opportunities in heterogeneous classrooms. *International Journal of Science and Mathematics Education*, *4*, 117–143. https://doi.org/10.1007/s10763-005-9002-y

Sullivan, P., Zevenbergen, R. L., & Mousley, J. (2001). Open ended tasks and barriers to learning. *Australian Primary Mathematics*, *6*(1), 4–9.

Turing, A. (1950). Computing machinery and intelligence. *Mind*, *58*(236), 433–436. https://www.jstor.org/stable/2251299

Wright, R. J. (1998). An overview of a research-based framework for assessing and teaching early number learning. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times. Proceedings of the 21st annual conference of the Mathematics Education Research Group of Australasia*, (pp. 701–708). Brisbane: MERGA.

Keynote Presentations

Interventions and Development of Mathematics Education for Primary Schools

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In this presentation I discuss the work of two research and development projects in my country, Malawi, which I coordinated since 2014, and how the work responded to the needs and context of mathematics education in Malawi. For more than two decades, Malawi has been concerned about low learner achievement in mathematics as indicated by national and international assessments. The concerns are similar to other countries in sub-Sahara Africa, where the mathematics achievement of most learners in primary schools is below their grade level. There have been some interventions aimed at addressing this concern by targeting schools, teachers and learners. I share these briefly as part of the background. Then I share the work of 'my' two projects called *Improving quality and capacity of mathematics teacher education in Malawi project (2014–2019)* and *Strengthening numeracy in early years of primary school through professional development of teachers project (2017–2022)*. I focus on the research and interventions by the projects - in particular the intervention on counting in the first two grades of primary school. Finally, I discuss the findings and implications to mathematics education research in Malawi and other similar contexts.

It is a great honour to be invited as keynote speaker, many thanks to the MERGA 2024 organising committee. My talk is drawn from my work in Malawi but can apply to other similar contexts. First, I share the context of Malawi and three examples of interventions that have taken place in primary schools within the last decade. Then I discuss the two projects that I coordinated since 2014. I share the motivation for projects, a summary of the work and interventions by projects, focusing on an intervention on counting in the first two grades of primary school. Finally, I highlight the findings and implications to mathematics education research.

Introduction and Background

Malawi Context

Malawi is a small low-income country in southeast Africa with an area of 118,484 square kilometres and population of about 21 million. It is a republic that gained its independence from Britain in 1964. The official language is English and is used in schools from Grade 5 onwards. Grades 1–4 use Chichewa, the national language, or other local language. The school structure has 8 years of primary school and 4 years of secondary school. Primary teacher education takes place in Teacher Training Colleges (TTC) under the Ministry of Education, and the qualification is a Teacher's Certificate. The duration is two years covering 6 terms; four terms for taught courses at the TTC and two terms for school-based training in form of teaching practice. The initial teacher education programme prepares generalist primary school teachers with no specialisation in terms of teaching subjects or year level. Teacher education for secondary school is offered at universities and has a duration of four years of taught courses and one school term of teaching practice. The university teacher education programmes have specialisations for teaching subjects; often one major and one minor teaching subject (e.g., Mathematics major and Physics minor).

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 13–20). Gold Coast: MERGA.

Teaching and Learning Mathematics in Malawi

The teaching and learning of mathematics in Malawi is considered important, hence mathematics is a compulsory subject in primary and secondary schools, and a core subject for all science related programmes in post-secondary education. In primary and secondary schools, mathematics is allocated more time than all other subjects except English, which is also allocated about the same time. For example, 40% of teaching time in Grade 1, 34% in Grade 2 and 31% Grade 3 (Ministry of Education, 2006). While there is all this emphasis on mathematics in schools, the learning seems to be limited as evidenced in low learner achievement. Assessments such as the Early Grade Mathematics Assessment in 2010 and the Monitoring Learning Achievement survey in 2012 reported that more than 50% of the learners achieved below the expectation of the Malawi primary school mathematics curriculum (Brombacher, 2011; Ravishankar et al., 2016). In all the assessments by the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ) in 1995, 2000, 2007, and 2013, Malawi was one of the two lowest performing countries in mathematics, with more than 90% of learners performing at or below the basic numeracy level (Milner et al., 2011; Brombacher, 2019).

The concern in low learner achievement in mathematics led the government of Malawi through the Ministry of Education to commission a study to investigate the problem. This was a large-scale study from 2018 to 2019 that investigated what was inhibiting student learning of mathematics. The study focused on the first four grades of primary school and scrutinised the curriculum materials, lesson observations, and the various assessments (Brombacher, 2019). Findings of the study include that the Malawi curriculum had low expectations of learners, for example, it had low number range for calculations (0–9 for Grade 1, 0–99 for Grade 2, and 0–999 for Grade 3); the focus of teaching was on procedures not understanding; and the approach to calculations was the same from Grade 1 to Grade 4. Same single-digit arithmetic using counters and the "combine and count all" strategy was observed. There was no increase in efficiency or development of number range appropriate strategies (Brombacher, 2019). There are other factors that contribute to the learners' low achievements, including large classes, limited teaching and learning resources, and problems of learning mathematics in English while not competent in the language (Kazima, 2008, 2014).

Interventions in Primary Mathematics

The concerns in teaching and learning mathematics have led to some interventions in primary mathematics. These are mostly small-scale interventions targeting schools, teachers, and learners. Below are three examples, selected for their relatively good impact. The first example is *Unlocking Talent* which focuses on learners, the second is *Numeracy Boost* which focuses on the community, and third is *Japan International Cooperation Agency (JICA) Numeracy* which focused on teachers.

Unlocking Talent Intervention

This is an intervention by an international project aimed at improving numeracy and literacy for marginalised children in various countries (Pitchford, 2018). In Malawi, the intervention started in 2014. It is funded by the Norwegian Embassy and implemented in partnership with the Voluntary Services Oversees, the Malawi Ministry of Education, a non-profit organisation called Onebillion, The United Kingdom's Department for International Development (DFID), United Nations International Children's Emergency Fund (UNICEF), and the Scottish Government (Royal Norwegian Embassy, 2017). The intervention offers children in early years of primary school individualised learning using tablets. The tablets are installed with applications that were developed based on the content of the country's primary school curriculum (Pitchford, 2018). The applications focus on basic numeracy and are designed in such a way that they do not rely heavily on a quality teacher. This was done to overcome the challenge of limited resources including qualified teachers (Hubber et al., 2016).

Each intervention school has a classroom reserved for tablet use and is provided with 30 tablets. During school time, groups of 29 learners at a time are taken out of their regular class to the tablet class with one supervising teacher. All the applications are in local language, are easy for the learners to use and give feedback to the learner. For example, the tablet can display 10 circles and ask the child to count how many. The tablet would also display some numbers for the child to select the answer by touching the number. Verbal and visual feedback is given to the child before moving to the next item. The supervising teacher only acts as a facilitator and solves technical problems (Pitchford et al., 2019).

The intervention has registered some successes including that it has helped to support development of basic numeracy skills because it allows learners to repeat activities when needed (Outhwaite et al., 2017). Furthermore, it promotes inclusiveness of both boys and girls, and of special needs learners (Pitchford et al., 2019).

Numeracy Boost Intervention

The Numeracy Boost intervention is supported by Save the Children. It started in 2012 in selected rural school communities in one district. The intervention has three components. The first component is community camps which are the core feature of the intervention. The camps are located in the school communities and run by volunteers from the community, referred to as camp facilitators. The camps operate after school and offer a learning through play environment. Some school teachers have the responsibility of camp supervisor which involves suggesting mathematics content areas to be covered at the camp (Mbendera, 2019).

The second component is capacity building of the volunteer camp facilitators and teacher camp supervisors in form of brief training. The intervention works collaboratively with a primary teacher training college, where the mathematics teacher educators are offered training and then participate in developing teaching and learning materials for the camp and training the camp facilitators and supervisors (Mbendera, 2019). The third and final component is learner assessment. The intervention administers baseline and endline assessments each academic year in the intervention schools and some control schools. So far, the intervention schools have registered better achievement than the control schools, and have shown a more positive attitude towards mathematics (Mbendera, 2019).

JICA Numeracy Intervention

This intervention was conducted in 2016 in two schools only and focused on teachers of Grades 1–3. The aim was to equip teachers with knowledge and skills to help learners progress from unit counting to composition and decomposition when adding and subtracting numbers. For example, when addition of numbers crosses a 10 such as $8 + 5$, they should be able to see $8 + 5$ as $(8 + 2) + 3$ and obtain the answer, 13, without unit counting. The teachers were offered three weeks of training before implementing the composition and decomposition strategies in their classrooms. A pre-test and post-test were administered to learners in the classes. It was observed that the learners that used unit counting in the pre-test were able to work out the problems without unit counting during the post-test, indicating that they were able to progress to composition and decomposition strategies of addition and subtraction (Kazima et al., 2022).

Successes and Challenges of the Interventions

Each of the interventions registered some successes which is commendable. However, they also registered some challenges which presented limitations. For example, the Numeracy Boost camps depend on volunteers, which might pose problems if the mathematical knowledge is not at the desired level. The Unlocking Talent takes groups of children out of class for the tablet sessions, which means the children miss some of the regular learning. A common limitation across many interventions is that there is not much possibility for scaling up. Most start and end with the project life; like small experiments in the schools, then back to usual. The good outcomes are rarely shared widely to apply to others, and possibly there are some bad outcomes for some experimental groups.

Projects I Coordinated

Since 2014, I have coordinated two large projects which I refer to as project 1 and project 2. Both projects were in collaboration with the University of Stavanger, Norway, and with funding from the Norwegian Agency for Development cooperation (Norad).

Project 1: Improving Quality and Capacity of Mathematics Teacher Education in Malawi (2014–2019)

In 2013, prior to the project, we started a collaboration between the University of Malawi (UNIMA) and the University of Stavanger (UiS). We were successful in getting seed funding to develop a 5-year project proposal. Our overall goal was to strengthen the teaching and learning of mathematics in Malawi. We used the seed funding for a baseline study to identify needs that would guide our focus. We identified many needs at different levels; schools, learners, teachers, and teacher education. We decided to focus on teacher education for long term effect. Specifically, we focused on capacity building at the University of Malawi, and in TTCs.

We received a grant of 16.5 Million NOK and we designed a project with five components (Kazima & Jakobsen, 2019) based on the identified needs, as shown in Table 1.

Table 1

Components of Project 1

Component	Needs
PhD programme (developed for UNIMA and offered jointly with UiS)	Capacity building for UNIMA: to have more faculty with PhD, to offer PhD programme, (increase number of well qualified mathematics educators in Malawi)
Master course (developed for UNIMA and offered jointly with UiS	Capacity building for UNIMA to offer course Target TTC lecturers to strengthen their capacity in preparing primary school teachers Increase number of well qualified mathematics educators in Malawi
Research	To learn from Malawi context and inform professional development Capacity building for UNIMA To increase research in mathematics education in Malawi
Infrastructure development	Capacity building for University of Malawi to offer the new courses (Master & PhD) and improve the existing one (Bachelor)
Professional development	Capacity building: for TTC lecturers to offer better quality mathematics education for UNIMA to offer the PD

Research and Intervention by Project 1

The project intervention was to introduce *Lesson Study* in public TTCs. Lesson Study originated in Japan where teachers worked together to plan, teach, and observe a lesson with the aim of deepening their understanding of how students learn, and how to use that understanding to improve their teaching (Dudley, 2014; Fauskanger et al., 2019). In Malawi, lesson study is relatively new. The project introduced it with the aim for mathematics teacher educators to learn about their own teaching and their students' learning, to inform improvement in their practice. Research informed the intervention, both prior and during the intervention.

The intervention was carried out from 2016 to 2018 through a series of activities; designing the lesson study programme, conducting workshops with teacher educators before and after the lesson study, and working with teacher educators on their lesson plans for lesson study. Each year, the programme started with a 3-day workshop in May, then lesson study in the TTCs from May to November, and finally another 3-day workshop in November. The intervention was conducted in all eight public primary teacher education colleges and reached all mathematics teacher educators in the colleges, 106 educators at the time. The intervention registered success in terms of mathematics teacher educators' learning and uptake of lesson study as model for their professional development. Challenges include the demand on time of the teacher educators. However, it was observed that advance planning reduced the challenge of time. The project's research and intervention revealed that the mathematics teacher educators need support, suggesting that teacher education should not be ignored as is often the case in many of the interventions.

Project 2: Strengthening Numeracy in Early Years of Primary School Through Professional Development of Teachers (2017–2022)

Midway through project 1, we applied for project 2 to continue the collaboration between UNIMA and UiS. We received a grant of about 15 Million NOK. Through our work with teacher educators in project 1 and our earlier baseline study, we identified the need for professional development of teachers. Thus the project 2 focus was on primary school teachers, and the overall aim was to improve Numeracy in early years of primary school in Malawi (Kazima & Jakobsen, 2021). The project had three main components based on the needs we identified, as shown in Table 2.

Table 2

Components of Project 2

Component	Needs
Research (including Master, PhD, and PostDoc)	To learn from Malawi context and inform professional development Capacity building for UNIMA To increase research in mathematics education in Malawi
Infrastructure development	Capacity building for University of Malawi to offer PD
Professional development	Capacity building: for teachers to strengthen their teaching for UNIMA to offer the PD to provide TTCs with a possible model of working with schools

Research and Intervention by Project 2

In project 2, we planned to introduce an intervention on something teachers already do in their teaching and strengthen it so that they can do it better. Based on our research and observations, we identified counting in the first two grades of primary school as a good focus. Almost all lessons started with counting activities usually through play and song which was motivating and exciting for young learners. However, the counting was all forward counting in ones starting from one. We planned an intervention that would extend the teachers' work of counting to more mathematically meaningful counting activities.

The intervention was conducted in form of professional development for the teachers through lessons study. We adapted the structure of lesson study from project 1 where we started with a 3-day workshop with teachers in May, followed by lesson study in schools from May to

November, then another workshop for 3 days in November. We repeated this from 2019 to 2022, with different schools each year except 2020 due to COVID. We managed to work with a total of 11 schools and all the 136 teachers of grade 1 and 2 in the schools.

Workshop 1

The first workshop included (i) what is teaching and what is learning? (ii) discussion of how the participants teach counting in their classrooms; (iii) counting activities—choral counting and counting collections (Franke et al., 2018)—and why they are important; and (iv) an introduction to lesson study. On the last day, teachers from each school were encouraged to work together to conduct lesson study on counting based on what they learnt during the workshop. Each school drafted their initial ideas then shared with the rest of the group by the end of the workshop. Teachers were also encouraged to use the counting strategies explored during the workshop, specifically counting forwards and backwards, counting in groups, skip counting, counting collections, and choral counting.

Lesson study in schools started with developing draft lesson plans for the research lesson. The focus of the lesson study was for teachers to learn about their students' learning. As such, the research lesson plans needed to make clear what it was about their students' counting that the teachers wanted to learn and how the teachers would observe it. The draft lesson plans were submitted to the project team at UNIMA and UiS. Each draft was reviewed by at least two project team members who provided written feedback to the teachers. The UNIMA project team visited each school to discuss the feedback with the teachers and provide support where needed. The schools revised their lesson plans following the feedback, then identified one of the teachers to teach the research lesson while the others observed. Finally, the teachers reflected on the lesson, discussed their learning from it, and prepared a report to share at workshop 2.

Workshop 2

The second workshop again lasted 3 days, and it covered (i) reports form all schools on their lesson study, what they learned and discussions; (ii) reflections and more lesson plans on counting; (iii) talk moves (Kazemi & Hintz, 2014); and (iv) the mathematics teaching framework (Adler, 2021).

The UNIMA project team visited all schools during the lesson study and later followed up with some of the schools after the second workshop. The findings that follow are from both the lesson observations and reports from the teachers.

Findings and Implications

Findings

I highlight three findings, then discuss their implications to mathematics education research. First, we observed that teachers included more meaningful counting activities in their teaching besides the rote counting through song and play. Teachers adapted skip counting in their play and songs. However, there were reports of challenges in implementation in large classes. Second, almost all the teachers reported surprise at what their learners could do, for example, counting in groups and identifying number patterns on a 100 square or other grid arrangement of numbers. There seemed to be a general underestimation of what grade 1 and 2 learners can do. This supports the argument by Brombacher (2019), that the primary mathematics curriculum has a low expectation of the learners. It seems that the teachers also shared the low expectation. However, it could be argued that the curriculum influenced the teachers' low expectations of their learners. For example, the curriculum's low number range of 0-9 for Grade 1 might have limited teachers' expectation of Grade 1 and 2 learners identifying number patterns. Third, most teachers were able to link counting to mathematical concepts (e.g., linking counting in groups to multiplication). However, some teachers might have benefited from more workshops and activities to see the mathematical connections.

A general finding from the two projects was that Lesson Study is an effective method of professional development of teacher educators and teachers. However, there was need for support from ‘experts’ in the field, referred to as knowledgeable others in lesson study terms.

Implications For Mathematics Education Research

The findings have implications for teaching, for teacher education, for curriculum development, and others. I focus only on implications for research in mathematics education. One implication is a call for research on implementing the counting strategies in large classes. During the lesson study, some schools split the classes into two so that they could observe all learners easily. While this was a good solution, it is not practical to do so in their everyday teaching. Hence the need for research to inform our practice. Another call for research is on what Malawi children bring to the classroom as they start primary school mathematics and how to use that in teaching. Many children in Malawi do not have access to preschool, and there is no reception class prior to the first grade at age 6. Studies of the curriculum have shown that it has low expectations (Brombacher, 2019), probably because the assumption of what children bring to the classroom was too low. Research to explore this would greatly inform the curriculum and teaching.

Teacher education is another area that needs research to inform practice, specifically research that addresses the knowledge teachers need to teach counting effectively and make mathematical connections with other concepts. What role can initial teacher education play in developing counting skills and teaching strategies? And what role can school-based professional development play?

A general implication is the need for more lesson study with a research component on how to learn from teachers’ own teaching and how to apply their learning in teaching to improve their practice.

We found research within the Malawi context crucial. Research prior and during intervention informs design and ongoing improvements. Research from other contexts informs us better when we understand our own context.

Concluding Remarks

I have shared my work and how interventions can contribute to development. Malawi is an illustrative example, the discussion can apply to other similar contexts. We learnt from project 1 that teacher education needs support and should be included in research and interventions in mathematics education. From project 2, among other things, we learnt that there are many things that the teachers do well within the constraints of the context. Interventions need to acknowledge and strengthen these existing skills while developing new skills. For example, the teachers were good at counting with young learners through songs and play in a motivating way. The intervention needed to strengthen their approach, by keeping the counting songs and extending from the rote counting to more mathematically meaningful counting. There is a lot we can learn from such contexts to complement our understanding of mathematics education that we learn from other ‘more visible’ contexts. There is need for more research and dissemination from different contexts, especially the low-income countries whose work is rarely disseminated beyond their national boundaries. Mathematics Education research associations in the various regions such as MERGA and SAARMSTE have a role to play.

Thank you.

References

Adler, J. (2021). Levering change: The contributory role of a mathematics teaching framework. *ZDM Mathematics Education*, *53*, 1207–1220. https://doi.org/10.1007/s11858-021-01273-y

Brombacher, A. (2011). *Malawi early grades mathematics assessment (EGMA): National baseline report 2010*. Malawi: USAID/Malawi and the Ministry of Education, Science and Technology.

Brombacher, A. (2019). *Research to investigate low learning achievement in early grade numeracy (standards 1–4) in Malawi: The victory of form over substance*. Oxford, United Kingdom: HEART.

Franke, M. L., Kazemi, E., & Turrou, A. C. (Eds.). (2018). *Choral counting & counting collections. Transforming the PreK–5 math classroom*. Portmouth, NH: Stenhouse Publishers.

Hubber, P. J., Outhwaite, L. A., Chigeda, A., McGrath, S., Hodgen, J., & Pitchford, N. J. (2016). Should touch screen tablets be used to improve educational outcomes in primary school children in developing countries? *Frontiers in Psychology*, *7*, 1–4. https://doi.org/10.3389/fpsyg.2016.00839

Kazemi, E. and Hintz, A. (2014). *Intentional talk: How to structure and lead productive mathematical discussions*. Portland, Maine: Stenhouse Publishers.

Kazima, M. (2008). Mother tongue policies and mathematical terminology in the teaching of mathematics. *Pythagoras*, *2008*(1), 53–63.

Kazima, M. (2014). Universal basic education and the provision of quality mathematics in Southern Africa. *International Journal of Science and Mathematics Education, 12*(4), 841–858.

Kazima, M., & Jakobsen, A. (2019). Improving quality and capacity of mathematics teacher education in Malawi project: Collaboration between University of Stavanger and University of Malawi. In E. Kirumira, K. S. Orgeret, R. Krøvel & T. Halvorsen (Eds.), *Sharing knowledge—Transforming societies. The NORHED Programme* (pp. 268–282). Cape Town: African Minds.

Kazima, M., & Jakobsen, A. (2021). Strengthening numeracy through professional development of teachers [Conference presentation]. *Sixth Africa regional congress of the International Commission on Mathematics Education (AFRICME 6*) (pp. 25–27 October 2021), University of Rwanda, Rwanda.

Kazima, M., Longwe, J., & Gobede, F. (2022). Improving teaching and learning mathematics in Malawi primary schools: A review of reforms, interventions, successes and challenges [Special issue]. *Journal of Applied Learning and Teaching*, *5*(2). https://doi.org/10.37074/jalt.2022.5.S2.3

Mbendera, M. (2019). *Exploring the contribution of community-based camps to the reduction of repetition in lower primary grades: The case of primary schools in Zomba rural education district* [Unpublished Master's thesis, University of Malawi].

Milner, G., Mulera, D., Banda, T. C., Matale, E., & Chimombo, J. (2011). *Trends in achievement levels of grade 6 learners in Malawi: Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ) III Reports (Policy Brief No. 1)*. SACMEQ.

Outhwaite, L. A., Gulliford, A., & Pitchford, N. J. (2017). Closing the gap: Efficacy of a tablet intervention to support the development of early mathematical skills in UK primary school children. *Computers & Education*, *108*, 43–58. https://doi.org/10.1016/j.compedu.2017.01.011

Pitchford, N. J. (2018). Unlocking talent through technology: The use of touch-screen tablets and interactive apps to raise early learning outcomes in marginalized children worldwide [Conference abstract]. *Third international conference on educational neuroscience, United Arab Emirates*. https://www.frontiersin.org/10.3389/conf.fnhum.2018.225.00009/event_abstract

Pitchford, N. J., Chigeda, A., & Hubber, P. J. (2019). Interactive apps prevent gender discrepancies in early-grade mathematics in a low-income country in sub-Sahara Africa. *Developmental Science*, *22*(5), 1–14. https://doi.org/10.1111/desc.12864

Pitchford, N. J., Kamchedzera, E., Hubber, P. J., & Chigeda, A. L. (2018). Interactive apps promote learning of basic mathematics in children with special educational needs and disabilities. *Frontiers in Psychology*, *9*(262), 1–14. https://doi.org/10.3389/fpsyg.2018.00262

Ravishankar, V., El-Kogali, S. E.-T., Sankar, D., Tanaka, N., & Rakoto-Tiana, N. (2016). *Primary education in Malawi: Expenditures, service delivery, and outcomes*. The World Bank. https://doi.org/10.1596/978-1-4648-0794-7

Royal Norwegian Embassy. (2017). *Unlocking talent through technology: Improving learning outcomes of primary school children in Malawi*. Norgesportalen. https://www.norway.no/en/malawi/norway-malawi/development-cooperation2/education/

Towards Embodied Validity in Mathematics Education Research

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In 1993, Patti Lather proposed the notion of voluptuous, or embodied, validity. As part of her broader post-structural methodological inquiry, embodied validity was a way of speaking to the importance of lived experience. An author can describe something that happened, but that is significantly different than experience what happened oneself. Consider the transcript that includes a parenthetical remark indicating that a student has laughed. Now start laughing yourself: part your lips and stretch them out, feel the air coming out of your nose and the contraction of the chest; notice your eyes squinting and the repetitive heaving that is accompanied by unusual sounds; all the while, endorphins are released and others around you might be sympathetically laughing too. In my talk, I will be going on an adventure in thinking, one that explores why embodied validity matters in mathematics education research, both as a method for creating and interpreting data and as a method for communicating research. One of my premisses is that despite the growing interest in theories of embodiment, the methods that have been used in mathematics education research are still dominated by static, language-dominant and cognitively focused methods that often fail to adequately understand and express what is at stake in mathematics teaching and learning. The stakes are political because focusing only on what is said or what can be objectively seen and heard, frames both mathematics and knowledge in exclusive ways (de Freitas & Sinclair, 2017).

My talk will proceed in three parts, each motivated by a re-thinking of widely held assumptions about the role of the senses in knowing, the kinds of evidence that can give rise to knowledge, and the forms of experience required to communicate knowledge. Although this adventure began from my own inclusive materialist commitments (de Freitas & Sinclair, 2014), I hope this adventure might also be enriching for those of different theoretical persuasions.

Part 1: What Senses Matter in Mathematics?

When, as researchers, we look at a video clip, we often attend to what is said, both by the teachers and the students. If we are using theories of embodiment, we might also attend to student gestures, or even to their postures or gaze. If they are working with tools of any kind, we might also attend to their actions. Essentially, we are focused on what we can see and what we can hear. Howes (2022) brings together researcher from anthropologists, biologists, neuroscientists, and artists to propose that our typical focus on the five “cardinal” senses is quite restricted and that we may have many, many more (up to 32, according to Young (2021)). These include senses such has proprioception, pressure, rhythm, as well as mixed senses such as hand-eye. If we subscribe to theories of embodiment, surely these senses should matter in our research. However, as researchers, how can we “get at” these senses if we are only watching and listening to videos? Is it enough to be aware of these senses to see if they matter at all in mathematics teaching and learning? Can we feel them by watching videos? These questions are perhaps most important in contexts where students and teachers are using tools or materials that engage these other senses, such as touching screens or weaving thread, but they are also relevant to collective interactions, where sympathy and proprioception are often significant.

Part 2: Aestheticising Experience

Aestheticizing is a term I borrow from Fuller and Weizman (2021), who draw on the Ancient Greek concept of sensory knowing, so that aestheticizing experience is about rendering it “more attune to sensing” (Fuller & Weizman, 2021, p. 34). They use this idea of aestheticizing

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experience to describe the work of forensic architects, which aims "to uncover hidden, obfuscated facts" through tracking movements, disentangling "the making of a situation", working out "the genesis of an incident" and "combining and interpreting clues that are already in the open" (p. 107). In contrast to representational research, which aims to account for what happened, or critical research, which is interested in the hidden formative forces of representations, their investigations are interested in the formation of the representations to propose "new conditions of knowing, seeing and doing" (p. 111). Fuller and Weitzman use simulations to do their work. They will recreate a crime scene, for example, and run through the crime multiple times in order to create new conditions of knowing: for example, what could be heard or seen or smelt from different positions? What if we as researchers also conducted simulations, by re-enacting the videos from research sites (see Günes et al., 2024)? Would we be able to feel the pressure required to make something move on the screen or the changing nature of hand-eye coordination as screen touches are made? In other words, rather than seeing bodily movements only as the representation of mathematical thinking, what new conditions for knowing and doing mathematics might arise?

Part 3: *Not* Obviating the Necessity for Direct Experience

Shapin (1984) traces the technologies of scientific research that were devised by Robert Boyles during his air pump experiments in the 1650s, and that have since become normative in empirical research (including mathematics education research). Shapin argues that Boyles created a literary technology of virtual witnessing, which was "the production in a reader's mind of such an image of an experimental scene as obviates the necessity for either its direct witness or its replication" (p. 491). The researchers must provide details of what happened, in objective ways, so that the reader finds the experiment credible (as well as potentially replicable). As a consequence, writing the experimental report became just as important as the experiment itself, since it enabled the establishment of matters of fact by the public.

This is what we do in our journal publications. Our writing is circumstantially dense as we provide descriptions of the research setting and even more specific and detailed accounts of the events that occurred, including the things said as well as the actions made. Transcripts are part of this technology of knowledge production, enabling the kind of virtual witnessing that Shapin identifies (Sinclair, 2024). However, I wonder whether transcripts really obviate the necessity for direct experience? In the case of accounting for experience in which multiple senses are at stake, like the kinetic and the tactile, witnessing seems insufficient. What new literary technologies might render possible direct experience?

References

de Freitas, E. & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.

de Freitas, E. & Sinclair, N. (2014). The politics of the mathematic aesthetic: Curricular con(sens)us and acts of dissensus. *Philosophy of Mathematics Education Journal, 28*, 1–21.

Günes, C., Paton, K., & Sinclair, N. (2024). The sensory politics of mathematics: Aestheticising multiplication. *Educational Studies in Mathematics*, (in press).

Fuller, M., & Weizman, E. (2021). *Investigative aesthetics: Conflicts and commons in the politics of truth*. Verso Books.

Lather, P. (1993). Fertile obsession: Validity after poststructuralism. *Sociological Quarterly, 34*(4), 673–693.

Shapin, S. (1984). Pump and circumstance: Robert Boyle's literary technology. *Social Studies of Science, 14*(4), 481–520. https://doi.org/10.1177/030631284014004001

Sinclair, N. (2024). Knowing as remembering: Methodological experiments into the embodied experiences of number. *Digital Experiences in Mathematics Education, 10*(1), 29–46.

Young, E. (2021). *Super senses: The science of your 32 senses and how to use them*. John Murray Publishers.

Beth Southwell Practical Implications Award

The Beth Southwell Practical Implications Award was initiated and sponsored by the National Key Centre for Teaching and Research in School Science and Mathematics, Curtin University, Perth, Western Australia. Curtin sponsored the “Practical Implications Award” (PIA), as it was then known, for the first 10 years. The Australian Association of Mathematics Teachers (AAMT) now sponsors the Award. In 2008, MERGA was honoured to be able to rename the PIA as the Beth Southwell Practical Implications Award (BSPIA), in honour of MERGA’s and AAMT’s esteemed late member, Beth Southwell. The award is designed to stimulate the writing of papers on research related to mathematics teaching or learning or mathematics curricula. Application for the award is open to all members of MERGA who are registered for the conference. Applications for the BSPIA are judged against specific criteria set by a four-member panel. The panel consists of two members from MERGA, two from AAMT, and is chaired by the MERGA Vice President (Development).

Turn Left, Turn Right: An Embodied Perspective on Children's Difficulties With Left/Right Spatial Orientations

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In this paper we problematise the expectation that Year 1 (6/7 years) children can effectively discriminate left/right, enact left/right-turning directions and use the language of left/right to give directions. Results from task-based interviews with 36 Year 1 children are interpreted through the lens of embodied cognition and spatial frames of reference to reveal some of the complexities and cognitive demands of learning what it means to 'turn left' or 'turn right', as a basis for further investigation.

The Australian Curriculum: Mathematics includes the following achievement standard for Year 1 (the second year of schooling), "Children give and follow directions to move people and objects within a space." Further elaboration of the learning expectation emphasises the role of language and "understanding the meaning and importance of the words when giving directions: for example, using words like 'forwards' and 'backwards', straight ahead', 'left or right' to describe movement" (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2022). The learning requirement is mirrored in the NSW state Mathematics K–10 syllabus with the statement, "Give and follow directions, including directions involving turns to the left and right" (NSW Education Standards Authority [NESA], 2022). The presence of this learning expectation for 6/7-year-olds implies that the meaning of turning left and right is not innately developed and needs to be taught, and that it is reasonable to expect children of this age group to achieve competence in both following and giving such directions. Yet left/right confusion in people is common and research to inform the teaching of left/right discrimination and its application to navigating the environment is scarce.

The current curriculum clearly specifies the involvement of moving people, yet much of what we know about children's lateral left/right discrimination abilities comes from psychological research conducted decades ago involving inanimate objects and static tasks rather than dynamic tasks involving moving the body and turning left/right. The curriculum also emphasises the importance of understanding the meaning of *relative* directional words. Humans, like many other animals, have a natural 'front-facing' perspective that assists us in orienting ourselves in space. In the English language, a sense of 'front-facing' is essential for the understanding the *viewer-centric* terms left/right and turn-left/turn-right. However, some other cultures and languages make greater use of an allocentric frame of reference (FoR) in which locational and directional terms refer to objects and landmarks (Abarbanell & Li, 2021).

In this study we focused on situations where the viewer-centric perspective is central to the enactment of spoken directions and the production of spoken directions by children. We concentrated on two scenarios for utilising the viewer-centric perspective; one involved the egocentric FoR relating the child's own body and the other was where the child had to switch the FoR to the viewpoint of another front-facing entity. In the former scenario the child would be responding to verbal instructions to move their own body, and in the latter scenario, the child would be directing someone else to move. To reflect the exploratory nature of the investigation, the study was guided by the open-ended question: How do Year 1 children respond to, and give, verbal instructions to turn-left and turn-right?

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 25–32). Gold Coast: MERGA.

Background Research

In previous research an important distinction has been made between the *discrimination and recognition* (awareness) of left and right and the *verbal identification* of left and right (Rigal, 1994; Roberts & Aman, 1993). Discrimination and recognition of the left and right of one's own body develops much sooner than verbal identification. Some children may correctly verbally identify their own left and right by 7 years, but verbal identification of someone else's left and right takes longer to develop. Half of the 11-year-olds in Rigal's (1994, 1996) study still could not correctly apply the words left and right to someone else's perspective. The long development period was attributed to the children's persisting egocentric spatial FoR and the cognitive demands of performing mental rotations to imagine other perspectives (e.g., Rigal, 1996; Roberts & Aman, 1993). Notably, studies in this era consistently used static two-dimensional displays of geometric shapes rather than dynamic and/or embodied situations in the three-dimensional world.

More recently, the broader acceptance of embodied cognition theory (e.g., Dackermann et al., 2017; Keifer & Trumpp, 2012) and evidence from neuroscience about spatial processing and memory (e.g., Ruggiero et al., 2016), has led to renewed interest in the development of spatial FoR, the role of body movement and the implications for education. A modest quantity of research on 'turning the body' has arisen from the field of technology and robotics education with children (e.g., Clements et al., 1996; Kocher et al., 2020).

Navigating Without Language

At a basic level, physically navigating oneself around a spatial environment involves an egocentric spatial FoR in which all orientations, positions, directions and movements are processed in relation to one's own body. We can achieve tasks such as walking across a room while navigating around tables and chairs, at an embodied level, with minimal cognitive demand. In essence, we can move forward and turn left and right, whenever required, without actually thinking about it or needing to connect our actions with language—we can just do it. Similarly, we could push an object around a structured spatial environment, much like a chess piece on a chess board, without connecting language to the movements. Perhaps it could be argued that the latter scenario also involves egocentric referencing because of its proximity (Ruggiero et al., 2016), and the direct manipulation of the 'object' may be perceived as an extension of oneself.

If we wanted to guide another person across the room following the same path (and they could see us) we could again use embodied modes to communicate by gesturing and pointing. Such a task appears to use an *allocentric spatial FoR*, where we consider the location of objects in relation to other objects. In this sense, we perceive the other person as an 'object' that we can move without actually touching because 'it' responds to our hand signals. Consider the effectiveness of directing a driver trying to reverse a vehicle into a parking space using gestures rather than calling out left and right directions. Without the use of language, it is not necessary to use a viewer-entered perspective and switch to the other's perspective.

Navigating with Language

In each scenario above, embodied modes are dominant in the navigation of the three-dimensional environment and language is redundant. The encoding of spatial frames of reference—the attachment of specific language to spatial and directional concepts—creates another layer of complexity. While some research suggests that increased exposure to spatial language supports the development of spatial skills (Casasola et al., 2020), we know little about how children learn frame-of-reference words (Shusterman & Li, 2016) and even less about how to effectively teach this aspect of spatial competence.

Being able to give left/right directions to someone else appears to be more difficult than following left/right directions. Waller (1986) reported that 5 to 6-year-olds could differentiate left/right but had difficulty in remembering left/right instructions and giving appropriate left/right directions to others. Giving effective directions requires the cognitive flexibility to utilise an allocentric spatial FoR in a way that requires switching spatial perspectives from one's own perspective to the perspective of the other person—plus an awareness of the need for accurate instructions (Waller, 1986). Cognitive science research provides at least a partial explanation for different levels of difficulty through establishing that spatial memory of egocentric-based experiences is easier to retrieve than spatial representations that involve allocentric referencing, which involves a different part of the brain (Ruggiero et al., 2016).

In recent years, educational research regarding children's interactions with computers and robotics has produced some insights into the development of navigational language through the use of verbal commands (Kocher et al., 2020). In their research with 4-to-9-year-olds, Kocher et al. (2020) asked the children to verbally guide an adult (acting as a robot) to find a 'treasure' the child had hidden in the room. The researchers categorised the navigational commands the children naturally used into three levels that illustrate increasing precision in spatial language, alongside decreasing dependence on embodied communication. Explication of the levels was intended to inform teaching and the choice of types of digital robot suitable for use with the children. The *Beginner level* featured the use of landmarks and gestures. For example, 'go to the chair' or 'walk round the table'. Such direction might utilise gesture to reduce the demand for language, and vague terms such as 'Go there' and 'Turn' needed to be accompanied by gesture to be effective in producing the desired movement by the robot-actor. At the *Intermediate level* more specific spatial terms were used such as 'left', 'right', 'turn-stop', 'forward' and 'diagonal'. Self-corrections were made following the feedback afforded by the robot-actor's response to a command (e.g., 'the other left'). *Advanced level* commands were more precise, such as 'turn-left' (sometimes with the angle specified), or the inclusion of distance (e.g., forward 5 steps) (Kocher et al., 2020).

An important language consideration is the meaning of 'turn' in different contexts, particularly in relation to the context of robotics. In most circumstances the command 'turn left' would be interpreted as rotating the front-face until the new front-face is 'looking' directly to the left-side of the original position, that is, a turn of 90º. However, depending on the context, the movement that achieves the turn can be quite different. For example, in the contexts of walking or driving a car, the command 'turn left' would be enacted gradually by moving forward simultaneously with the left-turn motion, creating a curved pathway (Clements et al., 1996). So, there is a change in both location and the front-face orientation. However, in the context of robotics the left turn would be achieved by a rotation of 90º on the spot, without any forward movement. The robot's location remains the same and only the orientation of the front-face changes. Bakala et al. (2021) suggest the method of turning in robotics is counterintuitive for children and so would require special attention in teaching.

Methods

Context and Participants

This study is situated within the *Embodied Learning in Early Mathematics and Science* project (2021–2024), the aim of which is to translate research on embodied cognition into classroom practice for Preschool through to Year 2 (4- to 8-year-olds). The participants for this study were randomly selected Year 1 children from three different schools (New South Wales, Australia) which had not been part of the professional learning intervention aspect of the larger project. A total of 36 children returned consent forms from their parents. School 1 is a medium-sized primary school with 409 children, 29% of children are from a LBOTE (Language Background other than English). School 1 has an ICSEA (Index of Community Socio-

Educational Advantage) score of 1121 indicating a higher SES (socio-economic status) compared with other schools in Australia. [Note: An ICSEA score of 1000 is set as the average benchmark with which other schools are valued as lower or higher levels of educational advantage of student populations (ACARA, 2020)]. School 2 is a medium-sized primary school with 301 children, 33% of children are from a LBOTE. School 2 has an ICSEA score of 1028 indicating it has a higher SES. School 3 is a medium-sized school with 553 children. School 3 is identified as having a high, 94%, LBOTE child population and lower level of educational advantage with an ICSEA of 950.

Task-Based Interviews

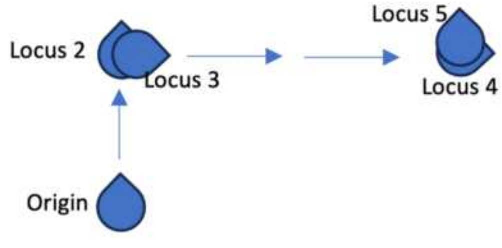
Task-based interviews (Goldin, 2000) were carefully designed for the purpose of the study and to elicit each child's embodied "representations of particular mathematical ideas" (Maher & Sigley, 2014, p. 821), in this case the concepts of right-turn and left-turn. The one-to-one task-based interviews were conducted by five different researchers. The delivery of the tasks was scripted and practiced for consistency of question-asking by the interviewers. However, it is acknowledged that some variations still occurred within the child-interviewer interactions, mainly due to the unpredictability of the child responses. The interviews were video recorded, with additional permission given by the children themselves before commencing the tasks.

Task 1: You be the Robot

Task 1 required the child to respond to spoken orientation and directional terms (turn-right, turn-left) by moving their own bodies, and therefore utilised a viewer-centric perspective and an egocentric spatial FoR. The task began by inviting the child to stand in a clear floor space, then the interviewer followed the script (Table 1), delivered one part at a time. No feedback was given to the child during the task and incorrect moves were simply followed by the next instruction. In the mapping of the correct movement responses (Table 1) the right-turn is represented by Locus 2 and 3, and the left-turn by Locus 4 and 5.

Table 1

Task 1 Details

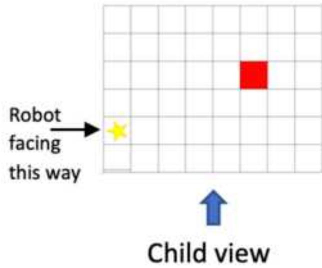
Task 1 script	Mapping of correct movements
Let's play a robot game. You be the robot. Stand up like a robot. I'm the robot controller. Robot, take 1 step forward (*pause*) Robot, turn right (*pause*) Robot, take 2 steps forward (*pause*) Robot, turn left (*pause*) Robot, sit on the chair (*point to child's interview chair*)	

Task 2: Direct the Robot

Task 2 required the child to give verbal commands to a toy robot (animated by the interviewer), and therefore required an allocentric spatial FoR and switching the viewer-centric perspective to the robot's viewpoint. The task began by placing a grid on the desk in front of the child (see Table 2 for the orientation) and showing the child the toy robot, followed by scripted instructions, all delivered at the beginning. The child was not permitted to touch the robot and the interviewer moved the robot as if it could hear the child's verbal instructions. Therefore, if the child did not give a verbal instruction the robot did not move, and the child was prompted to try again. Feedback was often (but not consistently between interviewers) provided if the child did not use the verbal prompt related to a specific direction. For example, if the child said, 'go there' or 'take one step', a prompt might have been 'which way?' Unavoidable feedback was also provided by the robot's response to a command—which may or may not be the movement envisaged by the child.

Table 2

Task 2 Details

Instructions	Robot grid and placement
Now you be the controller for this little robot. The robot can only move one square at a time. It won't turn left or right unless you tell it to. Your job is to tell the robot how to move from here (place robot on star) to here (point to red square). What's your first order for the robot? (Interviewer moves the robot according to child's instructions). (If child does not commence, demonstrate) Robot, take 1 step forward (move the robot one square). Your turn to tell the robot	 Toy robot 

Analysis

Both tasks included other concepts such as forward movement and distance, but the focus of analysis was narrowed to the instances of right (R) and left (L) turns for the purposes of this paper. Two analysis approaches were applied to viewing the videos. The deductive approach recorded whether each child could produce a 'correct' response (yes/no). The inductive approach recorded more information about the nature of each child's response using descriptive text. An analysis spreadsheet was constructed and two members of the research team who had conducted the interviews analysed the videos, cross-checked each other's analyses, and resolved any difficult interpretations through discussions.

Correct for Task 1: You be the Robot

Correct body movements in response to the verbal instruction Turn right (R), or Turn left (L):

- R/L awareness: Gave some indication that they know which is their R/L side, such as a gesture or other movement to the R/L (including turning);
- R/L Turn: Turned R/L a quarter turn (90º) to reorient their body to R/L of their original 'front-facing' position—and remains in that position and orientation.

Correct for Task 2: Direct the Robot

Correct verbal commands (on first attempt) given to the robot relative the robot's Right (R) or Left (L).

R/L awareness: Gave some indication that they know which is the robot's R/L side. To determine correctness the researcher must make a judgement about the child's intent. For example, if a child says 'Turn left' but shows surprise when the interviewer turns the robot to the robot's left, then the spoken direction did not match the child's intent, so was recorded as incorrect. When incorrect, the command was classified as R or L according to the intended direction. Pointing to the grid square to the left of the robot or using a left gesture (with no accompanying verbal 'left') were recorded as incorrect, as these embodied actions 'may' have indicated L/R awareness, however they may also have simply indicated the child knew which way or path to take to the target square.

R/L Turn: Gave a verbal command to Turn R/L. Verbal instructions such as 'Go Left', or non-verbal instructions such as pointing were not recorded as correct.

Results

The results of the two Robot Tasks are first presented separately with a focus on specific findings in relation to egocentric (Task 1) and allocentric/view-switching (Task 2) spatial FoR. Second, findings from across the two tasks are presented as general findings highlighting connections and comparisons. Table 3 provides a summary of correct responses for L/R awareness and turn for each task. This table will be referred to throughout the results section in conjunction with the descriptive text regarding children's alternate incorrect responses.

Table 3

Deductive Analysis Summary: Number of Children with Correct Responses

	Task 1 (egocentric)					Task 2 (allocentric/perspective switching)				
School (*n*)	R aware	R turn 90º	L aware	L turn 90º	All yes	R aware	R turn verbal	L aware	L turn verbal	All yes
1 (10)	7	6	7	3	2	3	3	7	6	2
2 (15)	12	9	12	5	5	5	3	12	6	3
3 (11)	9	6	8	2	1	1	1	3	1	1
Total (36)	28	21	27	10	8	9[α]	7[α]	21	13	6[β]

[α]Some children did not have the opportunity to make a right turn command because the pathway they chose for the robot only required left turn/s.

[β]All yes in Task 2 only refers to children who provided commands for both R/L.

Task 1 Results

Across the three schools eight children obtained a complete correct (R and L) score for the egocentric task, and eight children we unable to complete any aspect correctly. Of the eight children who were unable to complete Task 1, most (7) turned, but turned the opposite direction, for example turned L for R, therefore they scored incorrectly on both awareness and turn. The remaining 20 children's responses were mixed. All except one showed both R and L awareness. Of the 19 that showed R and L awareness, 10 children could also turn R and turn L–but turned L 180° instead of 90°. It was noted that turning L from Locus 4 (see Table 1) would place children facing L if a L turn was completed facing the 'original' front (starting direction) of Task 1. Overall, children were more competent with R and L awareness than physically turning R or L 90° as indicated in Table 3. However, if interpreting correct 'turning' as correct directional turn, not necessarily correct rotation amount, then 18 children could be considered as scoring complete correct on Task 1. Where children scored incorrect for R or L turn, they either turned in the opposite direction or stepped sideways in the correct direction.

Task 2 Results

Six children obtained a complete correct (R and L) score for the allocentric task. As noted in the analysis, Task 2 could be completed without a R turn, therefore, an additional four children were correct for the turns they use to direct the robot. Observational notes made about these children revealed; two children made 'Ls' with their hands/fingers to know which way was left before stating their directions, and one child moved around the desk to position their body behind the robot's starting position. There were 13 children unable to give any correct verbal command for the robot to turn R and/or L. Eight of these children indicated to move/turn but said the opposite direction (e.g., R for L), half (4) then self-corrected after seeing the robot move/turn the opposite direction to what they intended. Table 3 shows that children from School 3 (high NESB, low SES school context) had the most difficulty in completing the allocentric/view-switching task. The majority (8) of the 13 children unable to complete the allocentric task were from School 3. Some children (16) used gestures or co-speech gestures. Gesturing alone was used to indicate/point to the square to move to or the possible direction—not specifically linked to L or R as no verbal 'left' or 'right' were uttered. Co-speech gesture was used in children's commands, for example 'go left [while pointing in a L direction]', or 'turn up [while gesturing L]'. Gesture or co-speech gesture were used more frequently by children from School 3 (8 of the 11 children from School 3).

General Findings

Four of the 36 children scored a completely correct score on both tasks. Five children who were unable to follow any directions in Task 1 (egocentric), could give some correct directions in Task 2 (allocentric/switching). One of those children made no errors in the directions they provided in Task 2. Six of the 10 children who correctly completed Task 2 had mixed results in the egocentric task. In the allocentric task, like the egocentric task, more children were L aware than able to provide a verbal L turn command. Children would often use alternate words for turn including 'go left', 'turn up', 'this way', 'turn that way [gesture]', 'go right', 'turn straight', 'one step left'.

Discussion

Our findings aligned with previous research related to the 5-to-7-year-old age range in several ways. The majority of children were successful in responding to the viewer-centric terms 'left and right' when operating with an egocentric FoR (Waller, 1986; Ruggiero et al., 2016). We suggest there is a relationship between the lower success rate in School 3 and the school's social/educational disadvantage and high non-English speaking population. It is possible that some of the children's home languages preference allocentric terminology based on landmarks and object-to-object relationships (with little use of left/right), rather than viewer-centric navigational terms (as in English), as discussed by Abarbanell and Li (2021).

The children found following left-right directions easier than giving left-right directions, aligning with Waller (1986). Compared with Task 1, fewer children were able to successfully take the viewer-centric perspective of the robot in Task 2, though the difference was not large. The success of some students in Task 2 could possibly be attributed to the availability of feedback through seeing the response of the robot to a command, and the availability of the grid for visualising a route to the target location. However, there were a few children who, surprisingly, performed better in Task 2 than Task 1, and this requires further investigation. Like Kocher et al. (2020), we found that the children tended to use gesture and vague directional terms rather than the precise commands of left/right or turn. Some children managed to produce the appropriate directions when prompted, but others found it beyond their current capability. The main focus of the study was the children's responses to the turning aspect of the tasks, and we found that the children interpreted the meaning of 'turn' in a variety of ways, with some children even responding in different ways within the same task. For example, in Task 1, 10 children correctly turned right 90°, but when asked to turn left inexplicably turned 180°. We are yet to find any explanation for this in previous research. Some children responded by turning 90°, but in the opposite direction, while others did not rotate their bodies at all, instead stepping sideways or just turning their heads or shoulders then resuming their front-facing orientation. In Task 2, many children were not able to give the verbal command 'turn' to the robot, which perhaps can be partly explained by the requirement in the robotics context to change the orientation of the robot by rotating on-the-spot before moving to a new location, as indicated by Clements et al. (1996) and Bakala et al. (2021).

Conclusion

This study investigated the question, 'How do Year 1 children respond to, and give, verbal instructions to turn-left and turn-right?' Keeping in mind the limitations of a small sample of children, we offer some tentative propositions for further investigation. First, the mastery of the viewer-centric terms of left, right, and turn is problematic for both spatial/cognitive and cultural/linguistic reasons. Second, the tendency of children to preference embodied representations and 'landmark' cues may offer a starting point for instructional practices. Third, further attention should be given to the differing meanings of 'turn' when enacted in different contexts. We refrain from making more specific recommendations until analysis of the much larger data set from the project has been completed.

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References

Abarbanell, L., & Li, P. (2021). Unraveling the contribution of left-right language on spatial perspective taking. *Spatial Cognition & Computation*, *21*(1), 1–38.

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: Mathematics (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-1

Bakala, E., Gerosa, A., Hourcade, J. P., & Tejera, G. (2021). Preschool children, robots, and computational thinking: A systematic review. *International Journal of Child-Computer Interaction*, *29*, Article 100337.

Casasola, M., Wei, W. S., Suh, D. D., Donskoy, P., & Ransom, A. (2020). Children's exposure to spatial language promotes their spatial thinking. *Journal of Experimental Psychology: General*, *149*(6), 1116.

Clements, D. H., Battista, M. T., Sarama, J., & Swaminathan, S. (1996). Development of turn and turn measurement concepts in a computer-based instructional unit. *Educational Studies in Mathematics, 30,* 313–337.

Dackermann, T., Fischer, U., Nuerk, H. C., Cress, U., & Moeller, K. (2017). Applying embodied cognition: From useful interventions and their theoretical underpinnings to practical applications. *ZDM Mathematics Education*, *49*, 545–557. https://doi.org/10.1007/s11858-017-0850-z

Goldin G (2000). A scientific perspective on structures, task-based interviews in mathematics education research. In: R. Lesh, A. E. Kelly (Eds.), *Research design in mathematics and science education*. Erlbaum, (pp. 517–545).

Kiefer, M., & Trumpp, N. M. (2012). Embodiment theory and education: The foundations of cognition in perception and action. *Trends in Neuroscience and Education*, *1*(1), 15–20.

Kocher, D., Sarmiento, L., Heller, S., Yang, Y., Kushnir, T., & Green, K. E. (2020, October). No, your other left! Language children use to direct robots. *Joint IEEE 10th international conference on development and learning and epigenetic robotics* (pp. 1–6). IEEE.

Maher, C. A., & Sigley, R. (2020). Task-based interviews in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*, (pp. 821–824). Springer.

NSW Education Standards Authority. (2022). *Mathematics K–10 syllabus*. NSW Education Standards Authority.

Rigal, R. (1994). Right-left orientation: Development of correct use of right and left terms. *Perceptual and Motor Skills*, *79*(3), 1259–1278.

Roberts Jr, R. J., & Aman, C. J. (1993). Developmental differences in giving directions: Spatial frames of reference and mental rotation. *Child Development*, *64*(4), 1258–1270.

Ruggiero, G., D'Errico, O., & Iachini, T. (2016). Development of egocentric and allocentric spatial representations from childhood to elderly age. *Psychological Research*, *80*, 259–272.

Shusterman, A., & Li, P. (2016). Frames of reference in spatial language acquisition. *Cognitive Psychology*, *88*, 115–161.

Waller, G. (1986). The use of 'left' and 'right' in speech: the development of listener-specific skills. *Journal of Child Language, 13*(3), 573–582.

Research Symposia

Symposium: From Tensions to Opportunities: Evidencing Mathematics Leadership

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This symposium offers insights into the leadership enacted by those who lead the mathematics education professional learning of in-service teachers in schools. We provide evidence of mathematics leadership practice as a way of contributing knowledge to this undertheorised area of mathematics education research. Three separate accounts of mathematics leadership are reported, with two focused on leading enacted in primary school settings, whilst the third paper highlights the support offered to rural and regional mathematics leaders through a sector-wide leadership network initiative.

Although separate accounts of leadership are presented, each paper is connected through the ways that tensions in practice provided opportunities for mathematics leaders to develop leading practices within the spaces in which their leadership was enacted. In this symposium, the relational dimension of mathematics leadership is highlighted, providing evidence of the critical role that relationships play in the ways that mathematics leadership responds to tensions as opportunities for practice development.

The format of the symposium is as follows:

Chairs: Matt Sexton and Ann Downton.

Paper 1: *Evidencing How Primary Mathematics Leaders Balance the Supports and Challenges of Their Role.*
Kate Copping & Natasha Ziebell.

Paper 2: *Evidencing Mathematics Leadership as Relational and Developmental Activity*.
Matt Sexton & Ann Downton.

Paper 3: *Evidencing Sector Leadership for Mathematics Leaders Working in Rural and Regional Schools.*
Bernadette Pearce, andrea O'Connor, & Lauren Gould.

Discussant: Peter Grootenboer.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 35–47). Gold Coast: MERGA.

Evidencing How Primary Mathematics Leaders Balance the Supports and Challenges of Their Role

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Primary mathematics leaders work together both with and between school leadership and teachers as middle leaders, balancing expectations and responsibilities of themselves, school leadership, and teachers. This paper presents a case study of mathematics leaders in a school with a diverse community and frequently changing staff. It explores the tensions in this school and the supports that help the mathematics leaders find opportunities to respond. Findings show that building relationships and trust with staff were essential to address challenges and meet the needs of the school.

Primary mathematics leaders (PMLs) are middle leaders working with and between school leaders and teachers with a recognised responsibility for improving student learning (Copping, 2022). Mathematics leadership is a multi-faceted and complex role that includes balancing both management and leadership responsibilities to improve teacher impact on student learning (De Nobile, 2017; Gurr & Drysdale, 2013). This paper focuses on the following research question: How do PMLs balance the challenges and supports within their role at a metropolitan school in Melbourne? A case study is reported from one school, Wattle Tree Primary School (WTPS), which has the unique situation of having two mathematics leaders. The narrative incorporates viewpoints of mathematics leaders and others to convey the challenges and supports for primary mathematics leadership at WTPS. Pseudonyms have been used throughout this paper.

Literature Background

Primary mathematics leaders play a strategic role in the improvement of mathematics practices as the critical link between a school's vision and the work enacted in classrooms, working with school leaders and directly with teachers (Leithwood, 2016). In Australia, middle leaders often have a teaching aspect to their role, maintaining current classroom practice, while also working with teachers in other classrooms (Grootenboer et al., 2015). Teachers therefore view middle leaders as still being connected to the classroom and practising alongside them. Concurrently, middle leaders work with school leadership guiding a strategic, whole-school approach to teaching and learning (De Nobile, 2017). A significant role associated with PMLs is ensuring that professional development is localised and targets the specific needs of the school, teachers, and students. Leading this professional learning (PL) in a school requires the development of relational trust with all participants (Grootenboer & Edwards-Groves, 2020).

Common practices of successful middle leaders focus on student learning and teacher development through fostering a clear vision and strategy, collective responsibility, and trust, with high expectations of themselves and others (De Nobile, 2017; Gurr & Drysdale, 2013). These ideas were developed further in a framework of middle leadership, which underpins this research exploring the roles and responsibilities of PMLs. The framework is designed for exploring the roles of middle leaders, as distinct from senior leadership positions, such as a principal's role (De Nobile, 2019). It is important to note that this framework is not a discipline-based model of leadership, but it can be applied to the roles and responsibilities of a primary mathematics leader. The six role aspects are "Student focussed, Administrative, Organisational, Supervisory, Staff development, and Strategic" (De Nobile, 2019, p. 3). This framework of role aspects is situated along a continuum from predominantly managerial tasks at one end to leadership aspects on the other end, reflecting the diverse responsibilities of the role. Each stage supports and enables more effective enactment in the subsequent level.

A key aspect of success in the leadership role is attributed to the principal's support for a PML by showing trust in the middle leader's expertise, as well as providing the time, organisation, and resourcing to support change and professional learning (Grootenboer et al., 2020). A shared understanding by all staff of their roles and responsibilities is important to attain a school's goals. However, there are contextual factors that can influence or challenge the success of this collective understanding, including time allocation for the role and teachers' experience (Copping, 2023). Challenges for PMLs identified by Driscoll (2017), included: time, leader's expertise, teacher knowledge, and funding. Furthermore, Sexton and Downton (2014) noted the impact of time constraints on the ability to fulfil leadership duties and the difficulty faced in sustaining improvements in changed practices. The research in this paper investigated supports and challenges, such as these, experienced by PMLs at WTPS.

Methodology

This case study forms part of a PhD research project examining how primary mathematics leadership is conceptualised and experienced. A phenomenological approach (Heidegger 2008/1928) has been utilised to acknowledge the experiences, perceptions, and understandings from the participants. The wider project included eight schools within the state of Victoria. Interviews were conducted with the PML/s, a school leader, and two teachers at each school to investigate how primary mathematics leadership was conceptualised and experienced by those in the role, and by those with whom they work. This paper reports on interviews from WTPS, a Government, Foundation to Year 6 school in the northern suburbs of metropolitan Melbourne. At the time of data collection WTPS had an enrolment of ≈400 students. The diverse school community was well below average in socio-educational advantage (>90% English as an Additional Language).

Semi-structured interviews were conducted with five participants at WTPS: Mia, PML for Years 3–6, Zara, PML for F–Year 2 (both worked three days as mathematics leaders and two days as classroom teachers), Assistant Principal Holly, experienced teacher Anna, and graduate teacher Justin. The semi-structured interviews varied slightly depending on the role of the participant. Participants were asked what challenged or supported the PMLs in their role. Interview data was analysed using an inductive approach to identify, summarise, and refine themes (Thomas, 2006), which was then applied to the Framework for Middle Leadership (De Nobile, 2019) to support interpretation of the analysis.

Results and Discussion

Zara (PML F–2) was new to the school and role but had been a primary mathematics specialist at her previous school. Mia (PML 3–6) had been the mathematics leader in the school for more than 5 years. Tensions faced at WTPS by the PMLs were recognised by all participants, as well as supports available which presented opportunities for action.

Consistency of the Instructional Program

Staff turnover at the school was high and there was a large proportion of graduate teachers. The school population was changing with families moving in and out frequently with some students attending an intensive English language school part time on site. Due to these regularly changing circumstances, the school's focus was on consistency of the instructional program. The PMLs were responsible for supporting staff to implement the instructional program, acting in a *Supervisory* role (De Nobile, 2019). This was identified as a clear expectation by all participants, and an area of tension for the PMLs, as Anna stated:

> It's been quite a high turnaround of staff ... there is an instructional model in place in terms of how mathematics is expected to run. But because of Covid, because of the high turnaround of staff, it's hard to then, disseminate it to others, when you're constantly having to start again, start from fresh.

Sustaining and maintaining an instructional model with a consistent approach has been previously recognised as a challenge for PMLs (Sexton & Downton, 2014). Both Zara and Mia acknowledged this as an issue. In response, they enacted their *Staff development* role (De Nobile, 2019), and conducted whole school PL sessions together and participated in coaching sessions with individual teachers, on a needs-based approach. Crucially, Zara and Mia harnessed the opportunity to focus on the development of supportive, trusting relationships (Grootenboer & Edwards-Groves, 2020) to aid in the implementation of the instructional model and address the challenge of changing staff. Mia stated:

> I also think it's the good relationships that I have with the teachers. Because the teachers trust me and especially when we do coaching...it's personal conversation, and I'm not going to be sharing this to other people, and it's confidentiality and non-judgmental. So, teachers give me their trust.

Consistency of Timetabling

Time was another recognised area of tension by all participants. Covid implications resulted in days timetabled for leadership work being used to cover teacher absences. Timetabling and coaching sessions were affected and the *Staff development* role (De Nobile, 2019) was not able to be implemented as planned. Holly stated:

> It's [the PMLs] being used as CRTs [Casual Relief Teachers] a little bit at the moment, and then the coaching disruptions. The inability sometimes to do that part around the conversation after coaching. That's a challenge, because then I'm not able to support with that time.

Zara and Mia's time for mathematics leadership was affected by teacher absences and staffing issues, as Zara said, "Obviously the absences and not being able to do the coaching is a massive challenge." This impacted their opportunity to get into classrooms and to enact their plans as intended. They experienced frustration due to time constraints (Driscoll, 2017). However, this challenge was recognised by the leadership team, as Holly said, "Also giving them that time out of the classroom, I think is essential." The support of school leadership was evident (Grootenboer et al., 2020) with extra time provided where possible. The PMLs also adjusted their schedules to find opportunities to support teachers, as Mia explained:

> I try to, even though time is a challenging issue for me, but I still try to, whenever I'm available, I get to make-up my coaching or I get to meet this teacher catch up with teacher ... even though sometimes I lose my lunch or I lose my time, but I still, I still want to support people.

Being a Middle Leader

Tension was noted by participants that working with and between school leadership and teachers requires balancing leadership, *Administrative, and Organisational,* responsibilities (De Nobile, 2019). Justin stated, "Being part of leadership here, I know you do get roles outside your maths domain." Zara also noted that managing leadership duties was challenging, "It is hard, and I find I'm unfortunately the kind of person that just, especially when you're in leadership, I take on all those things that I can't control." These extra duties can be draining on middle leaders and it is important that PMLs do not get overwhelmed by administrative responsibilities (Grootenboer et al., 2020). However, the support between Mia and Zara to help manage their roles was widely noted by the participants, particularly Zara:

> Mia is an amazing support because she's, one, been at the school for, for longer than me, but also she's got great maths knowledge. But also just, she's further along in her leadership journey... So, um, she's been great to talk to about what's going on and how possibly to approach.

Additionally, participants discussed the support the PMLs had from school leadership noting the principal trusted the PMLs and valued their role (Grootenboer et al., 2020). Zara discussed the possibilities of her role, saying, "I think that the, the principal is very supportive of the role and that is massive ... I know that it's really valued." While Mia stated, "My principal trusts that I'm doing my job well ... I get my opportunity to work with teachers and work with students." Both identified the opportunities present while acknowledging the challenges they face and that supportive relationships built on trust are essential to their role.

Conclusion

Balancing the responsibilities and roles of PMLs, such as leadership, staff development, supervisory, administration, and organisation (De Nobile, 2019) was impacted by the school's needs and resources (e.g., human, time). However, supports were in place which afforded opportunities to address these tensions. One of the most significant findings was the development of trust between the two PMLs, between school leadership and PMLs, and between teachers and PMLs. The nurturing of supportive relationships enabled the PMLs to meet challenges with the development of relational trust being central to their success. Support and trust were not one-way, but multi-layered and multi-directional. For PMLs and school leaders, the relationships between and within PMLs, school leadership, and teachers are important to build and maintain, as it is the relationships and the support offered through them which provide opportunity to address tensions within schools.

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References

Copping, K. (2022). Perceptions of the role of primary mathematics leaders. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Group of Australasia*, (pp. 146–153). Launceston: MERGA.

Copping, K., Ziebell, N., & Seah, W.T. (2023). Shared expectations? An exploration of the expectations between primary mathematics leaders and teachers. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel., & M. Tabach (Eds.), *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol 2, pp. 195–202). PME.

De Nobile, J. (2017). Towards a theoretical model of middle leadership in schools. *School Leadership & Management*, *38*(4), 395–416. https://doi.org/10.1080/13632434.2017.1411902

De Nobile, J. (2019). The roles of middle leaders in schools: Developing a conceptual framework for research. *Leading & Managing*, *25*, 1–14.

Driscoll, K. (2017). Primary school mathematics leaders' views of their mathematics leadership role. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning. Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 213–220). MERGA.

Grootenboer, P., & Edwards-Groves, C. (2020). Educational middle leading: A critical practice in school development. *Leading & Managing*, *26*(1), 23–30.

Grootenboer, P., Edwards-Groves, C., & Rönnerman, K. (2015). The practice of "middle leading" in mathematics education. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins. Proceedings of the 38th annual conference of the Mathematics Education Group of Australasia*, (pp. 277–284). MERGA.

Grootenboer, P., Edwards-Groves, C., & Rönnerman, K. (2020). *Middle leadership in schools: A practical guide for leading learning*. Routledge.

Gurr, D., & Drysdale, L. (2013). Middle-level secondary school leaders: Potential, constraints and implications for leadership preparation and development. *Journal of Educational Administration*, *51*(1), 55–71. https://doi.org/10.1108/09578231311291431

Heidegger, M. (2008/1928). *Ontology: The hermeneutics of facticity*. Indiana University Press.

Leithwood, K. (2016). Department-head leadership for school improvement. *Leadership and Policy in Schools, 15*(2), 117–140. https://doi.org/10.1080/15700763.2015.1044538

Sexton, M., & Downton, A. (2014). School mathematics leaders' perceptions of successes and challenges of their leadership role within a mathematics improvement project. In J. Anderson, M. Cavanagh & A. Prescott (Eds.), *Curriculum in focus: Research guided practice. Proceedings of the 37th annual conference of the Mathematics Education Group of Australasia* (pp. 581–588). MERGA.

Thomas, D. R. (2006). A general inductive approach for analyzing qualitative evaluation data. *American Journal of Evaluation, 27*(2), 237–246. https://doi.org/10.1177/1098214005283748

Evidencing Mathematics Leadership as Relational and Developmental Activity

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We report the leadership of mathematics leaders who participated in a leadership intervention. Participation in the intervention was provoked by a tension in teaching practice concerned with a lack of challenge in mathematics teaching in the leaders' schools. We evidence how the mathematics leaders sought to address the tension they faced through their relational and developmental leadership activity.

Using cultural-historical activity theory (CHAT), we report on the collective activity of eight primary school mathematics leaders who participated in a leadership intervention. The intervention focused on expanding mathematics leadership activity. The intention of activity expansion focused on ways that leaders influenced teachers' use of pedagogical approaches that facilitated greater challenge for student learning in mathematics lessons. We attend to the motive-objects and the mediational means that facilitated the leaders' activity with the purpose of responding to this research question: At what do mathematics leaders direct their activity and how is their leadership mediated?

Literature Background

Teaching practice that challenges learning can support students in becoming more autonomous learners as they rely less on teacher support and take greater ownership of their learning (Ingram et al., 2020). However, a tension that faces many teachers concerns the enactment of practices that incorporate appropriate challenge for all students in mathematics lessons (Russo & Hopkins, 2017; Sullivan, 2018). Sullivan (2018) stated reasons for this tension including teachers' beliefs about challenge in mathematics (Russo et al., 2020), and infrequent use of pedagogies that incorporate and maintain challenge. Another reason is teachers' limited mathematical content knowledge (MCK). The ways in which mathematics leadership responds to this tension in teaching for challenge remains undertheorised.

Mathematics leadership is understood as a form of middle leading, enacted by a staff member who leads mathematics education whilst undertaking teaching responsibilities (Sexton, 2019). Working in the space *between* executive leadership and classroom practice mathematics leaders enact leadership that develops dispositions, practices, and knowledge for mathematics education (Grootenboer, 2018). Sexton (2019) outlined how they lead that developmental work as a form of activity, through their leadership of school-based professional learning (PL). CHAT understands activity as *object-orientated* meaning that psychological and practical activity is concurrently advanced through attention to *motive-objects*, pursued in dynamic ways to achieve desired outcomes (Engeström, 2015). Motive-objects drive activity and are vital when interpreting the 'what' and 'why' of activity (Kaptelinin, 2005). Motive-objects are realised through a hierarchical organisation of activity, implying that individuals engage in action sequences to achieve the motive-objects they pursue (Leont'ev, 1978).

With *mediation* as core to CHAT (Miettinen, 2006), activity is facilitated through mediational means. Meditational means are understood as *cultural tools* (psychological tools like concepts and physical ones such as laptops), *rules* that explicitly and implicitly govern behaviours of those involved, and a *division of labour* that mediates the distribution and organisation of roles and responsibilities among those engaged in activity (Engeström, 2015). In CHAT, tensions are catalysts for change and the adoption of new mediational means can facilitate change in activity through the process of *remediation* (Miettinen, 2006).

Methodology

The context of the study was a mathematics leadership intervention, focused on leading the development of teaching practice that incorporated greater opportunities for challenge in mathematics lessons. The intervention was funded by a Catholic education system in the Sydney region during 2023. Eight mathematics leaders, working in five schools involved in the intervention, participated in our study. Three schools had two staff members undertaking the mathematics leadership role, whilst the other two schools nominated one leader. Each participant was required to enact mathematics leadership whilst undertaking teaching responsibilities, meaning they were middle leaders in their schools (Grootenboer, 2018).

In August 2023, data were generated using semi-structured interviews with each of the eight mathematics leaders who volunteered to take part in the study. Interview questions were informed by CHAT concepts, specifically motive-object, cultural tool, rules, and division of labour, to evidence the what, the how, and the why of the collective mathematics leadership activity enacted across the schools. Documents identified by the leaders, along with photographs of leadership activity were also collected. We used the aforementioned CHAT concepts when deductively analysing data, searching for evidence of them in the dataset. Inductive analysis involved asking these questions of the data: What are the mathematics leaders working on? How are they working on that? What is mediating their work?

Results and Discussion

We found that the mathematics leaders worked on two main motive-objects of activity: the development of relational trust and the development of mathematics planning practices.

Development of Relational Trust

The development of relational trust through collaborative approaches to PL surfaced as a motive-object of mathematics leadership activity. Table 1 provides examples of evidence.

Table 1

Evidence Examples of the Development of Relational Trust Motive-Object of Activity

Motive-object	Key leadership actions	Adoption of mediational means
Development of relational trust	Making the developmental work focus explicit and shared amongst staff	**Cultural tool** Mathematics Leadership Activity Plan (MLAP) **Rule** Everyone knows what we are working on and why
	Creating dialogical spaces for pedagogical discussions about teaching practice development	We are all in this together, so everyone has to trial ideas in classrooms **Division of labour** Leaders share their own PL with staff along with their own experiences of teaching for challenge Leaders and teachers engage in pedagogical discussions to create shared commitment to new practices

The mathematics leaders pursued relational trust development when they nurtured shared understandings about the purpose and content of the leadership intervention through the deliberate opening of spaces for pedagogical dialogue (Grootenboer, 2018). Developing relational trust through a commitment to collaboration supported the adoption of new rules (Miettinen, 2006). Those new rules concerned shared understanding of reasons for teaching practice development and the expectation that teachers will trial new pedagogical practices.

One critical mediator that supported the pursuit of the relational motive-object was the *Mathematics Leadership Activity Plan* (MLAP), a leadership planning resource introduced within the intervention. The MLAP, which documented their work plans for influencing teachers' PL, was adopted as a cultural tool by the leaders (Miettinen, 2006). They used the MLAP to develop *interactional trust* by creating communicative spaces about reasons for teaching practice development, as well as *pragmatic trust* by highlighting how that development was linked to teachers' work in reasonable and practical ways (Grootenboer, 2018). This was highlighted by Cathy's (a pseudonym) statement about the use of the MLAP:

> Because we had to make the plan on the MLAP. So, then we shared with the staff what we were doing and why we were doing it. And that seemed to be a big shift with how teachers were participating in the planning (meetings), but also what we were doing in the classroom.

Development of Mathematics Planning Practices

The mathematics leaders engaged in developmental activity by working on improvement of mathematics planning practices in their schools. Table 2 presents examples of evidence.

Table 2

Evidence Examples of the Development of Planning Practices Motive-Object of Activity

Motive-object	Key leadership actions	Adoption of mediational means
Development of mathematics planning practices	Using challenging tasks with teachers in planning meetings to develop teachers' MCK Influencing teachers' use of anticipation, questioning, and extending prompts during planning meetings	**Cultural tools** Mathematics Leadership Activity Plan (MLAP) Planning documentation includes sections about anticipation, questioning, and extending prompts for tasks Mathematical task analysis document with pedagogical discussion prompts Challenge task sequences **Rule** Planning meetings are teacher PL opportunities. Teachers do the maths tasks in planning meetings and specifically plan for the use of extending prompts **Division of labour** Mathematics leaders facilitate planning meetings Teacher share responsibility for the design of planning documentation

The leaders decided that to address tensions about the lack of challenge in mathematics, they needed to create opportunities that influenced how teachers were prepared to teach for challenge. This saw the surfacing of the developmental motive-object, realised through a repositioning of planning meetings. This developmental motive-objects was mediated by the leaders' own PL within the leadership intervention within which examples of challenging task sequences, advice concerning the use of anticipation and extending prompts (Sullivan, 2018), and a mathematical task analysis tool were provided. The leaders claimed they used those as planning resources, evidencing an example of cultural tool adoption intended to remediate teaching practice (Miettinen, 2006).

Remediation of planning practice was further evidenced through the adoption of new rules (Miettinen, 2006). This was realised through the leaders' co-option of teachers' planning meetings, using them as opportunities for PL to improve teachers' MCK (Sullivan, 2018) and to develop beliefs about challenge (Russo et al., 2020). MCK development was worked on when leaders used challenging tasks along with the mathematics task analysis document in planning meetings. They also encouraged teachers to adapt planning tool to include sections for anticipation, questioning, and extending prompts as ways of preparing teachers to teach for challenge. Claire (a pseudonym) evidenced this when she shared:

> It is about them (teachers) actually taking the time to do the tasks themselves, to work it out, because then that's going to lead them to being more confident in presenting it and working with their students. But it's also giving them ideas, and think, 'oh, what kind of questions might the kids ask in response to this?'

It is important to acknowledge the crucial support of principals. The mathematics leaders reported how principals created conditions for their leadership. An example of this support was how principals advocated for the introduction of facilitated planning meetings and arranged the spatial and temporal resources for those meetings to take place. This highlights again the role of principal leadership in mediating mathematics leadership activity (Sexton, 2019).

Concluding Remarks

We do not know yet the influence on teachers' practice, but we highlight that the mathematics leaders responded to the practice tension by directing their activity at relational and developmental motive-objects. We evidenced the adoption of cultural tools, rules, and divisions of labour by the leaders and how they acted as mediational means that mediated the motive-objects of their leadership activity with the intention of remediating teaching practice. Although presented separately, the mathematics leaders pursued their motive-objects in simultaneous and dynamic ways (Engeström, 2015). For example, as they engaged teachers in challenging tasks during planning meetings to develop MCK (Sullivan, 2018), they also worked on relational trust development by opening dialogical spaces for teachers to engage in pedagogical discussions (Grootenboer, 2018). This highlights the dynamism of mathematics leadership as a relational and developmental activity.

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References

Engeström, Y. (2015). *Learning by expanding: An activity-theoretical approach to developmental research* (2nd ed.). Cambridge University Press.

Grootenboer. P. (2018). *The practices of school middle leadership: Leading professional learning*. Springer.

Ingram, N., Holmes, M., Linsell, C., Livy, S., McCormick, M., & Sullivan, P. (2020). Exploring an innovative approach to teaching mathematics through the use of challenging tasks: A New Zealand perspective. *Mathematics Education Research Journal, 32*, 497–522.

Kaptelinin, V. (2005). The object of activity: Making sense of the sense-maker. *Mind, Culture, and Activity, 12*(1), 4–18.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality.* Prentice-Hall.

Miettinen, R. (2006). Epistemology of transformative material activity: John Dewey's pragmatism and cultural-historical activity theory. *Journal for the Theory of Social Behaviour, 36*(4), 389–408.

Russo, J., & Hopkins, S. (2017). How does lesson structure shape teacher perceptions of teaching with challenging tasks? *Mathematics Teacher Education and Development*, *19*(1), 30–46.

Russo, J., Bobis, J., Downton, A., Hughes, S., Livy, S., McCormick, M., & Sullivan, P. (2020). Elementary teachers' beliefs on the role of struggle in the mathematics classroom. *Journal of Mathematical Behavior, 58*, 1–11. https://doi.org/10.1016/j.jmathb.2020.100774

Sexton, M. (2019). Object-motives of mathematics leaders' professional learning leadership during participation in a mathematics project. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 466–473). MERGA.

Sullivan, P. (2018). Supporting teachers in improving their knowledge of mathematics. *Journal of Mathematical Behavior*, *51*, 161–166. https://doi.org/10.1016/j.jmathb.2017.08.006

Evidencing Sector Leadership for Mathematics Leaders Working in Rural and Regional Schools

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We share our experience of establishing a network for primary and secondary mathematics leaders working in rural and regional Catholic schools in Victoria. We evidence the influence of our sector leadership that addressed a tension concerning the leaders' work isolation through a network initiative. This network initiative was in response to the leaders demands for establishing a way to connect and to learn from and with one another using evidence-based mathematics learning and teaching practices.

For decades, a tension in mathematics education has existed that sees urban school students outperforming rural and regional students in mathematics (McConney et al., 2018). The Organisation for Economic Co-operation and Development (OECD, 2013) reported that according to the Programme for International Student Assessment (PISA) 2009 results, urban students outperformed rural students in every country that participated. On average urban students outperformed rural students by up to half a year of schooling (Lamb et al., 2014). Contributing factors to the widening gap include staff access to quality professional learning and educational resources (Murphy, 2018). For staff working in rural and regional schools, these factors cause another tension that problematises this situation further. These schools are faced with the challenge of attracting talent from urban communities, retaining the best talent over time, and conquering the tension of distance (Hargreaves et al., 2015).

Literature Background

The greatest in-school factor for improving student learning and achievement is teacher quality (Hattie, 2009). Fullan and Hargreaves (2012) discussed the benefit of building professional capital of teachers and leaders by investing in leadership through the provision of high-quality professional learning (PL). Opportunities are required to build and share knowledge, and provide feedback in communities characterised by relational trust, strong collaboration, and a shared vision to improve student outcomes. Unfortunately, access to quality PL and opportunities to connect and network with colleagues from other schools is a significant challenge for rural and regional teachers and leaders (Hargreaves et al., 2015).

Networks provide a platform for educational leaders to connect and collaborate through the delivery of high-quality PL to build professional leadership practice (Hargreaves et al., 2015). Rincón-Gallardo and Fullan (2016) highlighted the importance of three shifts needed in relationship between network participants and sector leaders for the establishment of successful networks. These shifts are critical to ensure the sustainability of any network. Figure 1 captures the shifts as reported by Rincón-Gallardo and Fullan (2016).

Figure 1

Required Shifts in the Relationships Between Networks and Sector Leadership

However, the effectiveness of networks measured through improved system-wide student learning outcomes is varied and is dependent upon how the network facilitates effective collaboration and relationships driven by a shared purpose (Rincón-Gallardo & Fullan, 2016). Figure 2 presents a summary of the eight Essential Features for effective networks that support the successful implementation and impact of professional learning features.

Figure 2

Eight Essential Features Required for Effective Networks (Rincón-Gallardo & Fullan, 2016, p. 10)

The eight features provide guidelines that support the development of an educational network by building relational trust that facilitates effective collaboration between system and school leaders. Relational trust is especially highlighted in features 2, 3, 4, 5, 6, and 7.

Our Response to the Tension

Catholic Education Sandhurst (CES) has been intentional in engaging with research literature concerning middle leadership and mathematics education as a tool to support innovation and improvement in student learning outcomes. In response to that literature and using the work of Rincón-Gallardo and Fullan (2016), CES sought to create a network for mathematics leaders working in the diocesan primary and secondary schools. One intention of the network was to provide opportunities that developed mathematics leadership practice through networking and relationship building. This was done in response to the tension of work isolation, which was compounded by the effects of the COVID-19 pandemic that faced the mathematics leaders working in schools within the CES diocese.

The establishment of the Sandhurst Numeracy Leader Network (SNLN) in 2021 aimed to provide CES mathematics leaders access to quality mathematics PL and to support the formation of a professional network between leaders, professional organisations, and mathematics educators working in universities. In essence, the SNLN intended to impact students' mathematical achievement by developing the professional practice of mathematics leaders so they could lead improvement in mathematics teaching practice in their schools.

During the formation of the SNLN, a steering committee was developed to set goals and develop a shared vision for the network. The steering committee included CES staff members, mathematics leaders working in diocesan schools and a mathematics educator working in a Victorian university. The formation of this steering committee was informed by Essential Features 6 and 7 for effective networks (Rincón-Gallardo & Fullan, 2016).

As a way of supporting the implementation of the SNLN, mathematics leaders were included in nominating content of the PL, making the content more demand driven and learning-oriented (Rincón-Gallardo & Fullan, 2016). CES staff developed a needs analysis tool as a way of collecting data, used to inform the content and implementation of the SNLN workshops. Workshops were a combination of virtual and face-to-face PL opportunities, used to address the tension of work isolation whilst focusing on mathematics leadership and teaching practice development. In this paper, we evidence the influence of the SNLN drawing on data about the perceptions of mathematics leaders who have participated in the network.

Methodology

We draw on data from workshop feedback provided by 30 primary and secondary network participants who engaged in middle leadership of mathematics in CES rural and regional schools. Data were gathered using open response questionnaires. Participants were provided with prompts asking them to comment on new learning and actions they will take up as mathematics leaders in their schools. Using the features of effective networks (Rincón-Gallardo & Fullan, 2016), a coding scheme was created to support the deductive analysis approach that was used to generate evidence of those features in the mathematics leaders' responses. Workshop data were also read and coded using an iterative inductive approach that supported the development of themes that captured the leaders' perceptions.

Results and Discussion

Four key themes were generated from the data analysis which are presented in Table 1.

Table 1

Data Excerpts of the Mathematics Leaders Responses Aligned to the Themes

Themes	Essential network feature	Evidence of responses from mathematics leaders
Engaging in interactive opportunities to learn with other leaders	5, 6 & 7	Being able to engage with a [mathematical] task with other [mathematics] leaders from different settings
		Having time to dissect task with others to identify the maths involved, the possibilities for extension and enabling
		I was able to collaborate with a colleague that is further along the journey than I! Helped clarify next steps
		Great to listen to other leaders explain their journey with their MLAP [mathematics leadership activity plan]. Gave us some ideas
Developing strategies for mathematics leadership	4	Looking at the Data to help guide our planning, it's ok to start with a small team [of teachers] before trying to implement whole school [improvement]
		In planning sessions, being mindful to spend time explicitly planning for [differentiation] prompts. Also have teachers 'become the learner' with rich tasks so they feel confident to transfer this to the classroom
		Ensuring I am continuously meeting with my [executive leadership] team and referring back to data to inform our practice
Planning for leadership of school-based professional learning	1 & 4	Think more deeply about common student misconceptions with tasks and plan for them more explicitly
		Continue to work with staff on developing the use of open/challenging tasks and how we can enable and extend students
		Model/use the differentiation planning sheet to support staff PCK
		Using MLAP to develop goals and planning
Having access to resources to support leadership	8	The MAAP tool [mathematics task analysis document] is so useful for discussing the possible avenues that the task can take a learner
		I really enjoy activities where we get to trial a task. I want to do more of this with my teams at school
		Love the Jigsaw protocol, something I wish to incorporate into my learning leader meetings moving forward
		Add the MLAP tool to team meeting

Along with those themes, links to network features (Rincón-Gallardo & Fullan, 2016), and data excerpts (quotes from mathematics leaders) are also reported. The responses are illustrative of how the SNLN provided opportunities for mathematics leaders to interact with and learn from each other to build their skills and expertise to enact this new learning with the teachers in their schools. This is evidenced in the theme of *Interactive opportunities to learn with other mathematics leaders,* suggesting to us that the SNLN responded in some way to the tension of work isolation by building relationships through networking. This highlights the importance of interacting and learning with others within network settings (Rincón-Gallardo & Fullan, 2016).

The data also suggest that the SNLN facilitated opportunities for mathematics leaders to focus on developing leadership strategies, allowing them to engage with executive leaders and teachers in their schools through leadership of mathematics PL. Focusing leadership work on school-based PL for teachers is central to the work of mathematics leaders (Sexton, 2019). It was also evident how the SNLN provided resources that mathematics leaders claimed they would use as part of their leadership. We interpreted this as another way of addressing the tension of work isolation as rural and regional staff members may have limited access to such resources compared to colleagues working in urban and metropolitan schools.

Conclusion

The establishment of the SNLN stemmed from a tension brought on by work isolation. It presented opportunities for our CES leadership to bring together geographically diverse mathematics leaders through networking as a relationship building initiative. We have evidenced enactment of the essential features of networks using data concerning perceptions held by the mathematics leaders. This small study provides opportunities for future research into the key features of successful networks and how they address tensions that exist in rural and regional settings. We had the opportunity to support mathematics leaders to connect, collaborate, and engage in mathematics leadership PL, but more importantly, influence relationships between the leaders as a response to the work isolation tension they faced.

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References

Fullan, M., & Hargreaves, A. (2012). Reviving teaching with ‘professional capital’. *Education Week, 31*(33), 30–36.

Hargreaves, A., Parsley, D., & Cox, E. K. (2015). Designing rural school improvement networks: Aspirations and actualities. *Peabody Journal of Education, 90*(2), 306–321.

Hattie, J. (2009). *Visible learning: A synthesis of over 800 meta-analyses relating to achievement.* Routledge

Lamb, S., Glover, S., & Walstab, A. (2014). Educational disadvantage in regional and rural schools [Conference presentation]. *Quality and equity: What does research tell us* https://research.acer.edu.au/research_conference/RC2014/4august/14

Murphy, S. (2019). School location and socioeconomic status and patterns of participation and achievement in senior secondary mathematics. *Mathematics Education Research Journal, 31*(3), 219–235.

Rincón-Gallardo, S., & Fullan, M, (2016). Essential features of effective networks in education. *Journal of Professional Capital and Community 1*(1), 5–22.

Sexton, M. (2019). Object-motives of mathematics leaders’ professional learning leadership during participation in a mathematics project. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 466–473). MERGA.

Smith, W., Lewis, W.J., Heaton, R. (2013). Ensuring mathematical learning in rural schools: Investing in teacher knowledge. *Great Plains Research, 23,* 185–197.

Sullivan, K., McConney, A., & Perry, L. B. (2018). A comparison of rural educational disadvantage in Australia, Canada, and New Zealand using OECD’s PISA. *Sage Open*, *8*(4).

Symposium: Attending to Student Diversity in Mathematics Education in Inclusive Settings

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Classrooms worldwide are becoming increasingly diverse. The term 'diversity' is contextual and often ambiguous. At a foundational level, 'diversity' is a descriptive term that refers to individual differences and needs (Forghani-Arani et al., 2019). The type of individual differences varies to include the following dimensions "migration, ethnic groups, national minorities and Indigenous peoples; gender; gender identity and sexual orientation; special education needs; and giftedness" (OECD, 2023, About us section). The OECD definition captures a range of individual differences, but it is essential to recognise that these differences can occur simultaneously, be intersecting, and often inseparable. In this way, an individual could have multiple dimensions of diversity in which they differ from others.

The multi-dimensionality or 'hyper-diversity' recognises the "intense diversification of the population, not only in socio-economic, socio-demographic and ethnic terms, but also with respect to lifestyles, attitudes and activities" (Tasan-Kok et al., 2013, p. 8). We adopt the term 'hyper-diversity' to refer to students who have multiple dimensions of diversity. In light of growing student diversity, there is a need for more research (Rigney & Rinaldi, 2023). We would extend this claim to students who are 'hyper-diverse'. This symposium showcases different dimensions of diversity, focusing on students with diverse needs in inclusive mathematics education. The papers explore students with diverse needs from the early primary years to post-secondary schooling, highlighting the importance of inclusiveness across the lifespan.

Chair: Kate Quane.

Paper 1: *Reflecting on the school mathematics experiences of adults with Down Syndrome.*
Matt Thompson, Catherine Attard and Kathryn Holmes.

Paper 2: *"Look at solutions": Differentiated instruction (DI) in senior secondary mathematics.*
Lorraine Gaunt and Tom Porta.

Paper 3: *Participation in mathematics for a student with blindness or low vision in Australian mainstream schools: A longitudinal case study.*
Melissa Fanshawe and Melissa Cain.

Paper 4: *Opportunities for hyper-diverse students to communicate their mathematical thinking in multi-year classes.*
Kate Quane and Bec Neill.

References

Forghani-Arani, N., Cerna, L., & Bannon, M. (2019). *The lives of teachers in diverse classrooms*. OECD Education Working Paper No. 198. OECD Publishing, Paris. https://doi.org/10.1787/8c26fee5-en

OECD. (2021). *TALIS 2018 results* (Vol. I). OECD. https://doi.org/10.1787/1d0bc92a-en

Rigney, L.-I., & Rinaldi, C. (2023). Teaching in cultural and linguistic super-diverse Australian classrooms: A north–south exploration of Reggio Emilia. In B. Fyfe, Y. L. Lee-Johnson, J. Reyes, & G. Schroeder Yu (Eds.), *Affirming the rights of emergent bilingual and multilingual children and families: Interweaving research and practice through the Reggio Emilia Approach* (pp. 209–225). Routledge.

Tasan-Kok, T., Kempen, R., Raco, M., & Bolt, G. (2013). *Towards hyper-diversified European cities: A critical literature review*. Utrecht: Faculty of Geosciences, Utrecht University.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 49–65). Gold Coast: MERGA.

Reflecting on the School Mathematics Experiences of Adults with Down Syndrome

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This paper reports on a section of a larger study, investigating the mathematics experiences of Down syndrome (DS) learners in Australian Primary Schools. Developing the numeracy skills to experience independence in post school settings is crucial for individuals with DS. The aim of this paper is to share the school mathematics experiences of six DS adults and their parents/carers to ascertain if their experiences (DS adults) with mathematics when they were at school have had consequences for how they engage and participate in society as adults. Initial findings suggest that DS adults' mathematics experiences have impacted on their quality of life in a post-school setting.

Down syndrome (DS) is the most commonly occurring chromosomal disorder in Australia with 1 in every 1100 births resulting in a DS diagnosis (Miller, 2015). Individuals with DS often have an intellectual disability and experience significant developmental delays. At the beginning of the 21st century, research alluded that most individuals with DS would struggle and have difficulties with learning mathematics (Bird & Buckley, 2001). Fortunately, contemporary research highlights the need for encouraging further mathematics education research, aiming to find new ways to bring positive mathematics experiences to this population (Faragher & Gil Clemente, 2019). This need is grounded in the fact "that mathematics can contribute, like other disciplines (language, theatre, sports) to the holistic development of people with DS especially with respect to thinking skills and awareness of the world" (Faragher & Gil Clemente, 2019, p. 112).

Importance of Positive Mathematics Experiences

Many individuals in society have a negative disposition towards mathematics as a result of their experiences with the subject at school. These experiences can be the result of teachers not having the pedagogical content knowledge in mathematics to design mathematics experiences for students that are substantively engaging, purposeful, and relevant to students' lives, and that are reflective of the individual needs of each student (Martin et al., 2009). Consequently, sustained student engagement, attainment of the subject in the later years of schooling as well later career choices are all factors that have the potential to be negatively impacted as a result of students' adverse experiences when learning mathematics (Bourgeois & Boberg, 2016). Interestingly, this research is usually only undertaken with students without intellectual disabilities in mainstream schools.

However, it can be argued that negative experiences in mathematics, leading to negative consequences in adult life could equally be as applicable and devastating to students with intellectual disabilities. Specific to individuals with DS, developing the appropriate numeracy skills needed to be able to function in, contribute to, and make sense of the world in which they live is crucial for them to experience independence, develop a sense of purpose, and function as a member of wider society (Faragher, 2019).

Methodology

Snowball sampling was used to recruit six DS adults and their parents/caregivers to participate in a joint semi-structured interview. Interview questions were designed to garner the perspectives of these individuals and their parents/caregivers about their mathematics experiences when they were at school. The methodology literature that explores research in marginalised contexts, highlight the importance of not interviewing those with disabilities in isolation. Kelly (2007) attests to this notion and states that "gaining access to marginalised groups may be difficult, and that, in the case of those with disabilities, it is likely to be necessary to gain access through gatekeepers" (p. 24) such as parents/caregivers. The DS adult participants were given the option to provide their own informed consent (n = 1) or have their parents consent for them (n = 5) and to participate in their own separate interview (n = 0) or to be interviewed with their parents (n = 6).

Findings

This section reports on the findings from the interviews of each participant group (pseudonyms used), made up of the DS adults and their parents/caregivers. Thematic analysis underpinned by Brofenbrenner's ecological model (1994) was used to analyse interview data.

Participant Group One—Michael and Tania

Michael is twenty-two years old and completed school four years ago. He was in a support unit that was in a mainstream public school for both primary and secondary school. When reflecting on Michael's school experience with mathematics, his mother, Tania said, "They [school] probably could have done a bit more … expanded on it [mathematics] instead of just limiting him, because now he is probably working at a year two level … I can see it now … his understanding is still not great." Tania also acknowledged in the interview that she felt that Michael was never challenged with his learning, "I found that they [school] almost expected him to be behind and not be able to do things, this was completely the opposite for my other kids, if they were struggling with something at school, I was told straight away."

Participant Group Two—David and Kate

David is 24 years old and completed school six years ago. He was home-schooled for his first year of school, then attended a support unit in a public school for Year 1 and Year 2. Due to negative experiences with his teacher and class during this time, David's mother, Kate enrolled him in a mainstream Catholic school where he attended from Year 3 until Year 12. When asked about David's school experiences with mathematics and how his learning in mathematics has impacted on his life now in a post school setting, Kate said, "If I hadn't got myself educated, he'd still be struggling … not knowing the days of the week … not knowing the time of day … these are the things that are foundational to be able to function in the world." It was evident when interviewing David and Kate at their home, that Kate has invested an enormous amount of time since David has finished school, trying to teach David basic mathematics concepts such as time, money, addition and subtraction so that he can experience some independence in society.

Participant Group Three—Cooper, Lauren and Renee

Cooper is 23 years old and completed school five years ago. He attended a support unit in a mainstream public school for kindergarten; however, like David, encountered negative experiences with his teacher. As a result, his mother, Lauren enrolled him in a support school for the remainder of his school career. Lauren stated in her interview that Cooper "loved school, but it wasn't until later on that we realised that he didn't learn a lot at school … my daughter, Renee taught him how to read and write … he learned so much after he left school." Lauren also stated that "he [Cooper] just never understood anything to do with maths. Counting, money, time he doesn't understand that." Cooper's sister, Renee spoke about Cooper not understanding weather predictions. She gave the example of when it is forecast to be cold and raining, Cooper would get himself dressed in shorts and a t-shirt. She said, "we've had big issues with this … it is really frustrating for him now that he is an adult.” Lauren also stated in the interview when discussing the basic numeracy skills needed to be able to independently function in society, “all those skills, he [Cooper] really didn’t get them at school, which is a shame, because now he really does struggle.” Like David, it appears that Cooper’s experiences with mathematics at school, have also impacted on his ability to be able to function independently, now that he is an adult.

Participant Group Four—Kim, Joanne and Robert

Kim is 35 years old and completed school 17 years ago. She was in a mainstream class in a mainstream public school for her primary years of schooling and was enrolled in a support unit in a mainstream public high school for her secondary schooling. When asked about the importance of mathematics in Kim’s life now that she has finished school and lives independently, Joanne and Robert spoke about the important role that Kim’s schooling had in her learning of mathematics. Joanne said, “we were lucky, in those days we had lots of support in the classroom, we only had to get on the phone and say, Kim needs help with this, and the support was there.” Kim spoke about her ability to be able to independently budget her money each week and proudly showed her weekly budgeting folder. When asked if these skills were something that Kim learnt at school, Joanne said “it was learnt at home, Kim taught herself how to do those things because she wanted to be independent.”

Participant Group Five—Evelyn and Alison

Evelyn is 30 years old and completed school 12 years ago. She was enrolled in a support unit in a mainstream public school for both her primary and secondary schooling. Alison spoke of a teacher that Evelyn had in her primary school years, Mrs Peterson. She said “Mrs Peterson, is one of those people that should be cloned … she was unbelievably fantastic, I felt like between them [Evelyn and Mrs Peterson] that they reached so many goals.” Interestingly, Alison also said when discussing Evelyn’s mathematics experiences, “I felt like it [mathematics] didn’t get anywhere, but I didn’t expect it to because there was so much other stuff to concentrate on … we were trying to get language and reading happening.” Alison also said that she felt that mathematics plays a rather large role in Evelyn’s life now, that she is in a post school setting. Similarly, to David, Cooper, Kim and Noah, Alison stated that “most of her [Evelyn’s] mathematics has been learnt since she finished school.”

Participant Group Six—Noah, Melissa and Jackson

Noah is 36 years old and completed school 18 years ago. He attended a mainstream class in a Catholic school for his primary school years; however, due to negative experiences with teachers and other students, Noah’s parents, Melissa and Jackson decided to enrol him in a supported setting in a Catholic school for Year 7 through to Year 12. Noah’s parents spoke about their negative experiences encountered when Noah was in primary school. They said they “felt pressured to send Noah to a mainstream school” and stated when talking about the teachers’ aide that supported Noah that “they were there to take him out of class, so that the rest of the class could get on with it and he did kind of whatever.” Like Michael and Cooper, it appears that Noah also experienced limited opportunities. Similarly, to Cooper and David’s experiences, Noah’s parents also expressed the amount of work they have done since Noah has left school, to try and support him in developing his independence, “even at the start of this year, we were trying to teach Noah how to read and understand time.” When asked, if they thought if Noah’s mathematics skills were the result of what he learned at school or at home, Noah parents said, “it’s definitely been more home-based than school-based.”

Discussion and Conclusion

Analysis of interview data has revealed that irrespective of the type of school that was attended by the adult participants with DS when they were at school, negative mathematics experiences at school were commonplace, except in the case of Kim. These negative experiences appear to have manifested themselves into ongoing problems for this population and their parents/caregivers, navigating the world in a post-school setting. It appears that low teacher and school expectations in relation to mathematics were evident among most participants. Contrary to this were Kim's experiences; however, she too has had to learn the numeracy skills needed to experience independence, since leaving school. Similarly, all parent/caregiver participants expressed similar experiences relating to having to teach their child mathematics at home, after finishing school, to try and give them the opportunity to independently access society. Not having the opportunity at school to develop the skills needed to be prepared for their role as a community and workforce members (Education Council, Australia, 2019) is "disheartening and frustrating" as expressed by one parent. This paper contributes to our knowledge of the impact of negative school mathematics experiences on the quality of life of adults with DS navigating the world in post-school settings. It is imperative that more research be conducted to ensure that current and future DS learners in Australian schools, be given the opportunity to experience positive mathematics throughout their school career, developing the appropriate skills needed to prepare them to be able to engage, post school, as informed numerate citizens.

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References

Bird, G., & Buckley, S. (2001). *Number skills for individuals with DS an overview*. Hampshire: The DS Educational Trust.

Bourgeois, S., & Boberg, J. (2016). High-achieving, cognitively disengaged middle level mathematics students: A self-determination theory perspective. *Research in Middle Level Education), 39*(9), 1–18. https://doi.org/10.1080/19404476.2016.1236230

Bronfenbrenner, U. (1994). Ecological models of human development. *International Encyclopedia of Education, 3*(2), 1643–1647.

Education Council, Australia. (2019). *Alice Springs (Mparntwe) Education Declaration*. Australian Government.

Faragher, R. (2019). The new 'functional mathematics' for learners with Down syndrome: Numeracy for a digital world. *International Journal of Disability, Development and Education, 66*(2), 206–217. https://doi.org/10.1080/1034912X.2019.1571172

Faragher, R., & Gil Clemente, E. (2019). Emerging trends in mathematics education for people with DS: Current research and future directions. *International Journal of Disability, Development, and Education, 66*(2), 111–118. https://doi.org/10.1080/1034912X.2019.1579891

Kelly, B. (2007). Methodological issues for qualitative research with learning disabled children. *International Journal of Social Research Methodology, 10*(1), 21–35. https://doi.org/10.1080/13645570600655159

Martin, A., Marsh, H., McInerney, D., & Green, J. (2009). Young people's interpersonal relationships and academic and non-academic outcomes: Scoping the relative salience of teachers, parents, same-sex peers, and opposite-sex peers, *Teachers College Record, 111*, Article 15593. http://www.tcrecord.org/

Miller, P. (2015). *DS*. Cavendish Square Publishing LLC.

"Look at Solutions": Differentiated Instruction (DI) in Senior-Secondary Mathematics

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Differentiated Instruction (DI) is a philosophical and pedagogical approach supporting diverse student engagement in learning, but limited research exists in DI in senior-secondary mathematics. Using semi-structured interviews, the perceived use of DI of two senior secondary mathematics teachers was investigated. One of three themes is discussed in this paper; using strategies to enable student choice and voice. Results indicated teachers used various DI strategies to support students to understand their current levels of need, allowing student choice in their tasks, and supporting student reflective practice. DI in senior-secondary mathematics is complex, but achievable.

Differentiated Instruction (DI) is a widely researched framework supporting teachers to attend to student diversity. Teachers do so by adjusting instruction to suit student need, taking a proactive (Tomlinson, 2014) and responsive approach (Tomlinson, 2022) based on data-driven teaching. There is a paucity of research, however, in DI practices and implementation in mathematics classrooms, specifically in senior-secondary mathematics. Australian teachers are required to differentiate for their students (Australian Institute of Teaching and School Leadership [AITSL], (2017). Thus, understanding how successful teachers use DI in senior secondary mathematics is imperative. DI is not reactive but takes a responsive approach to meeting the needs of diverse learners in one's classroom (Tomlinson, 2022). While research on DI has increased in the last 20 years (Sun & Xiao, 2021), research on DI and mathematics in senior-secondary education is scarce, with studies focusing on DI in primary (Fitzgerald et al., 2021) middle or lower secondary education (Pozas et al., 2023). In a special issue of *Mathematics Teacher Education and Development* (Russo et al., 2021) that focused on differentiating instruction in mathematics, only two of 11 articles were on senior secondary mathematics differentiation (Coles & Brown, 2021; Mellroth et al., 2021). While Coles and Brown (2021) discussed three teacher's reflections of the process of DI in their senior secondary classes, Mellroth et al. (2021) studied eight mathematics teachers in Sweden who taught university mathematics preparation courses in years 10–12. Mellroth et al. (2021) discussed how teachers collaboratively planned a problem bank of challenging tasks to be implemented in their classes, but no discussion of classroom implementation was provided in either paper. Given van Geel et al. (2019) outlined that teachers struggle to differentiate instruction, research highlighting effective DI practices in senior-secondary mathematics classrooms is timely. Student enrolment in Year 12 mathematics subjects has dropped in Australia (Australian Mathematical Sciences Institute [AMSI], 2020). Given research outlined that DI is one way to engage students in learning, determining how teachers use DI for learners in senior-secondary mathematics may support more teachers in implementing DI, ensuring success of mathematics students. This study investigated how two teachers implemented strategies in their mathematics classrooms, to answer: What teacher-developed DI strategies are senior-secondary mathematics teachers utilising in their classroom, to cater for learner diversity?

Methodology

This study was conducted at two independent schools in Australia, Adelaide, South Australia, and Brisbane, Queensland. Part of a wider study, this paper reports on two senior secondary mathematics teachers. Maria (pseudonym) with 20+ years' experience, taught Essential Mathematics in South Australia (Government of South Australia, 2016). Julia (pseudonym) with 10 years' experience, taught Mathematical Methods in Queensland

(Queensland Curriculum & Assessment Authority, 2019). This study aims to compare the DI strategies used in senior-secondary mathematics. Case studies allowed the first author to gain an understanding of the practice of DI (Creswell, 2012). Using purposeful sampling, data were collected using semi-structured interviews to elicit detail of DI, as a philosophical practice, in classrooms. Data were analysed according to the six steps reflexive thematic analysis (Clarke & Braun, 2021), which included researchers familiarising themselves with data, reading and conducting member checks. Researchers coded data inductively and deductively, according to the framework by Tomlinson (2014), ensuring interrater reliability during joint coding.

Results

Three themes constructed from data were (1) Strategies to enable student voice and choice in mathematics; (2) Supporting the process of learning, not just content of mathematics; and (3) DI is for all students and takes time to master. Here, the first theme will be explored. Both teachers were efficacious in using DI, both identifying several effective strategies. These teachers taught two mathematics subjects individually, that both target different levels of mathematical ability, but data from interviews with both teachers showed similarity in their strategies to implement DI, even within the difficult confines of the inflexibility of senior secondary mathematics. Strategies identified by both teachers within this theme have been delineated into three sub themes; (1) Strategies supporting students to see themselves as mathematics learners; (2) Strategies giving students choice that led to student success; and (3) Strategies supporting student voice using reflections and feedback.

Theme 1.1: Strategies Supporting Students to see Themselves as Mathematics Learners

Both teachers felt supporting students to see themselves as capable mathematics learners was vital. Julia said, “every kid walks out of my classroom feeling like they can do something” and from Maria, they “don’t feel like maths failures anymore. I can successfully convince them that they do have a mathematical brain”. In Mathematical Methods, Julia created a safe space where students feel empowered to try even if they were wrong. She said students were not afraid to be wrong because “that’s the place [the classroom] to be wrong, and who cares if you’re wrong? We can fix wrong”. To cultivate this safe space, Julia encouraged student collaboration, stating “it gives the students who are able to carry on, on their own, access to each other as well to push each other, and I think that is almost more important than any teacher driving anything”. Similarly, Maria encouraged collaboration in her classroom stating that when students learn from each other, “they can reinforce their own learning in class”. This leads to “that beautiful moment, when you teach something to a kid and they’re really enjoying learning it” and when they share that learning with others, “it empowers them”. Hence, teachers felt supporting students to see themselves as mathematics learners was empowering.

Theme 1.2: Strategies Giving Students Choice That led to Student Success

Both Julia and Maria differentiated instruction within their classrooms by providing choice for students. Strategies included starter questions, colour coding questions into different levels, providing extra resources such as videos and weblinks, formative assessment tasks, regular feedback, exercises with multiple destinations, group work, and students teaching each other. Both teachers saw value in providing student choice. Julia indicated formative assessment tasks and starter questions enabled students to select the most appropriate level to work, stating “lessons I make are created so that the kids have a say in what they’re doing and … help them decide what levels they’re at”. Maria would “*look at solutions*, ways of presenting things to kids that enable them”. She suggested “having multiple destinations in the exercise, they make the choice. They push themselves to their limit”. Both teachers outlined students responded to choice in their learning by working harder and they had seen improvements in students’ success.

Julia stated she saw "what they're doing very regularly", and Maria said "they came away with extra skills themselves. They really enjoyed learning from one another". Thus, teachers believed, strategies which gave students choice, led to student success.

Theme 1.3: Strategies Supporting Student Voice Using Reflections and Feedback

A more recent strategy that both teachers had employed was student self-reflection, which allowed students to deepen their own understanding. Julia provided students with feedback on work booklets and students then completed a self-reflection sheet. Julia stated that self-reflection was "helping them develop their own understanding of exactly where they are". Maria used open ended problem tasks and asked students to reflect on the process. She felt that this deepened student learning. For example, in an open-ended geometry task where students needed to develop a product, Maria asked students "to explain why the construction worked". Additionally, Julia differentiated her instruction by facilitating classroom discussions where she could support or extend student thinking "because I think for maths particularly it's incredibly important for development of understanding". Therefore, self-reflections and feedback were identified as supporting student voice.

Discussion

We recognise that DI is more than just a series of strategies, however, one must have a repertoire to differentiate effectively. The results highlighted that these two efficacious senior secondary mathematics teachers used a variety of differentiation strategies that focused on supporting students to understand the current levels of need, allowing student choice in their tasks and approaches, and in supporting student reflective practice to deepen student learning. The use of a variety of strategies aligns with the results from Smets and Struyven (2020) who found that DI can quickly be seen as just a series of strategies. While true in the case of these two teachers, the use of a variety of strategies contributed to greater student voice being included in the differentiated classroom. Importantly, teachers were asked questions beyond DI strategies, including, for example, using differentiated resources and tasks. The use of tiered assessments, one of the most applied DI practices (Smit and Humpert, 2012), were not highlighted. Therefore, tiering may not be as applied in senior-secondary mathematics classrooms. Julia and Maria both cultivate supportive classroom climates where students felt comfortable to "have a go" and it was "okay to make a mistake". Students collaborated, often participating in group problem solving tasks, teaching each other, or revising and testing each other. Teacher planning supported students to make choices and work through the material at their own pace and students chose materials and resources that best supported their learning. Even within the perceived inflexibility of senior-secondary mathematics curriculum, both teachers stated they found ways to use DI and support student learning. Hence, while teachers worldwide are struggling to differentiate (van Geel et al., 2019), the results from this study extend the findings of van Geel et al. (2019), by outlining that teachers make DI work, within the constraints they perceive they have. As Julia stated, with the restrictions on assessment and content, "the only thing we can change is the instruction and the support behind it" and in that regard, both Julia and Maria successfully used DI and supported their students. Limitations in this research included the small sample size, teacher participants were female, taught in independent, all-girls schools, and self-reported DI strategies. The teachers taught different levels of mathematics, but demonstrated remarkably similar approaches to DI. Future research warrants a broader sample across school systems.

Conclusion and Recommendations

As student numbers in senior secondary mathematic decline, it is possible students may not be engaging in higher level mathematics because teaching does not meet their needs. These two exemplary teachers demonstrated that DI in the senior-secondary classroom is both possible and necessary to improve student outcomes. This calls for further investigation into how exemplary practices like these can be shared through professional learning activities to ensure both greater enrolment in senior secondary mathematics and better student support.

Acknowledgments

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References

Australian Institute of Teaching and School Leadership (AITSL). (2017). *Australian professional standards for teachers*. AITSL. https://www.aitsl.edu.au/standards

Australian Mathematical Sciences Institute (AMSI). (2020). *Maths crisis: Year 12 maths enrolments reach all-time-low*. AMSI https://amsi.org.au/2022/04/27/maths-crisis-year-12-maths-enrolments-reach-all-time-low/

Clarke, V., & Braun, V. (2021). *Thematic analysis: A practical guide*. London: Sage

Coles, A., & Brown, L. (2021). Differentiation from an advanced standpoint: Outcomes of mathematics teachers' action research studies aimed at raising attainment. *Mathematics Teacher Education and Development*, *23*(3), 166–181.

Creswell, J. (2012). *Educational research planning: Planning, conducting, and evaluating quantitative and qualitative research*. Boston, MA: Pearson.

Fitzgerald, L., Hunter, J., & Hunter, R. (2021). Shifting teacher practices in relation to grouping: Gap gazing or strengths focused approaches. *Mathematics Teacher Education and Development*, *23*(3), 97–110.

Government of South Australia. (2016). *Essential mathematics 2023 subject outline stage 1*. https://www.sace.sa.edu.au/web/essential-mathematics/stage-1/planning-to-teach/subject-outline

Mellroth, E., Bergwall, A., & Nilsson, P. (2021). Task design for differentiated instruction in mixed-ability mathematics classrooms: Manifestations of contradictions in a professional learning community. *Mathematics Teacher Education and Development*, *23*(3), 78–96.

Pozas, M., Tovar, J., Guerra, L., Armendariz, L., & Zubiria, A. (2023). Differentiated instruction in mathematics according to lower secondary school Mexican students. In V. Letzel-Alt & M. Pozas (Eds.), *Differentiated instruction around the world: A global inclusive insight*. Waxmann.

Queensland Curriculum & Assessment Authority (QCAA). (2019). *Mathematical methods 2019 v1.2*. QCAA. https://www.qcaa.qld.edu.au/downloads/senior-qce/syllabuses/snr_maths_methods_19_syll.pdf

Russo, J., Bobis, J., & Sullivan, P. (2021). Differentiating instruction in mathematics. *Mathematics Teacher Education and Development*, *23*(3), 1–5.

Smets, W., & Struyven, K. (2020). A teachers' professional development programme to implement differentiated instruction in secondary education: How far do teachers reach? *Cogent Education*, *7*(1). https://doi.org/10.1080/2331186X.2020.1742273

Smit, R., & Humpert, W. (2012). Differentiated instruction in small schools. *Teaching and Teacher Education*, *28*(8), 1152–1162. https://doi.org/https://doi.org/10.1016/j.tate.2012.07.003

Sun, Y., & Xiao, L. (2021). Research trends and hotspots of differentiated instruction over the past two decades (2000–2020): A bibliometric analysis. *Educational Studies*, 1–17. https://doi.org/10.1080/03055698.2021.1937945

Tomlinson, C. A. (2014). *The differentiated classroom: Responding to the needs of all learners*. Association for Supervision & Curriculum Development. http://ebookcentral.proquest.com/lib/usq/detail.action?docID=1709534

Tomlinson, C. A. (2022). *Everybody's classroom: Differentiating for the shared and unique needs of diverse students*. Teachers College Press.

van Geel, M., Keuning, T., Frèrejean, J., Dolmans, D., van Merriënboer, J., & Visscher, A. J. (2019). Capturing the complexity of differentiated instruction. *School Effectiveness and School Improvement*, *30*(1), 51–67. https://doi.org/10.1080/09243453.2018.15390133

Participation in Mathematics for a Student With Blindness or Low Vision in Australian Mainstream Schools: A Longitudinal Case Study

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Students with blindness and low vision (BLV) are less likely to choose mathematics as a subject in the senior secondary years which may negatively impact future employment opportunities. Using a longitudinal qualitative methodology, three interviews were recorded with a student who is legally blind over a six-year period. Findings suggest that access to mathematics curriculum and assessment was significantly impacted. Use of assistive technology and support from others enabled increased participation and achievement in this subject. Independent access to the curriculum and use of assistive technology may lead to students with BLV choosing mathematics in senior secondary.

Participation in mathematics for students in schools has been previously found to develop the required attitudes, knowledge, and skills to gain entry to Science, Technology, Engineering, and Mathematics (STEM) careers (Nitzan-Tamar & Kohen, 2022). This is important as STEM offers some of the highest employment opportunities within the world labour market. Further, mathematics is an essential component of many professions which require employees to understand numbers, solve problems, and apply critical reasoning (Just & Siller, 2022). Barriers to accessing mathematics learning have been previously documented based on gender, ethnicity, and disability. Students with blindness or low vision (BLV) face unique challenges within classrooms, due to the curriculum being designed for those who can see (Fanshawe & Jones, 2021). Curriculum materials such as writing on the board, handouts in class, textbooks images, and videos can be inaccessible, along with symbolic representations such as graphs, diagrams, symbols, shapes, and patterns. Inability to access classroom materials presents as challenges for BLV students, if students are not given agency to access and participate in learning (McLinden et al., 2016). With approximately 4,000 Australian students with BLV, this paper is guided by the research question: What factors does a student with BLV perceive as impacting participation in mathematics learning?

Methods

Qualitative methodology was considered the most useful to answer the research question, to examine participation in mathematics in Australian Schools for students with BLV (Calman et al., 2013). The longitudinal case study presented in this paper concerned one student with the pseudonym of Kye, who was known to the researchers and thus recruited through convenience sampling. Kye was interviewed once at the end of primary school (Year 6), once in secondary school (Year 9) and again in senior secondary school (Year 11). Three different interviewers were used over the project and member checking was used to ensure validity in reporting of results. In each interview, Kye was asked questions which aimed to elicit information about participation in mathematics in the classroom. Transcripts of the recordings were checked for accuracy by Kye and his parents and uploaded into NVivo. Tools within NVivo were used to identify main themes using inductive category development (Mayring, 2000) and then to categorise qualitative data examining the contextual changes over time.

Results

Kye attended a local government primary school and a Catholic secondary school. Kye had a congenital vision condition with clouding on his cornea and nystagmus with a medical diagnosis of counting fingers at 70cm, which meant he was legally blind. Kye described his vision "the most I can see is one metre away, but it's cloudy and shaky" and in a later interview, "I like to say I'm legally blind so other people know I can't see much at all. If I say I have low vision, they think I just need glasses or something".

Accessibility of Mathematics in Primary School

Concrete materials were used in primary school which Kye reported helped aid his understanding of mathematical concepts, "if my teachers are talking about a cube, they give me a cube so that I can feel the sides and corners and angles". He also received worksheets "blown up in large print". He stated, however, he still needed magnification such as a handheld dome magnifier, a digital magnifier, and CCTV camera to enlarge the worksheets. Furthermore, he had an iPad, which he used to take photos which he used to magnify images on the board and worksheets. When completing tests, Kye reported that it took him a lot longer than his peers. The "hardest part is that I can't see all the numbers in one spot, so I have to look at one number and go back and look at the next". He explained that he would have a teacher read out the question, then "I write the numbers down really big so I can remember them". Despite using technology, he shared that mathematics was "difficult to access information". He would ask friends to read out questions and "I'd remember numbers in my head".

Technological devices such as a screen reader and an electronic braille device were used by Kye to access information in subjects such as English, however, he reported that these technologies were not as useful in mathematics as "there are lots of pictures and shapes". He explained that his friends and teachers could not use screen readers or read braille and it took a long time to find information; "and it's difficult to go back in braille to find something, you have to read it all again". To access images, shapes, and graphs, Kye was provided with tactile diagrams, which were created on a "PIAF which gets all the ink on the paper that is black and raises it up so you can feel it". Teachers and support staff assisted Kye to access mathematics in diverse ways including using thick pens on the board which made the magnified image on his iPad clearer, giving verbal instructions, and reading out what they wrote on the board. Kye spoke of an advisory teacher who supported his classroom teachers and a teacher aide who taught him braille and created his enlarged worksheets. Despite not all aspects of the mathematics curriculum being accessible, and exams taking additional time, Kye shared his love of mathematics and provided examples of ways he was able to access mathematics.

Accessibility of Mathematics in Secondary School

Microsoft OneNote, which is a digital note-taking tool used by Kye's school to store and organise class content, increased independent access to mathematics in secondary school for Kye, "the teacher in maths, instead of writing on the board, he does it in OneNote". He explained that teachers uploaded mathematics curriculum materials into OneNote, which meant Kye could use a braille device, magnification, or a screen reader to read out the online content at the same time as his peers. Kye stated the benefit was that "everybody has OneNote, not just me". The benefits to the class were further acknowledged "the teacher writes on OneNote, and we all have access, but [the teacher] isn't standing in front of the board, so everyone can see. Then I sit at the back and zoom in on my iPad". Kye's textbook was downloaded in digital format, accessible on iPad Pro. Tactile diagrams were common again in secondary school, with many PIAFs being created by the teacher aide, particularly for graphs and images. Kye was also provided with 3D models created on the school's 3D printer.

Internal examinations were created in a word document using a table and equation editor, which was accessible with braille, magnification, and a screen reader, and allowed Kye to tab through the fields to access content. Kye explained that he received an additional half an hour in time for every hour of exam, "but I need every minute of it to access the test". Kye preferred to work independently in class. Support from teachers was appreciated in creating accessible materials for OneNote and consistent training in assistive technology was provided by the advisory teacher. Kye reported that he performed well in the exams and was selected in an advanced enrichment mathematics class.

Accessibility of Mathematics in Senior Secondary School

At the time of the third interview, Kye was engaged and performing well at school, however, it was reported he was no longer using braille and had withdrawn from Senior Maths Methods. He reported, "OneNote kept glitching. I had no idea what was on the board. By the time the teacher came to help me, I had missed a double lesson of information and had no idea how to catch up". When asked why the OneNote had changed from previous years, Kye suggested ICT support didn't prioritise his access. Further, Kye shared senior secondary examinations were created externally which meant that previous accessibility was no longer available. Teachers in the school would make the examinations accessible for Kye by providing equations, graphs, and images in alternate formats. However, despite additional time provided, Kye reported exams being "too frustrating as I just ran out of time". He said this reflected in his marks. Kye shared that while he had tried to remain enrolled in Maths Methods, it "took me too long with all of my other subjects", resulting in Kye withdrawing midway through the year.

Discussion

This case study identified that the mathematics curriculum and assessment was not inherently accessible for a student BLV, which is concerning given the goal for the Australian education system to promote equity and excellence. Successful adjustments included accessing information from the board, textbooks, and worksheets through simple solutions such as verbal descriptions of what were on the board, along with assistive technology including magnification, braille, and screen readers to access digital information. McLinden et al., (2016) shared the importance of students being able to access materials independently, to ensure agency in learning, such as Kye's use of OneNote. When items were not able to be accessed independently, support people within the school assisted to create accessible content.

Advances in assistive technologies have increased opportunities for students with BLV to independently access the mathematics curriculum. For Kye, digital technology provided ways for digital textbooks and documents to be accessed using screen readers, magnification tools, and an electronic braille device. When digital documents were accessible and the technology was working, Kye's participation was enabled. However, inequities in access to assistive technologies for students with BLV (Fanshawe et al., 2023) can further emphasise the gap in achievement in mathematics for these students. In Kye's case, the secondary school seemed willing, capable, and equipped to use inclusive technologies, and implemented recommendations to support access from an advisory teacher from outside the school. However, when technology was not connecting, and documents were not formatted Kye experienced barriers to participation in mathematics. Although additional time was provided in examinations this was not always sufficient for Kye and as a result his grades were not necessarily indicative of his knowledge. Similarly, Al-Dababneh et al. (2015) found that barriers to participation in mathematics could mask academic potential. This is indeed problematic, as Nitzan-Tamar & Kohen (2022) report that learning experiences in classroom mathematics have a potential to impact university selection and employment outcomes.

Support from others also positively impacted Kye's participation. Kye reported the school utilised an advisory teacher and all his teachers were willingly trying to make access inclusive. Whitburn (2014) asserted that pedagogical practices were important for inclusion in schooling as the teacher's ability to normalise differences within a class could decrease the stigma if all students had access to the curriculum and pedagogy. Kye reported that he preferred the teacher aide to support his access through the creation of accessible curriculum and assessment, rather than sit with him in class. This aligns with the study of Byrne (2014) who found students resisted the assistance of a teachers' aide as this meant they had decreased control over their learning or looked different to their peers. Independence in accessing mathematics information was an important enabler for the student in this study.

Conclusion

The qualitative case study highlighted that the standardised curriculum and assessment provided was not always accessible for all students in Australian mainstream schools. The longitudinal data from three interviews of one student in in primary, secondary, and senior secondary school contexts showed barriers to accessing mathematics for BLV students. While materials which were formatted digitally and use of assistive technology provided greater independence to the student, many essential elements of mathematics, such as images, diagrams and graphs were not able to be accessed independently. When the student not able to independently participate in mathematics learning, teachers and external experts provided modification to access learning and teaching materials. Unfortunately, this case study also showed, that barriers in access for students can take additional time and frustration, which ultimately led to attrition from mathematics subject by student who is blind. Advances in technology and reconsideration of accessibility embedded in the curriculum is essential to provide excellence and equity in education for all students. It is hoped that further research will identify needs of BLV students in the classroom, resulting in increased participation in mathematics and more equitable access to STEM careers.

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References

Al-Dababneh, K., al-Masa'deh, M., & Oliemat, E. (2015). The effect of a training programme in creativity on developing the creative abilities among children with visual impairment. *Early Child Development and Care, 185(2*), 317–339.

Byrne, B. (2014). Getting in and getting on? The experiences of young people with visual impairments and hearing impairments in third-level education. *International Journal of Disability, Development and Education, 61*(2), 119–133. https://doi.org/10.1080/1034912X.2014.905057

Calman, L., Brunton, L., & Molassiotis, A. (2013). Developing longitudinal qualitative designs: Lessons learned and recommendations for health services research. *BMC Medical Research Methodology, 13*(1), 14.

Fanshawe, M., Barton, G., Mandarakas, M., Cain, M., Todd., N. (2023). Enablers and barriers to equitable participation for students with blindness or low vision. *International Journal of Inclusive Education*, 222652991. https://doi.org/10.1080/13603116.2023.2265915

Fanshawe, M., & Jones, A. (2021). Accessible mathematics for students with vision impairments. *Teaching Mathematics*, *46*(3/4), 14–16.

Just, J., & Siller, H.-S. (2022). The role of mathematics in STEM secondary classrooms: A systematic literature review. *Education Sciences*, *12*(9), 629. https://doi.org/10.3390/educsci12090629

Mayring, P. (2000). Qualitative content analysis. *Qualitative Social Research*, *1*(2). http://www.qualitative-research.net/index.php/fqs/article/view/1089/2385

McLinden, M., Douglas, G., Cobb, R., Hewett, R., & Ravenscroft, J. (2016). 'Access to learning' and 'learning to access': Analysing the distinctive role of specialist teachers of children and young people with vision impairments in facilitating curriculum access through an ecological systems theory. *British Journal of Visual Impairment, 34*(2), 177–195. https://doi.org/10.1177/0264619616643180

Nitzan-Tamar, O., & Kohen, Z. (2022). Secondary school mathematics and entrance into the STEM professions: A longitudinal study. *International Journal of STEM Education, 9*, Article 63.

Whitburn, B. (2014). Accessibility and autonomy preconditions to 'our' inclusion: a grounded theory study of the experiences of secondary students with vision impairment. *Journal of Research in Special Educational Needs*, *14*(1), 3–15. https://doi.org/10.1111/1471-3802.12014

Opportunities for Hyper-Diverse Students to Communicate Their Mathematical Thinking in Multi-Year Classes

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This paper examines the mathematical experiences of students with additional and diverse needs in multi-year classes within the educational context of small regional South Australian (SA) schools. Qualitative research methods were used to collect data about how students communicate their mathematical thinking. Opportunities for students to communicate their mathematical thinking were categorised using Bruner's experiential stages of learning. Findings suggest that the collaborative nature of multi-year classes fosters peer learning and cooperation, enabling students to share and build upon each other's mathematical thinking.

One way that student diversity is evident is in students attending small schools. Students attending small schools can be arranged in composite, multi-age, multi-year, or stage classes (Cornish, 2010). Often, these terms are used interchangeably but each refers to a specific structure and rationale for the organisation of students. Cornish (2010) describes the formation of multi-grade (or multi-year) classrooms due to low student and/or teacher numbers which are predominately found in rural locations. The number or years within one class will be determined by the number of students and can include all students from all primary years within the one class. In contrast, Cornish (2010) describes composite classes "formed by necessity" due to the annual variation in student enrolment within particular year groups (p. 8). In this way, composite classes are a result in fluctuations in student numbers and are temporary. A third way to organise students is in multi-age classes whereby students are not associated with a year, rather the classes are flexible based on choice (Cornish, 2010). According to Cornish (2010) multi-age classes can be known as "non-graded classes" or "family classes" and students "have no association with a grade, nominal or otherwise" (p. 8). Rather the classes are structured so that they are "developmentally appropriate" tailoring the curriculum to "allow for continuous progress in learning" (p. 8).

The paper reports on an aspect from a larger study that explored optimising early mathematical learning experiences and establishing positive attitudes towards and experiences of mathematics for young South Australians attending small regional schools. In SA, 70% of small schools are in regional or remote locations, typically characterised by multi-year classes whereby two or more consecutive curriculum-year levels are within the one class. This focus responds to enduring inequalities in academic success and lower life opportunities experienced by young Australian children living in regional and rural locations (Thomson et a., 2019). This paper seeks to answer the research question, "What opportunities do students have in multi-year classrooms to communicate their mathematical thinking to others?"

Communicating Mathematical Thinking

Communication is a participatory and cultural process, that is a significant practice in mathematics (van Oers, 2013). The ways in which mathematical thinking is communicated are diverse, interrelated, and can often go unnoticed. Freitas and Walshaw (2016) argue that thinking can happen "without language" but through language, thinking becomes more sophisticated (Freitas & Walshaw, 2016, p. 20). Mathematical thinking is often communicated verbally or in written form. While these two forms of communication are predominant features in communicating mathematical thinking, there are other multi-modal forms that students can use to communicate their thinking (Quane & Booth, 2023). Numerous curriculum documents and standards emphasise the importance of developing student's mathematical thinking. One

Australian jurisdiction that elaborates on the communication of mathematical thinking is the New South Wales Education Standards Authority (NESA, 2021), which provides a general description of students communicating mathematical by describing, representing, and explaining "mathematical situations, concepts, methods, and solutions to problems, using appropriate language, terminology, tables, diagrams, graphs, symbols, notation, and conventions". From this description, we can see that communicating mathematical thinking involves several processes including representing, describing, and explaining through using appropriate language, terminology, and conventions.

Method

The participatory action research was conducted in two small regional South Australian state schools with two junior primary teachers. Site 1 had 35 student enrolments with 17% of students (six) identifying as Aboriginal or Torres Strait Islander. The 35 students were arranged in two classes, a junior class comprising of Reception (first year of school), year 1, and year 2 (n = 15), and an upper class of students in years 3 to 6 inclusive. Site 2 had 45 students enrolled and arranged in three classes, a Reception and year 1 (n = 13); a year 2 and 3; and a year 4–6 inclusive. Both sites had high proportions of students with additional learning needs and diagnosed disabilities.

Data was collected using children's drawings, semi-structured interviews (n = 16), and 20 classroom observations to ascertain how students in multi-year classrooms communicated their mathematical thinking. The participating students were diverse in terms of their gender (5 females, 11 males), their special education needs (50% of students with a formal diagnosis of a form of neurodivergence, 1 child with an intellectual disability, 3 children with delayed speech) or their giftedness with 1 child from site 1 being accelerated in mathematics, attending the year 3–6 class. In addition, students at both sites experienced higher levels of socio-economic disadvantage adding another dimension of diversity. Further, children also completed a task co-designed between the two teachers and two researchers. Data analysis occurred atomically and holistically, initially viewing each data source separately and then collectively using an open-coding process to identify emerging themes. Bruner's (1966) experiential stages of learning classified as enactive, iconic, and symbolic were used to analyse the opportunities for students to communicate their mathematical thinking.

Findings

Opportunities for students to communicate their mathematical thinking were classified into three broad themes: (1) opportunities to represent mathematical thinking; (2) opportunities to describe and explain their thinking; and (3) opportunities to use appropriate language, terminology, and conventions. In this paper, the first theme of representing mathematical thinking will be discussed. Bruner's (1966) experiential stages of learning have been used to elaborate on the first theme with this paper paying particular attention to the iconic stage.

We report on an observation conducted at Site 1 where all students engaged in the same mathematical task and the resultant opportunities provided by the teacher for students to communicate their mathematical thinking. Students were given numerous opportunities to represent their mathematical thinking during all three experiential stages, with greater opportunities for representations occurring during the enactive and iconic stages. The enactive representations lead to students describing what they have represented which aided the transition from the enactive to iconic representations. The task (Figure 1) explored the concept of additive patterns and was displayed on the interactive board. The teacher organised a range of iconic representations including buttons and pom poms to represent the decorations. The teacher read the task and gave explicit instructions on the materials and group expectations. Students were grouped into three mixed-year-level groups. Figures 2 and 3 show two different student group representations created collaboratively. At the mid-point of the learning experience, the teacher gathered the students around each group's representations and asked students to share what they had done and why (Figure 6). Students were then encouraged by the teacher to use the strategies that were shared by their peers. Students had further opportunities to develop and refine their representations (Figures 5 and 6) before a final sharing opportunity was instigated.

Figures 1–6

Task Description (1), Students Representing their Thinking (2 and 3), Sharing their Thinking (4), and Final Representations (5 and 6)

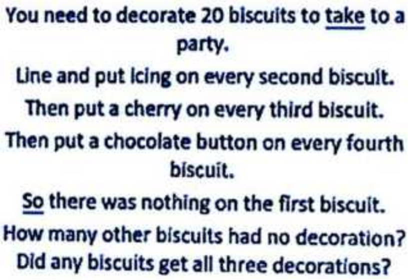

Figure 1

Figure 2

Figure 3

Figure 4

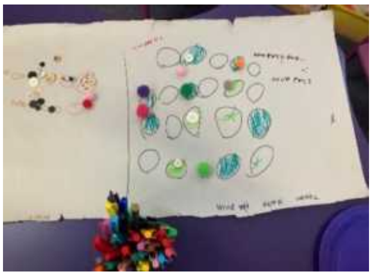

Figure 5

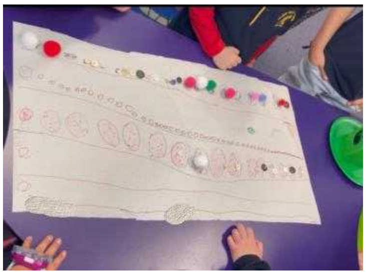

Figure 6

Discussion

The student-co-constructed representations were great opportunities for students to communicate their mathematical thinking to their peers and their teacher. In developing their constructions, students used multiple forms of communication including gestures, the use of manipulatives, and assistive technology. The familiar objects (Larkin, 2016) of buttons and pom poms provided opportunities for students to create representations that could then be later used to aid students in describing and explaining their mathematical thinking. Students combined iconic representations, drawing circular shapes for the biscuits with the familiar objects. The use of diagrams (Larkin, 2016) was a key feature of all iconic representations providing further opportunities to share mathematical thinking. The student-generated diagrams led to a class discussion about the suitability of the diagrams, with one student adamant that the only way to represent the 20 biscuits was in lines.

The two multi-year classes that were the focus of this study were rich in examples of the many opportunities afforded to students to communicate their mathematical thinking. The engagement with peers of different ages provided numerous opportunities to use and engage in multiple representations and hear and see different descriptions and explanations. Peers acted as enablers, motivators, and collaborators (Quane, 2021) whereby students sought help from their peers and to share ideas. Students actively contributed to the ideas of other students, building on or adapting their thinking or adopting the thinking of others, thereby, enabling and fostering students' mathematical thinking. Through facilitating sharing opportunities, the teacher provided further opportunities for students to observe and listen to other students' thinking, cultivating the classroom norms to share and collaborate. The multi-year structure of the classes provided opportunities for students to hear a variety of explanations that may be beyond their curriculum year level. As such, they provided opportunities for greater exposure to how mathematics develops including but not limited to mathematical language which in turn provides further opportunities for students to communicate their mathematical thinking. In selecting the task (Figure 1), the teacher provided students with an authentic scenario that was relatable to the children. Further, the investigation style of the lesson provided another layer of opportunities for students to communicate their mathematical thinking. Students were given opportunities that were planned by teachers while other opportunities to represent mathematical thinking spontaneously occurred, introduced by both teachers and students.

Conclusion

The paper highlights the invaluable opportunities for hyper-diverse students to communicate their mathematical thinking within the unique context of multi-year classes in small regional SA Schools. The findings underscore the significance of providing students with multiple opportunities to represent their mathematical thinking using various modalities together with the nature of multi-year classes fostering peer learning and collaboration, enabling students to share and build upon each other's mathematical thinking. Teachers play a crucial role in facilitating these opportunities by creating a supportive learning environment and selecting authentic tasks that resonate with students' lived experiences. By harnessing the power of multi-modal communication and peer interaction, educators can empower hyper-diverse students to develop confidence, agency, and a deeper understanding of mathematics, ultimately promoting equitable access to mathematical learning and success. Moving forward, continued research is required into the practices used in multi-year classes to better understand the complexities of these settings.

Acknowledgments

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References

Bruner, J. S. (1966). *Toward a theory of instruction*. Harvard University Press.

Cornish, L. (2010). Multiage Classes—What's in a name? *Journal of Multiage Education, 4*(2), 7–11.

Freitas, E., & Walshaw, M. (2016). *Alternative theoretical frameworks for mathematics education research*. Springer. https://doi.org/10.1007/978-3-319-33961-0

Larkin, K. (2016). Mathematics education and manipulatives: Which, when, how? *Australian Primary Mathematics Classroom, 21*(1), 12–1.

New South Wales Education Standards Authority. (2021). *Working mathematically*. NSW Education Standards Authority (NESA). https://educationstandards.nsw.edu.au/wps/portal/nesa/k-10/learning-areas/mathematics/mathematics-k-10

Quane, K. (2021). Curriculum implications of understanding the influence of peers on the nature of young children's attitudes towards mathematics. *Curriculum Perspectives*, *41*, 245–255.

Quane, K., & Booth, H. (2023). Unpacking mathematical thinking: The case of describing and explaining thinking. *Australian Primary Mathematics Classroom*, *28*(2), 21–28.

Thomson, S., De Bortoli, L., Underwood, C., & Schmid, M. (2019). *PISA 2018: Reporting Australia's results. Volume I student performance*. ACER. https://research.acer.edu.au/ozpisa/35

van Oers, B. (2013). Communicating about number: Fostering young children's mathematical orientation in the world. In J. Mulligan, & L. English (Eds.), *Reconceptualising Early Mathematics learning* (pp. 183–203). Springer 183–203.

Symposium: Effective Mathematics Teaching: Building Partnerships to Co-Develop Evidence-Based Capability

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Providing professional development at scale requires engaging diverse stakeholders to ensure support is based on research evidence and meets a range of teachers' needs. This symposium outlines research, partnerships and initiatives undertaken by a mathematics team in a state department of education to build a cohesive network of resources and professional learning to improve mathematics teaching and learning across the state.

Supporting teachers with relevant resources and professional learning is a priority to promote improvement in mathematics teaching and learning. At a systemic level, providing support at scale while recognising the highly diverse needs of teachers and schools is a well-documented challenge. A significantly revised mathematics curriculum has heightened the need for timeliness and range of expertise and perspectives. Collectively, the papers in this symposium tell a story of how a state department of education strategically partnered with mathematics education researchers, teachers and schools to design and implement a range of co-ordinated initiatives to support teachers and improve students' learning in mathematics.

In the first paper, Wood and her colleagues outline the history and background of ways that the Queensland Department of Education (the Department) have sought to support teachers to develop their mathematics pedagogy through a range of strategic partnerships across two decades. In *Building system-wide mathematics pedagogy through collaborative partnerships*, the authors discuss the impetus behind building teachers' pedagogical expertise in guided mathematical inquiry by working with mathematics education researchers as critical friends and developing resources at scale. In the second paper, *Designing curriculum resources to support teacher learning,* Goos details her theoretical analysis of the design of resources supporting teachers to "learn how to learn" to teach content that was new to them in the Queensland senior secondary mathematics syllabuses. Her paper exemplifies the Department's initiative to create a suite of professional learning materials for teachers designed by mathematics education researchers in a range of topics in mathematics curriculum, pedagogy, and classroom strategies. In the next paper, *Building capability: What to do when you don't know what to do,* school practitioners Moran and Lambie discuss how their school worked with a mathematics education researcher as a critical friend to address a problem of practice: improving students' performance on a new state assessment using complex, open-ended problems. They provide school-based evidence of how the using a research-based framework supported students to build confidence in addressing these tasks. Finally, in *Building capability for teachers of mathematics,* Horne and Hillman outline the partnership between the Department and an experienced teacher to develop resources that build teachers' capabilities in teaching mathematics. The 'How to Teach Mathematics Toolkit' seeks in particular to support beginning teachers and those teaching mathematics out-of-field in an online resource.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education*. *Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 67–83). Gold Coast: MERGA.

Building System-Wide Mathematics Pedagogy Through Collaborative Partnerships

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Research shows that when students engage with mathematical inquiry their problem-solving skills are strengthened. Demands in the revised Australian curriculum raised problem-solving of new senior secondary mathematics assessment, specifically in Queensland Problem-solving and Modelling Tasks (PSMTs). The challenge for Queensland was to scale inquiry pedagogies through secondary state schools in a way that was age appropriate and curriculum-aligned. A system, researcher and teacher collaboration produced a suite of resources and capability materials to build inquiry pedagogies of secondary teachers and ultimately students' problem-solving skills.

When students engage with guided mathematical inquiry, their problem-solving skills are strengthened (Lazonder & Harmsen, 2016). In addition, students experience greater engagement, enjoyment, and achievement (Collie & Martin, 2017). However, research outlines the difficulties that teachers experience in learning to adopt and appropriately guide their students' learning to engage in mathematical inquiry (Makar, in press; Munter, 2014). Facilitating system-wide pedagogical change in mathematics classrooms requires a multi-faceted, scalable approach (Roesken-Winter et al., 2021; Spillane et al., 2018).

This paper outlines the direction, history and development that the Queensland Department of Education made over 15+ years in supporting teachers to engage in adopting guided mathematical inquiry and problem-solving state-wide across all levels of schooling. The implications of this journey can provide guidance for system-level change over time in other jurisdictions. The outcomes highlight the importance of vision, partnerships, resource development and time in seeing systemic improvement in mathematics pedagogy from primary through secondary.

Queensland's Journey

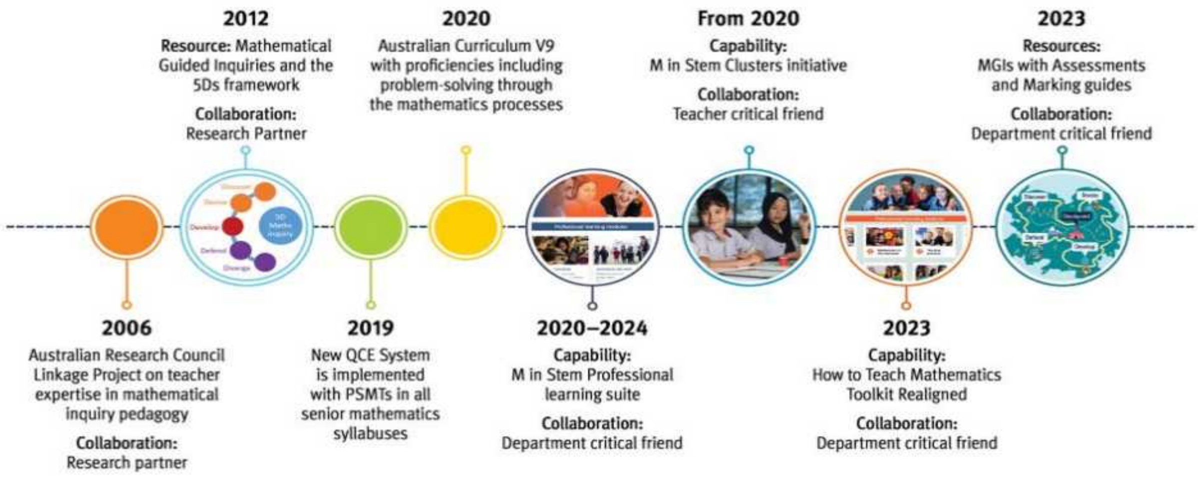
In 2023 the Queensland Department of Education (the Department) reaffirmed its commitment to prioritising achievement in mathematics in the education strategy, Equity and Excellence—a progressive, high performing education system realising the potential of every student—and continued a systematic approach to lifting outcomes for students in mathematics through building teacher capability. The Department has established a Prep to Year 12 approach to developing students' problem-solving skills with a focus on building teacher capability in inquiry pedagogies particularly in the secondary phase of schooling. Central to this has been collaborative partnerships with mathematics education researchers (Rosenquist et al., 2015). The Department's journey of valuing evidence-based practice through partnering with researchers has spanned almost two decades. In 2006, a formal partnership was initiated between the Department and a university in the form of an Australian Research Council (ARC) Linkage Project. The aim of the project was to study teachers' evolving experiences as they developed expertise in teaching mathematics through inquiry. The partnerships expanded to other universities over the years.

A Strategic Approach to Building Teacher Capability in Mathematics Pedagogy

The M-in-STEM initiative of the Queensland Department of Education was established in 2019 to strengthen age-appropriate and curriculum-aligned pedagogical practices of teachers of mathematics in Prep to Year 12. Through the STEM team, the Department engaged with fourteen mathematics education researchers from seven universities across Queensland and beyond to co-develop resources and capability programs to support teachers in adopting inquiry and problem solving pedagogies. The partnerships contributed to a range of co-ordinated and differentiated resources to build teacher capability (Figure 1), with many of these partnerships continuing.

Figure 1

Nature and Outcomes of Research Collaboration



Professional learning resources were developed for teachers across a range of experiences. The *How to Teach Mathematics Toolkit* was designed to support beginning, returning and out of field teachers of mathematics and is self-paced and online. Mathematical inquiry is addressed in two modules: Teaching practices—the pedagogy of inquiry, and Problem-solving—using the mathematical guided inquiries (MGIs). A partnership with a mathematics education researcher saw the adaptation of a framework for effective teaching of mathematics (National Council for Teachers of Mathematics [NCTM], 2014) supported with video content unpacking the framework and its implementation in classroom practice.

The M in STEM professional learning suite was developed to support experienced teachers of mathematics, particularly in secondary. Researchers co-developed professional learning resources with evidence-based pedagogical approaches across a range of topics supported with video content of classroom implementation. For example, in the inquiry module, the video case study *Effective inquiry strategies: Implementing a mathematical inquiry framework* documents how a secondary school adopted the 5Ds approach (Allmond et al., 2010) by implementing MGIs by collaborating with a mathematics education researcher. The partnership supported a consistent approach to inquiry across Years 7 to 10 in the school to ensure their students developed the age appropriate and curriculum aligned skills they needed to meet the demands of the PSMT into Years 11 and 12.

M in STEM Clusters Initiative

Building capabilities at scale required systemic partnerships between the Department, schools, and mathematics education researchers to support teachers with identified problems of practice (e.g., Koichu & Pinto, 2018). More direct collaboration between researchers and teachers was facilitated through the M in STEM clusters initiative. Researchers walked the journey of school improvement in mathematics with middle and aspiring leaders working in a cluster with a similar problem of practice. Researchers played the role of critical friend and provider of professional learning.

Case Study—M in STEM Collaborative Mathematics Inquiry

In 2021 a cluster of six state secondary schools undertook a collaborative inquiry to investigate how to improve implementation and outcomes in the PSMT requirements of the senior secondary mathematics syllabuses. PSMTs were prioritised as they contributed 20% of the final mathematics grade in Year 12. Pre-intervention data analysis across the six schools showed that students achieved lowest in the *Solve* and *Evaluate and Verify* criteria of the PSMT (compared with, for example, *Communicate*). The working hypothesis was that low performance in Solve was due to difficulties in the *Formulate* criterion, and, that low performance in Solve led to difficulties in the Evaluate and Verify criterion. Furthermore, both teachers and students were challenged by the language of the PSMT criteria.

In discussing the problem of practice with a content expert researcher as critical friend, the cluster agreed to use brainstorming in senior secondary (Years 11 and 12) to support student confidence in the Formulate and Evaluate and Verify stages of PSMTs. Taking an inquiry approach, the cluster backward mapped from the intended student outcomes and class behaviours, requisite teacher practice and professional learning, and the expected evidence of these anticipated changes, to understand how the cluster leaders needed to create the conditions for the change to occur. The M in STEM initiative provided the professional learning in instructional leadership and opportunities for intentional collaboration.

Schools in the cluster introduced brainstorming in junior secondary to strengthen problem solving skills and a whole school approach to language and pedagogy for problem solving tasks. Co-developed lesson plans and teaching resources were developed for implementation across the schools that developed brainstorming skills. Brainstorming encourages people to think in a free and open way with no restrictions. As a result, they often generate more possibilities than they would using a structured approach (Dugosh et al., 2000). The shared lessons were implemented with classroom routines and norms established to build students' confidence in the process and a safe environment for sharing ideas. *Fermi* problems were also used to stimulate ideas to evaluate and make assumptions. Fermi problems are miniature modelling problems that emphasise estimation (Albarracín & Ärlebäck, 2019).

Monitoring and reviewing activities including classroom walk-throughs, feedback from teachers and students, and pre- and post-intervention data analysis, showed an increase in confidence in both brainstorming and PSMT processes for students and teachers, improved disposition towards PSMTs for teachers and students and improved assessment literacy. The school found evidence of its effectiveness not only in the initial criterion (Formulate), but gave students confidence to proceed across all four criteria (Formulate, Solve, Evaluate and Verify, Communicate). A video capturing the case study was provided to all schools as an example of a high quality strategy and benefit of engaging a critical friend. The benefits of working in a cluster were identified as access to critical friend, opportunities to collaborate with other schools, and professional learning around an inquiry approach to school improvement.

Next Steps

The quality assured and curriculum-aligned materials developed through system-researcher partnerships are provided to all Queensland state schools to lift mathematics outcomes for students. In addition system-researcher-teacher relationships have further strengthened the network of mathematics educators across the state. While we have significant case studies demonstrating impact (see for example, Moran & Lambie, 2024), next steps are to gather further evidence of reach and impact of these materials. This includes evidence of downloads and access to online resources and professional learning, localised case studies of impact in schools and clusters, opt-in surveys of teacher feedback after engaging with and implementing the materials, and monitoring trends in system-wide mathematics reporting data.

This paper outlined strategies that enact the Department's commitment to strengthening teaching and learning in mathematics at scale to ensure every student realises their potential. It exemplifies the Department's focus on fostering collaborative partnerships to build evidence-based, inquiry pedagogies in mathematics across the system to build problem-solving skills.

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References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: Mathematics (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-1

Albarracín, L., & Ärlebäck, J. B. (2019). Characterising mathematical activities promoted by Fermi problems. *For the learning of mathematics, 39*(3), 10–13. https://www.jstor.org/stable/26854427.

Allmond, S., Wells, J., & Makar, K. (2010). *Thinking through mathematics: Engaging students in mathematical inquiry*. Curriculum Press.

Collie, R. J., & Martin, A.J. (2017). Students' adaptability in mathematics: Examining self-reports and teachers' reports and links with engagement and achievement outcomes. *Contemporary Educational Psychology, 49,* 355–366. https://doi.org/10.1016/j.cedpsych.2017.04.001

Dugosh, K., Paulus, P., Roland, E., & Yang, H.-C. (2000). Cognitive stimulation in brainstorming. *Journal of Personality and Social Psychology, 79*(5), 722–735. https://doi.org/10.1037//0022-35I4.79.5.722

Koichu, B., Pinto, A. (2018). Developing education research competencies in mathematics teachers through TRAIL: Teacher-researcher alliance for investigating learning. *Canadian Journal of Science, Mathematics and Technology Education, 18*, 68–85. https://doi.org/10.1007/s42330-018-0006-3

Lazonder, A. W., & Harmsen, R. (2016). Meta-analysis of inquiry-based learning: Effects of guidance. *Review of Educational Research, 86*(3), 681–718.

Makar, K. (2012). The pedagogy of mathematical inquiry. In R. Gillies (Ed.), *Pedagogy: New developments in the learning sciences* (pp. 1–23). Nova Science.

Makar, K. (2024). Primary teachers' early and retrospective instructional vision of mathematical inquiry. *Journal of Educational Change, 25*, 173–196. https://doi.org/10.1007/s10833-023-09487-5

Moran, T., & Lambie, K. (2024). *Building capacity: What to do when you don't know what to do.* In J. Višňovská, E. Ross, & S. Getenet (Eds), Surfing the waves of mathematics education. *Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (in press). MERGA.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education, 45*(5), 584–635.

National Council for Teachers of Mathematics (NCTM). (2014). *Principles to actions: Ensuring mathematical success for all*. NCTM.

Roesken-Winter, B., Stanke, R., Prediger, S., & Gasteiger, H. (2021). Towards a research base for implementation strategies addressing mathematics teachers and facilitators. *ZDM Mathematics Education, 53*, 1007–1019. https://doi.org/10.1007/s11858-021-01220-x

Rosenquist, B., Henrick, E., & Smith, T. (2015). Research-practice partnerships to support the development of high quality mathematics instruction for all students. *Journal of Education for Students Placed at Risk, 20*(1-2), 42–57. https://doi.org/10.1080/10824669.2014.988335

Spillane, J. P., Hopkins, M., & Sweet, T. M. (2018). School district educational infrastructure and change at scale: Teacher peer interactions and their beliefs about mathematics instruction. *American Educational Research Journal, 55*(3), 532–571. https://doi.org/10.3102/0002831217743928

Queensland Curriculum and Assessment Authority (QCAA). (2021). *Problem-solving and modelling task. QCAA*. www.qcaa.qld.edu.au.

Designing Curriculum Resources to Support Teacher Learning

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This paper presents an analysis of how resources were designed to support implementation of the new Queensland senior secondary mathematics syllabuses. The analysis draws on the concept of educative curriculum materials that build teachers' subject matter knowledge and pedagogical content knowledge. Such resources are intended to help teachers "learn how to learn" to teach mathematical content that is new or unfamiliar to them.

Introduction of the Australian Curriculum: Mathematics at F–10 and senior secondary mathematics levels has led to diverse initiatives by state and territory education jurisdictions to provide resources and support for teachers. This paper examines one aspect of the Queensland Department of Education's *M in STEM* initiative, which involves collaboration with university-based mathematics education researchers to develop a professional learning suite for secondary school mathematics teachers. The resources are intended to strengthen teachers' curriculum knowledge and pedagogical practices across six topics: productive dispositions, problem solving and inquiry, modelling, reasoning, new content in the senior mathematics syllabuses, and strategies for long term retention of knowledge and preparing students for assessment.

The aim of this paper is to analyse the process used to develop resources for one of the focus topics in the professional learning suite: learning to teach new content in the senior secondary mathematics syllabuses. The analysis draws on the concept of *educative curriculum materials*, that is, curriculum resources that are designed to promote teachers' learning of mathematical content and pedagogy as well as student learning (Ball & Cohen, 1996; Davis & Krajcik, 2005). The paper addresses the following research question:

- How can curriculum resources be designed to support mathematics teacher professional learning in the context of curriculum reform?

Curriculum Context

In 2019, the Queensland Curriculum and Assessment Authority (QCAA) introduced new syllabuses for senior secondary mathematics based on the subjects developed by the Australian Curriculum, Assessment and Reporting Authority (ACARA, n.d.): General Mathematics, Mathematical Methods, and Specialist Mathematics (version 8.4). The new syllabuses included mathematical content that was either new or at a higher level of difficulty than in the previous Queensland syllabuses for the equivalent subjects of Mathematics A, Mathematics B, and Mathematics C respectively. The new suite of subjects represented the most significant change to senior secondary mathematics curriculum in Queensland since the previous syllabuses were launched in 1992.

The support offered to teachers for implementing a new syllabus is often in the form of instructional materials that help teachers interpret the official curriculum and create their own personal plans for teaching specific groups of students (Remillard & Heck, 2014). One challenge in designing such resources is to find a realistic balance between pedagogical prescription and professional autonomy (Davis & Krajcik, 2005). Determining the appropriate amount of guidance needed by senior secondary teachers was a particular challenge for General Mathematics, the subject most likely to be taught by out-of-field teachers who have undertaken limited advanced studies of mathematical content and little or no formal preparation in teaching mathematics. Text-based and online curriculum materials are also more educative for teachers if combined with in-person social support (Robutti et al., 2016). However, delivering a state-wide professional learning program containing face-to-face elements is challenging in Queensland, the Australian state with the most decentralised population spread over a very large area (see Australian Bureau of Statistics, 2022; Geosciences Australia, 2023). Each of these constraints influenced the design of the *M in STEM* professional learning suite.

Theoretical Background

The design of resources for one of the *M in STEM* focus topics is analysed by reference to five high-level guidelines for educative curriculum materials set out by Davis and Krajcik (2005). They proposed that educative curriculum materials should:

- Develop teachers' capacity to anticipate and interpret student thinking during instructional activities, as well as how to respond to student thinking (e.g., by using appropriate examples or instructional representations);
- Support teachers' learning of the subject matter and related disciplinary practices;
- Help teachers recognise how a learning objective, instructional activity, or lesson Sequence is related to the curriculum as a whole;
- Make visible the resource developer's pedagogical reasoning, thus enabling teachers to integrate this knowledge into their own repertoire;
- Promote teachers' pedagogical design capacity so they are able to make principled adaptations to the original curriculum materials.

In these ways, educative curriculum materials build teachers' subject matter knowledge and pedagogical content knowledge.

Designing Curriculum Resources for General Mathematics

In the curriculum context outlined above, consultation with the Queensland Department of Education led to a decision to focus on teaching new content in the General Mathematics syllabus (QCAA, 2019). It was not feasible to design curriculum resources for every topic in the syllabus that was likely to be new or unfamiliar to teachers. Instead, three syllabus topics considered to be most demanding for inexperienced or out-of-field teachers were selected: linear equations and their graphs; geometric sequences; and planar graphs, paths, and cycles.

Teachers are also time poor and not always willing to engage with extensive materials. Thus, the resources needed to concisely address key ideas for teaching while simultaneously illustrating how teachers could "learn how to learn" to teach other new topics in the syllabus. This was done by creating, for each topic, a series of three recorded PowerPoint presentations outlining evidence-based pedagogical strategies (1 hour total) and a placemat that defined the topic together with planning and teaching principles. These static resources were supplemented by an interactive 40-minute online professional discussion with teachers from around the state.

The intention was to develop a consistent structure for the recorded presentations that would expose the pedagogical decision-making underpinning the design. The rationale for these decisions was also made explicit in the placemat representing the design process. The design process moves through three stages: (a) interrogating the senior syllabus to identify and understand the subject matter; (b) mapping connections backwards, forwards, and across the Australian curriculum; and (c) designing pedagogy by selecting appropriate representations and real-life examples, and addressing common misconceptions that hinder student learning. The design process for the geometric sequences PowerPoint presentation is illustrated in Table 1 and mapped against Davis and Krajcik's (2005) guidelines for educative curriculum materials.

The principles underpinning the design process illustrated in Table 1 were articulated in the topic placemat, which is presented in abbreviated form in Figure 1. The placemat highlights the teacher's role in bringing the curriculum to life for students, by moving back and forth between the curriculum world, real world, and classroom world.

Table 1

Design Process for Geometric Sequences PowerPoint Presentation

(a) Interrogate the syllabus

Educative curriculum materials guideline 2: Support teachers' learning of subject matter

What subject matter is included?

- Generating sequences using recursion or rule for the n^{th} term
- Displaying the terms of a sequence in tabular or graphical form
- Using geometric sequences to model and analyse (numerically or graphically only) practical problems involving geometric growth and decay

How are key terms defined?

- Syllabus glossary definition of a geometric sequence, "a sequence of numbers where each term after the first is found by multiplying the previous term by a fixed non-zero number (excluding ±1) called the common ratio" (QCAA, 2019, p. 59)
- Common ratio > 1 → exponential growth; Common ratio < 1 → exponential decay
- Illustrate two methods of generating the geometric sequence 2, 6, 18, 54, …:
- Recursion relation $t_1 = 2, t_{n+1} = 3t_n$ for $n \geq 1$
- Rule for the n^{th} term $t_n = 2 \times 3^{n-1}$ for $n \geq 1$

Why is the topic important?

- Geometric sequences are used to understand real life situations and solve real life problems involving exponential growth and decay

(b) Map the curriculum

Educative curriculum materials guideline 3: Relate the topic to the curriculum as a whole

What prior learning have students experienced from the F–10 mathematics curriculum?

- Understand the connection between algebraic and graphical representations
- Solve basic problems involving simple and compound interest

What other topics in the General Mathematics syllabus connect to this topic?

- Loans, investments, and annuities: Use a spreadsheet to investigate the effect of interest rate on the future value of an investment

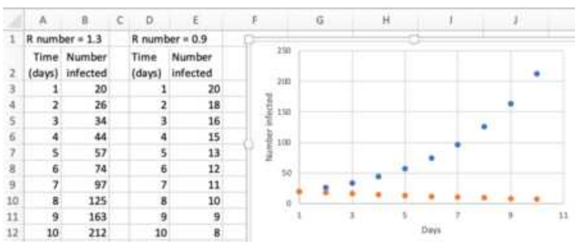
What other curriculum areas connect with this topic?

- Physics and ancient history: radioactive decay, carbon dating
- Biology: population growth, bacterial growth, spread of infectious diseases such as COVID-19

(c) Design pedagogy

Educative curriculum materials guideline 1: Anticipate, interpret, and respond to student thinking

- Select appropriate representations and link to real life examples.

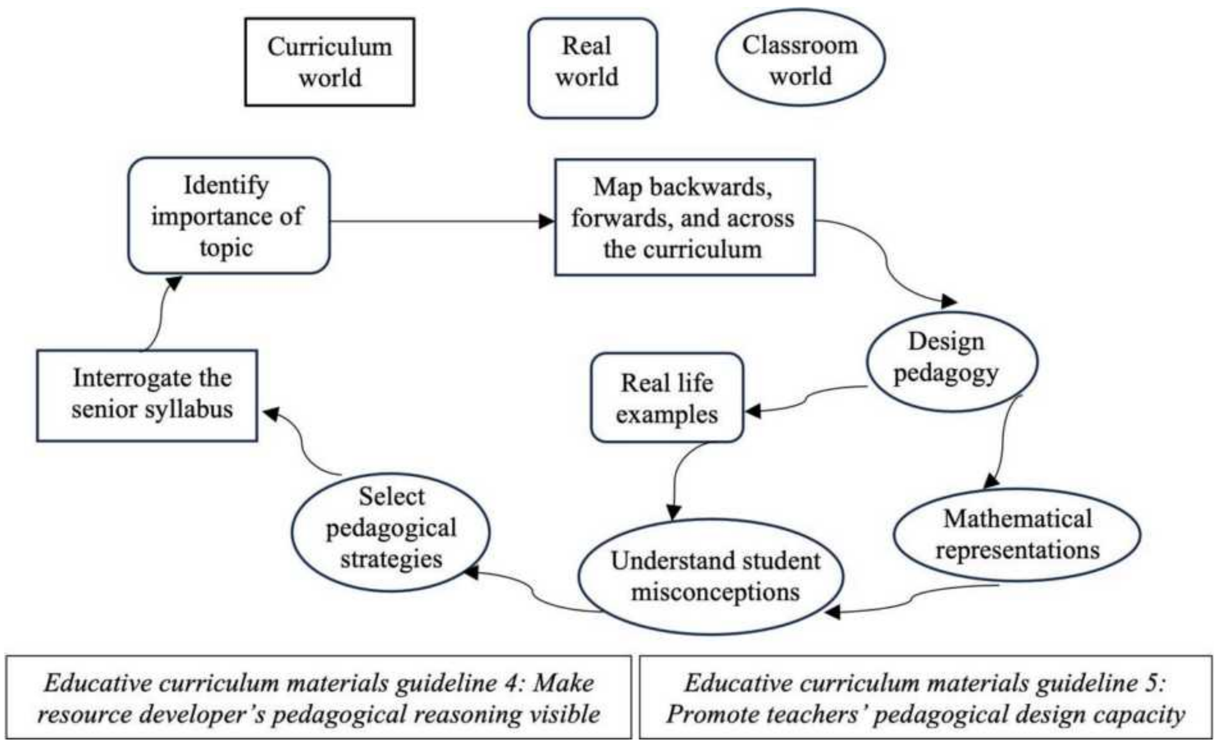


	A	B	C	D	E
1	R number = 1.3			R number = 0.9	
2	Time (days)	Number infected		Time (days)	Number infected
3	1	20		1	20
4	2	26		2	18
5	3	34		3	16
6	4	44		4	15
7	5	57		5	13
8	6	74		6	12
9	7	97		7	11
10	8	125		8	10
11	9	163		9	9
12	10	212		10	8

- Understand and respond to students' thinking
- Recognise misconceptions about working with numerical expressions in exponential form.
- Encourage students to explore both recursion relationships and the rule for n^{th} term to define sequences
- Provide experiences for students to explore and compare additive and multiplicative patterns that model arithmetic and geometric sequences respectively
- Use technology so students can investigate patterns of growth and decay in sequences, generating both tables of values and graphs

Figure 1

Placemat for Teaching New Content in Senior Mathematics



Concluding Remarks

This paper illustrates one approach to designing educative curriculum materials that support teacher learning as well as student learning. In principle, the resources developed for Queensland senior secondary mathematics teachers align with the guidelines proposed by Davis and Krajcik (2005). However, little is known about teachers' uptake of these resources and what difference this makes to their professional knowledge and classroom practice. These promise to be fruitful areas for future research on teachers' learning in times of curriculum reform.

References

Australian Bureau of Statistics. (2022, December 20). 50 years of capital city population change. ABS. https://www.abs.gov.au/articles/50-years-capital-city-population-change

Australian Curriculum, Assessment and Reporting Authority (ACARA). (n.d.). Senior secondary curriculum: Mathematics (version 8.4). ACARA. https://www.australiancurriculum.edu.au/senior-secondary-curriculum/mathematics/

Ball, D. L., & Cohen, D. (1996). Reform by the book: What is—or might be—the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher, 25*(9), 6–8, 14. https://doi.org/10.3102/0013189X025009006

Davis, E., & Krajcik, J. (2005). Designing educative curriculum materials to promote teacher learning. *Educational Researcher*, *34*(3), 3–14. https://doi.org/10.3102/0013189X034003003

Geoscience Australia. (2023, July 26). Area of Australia—States and territories. Geoscience Australia. https://www.ga.gov.au/scientific-topics/national-location-information/dimensions/area-of-australia-states-and-territories

Queensland Curriculum and Assessment Authority (QCAA). (2019). General mathematics 2019 v1.2. QCAA. https://www.qcaa.qld.edu.au/downloads/senior-qce/syllabuses/snr_maths_general_19_syll.pdf

Remillard, J., & Heck, D. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM Mathematics Education, 46*(5), 705–718. https://doi.org/10.1007/s11858-014-0600-4

Robutti, O., Cusi, A., Clark-Wilson, A., Jaworski, B., Chapman, O., Esteley, C., Goos, M., Isoda, M., & Joubert, M. (2016). ICME international survey on teachers working and learning through collaboration. *ZDM Mathematics Education, 48*(5), 651–690. https://doi.org/10.1007/s11858-016-0797-5

Building Capability: What to do When You Don't Know What to do

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The Queensland Certificate of Education (QCE) system is Queensland's senior school qualification. To support the introduction of the system in 2019, existing senior syllabuses were redeveloped and a new senior assessment model was established, this included the implementation of a mandatory high-stakes assessment task, the Problem-solving and Modelling Task (PSMT) in all four mathematics syllabi. The PSMT required new skills from both students and teachers to manage complex, open-ended investigations. In this paper, we reflect on our school's approach to build teacher capability in designing PSMTs and supporting student engagement with PSMTs.

In 2019, the new Queensland Certificate of Education (QCE) system was introduced. To support the introduction of the system, existing senior syllabi were redeveloped and a new senior assessment model was established to strengthen the quality and comparability of school-based assessment. These changes included the introduction of a mandatory high-stakes assessment—the Problem-solving and Modelling Task (PSMT)—to four senior mathematics syllabi: Essential Mathematics, General Mathematics, Mathematical Methods and Specialist Mathematics. PSMTs are designed by schools and then must be approved by the state assessment authority. Designing these tasks and supporting students to successfully engage with them required new skills. The PSMT was designed to evaluate a student's ability to respond to an investigative mathematical scenario or stimulus in relation to the mathematical concepts they have learned against four assessment criteria: Formulate, Solve, Evaluate and verify, and Communicate (QCAA, 2021). In most cases, the key feature of this task has been to provide for a response that addresses a real-life application of mathematics. Indeed, rich mathematical understanding goes beyond being able to correctly complete mathematical exercises, but also to make connections and transfer learning to unfamiliar problems (Skemp, 1978; Sullivan, 2011). As Peter Sullivan (2011) advised, "One of the major constraints that teachers experience when utilising such tasks is that many students avoid risk taking and do not persist with the challenges that are required in order to complete the task." (p. 38).

In this paper, we reflect on how our department in a state secondary school, in which the two authors were teacher leaders at the time, sought to facilitate this process for teachers and students. We drew on research and partnered with a university researcher, using a framework and seeking strategies to support students to build problem solving competence on complex, unfamiliar problems to allow us to reflect on, improve, and evaluate our progress.

Initial Practice of PSMTs

With the introduction of the new senior syllabi, our mathematics department wanted to ensure students were given every opportunity to perform well in the PSMT. A decision was made to introduce a practice PSMT as part of the problem-solving proficiency of the Australian curriculum at Year 10 to strengthen transition from junior to senior secondary mathematics courses. While we gave plenty of scaffolding on how to set out the response and developed understanding of the valued features outlined in the marking criteria, many students struggled to complete the task and wanted to be shown how to find the answer (Sullivan, 2011). This expectation was in line with the students' experiences of class work, but could not be provided in a PSMT setting where students are required to develop their own unique response to the set problem.

We noticed that our initial PSMT results in 2019 indicated that our high achieving students, who were preparing to study Mathematical Methods (Cohort A), generally performed well, with

88% passing (grade of C or better); whereas students who were preparing to study General Mathematics (Cohort B) did not have the same level of success, with only 63% passing. To better understand where our students were struggling, we followed up with some students to seek their feedback. Many told us that they simply did not know what to do and the teachers could not tell them, so they gave up. Essentially, they did not know how to start, and if they did attempt the problem and got stuck, they struggled to look for alternative approaches. We wondered how to help students with the challenge, 'What do you do when you don't know what to do?', so that they would have the confidence and skills to solve the unfamiliar problems posed by PSMTs.

A New Approach

We realised that it was necessary to employ a different pedagogical approach to improve student success and disposition. We were teaching mathematics using direct instruction with a gradual release of responsibility and a move from simple to complex questions. We also had an established problem-solving approach. However, these approaches were not helping students address unfamiliar, open-ended tasks like PSMTs. That is, our pedagogies were based on students being modelled how to do the mathematics first, before moving to a related application or a question they were less familiar with; however, even in these instances, the problems we gave students were fairly well-defined and typically had a single correct answer. The difference the PSMT posed was that students needed to come up with a way of responding to the task that was not directly modelled first by the teacher; and to find a solution to a task that did not come with an answer page to reassure them of their accuracy.

We decided to explore different pedagogical approaches that may help address students' unwillingness to make a start on the PSMT and complete it without teacher guidance. We conducted a search for solutions and identified research on the pedagogy of mathematical inquiry. Mathematical inquiry is an approach to solving complex, open-ended problems that relied on mathematical evidence (Makar, 2012). (Excerpts are from a video case study created about our journey by the Queensland Department of Education for the M-in-STEM Professional Learning Suite.)

> Guided inquiry [is] responding to the question, 'What do you do when you don't know what to do?' Complex open-ended problems like problem-solving and modelling tasks ask students to negotiate, adapt and revise their solution. This can be a real challenge for students. (K. Makar, quoted in Queensland Department of Education, 2022, 02:56)

We contacted the researcher, who agreed to discuss their research with us and act as a critical friend. We were particularly interested in their 5D model (Discover, Devise, Develop, Defend, Diverge; Allmond et al., 2010), which seemed to align well with the criteria already used with PSMTs. The Discover phase in the 5D model extends the QCAA modelling criteria by providing students with extra scaffolding to get started with the problem. It also engaged students with the problem context in a low-stakes setting before starting to plan a possible solution. This seemed particularly helpful when students did not know how to start the PSMT.

Together with the researcher, we (second author) designed an inquiry approach to support students in approaching the PSMT confidently and appropriately, using the 5D model as a guiding framework. We observed how students responded and plans to evaluate student data on improvement:

> In the past, our students have struggled to engage because of the size of the [PSMT] task. What we are seeing now, is that by using the 5Ds, students are less stressed and more engaged with the problem-solving and modelling tasks, and in turn, more willing to engage with [the] assessment. Next, would be a checkpoint at the halfway mark to see how much of the task students have completed. We can compare this directly with previous assessment pieces. Thirdly, we'll be comparing student achievement data. (K. Lambie, quoted in Queensland Department of Education, 2022, 05:54)

From 2020, we also began giving our Year 9 students inquiry tasks similar to PSMTs to give them additional practice, each time using the 5D framework as a scaffold. In particular, five lessons were developed to assist Year 9 students in unpacking a PSMT-like task using the 5D inquiry model. This approach sought to:

- Give students an opportunity to experience risk in a low stakes environment;
- Reduce the size of the task;
- Reduce student stress;
- Include a ‘checkpoint’ opportunity, where students shared their interim progress and discussed difficulties they were encountering with peers to generate possible ideas.

The benefit of providing students with an age-appropriate and curriculum-aligned inquiry task in Year 9 was that the students became much more confident by Year 10 because they had gone through a similar process the year before. As students were initially unfamiliar with this assessment style, the Year 9 task was designed to give students confidence moving forward and from the beginning, the Year 9 cohort demonstrated a high passing rate (93%). They also transitioned well into the more challenging Year 10 practice PSMT. We looked at the practice PSMTs again to see whether our Year 10 students felt more confident. Indeed, within these two years, students’ practice PSMTs significantly improved (Table 1), particularly the Year 10 lower-performing students (Cohort B) who increased their passing rate from 63% in 2019 to 96% in 2021.

Table 1

Practice PSMT Aggregated Results From our Year 10 Cohort

	PSMT percentage of students passing (grade C or better)	
Subject	**2019**	**2021**
Cohort A	88%	99%
Cohort B	63%	96%

Building Thinking Classrooms

Following our work above, we have continued to seek ways to improve students’ learning and their confidence to tackle complex, unfamiliar tasks. We recently attended a two-day professional development workshop provided by the Department on how to build thinking classrooms in mathematics (Liljedahl, 2020). The *Building Thinking Classrooms* workshop, delivered by Professor Peter Liljedahl represented the next step in our journey to increase problem-solving opportunities and success for students in our classrooms. Liljedahl’s approach to mathematics teaching emphasised a combination of group work, open-ended questions and opportunities to explore and test understanding in a low-risk environment. The combination of random grouping, the use of vertical whiteboards and the collaborative solving of problems that had not been previously modelled provided a fun-filled and engaging opportunity to solve mathematics problems in a safe and supportive environment. One of the difficulties we continue to have is getting students to make a start on solving problems when they are not sure if their selected methods will lead them to the answer. Liljedahl’s work provided us with further strategies to address this challenge with our students.

Following the workshop, we shared two different activities with the other teachers at our mathematics meetings and we have written a number of *Building Thinking Classrooms* activities into our mathematics plan. As we move towards the implementation of Australian Curriculum: Mathematics (Version 9), we have identified opportunities to embed these practices across all year levels, and our teachers are on board with this new approach, already incorporating the activities in their classrooms across all year levels. Teachers are motivated to learn more about the approach and how they can continue to support student engagement and success in mathematical problem-solving.

Conclusion

As a school, we are always interested in improving student learning. In particular, we were wanting to support students to respond to the challenge of "What do you do when you don't know what to do?" The inclusion of mathematical processes in the newly revised Australian Curriculum: Mathematics (v9.0) (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2022) provides increased opportunities for students to address this challenge. "The mathematical processes ... are mathematical problem-solving and investigation processes that students learn to use in mathematics, and that draw upon students' mathematical process skills and proficiency in mathematics in an interconnected way" (ACARA, 2022, Mathematics, Key Considerations, Mathematical Processes section). We have been encouraged by the Year 10 practice Problem-solving and Modelling Tasks (PSMTs) in their capacity to increase student confidence, persistence, and skills in addressing complex, unfamiliar problems. Three approaches have assisted us to support students in this way: Drawing on research and working with a researcher as critical friend to guide the direction of our improvements to be evidence-based; engaging with the 5D framework (Allmond et al., 2010) and Thinking Classrooms material (Liljedahl, 2020) to improve our problem-solving pedagogy; and using data to reflect on, improve and evaluate our progress.

References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: Mathematics (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-1

Allmond, S., Wells, J., & Makar, K. (2010). *Thinking through mathematics*: *Engaging students in mathematical inquiry*. Curriculum Press.

Liljedahl, P. (2020). *Building thinking classrooms in mathematics, grades K–12: 14 teaching practices for enhancing learning*. Corwin Press.

Makar, K. (2012). The pedagogy of mathematical inquiry. In R Gillies (Ed.), *Pedagogy: New developments in the learning sciences* (pp. 371–397). Nova Science.

Queensland Department of Education (2022). *M in STEM—Effective inquiry strategies: Implementing a mathematical inquiry framework* [Video]. https://www.youtube.com/watch?v=hS0YZw7s9lk

Skemp, R. (1978). Relational understanding and instrumental understanding. *The Arithmetic Teacher, 26*(3), 9–15.

Sullivan, P. (2011). *Teaching mathematics: Using research-informed strategies*. ACER. https://research.acer.edu.au/cgi/viewcontent.cgi?article=1022&context=aer

Queensland Curriculum and Assessment Authority (QCAA). (2021). Problem-solving and modelling task. QCAA. www.qcaa.qld.edu.au

Building Capability for Teachers of Mathematics

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Teaching resources and professional development based on mathematics education research have the potential to support teachers to develop and sustain improved pedagogies. The Queensland Department of Education provided online professional learning modules for teachers of Prep (Foundation) to Year 10 mathematics . To support implementation of the Australian Curriculum: Mathematics this evidence-based resource would assist teachers in understanding the curriculum and providing support for quality teaching and learning. This resource exemplifies the partnerships between the department, researchers and teachers in building capability in mathematics teaching.

The notion of curriculum is not static, with distinctions made between what is intended by curriculum writers and how curriculum is enacted in the classroom (Remillard & Heck, 2014). Teacher resources can greatly influence how teachers interpret the curriculum, and innovative resource materials have long been used to support teachers adopt more effective pedagogies (Ball & Cohen, 1996; Pepin, 2018). "The characteristics of these innovative materials ultimately influence teachers' instructional practices, including their use of curriculum materials" (Choppin, 2011, p. 332). Because of their potential impact on teachers' pedagogy, it is essential for resource materials draw on evidence of effective practice and based on contemporary research (Irgens et al., 2023; Munter, 2014; Roesken-Winter et al., 2021; Sullivan, 2011).

In this paper, we outline how a state education department developed an evidence-based online teacher capability to support beginning, returning and out of field teachers in diverse school and community contexts to implement Australian Curriculum: Mathematics by partnering with university researchers and teachers as co-constructors.

Promoting System-Wide Capability Development in Mathematics Pedagogy

The Queensland Department of Education (the Department) outlines its commitment to realizing the potential of every student, including the prioritising achievement in mathematics, in the education strategy, Equity and Excellence. Version 9.0 of the Australian Curriculum highlighted the need for state-wide capability-building for teachers of mathematics. The challenge was also identified in providing systematic and contextualised capability development in mathematics pedagogy for a range of teacher needs that draws on contemporary research. These challenges have been identified in research beyond Australia as well:

> Mathematics teachers face challenges in modifying their teaching to incorporate effective pedagogical practices, technology tools, and new curricula resources. They also face challenges in making changes to address updated standards and expectations for mathematics. … Teachers often have limited resources to support professional development to learn how to make these changes. Many teachers are seeking out online professional development opportunities. … Evidence suggests online professional development (PD) that is accessible, meaningful, collaborative, and addresses varied needs and abilities of participants can lead to changes in teachers' instructional practices. (Hollebrands & Lee, 2020, pp. 859–860)

A universal, online resource would provide foundational instruction tailored to beginning, returning, and out of field teachers of mathematics, and accessible to all state school teachers in Queensland. This was realised through the redevelopment of the 'How to Teach Mathematics Toolkit' (the toolkit). To maximise relevance and engagement for teachers, the resource offers online, self-directed modules with research-validated information and advice to build teacher knowledge, skills, and understanding of mathematics curriculum and pedagogy embedded in the context of teachers' classrooms and supported with peers (Kleiman et al., 2015; Powell & Bodur, 2019). Teachers' engagement in substantial professional learning resources such as this has been shown across multiple studies to have substantial improvement in student learning (Yoon et al., 2007).

The Department has a long history of creating and sustaining partnerships with researchers on the improvement journey of mathematics curriculum and pedagogy (Horne & Makar, 2013). Existing and new research partnerships were activated to ensure the resource was informed by leading edge evidence of effective pedagogy in mathematics (cf. Berger & Baker, 2008; Lillejord & Børte, 2016). Collaborative research partnerships were instrumental to this resource in three ways: knowledge translation, critical friends, and content co-developers (Irgens et al., 2023). Collaborative teacher partnerships were equally important providing lesson plans and videos demonstrating examples of content, mathematical guided inquires and related assessment.

Designing a Capability Resource Using Evidence-Based Practices

The toolkit is focused on evidence-based research and is organised over eight modules. The modules address current teaching and learning, the structure of the Australian Curriculum: Mathematics—including the mathematical proficiencies (Understanding, Fluency, Problem solving, Reasoning), and the importance of ongoing teacher personalised learning (Figure 1).

Figure 1

How to Teach Mathematics Toolkit Professional Development Modules

Research on teacher professional learning has highlighted that teachers value professional development that includes a focus on content, active learning, alignment with curriculum, and engagement over time (Haug & Mork, 2021). In the Teaching and Learning modules (1–3), teachers will explore:

- *Teaching Mathematics*: Mathematical opportunities, knowing and planning the curriculum and mathematical content, knowing how to plan a lesson;
- *Mathematics in the classroom*: Addressing you own self-efficacy, understanding your school's context, knowing your students with a focus on assessment and differentiation and how to support mathematical language within the classroom;

- *Teaching practices*: Understanding effective teaching of mathematics, focusing on positive dispositions, orchestrating classroom discourse, supporting student engagement and pedagogy in the mathematics classroom.

Modules 4–7 model the structure of the Australian Curriculum: Mathematics. By unpacking the mathematical proficiencies (*Understanding, Fluency, Problem solving, Reasoning*). Within each proficiency, the modules use content strands in teaching, learning, and assessing the proficiency with lessons and assessment ideas for each phase of schooling (Prep to Year 2, Year 3 to 6 and Years 7 to 10).

Finally, *Personalised learning* supports teachers in their understanding of continued learning in teaching and learning mathematics and reflect on their own self-efficacy.

The toolkit modules are self-paced and combine online learning and offline self-reflection and practical application. Importantly, there are opportunities for participants to consolidate and extend their learning through collaborative activities with a mentor, thus allowing a contextualised approach (Fantilli & McDougall, 2009). The toolkit encourages further reading and engagement with research through the resources lists included at the end of each module.

The toolkit modules are designed to align to the Australian Institute for Professional Standards (AITSL) so that completion of the course contributes to teachers' continuing professional development requirement for registration. This validates participants' investment of time in completing the toolkit. Teachers have the opportunity to collect evidence of their participation by recording reflections, mentoring discussions, implementation trials in the classroom, peer observations and student observations and work samples.

Next Steps—Supporting Mentoring and Scaling up Effective Practice

In the context of widespread teacher shortages the Department recognises the critical importance of universal access to high quality professional learning in mathematics, scaling up effective practice, sharing expertise through clusters and attending to teacher wellbeing (Haug & Mork, 2021; Irgens et al., 2023; Powell & Bodur, 2019). There are opportunities to support mentoring partnerships through:

- Mobilising suitable expertise as mentors within/across the department and in research organisations;
- Facilitate clusters to share expertise—strengthen and expand the network of mathematics educators by supporting partnerships between teachers and researchers as mentors.

There are opportunities to embed toolkit modules in initial teacher education programs to support transition of beginning teachers into mathematics classrooms in Queensland state schools.

Conclusion

The *How to Teach Mathematics Toolkit* is an evidence-based resource to promote system-wide capability development in mathematics pedagogy in Queensland state schools. It supports the implementation of Australian Curriculum: Mathematics in F–10, and combines both universal access and contextualised implementation to maximise reach and impact. It builds and shares expertise in mathematics pedagogy through collaborative partnerships to build capability of teachers to deliver quality teaching and learning in mathematics.

References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: Mathematics (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-1

Australian Government Department of Education. (2019). *The Alice Springs (Mparntwe) Education Declaration*. Australian Government. https://www.education.gov.au/download/4816/alice-springs-mparntwe-education-declaration/7179/alice-springs-mparntwe-education-declaration/docx

Australian Institute for Teaching and School Leadership. (2017). Australian Professional Standards for Teachers. AITSL.

Ball, D. L., & Cohen, D. (1996). Reform by the book: What is - or might be - the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher, 25*(9), 6–8, 14.

Berger, J. G., & Baker, R. (2008). Developing new knowledge and practice through teacher-researcher partnerships? *Paper presented at the International Congress for School Effectiveness and Improvement (ICSEI)*, Auckland, January 6–9, 2008. New Zealand Council for Educational Research. http://h41-239.catalyst.net.nz/sites/default/files/pages/developing-new-knowledge.pdf

Choppin, J. (2011). Learned adaptations: Teachers' understanding and use of curriculum resources. *Journal of Mathematics Teacher Education, 14,* 331–353.

Department of Education, Queensland. (2023). Equity and excellence: Realising the potential of every student. Brisbane, Queensland: Department of Education.

Fantilli, R., & McDougall, D. (2009). A study of novice teachers: Challenges and supports in the first years. *Teaching and Teacher Education, 25*(6), 814–825. https://doi.org/10.1016/j.tate.2009.02.021

Haug, B., & Mork, S. (2021). Taking 21st century skills from vision to classroom: What teachers highlight as supportive professional development in the light of new demands from educational reforms. *Teaching and Teacher Education*, *100,* Article 103286. https://doi.org/10.1016/j.tate.2021.103286

Hollebrands, K. F., & Lee, H. S. (2020). Effective design of massive open online courses for mathematics teachers to support their professional learning. *ZDM Mathematics Education, 52,* 859–875. https://doi.org/10.1007/s11858-020-01142-0

Horne, R., & Makar, K. (2013). Building relationships between state stakeholders and researchers: People, persistence and passion. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow: Proceedings of the Mathematics Education Research Group of Australasia* (pp. 767–770). MERGA.

Irgens, G., Hirsch, S., Herro, D., & Madison, M. (2023). Analyzing a teacher and researcher co-design partnership through the lens of communities of practice. *Teaching and Teacher Education, 121,* Article 103952. https://doi.org/10.1016/j.tate.2022.103952

Kleiman, G., Wolf, M. A., & Frye, D. (2015). Educating educators: Designing MOOCs for professional learning. In P. Kim (Ed.), *Massive open online courses: The MOOC revolution* (pp. 117–144). New York: Routledge.

Lillejord, S., & Børte, K. (2016). Partnership in teacher education—A research mapping. *European Journal of Teacher Education, 39*(5), 550–563. https://doi.org/10.1080/02619768.2016.1252911

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education, 45*(5), 584–635.

National Council for Teachers of Mathematics (NCTM). (2014). *Principles to actions: Ensuring mathematical success for all*. NCTM.

Pepin, B. (2018). Enhancing teacher learning with curriculum resources. In L. Fan, L. Trouche, C. Qi, S. Rezat, & J. Višňovská (Eds.), *Research on mathematics textbooks and teachers' resources: Advances and issues* (pp. 359–374). Springer.

Powell, C. G., & Bodur, Y. (2019). Teachers' perceptions of an online professional development experience: Implications for a design and implementation framework. *Teaching and Teacher Education, 77*, 19–30.

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM Mathematics Education, 46*(5), 705–718.

Roesken-Winter, B., Stanke, R., Prediger, S., & Gasteiger, H. (2021). Towards a research base for implementation strategies addressing mathematics teachers and facilitators. *ZDM Mathematics Education, 53*, 1007–1019. https://doi.org/10.1007/s11858-021-01220-x

Sullivan, P. (2011). *Teaching mathematics: Using research-informed strategies*. ACER. https://research.acer.edu.au/cgi/viewcontent.cgi?article=1022&context=aer

Yoon, K., Duncan, T., Lee, S., Scarloss, B., & Shapley, K. (2007). *Reviewing the evidence on how teacher professional development affects student achievement*. US Department of Education, Regional Educational Laboratory Southwest. https://ies.ed.gov/ncee/edlabs/regions/southwest/pdf/REL_2007033.pdf

Research Papers

Beyond Qualifications: Identity of Out-of-Field Teachers in Years 7–10 Mathematics in South Australia

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Concern regarding the prevalence of out-of-field teaching continues to grow, as does the body of literature calling for a better understanding of the nuanced complexities associated with out-of-field teaching. In this paper, we extend our previous analysis of data from a survey of the profession teaching Years 7–10 mathematics in South Australia. Analysing the data through an identity lens indicates that out-of-field teachers who self-identify as mathematics teachers and express a preference for teaching mathematics are more likely to exhibit levels of interest, enjoyment, confidence, and commitment that align with those of in-field teachers.

Out-of-field teaching, where educators are assigned subjects beyond their qualifications or training, is a phenomenon observed in many countries around the world. In Australia, data from 2013 indicates 17% of mathematics classes in Years 7–10 were taught by out-of-field teachers, intensifying in remote areas where the percentage rises to 26% (Weldon, 2016). Figures like these are often reported in the media to imply that out-of-field teaching is responsible for poor outcomes in standardised assessment such as NAPLAN, PISA, and TIMSS. However, definitions of out-of-field teaching that focus on the criteria used to qualify and/or register teachers (often describing them at the start of their careers) fail to recognise that teachers develop and grow throughout their career as they gain confidence, experience, and perhaps further qualifications. For example, a teacher might not have been initially trained in mathematics but may have subsequently engaged in self-study and attended professional development, becoming comfortable and competent in the subject. This evolution challenges the appropriateness of the 'out-of-field' label and its implications for teaching efficacy.

Hobbs et al. (2022) designed a multi-faceted definition of out-of-field teaching to help better understand and manage the out-of-field phenomenon, informed by the existing literature on in-field and out-of-field teaching. The definition has four dimensions: out-of-field by qualification (a mismatch between current teaching and discipline qualification, school level qualification, or both), out-of-field by specialisation (a misalignment at the sub-discipline level), out-of-field by workload (proportion, stability, and type of load), and out-of-field by capability (recognising that teachers may feel out-of-field depending on factors including experience and identity).

The approach by Hobbs et al. (2022) shifts the focus of 'out-of-fieldness' from the teacher to the context of teaching. For example, being out-of-field by workload recognises that it is the assignment of work to the teacher, not the teacher themselves, that is mismatched. Out-of-field by capability refers to a teacher's growing identity within a new context. The authors define a highly capable out-of-field teacher as one who is capable in the out-of-field subject, has a high degree of confidence, has personal interest in the subject, is professionally committed to developing and reflecting on their practice, self-identifies as proximal to the subject, and has accepted the role long-term, expanding their professional identity to include the role. In earlier work, Hobbs (2013) refers to this as 'boundary crossing', suggesting that teachers who are technically out-of-field can identify as in-field if they have sufficient support to enable them to feel confident and competent in their teaching. Conversely, teachers who are in-field by qualification can feel out-of-field by capability when placed in a new context. Ingersoll (2019) highlights how experienced and qualified teachers may become "highly unqualified if they are assigned to teach subjects for which they have little training or education" (p. 22).

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 87–94). Gold Coast: MERGA.

The identity of teachers is significant and has been shown to influence their practice. For instance, the literature review by Heyd-Metzuyanim et al. (2016) revealed that most of the studies reviewed demonstrated an interaction between teachers' identities and their practices, indicating that changes in one can affect the other. Similarly, Willis et al. (2023) found that identity as a teacher of mathematics and participation in professional learning were correlated.

A number of studies have demonstrated the significance of teacher identity for out-of-field teachers of mathematics (e.g., Hobbs, 2013; Ní Ríordáin et al., 2022). Goos et al. (2019) emphasised the importance of further research into understanding the development of teacher knowledge and its influence on the identities of out-of-field teachers of mathematics. Such research will create a better understanding of teacher learning, which in turn can better inform approaches to professional learning. For instance, while studies such as Ní Ríordáin et al. (2019) highlight the importance of professional learning aimed at out-of-field teaching to enhance teachers' content knowledge alongside their pedagogical content knowledge, other research, such as that by Bosse and Törner (2015) highlight the importance of attending to teachers' "subject-related identity" (p. 8).

Research Background

This section summarises previously reported research to contextualise the research questions asked in this paper; see O'Keeffe & Albrecht (2023), Albrecht et al. (2023), and Albrecht and O'Keeffe (2024) for more details.

An anonymous online survey was distributed in late 2022 by the SA Department for Education for one month, with all Years 7–10 teachers invited to participate. The survey design was informed by Hobbs et al.'s (2022) classifications of 'out-of-field', AITSL's (2021) report on the SA teacher workforce, and Weldon's (2016) study of out-of-field teaching in Australian secondary schools. Of the 232 survey participants, 196 had taught Years 7–10 mathematics at some stage. The survey collected information on teacher demographics, teaching qualifications, professional learning, teaching context and experience, employment status, and workload. It also probed teacher identity in multiple ways, including two questions: (1) Do you consider yourself a mathematics teacher? (In other words, are you comfortable calling yourself a mathematics teacher?); and (2) Do you consider yourself an out-of-field mathematics teacher?

Of the 165 teachers who responded to both questions, 60 were classified as in-field by qualification (QIN) while the remaining 105 were deemed out-of-field by qualification (QOOF). As might be expected, nearly all QIN teachers identified as mathematics teachers (98%, $n = 59$), whereas only 68% ($n = 71$) of QOOF teachers did. Similarly, almost all QIN teachers identified as 'not out-of-field' (which we refer to as identifying as in-field, for convenience). Surprisingly, about 50% of QOOF teachers ($n = 52$) also saw themselves as in-field, despite being out-of-field by qualification.

The survey also collected data on affective dimensions such as teachers' personal interest in mathematics, enjoyment teaching mathematics, confidence in their mathematical content knowledge (CK), confidence in their pedagogical approaches for teaching mathematics (PCK), and personal commitment to developing their CK and PCK. Teachers gave responses on a scale from 0 (low) to 5 (high). Teacher identity was shown to have an impact on interest, enjoyment, confidence, and commitment to teaching mathematics with those not identifying as mathematics teachers reporting the lowest means across all categories (Albrecht et al., 2023).

Finally, the survey explored confidence in teaching various aspects of the Australian Curriculum: Mathematics (AC:M) with respondents rating their confidence as low, medium, or high for: teaching each year level (Years 7–10), teaching each strand in each year level, integrating each of the four mathematical proficiencies, and integrating each of the four mathematical processes. Again, those not identifying as mathematics teachers reported the lowest means across all categories (O'Keeffe & Albrecht, 2023).

A third aspect of teacher identity, preference for teaching mathematics, was examined by Albrecht and O'Keeffe (2024) and gathered from the question 'What is your preferred learning area(s) to teach?' by including those that mentioned mathematics. Overlaying these indicators onto workload indicated that QOOF teachers with more than 50% of their teaching in mathematics predominantly saw themselves as mathematics teachers and with a preference for teaching mathematics. This led us to speculate that these 'out-of-field' teachers may have acquired the personas of in-field teachers of mathematics.

Research Question

The aim of the research reported in this paper is to further explore aspects of teacher identity, in particular: self-identifies as in-field, self-identifies as a mathematics teacher, and preference for teaching mathematics. The research question guiding the aspects reported in this paper is:

- Are various cohorts of QOOF teachers statistically indistinguishable from QIN teachers when considering factors such as interest, enjoyment, confidence, and commitment to teaching mathematics among Years 7–10 mathematics teachers in South Australia?

Findings

Comparing QIN and QOOF Teachers

We used two-tailed Mann-Whitney U tests (equal variance not assumed) to determine if the mean differences between QIN and QOOF teachers were statistically significant. Table 1 presents mean responses and p-values across eighteen dimensions, with $p \leq 0.05$ shaded grey to indicate statistical significance. The data indicates statistically significant differences between the means of QIN and QOOF teachers who self-identify as in-field for all dimensions except for enjoyment in teaching mathematics and confidence teaching Year 7 mathematics. QIN teachers demonstrated lower confidence in Year 7 mathematics, which is mostly likely attributed to the transition of Year 7 to secondary schools in South Australia occurring just a year prior to the survey being distributed.

Table 1 reveals an interesting pattern in the number of statistically significant differences between the means of various teacher cohorts. When comparing QIN teachers to QOOF teachers who self-identify as in-field, 16 out of the 18 dimensions show significant differences. However, this number decreases when comparing QIN teachers to QOOF teachers who self-identify as mathematics teachers (14 out of 18) and further reduces when comparing QIN teachers to QOOF teachers with a preference for teaching mathematics (10 out of 18). For those who self-identify as mathematics teachers, no statistically significant differences in means were observed for enjoyment teaching mathematics and commitment to developing CK and PCK. For those preferring to teach mathematics, no statistically significant differences in means were observed regarding their personal interest in mathematics, enjoyment teaching mathematics, commitment to developing CK and PCK, confidence teaching Years 7 and 8, or the problem solving and reasoning proficiencies.

Within the QOOF teachers, comparing those who self-identify as mathematics teachers to those who self-identify as in-field reveals that the former group has higher means for 16 of the 18 dimensions, with the only exceptions being confidence in CK (both groups have means of 4.10) and confidence in teaching Year 10 (the in-field group has a marginally higher mean of 3.10 compared to 3.08). A similar pattern emerges when comparing QOOF teachers with a preference for teaching mathematics to those who self-identify as in-field, with the group preferring to teach mathematics having higher means for 15 of the 18 dimensions. The exceptions in this case are confidence in teaching Year 7 (both groups have means of 4.21), confidence in teaching Year 9 (the in-field group has a slightly higher mean of 3.74 compared to 3.71), and confidence in teaching Year 10 (3.10 compared to 2.93).

In summary, the analysis reveals that QOOF teachers who self-identify as mathematics teachers or prefer teaching mathematics have more similarities to QIN teachers across various dimensions compared to those who self-identify as in-field. Within the QOOF cohort, those who self-identify as mathematics teachers or prefer teaching mathematics generally exhibit higher means across most dimensions compared to those who self-identify as in-field.

Table 1

QIN and QOOF Teachers' Mean Interest, Enjoyment, Confidence, and Commitment by Identity Grouping

Dimension	**Identifies as in-field**			**Identifies as a maths teacher**			**Prefers to teach mathematics**		
	QIN (59)	**QOOF (52)**	***p*-value**	**QIN (59)**	**QOOF (71)**	***p*-value**	**QIN (51)**	**QOOF (57)**	***p*-value**
Personal interest math	4.59	4.00	0.015	4.61	4.29	0.045	4.69	4.55	0.209
Enjoy teaching math	4.37	3.96	0.213	4.41	4.33	0.592	4.47	4.69	0.415
Confidence in CK	4.73	4.10	0.000	4.71	4.10	0.000	4.78	4.22	0.000
Confidence in PCK	4.27	3.69	0.019	4.25	3.94	0.036	4.33	4.09	0.073
Commit to develop CK	4.37	3.86	0.027	4.41	4.29	0.244	4.51	4.51	0.548
Commit to develop PCK	4.54	3.82	0.012	4.59	4.35	0.227	4.67	4.60	0.740
Year 7 mathematics	4.16	4.21	0.736	4.16	4.37	0.318	4.23	4.21	0.923
Year 8 mathematics	4.68	4.16	0.009	4.68	4.41	0.048	4.63	4.24	0.054
Year 9 mathematics	4.57	3.74	0.001	4.53	3.85	0.001	4.50	3.71	0.006
Year 10 mathematics	4.59	3.10	0.000	4.55	3.08	0.000	4.62	2.93	0.000
Problem solving	4.39	4.00	0.031	4.37	4.01	0.012	4.37	4.11	0.059
Understanding	4.73	4.06	0.000	4.71	4.19	0.000	4.73	4.25	0.000
Reasoning	4.37	3.86	0.009	4.37	3.97	0.007	4.37	4.07	0.069
Fluency	4.71	4.24	0.002	4.69	4.36	0.003	4.69	4.36	0.020
Math. modelling	4.20	3.63	0.009	4.17	3.64	0.004	4.22	3.71	0.011
Comp. thinking	4.08	3.48	0.013	4.05	3.59	0.010	4.10	3.71	0.035
Stat. investigations	4.36	3.57	0.001	4.34	3.71	0.000	4.35	3.69	0.001
Prob. experiments	4.25	3.55	0.002	4.24	3.72	0.004	4.22	3.76	0.016

Cohorts Within QOOF Teachers

To better understand the importance of each identity factor, we looked closer at the data for QOOF teachers only. Two-tailed Mann-Whitney U tests (equal variance not assumed) were used to determine statistically significant differences between the means. Table 2 presents mean responses and *p*-values across the eighteen dimensions, with $p \leq 0.05$ shaded grey to indicate statistical significance. The analysis in Table 2 confirms our hypothesis that self-identification as a mathematics teacher amongst QOOF teachers is significant, with statistically significant differences in means across all dimensions. A similar impact is observed for those with a preference for teaching mathematics. However, self-identity as in-field seems less influential, showing no significant differences in commitment to developing CK and PCK, confidence in teaching Years 7 and 8, or integrating the reasoning proficiency into their teaching.

Table 2

QOOF Teachers' Mean Interest, Enjoyment, Confidence, and Commitment by Identity Grouping

Dimension	**QOOF identifies as in-field**			**QOOF identifies as a maths teacher**			**QOOF prefers to teach mathematics**		
	Yes (52)	**No (53)**	***p*-value**	**Yes (71)**	**No (34)**	***p*-value**	**Yes (57)**	**No (48)**	***p*-value**
Personal interest in math.	4.00	3.19	0.003	4.29	2.18	0.000	4.55	2.50	0.000
Enjoyment teaching math.	3.96	3.35	0.014	4.33	2.26	0.000	4.69	2.46	0.000
Confidence in CK	4.10	2.98	0.000	4.10	2.38	0.000	4.22	2.75	0.000
Confidence in PCK	3.69	3.00	0.006	3.94	2.12	0.000	4.09	2.48	0.000
Commit. to develop CK	3.86	3.33	0.069	4.29	2.18	0.000	4.51	2.54	0.000
Commit. to develop PCK	3.82	3.27	0.097	4.35	1.91	0.000	4.60	2.33	0.000
Year 7 mathematics	4.21	3.86	0.151	4.37	3.34	0.000	4.21	3.82	0.147
Year 8 mathematics	4.16	3.77	0.079	4.41	3.04	0.000	4.24	3.65	0.011
Year 9 mathematics	3.74	2.71	0.005	3.85	1.90	0.000	3.71	2.64	0.001
Year 10 mathematics	3.10	1.97	0.006	3.08	1.37	0.000	2.93	2.05	0.025
Problem solving	4.00	3.53	0.029	4.01	3.22	0.002	4.11	3.35	0.001
Understanding	4.06	3.45	0.001	4.19	2.81	0.000	4.25	3.15	0.000
Reasoning	3.86	3.45	0.094	3.97	2.97	0.000	4.07	3.15	0.000
Fluency	4.24	3.51	0.001	4.36	2.77	0.000	4.36	3.27	0.000
Mathematical modelling	3.63	2.92	0.008	3.64	2.41	0.000	3.71	2.72	0.001
Computational thinking	3.48	2.88	0.019	3.59	2.23	0.000	3.71	2.52	0.000
Statistical investigations	3.57	3.10	0.047	3.71	2.45	0.000	3.69	2.88	0.012
Probability experiments	3.55	3.12	0.089	3.72	2.41	0.000	3.76	2.79	0.002

We then grouped respondents into one of eight 'identity' cohorts (summarised in Table 3) and conducted chi-square tests of independence to assess the strength of association between the three variables. This analysis revealed statistically significant associations as follows:

- Identifying as a mathematics teacher and a preference for teaching mathematics: $\chi^2(1, n = 105) = 39.211$, $p < 0.001$. The effect size, measured using Cramér's V, was found to be V = 0.611, indicating a strong association;
- Identifying as in-field and identifying as a mathematics teacher: $\chi^2(1, n = 105) = 6.990$, $p = .008$ with effect size of V = 0.258 indicating a weak to moderate association;
- Identifying as in-field and a preference for teaching mathematics: $\chi^2(1, n = 105) = 6.038$, $p = .014$ with effect size of V = 0.240 indicating a weak to moderate association.

Based on the analysis thus far, we exclude 'self-identifies as in-field' from the remainder of this paper for three reasons: (1) its less significant impact, as observed in Table 2, (2) the wide variety of QOOF teachers' definitions of out-of-field (see Albrecht and O'Keeffe, 2024), and (3) chi-square tests of independence revealing only weak to moderate associations with the other two variables. In contrast, self-identification as a mathematics teacher and a preference for teaching mathematics are strongly associated and significantly influence interest, enjoyment, confidence, and commitment.

Table 3

Eight Cohorts of QOOF Teachers by Self-Identity as In-Field, Self-Identity as a Mathematics Teacher, and Preference for Teaching Mathematics

Self-identifies as in-field or out-of-field	Self-identifies as a mathematics teacher	Prefers to teach mathematics	Total
In-field	Mathematics teacher	Yes	35
In-field	Mathematics teacher	No	7
In-field	Not mathematics teacher	Yes	0
In-field	Not mathematics teacher	No	10
Out-of-field	Mathematics teacher	Yes	19
Out-of-field	Mathematics teacher	No	10
Out-of-field	Not mathematics teacher	Yes	3
Out-of-field	Not mathematics teacher	No	21

Table 4

Four Cohorts of QOOF Teachers by Self-Identity as a Mathematics Teacher and Preference for Teaching Mathematics

	Self-identifies as a mathematics teacher	Prefers to teach mathematics	Total
Cohort 1	Mathematics teacher	Yes	54
Cohort 2	Mathematics teacher	No	17
Cohort 3	Not mathematics teacher	Yes	3
Cohort 4	Not mathematics teacher	No	31

Pairwise testing of the four cohorts in Table 4 with two-tailed Mann-Whitney U tests (equal variance not assumed) was used to check for statistically significant differences between means. Unsurprisingly, Cohort 1 exhibited the highest means across 16 of the 18 factors. The two exceptions are confidence in teaching Year 7 and Year 8. Cohort 2, which self-identifies as mathematics teachers but does not prefer teaching mathematics, exhibits the highest means for both Year 7 and Year 8. Cohort 3, consisting of only three teachers, is intriguing as they do not consider themselves mathematics teachers (and view themselves as out-of-field) and yet express a preference for teaching mathematics. With only three teachers in this group, it is difficult to draw conclusions. It is unsurprising that Cohort 4, which does not self-identify as mathematics teachers and does not prefer teaching mathematics, demonstrated markedly low means (ranging from 1.71 to 2.29) across personal interest in mathematics, enjoyment in teaching mathematics, and confidence in and commitment to developing CK and PCK.

A comparison of Cohorts 1 and 2, which self-identify as mathematics teachers but differ in their preference for teaching mathematics, showed statistically significant differences in 12 of the 18 dimensions. Notably, there were no statistically significant differences in confidence in teaching Years 7 to Year 10, nor in statistical investigations and probability experiments. A comparison of the two diametrically opposed cohorts, 1 and 4, revealed statistically significant differences across all 18 dimensions, as might be expected.

A comparison of Cohorts 2 and 4, which do not prefer teaching mathematics but differ in their self-identification as mathematics teachers, showed statistically significant differences in 14 of the 18 dimensions. There were no statistically significant differences in problem solving, reasoning, mathematical modelling or computational thinking. Cohort 3 was excluded from the pairwise comparisons due its small size of three teachers.

Comparing Cohort 1 With QIN Teachers

The pairwise comparisons suggest that Cohort 1 is distinct from the other cohorts, so we compared Cohort 1 with QIN teachers. Table 5 indicates that Cohort 1 have mean scores similar to, and sometimes higher than, QIN teachers. The only statistically significant differences relate to confidence in CK, Years 9 and 10 (arguably when content becomes more challenging), the understanding proficiency (arguably tied to content knowledge) and three of the four processes. (Note that QIN teacher confidence in the fourth process, computational thinking, is also low.) The data suggests that Cohort 1 shares identity characteristics with their in-field colleagues.

Table 5

QIN and QOOF Teachers Identifying as Mathematics Teachers and Preferring to Teach Mathematics

Dimension	**Identifies as mathematics teacher and prefers to teach mathematics**		
	QIN (51)	**QOOF (54)**	***p*-value**
Personal interest in mathematics	4.69	4.58	0.303
Enjoyment teaching mathematics	4.47	4.69	0.406
Confidence in CK	4.78	4.27	0.000
Confidence in PCK	4.33	4.15	0.135
Commitment to developing CK	4.51	4.54	0.686
Commitment to developing PCK	4.67	4.63	0.961
Year 7 mathematics	4.23	4.23	0.754
Year 8 mathematics	4.63	4.36	0.134
Year 9 mathematics	4.50	3.87	0.018
Year 10 mathematics	4.62	3.05	0.000
Problem solving	4.37	4.15	0.108
Understanding	4.73	4.33	0.001
Reasoning	4.37	4.10	0.083
Fluency	4.69	4.48	0.059
Mathematical modelling	4.22	3.79	0.024
Computational thinking	4.10	3.81	0.076
Statistical investigations	4.35	3.75	0.002
Probability experiments	4.22	3.83	0.031

Summary and Conclusion

The importance of teacher identity in influencing and shaping teacher practice is well established. The findings in this study align with Neumayer-Depiper's (2013) assertion about the importance of understanding how identity is situated and negotiated by teachers in different contexts. While the QOOF teachers in this study may not be in a position to change their initial qualifications or access additional ones, this does not preclude them from positioning themselves as competent mathematics teachers.

As discussed, QOOF teachers who self-identify as mathematics teachers and prefer teaching mathematics have a high degree of confidence across many dimensions, are interested in the subject, and are professionally committed to developing and reflecting on their practice, sharing many identity characteristics with their in-field colleagues. This suggests they should not be considered out-of-field by capability. What remains unclear at this stage of our analysis are the factors that contribute to teachers identifying in this way. Nonetheless, the analysis to date shows the significance of identity in reauthoring oneself as a teacher of mathematics.

It would be fair to say that there are still very important questions to be asked about out-of-field teachers' competence and effectiveness, but those questions should be asked about all teachers. Professional learning and support is relevant to all teachers; it might just need to look, sound, and feel different depending on the individual teacher. Understanding teacher identities can help inform approaches to professional learning for these different cohorts of teachers.

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References

AITSL. (2021). *Australian teacher workforce data: National teacher workforce characteristics report: South Australia. Melbourne*, Vic: Australian Institute for Teaching and School Leadership.

Albrecht, A. & O'Keeffe, L. (2024). An exploration of teacher identity and out-of-field teaching in years 7 to 10 mathematics in South Australia. *The 15th International Congress on Mathematical Education*, (*in press*).

Albrecht, A., O'Keeffe, L., & Morrison, A. (2023). What matters with out-of-field teaching: A preliminary analysis of middle years teachers of mathematics in South Australia. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes & A. Hughes (Eds.), *Weaving mathematics education from all perspectives. Proceedings of the annual Mathematics Education Research Group of Australasia* (pp. 81–89), Newcastle: MERGA.

Bosse, M., & Törner, G. (2015). Teacher identity as a theoretical framework for researching out-of-field teaching mathematics teachers. In C. Bernack-Schüler, R. Erens, T. Leuders, & A. Eichler (Eds.), *Views and beliefs in mathematics education* (pp. 1–13). Springer.

Goos, M., Bennison, A., Quirke, S., O'Meara, N., & Vale, C. (2019). Developing professional knowledge and identities of non-specialist teachers of mathematics. *International Handbook of Mathematics Teacher Education* (Vol. 1, pp. 211–240). Brill. https://doi.org/10.1163/9789004418875_009

Heyd-Metzuyanim, E., Lutovac, S., & Kaasila, R. (2016). Identity. In M. S. Hannula, G. A. Goldin, E. Heyd-Metzuyanim, A. Jansen, R. Kassila, S. Lutovac, P. Di Martino, F. Morselli, J. A. Middleton, M. Pantziara, & Q Zhang (Eds.), *Attitudes, beliefs, motivation, and identity in mathematics education: An overview of the field and future directions.* New York: Springer Open.

Hobbs, L. (2013). Teaching 'out-of-field' as a boundary-crossing event: Factors shaping teacher identity. *International Journal of Science and Mathematics Education*, *11*(2), 271–297.

Hobbs, L., Campbell, C., Delaney, S., Speldewinde, C., & Lai, J. (2022). Defining teaching out-of-field: An imperative for research, policy and practice. In L. Hobbs & R. Porsch (Eds.), *Out-of-field teaching across teaching disciplines and contexts* (pp. 23–48). Springer.

Ingersoll., R. (2019). Measuring out-of-field teaching. In L. Hobbs & G. Törner (Eds.), *Examining the phenomenon of 'teaching out-of-field': International perspectives on teaching as a non-specialist* (pp. 21–52). Springer.

Neumayer-Depiper, J. (2013). Teacher identity work in mathematics teacher education. For the Learning of Mathematics. *For the Learning of Mathematics*, *33*(1), 9–15.

Ní Ríordáin, M., Goos, M., Faulkner, F., Quirke, S., Lane, C., & O'Meara, N. (2022). Eliminating the fear of getting 'caught out': An examination of the development of out-of-field mathematics teachers' professional self-understanding. In L. Hobbs & R. Porsch (Eds.), *Out-of-field teaching across teaching disciplines and contexts* (pp. 241–259). Springer Nature.

Ní Ríordáin, M. N., Paolucci, C., & Lyons, T. (2019). Teacher professional competence: What can be learned about the knowledge and practices needed for teaching? In L. Hobbs & G. Törner (Eds.), *Examining the phenomenon of 'teaching out-of-field': International perspectives on teaching as a non-specialist* (pp. 129–149). Springer.

O'Keeffe, L. & Albrecht, A. (2023). *What matters with out-of-field teaching: Teachers of Years 7–10 mathematics in SA department schools*. University of South Australia. https://doi.org/10.25954/efcp-1y11

Weldon, P. R. (2016). *Out-of-field teaching in Australian secondary schools (Policy Insights, Issue #6).* Camberwell, Vic: Australian Council for Educational Research.

Willis, R., Lynch, D., Peddell, L., Yeigh, T., Woolcott, G., Bui, V., Boyd, W., Ellis, D., Markopoulos, C., & James, S. (2023). Development of a teacher of mathematics identity (ToMI) scale. *Mathematics Education Research Journal*, *35*(S1), 107–132. https://doi.org/10.1007/s13394-021-00391-w

An Analysis of Multiplicative Thinking Development in Years 3 to 6

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Research has shown that many primary students experience transition barriers between additive and multiplicative thinking. This paper analysed responses from 253 Years 3 to 6 students to a diagnostic assessment which consists of whole number multiplication and division problems involving equal groups, arrays, multiplicative comparison and Cartesian product situations. Based on the Rasch analysis, item responses were differentiated into five developmental Stages indicating a wide range of understanding and pointing to different transition barriers that students experience. The reasons for these are discussed in the paper and some advice is presented for teachers.

This paper intends to share some of the findings from a larger study aimed at identifying Years 3 to 6 students' transition barriers between additive and multiplicative thinking by analysing students' written responses from the *Multiplicative Thinking Diagnostic Assessment* through Rasch analysis. The study was stimulated by previous research findings indicating that students in the middle years of schooling experience transition barriers while moving from additive thinking to multiplicative thinking (e.g., Bao, 2023; Hurst & Hurrell, 2016).

Introduction

The transition from additive to multiplicative thinking is a slow climb, involving a conceptual leap which constitutes an obstacle for many students (Siemon et al., 2006). There is a considerable body of research pointing to the transition barriers students experience during their development. Early studies (e.g., Jacob & Willis, 2003; Mulligan & Mitchelmore, 1997) claimed that recognising the equal grouping structure (groups of equal size and the number of groups) is a barrier for many students. Others such as Sophian and Madrid (2003) and Downton et al. (2022) pointed out that students have difficulty in understanding the abstract concept of *many-to-one correspondence*. For example, to understand 5×3, students need to understand five units of one is distributed over the elements of one unit of three (Steffe, 1994). Studies such as Larsson (2016) suggested that repeated addition as a procedure to solve whole number multiplication problems can potentially hinder students' development. Larsson (2016) also reported another barrier for making inappropriate generalisations while solving two-digit by two-digit multiplication where students solve 19×26 as 10×20+9×6. Other studies by, for example, Hurst and Hurell (2016), added that procedural based learning could limit students' ability to recognise multiplicative relationships and apply their properties to solve problems. Research by Downton and Sullivan (2017) and Bao (2023) also noted that students have difficulties in solving multiplicative problems involving Cartesian product situation.

Transition from early notion to sophisticated thinking about a targeted concept is a process of cognitive change along a developmental stage where cognitive difficulties are encountered (Tzur, 2019). Cognitive difficulties are critical transition points throughout the stages of concept development (Griffin, 2020). Many researchers (e.g., Tzur, 2019; Griffin, 2020) have argued that these transition points appear to be boundaries between two stages of a developmental progression. Therefore, the construction and testing of a developmental progression for multiplicative thinking is a vital step to reveal students' transition barriers.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 95–102). Gold Coast: MERGA.

The Research Project

The research reported here aimed at investigating Years 3 to 6 students' understanding of multiplicative thinking to reveal their transition barriers. This study is framed in terms of the need to identify students' transition barriers and build on what is known. It used a design-based research approach involving an iterative process of construction, evaluation, and refining of assessment (e.g., Plomp & Nieveen, 2013) and the use of Rasch modelling (Masters, 1982) for the pilot study in phase 1 and implementing the assessment in the current phase. This paper reports some key findings indicating what transition barriers students across Years 3 to 6 experience during their development of multiplicative thinking.

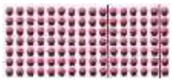
Method

The *Multiplicative Thinking Diagnostic Assessment* (Table 1) was tried out on a purposeful sample of 253 Years 3 to 6 students from six government schools in the Geelong region, Victoria, Australia. There were 42 Year 3 students, 71 Year 4 students, 68 Year 5 students and 72 Year 6 students. A sample of this size provided a substantive amount of data on students' responses to the test and allowed important generalisations to be made about students' performance (Rogers, 2014). Students were from a mixed of high, medium, and low social-demographic backgrounds which are a representation of the key characteristics of the population. This adds weight to the generalisability of the results (Rogers, 2014).

Between November 2022 and March 2023, teachers from research schools administered the assessment based on instructions provided by the researcher. Students completed the assessment within 45 minutes during class time. Their responses were collected, coded, and scored by the researcher to ensure the process was as systematic and objective as possible.

Table 1

Multiplication and Division Word Problems

Multiplicative thinking diagnostic assessment items
1a. The value of five 20c coins is same as one $1 coin. How many 20c coins are the same as $4? 
1b. If you have 45 20c coins, how many $1 coins can you make?
2a. Michelle bakes 40 biscuits. She puts them in rows of 8 biscuits on a baking tray. How many rows of biscuits does Michelle bake? 
2b. Michelle puts party pies on a baking tray like in the picture and fills the tray. How many party pies does Michelle bake? 
2c. Michelle bakes 18 pies. She also bakes 4 times as many sausage rolls as pies. How many sausage rolls does Michelle bake?
2d. Michelle sold 15 pies on Friday and 60 pies on Saturday. How many times as many pies were sold on Saturday?
2e. Michelle needs 25 boxes to pack 200 pies. How many pies are in each box?
3a To work out the total number of cupcakes, Beth divided the cupcakes into 3 sections like in the picture. How did Beth work out the total number of cupcakes? 
3b. Beth baked 12 rows of cookies with 15 cookies in each row. To work out the total number of cookies, Sam did 12 × 15 = 10 × 17 = 170. Tom did 12 × 15 = 10 × 10 +2 × 5 = 110. Emily did 12 × 15 = 6 × 30 = 180. Who do you think is correct? Why?
4a. Sam has 4 jumpers and 3 shorts. If Sam chose a blue jumper, what might be Sam's choice of outfits? 
4b. Sam has 4 jumpers and 3 shorts. How many different outfits are there in total?
4c. Sam has 5 jumpers and 30 outfits. How many shorts does Sam have?

Multiplicative thinking diagnostic assessment items
4d. Sam's Dad has 18 jumpers and 13 shorts so he has 234 different outfits. Sam's younger brother has 13 jumpers and 18 shorts. How many different outfits does Sam's younger brother have?
4e. Sam's older brother has 13 jumpers and 19 shorts. How many different outfits does Sam's older brother have?

An earlier study Bao (2023) provided a detailed rationale for each of the above items showing how each is supported by relevant research literature. Following established procedures for using Rasch analysis (Bhatti, et al., 2023; Callingham & Watson, 2005; Siemon et al., 2021), each item has two scoring rubrics: one was designed to score dichotomous data where the item was scored 0 or 1(correct or incorrect) and another was to score polytomous data where the same item was scored 0, 1, 2 etc. for different levels and types of thinking. The numerical scoring code assigned to each response ranged from 0–1 to 0–4. The number of codes for each item depended on the complexity of responses to particular items which was determined by qualitative analysis of students' responses taken from previous studies, as well as Rasch analysis in the pilot study in Phase 1. Scoring rubrics designed to fit Rasch Partial Credit Model did not need to have the same number of categories (Callingham & Watson, 2005). Scoring rubrics aim not only to differentiate cognitive skills between answering the item correctly with a correct reasoning and giving a correct answer without explanation but are also able to acknowledge students' partially correct thinking (Arieli-Attali & Liu, 2016). Scoring rubrics for Task 2d are shown in Figure 1 below.

Figure 1

Task 2d and Scoring Rubrics in Michelle's Bakery

d. Michelle sold 15 party pies on Friday and 60 party pies on Saturday.

How many times as many party pies were sold on Saturday? [2d1]

Scoring rubrics for [2d1]

Scores	Description
0	No response or irrelevant response or incorrect answers
1	Correct answer (4)

You can write or draw to show how you worked out your answer.

Scoring rubrics for [2d2]

Scores	Description
0	No response or irrelevant response or no indication of strategies, e.g. an incorrect response resulting from adding or subtracting two given numbers.
1	A response reflects an error of overreliance of additive thinking, e.g., 4 -1 = 3
2	A response based on using additive strategies such as skip counting or repeated addition/subtraction.
3	A response based on a multiplicative approach.

Analysis

The data was calibrated using the Winsteps 5.3.3.1 (Linacre, 2022) to fit the Rasch Partial Credit Model (Masters, 1982) based on the scoring rubrics for each item. This allowed both students' performance and item responses' difficulties to be measured and placed on the same scale. The Winsteps program evaluates the fit of the data to the Rasch model, showing the mean Infit MNSQ for each item value is between 0.73 and 1.26 which are within the default acceptable value (between 0.7 and 1.3) and indicates that the data fit to the model. The values of the Separation Reliability for both items and persons were high, indicating consistent behaviours of both items and persons.

Figure 2

The Variable Map Based on Rasch-Thurstonian Thresholds

	#	|									
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4		+				3b2.4					
	.###	|									
		|									
	.##	T|									
3	.	+									
	#	|			4e2.3					
	.#	|T									Stage 5
	.##	|			4d2.3						
2	###	+									
	.##	S|	4e1.1	4a2.2	2d2.3						
	.#######	|			4b2.3		4c2.3				Stage 4b
	######	|S	4c1.1	4c2.2	3b2.3	1b2.4	4e2.2	4b2.2	4d2.2	Stage 4a
1	#########	+	3b1.1	2a2.3	2e2.2	4c2.1	4d2.1			
	.######	|	4b1.1	4e2.1	1b2.3	3a2.4	2c2.4	4d1.1	4b2.1	Stage 3b
	####	|		3b2.2	1a2.3					
	.######	|	4a2.1		2b2.3					Stage 3a
0	###########	M+M	2e1.1	3a2.3	4a1.1	2c2.3				
	#######	|	2d1.1	2a2.2		3a1.1	2d2.2			
	######	|	2e2.1	1b2.2	2d2.1					Stage 2
	#####	|	3b2.1	3a2.2	2c1.1	2b2.2				
-1	############	+		1a2.2	2c2.2					
	.#####	|S	3a2.1							
	##	S|	1b1.1							Stage 1b
	#########	|	1b2.1	2c2.1						
-2	###	+								
	###	|	1a1.1							Stage 1a
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-3	#	+	2a2.1							
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The variable map based on Rasch-Thurstonian thresholds is shown in Figure 2. It produced an overall list of item thresholds which differentiated items based on students' responses. In Rasch analysis, students who appear on the scale at about the same logit value as an item response have a 50% chance of exhibiting that level of thinking. For an item response above the student's position on the scale there is less than 50% chance of demonstrating that level of thinking for that item and for item response below the student's position the chance is greater than 50%. The '#s' represent the distribution of the students according to their ability estimate. Each person is shown by '#' and the position on the scale indicates the ability measure. The right-hand side of the figure shows the distribution of the test items with responses on the same scale in reference to their difficulty estimates. Easier and more accessible items had relatively low item thresholds. For example, the item threshold (1a2.1) associated with a score 1 for part 2 of the item 1a of the *Australian Coin* Task (possible scores 0, 1, 2 or 3) was -3.16 where students used drawing and *one-to-one correspondence*. The item threshold (3b2.4) associated with a score of 4 on part 2 of the item 3b of the *Beth's Cupcakes* task (possible scores 0, 1, 2, 3 or 4) was 4.4 where students needed to demonstrate understanding of associative property.

The Rasch model used has the capacity to order assessment items from the least difficult to the most difficult with logit scores along the scale. According to Griffin (2020), locating substantive change in item difficulty based on logit scores can determine the transition points or "natural breaks" where a change in cognitive skills occurs. Griffin (2020) described this method as criterion-referenced interpretation. This is an empirically grounded method in many different mathematical research topics (e.g., Griffin, 2020; Rogers, 2014; Siemon et al., 2021). Using this method, the test data were used to derive the developmental progression for multiplicative thinking. Since Rasch analysis connects differences in the logit scores of related items to increased cognitive demands imposed on students (Griffin, 2020), transition barriers can be identified where Rasch analysis shows a substantive difference in the logit scores of related items between stages along a developmental progression.

Firstly, each item response was ordered from the least to the most difficult according to the Rasch data. It was classified according to the underpinning cognitive skills that the item exhibits (Griffin, 2020). Next, the difference between the difficulties of adjacent item response was calculated to determine the location of substantive changes in item difficulty as "natural break" where a change in the cognitive skills appears to be required to correctly answer items (Griffin, 2020). Based on logit difference, substantive changes between adjacent item response were identified: 4d2.3 and 4a2.2 (0.45); 4c2.3 and 1b2.4 (0.15); 4d2.2 and 4c2.1 (0.07); 3b2.2 and 4a2.1 (0.2); 2c2.2 and 3a2.1 (0.28); 2c2.1 and 1a1.1 (0.57). Locating substantive changes allows item responses to be clustered together in Stages (Griffin, 2020) which is shown in Figure 2. The horizontal double lines on the map indicate the points where there appeared to be some significant change in the cognitive demands of the item responses and the single horizontal lines indicate a less pronounced change (in the judgement of the researcher). These points were confirmed after considering the content of the items, the skills required for correctly answering the items, and apparent discontinuities in the difficulty levels. Then, item responses within each stage need to be analysed qualitatively to determine if they exhibit a common substantive interpretation (Griffin, 2020). According to Rogers (2014) the clustered item responses at each stage should differ distinctly from those at other stages so that a hierarchy of students' understanding can be implied and allow exploration of transition barriers between the developmental stages of multiplicative thinking.

Results

Identifying where students lie on the scale and interpreting the nature of the item responses around the same location, common skills and similar demands from the items can be identified (Griffin, 2020; Siemon et al., 2021). A detailed content analysis of item responses led to the identification of eight substantive changes consisting of several sub-stages which sit within five relatively discrete Stages. Table 2 provides a succinct summary of item responses and their underpinning multiplicative knowledge and thinking skills required for each Stage.

In Stage 1 students generally relied on drawing and *one-to-one correspondence* to solve problems involving familiar situations such as equal groups and arrays. Stage 1 includes two sub-stages: 1a and 1b. From sub-stage 1a to 1b, there is a change in size of numbers in item response 3a2.1 and 2c2.1 (Figure 2) which involve single and two-digit factors. Students classified in Stage 1 had difficulty in recognising the equal grouping structure. In sub-stage1b, students showed overreliance of additive thinking while dealing with multiplicative comparison situation involving single-digit by two-digit multiplication where students had difficulty in understanding the notion of "times as many". According to Table 2, item responses with thresholds ranging from -1 to 0 were grouped together to form a discrete Stage 2 where students used skip counting or repeated addition to solve single-digit by two-digit multiplication problems involving equal groups, arrays and multiplicative comparison situations, but switched to *one-to-one correspondence* for division problems with single and two-digit factors involving equal group situation. In Stage 2, even though students recognised groups of equal size and the number of groups, they still failed to recognise the abstract multiplicative relationship, *many-to-one correspondence*. Students still showed overreliance of additive thinking in item 2d2 (Figure 1) for multiplicative comparison situation involving division operation. Many responses in Stage 2 showed inappropriate generalisation of additive thinking (Squire et al., 2004; Larsson, 2016). For instance, in item response 3b2.1, students wrote $12 \times 15 = 10 \times 17$ by moving 2 to 15 or $12 \times 15 = 10 \times 10 + 2 \times 5$ by splitting 12 into 10 and 2, and splitting 15 into 10 and 5. The barrier for these students is that they apply place value partitioning incorrectly for multiplication problems. The Rasch analysis confirms that inappropriately using partitioning is a serious barrier to the development of multiplicative thinking.

Table 2

The Developmental Stages for Multiplicative Thinking

Stages	Thinking skills and knowledge
1a	Using drawing and one-to-one correspondence to solve single-digit by single-digit multiplication problems involving arrays and equal groups situations
1b	Using drawing and one-to-one correspondence to solve multiplication and division problems with single and two-digit factors involving arrays and equal groups but showing an error of overreliance on additive thinking while attempting multiplicative comparison situations problem involving multiplication operation
2	Using skip counting or repeated addition to solve single-digit by two-digit multiplication problems involving equal groups, arrays and multiplicative comparison situations, switching to one-to-one correspondence for division problem with single and two-digit factors involving equal groups situation; showing an error of overreliance of additive thinking for division problem with single and two-digit factors involving multiplicative comparison situation; showing an error of inappropriate generalisation of additive thinking for two-digit by two-digit multiplication problems involving arrays situation
3a	Using multiplicative strategies such as spitting or doubling, multiplication facts to solve single-digit by two-digit multiplication problems involving equal groups, arrays and multiplicative comparison situations; switching to one-to-one correspondence for single-digit by single-digit multiplication problem involving Cartesian product situation with a visual diagram
3b	Using multiplicative strategies to solve multiplication and division problems with single and two-digit factors involving equal groups, arrays and multiplicative comparison situations; switching to skip counting or repeated addition for two-digit by two-digit multiplication involving equal groups situation and one-to-one correspondence for multiplication and division problems with 2 single-digit factors involving Cartesian product situation; showing an error of inappropriate generalisation of additive thinking for two-digit by two-digit multiplication problems involving Cartesian product situation
4a	Using procedural based multiplicative approaches to solve multiplication and division problems with 2 two-digit factors involving equal groups and arrays situations; switching to additive strategies for Cartesian production situation. Showing a procedural understanding of commutativity
4b	Using procedural based multiplicative approaches to solve multiplication and division problems involving various multiplicative situations
5	Applying properties of multiplication such as commutativity, distributivity and associativity to solve multiplicative problems involving various multiplicative situations

According to Table 2, students experienced more difficulty in understanding multiplicative relationships involving division operations than multiplication operations; and identifying the equal grouping structure in division items appears to be a barrier for students. Hurst and Linsell (2020) also draw attention the fact that many students in the middle primary years fail to notice the inverse relationship between multiplication and division. In Stage 3 students also experienced more difficulty in understanding multiplicative comparison situation than equal groups and arrays as students had to switch multiplicative strategies to repeated addition or skip counting. The analysis shows that Cartesian product situations present a clear barrier for many students who could not access simple Cartesian product situation items until Stage 3a with item responses 4a1.1 and 4a2.1 where they relied on drawing and *one-to-one correspondence* strategy. Another key transition barrier between Stage 3 and 4 relates to students' ability to deal with two-digit by two-digit multiplication problems. This is evident in item response 4e2.1 when students wrote $13 \times 19 = 234 + 1 = 235$ because in item 4d $13 \times 18 = 234$ and

19 is 1 more than 18. In Stage 4, even though students were fluent with procedural methods to solve multiplication and division problems, many failed to understand the relationship between 13 × 18 and 13 × 19 which requires adding another group of 13. Instead, these students used vertical multiplication to solve 13 × 19 as a new problem. Similarly in item 3b2, further barrier was evident when students did not see the relationship between12 × 15 and 6 × 30, instead relying procedural methods to solve the problem. The Rasch analysis shows that Stage 5 requires an understanding of properties of multiplication such as distributivity and associativity.

Conclusion

Understanding areas where students encounter difficulties should inform ways and means to overcome these barriers (Hurst & Hurell, 2016). Siemon et al. (2021) argue that identifying students' transition barriers and building on their prior knowledge is the key to improve learning outcomes. Based on the Rasch analysis, the detailed item response analysis reported in this paper identified eight substantive changes, comprising five developmental Stages of multiplicative thinking which provides a more detailed and richer description of transition points between Stages along the developmental progression of multiplicative thinking.

The Rasch analysis also shows the importance of recognising the equal grouping structure and understanding of the concept of *many-to-one correspondence* at the early Stages of the development. Also important are size of numbers and multiplicative contexts such as two-digit by two-digit multiplication problems and Cartesian product situation during students' development for multiplicative thinking. It is possible that some students had limited encounters with Cartesian product situations which involve repeated equal sets, unlike equal groups and arrays. This may create a barrier for these students. The Rasch analysis clearly confirms students' lack of knowledge in the properties of multiplication. Understanding of the distributive and associative properties of multiplication clearly remains a barrier for middle and upper primary students. This study shows that reliance on procedural based methods may provide correct answers but limit students' ability to see underlying multiplicative relationships, including properties of multiplication between the pairs of quantities involved in the operation. It is important to assist students to see the structural properties of multiplication and to be able to identify and explain correct and incorrect solutions to multiplicative situations. Equally important is that for students being able to use the known results to arrive at a new result without having to do a full calculation, which remains a challenge for these students. The *Multiplicative Thinking Diagnostic Assessment* test was designed to explore these barriers and transition points. The Rasch analysis has confirmed the presence of these barriers and draws attention of teachers and educators to key transition points in the development of multiplicative thinking.

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References

Arieli-Attali, M., & Liu, Y. (2016). Beyond correctness: development and validation of concept-based categorical scoring rubrics for diagnostic purposes. *Educational Psychology*, *36*(6), 1083–1101.

Bao, L. (2023). Identifying and assessing students' transition barriers between additive and multiplicative thinking. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 91–98). Newcastle: MERGA.

Bhatti, H. A., Mehta, S., McNeil, R., Yao, S. Y., & Wilson, M. (2023). A scientific approach to assessment: Rasch measurement and the four building blocks. *Advances in applications of Rasch measurement in science education* (pp. 157–187). Springer, Cham.

Callingham, R., & Watson, J. M. (2005). Measuring statistical literacy. *Journal of Applied Measurement*, *6*(1), 19–47.

Downton, A., & Sullivan, P. (2017). Posing complex problems requiring multiplicative thinking prompts students to use sophisticated strategies and build mathematical connections. *Educational Studies in Mathematics*, *95*, 303–328.

Downton, A., Russo, J., & Hopkins, S. (2022). Students' understanding of the associative property and its applications: Noticing, doubling and halving, and place value. *Mathematics Education Research Journal, 34*(2), 437–456.

Griffin, P. (2020). Competence Assessment. In P. Griffin & K. Woods (Eds.), *Understanding Students with Additional Needs as Learners* (pp. 9–24). Springer, Cham.

Hurst, C., & Hurrell, D. (2016). Investigating children's multiplicative thinking: Implications for teaching. *European Journal of STEM Education*, *56*, 1–11.

Hurst, C., & Linsell, C. (2020). Manipulatives and multiplicative thinking. *European Journal of STEM Education*, *5*(1), 04. https://doi.org/10.20897/ejsteme/5808

Jacob, L., & Willis, S. (2003). The development of multiplicative thinking in young children. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity. Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia* (pp. 460–467). Geelong, Vic: MERGA.

Larsson, K. (2016). Students' understandings of multiplication [Unpublished doctoral dissertation]. Stockholm: Stockholm University.

Linacre, J. M. (2022). R Statistics: Survey and review of packages for the estimation of Rasch models. *International Journal of Medical Education*, *13*, 171. https://dx.doi.org/10.5116/ijme.629d.d88f

Masters, G. N. (1982). A Rasch model for partial credit scoring. *Psychometrika*, *47*(2), 149–174. https://doi.org/10.1007/BF02296272

Mulligan, J. T., & Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28*(3), 309–330.

Plomp, T., & Nieveen, N. (Eds). (2013). *Educational design research*. Enschede: Netherlands Institute for Curriculum Development (SLO).

Rogers, A. (2014). *Investigating whole number place value in Years 3–6: Creating an evidence-based developmental progression* [Unpublished doctoral dissertation]. Melbourne: RMIT University.

Siemon, D., Izard, J., Breed, M., & Virgona, J. (2006). The derivation of a learning assessment framework for multiplicative thinking. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Mathematics in the centre. Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 113–120). PME.

Siemon, D., Callingham, R., & Day, L. (2021). The development and validation of two new assessment options for multiplicative thinking. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & L. F. Wong (Eds.), *Excellence in mathematics education: Foundations and pathways. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 369–378). Singapore: MERGA.

Sophian, C., & Madrid, S. (2003). Young children's reasoning about many-to-one correspondences. *Child Development, 74*(5), 1418–1432. https://www.jstor.org/stable/3696186

Squire, S., Davies, C., & Bryant, P. (2004). Does the cue help? Children's understanding of multiplicative concepts in different problem contexts. *British Journal of Educational Psychology*, *74*, 515–532.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (1 ed., pp. 3–39). Albany, NY: State University of New York Press.

Tzur, R. (2019). Hypothetical learning trajectory (HLT): A lens on conceptual transition between mathematical "markers". In D. Siemon, T. Barkatsas, & R. Seah (Eds.), *Researching and using progressions (trajectories) in mathematics education* (pp. 56–74). Brill. https://doi.org/10.1163/9789004396449_003

Beliefs About the Active, Bodily Experience Mathematics Learning Activities: An Explorative Teacher Survey in Australia

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The exploratory study combined an online survey and interviews with Australian primary and secondary mathematics teachers to examine their views on *Active, Bodily experience Mathematics learning activities*. The research sought to identify teachers' expectations, challenges, and limitation for their implementation. Initial results indicate teachers recognise the value of such activities in deepening students' understanding and increasing engagement but encounter obstacles such as time constraints, insufficient resources, and difficulties in managing the classroom, underscoring the need for further investigation into effectively incorporating these activities into teaching practices.

The educational field broadly recognises the importance of integrating physical movement and bodily experience into mathematics education, drawing on various research backgrounds (Abrahamson et al., 2020). Over the years, both theoretical and experimental studies coming from general pedagogical and mathematics education traditions, the philosophy of mathematics, neurosciences, and cognitive psychology research fields have consistently shown the benefits of actively engaging students in experiential activities and the significant impact of movement and sensory experiences on learning math. Theories of embodied, enactive, embedded, and extended cognition (Lakoff & Núñez, 2000; Varela et al., 1991) have inspired diverse research directions in mathematics education, and multiple research groups have designed learning activities, materials, and tools to enhance the teaching and learning strategies in this direction (Palatnik et al., 2023). Although there is not an explicit reference to the implementation of activities designed from an *enactive-embodied* perspective in the Australian curriculum (ACARA, 2022), there are some aspects that can be linked to them. Indeed, within the curriculum, a specific focus is on the dynamic use of digital tools for virtual manipulation, and some general references to "experience with mathematical concepts using multisensory methods to stimulate thinking skills" and "access to familiar objects to represent and solve mathematical problems", in order to meet the needs of diverse learners. Looking at specific strands within the curriculum, in the strand Space it is mentioned "the ability to make pictures, diagrams, maps, projections, networks, models, and graphics that enable the manipulation and analysis of shapes and objects through actions and the senses". In the strand Probability, something is said about learning that is based on "experimentation through exploration and play-based learning in the early years". Even if some general guidelines, possibly relating to the implementation of activities *embodied designed* (Abrahamson et al., 2020), are illustrated in the curriculum, more has to be inquired on the school realm and the integration of these approaches in teaching practice (Boscolo, 2023). In line with the approaches of *Implementation Research in Mathematics Education* (IRME) (Jankvist et al., 2017), our study intends to make an initial exploratory move in this direction. In our research, we refer to activities designed for enactive-embodied learning or, more broadly, any activities that involve students actively exploring mathematical concepts through the use of manipulatives, tools (either virtual or physical), or simply body movements, as *Active, Bodily Experience Mathematics Learning* (ABM) activities. These ABM activities are grounded in the belief that experiencing mathematical meanings with the body and senses can enhance their understanding and retention. The use of an all-embracing construct, which includes a wide range of activities, and has also roots in different research perspectives, is essential for closely aligning with and

(2024). In J. Víšňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 103–110). Gold Coast: MERGA.

investigating the realities of the school practice (Boscolo, 2022). Particularly, due to teachers' pivotal role in the implementation of educational innovation (Century & Cassata, 2016), we focus on the investigation of teachers' perspectives on the implementation of ABM activities in their teaching practice. We aim to uncover what teachers expect from integrating ABM activities, as well as the main challenges, constraints, and limitations they anticipate when integrating them into educational settings.

Research Design and Methodology

The explorative study consists of a teacher survey. Primary and secondary school mathematics teachers were invited to fill out an online questionnaire and, after completing it, they were asked if they would be willing to participate in an individual interview intended to elaborate on the issues raised in the survey responses. The administration of the questionnaire took place between November 2021 and May 2022. Primary and secondary schools' mathematics teachers were recruited from across Australia, via National and State mathematics teacher professional associations' Facebook pages/groups, or associations' newsletters. This entailed posting an anonymous link to the questionnaire on Facebook pages/groups/newsletters with a request for it to be completed by participants. In addition to advertising via Australian mathematics associations' Facebook organisations newsletters, we sought to advertise through umbrella organisations (for example, the Independent Schools Queensland; Catholic Education Offices), and broader teacher organisations bringing together Australian teachers (e.g., the Australian Teachers Association). Furthermore, we sent the questionnaire to a list of mathematics educators who have indicated their interest in participating in ILSTE (Institute for Learning Sciences & Teacher Education) mathematics research projects. Following multiple strategies, we reach a convenience sample of 81 mathematics teachers. Upon completion of the questionnaire, 16 teachers provided their email indicating that they were willing to be contacted again by the researchers to conduct an individual follow-up interview. With nine of them, it was possible to arrange a 30-minute interview between April 2022 and May 2022.

The Questionnaire

The survey items cover dimensions derived from the literature concerning teachers' beliefs on mathematics teaching and learning (Van Zoest et al., 1994; Dionne, 1993; Ernest, 1989), conceptions of educational material usage (Skoumios & Skoumpourdi, 2021), and beliefs and instructional practices with manipulatives (Carbonneau & Marley, 2015; Golafshani, 2013). Other items were adapted from items on existing surveys (e.g., OECD TALIS 2018, IEA TIMSS 2019), along with new items concerning explorative dimensions from our research on ABM activities, framed on the results of a former study in which international researchers were interviewed to explore their beliefs about ABM activities implementation and research direction for the teacher survey (Boscolo, 2022). The survey, tailored for both primary and secondary school teachers through two parallel versions with slight modifications to better align with their teaching contexts, is structured into five main sections:

1. *The School*—Collects basic information about the teacher's current school, including its type (government/non-government), educational approach, as well as the levels taught.
2. *General*—Gathers details on the teacher's educational background and experience.
3. *Beliefs (general)*—Explores general beliefs about teaching and learning mathematics, such as the role of teachers and peers.
4. *Beliefs (ABM activities)*—Focuses on specific beliefs about ABM activities, including their appropriateness for different school levels, expected impact, limitations and constraints, and assessment strategies.
5. *Implementation Inquiry*—After a filter question about the use of ABM activities in teaching, the survey splits based on the response (Yes/No), asking for reasons, other effective strategies, and comments on implementation if applicable.

The questionnaire is a web-based tool created with Qualtrics software, and it combines Likert-type, multiple-choice, short open-response, and vignette items; a copy of the questionnaire can be found online (Boscolo, n.d.). The estimated time for filling out the questionnaire is around 20 minutes.

The Sample

The convenience sample consists of 81 respondents, however only 39 completed the entire questionnaire. Among the ones who answered the first part, 15 were primary school teachers and 64 were secondary school teachers. Most of them work in Government schools (45), while more than a third (29) are in Catholic or Independent schools. While 58 teachers work in Comprehensive (open) schools, 11 respondents were working in schools with streamed classes into attainment groupings, and 5 were in selective, special, specialist, or international schools. Almost all (71) work in traditional method schools. Concerning their expertise, the majority (43) are expert teachers (more than ten years of experience), 15 are middle-expertise teachers (4–10 years of exp.), and 15 are new to the teaching profession (1–3 years of exp.).

Follow-Up Interviews

Although the online questionnaire enabled us to reach a certain sample of teachers, the information gathered, especially through closed-ended items, deserved to be further investigated with follow-up interviews, in order to deepen the interpretation of the main themes and trends that emerged. For this reason, we sought to explore in depth some significant issues with some of the teachers who completed the questionnaire, through a semi-structured online interview. These interviews began with an introduction of the respondents, including their teaching expertise and their school contexts, followed by a discussion on their experiences of completing the questionnaire. The conversation then shifted to a comprehensive investigation of the teacher's experience and key beliefs concerning ABM activities. This included inquiries into examples of ABM activities they had implemented (if any), their reasons for valuing these activities, and the outcomes they anticipated. Additionally, we explored the challenges and limitations they encountered in integrating ABM activities into their classrooms, the difficulties faced during implementation (both from the students' and the teacher's perspectives), and the strategies employed to address these challenges. The discussion also covered instances of unsuccessful implementation, teaching strategies, and instructional guidance believed to be crucial for the effectiveness of ABM activities. Finally, we discussed factors that hindered or facilitated the integration of ABM activities, personal motivations for adopting such activities, constraints that may have limited their implementation, and the role of collaboration and support in this context.

Description of Interviewees

Table 1 shows a brief description of the interviewed teachers' profiles. The majority of the interviewees are highly experienced teachers currently working in non-government secondary schools, in different states of Australia (with a prevalence from Queensland). Some of them work with students with a middle-high socio-economic background but others with students of low socio-economic situation and non-English speaking (e.g., O).

Discussion of Results

Due to space constraints, this section focuses on key insights from the survey, concerning exclusively the beliefs on expected outcomes of ABM activities implementation and the perceived obstacles to their integration in classrooms. Additionally, it emphasises how follow-up interviews have further delved into these topics, offering deeper understanding and perspectives.

Table 1
Teachers' Profiles Description

Teacher	Where is the school	Type of school	Year levels taught	Teaching experience
G	Canberra	Government, Traditional, Open entry, Secondary school	11–12	38 years
K	Gunnedah	Non-Government (Christian Catholic), Secondary school	7–10	30 years
R	Brisbane	Non-Government, Single sex (girls), Open entry, from Prep to 12 school	7–12	25 years
St	Sydney	Independent (Roman Catholic), Single sex (girls), Boarding and day, Secondary school	7, 8 10 (high achievers) 11,12 (specialists)	37 years
Su	Queensland	Non-government (Catholic school), Single-sex (boys), Secondary school	7–12	40 years
T	Brisbane	Independent, Boarding and day, Single sex (girls) school, from Kindergarten to Year 12	11–12	Long-time experience
X	Canberra	Government, Traditional, Streamed classes, Secondary school	7–12	Over 20 years
O	Melbourne	Non-Government (Catholic School), Open entry, Secondary school	11–12	25 years
J	Near Brisbane	Non-Government, Traditional school, from Prep to 12	5, 10, 12	21 years

Importance and Expected Outcomes

Examining the responses to Likert-type item Q16, teachers in our study report that ABM activities significantly enhance *deep conceptual understanding*, improve *mathematical visualization skills*, and effectively engage *students' interest*. However, there is less consensus on the effectiveness of ABM activities in boosting students' *performance in standardised tests*. Additionally, when considering other anticipated impacts (Q17), most teachers believe ABM activities enrich their understanding of students' learning processes but are less effective in including students with special educational needs. This perspective is further supported by a response to Q18, where one teacher (in the alternative *Other*) noted, "Students with special needs are often 'lost' in these tasks even with peer and teacher support. They benefit more from direct instruction and explaining". In the follow-up interviews, similar themes recur and are further explored. For teachers, promoting deep learning involves having a long-lasting imprint in students' minds, "I think the more senses and the more parts of your body you use, the greater the, the retention of the knowledge, and the skills is going to be" (Teacher Su).

> I also just think that the more senses that are involved in your learning, which, moving … involves your feelings and stepping out something. I just think you are involving more of your brain … it's more memorable that we talked about. So … it stays deeper in their consciousness, I think. (Teacher K)

> They will learn it a lot better and, basically, rather than just writing something down, and trying to learn it by rote, if they're actually involved in carrying out the activity, it will have a lasting imprint on their minds and all that makes you remember it a lot more clearly. And, yeah, it should be more relevant, too. (Teacher St)

Further, Teacher R emphasised how, in experiencing ABM activities, students can build cognitive roots, essential to recalling mathematics concepts:

> I think muscle memory is very important. … I think also the experiential, learning experiences that kids have, where it's embedded in their memory, they can recall it, and then the teacher, at a later time, can also then recall it, and, talk about "well, remember when we did this" … So, you're drawing on their personal account of their personal experience, and I think that's a very valuable learning experience for the kids. (Teacher R)

Finally, having a deep understanding includes having a more meaningful learning:

> Give meaning. So they get a sense of either number, measurement, umm, what these abstract things are … understand the concepts better … avoid misconceptions. So, if they, for example, drew an angle and they, they turn, you know, and they realise that the angle is a turn, it's the size of the turn, it's not being anything to do with the lines. They always think about the lines—as the lines are big and all that. So, I hope that it gives them an understanding of the concept. Initial concept, yeah. (Teacher O)

Even in interviews, expected outcomes also concerns to enhance students' interest, "learning should be, engage you, should be interesting, should be fun and the more senses that you can use, the more you involve the student in any learning, the better they will learn" (Teacher K), for instance showing the relation of mathematics and the world, "it also helps them see that maths is connected to life, to the world" (Teacher K), "they can see the real world—you know, like why they're doing something. So, I think that's important for students to understand that, yeah, it's not just pen and paper" (Teacher J).

Limitation and Constraints

Looking at question Q18: *In your experience, what are the most relevant limitations for this type of learning activities' implementation?* (to answer, respondents could select up to 3 options from a list, the total number of answers was 157), the main limitations highlighted in the questionnaire were *classroom management* (42 respondents), *time factors* (41 respondents), and *lack of resources and appropriate spaces* (36 respondents). These themes also recur in follow-up interviews. Concerning classroom management, the interviewees mentioned, in particular, the number of students in classrooms and the diversity of students within the same classroom, "I think a big issue is the, ah, difference in understanding within a class" (Teacher X), and the behaviour of students, such as loudness during ABM implementation, "Because they're noisy that they're not on task" (Teacher St). The class size issue is also related to the availability of appropriate spaces and resources:

> When you first do this with students it can get noisy, it can get a bit chaotic, and it also depends on the class size, … So how can we approach this? Maybe having a classroom designated for this, where you can have some time al-, where you're away from the rest of maths classrooms, where the noise is not going to be an issue—next to the drama room, or something—dance room or whatever. (Teacher Su)

> And the other limitation is cost because you can propose lots of activities that, that require costly materials or costly equipment to implement, and so, we're restricted by budget to some degree as to how far we can go with activities, but we can do a lot with what we've got. (Teacher G)

One other main limitation identified in the responses to the questionnaire was the time factor. This theme emerged in the wide majority of interviews, concerning two different time constraints. On the one hand, it concerned the time required for the activities preparation (i.e., designing the activities, searching for resources, structuring, and integrating the activity into the curricular program), e.g., "A lot of preparation to adapt to the class that is in front of the teacher" (Teacher G), "I think it's mainly the time to talk about ideas and map them into the curriculum, into lessons. So, having time to share best practices … One of the biggest barriers is just communication with each other and time to be able to collaborate and come up with ideas" (Teacher J). On the other hand, the time factor concerned class time spent in performing the activity (i.e., the ABM activities are quite time-consuming), "the time it takes. We don't have a lot of time to get through a lot of content" (Teacher J). Indeed, time considerations were almost always related to the difficulty of addressing all curricular contents:

> The theoretical limit is you still have a certain amount of curriculum to cover in the time. And some of these activities take more time than traditional teaching, so there's a limitation on how many of these activities you could fit into a term or a semester. (Teacher G)

> Here in Australia, given that we have a very content driven curriculum, that is extremely challenging, and particularly in secondary. And it's almost imposs-, I would go so far as to say that in senior secondary, in our Year 11 and our Year 12, here in Queensland, it's virtually impossible to have hands-on activities and sort of activity-type situations. Because it's just, it's s- so heavily content-driven. (Teacher R)

Along with a content-driven curriculum, preparation for mathematics tests was identified as a fostering factor:

> But I often have this tension between time and the curriculum content and making up new activities and engaging activities that might take us even off track from that content. I think most maths teachers would say that there is some kind of tension there with teaching to the tests that we know children are going to encounter, but trying to engage them in thinking and learning and being involved in their learning. That's a bit more open-ended. … We seem to be very locked in to pen and paper formative testing. (Teacher K)

A further constraint, mentioned in interviews, is an unsupportive school culture with mathematical views on what is expected of a math lesson, which can be an obstacle:

> Parents' and kids' views of what mathematics is [is a limitation] as well. Umm, they have a particular view of what mathematics is and if what you do doesn't fit within that view, then (in our school anyway) they're very vocal about, you know, this, this is not what it should be. So, there's a fine line that you've got to walk with, getting kids involved, and also making sure that, umm, see the other side of it is that, whatever I do, everybody's got to do it. (Teacher T)

Implementation

Answering Q21: *Do you include the ABM activities in your instructional practice?*, 41 teachers stated that they included such activities (eight from primary and 33 from secondary school) while 16 answered no (only one from primary and 15 from secondary school). Asking (Q22 Alternative) the reasons why they did not implement, the principal factors were the *lack of time*, the lack *of resources, tools and materials*, the *difficulties in classroom management*, and the belief *they are not appropriate for one's school level*. Looking at the interviews, some teachers indicated that they do have wide expertise with ABM activities, while others (Teachers St, J, X) stated that they were only familiar with such activities on a theoretical level:

> Far removed from my reality. In the ideal world, it would be wonderful to be able to offer that. There are lots of working parts, there are lots of activities going on that impact on the students. Umm, lots of little things that interrupt throughout. [..] It's not taking into consideration the reality of my experience of schooling. (Teacher X)

The answers in the questionnaire showed a tendency for greater resistance to the proposal of ABM activities for secondary school teachers than for primary school teachers. Indeed, the general belief that these activities may be suitable only for children in the lower grades emerged also in follow-up interviews, "In the high school setting, the active body idea really makes it tokenistic. … I think it's more for the early conceptualisation of, basic ideas, in the primary years" (Teacher X). Concerning primary school, teachers described a different situation, "In primary school, they do lots of—I would say they use lots of materials and movement in maths, but then when they get to high school, it just stops, that's it. And it's just books, writing, reading" (Teacher O). Teachers pointed out that in secondary school ABM activities were generally rarer and less suitable, for example, not necessarily applicable to much content, "In my experience, there are few topics that high schools allow themselves to be presented that way without it becoming tokenistic" (Teacher X), "being honest, two lessons out of the 50 are hands-on and the rest is traditional" (Teacher J). Generally, teachers claimed that they did not carry out ABM activities as often as they would like or give as much space for exploration as desired:

> I think we don't do enough movement. … Occasionally we'll do something with movement in the school, but Yeah, … I can remember reading it and thinking 'oh, we really need to do more of this' of what you could see what you were asking in the questions and I was thinking 'yeah, we don't do enough movement'. (Teacher J)

Conclusion

The survey highlighted teachers' beliefs in the positive impacts of ABM activities, particularly in promoting deep conceptual understanding, enhancing mathematical visualization capabilities, and fostering student interest. Follow-up interviews delved deeper, revealing that teachers believe ABM activities have a long-lasting impact on students' memory by involving multiple senses. Teachers also emphasised the importance of experiential learning for building cognitive roots essential for recalling mathematical concepts and the role of ABM activities in making learning meaningful and engaging, connecting mathematics to the real world. However, several limitations and constraints for implementing ABM activities were identified. One of the main limitations pointed out by teachers is the lack of available time. In the follow-up interviews, we were able to get a better understanding of what teachers meant: on the one hand, the activities require a lot of time both in the classroom and in research and planning; on the other hand, with the time available for face-to-face instruction it is difficult to implement such activities, more time-consuming than traditional transmissive approaches, aiming to cover curricular contents. These statements highlight many subtexts, which can also be deduced from the analysis of other issues. Indeed, teachers do not seem to be so convinced that these activities bring results that are then reflected in standardised tests. Instead, good results in these tests are often the main goal of schools, which tend to measure themselves against NAPLAN assessments and align with the view of mathematics that students and parents have. It is therefore clear that the proposal is perceived by many teachers as ancillary to the planning and goals they are called upon to achieve, particularly in secondary school. Indeed, despite the recognised benefits, there was a tendency among secondary school teachers to view ABM activities as less appropriate compared to their primary school counterparts. This perception was partly due to the curriculum demands. Nevertheless, some teachers expressed a desire to incorporate more ABM activities into their teaching but felt hindered by various practical constraints, such as classroom management and resource availability. This relates both to affordability and, again, to the availability of materials and resources, without spending a lot of time looking for them.

In summary, the exploratory study highlighted that while ABM activities are believed by teachers to be of value for their potential to enhance learning experiences, their integration into classroom practice faces significant barriers, including logistical challenges, curriculum pressures, and differing perceptions of their suitability across educational levels. While the context may therefore limit the implementation of ABM activities, the beliefs of teachers in prioritizing more traditional teaching methods geared toward content transmission, to cover the curriculum and prepare for mathematics tests, should not be underestimated. Although about one-third of students in Australia attend non-government schools, the over-representation in the sample of non-government, especially Catholic, school teachers, as well as highly experienced teachers, must be taken into account for the exploratory teacher survey results interpretation.

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References

Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning. *Frontiers in Education* (Vol. 5, p. 147). Frontiers Media SA.

Australian Curriculum, Assessment, and Reporting Authority (ACARA). (2022). *Australian Curriculum*. ACARA. https://v9.australiancurriculum.edu.au/teacher-resources/understand-this-learning-area/mathematics

Boscolo, A. (2022). Active, bodily experience mathematics learning activities. Looking at implementation from the teachers' perspective. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the twelfth congress of European Research in Mathematics Education (CERME12)* (pp. 4103–4110). ERME / Free University of Bozen-Bolzano.

Boscolo, A. (2023). Cultural aspects in the conceptualization of active, bodily experience mathematics learning activities. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel, & M. Tabach (Eds.), *Proceedings of the 46th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 139–146). University of Haifa, Israel: PME.

Boscolo, A. (n.d.). *Body movement and active learning in mathematics: Questionnaire* [Online]. https://drive.google.com/file/d/1_QnBnmAxLCyKXWrA2ZWgMvvgwC2Yidv5/view

Carbonneau, K. J., & Marley, S. C. (2015). Instructional guidance and realism of manipulatives influence preschool children's mathematics learning. *The Journal of Experimental Education*, *83*(4), 495–513.

Century, J., & Cassata., A. (2016). Implementation research: Finding common ground on what, how, why, where, and who. *Review of Research in Education*, *40*(1), 169–215.

Dionne, J. (1993). Modifying elementary school teachers' conceptions of mathematics and mathematics teaching and learning: A strategy based on conceptual analysis. *The proceedings of the third international seminar on misconceptions and educational strategies in science and mathematics*. Ithaca.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249–253). Falmer.

Golafshani, N. (2013). Teachers' beliefs and teaching mathematics with manipulatives. *Canadian Journal of Education*, *36*(3), 137–159.

Jankvist, U. T., Aguilar, M. S., Ärlebäck, J. B., & Wæge, K. (2017). Introduction to the papers of TWG23: Implementation of research findings in mathematics education. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the tenth congress of the European Society for Research in Mathematics Education* (pp. 3769–3775). Dublin, Ireland: DCU Institute of Education; ERME.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from.* Basic Books.

Palatnik A., Abrahamson D., Baccaglini-Frank A., Ng O., Shvarts A., & Swidan O. (2023). Theory and practice of designing embodied mathematics learning. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel., & M. Tabach (Eds.), *Proceedings of the 46th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 159–189). PME.

Skoumios, M., & Skoumpourdi, C. (2021). The use of outside educational materials in mathematics and science: Teachers' conceptions. *International Journal of Education in Mathematics, Science, and Technology*, *9*(2), 314–331.

Van Zoest, L. R., Jones, G. A., & Thornton, C. A. (1994). Beliefs about mathematics teaching held by preservice teachers involved in a first grade mentorship program. *Mathematics Education Research Journal*, *6*(1), 37–55.

Varela, F. J. Thompson, E. & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. MIT Press.

Classroom Expectations: Listen to the Maths

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For students to engage in mathematical thinking, a greater emphasis is needed on the expectations of students, by themselves and others, as active listeners and explainers. As student responsibility for mathematical thinking and participation increases, the roles of the teacher and learner change. A qualitative study of changing classroom expectations in a Grade 2 class was conducted. The teacher saw her role shifting to that of facilitator and the students' expectations were as active listeners and explainers. The type of questions asked by the teacher was instrumental in whether justification and mathematical reasoning was elicited from all learners.

With a focus on classroom expectations of the *teacher and students* in primary classrooms, Wood (2002) argued that differences in teaching impact what mathematics students value and learn. From observing classrooms, Wood's research group were led to appreciate the importance of the quality of students' "explanations and justifications they were required to give for their solution processes. Furthermore, these differences were found to affect the quality of [students'] thinking and reasoning about mathematics" (Wood, 2002, p. 61). More recently, Dorier and Maass (2020) have reiterated the necessity for this focus on student-centred learning and the critical role of the teacher. They describe student-centred learning as being when students work in ways similar to mathematicians, including asking questions, focussing on mathematical ways of thinking about questions, discussing, interpreting, evaluating, and communicating solutions in effective ways. Following Wood (2002), the focus of this paper is on the expectations of teachers and students in such classrooms and what role the students perceive and enact in such a classroom.

A Mathematical Community of Inquiry

Drawing on extensive video-recordings of Grade 2 and 3 mathematics classes in the USA, Wood and Turner-Vorbeck (2001) closely examined *teacher questioning and students' mathematical reasoning*. Wood (2002) reported that differences in students' reasoning were correlated with the type of questions teachers asked, the cognitive demand these questions placed on students' mathematical reasoning, and the expectations for students' participation. Dorier and Maass (2020) note the role of the teacher in such classrooms, "includes making constructive use of students' prior knowledge, challenging students through effective, probing questions, managing small group and whole class discussions, encouraging the discussion of alternative viewpoints, and helping students to make connections between their ideas" (p. 385). According to Wood (2002), the key is not so much on the style of question, as it is on the expectations placed on students, as listeners and explainers, by themselves and others, in particular, the expectation that they be actively listening and explaining, that bring this change in student thinking and reasoning. Makar et al. (2015) report that the uptake of inquiry focused lessons in mathematics has been slow. They suggest this is a result of the challenges for both teachers and students.

In setting up a classroom culture fostering the development of mathematical thinking, the responsibility for thinking mathematically by students can be increased as the discussion context implemented by the teacher changes according to Wood (2002). Wood described three discussion contexts (above and beyond the traditional reporting answers context), as reporting strategies → inquiry → argument whereby student responsibility for thinking and participation

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 111–118). Gold Coast: MERGA.

increases. Similarly, responsibility for participation increases as the expectations of the role of the student changes from being a student explainer to an active student listener.

The first row of Table 1 is described as a conventional classroom culture (Wood, 2002). The following rows describe these increasingly sophisticated discussion contexts: strategy reporting, inquiry, and argumentation, described by Wood as reform classrooms. As the discussion context increases in complexity, the roles of explainers and listeners expand (e.g., an inquiry discussion context includes reporting of strategies and more). Furthermore, the *responsibility for mathematical thinking* and *responsibility for participation* increasingly shift from the teacher to the students. The role of the teacher does not diminish, rather it shifts to becoming a key member of the mathematical community of inquiry (Dorier & Maass, 2020). Expectations for the role of the student shift from being a passive to an active listener and having a vital part to play in discussions. Providing genuine opportunities for mathematical reasoning "requires a shift of role for both educators and students" (Hunter et al., 2020, p. 312).

Table 1

Theoretical Framework for Teaching and Learning Based on Wood (2002)

Discussion context	**Mathematical thinking**	**Role(s) of explainers (student)**	**Role(s) of listeners**	
			Teacher	**Student**
Reporting of correct answers	Recall answers Recall procedures	Tell answers Tell procedures	Evaluate Ask testing questions	Pay attention (passive)
		Responsibility for Thinking ↓	Responsibility for Participation →	
Reporting of Strategies	Comparing Contrasting	Tell different ways	Accept solutions Elaborate solutions	Compare/ contrast solutions
Inquiry	Comparing Contrasting Reasoning Questioning	Tell different ways Clarify solutions Give reasons	Accept solutions Elaborate solutions [Ask questions] [Provide answers]	Compare/ contrast solutions [Ask questions]
Argument	Comparing Contrasting Reasoning Questioning Justifying Challenging	Tell different ways Clarify solutions Give reasons Justify Defend solutions	Accept solutions Elaborate solutions [Ask questions] [Provide answers] Make challenges	Compare/ contrast solutions [Ask questions] Disagree Make challenges

Questioning

Teacher questioning is thus an important aspect of shifting responsibility for mathematical thinking in the classroom from the teacher to the students. The NCTM (2014, p. 35) elaborates that effective teaching is associated with purposeful teacher questions that both advance and assess reasoning and sense by students. Bauserfeld (1992) argued that to generate the flexibility of thinking necessary for problem solving, "it would seem more promising to develop the art of mathematical interpreting, rather than the usual funnel-like limiting to prepared, unequivocal reactions" (p. 469). Furthermore, to develop such flexibility students need to reflect on their own mathematical thinking. Hence, teachers need to provide opportunities whereby students are expected to make explicit their ways of thinking as they engage with complex tasks.

Wood (1998) noted that although well intentioned, many teachers—once a student offers a method representative of what they intend students to use—emphasise their preferred method thus inadvertently communicating to students a single approach to obtaining a desired solution. Bauersfeld (1992) described this as *funnelling*. The need to become more purposeful in mathematical classroom discourse remains an issue (Hunter et al., 2020). In contrast to the funnelling approach, Wood (1998) argued for a more dialogical approach whereby “a variety of solutions are accepted and valued … [with the intention being] to help students notice an idea” (p. 168). In this *focussing* pattern of discourse, students participate more equally in the discussion. Focusing occurs when “the teacher expects the students to think about the mathematics. To figure things out for themselves, and to discuss their ideas with others” (p. 176) and the students are aware of this expectation. A study by Hagenah et al. (2018) in Year 9 science classrooms found focussing (rather than funnelling) question use to be aligned to increased student learning outcomes and confidence.

Reasoning

Mathematical content knowledge (MCK) contains what Kilpatrick et al. (2001) describe as five strands that are the basis of students’ mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. In the Australian (and Victorian) Curriculum: Mathematics (ACARA, 2022) these are modified to understanding, fluency, problem solving, and reasoning. One important point is that the ‘content’ of MCK extends beyond number, algebra, geometry, measurement, probability, and statistics to ways of thinking mathematically. The proficiencies describe what students ‘do’ including compare, contrast, explain, identify, justify, and reflect. Kilpatrick et al. (2001) stressed the proficiency strands “are not independent; they represent different aspects of a complex whole … [they are] interwoven and interdependent” (p. 115).

The interwoven nature of the proficiencies in the enacted Australian curriculum and emphasis on each has been challenged by Australasian researchers, as has student capacity to reason. Analysis of the Australian Curriculum: Mathematics by McCluskey et al. (2016) suggests the intentions of the curriculum writers for the proficiencies to be integrated with conceptual understanding has not been met, and reasoning is the least emphasised proficiency. Given that Ball and Bass (2003) argue that understanding without reasoning is “meaningless” (p. 28), this apparent lack of emphasis on reasoning is a concern. International studies consistently show that Australian Year 4–9 students have difficulty explaining and justifying their mathematical thinking.

The Study

The data for the analysis of the primary mathematics class presented here were collected as part of the *Teacher as Learner Research* (TALR) project that examined teaching and learning in one Victorian state primary school. The project objectives were for teachers to promote the understanding of mathematical concepts through student discussion and reflection following Wood (2002). The development of *mathematical content knowledge* and mathematical *pedagogical content knowledge* (building on the work of Shulman, 1986) by the teachers, and hence classroom practices, including establishing classroom expectations of teachers to enhance student learning, were the focus. Tasks and approaches were developed collaboratively emphasising both the content and processes of mathematics, including ways of working mathematically such as conjecturing, convincing, and justifying that underpin mathematical thinking. Both questioning style and eliciting reasoning were of high importance in the project.

Key elements of the project included: *demonstration days* (each term the researcher taught several lessons observed by some teachers and discussed by all teachers at the end of the school day, following Clarke et al., 2013), *professional learning* (a two hour session each term focusing on elements of MCK and PCK including task design and development, questioning, and reasoning), *teacher lesson diaries* (TLD) documenting two focus lessons per term taught by the teacher), and *reflection.* The lesson diary included a record of the lesson intention, the lesson enacted, and a reflective component focussing on what the teacher learnt from the lesson.

Before these lessons were implemented, teachers selected three ‘typical’ students to document their thinking. The students selected were not the ‘best’ three students but from those expected to have difficulties with the learning focus. The diary included specific details regarding these students’ contributions to classroom discussions and their progress on tasks. To facilitate articulation of student thinking and provide enhanced opportunities for teacher reflection, teachers used the iPad *ShowMe* app. This app allowed a photograph of student work to be taken and then used to facilitate students’ explaining their thinking. The photograph was annotated during this reflective discussion with the teacher, which the app also audio-recorded. Post-lesson viewing facilitated the teacher’s reflective process—a key part of the TLD.

A qualitative approach was followed to provide a comprehensive picture of what was occurring in the classroom. Following Stake (1995), a case study using an instrumental approach was undertaken, as the interest is what we can learn from this case about other classrooms. Data were coded from the perspective of teacher and student expectations for roles in discussion from Wood’s Framework and mathematical thinking as specified via the proficiencies. The analysis was similar to that of Strauss and Corbin (1990) in developing categories from the data. A rich description was essential for the expectations of the classroom to be evident. The research questions addressed were: (1) *From participation in a ‘teacher and researcher’ community of inquiry, what expectations does a teacher develop in a classroom community of mathematical inquiry*? (2) *From these expectations, what do students perceive as their role in the mathematics classroom and to what extent is this realised*?

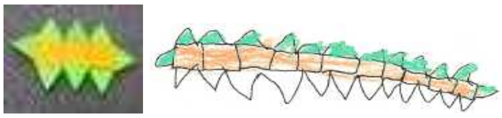
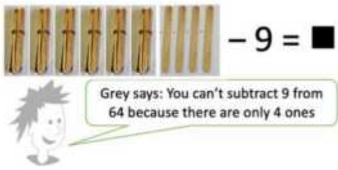
At the end of year, as part of the reflective aspect of the TALR project, all teachers were asked to implement a short survey with their class. Students responded to the question, “What do you think your role or job in maths is?” with the prompt, if needed, “What things are you expected to do in maths?” From these responses, one Year 2 class was chosen for analysis based on the response of one student. Muridi wrote, “My job is to listen to the maths, … listen to the shape figures and non-examples and the maths”. Not only had 8-year-old Muridi (all names are pseudonyms) recognised the importance of listening in his learning of mathematics, but also, he used the geometric language of figure and identified the importance of non-examples in his developing understanding. This response prompted an analysis of the classroom experiences and expectations that led him to articulate this view. Thus the case is the classroom expectations of the teacher and students in Muridi’s mathematics class.

Muridi, along with his classmates, attended three demonstration day lessons (taught by the researcher) during the year. Over the year, seven TLD lessons were taught and documented by Muridi’s teacher. Eleven of the 23 students in the class were selected as the focus student over this time for her diary. Muridi was a focus student on three occasions. To add to the richness of the data, two additional students, Ada and Madi, were selected as this meant every TLD included at least one of these students. In describing her “role or job in maths”, Ada nominated solving problems, figuring it out, understanding, show you know, and checking. She also noted “you could get it wrong” and suggested using “the maths equipment if it is hard or you get confused”. Madi included answering hard questions and solving challenging activities. Hence, three Grade 2 students, Muridi, Ada, and Madi are the main focus of the analysis reported. Whilst Muridi’s words prompted this class’s selection, it is critical to remember he is one student in a mathematical community of inquiry with other students and his teacher.

The data used were collected over four consecutive school terms and drew mainly on the TLDs. Table 2 summarises the lesson focus. Teacher reflection during and after demonstration days, and teacher reflection after professional learning sessions were used to substantiate the analysis as needed. For example, an analysis of student work collected by teachers during the year from the TLD lessons and the demonstration day lessons confirmed the three selected students were not the ‘best three’ in the grade.

Table 2

Overview of Teacher Lesson Dairies Lessons and Selected Focus Students

Lesson diary$^{\alpha}$	Focus students			Task title and brief details
	Muridi	Ada	Madi	
1		✓		*One is a snail, ten is a crab*: I saw 10 feet, what might I have seen …
2a		✓		*One change train*: Use attribute blocks to create a 'train' where adjacent pieces differ by one attribute
2b		✓		*Caterpillar Problem*: Investigate caterpillar growth over time 
3a			✓	*The Rubbish Bin Problem1 Convince Me*: Some digits from a rubbish bin number were found. What house number could the bin have come from? 
3a	✓			*Equality*: ■ = 64 − 9 Reasoning Cartoon 
4a	✓			Minibeasts What's the favourite minibeast in our class?
4b	✓	✓	✓	Which is the Odd One Out? Why? 39 37 79

$^{\alpha}$The number in column 1 indicates the term (e.g., 3b is the 2nd Lesson Diary in term 3).

Results and Discussion

The lesson in focus for the first teacher lesson diary (TLD1) is presented in sufficient detail for the reader to gain a good sense of the classroom expectations that are the focus of the research. Student comments are presented as recorded by the classroom teacher. The lesson for the Grade 2 class occurred in week six (of eight) of term 1. The lesson focused on problem posing and solving based on the picture book *One is a snail, ten is a crab* (Sayre & Sayre, n.d.). The focus task was 'I saw a total of ten feet. What animals might I have seen? How many possible combinations of animals are there? How are you going to know when you have them all?' The teacher expected students would use different ways of recording and documenting their understandings—drawings, words, numbers, equations, or a combination; make use of a range of strategies to solve the problem; apply thinking strategies; and articulate their learning approach for the problem using partner sharing and the *ShowMe* app if selected as a focus student. These expectations remained the same throughout term two as well.

When the storybook cover was presented at the start of the whole class discussion, the students recognised the type of task to come which they said was "Jill maths" drawing on their participation in a lesson on Demonstration Day 1 earlier in the term. When asked to explain, comments included "Jill gives us lots of maths things to do" (Madi), "challenges and questions" (Ada), "she gives us all sorts of new things that we don't know yet" and "gives us a problem and we try and work it out". According to her diaries the teacher used this connection regularly to how the students' experienced the demonstration lessons to reinforce her classroom environment expectations for students and to convey that their role included explaining and engaging with mathematical thinking as they worked on similar tasks.

After reading the book, checking for understanding ("a crab has 10 feet and a snail only one foot"), and discussing a simpler version of the task for 6 legs, the students set to work individually for 30 minutes, followed by ten minutes where they partnered to share their work, and then returned to individual work on the task. The teacher noted (TLD1) that throughout the lesson she frequently asked students, “Have you found all the possibilities? How do you know?” and implored them to “Convince us! Check for duplicates”.

Muridi wrote, “I had eight combinations. I putted numbers on everything I drawed.” During the lesson his teacher also recorded some of Muridi’s thoughts on his sheet. These included “I counted by twos so that is 10 altogether” in explaining his solution using 10 people. For his solution of 2 people, 2 snails and a dog, his teacher recorded “I did 2 people for 2, 4 then I counted 2 more, then I did 4 then 8. 2 people and 2 snails equals 6 then I added the dog, 6 + 4 = 10.” Muridi was the only student who recorded the number of combinations he found. His comment regarding putting numbers with his drawings is inferred as an attempt to explain his solutions were correct rather than to argue he had found all possibilities. Although it is impossible to tell the order in which Muridi recorded his solutions, his work showed evidence of a systematic approach. He clearly was also able to keep in mind a focus on how many possibilities there were and that this required him as the task solver to make a note of this.

Five students appreciated the need to argue they had found all combinations. None stated how many they found and their argumentation skills were still developing. Madi, for example, found eight solutions and wrote “I think there are know (sic) more combinations”. Ada noted she “counted by twos because ten is even”, checked for duplicates and “there were only two”, and used her facts to ten. She claimed, “I know I have them all. I tried thinking and thinking and I couldn’t find anything more. I looked at the animals I looked [at the] legs and feet and added on numbers”. She recorded drawings with numerical equations to represent her findings. She thus fulfilled her teacher’s expectation that multiple representations would be used.

Data presented from this term 1 lesson (TLD1) clearly indicated that Muridi, Ada, and Madi were in a classroom where students were expected to ‘have a go’ and to try different approaches, in this case different representations—drawings, numerals, words, equations, to look for duplicates and find more than one solution to a problem. As has been shown these expectations were clearly discerned and students were seeing their role as doers of mathematics, expressers of mathematical thinking, and documenters of that thinking even if at this point in the school year they were still developing their skills to do so.

Given space restrictions, the remaining teacher lesson diaries (two per term for terms 2–4) will not be detailed in full. In each lesson diary there was evidence of students realising their role included explaining both task and solution. After making and drawing a “train” with his partner for the *One Change Train Task* (noted in TLD2a), Muridi recorded his reasoning as “it is a one change train, one was skinny and one was fat, one was big and one was short” as he described his train of alternating rhombuses using pattern block pieces. He and his partner were one of three pairs that recognised a simple train could be made using an ABAB pattern. When asked about task difficulty, he said the task was “easy because I used the same shape”. Ada articulated her approach clearly, “I am not changing anything except the colour—not the size, the length or anything except the colour”. Madi also described the task as easy clearly believing that identifying mistakes and revisiting the task were part of learning, “because I worked with a partner, and we started it with getting some blocks and made it into a train and figured out it wasn’t right and then made another one that was right”.

In her term 2 professional learning reflection, the teacher noted her intention for an increased focus on “continually encourage[ing] students to unpack what they already know and discuss/elaborate/convince peers of their thinking”. By term 3, the teacher had added additional expectations to those she expressed in TLD1 and specifically noted her own changed role. In her first lesson diary for term 3 (TLD3a), she was explicit about her classroom expectations progressing to “setting up community of learners (justifying, reasoning, questioning, learning from each other’s [questioning]—changed teacher role as facilitator)”. In addition to the other expectations, she had identified previously, she expected students to be articulating the way they solved the problem. In the second lesson diary (TLD3b) she reiterated, she expected students to be actively listening to, and to be open to, the thought processes of peers, articulation of their thinking—orally and written, and being able to write mathematical understandings.

Challenging other students was also part of the classroom culture. Reasoning cartoons were often used in demonstration lessons to support this, where challenging an unknown student (e.g., Grey TLD3b) supported subsequently challenging peers and explaining the thinking of others. During the *Caterpillar Task* (TLD2b), at one stage after making the Day 6 caterpillar, students were discussing the Day 5 caterpillar. They were asked to explain their thinking to a partner "to justify their reasoning". Muridi's partner, Hugo, proposed there were 11 triangles and 5 squares, but he was unable to explain his thinking. Muridi immediately challenged this thinking, demonstrating his preparedness to disagree, claiming, "It is not 11, it is 12."

In some tasks, students chose the level of challenge. In the *Caterpillar Task* (TLD2b), after considering the Day 6 caterpillar, Muridi chose the Day 11 caterpillar. The colouring of his drawing (shading triangles only on one side—see Table 2 row 3) suggests he recognised a doubling pattern as part of his reasoning, but he did not articulate this. Madi working with the Day 10 caterpillar was better able to explain how, if not why. She wrote, "The rule is <u>10 + 10</u> = <u>20</u> so when you add the <u>2</u> more all together there is 22 [triangles]". She also showed evidence of checking using an alternative approach (i.e., count all). Ada was able to articulate her reasoning including relating her rule to the day (15), "The rule is you double the amount of days the caterpillar is … and add two", and correctly identified the number of triangles required.

In the *Minibeasts Task* (TLD4a), 17 of the 23 students contributed to a discussion in response to *What do you think the task is about*? Madi suggested, "data on most popular out of lots of minibeasts". Muridi pointed out, "[we] need an axis so we could write about what kind of minibeasts so we could go around and see what is the most". Post-lesson Show-me questions to Muridi (and another boy), focused on telling type questions (e.g., What did you do? Tell/show me), sufficed to allow articulation of the way the problem was solved but elicited no evidence of reasoning although Muridi did use comparative language. For the third student, the teacher took on a facilitator role deftly asking questions that elicited reasoning and justification.

Whether in a discussion context with the whole class, talking to a partner, or responding to a task using concrete materials, diagrams, words or symbols, the students in this class, as evidenced by the experiences of Muridi, Ada, and Madi seemed to view *their role* in the mathematics classroom was to actively listen, tell different ways, clarify, reason and to convince others that their mathematical thinking was correct as expected by Wood (2002) and others (e.g., Hunter et al., 2020; Kilpatrick et al., 2001; McCluskey et al., 2016). This was true even when they found a task challenging and/or struggled to explain their thinking.

Concluding Remarks

In this paper, a qualitative study has reported on one classroom where the teacher articulated her role in developing a community of mathematical inquiry. The data indicated she saw her role had to change to facilitator rather than evaluator of student learning and the learners should be increasingly responsible for active participation in their learning. The expectations of Muridi and his classmates, both by the teacher and of each other, were as active listeners and explainers with responsibility for mathematical thinking and participation, although this was not always realised. Teachers can bring about change in students' mathematical thinking by increasing expectations of learners as active listeners and explainers. Students in such environments perceive themselves as having responsibility for participation and mathematical thinking. To sustain these expectations, teachers need to focus on asking *inquiry questions* (that require explainers to clarify their solution and provide reasons for their decisions and actions) that position learners as mathematical reasoners and *argumentation questions* (that disagree with, or challenge, the explainer) that position explainers as justifiers and defenders of their solution to all students. The type of question asked positions the explainers in the different roles (Wood, 2002). Questions requiring explainers only to tell answers and procedures do not encourage mathematical reasoning nor indicate expectations for this. However, "powerful and innovative pedagogies such as rich and robust dialogical interaction are challenging for teachers to develop in their classroom" (p. 296). The role of the teacher in asking questions of all students that require clarification, reasoning, and justification cannot be underestimated, nor the challenge for teachers in sustaining such practice. This study contributes to a greater understanding of the challenges for both teachers and students in describing and sustaining classroom expectations for mathematical thinking.

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References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: Mathematics (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/

Ball, D., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27–44). National Council of Teachers of Mathematics.

Bauersfeld, H. (1992). Classroom cultures from a social constructivist's perspective. *Educational Studies in Mathematics*, *23*, 467–481.

Clarke, D. et al. (2013). Demonstration lessons in mathematics education: Teachers' observation foci and intended changes in practice. *Mathematics Education Research Journal*, *25*(2), 207–230.

Dorier, J. L., & Maass, K. (2020). Inquiry-based mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 384–388). http://doi.org/10.1007/978-3-030-15789-0

Hagenah, S., Colley, C., & Thompson, J. (2018). Funneling versus focusing: When talk, tasks, and tools work together to support students' collective sensemaking. *Science Education International*, *29*(4), 261–266.

Hunter, J. Miller, J., Choy, B.H., & Hunter, R. (2020). Innovative and powerful pedagogical practices in mathematics education. In J. Way, C. Attard, J. Anderson, J. Bobis, H. McMaster, & K. Cartwright (Eds.), *Research in mathematics education in Australasia 2016–2019* (pp. 293–318). Springer.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.

Makar, K., Bakker, A. & Ben-Zvi, D. (2015). Scaffolding norms of argumentation-based inquiry in a primary mathematics classroom. *ZDM Mathematics Education, 47*(7), 11077–1120. https://doi.org/10.1007/s11858-015-0732-1

McCluskey, C., Mulligan, J., & Mitchelmore, M. (2016). The role of reasoning in the Australian Curriculum: Mathematics. In B. White, M. Chinnappan, & S. Trenholm (Eds.), *Opening up mathematics education research. Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australasia* (pp. 445–452). Adelaide: MERGA.

National Council of Teachers of Mathematics (NCTM). (2014). *Principles to actions: Ensuring mathematical success for all*. NCTM.

Sayre, A. P., & Sayre, J. (n.d.). *One is a snail, ten is a crab*. Illustrated by R. Cecil. Walker Books.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 47–14.

Stake, R. (2005). Qualitative case studies. In N. Denzin & Y. Lincoln (Eds.), *The Sage handbook of qualitative research* (3rd ed., pp. 443–466). Sage.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research*. Sage.

Wood, T. (1998). Alternative patterns of communicating in mathematics classrooms: Funneling or focusing? In H. Steinbring, M. Bartolini Bussi, & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp. 167–178). Reston, VA: National Council of Teachers of Mathematics.

Wood, T. (2002). What does it mean to teach mathematics differently? In B. Barton, K. C. Irwin, M. Pfannkuch & M. Thomas (Eds.), *Mathematics education in the South Pacific. Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 61–67). Sydney: MERGA.

Wood, T., & Turner-Vorbeck, T. (2001). Extending the conception of mathematics teaching. In T. Wood, B. S. Nelson & J. Warfield (Eds.), *Beyond classical pedagogy* (pp. 185–208). Routledge.

Using the McNamara Fallacy to Critique (Mis)representations of "Success" in Mathematics Education

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This position paper examines the phenomenon of the McNamara Fallacy to analyse flawed conceptions of "success" in mathematics learning, normalised assessment structures and their implications for mathematics education. The established presence of the McNamara Fallacy and the ramifications of this statistical fallacy provide a foundation to demonstrate its existence and parallels within mathematics education, including NAPLAN and other common assessment structures. Viewing such assessments through the lens of the McNamara Fallacy allows educators to recognise, explain, and potentially address their negative consequences.

Various forms of formal and informal assessment play a prominent role in enabling judgments to be made about students' success in learning mathematics. For example, national and international assessments such as NAPLAN tests of numeracy achievement and PISA tests of mathematical literacy provide high-level information about the effectiveness of education provision across a country or for demographic sub-groups, while classroom assessment can monitor the progress of individual students and generate feedback to inform teaching. However, the results of mathematics assessments are often used for other unintended purposes—often with damaging consequences for students, teachers, and schools.

In this paper, we apply the McNamara Fallacy to critique such assessments in mathematics education. This term was coined by sociologist Daniel Yankelovich in 1972 as an eponym for US Secretary of Defence Robert McNamara's (1961–1968) statistical approach to the Vietnam War. The McNamara Fallacy refers to the logical yet flawed tendency to draw conclusions or reach a hypothesis derived from easy-to-measure quantitative data while also disregarding important variables that are nevertheless more difficult to quantify. This statistical fallacy can lead to misleading conclusions, oversimplified perspectives, cognitive bias, distorted truth, and overconfidence in unfounded decision-making. The aim of this position paper is to use the McNamara Fallacy to analyse flawed conceptions of "success" in mathematics learning and assessment and their implications for mathematics education. This paper will first explore the origins of the McNamara Fallacy before recognising its established presence within the healthcare fields to demonstrate parallels with mathematics education. This paper will continue to investigate other possible areas of mathematics education assessment impacted by the McNamara Fallacy.

The Origins of the McNamara Fallacy

Robert McNamara (1916–2007) is noted as a great logical economist and a remarkable statistical mind with an impressive resume and many notable achievements. In 1943, McNamara enlisted as a Captain in the United States (US) Army's Department of Statistical Control, where his profound statistical methodology dramatically improved the planning and execution of aerial bombing missions during World World II (Kelleher, 2021). After three years of active duty and an elevated rank of Lieutenant Colonel, McNamara and other ex-military statisticians were recruited by the troubled giant Ford Motor Corporation (United States Department of Defence, n.d.). McNamara's considerable talents in statistical analysis and financial management were critical to the triumph of Ford's expansion and restored profit margins in post-war times (United States Department of Defence, n.d.). McNamara's planning

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 119–126). Gold Coast: MERGA.

and management prowess resulted in his advancement through various high-level management positions, including a short stint as the President of the Ford Corporation (Kelleher, 2021; United States Department of Defence, n.d.). Because of McNamara's prominent reputation, then US President-Elect John F. Kennedy offered McNamara a cabinet position, which saw McNamara reject the initial offer of Treasury Secretary but accept the Secretary of Defence post (O'Mahony, 2017). Despite his distinguished career up to this point, McNamara's rational and statistical approach to 'problems' is now more commonly attached to costly decisions in prosecuting the Vietnam War.

Following his former successes, McNamara continued to demonstrate his aptitude for statistical analysis and quantitative decision-making in the approach of the US to the Vietnam War. A key measure of the success of McNamara's military strategy was ensuring the number of enemy casualties and fatalities exceeded those of the US. This strategy of taking casualty and fatality figures as the measure of success led to a massive escalation of the number of US soldiers in Vietnam. However, the data used by McNamara were flawed. O'Mahony (2017) explains how "the South Vietnamese army reported what they thought the Pentagon wanted to hear—they were 'gaming' the figures—and the US did not question the numbers" (p. 281). Kelleher (2021) writes how, increasingly, the 'body count' metric became the "preferred way for US Generals to rank the effectiveness of different American combat units" (p. 260), including the determination of promotions, ensuing 'gaming' data practices in which previously recorded fatalities were re-tallied to inflate numbers.

Later, McNamara exhibited remorse and doubt in his military strategy and, as O'Mahony (2017) writes, "conceded that excessive emphasis on a single crude metric over-simplified the complexities of conflict" (p. 281). The Vietnam War was the first modern-day and significant conflict that depended upon guerrilla warfare. Guerrilla warfare refers to accumulative small-scale yet intense acts of combat via irregular methods or unconventional tactics and is typically carried out by grassroots or irregular members fighting against larger and more traditional powers (Lebo et al., 2021). Consequently, there were limitations in relying on quantitative measurements to evaluate large actions, especially when boundaries are indistinguishable, unpredictable or unstable. The complexities of people and their actions do not always fit into formerly defined boxes as times change, as do technologies, resources, and the evolution of people's motivations (O'Mahony, 2017). As the war stretched, then U.S. President Lyndon B. Johnson and McNamara's views on further strategies did not align, which was amplified by the public opposition (Kelleher, 2021; O'Mahony, 2017). McNamara resigned in 1967 and, within six months, became the President of the World Bank, where he remained until the early 1980s (United States Department of Defence, n.d.).

Despite being a capable and clever man with many prestigious posts, McNamara's reputation has been forever linked to the failure of military strategy in the Vietnam War, a "problem that did not submit itself to numerical analysis" (O'Mahony, 2017, p. 281). In coining the term, '*the McNamara Fallacy*', Yankelovich (1972) succinctly described its flawed logic as follows:

> The first step is to measure whatever can easily be measured. This is OK as far as it goes. The second step is to disregard that which can't be easily measured or to give it an arbitrary quantitative value. This is artificial and misleading. The third step is to presume that what can't be measured easily really isn't important. This is blindness. The fourth step is to say that what can't be easily measured really doesn't exist. This is suicide. (Yankelovich, 1972, as cited in O'Mahoney, 2017 pp. 281–282)

We argue that the flawed logic inherent in the McNamara Fallacy is readily observable in approaches to assessment in mathematics education and the educational practices and decisions informed by the results of these assessments. However, despite its relevance and prevalence, little literature currently connects the McNamara Fallacy to mathematics education. Nevertheless, in fields such as medicine and medical education, the McNamara Fallacy is connected to worrying trends towards inappropriate reliance on quantitative measures. In the next section, we briefly review the literature on the McNamara Fallacy from the medical field as a starting point for considering its implications for mathematics education.

The McNamara Fallacy in Medicine and Health Care

Literature sources (e.g., Hirkani, 2023; O'Mahony, 2017; Singh & Shah, 2023) within the medical field have applied the McNamara Fallacy to demonstrate the negative consequences of over-dependence on quantitative measures, including the decline of patient care, professionalism, and appropriate objectives. They contend that the complexities of medicine cannot be accounted for and controlled by crude numerical analysis. They stressed that setting harmful "arbitrary targets" (O'Mahony, 2017, p. 282) does not improve care for patients and consequently results in the neglect of other important unquantifiable attributes such as communication and compassion (O'Mahony, 2017). Similarly, Kelleher (2021) describes the dismissed 'immeasurables' within dentistry, such as clinical judgment, artistry, consistency, and manual gentleness of touch. He describes how, instead of prioritising a patient's long-term dental health, UK dentists are pressured to meet specific quotas or risk financial penalties (Kelleher, 2021). O'Mahony (2017) also describes how doctors could be pressured by "audit and quality assurance programmes … to carry out treatments that are not in the patient's best interest" (p. 282). A similar UK initiative aims to judge the overall performance of British hospitals based on overall mortality rates gleaned through specific calculations, which are subject to distortion and reality bias (O'Mahony, 2017).

All of these studies applied the McNamara Fallacy to demonstrate the propensity to select easy-to-measure variables—that disregard many other more difficult-to-measure or unconsidered variables—in the effort to make conclusive judgements about performance and professionalism. In doing so, the authors highlighted the unintended consequences that have repercussions for the everyday people cared for by health professionals. Discussing the McNamara Fallacy, Singh & Shah (2023) also describe the medical education of future doctors in India and how assessment structures that "focus on numbers alone gives the wrong message to [medical] students" (p. 3) about the profession, their medical training, and the characteristics of competent and compassionate doctors of India. For example, marks on an examination do not equate to medical students' ability to ethically "apply knowledge appropriately in the given context" with honour, integrity, or emotional intelligence, nor the ability to work in a team (Singh & Shah, 2023, p. 3). Although drawing too many literal comparisons between the fields of medicine and mathematics education is unwise, there are obvious parallels between these fields in the unintended consequences of making decisions about "success" based solely on quantitative observations of one or a few variables.

Applying the McNamara Fallacy to Mathematics Education

Many examples from the medical field share parallels with factors and experiences within current educational assessments and success metrics within mathematics education. The most obvious parallel is the continued misuse of Australia's National Assessment Program—Literacy and Numeracy (NAPLAN) regime. Since 2008, Australian students in Years 3, 5, 7, and 9 have participated in NAPLAN, a series of nationwide mandatory standardised tests that assess performance in Mathematics and English aligned with the content strands of the Australian curriculum. Though standardised assessment is a common educational tool worldwide, many doubts have been raised about NAPLAN's purpose, implementation, and perverse consequences (Thompson & Cook, 2014; Wu, 2010).

The (mis)representation of NAPLAN data also carries traits of the McNamara Fallacy. Like other standardised assessments, the test design is based on a probabilistic psychometric analysis called the Rasch model, which determines students' performance based on the probability of answering questions strategically designed to demonstrate generalised underlying traits or presumed indicators (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.; Burtenshaw, 2022). NAPLAN's numeracy test contains approximately 40 questions to sample numerous mathematics concepts across multiple-year levels from the Australian curriculum. Despite ACARA (2010) telling parents that NAPLAN's data can "measure how their child is performing against the national average", these claims have been refuted due to statistical and measurement errors (e.g., expected margins of error) not illustrated within representations sent home to parents (Wu, 2010). Without proper explanation, the placement of dots and markers on a calibrated scale could be considered arbitrary, artificial, or misleading, as outlined by Yankelovich's (1972) descriptions of the McNamara Fallacy. These statistical considerations are better mitigated with larger collective data representations for government or system-level use. Nevertheless, similar to McNamara's strategies in the Vietnam War, this one-point-in-time data is being utilised in such a manner to draw sweeping conclusions about a student, a cohort, or an entire school, including assumptions about and factors that are not assessed.

Understanding the role of the McNamara Fallacy within NAPLAN's culture can aid in comprehending and distinguishing relevant criticisms and addressing such criticism, such as the overemphasis and overreliance on NAPLAN data to make consequential judgements. Though NAPLAN specifically focuses on mathematics, viewing its repercussions through the lens of the McNamara Fallacy shows that it prioritises particular aspects of mathematics and leads to practices that narrow the curriculum to fulfil NAPLAN's illusory objectives. Instead of supporting authentic and holistic explorations of mathematics concepts, NAPLAN preparation is often focused on fragmented content exposure, best achieved through rote-learning methods of instruction and drill-like practice (Serow et al., 2016; Thompson & Cook, 2014). Such approaches are typically void of creativity, higher-order or critical thinking, or ethical or social applicability (Klenowski & Wyatt-Smith, 2012). In many classrooms, preparation for NAPLAN also includes explicitly teaching students contingency strategies to approaching questions, such as how to narrow the multiple-choice options by removing apparent incorrect responses, and 'when in doubt, guess because at least you have a 25% chance of getting it right'. This raises questions about whether the data NAPLAN produces—though easy to measure in content and method—should be appropriately relied upon to draw broad conclusions, especially if the metrics are so readily perceived by schools as gameable. Furthermore, beyond mathematics education, critics of NAPLAN argue how its overemphasis has overshadowed and disregarded other learning areas, such as The Arts (Garvis & Pendergast, 2010). Acknowledging the impact of the McNamara Fallacy within NAPLAN's culture can help distinguish the actual challenges we face as a wider education system. The real focus of concern is not mathematics—or the perceived version of mathematics portrayed by NAPLAN interpretations—but the need to unite in addressing the flaws associated with the McNamara Fallacy.

The parallelism between the medical field and the education system continues with the findings and claims about the harmful or unintended consequences of flawed conclusions fuelled by the McNamara Fallacy. Much like NAPLAN, initiatives in health care were driven by expectations of accountability and measurements for comparison purposes to determine whether the country can have confidence in their systems (O'Mahony, 2017; Singh & Shah, 2023). However, simplified initiatives that could not capture the complexities of medical and patient care have provoked perverse professional cultures, misallocation of foci, and oversimplified statistics with far-reaching ramifications (O'Mahony, 2017; Singh & Shah, 2023). A 2015 U.K. report, 'Uses and Abuses of Performance Data in Healthcare', listed various consequences of metric-based clinical targets, including tunnel vision, inequity, bullying, erosion of professional motivation, gaming of data, and deflection from less-favourable data (Shaw et al., 2015). Mathematics education is witnessing similar claims in the narrowing of curriculum, the possibility of wasted resources, the commodification of schools, oversimplified league tables, unsubstantiated comparison disregarding natural cohort variances (e.g., per school, per cohort, etc.), and questioning whether teachers core business is education or to fulfil political or neoliberal agendas (Klenowski & Wyatt-Smith, 2012; Serow et al., 2016; Thompson & Cook, 2014; Wu, 2010). Both fields have seen the upsurge of manipulated data, the media's misguided attention, public "naming and shaming", obsessive data overuse, and small-unit data analysis unfitting to its collection's intent (O'Mahony, 2017; Klenowski & Wyatt-Smith, 2012; Serow et al., 2016; Singh & Shah, 2023). Serow et al. (2016) conclude that currently, "the value of mathematics to society is reflected in the extent to which it is externally accessed—that is assessed by agents external to the classroom and outside of the teachers' control" (p. 239). Daliri-Ngametua and Hardy (2022) write how NAPLAN's 'performativity' has resulted in the demoralisation of teachers, teacher dropout, and the 'naturalisation' of accountability discourses and 'dataveillance'. Medical and health professionals from multiple countries share similar concerns about the consequences of directives and initiatives that fall short to statistical fallacies. The same could be said about NAPLAN, as other countries (like the US and the UK) have noted similar unintended consequences from their standardised testing regimes years before Australia initially decided to go down this path (Hursh, 2005).

Examining the shared parallels between flawed measurement metrics in the medical field and NAPLAN demonstrates that the McNamara Fallacy exists within the landscape of mathematics education. Nevertheless, NAPLAN merely serves as a starting point. Other instances of the fallacy persist within conventional or widely accepted assessment structures in mathematics education. The following section aims to shed light on these additional examples for consideration.

Applying the McNamara Fallacy to Mathematics Education Assessment

The misleading logic of the McNamara Fallacy is evident within common assessment methods adopted in mathematics classrooms. This includes the tradition of mathematics exams that quiz students on fragmented mathematical content or selective mathematical activities yet are then used to form overarching conclusions about students' mathematical capabilities (Watt, 2005). Burtenshaw (2023) contends that such traditional approaches to mathematics learning and popular methods of data collection are possibly intertwined with deeply engrained beliefs or behaviourist ideologies, which suggest learning is achieved through the replication or reproduction of rules or procedures and demonstrated through performance that is "distinctly observable and objectively measurable" (p. 125). Objectivity continues to be a highly valued and widely acknowledged characteristic of reliable, "good" assessment (Watt, 2005; Shepard, 2000; Zane, 2009). However, objectivity may not necessarily warrant the degree of emphasis it receives. Nonetheless, an inherent aspect of objectivity lies in its facilitation of straightforward measurement, achieved through established limits, defined properties, and reduced chances for subjective complexities. The marking of right and wrong answers to exam questions could be considered an easy method to measure learning, especially if the purpose of such questions is to determine whether a specific skill or chunk of knowledge is demonstrated. However, this approach to measuring mathematics learning only skims the surface of what mathematics is and broader mathematical capabilities.

An inherent aspect of the McNamara Fallacy involves a cognitive bias wherein selected metrics are sufficient or appropriate enough to draw broad conclusions. This can be seen through narrow assessments like exams being relied upon too heavily to determine students' overall mathematical proficiency. Watt (2005) found an astounding overreliance on written tests despite research participants also indicating how "traditional mathematics tests cannot be used to infer more general mathematical ability" (p. 23). Not only does this continue to minimise what mathematics and mathematics learning entails, but the possible inaccuracies of such judgment formation can contribute to students' beliefs about themselves and their mathematical abilities, reinforcing values, attitudes, and opportunities in life beyond the mathematics classroom (Nardi & Steward, 2003; Thanheiser, 2023). Yet, such judgments and measures are often collated into assigning grades on report cards. These codified A–E grades only further contribute to oversimplified measures that construct broad assumptions with far-reaching effects. Clarke (1997) writes how grades are a "means of coding assessment information" and how the condensing and categorising into one single grade or score sacrifices the all-encompassed "detail that might contribute most constructively to the subsequent actions of teacher, student, or parent" (p. 65). Therefore, this also raises questions about whether the purposes of assessment are retained when afflicted by the McNamara Fallacy and if reporting structures might also be cultivating the McNamara Fallacy.

In Australia, teachers of mathematics are legally required to report to the Australian Curriculum's Achievement Standard for a student's specific year level (Department of Education, 2023). The Achievement Standards identify what must be measured, and—viewed through the lens of the McNamara Fallacy—what is therefore important. However, the mathematics Achievement Standards underscore a predominant focus on the output of skills and content, characterised by a limited range of objective-inclined verbs. A preliminary comparison with Achievement Standards from the English curriculum, for example, hints at chances for experimentation, the incorporation of dispositions, fostering critical orientations, enabling student autonomy in the exploration of content and a broader array of verbs, some of which are not purely objective (ACARA, 2022a). The choice of words to describe an Achievement Standard could subliminally reinforce actions and beliefs that prompt certain forms of assessment to ascertain student mathematical 'achievement'. For example, the verb 'experiment' is within the Years 9 and 10 English Achievement Standards but not within the Mathematics Achievement Standard (ACARA, 2022b). Across all year levels' Achievement Standards for Mathematics, the verb 'experiment' is only stated once within the Year 3 Achievement Standard.

Long since industrialised approaches to mathematics learning were the only option, cognitive science and mathematics education research has repeatedly found the importance of constructivist, explorative and meaning-making approaches to learning mathematics (Shepard, 2000; Thanheiser, 2023). This includes similar themes of experimentation, problem-solving, critical thinking, learning in communities of practice, and the impact of attitudes and dispositions (Thanheiser, 2023). These traits and characteristics of mathematics learning denote more complexity in thinking and acting and, therefore, more complexity in determining or assessing such learning. Teachers may find embedding research-informed pedagogical practices that require more complex and creative approaches to mathematics assessment more challenging if these approaches are not adequately represented in the wording of the curriculum's Achievement Standards. Here, we observe the potential effects of McNamara's Fallacy, which suggests "what can't be measured easily really isn't important" or does not exist. Mathematics education needs to move on from such flawed prioritisation of the easy-to-measure metrics to consider more accurate and more holistic alternative approaches to assessment.

Conclusions and Implications for Mathematics Education

Clarke (1997) writes how "assessment should model the mathematical activity we value" (p. 8). Mathematics is seldom performed in isolation without context, nor rarely is mathematics beyond the classroom approached in a singular way with a singular outcome—or alternatively, easy solutions for easy problems. Assessment—formative or summative—should be seen as the opportunity to gauge how well students can handle the messy, ill-structured real world in novel yet relevant ways, instead of atomised knowledge and decontextualised behaviours (Clarke, 1997; Zane, 2009). Therefore, it seems fit that measures of mathematics learning should be complex enough to capture the complexities of student thinking and acting. That is not to say such an assessment does not exist. There are many examples of good assessment practices in mathematics that measure the complexities of mathematical learning and qualitative elements of performance. This could include the *MCTP Assessment Alternatives in Mathematics* book written by David Clarke and published as part of the Mathematics Curriculum and Teaching Program (MCTP) in the 1980s. The Language of Functions and Graphs package was developed by the Shell Centre for Mathematical Education in Nottingham around the same time. These assessment examples allow for the complexities of mathematics learning to be captured. Clarke (1997) writes how, in the process of assessment, it is "the students' responsibility to demonstrate understanding and the teacher's responsibility to provide the opportunity and the means for that demonstration" (p. 24). However, we need to consider the barriers preventing this, such as the mathematics education assessments' susceptibility to the McNamara Fallacy.

The McNamara Fallacy plays a role within mathematics education assessment, and the ramifications stretch far beyond assessment, such as the worth placed on measurement and metrics, the illustration of values and the portrayal of messages—explicit or hidden. For example, suppose the selection of mathematical content and the methods of measuring content are grounded purely by the logical preference to measure what is easy to measure (or not). In that case, this communicates many hidden messages to students about mathematics, the classroom and mathematics education. Furthermore, narrow metrics provide narrow opportunities for success and narrow perceptions of success in mathematics education. Narrow opportunities and perceptions of success play a role in students' developing beliefs about themselves and their mathematical abilities, reinforcing values, attitudes, and opportunities in life beyond the mathematics classroom (Nardi & Steward, 2003; Thanheiser, 2023).

Assessment methods, such as exams and standardised testing, have their place in the education system's toolkits. However, the misuse and overconfidence in the measures and consequential data is a problem. We need to remind teachers that they have agency in their classrooms, and good examples of assessment practice will reveal the valuable and valued thinking that students do. But, for these alternatives to be considered, we must also become comfortable calling out the status quo and flawed statistical decision-making, such as what the McNamara Fallacy describes. As Thanheiser (2023) notes, this may include "dismantling structures that impede student success and participation rather than setting achievement, or lack thereof, at the feet of students" (p. 6). What is normalised is not necessarily right.

References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022a). *Australian Curriculum: English.* https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/english/foundation-year_year-1_year-2_year-3_year-4_year-5_year-6_year-7_year-8_year-9_year-10

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022b). *Australian Curriculum: Mathematics.* https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/foundation-year_year-1_year-2_year-3_year-4_year-5_year-6_year-7_year-8_year-9_year-10

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2010). *Submission to the senate education, employment and workplace relations committee: Inquiry into the administration and reporting of NAPLAN testing.* https://www.nap.edu.au/_resources/ACARA_(2010)_-_NAPLAN_senate_inquiry.pdf

Australian Curriculum, Assessment and Reporting Authority (ACARA). (n.d.). *ACARA facts: Naplan tests and results* [Online]. https://www.acara.edu.au/news-and-media/ acara-facts

Burtenshaw, R. (2022). The collateral of NAPLAN's false narratives. *Australian Mathematics Education Journal*, *4*(4), 267–32. 10.3316/informit.882302786988943

Burtenshaw, R. (2023). The deeply engrained behaviourist assessment ideologies constraining school mathematics. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 123–130). Newcastle: MERGA.

Clarke, D. (1997). *Constructive assessment in mathematics: Practical steps for classroom teachers*. Key Curriculum Press.

Daliri-Ngametua, R., & Hardy, I. (2022). The devalued, demoralized and disappearing teacher: The nature and effects of datafication and performativity in schools. *Education Policy Analysis Archives*, 1–24.

Department of Education. (2023). *P-12 Curriculum, assessment and reporting framework.* Education Queensland. https://education.qld.gov.au/curriculums/Documents/p-12-curriculum-assessment-reporting-framework-version9.pdf

Garvis, S., & Pendergast, D. (2010). Does arts education have a future in Australia against literacy and numeracy. *International Journal of Arts in Society*, *5*(2), 111–120.

Hirkani, M. A. (2023). The McNamara fallacy in medical education: Spot it, stop it. *Journal of Postgraduate Medicine*, *69*(1), 7.

Hursh, D. (2005). The growth of high-stakes testing in the USA: Accountability, markets and the decline in educational equality. *British Educational Research Journal*, *31*(5), 605–622.

Kelleher, M. (2021). 'McNamara's' and other fallacies in dentistry: Part 1. *Dental Update*, *48*(4), 260–262.

Klenowski, V., & Wyatt-Smith, C. (2012). The impact of high stakes testing: The Australian story. *Assessment in Education: Principles, Policy & Practice*, *19*(1), 65–79.

Lebo, D., Midhio, I. W., & Prakoso, L. Y. (2021). Comparison of the China-Vietnam-Indonesia guerrilla war in the perspective of the universal war. *Journal of Social Science*, *2*(3), 248–257.

McNamara, R. S., & VanDeMark, B. (1995). *In retrospect: The tragedy and lessons of Vietnam*. New York.

Nardi, E., & Steward, S. (2003). Is mathematics TIRED? A profile of quiet disaffection in the secondary mathematics classroom. *British Educational Research Journal*, *29*(3), 345–367. https://doi.org/10.1080/01411920301852

O'Mahony, S. (2017). Medicine and the McNamara fallacy. *Journal of the Royal College of Physicians of Edinburgh*, *47*(3), 281–287.

Serow, P., Callingham, R., & Tout, D. (2016). Assessment of mathematics learning: What are we doing? In K. Makar, S. Dole, J. Višňovská, M. Goos, A. Bennison, & K. Fry (Eds.), *Research in mathematics education in Australasia 2012–2015* (pp. 235–254). Springer.

Shaw, J., Taylor, R., & Dix, K. (2015). *Uses and abuses of performance data in healthcare*. Dr Foster, 1–40. https://www.patientlibrary.net/cgi-bin/download/file/29702

Singh, T., & Shah, N. (2023). Competency-based medical education and the McNamara fallacy: Assessing the important or making the assessed important? *Journal of Postgraduate Medicine*, *69*(1), 35.

Thanheiser, E. (2023). What is the mathematics in mathematics education? *The Journal of Mathematical Behavior*, *70*, 101033.

Thompson, G., & Cook, I. (2014). Manipulating the data: Teaching and NAPLAN in the control society. *Discourse: Studies in the Cultural Politics of Education*, *35*(1), 129–142.

United States Department of Defence. (n.d.). *Office of the Secretary of Defence—Robert S. McNamara.* https://history.defense.gov/Multimedia/Biographies/Article-View/Article/571271/robert-s-mcnamara/

Watt, H. M. (2005). Attitudes to the use of alternative assessment methods in mathematics: A study with secondary mathematics teachers in Sydney, Australia. *Educational Studies in Mathematics*, *58*(1), 21–44.

Wu, M. (2010). The inappropriate use of NAPLAN data. *Professional Voice*, *8*(1), 21–25.

Yankelovich, D. (1972). *Corporate priorities: A continuing study of the new demands on business.* Yankelovich Inc. Stamford CT.

Zane, T. W. (2009). Performance assessment design principles gleaned from constructivist learning theory (Part 2). *TechTrends*, *53*(3), 86.

Changes in Year 11 Mathematics Students' Choices About the Use of a Computer Algebra System (CAS) to Solve Routine Problems

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This paper reports changes in the extent to which a group of Year 11 students used a Computer Algebra System (CAS), pen-and-paper (P&P), or a combination of both, when solving routine problems across seven months in four different topics. Comparing the frequency of CAS use across topics shows students made greater use of CAS in the topics of *Linear, quadratic, and cubic functions* and *Exponential and logarithmic functions* than in the topics of *Trigonometry and circular functions* and *Calculus*. These differences suggest that students either made different choices about the use of CAS in these topics or used CAS less frequently as they gained experience with CAS.

Students with access to a Computer Algebra System (CAS) need to make effective decisions about when to use CAS, pen-and-paper (P&P), or a combination of both, when solving mathematics problems (Pierce, 2001). These decisions could be based upon considerations such as their perceptions of the speed of CAS, the difficulty of the problem, and their P&P facility (e.g., Ball & Stacey, 2005). However, making effective choices about the use of CAS can be problematic for students who are new to working with CAS (Thomas et al., 2004). Despite this difficulty, few studies (e.g., Guin & Trouche, 1999; Orellana, 2016) have explored how students' choices about the use, or non-use, of CAS change as they gain experience.

For the students in this study, technology (i.e., a handheld CAS) was to be embedded into the teaching, learning, and assessment of mathematics (VCAA, 2015). Consequently, students needed to make decisions about when the use of CAS would be useful for supporting their learning of mathematics and solving of problems. This paper builds upon existing literature by exploring changes in students' use, or non-use, of CAS across seven months in one school year.

Literature Review

Artigue (2002) uses theory of instrumental genesis to explain how an artefact is transformed into an instrument through the dual processes of instrumentation and instrumentalisation. An artefact describes an object that can be used as a tool to complete tasks (e.g., a calculator, pencil, or algebraic symbol), while an instrument is "… a mixed entity, part artefact, part cognitive schemes" (p. 250). These cognitive schemes, formed through the instrumentation process, link an individuals' mathematical thinking and their gestures, where gestures describe the actions (i.e., functions used, buttons pressed) that are used to complete a given task (Trouche, 2005). The development of these schemes requires (i) technical understanding of how the technology can be used (e.g., knowing there is a *Solve* command that can solve equations); (ii) understanding of the affordances of the technology (e.g., CAS can perform routine calculations quickly and thus save time); and (iii) understanding of the constraints of the technology. Together, these understandings shape the use of an instrument.

The instrumental genesis of CAS has been described as a "time consuming and lengthy process" (Bukhove & Drijvers, 2010, p. 48), as students need to develop new cognitive schemes

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 127–134). Gold Coast: MERGA.

that allow them to complete tasks, while also considering how CAS should be used (Trouche, 2005). Through instrumentation, schemes are either developed personally or adopted from pre-existing social schemes, such as uses demonstrated by a teacher or peer (Artigue, 2002). While there can be a social element to instrumental genesis, the overall development of schemes and techniques is unique to an individual (e.g., van Dijke-Droogers et al., 2021). Consequently, different students may make different choices about whether to use CAS based on their cognitive schemes, knowledge of techniques, technical facility, and understanding of affordances and constraints of CAS.

When working with CAS, students can experience an "explosion of possible techniques" (Artigue, 2002, p. 260) that can be used to complete tasks due to the availability of a range of CAS techniques that can be added to existing P&P techniques. Where multiple techniques are available, students select one based on their evaluation of the pragmatic and epistemic value assigned to each technique (Artigue, 2002). In the early stages of instrumental genesis, some students tend to favour the use of CAS rather than P&P (Guin & Trouche, 1999). As students progress through instrumental genesis, they become more selective in their use of CAS and reintroduce P&P techniques (Guin & Trouche, 1999). The reintroduction of P&P techniques, and subsequent need to choose between CAS or P&P, requires judicious use of CAS which involves consideration of the effectiveness of CAS approaches for solving problems and making effective choices about the use of CAS based on these considerations (Pierce, 2001).

Several studies have explored students' CAS use from a range of perspectives. Analysis of self-reported frequency of CAS use suggests that students believe they frequently use CAS in class (Kissane et al., 2015; Orellana, 2016). Analysis of actual CAS use, through asking students to indicate where CAS has been used to perform a calculation, identified different ways that students used CAS when solving calculus problems (Thomas & Hong, 2005) and analysis of written responses to selected examination problems suggested that Year 12 students frequently used CAS to complete problems involving routine procedures (Ball, 2015). While these studies provide important insights into how students use CAS, existing studies were conducted in the context of a single unit of work (e.g., Thomas & Hong, 2005), with a single cohort of students who were experienced with CAS (e.g., Ball, 2015), or from different cohorts with different levels of experience (e.g., Orellana, 2016). These studies provide insight into how students use CAS in a particular topic, or how their frequency of CAS use changes with experience. This study builds on this literature by investigating whether students make different choices about the use of CAS when completing problems in different topics.

The research question for this study is: For Year 11 Mathematics students, how does their frequency of CAS use differ across the topics of: (i) *Linear, quadratic, and cubic functions*; (ii) *Exponential and logarithmic functions*; (iii) *Trigonometry and circular functions;* and (iv) *Calculus*? In this paper, "CAS" refers only to the symbolic functionalities of a CAS calculator and "P&P" refers to the completion of a calculation using mental or pen-and-paper procedures.

Methodology and Research Design

The participants for this study were the students ($n = 13$) of a single Year 11 Mathematical Methods (VCAA, 2015) class in a suburban school in Victoria, Australia. The school was selected for the larger study as students were not required to use CAS prior to Year 11. This provided a suitable context for a larger study which investigated changes in CAS use in different topics (reported here) and as students gained experience with CAS (see Cameron, 2023). All students were novice CAS users with eight not using CAS prior to Year 11, and the remaining five making limited use of CAS prior to Year 11. Data were collected through four worksheets, completed across a period of approximately seven months in one school year, to determine how CAS use differed with experience and across the four major topics studied (see Table 1). All students used a Texas Instruments (TI) Nspire CX series CAS calculator. The use of CAS was supported by the teacher who had extensive experience in teaching mathematics with CAS.

Table 1

Worksheet Topics and Timing of Data Collection

Worksheet	Topic	Date of data collection
1	Linear, quadratic, and cubic functions	Week 5 of Term 2
2	Exponential and logarithmic functions	Week 5 of Term 3
3	Trigonometry and circular functions	Week 2 of Term 4
4	Calculus	Week 6 of Term 4

Development of Worksheets

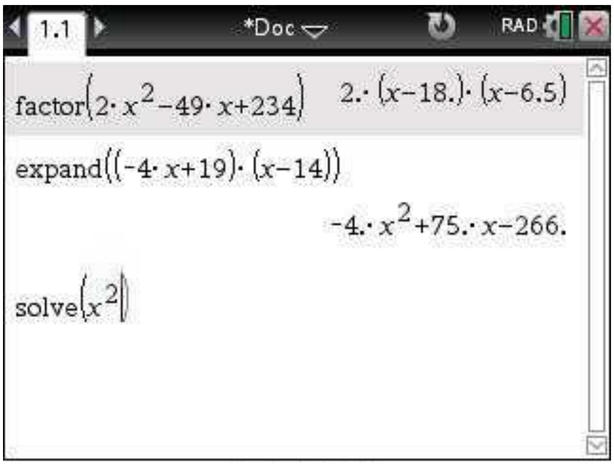
Four worksheets were developed (i.e., one per topic). The lead author developed worksheet items based on a review of curriculum documents and student textbooks; these items were reviewed by co-authors and the teacher. Items were predominantly routine and reflected the content that students had been learning in class. Worksheets included both problems where students could choose between CAS or P&P and others where it was expected that students could only solve the problem with CAS (i.e., where problems were outside the expected pen-and-paper range of students). The latter type of problems enabled investigation of the types of features of CAS used by students when solving problems within or outside the anticipated range of their P&P skills; this analysis is reported in Cameron (2023). Students completed the worksheets, under test conditions, at the conclusion of each topic. Students could choose to use CAS (or not) when completing items on each worksheet.

Collection of CAS Screenshots

While students were completing the worksheets, the TI Navigator system was used to record screenshots of the display of each student's CAS calculator (see Figure 1). The use of the Navigator system required the provision of a TI Nspire CX CAS calculator on each student's desk as the researcher needed to setup the Navigator system prior to students commencing the worksheet. While the provision of a calculator on the desk may have prompted CAS use, overall, this method was non-intrusive as it did not interfere with students' usual ways of working with CAS. Students were asked not to delete or clear any calculations so the researcher could download and review their CAS use following completion of the worksheet in the case where the Navigator system failed (as occurred when collecting data for Worksheets 2 and 3).

Figure 1

A Screenshot of a Student's CAS Display Collected With the Navigator System



Analysis of Worksheet and Screenshot Data

In the initial phase of analysis, written responses recorded on the worksheets and data collected from screenshots were analysed independently. Each student’s written response for each worksheet item was analysed using Ball’s (2015) indicators of CAS use. The indicators enabled CAS use to be inferred based on the appearance or absence of specific features in the written response (e.g., the appearance of CAS syntax, the absence of anticipated P&P working). This provided initial categorisation for each solution (Table 2). CAS screenshots were then reviewed for each student to identify evidence of a calculation being attempted with CAS (e.g., the use of CAS to solve in Figure 1). Following these independent analyses, evidence of CAS use in written responses and screenshots were compared for each student and for each problem to code the student’s approach to the problem as one of five calculation methods (see Table 2). Figure 2 provides an example of this analysis. This comparison provided a form of methods triangulation which serves to support the credibility and trustworthiness of the findings reported here (Hastings, 2014). Following coding of each students’ approach to each problem on each worksheet, the frequency, and percentage for each calculation method for all students on each worksheet were calculated; this enabled differences in use or non-use of CAS to be identified.

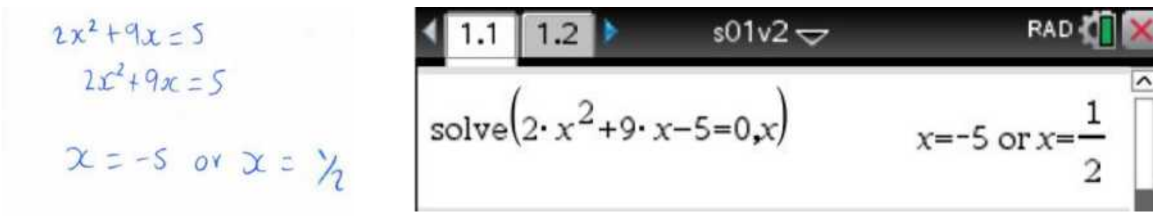
Table 2

Description of Calculation Methods

Calculation method	Description
CAS only	A written response was/was not provided There are indicators of CAS use in the written response and/or screenshots provide evidence of CAS use No intermediate P&P working was recorded
CAS with P&P	A written response was provided There are indicators of CAS use in the written response and/or screenshots provide evidence of CAS use Intermediate P&P working was recorded
P&P	A written response was provided There are no indicators of CAS use in the written response Screenshots do not provide evidence of CAS use Intermediate P&P working was recorded
Graphical	A written response was/was not provided There are no indicators of CAS use in the written response Screenshots provide evidence that graphical functionalities were used
No data	A written response was not provided Screenshots do not provide evidence of CAS use

Figure 2

An example of ‘CAS only'. Although a Written Response is Provided, the Absence of Intermediate Working Suggests CAS was Used; This was Confirmed With the CAS Screenshot



Results and Discussion

Table 3 provides the frequency and percentage of the calculation methods evident in students' approaches to the problems on each worksheet. In the first worksheet, 190 instances of *CAS only* were identified, and this is 56% of the total 338. Combining the top two rows for Worksheet 1 gives a total of 75% for *CAS [Total]*, as shown in the last row. Comparing the percentage of CAS [Total] across the four worksheets provides a broad understanding of how students' CAS use changed across the four worksheets. Despite the different topics, the frequency of CAS use was similar for problems from the topics of Linear, quadratic, and cubic functions and Logarithmic and exponential functions (75% cf. 73%). In contrast, CAS was used less frequently when completing problems from Trigonometry and circular functions and Calculus (19%, 27% respectively). The decreased percentage of CAS [Total] in the topics of Trigonometry and circular functions and Calculus is explained by an increase in the frequency of P&P only on these worksheets compared to the other two topics. Overall, this broad analysis suggests students less frequently used CAS as they gained experience, however, as discussed below, this change may be a result of the different topics. When comparing the frequency of calculation methods across worksheets, it is important to note the increased percentage of No data for Trigonometry and circular functions and Calculus. Many of these increases are attributable to student absences (rather than students not attempting problems) as two students did not complete Worksheet 3, and one did not complete Worksheet 4, so all calculation methods in these topics are underreported. Further, there was an unavoidable delay between students studying Trigonometry and circular functions and completing the worksheet; this delay may have resulted in an increased incidence of students not completing the more challenging problems due to a loss of proficiency.

Table 3

Summary of Calculation Methods for Four Worksheets

Calculation method	1. Linear, quadratic, and cubic functions ($n_1 = 338$)	2. Logarithmic and exponential functions ($n_2 = 182$)	3. Trigonometry and circular functions ($n_3 = 260$)	4. Calculus ($n_4 = 195$)
CAS only	190 (56%)	101 (55%)	50 (19%)	19 (10%)
CAS with P&P	64 (19%)	33 (18%)	0 (0%)	33 (17%)
P&P only	53 (16%)	41 (23%)	116 (45%)	105 (54%)
Graphical	0 (0%)	4 (2%)	2 (1%)	1 (0%)
No data	31 (9%)	3 (2%)	92 (35%)	38 (19%)
Total	338 (100%)	182 (100%)	260 (100%)	195 (100%)
CAS [Total]	254 (75%)	134 (73%)	50 (19%)	52 (27%)

Note. n is calculated by multiplying the number of items on a worksheet by 13. *CAS [Total]* is the sum of *CAS only* and *CAS with P&P*. Shading indicates the modal calculation method.

There are two different perspectives which may explain the variation in frequency of CAS use evident across the four worksheets, either (i) the mathematical content of each worksheet, or (ii) increased experience with CAS.

Potential Influence of the Mathematical Content of Each Worksheet

Differences in the frequency of CAS use across the four worksheets suggested that the topics Linear, quadratic, and cubic functions and Exponential and logarithmic functions (i.e., Worksheets 1 and 2) may have had more opportunities to use CAS than Trigonometry and circular functions and Calculus (i.e., Worksheets 3 and 4). All worksheets were designed to allow students to choose between CAS and P&P for all items, so the reduced frequency of CAS use on Worksheets 3 and 4 did not stem from the design of the worksheets.

A teacher's choices about the use of CAS or P&P, including the bounds placed around student CAS use, can impact student CAS use (Kendal & Stacey, 2001). Therefore, if the teacher made different choices about how they and the students should use CAS in different topics, it could be expected that students would make different choices about how and when to use CAS on each of the four worksheets. Although data were not collected from the teacher, students described the teacher demonstrating CAS features and supporting CAS use throughout the study (Cameron, 2023), so it is unlikely that different choices made by the teacher caused the differences observed on the worksheets.

Each of the topics studied by students required the learning and use of a range of different CAS commands and syntax. Learning how to use CAS commands and syntax can present barriers that impact students' CAS use. For example, Meagher (2012) reported students who avoided the use of CAS due to anxiety caused by difficulty in navigating and using CAS commands. Different topics may present different difficulties for students, with Pierce (2001) reporting that university students more frequently experienced difficulties with CAS when studying trigonometry than for other topics. Considering Meagher's and Pierce's findings, it is possible that the students in this study found the use of CAS more difficult in the topics of Trigonometry and circular function and Calculus (two topics that were new to students), and thus used CAS less frequently on these worksheets to avoid these difficulties.

Cameron et al. (2023) reported the students' beliefs about useful features of CAS (as part of a larger study). These beliefs reflected their overall experiences of working with CAS as a tool for the everyday learning of mathematics and many students believed that CAS was useful in mathematics across the study. Approximately half of the students believed that it was easier to use CAS than P&P to solve problems (5 of 11 at the start of the study cf. 7 of 11 at the end), so it is unlikely that students experienced greater difficulties when completing problems with CAS in Calculus compared to Linear, quadratic, and cubic functions. Overall, this suggests that the mathematical content of the worksheets did not influence students to make different choices about the use of CAS when solving problems from different topics.

Potential Influence of Experience on Frequency of CAS Use

An alternative explanation for the decreased frequency of CAS use on Worksheets 3 and 4 compared to Worksheets 1 and 2 is that students less frequently used CAS as they gained experience with CAS. It was expected that the P&P facility of these students would increase throughout the study as the development of P&P skills features explicitly in the outcomes of MM, with several key skills for Units 1 and 2 specifying the use of 'by hand' approaches (VCAA, 2015). Hence, a decrease in the frequency of CAS use could be expected as students develop facility with P&P skills. We also anticipated an increase in CAS facility throughout the study as students learnt more about CAS and how it could be used. Through the process of instrumental genesis, students are expected to become more selective in their choice of techniques (Artigue, 2002). As students gain experience with CAS and progress through the instrumentation process, they reintroduce P&P techniques despite displaying a propensity for working with CAS in the initial phase of the instrumentation process (Guin & Trouche, 1999). The reintroduction of P&P techniques may result in a decrease in the frequency of CAS use. The differences reported here suggest that, for many students, this change occurred approximately six months after commencing learning mathematics with CAS.

Students' Limited Use of CAS with P&P

Another trend event in the data was the comparatively infrequent use of CAS with P&P when compared to CAS only or P&P only in all topics except Calculus. The CAS with P&P

calculation method involved the use of both CAS and P&P when completing a problem (see Figure 2). These approaches are reported to be less common than the use of only CAS or P&P methods (Thomas & Hong, 2005; Weigand & Weller, 2001). Results from Worksheets 2 and 3 are consistent with the findings of Thomas and Hong (2005) and Weigand and Weller (2001), as CAS with P&P was evident in a smaller percentage of approaches than CAS only and P&P only (18% cf. 55% and 23% for Worksheet 2; 0% cf. 19% and 45% for Worksheet 3). However, results from Worksheets 1 and 4 provide a contrast with both CAS with P&P and P&P only evident in a similar percentage of approaches on Worksheet 1 (19% cf. 16%), and with CAS with P&P evident in a greater percentage of approaches than CAS only on Worksheet 4 (17% cf. 10%). It is important to note that few items, except for some on Worksheets 1 and 4, required the application of more than one mathematical procedure, so it was difficult for students to concurrently use CAS and P&P when completing a problem. Overall, when excluding Worksheet 3, CAS with P&P accounted for approximately one-fifth of approaches to problems on each worksheet, which contrasts with Weigand and Weller's finding that use of an "integrated working style" is "rare" (p. 99). In many cases, responses categorised as CAS with P&P reflect students completing a problem with P&P and using CAS to check their answer (Cameron, 2023), so CAS and P&P were used independently as students completed a problem.

Conclusion

In summary, *CAS only* was the most frequently identified method in students' responses to problems on Linear, quadratic and cubic functions and Exponential and logarithmic functions, while P&P only was the most frequently identified method for Trigonometry and circular functions and Calculus. This change corresponded with a reduction in the frequency of CAS use (as evidenced by a reduction in the percentage of CAS [Total] on Worksheets 3 and 4 when compared to Worksheets 1 and 2) and suggested a change from preferencing the use of CAS to the use of P&P. The most likely explanation for the differences in the frequency with which CAS was used on the worksheets was that students were becoming more selective in their CAS use as they progressed through the instrumentation process rather than due to the topic being studied. Guin and Trouche (1999) reported that students in the first phase of instrumental genesis often demonstrate a dependence on CAS, prior to the second phase where P&P methods are reintroduced and there is less reliance on CAS. In contrast to what was expected, students did not demonstrate a prolonged dependence on CAS, with the use of P&P calculation methods being more common than CAS within approximately six months of commencing to learn mathematics with CAS.

Prior studies have reported either increases in frequency of CAS use with experience (Weigand & Bichler, 2010) or no relationship between frequency of CAS use and experience (Orellana, 2016). In contrast, this study found a decrease in CAS use as students gained experience. Although there might be a concern that students working with CAS defer to CAS for solving all problems, rather than make choices about use of CAS or P&P, the results presented here show that for the small sample of 11 students in this study this concern would be unwarranted. Further research could investigate whether the results presented here, where use of CAS decreased as students developed P&P facility, was a wider phenomenon and applicable to other cohorts.

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References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*(3), 245–274. https://doi.org/10.1023/A:102210390

Ball, L. (2015). *Use of Computer Algebra Systems (CAS) and written solutions in a CAS allowed Year 12 mathematics subject: Teachers' beliefs and students' practices* [Doctoral dissertation, The University of Melbourne]. Minerva. http://hdl.handle.net/11343/42231

Ball, L., & Stacey, K. (2005). Students' views on using CAS in senior mathematics. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building Connections: Theory, research and practice. Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia* (pp. 121–128). MERGA.

Bokhove, C., Drijvers, P. (2010). Digital Tools for Algebra Education: Criteria and Evaluation. *International Journal of Computers for Mathematical Learning*, *15*, 45–62. https://doi.org/10.1007/s10758-010-9162-x

Cameron, S. (2023). *Computer algebra systems in a year 11 mathematics class: Students' use, attitudes, and factors percieved to influence use*. [Doctoral dissertation, The University of Melbourne]. Minerva. http://hdl.handle.net/11343/337647

Cameron, S., Ball, L., & Steinle, V. (2023). The stability of mathematics students' beliefs about working with CAS. *Mathematics Education Research Journal*, *36,* 471–491, https://doi.org/10.1007/s13394-023-00456-y

Guin, D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning*, *3*(3), 195–227. https://doi.org/10.1023/A:1009892720043

Hastings, S. L. (2012). Triangulation. In N. J. Salkind (Ed.), *Encyclopedia of research design*. SAGE. https://methods.sagepub.com/reference/encyc-of-research-design/n469.xml

Kissane, B., McConney, A., & Ho, K. F. (2015). *Review of the use of technology in mathematics education and the related use of CAS calculators in external examinations and in post school tertiary education settings*. School Curriculum and Standards Authority.

Meagher, M. (2012). Students' relationship to technology and conceptions of mathematics while learning in a computer algebra system environment. *International Journal for Technology in Mathematics Education*, *19*(1), 3–16.

Orellana, C. (2016). *Investigating the use of CAS calculators by senior secondary mathematics students* [Doctoral dissertation, Monash University]. Figshare. https://figshare.com/articles/Investigating_the_use_of_CAS_calculators_by_senior_secondary_mathematics_students/4696885

Pierce, R. (2001). *An exploration of algebraic insight and effective use of computer algebra systems* [Doctoral dissertation, The University of Melbourne]. Minerva. http://hdl.handle.net/11343/39022

Thomas, M. O. J., Monaghan, J., & Pierce, R. (2004). Computer algebra systems and algebra: Curriculum, assessment, teaching, and learning. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of Algebra* (pp. 153–186). https://doi.org/10.1007/1-4020-8131-6

Thomas, M. O. J., & Hong, Y. Y. (2005). Learning mathematics with CAS calculators: Integration and partnership issues. *Journal of Educational Research in Mathematics*, *15*(2), 215–232. https://koreascience.kr/article/JAKO200502637181734.page

Trouche, L. (2005b). Instrumental genesis, individual and social aspects. In D Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 197–230). https://doi.org/10.1007/b101602

VCAA. (2015). *Study summary: Mathematics 2006–2015*. VCAA. http://www.vcaa.vic.edu.au/Documents/vce/mathematics/mathematicsstudysum.doc

Weigand, H.-G., & Bichler, E. (2010). Symbolic calculators in mathematics lessons: The case of calculus. *International Journal for Technology in Mathematics Education*, *17*(1), 3–15.

Weigand, H.-G., & Weller, H. (2001). Changes of working styles in a computer algebra environment—The case of functions. *International Journal of Computers for Mathematical Learning*, *6*(1), 87–111. https://doi.org/10.1023/A:1011482007276

van Dijke-Droogers, M., Drijvers, P., & Bakker, A. (2021). Statistical modeling processes through the lens of instrumental genesis. *Educational Studies in Mathematics*, *107,* 235–260. https://doi.org/10.1007/s10649-020-10023-y

What Kind of Mathematics Teacher is ChatGPT? Identifying the Pedagogical Practices Preferenced by Generative AI Tools When Preparing Lesson Plans

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In their daily work teachers are responsible for several complex tasks; might AI be harnessed to support teachers in the challenging work of planning lessons? In this paper we investigate the use of an AI tool, namely ChatGPT, to generate a lesson plan that may be of use to teachers in their planning. A carefully worded prompt, informed by research, was used to generate four lesson plans for the teaching of division of fractions to students in years 7 or 8. We analysed the plans' structures and encoded practices. AI-generated lesson plans appeared suitable for identifying *what* should be taught but lacked detail of practices that support teachers to teach meaningfully.

Generative Artificial Intelligence (AI) tools (e.g., OpenAI ChatGPT, Google Gemini) have the potential to revolutionise teaching and learning through their capacity to generate a range of text types in a matter of moments (Sabzalieva & Valentini, 2023). When considering educational contexts, AI can be assigned a range of roles. AI can act as a 'guide on the side' when used to support teachers in the generation of classroom materials and advise on the learning sequence of specific concepts. AI can also act as a 'co-designer' of teaching materials when prompted to provide input into the design of curriculum materials (Sabzalieva & Valentini, 2023). Given the significant workload associated with the development of teaching materials (Hunter & Sonnemann, 2022), and the desire from teachers to have more time for lesson planning rather than using stock lesson plans (Stacey et al., 2023), the use of AI to generate teaching materials, including lesson plans, may be beneficial to teachers. We posit that teachers might be able to develop a lesson plan efficiently through using AI to develop a draft that is then refined for use in their classroom.

Although AI can generate a range of materials to support teaching and learning, the classroom teacher remains the expert in selecting instructional materials for use in their class. Often, teachers' choices are based on their pedagogical alignment with the content or focus of the instructional materials (Remillard, 2005). However, AI tools may be likened to a 'black box' where users cannot see or understand how outputs were generated, or the choices that resulted in a given output (Bearman & Ajjawi, 2023), thus potentially making it difficult for teachers to identify the pedagogies that informed the AI output. This paper reports a preliminary attempt to identify the preferred pedagogical practices of a generative AI tool.

Literature Review

Working with AI

AI tools use large language models to learn from a range of training material to complete tasks as requested by an end user. In the case of ChatGPT, the model has been trained on a range of material that is either publicly available on the internet, licenced from third parties, or provided by users and human trainers (Open AI, n.d.). Given the range of publicly available online teaching resources, we expect that this training material has captured an extensive array of lesson plans, curricula, and teaching materials. By analysing these data and identifying word associations, AI tools can generate text in response to a request by a consumer.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 135–142). Gold Coast: MERGA.

Generative AI tools produce text in response to a stimulus. The output contains "a judgement about an optimal course of action" (Bearman & Ajjawi, 2023, p. 1160) informed by the content of the training material. Given the complexity of the algorithms that result in such outputs, Bearman and Ajjawi argue that the decisions that underpin this course of action cannot be observed by the end user. Consequently, it is left to the consumer to develop an approach for working with AI tools. This view is shared by the Commonwealth of Australia (2023) who recognise the role of teacher expertise in using generative AI tools to support and enhance teaching and learning; teachers (and not AI) are recognised as the subject matter experts within the classroom. When working with AI, teachers will need to familiarise themselves with its potential affordances and constraints before they are able to capitalise its use (Su & Yang, 2023). This familiarisation involves new knowledge, such as the development of effective prompts for different tasks, and critical skills for interpreting and refining AI outputs (Commonwealth of Australia, 2023). The establishment of an evidence base, critical in supporting the work of teachers in using AI, is currently lacking across a range of educational contexts (Su & Yang, 2023). The preliminary work reported in this paper aims to assist teachers in understanding and interpreting AI-generated lesson plans.

Teachers as Pedagogical Experts

Alexander (2008) defines pedagogy as the *act* of teaching and connects this practice with discourses of educational theories, values, and evidence, "It is what one needs to know, and the skills one needs to command, in order to make and justify the many different kinds of decisions of which teaching is constituted" (p. 47). Shulman (1987) agrees that expert teaching is characterised by careful management of students and of *ideas* within classroom discourse. Together, these descriptions of teaching practice highlight the complexity of classroom practices where educational objectives are mitigated by specific skills and knowledge that are necessary requirements for the act of teaching. Alexander's metaphorical description of pedagogy as a "deep pool" (2015, p. 253) recognises the inherent challenge of capturing and defining these classroom practices that enable and support both teaching and learning processes.

Teachers develop sophisticated practices that represent the accumulated 'wisdom' of their professional experiences in the classroom (Shulman, 1987). We understand that this pedagogical expertise is formed, refined and powerfully influenced by one's experiences as learners (Schweisfurth, 2015), as teachers (Shulman 1987), and the professional practical experience in training developed to strengthen novices understanding of the nexus between theory and the practice of teaching (Darling-Hammond, 2017).

In contrast, the expected practices evident in AI generated lesson plans will be drawn from existing text-based resources and will be detached from any experiential learning. Consequently, there may be a disconnect between the pedagogical practices of AI and the pedagogical practices of an experienced classroom teacher.

Research Design

The following research question drove the preliminary research presented in this paper:

- What pedagogical practices are preferenced by ChatGPT in the development of lesson plans relating to the division of two fractions?

Rationale for Choice of AI Tool and Lesson Focus

ChatGPT was selected as the AI tool for this study as it is freely available and accessible to teachers. ChatGPT continues to be the most frequently visited AI tool on the internet (Similarweb, n.d.), thus suggesting that ChatGPT would be the most likely AI tool of choice if teachers were to use AI tools to support their lesson planning.

Division of fractions was chosen as the context for this preliminary research as this area of the curriculum is considered difficult for students to learn, and challenging for teachers to teach (e.g., Siemon et al., 2015). Consequently, division of fractions presents as a suitable topic for this analysis as teachers are more likely to draw upon a range of resources to develop their teaching ideas and pedagogical content knowledge; AI presents as one such resource.

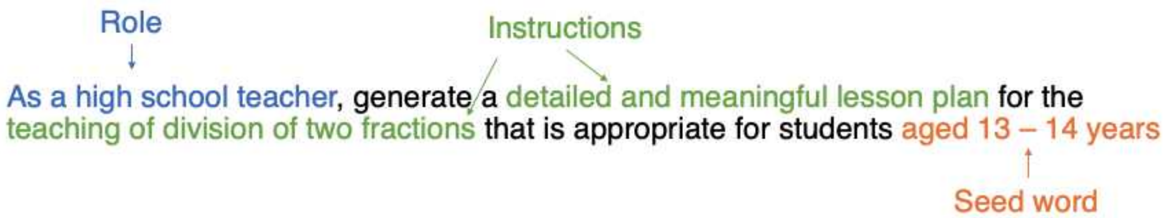
Data Generation

We used ChatGPT to prepare four lesson plans on the division of fractions. ChatGPT explains that its responses "can vary based on a variety of factors, including the context of our conversation, the details you provide in your questions, and any updates or changes to my training data" (OpenAI, 2024a). To account for this potential variation, the same prompt was used, and each lesson plan was requested in a different conversation. Lesson plans were generated on different days to allow for any differences in updates or training data to become apparent as it was expected that these changes may shift the preferred practices of ChatGPT.

Preliminary research by Spasić & Janković (2023) has explored how the structure of the input prompt shapes the quality of the generated lesson plan. Informed by their findings, we developed a prompt (see Figure 1) which specifies a **role** for ChatGPT and includes **instructions** and **seed words**. The inclusion of a role, instructions and seed words results in the generation of a lesson plan that is more detailed than if any of the three elements of the prompts are absent.

Figure 1

Prompt Used to Generate Lesson Plans



Coding of Lesson Plans

Four lesson plans were generated using the prompt in Figure 1. We used findings from The International Classroom Lexicon Project (Mesiti, Artigue, et al., 2021) that identified a set of pedagogical terms, the Australian Lexicon, which teachers use to describe the practices of mathematics classrooms (Mesiti, Hollingsworth, et al., 2021). When offered the 61 terms from the Australian Lexicon, 52 Victorian mathematics teachers were asked to reduce the terms to 'ten essential terms' (Mesiti et al., 2019). The responses were sorted according to frequency and a set of 15 terms were identified: *Assessment*, *Demonstrating*, *Differentiating*, *Engaging*, *Feedback*, *Formative Assessment*, *Group Discussion*, *Group Work*, *Modelling*, *Practising*, *Questioning*, *Reasoning*, *Reflecting*, *Scaffolding* and *Worked Example*. Each of these pedagogical terms have been operationalised with a short description, examples, and non-examples elsewhere (Mesiti et al., 2021b). These 15 terms were adopted as codes for the AI-generated lesson plans. The lesson plans that were generated in response to the prompt listed general 'instructions' under organisational headings. For example, the 'Warm Up' for Lesson Plan 1 stated:

- Review the concept of multiplying fractions briefly;
- Ask students to solve a multiplication problem involving fractions, such as $\frac{2}{3} \times \frac{3}{4}$;
- Discuss the steps involved in multiplying fractions (OpenAI, 2024b).

For two of the lesson plans, each instruction was coded by both researchers as they negotiated their understanding of the pedagogical terms. The other two lessons were coded independently. Coding was determined by the alignment of text from the lesson plan with the operationalised definition of the terms. For example, “Ask students to solve a multiplication problem involving fractions” was coded as Questioning and Practising.

Results and Discussion

The following sections report the results of two analyses related to the structure of, and pedagogical practices evident in, the four AI-generated lesson plans.

Lesson Plan Structure

The structure of each of the four lesson plans, determined from the section headings from the generated responses, is summarised in Table 1.

Table 1

Lesson Structure Headings in AI-Generated Lesson Plans

Heading	Lesson plan 1	Lesson plan 2	Lesson plan 3	Lesson plan 4
Pre-lesson plan headings		Title	Topic	
		Grade level	Grade level	
		Subject	Subject	
		Duration	Duration	
	Objective	Objective	Objective	Objective
	Materials needed	Materials	Materials needed	Materials needed
Lesson plan headings	Warm up	Warm up		
	Introduction to division of fractions	Introduction	Lesson introduction	Introduction
				Reciprocal fractions
		Direct instruction		Example problems
	Guided practice	Guided practice	Guided practice	Guided practice
	Independent practice	Independent practice	Independent practice	Independent practice
	Real-world application		Real-world application	
	Closure	Closure	Closure	Closure
Post-lesson plan headings	Extension activity	Extension (optional)	Extension	Extension activity (optional)
	Assessment	Assessment	Assessment	Assessment
	Differentiation	Differentiation	Differentiation	Differentiation
			Integration	
	Homework	Homework		Homework
	Reflection			Reflection

While the AI lesson plans appear to adhere to a structural formula, particularly in the pre- and post-lesson sections, there is some variation in the sequence and content of the main body of the lesson plan. The provision of *objectives* aligns with the advice to teachers about the importance of establishing and providing learning goals that define for students the purpose of the lesson and desired achievements (e.g., State of Victoria, 2017). Where a *warm up* was provided, these focussed on a review of multiplication of fractions, thus incorporating the practice of connecting to prior knowledge which has been shown to support learning (Lovitt & Clarke, 2011). Common to all lesson plans were *introductions* focussed on reviewing the concept of fractions and/or division. For example, particular emphasis was given to division of whole numbers in Lesson Plan 4 (LP4), and conceptual understanding of division in LP3. In contrast, LP1 and LP2 focussed on explaining how fraction division is similar to multiplication, albeit with a "slight twist" (LP2).

All four lesson plans outline a step-by-step approach for the division of two fractions, namely, the common rule 'invert and multiply' (Davis & Pearn, 2009). This approach was included in the *introduction* (for LP1 and LP3), in a section named *direct instruction* (for LP2), and in a section named *reciprocal fractions* (for LP4). Similarly, the inclusion of *example problems* in LP4 is encapsulated within the *guided practice* sections of the other lesson plans. These two examples suggest a default in the pedagogical approaches generated by ChatGPT despite the different language expressions and structures of the lesson plans. The inclusion of *example problems* and *guided practice* is consistent with the identification of worked examples as an important teaching strategy (e.g., State of Victoria, 2017) and reflects research findings that highlights the common use of worked examples in the mathematics classroom (Große, 2015). Notably, only LP1 and LP4 specified the examples that should be used ($\frac{2}{3} \div \frac{1}{4}$ in LP1; $\frac{1}{2} \div \frac{1}{4}$ in LP4). The absence of several worked examples in these lesson plans, possibly a ChatGPT default, indicates a significant aspect of the lesson plan which would require teachers to draw upon their expertise and understanding of the affordances of worked examples (e.g., Chick, 2007) before delivering the lesson. Indeed, the example offered to illustrate division of fractions in LP1, is not necessarily an example that would be used by an expert teacher in the very first instance (see also Davis & Pearn, 2009).

The inclusion of post-lesson materials was unprompted, and the consistency of structure again suggests a standard approach for the generation of lesson plans. In all lesson plans, the only evidence of *extension* and *differentiation* is in these post-lesson sections.

Practices Evident in the Lesson Plans

Table 2 and Figure 2 summarise the results of the coding of the four lesson plans against the *essential* terms of the Australian Lexicon, offering insight into the pedagogical practices preferenced in the development of these lesson plans. The most frequent code, *Group Discussion* ($n = 36$), reflects a significant proportion of instructions in the lesson plans that require either a whole classroom discussion or a teacher explanation. The instructions in the lesson plans are brief and do not identify an agent, so we were required to anticipate the most likely classroom activity and setting. Accordingly, it was assumed that such discussions and explanations would involve *Reasoning*, contributing to its high frequency ($n = 25$). Notwithstanding these, included in the lesson plans were specific instances where *Reasoning* was identified as the key code, "Encourage students to explain their reasoning as they solve each problem" (LP2). The frequency of codes *Demonstrating* ($n = 13$), *Modelling* ($n = 11$) and *Questioning* ($n = 15$), align with a pedagogical preference for the use of 'explicit teaching', a High Impact Teaching Strategy (State of Victoria, 2017), for classroom instruction of division of fractions.

Practising, the third most coded essential term ($n = 23$), identifies student activity. Instructions coded against this essential term involved instructing students to complete problems, as well as those classroom activities where both the teacher and the students complete problems together. The appearance of this practice is consistent with the notion of 'collaborative learning'; another High Impact Teaching Strategy (State of Victoria, 2017).

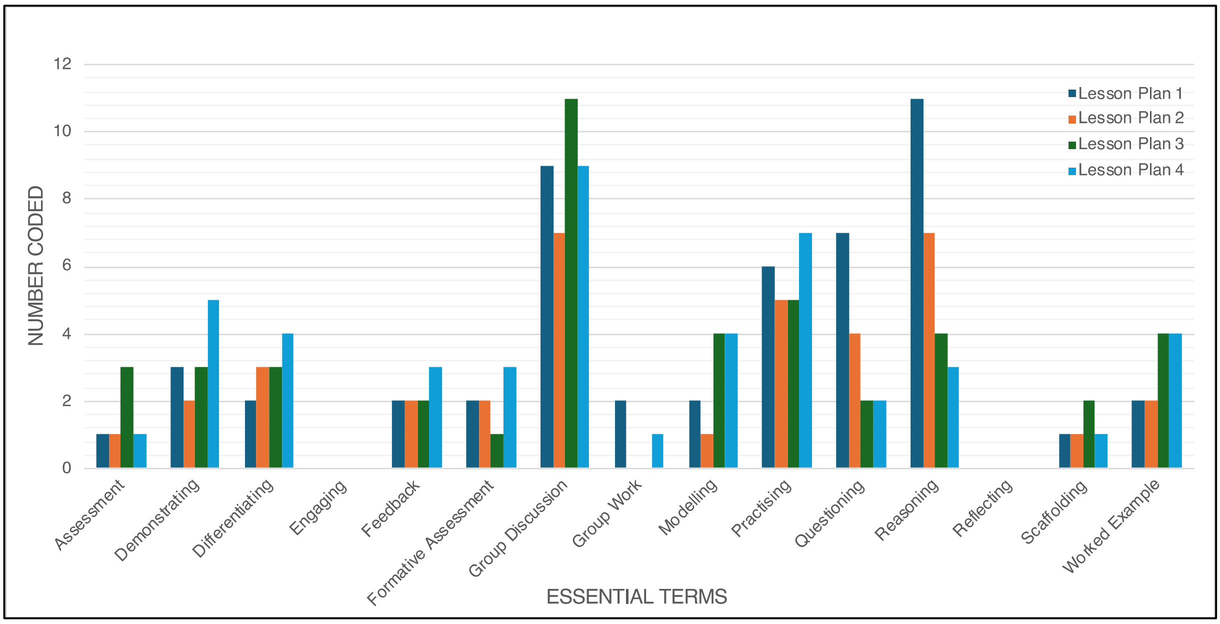
Table 2

Number of Essential Terms Coded in AI-Generated Lesson Plans

Essential terms	Lesson plan 1 (391 words)	Lesson plan 2 (298 words)	Lesson plan 3 (367 words)	Lesson plan 4 (406 words)	Total
Assessment	1	1	3	1	6
Demonstrating	3	2	3	5	13
Differentiating	2	3	3	4	12
Engaging	0	0	0	0	0
Feedback	2	2	2	3	9
Formative Assessment	2	2	1	3	8
Group discussion	9	7	11	9	36
Group work	2	0	0	1	3
Modelling	2	1	4	4	11
Practising	6	5	5	7	23
Questioning	7	4	2	2	15
Reasoning	11	7	4	3	25
Reflecting	0	0	0	0	0
Scaffolding	1	1	2	1	5
Worked example	2	2	4	4	12
Total	50	37	44	47	178

Figure 2

Number of Essential Terms Coded in AI-Generated Lesson Plans



Notably all but three *essential* terms were present in all lesson plans (*Engaging*, *Reflecting* and *Group Work*). The practice of *Engaging*, defined as "A student is actively involved with an educational experience, whereby he/she acts to maintain or extend their contact with the stimulus (typically, in order to increase their knowledge of it)" (Mesiti et al., 2021b, p. 44), is likely perceptible from teacher/student and student/student interactions rather than from lesson plans documenting the scope and structure of a lesson. The practice of *Reflecting*, defined as

"An activity in which students consider the effectiveness or progress of their learning (i.e., their developing knowledge, skills, and understandings)" (p. 49), was also absent, despite all lesson plans including a *closure*. The instructions related to the sections named *closure* generally involved reviewing and summarising what was covered in the lesson and assigning of homework. *Group Work,* whereby "Students work together to complete a given activity" (p. 45) was not evident in two of the four lesson plans. This absence may be due to the brevity of the lesson plans (ranging from 298 to 406 words) and although the instructions included in the lesson plans specified *what* should be taught, the *how* of teaching was more difficult to discern.

Conclusion

This paper sought to identify the pedagogical practices favoured by the AI tool, ChatGPT, when tasked with generating a lesson plan. The use of the fifteen *essential terms* (Mesiti et al., 2019), a subset of the Australian Lexicon (Mesiti, Hollingsworth, et al., 2021), proved useful in providing a framework through which the AI-generated lesson plans could be coded, and aided in the identification of pedagogical practices present and preferenced in the four lesson plans. Further analysis using the 61 terms of the entire lexicon would enable a more detailed description of the pedagogical practices preferenced in ChatGPT outputs and, we expect, further highlight the important role of teacher expertise in refining and implementing AI-generated lesson plans. It is worth noting the impact of prompts on outputs. Further research could explore the impact of different prompts on the development of lesson plans and provide guidance to teachers so they may develop effective prompts that support the generation of a lesson plan with greater detail with respect to explanations, examples, and suitable problems.

Despite slight differences in structure, the four lesson plans favoured the following pedagogical approach: key procedures or concepts are introduced, key steps and skills are demonstrated and illustrated with worked examples, and additional problems are set for students to complete. The key concept of 'division of fractions' was illustrated with the 'invert and multiply approach' separated from any meaningful representation. While this approach may be suitable in some contexts, it reflects a narrow view of the potential for mathematics teaching by appearing to favour traditional 'telling', stating of information or demonstrating of procedures (Smith, 1996), and practices for developing deep thinking, reflection and justification by students, remain absent. While several practices align with evidence-based approaches for teaching, these practices were general and did not incorporate evidence-based approaches that can support the learning of division of fractions (e.g., the use of a bar model; Yeap, 2011). The absence of such mathematical approaches, along with the absence of specific examples, explanations, and problems highlight the considerable expertise needed by teachers to refine and implement AI-generated lesson plans. In the case of lesson plans generated by ChatGPT, advice is provided for teachers on *what* to teach, but not *how* to teach it.

References

Alexander, R. (2008). *Essays on pedagogy.* Routledge. https://doi.org/10.4324/9780203609309

Alexander, R. J. (2015). Teaching and learning for all? The quality imperative revisited. *International Journal of Educational Development*, *40*, 250–258. https://doi.org/10.1016/j.ijedudev.2014.11.012

Bearman, M., & Ajjawi, R. (2023). Learning to work with the black box: Pedagogy for a world with artificial intelligence. *British Journal of Educational Technology*, *54*(5), 1160–1173. https://doi.org/10.1111/bjet.13337

Chick, H. (2007). Teaching and learning by example. In J. Watson, & K. Beswick (Eds.), *Mathematics: Essential research, essential practice. Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (pp. 3–21). Hobart: MERGA.

Commonwealth of Australia (2023). *Australian framework for generative artificial intelligence (ai) in schools.* Commonwealth of Australia. https://www.education.gov.au/download/17416/australian-framework-generative-artificial-intelligence-ai-schools/35400/australian-framework-generative-ai-schools/pdf

Darling-Hammond, L. (2017). Teacher education around the world: What can we learn from international practice? *European Journal of Teacher Education*, *40*(3), 291–309. https://doi.org/10.1080/02619768.2017.1315399

Davis, G. E., & Pearn, C. A. (2009). *Division of fractions.* Republic of Mathematics.

Große, C. S. (2015). Fostering modeling competencies: Benefits of worked examples, problems to be solved, and fading procedures**.** *European Journal of Science and Mathematics Education*, *3*(4), 364–375. https://doi.org/10.30935/scimath/9444

Hunter, J., & Sonnemann, J. (2022). *Making time for great teaching: How better government policy can help.* Grattan Institute. https://grattan.edu.au/report/making-time-for-great-teaching-how-better-government-policy-can-help/

Lovitt, C., & Clarke, D. (2011). The features of a rich and balanced mathematics lesson: Teacher as designer. *Educational Designer*, *1*(4), Article 15, 1–25.

Mesiti, C., Artigue, M., Hollingsworth, H., Cao, Y., & Clarke, D. J. (Eds.). (2021a). *Teachers talking about their classrooms: Learning from the professional lexicons of mathematics teachers around the world.* Routledge. https://doi.org/10.4324/9780429355622

Mesiti, C., Clarke, D. J., & van Driel, J. (2019). Describing and prescribing classroom practice: Do we have a common language? In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 492–499). MERGA.

Mesiti, C., Hollingsworth, H., Clarke, D. J., Reed, A.S., & Roan, K. (2021b). Australian Lexicon. In C. Mesiti, M. Artigue, H. Hollingsworth, Y. Cao, & D. J. Clarke (Eds*.)*, *Teachers talking about their classrooms: Learning from the professional lexicons of mathematics teachers around the world* (pp. 39–54). Routledge. https://doi.org/10.4324/9780429355622-3

OpenAI (2024a). *ChatGPT* (Feb 16 version) [Large language model]. https://chat.openai.com/chat

OpenAI (2024b). *ChatGPT* (Feb 18 version) [Large language model]. https://chat.openai.com/chat

OpenAI (n.d.). *How ChatGPT and our language models are developed.* https://help.openai.com/en/articles/7842364-how-chatgpt-and-our-language-models-are-developed

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*(2), 211–246. https://doi.org/10.3102/00346543075002211

Sabzalieva, E., & Valentini, A. (2023). *ChatGPT and artificial intelligence in higher education: Quick start guide.* UNESCO. https://unesdoc.unesco.org/ark:/48223/pf0000385146

Schweisfurth, M. (2015). Learner-centred pedagogy: Towards a post-2015 agenda for teaching and learning. *International Journal of Educational Development*, *40*, 259–266. https://doi.org/10.1016/j.ijedudev.2014.10.011

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–23. https://doi.org/10.17763/haer.57.1.j463w79r56455411

Similarweb (n.d.). *Top websites ranking*. https://www.similarweb.com/top-websites/

Smith, J. P. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education*, *27*, 387–402. https://doi.org/10.5951/jresematheduc.27.4.0387

Spasić, A. L., & Janković, D. S. (2023). Using ChatGPT standard prompt engineering techniques in lesson preparation: Role, instructions and seed-word prompts. In *58th international scientific conference on information, communication and energy systems and technologies*. IEEE. https://doi.org/10.1109/ICEST58410.2023.10187269

Stacey, M., Gavin, M., Fitzgerald, S., McGrath-Champ, S., & Wilson, R. (2023). Reducing teachers' workload or deskilling 'core' work? Analysis of a policy response to teacher workload demands. *Discourse: Studies in the Cultural Politics of Education*, *45*(2), 189–199. https://doi.org/10.1080/01596306.2023.2271856

State of Victoria (2017). *High impact teaching strategies; Excellence in teaching and learning*. Department of Education and Training, Victoria.

Su, J., & Yang, W. (2023). Unlocking the power of ChatGPT: A framework for applying generative AI in education. *ECNU Review of Education*, *6*(3). https://doi.org/10.1177/20965311231168423

Yeap, B.H. (2011). *Bar modeling–A problem-solving tool: From research to practice (an effective Singapore math strategy)*. Marshall Cavendish Education.

A School Mathematics Leader's Account of her Leadership

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Leading primary school mathematics involves stimulating teacher learning for the purpose of improving students' mathematical outcomes. Currently there are staff shortages and school mathematics leadership may have changed. The results of a semi-structured interview with an experienced School Mathematics Leader who reflected on her work are reported here. She was asked about her "classroom work" and to describe her goals for mathematics learning. Fullan's (2001) *Framework for Leadership* was used to shape conclusions to the study around: moral purpose, knowledge building and sharing, coherence making, understanding change and relationship building.

We hypothesise that teachers learn "on the job" and, in fact, the job changes as societies change. The recent experiences of teachers during the pandemic have shaped and impacted on their teaching and on the learning of mathematics by students. The consequences of these experiences for the leaders of mathematics teachers are largely undocumented. This paper documents the views of a single leader as she reflected on her current work.

The research literature notes that School Mathematics Leaders influence mathematics teaching and learning in classrooms and schools by improving teacher practice (Faragher & Clarke, 2014). These leaders are knowledgeable, on-site practitioners who support teachers in mathematics teaching with curriculum assistance, lesson planning, evaluation of student work and assessment (Campbell & Malkus, 2013). School Mathematics Leaders provide the necessary support for teachers to learn and develop high-quality teaching practices through on-site, job-embedded professional learning opportunities (Gibbons & Cobb, 2017). Additionally, these leaders have the capacity to create positive, practical, and sustainable change and as curriculum leaders they play a key role in providing in-class support and professional learning related to mathematics content and pedagogy (Grootenboer et al., 2015).

Authors of studies such as Ingvarson (2005) and Tharp and Gallimore (1989), have examined the types of support primary School Mathematics Leaders provide. Supportive techniques such as modelling, questioning, explaining, management strategies, feedback, demonstration lessons, professional readings and the "opportunity to observe effective examples and effective practitioners" (Tharp & Gallimore, 1989, p. 24) have been noted. Males, et al. (2010) also found that mathematics teachers benefited from the experiences of conversations focused on student learning and data. Further, Gibbons and Cobb (2017) listed activities leaders used to support teachers, including: working with groups of teachers, engaging in the discipline, examining student work, analysing classroom video, engaging in lesson study, and working with individual teachers by co-teaching, and by modelling instruction.

While it is generally accepted that continuous professional learning is key to improving practice, it is important to understand ways in which teachers learn to develop and refine their teaching practice (Hollingsworth & Clarke, 2017). Teachers in several studies indicated the importance of support from School Mathematics Leaders in their classrooms (e.g., Butler et al., cited in Clarke et al., 2013). Supportive practices such as observation, modelling, and lesson debriefing were viewed as some of the most valuable components of teachers' professional learning (Clarke et al., 2013). According to Ingvarson (2005) young teachers highly value the chance to see expert teachers at work and to get feedback from them about their own teaching. By working "shoulder to shoulder" the School Mathematics Leader can access and influence planning and teaching, model new practices, and provide informal feedback to teachers.

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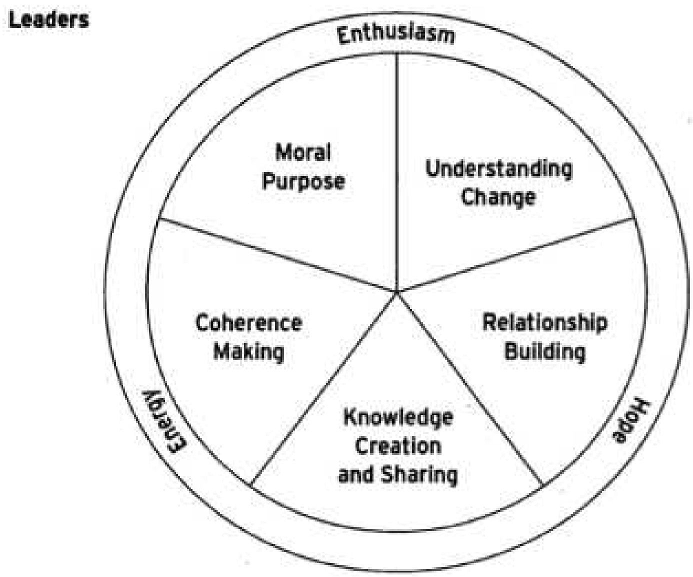
Leaders can create a protected environment where teachers feel confident to take risks and to learn. The intent of this study was to document details of a leader's work in the current environment.

Fullan (2001) claimed that ongoing improvement requires teachers and leaders work together as agents of change. The theoretical framework developed by Fullan (2001) and used to underpin large-scale reform of education in Canada and the United Kingdom and elsewhere, was used as a theoretical framework for the present study. In summary, the framework involving leaders is described as having five core components. These components include moral purpose which involves having a commitment to making a positive difference; understanding change which can be slow and difficult; relationship building whereby leaders foster purposeful interactions and the respect of others with diverse views; knowledge creation and sharing as an important component of effective leadership and coherence making which is complex and elusive. Pursuing moral purpose, understanding the change process, cultivating relationships, and building deep knowledge lead to greater coherence making (see Figure 1). These components are represented as a circular region of the framework with the personal characteristics of energy, enthusiasm and hope surrounding them as important characteristics of effective leaders.

Figure 1

A Framework for Leadership—Leaders (Fullan, 2001, p. 4)



The extent to which data reported in this paper can be interpreted with the Framework of Leadership will be considered as conclusions to this paper.

Method

Initially we had an informal professional conversation with a School Mathematics Leader, Kate (a pseudonym). We raised the possibility of conducting a formal interview with her to help us to understand her current work as a leader of mathematics in a primary school and Kate agreed to participate. The data reported here were collected using a semi-structured interview. The questions were designed to collect demographic details of the school, the length of Kate's experience as a School Mathematics Leader, and her approach to leading mathematics in her school. She was asked how she decided what to do and to describe her goal(s) for mathematics learning. Kate was asked what new learning she would like to see in her teachers and to describe any challenges she faced as the leader. The interview was conducted by the second author via Zoom. Video records and transcripts were produced by Zoom. The automated transcripts were checked and edited to remove repetitions of speech and provided the data that are described and discussed here. Each researcher read and coded the transcripts. Comments which detailed the leader's role, were highlighted and each researcher summarised the main point in a few words as margin notes. These brief notes were categorised according to themes which were used as headings in the Findings section here.

Our aim was to identify current features of the process of leading mathematics, collect an account of "classroom work", and create a narrative of the school mathematics leadership. The research question we sought to answer was:

- What can we learn from one leader about current leadership practices in primary school mathematics?

The focus of the interview was an element of her leadership that Kate called her "classroom work". We were interested to know about the components of practice called *teaching alongside* by Driscoll (2021). This practice involves mathematics leaders teaching with a teacher in the teacher's classroom with the teacher's students. Essentially, we hoped to have Kate describe what she did in classrooms to lead mathematics learning with teachers. Findings from the interview will be presented in a narrative style. The general points that Kate made will be presented succinctly with matched evidence from the transcript.

Findings and Discussion

Kate had told us in conversation, and later in interview, how important her "classroom work" was to her leadership of mathematics improvement in the school. She described how she had used assessment data available to the teachers to analyse the mathematical strengths and weaknesses of classes and Year levels across the school. In a large school of 480 students and 30 teachers she had to use her full-time mathematics leadership role strategically. Initially she targeted the classes in which students had very disappointing results on NAPLAN or PAT Maths measures. Having doubts about the mathematics content that the students were being offered, Kate decided to try to influence mathematics tasks through planning.

Support Through Planning

Planning meetings have been shown to be an effective vehicle for teacher professional learning (Driscoll, 2021). Kate described attending meetings to discuss "rich" tasks and making folders of activities available to teachers on shared drives:

> Planning—with the union [new enterprise bargaining agreements]—I've got to be invited into their planning time now rather than just come in at a specific time. So, there are some Year levels that are quite happy for me to come in. The ones where I want to make a difference, they want to get their planning done in their shared planning time … I find that challenging in itself. But I will drop things [task ideas] into their planning documents, I've got 'Kate folders' all over the place and some teachers will take the ideas up.

Kate was hoping that the examples of more open and more problem-solving tasks might inspire teachers to choose and use tasks that required deeper mathematical thought "I think in my mind it was let's try and get rich tasks in the classroom first. And then I was thinking the next step was to get into the classrooms a little bit more."

Kate found that while the tasks might be chosen by her teachers, they were not always implemented with the pedagogies that pressed and extended students' thinking. As Stein et al. (2009) had noted, teachers often simplified the problem to make it one that they felt confident to use with their students. Kate said, "I find you can do it in planning, and planning is good, but then also being able to do it in the classroom has just as much of an impact."

Demonstration Lessons

Developing their pedagogies to maximise student learning was considered difficult and teachers wanted demonstrations of teaching approaches. Kate commented:

> Sometimes the teachers want you to show them exactly how to do everything. So, then the real thinking, the evolving all the actual learning and doing it yourself disappears in a sense. … But getting the students to articulate what they did, and why they did what they did, is a big thing throughout the school.

Kate had realised that she needed to model tasks in teachers' classrooms with their students:

> It makes me stop and think and wonder. Teachers don't get enough opportunity to see other people in practice ... It's one thing (and don't get me wrong) planning is important and having a good task or a good lesson is very important. … But then at the same time, actually modelling that, is also very important.

Kate was aware that teachers were happy for her to take the responsibility for teaching their students. However, for them to teach in front of her, or with her, there needed to be a relationship built on trust, confidence, and secure mathematical knowledge. She realised moving from demonstrated lessons to co-teaching or teaching alongside others was a big step:

> I suppose you get a bit of "Oh, Kate did a really good job with this" and there's a bit of a vibe. So, they're quite happy for me to come and teach. I suppose the challenge is for them to teach in front of me and to get a little bit of feedback. It doesn't happen that naturally unless they're quite comfortable with me and confident in their own mathematical abilities.

Kate took heart from one of her teaching team who was a quiet source of encouragement:

> There was one teacher in particular and he was quite happy for me to observe him teach and I just worked with the kids as well. But he has been a maths leader before, so probably, his pedagogical content knowledge is high enough that he doesn't feel uncomfortable about me being there.

Teachers whose students' assessment results were not high, did not feel the same about teaching with Kate:

> The others [whose] PAT Maths data results didn't show as much growth in that Year cohort, they're more excited to see me teach … It's very awkward to find out what they're not doing because they don't like being observed. … Yeah, they don't want me to critique … they don't want me to even focus on one particular thing. When I go in … I'm trying to teach them something in a very subtle way.

Kate gave an example:

> The kids were sitting on set spots on the floor and I asked her to just cross everybody's name and as I asked questions, I said, "Look, I just want to check my own practice. I want to see if I'm covering lots of kids rather than just answering backwards and forwards with one. So, can you please mark who I'm actually talking to and how many times backwards and forwards." So, I do a few little things like that. In the end they actually think about it with their own practice as well.

Kate described the tasks she chose to implement as more challenging than those chosen by some teachers. She seemed to be trying to convince her teachers that their students are capable of more difficult mathematical thinking—that their expectations were too low. Perhaps straight grade school organisation can allow teachers to consider students as all at one place mathematically (in the intended curriculum).

> Some teachers still want to tell them [students] how to do it rather than asking them questions to get them to think. I get told that I choose tasks that are more challenging for the students than what the classroom teachers do. … I've noticed it's one of the challenges. I think teachers put things in boxes too much because I hear, oh, that's the Year 3 curriculum.

Targeting Parts of the Demonstration Lessons

A particular part of the lesson Kate mentioned was her work on the conclusion of the mathematics lesson where the intended learning is reviewed and the mathematical ideas that students have been grappling with are elicited and explained by the students and the teacher:

> I really focus on the *summarise* phase and allow that time. I say it's important and I suppose when I'm actually explaining what I'm doing with the students, because they're not used to me teaching. I use some strategies that I just naturally do. Like I say, "Turn and talk" and "Think, pair, share" and things like that because then they get an opportunity to talk amongst themselves before they answer back to me because that gives the kids confidence. So, I know I'm role modelling those strategies.

Modelling Curiosity and Experimentation as a Learner

Kate also modelled being willing to try new ideas by being inquisitive to learn more about teaching. She trialled an example of Liljedahl's (2020) management technique by bringing the students and the teacher along as participants:

> I just read something that's really good by Peter Liljedahl and we're all going to use the whiteboards … partners are going to be randomly selected and I actually tell the students, you know, this is this really good mathematician in Canada, and this is what he said and I'm going to give it a go with you guys. So, I'm really open with what I do … and we're going to see how well it works and I'm going to ask you what you think about it. So, I take them on the journey, but I'm also taking the teachers on the journey at the same time ... There's a purpose behind it.

Kate included the students in her willingness to evaluate new interesting tasks and to seek their views as well:

> I talk the students through it so that they understand, and I also tell them of tasks that are brand new because it's not like I just keep repeating what I'm doing. I'll tell the teacher. ... I've never taught this lesson before and I'll tell the students, [and ask] what did you think of that? Because they honestly sometimes think because I've been a maths leader for quite a while, that I'm the one with all the answers.

As well as being explicit about her role as a learner, Kate described some of the qualities she thought leaders need to have. She talked about the need to be curious, brave and vulnerable:

> You know, I'm still learning and growing too. So, there's a bit of that vulnerability but also that idea of I'm willing to try out something new in front of them as well so that they understand and explain, you know, sort of where it came from or the reasoning behind it.

Kate recognised that a reluctance to try new things may be due to fear of failure by very anxious teachers. She was aware that teachers need to come to terms with their need to build their knowledge for teaching:

> I'm the one that's going to fall or trip over or fail or whatever and especially the ones that are really anxious because they feel like, once they get over that fear that they see the students. Student's conversations with me or their responses [are seen] as a true reflection. That's the first fear. That their responses aren't good enough so it's a true reflection of their teaching straightaway. Once they get over that hurdle … then they can go, oh yeah, well, where do we move on to next?

Opportunities to Work in Classrooms

When asked about how often she worked in classrooms, Kate said that she had been in classrooms every day often teaching 7 or 8 lessons a week. She spoke about teaching versions of the same lesson 3 times across a Year level. Kate explained that, "the teachers are quite open to me coming in to teach lessons." However, Kate explained that on occasions:

> there have been times when I've tried to shift [responsibility for teaching], and it depends who the teacher is, … certain teachers are quite happy to teach and for me to support. If there was a bit of push back and I could see that a certain individual was tense or something I would go well I'm happy to come in and teach it for you and then you can see what I'm trying to explain.

This comment also shows Kate's awareness of the need to read people and be responsive. Kate was aware that change is slow and difficult and that individual teachers needed to feel strong and resilient and ready to take a risk. She explained, "The challenge is to work out when to press and initiate and suggest and then when to hold back." Kate said that she had worked with teachers long enough to realise that sometimes if they are feeling pressured "they're not willing to take a risk, or they're more insular or more protective of what they do, so you've just got to be really careful about how you go about the process and what you're willing to do."

Kate suggested that you need to “pick your moments and to know when to push and to know when to suggest … or keep trying with certain individuals, but the data’s helped me.” One of the strategies Kate used to develop improved teacher practice in the classroom was to have a conversation based on the student assessment data, “it’s a good thing to have a conversation … based on the students’ learning rather than the teachers’ teaching.”

Reflection on shared classroom experiences was not a systematic part of working in classrooms for Kate. Time was seen as a constraint except in her work with new graduates:

> It just depends on the day and what’s happening. I make the effort to go and see them and talk to them about it, but there’s no, you don’t get that extra time. The only time that we got to reflect was with the graduates with their programs because we just had more time allowed for that.

External Initiatives for Professional Learning

Kate noted the impact of participating in an action research project. Having external support for a short, focused professional project that looked at questioning and question types used in mathematics lessons was a successful initiative in her school. However, Kate said that using a similar action research cycle for general improvement was unlikely to be accepted by the staff:

> We did a little bit of focusing on questioning and the types of questions we used. It was one little project … I have been involved in a few action research cycles and that’s been really good. They’ve come up through my maths Masters [degree] … I feel like I need that support from the outside in to implement something like that and if I wanted to do an action research cycle, just purely to improve the quality of the teaching I think that I’d be hitting my head against brick wall, to be honest.

Making a Difference

Kate described teachers as at different points as learners and reflected on several occasions when she felt valued by teachers. She commented, “when they’re asking you for your advice and support, you’re halfway there.” Kate felt she was making a difference, particularly while working with a graduate teacher, who according to Kate “was absolutely brilliant to work with in maths and open tasks.” The influence of professional development and “seeing it [teaching] transformed [created] wonderful moments in that sense.” Kate also commented:

> I used to teach with her quite a bit. Yeah, that was good fun in that you could see her questioning, using tasks to be open enough to allow all students entry level, and then even having the discussions about the students work samples and what we could reflect on by looking at them and the students thinking in terms of, you know, the way that they were doing the task like that. …When they’re working with you to work out what the summary phase looks like, you know you’re getting somewhere.

Conclusion

To shape the narrative data, Fullan’s (2001) *Framework of Leadership* characteristics of leaders will be used as headings for some conclusions.

Personal constellation of energy, enthusiasm and hope. Kate’s personal characteristics of energy for her leadership through her “classroom work”, together with her enthusiasm for mathematics teaching and learning, and her hope for improvement in learning outcomes in mathematics, were plain.

Moral purpose. Kate referred to improving mathematics learning for all students saying, “the conversations are really important from within that sphere of how we can best support all students at the school.” However, it is not clear from the interview that the whole school team has developed a shared purpose of improving students’ learning in the manner Kate described, as the evidence suggests that some teachers are trying to be invisible through a process of change. That is not to say that they do not want the best for their students, it may be that they do not share Kate’s vision of what is “the best”.

Knowledge building and sharing. The central purpose of Kate's leadership was to improve the quality of mathematics teaching and learning in the school. To achieve this aim, she was intent on building the pedagogical content knowledge of the teachers. Initially she examined the assessment results to identify the Year level teachers most in need of support. She then collaborated with teams of teachers to improve mathematics planning by sharing more complex and challenging tasks. However, while teachers were willing to adopt the tasks, they had no confidence in implementing them. Kate was asked to demonstrate the open problems and trial investigative tasks in classrooms with teaching techniques that required students to reason and explain their mathematical thinking. Teachers noticed increased student interest and engagement and found students were capable of more complex mathematics than had previously been expected of them. One of the teachers observing Kate also became interested in the range of question types she used. These questions became a shared topic of discussion and focus amongst the staff more broadly. Kate used her involvement in a university project to initiate investigative teaching across the year levels with teacher-negotiated goals in mind. She used what Clarke and Hollingsworth's model of professional growth (2002) termed the *external domain of influence* to create a knowledge building environment in a short-term action research project.

Coherence making. It seems that teachers were being influenced by observing demonstration lessons with a focus on students and their thinking. Opportunities were created for discussion and reflection, learning and participating, by new graduate teachers. It could be beneficial to create more opportunities for reflection and discussion across Year levels with groups of teachers whose children have been taught the same lesson content. For example, several teachers' observations of their students' learning could promote an exchange of views and analysis of the pedagogical approaches that were demonstrated. An authentic part of teachers' teaching is reflecting (Shulman, 1986) and including opportunities for staff to reflect jointly could build coherence and shared knowledge.

The approaches to mathematics teaching, that were considered by Kate to be productive, were modelled for the teachers to observe. Presumably this practice was designed to build coherence by showing teachers what actions and knowledge were required to teach mathematics well. The questions this approach raises are: Whose methods are valued and privileged? Are teachers permitted to teach in different ways? A telling comment was Kate's view that teachers would not accept another action research project like the one they had undertaken previously. It seems that in a cost-benefit analysis those teachers had decided the "costs" were too great for the perceived benefits that resulted from the projects.

Understanding change. Change is messy and takes time according to Fullan (2001). Perhaps Kate's preference for using external projects to stimulate change is understandable. However, the stimulus to constant change is exhausting. The advantage of setting goals both long-term goals and short-term ones that can be readily achieved is that the teaching team has a sense of accomplishment in the short term and a sense of the long-term vision of success. Kate may reconsider her reluctance to specify goals.

Relationship building. It was clear from Kate's comments that she appreciated working in a classroom with her experienced and knowledgeable colleague where they shared the responsibility for mathematics teaching and learning. In addition, Kate described how she mentored the new graduates by having them assist in the classroom with her taking the lead. She encouraged their suggestions and spent time reflecting on events they had shared. She enjoyed using the experience she had gained developing novice teachers' skills as their mentor.

The findings of this study are limited to the reflections of only one leader and further research is needed to paint a broad current picture of primary school mathematics leadership. In addition, this study raised new issues for investigation. Two in particular-leaders' access to planning meetings under new enterprise bargaining agreements, and developing a shared purpose and coherence across teams of teachers where part-time appointments and job sharing is increasing the numbers of teachers and producing a lack of availability for team meetings. Apart from the changed work conditions, the results of this study outlined important elements of leadership which involved:

- Supporting quality mathematics through planning;
- Teaching students in classrooms to provide demonstration lessons for teachers;
- Modelling professional curiosity as a learner;
- Using external initiatives to stimulate willingness to change;
- Imagining ways to make a difference in the mathematical lives of students.

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References

Campbell, P. F., & Malkus, N. N. (2013). Elementary mathematics specialists: Influencing student achievement. *Teaching Children Mathematics, 20*(3), 198–205. doi: 10.5951/teacchilmath.20.3.0198

Clarke, D. & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*(8), 947–967.

Clarke, D., Roche, A., Wilkie, K., Wright, V., Brown, J., Downton, A., . . . Worrall, C. (2013). Demonstration lessons in mathematics education: teachers' observation foci and intended changes in practice. *Mathematics Education Research Journal, 25*(2), 207–230. doi:10.1007/s13394-012-0058-z

Driscoll, K. (2021). *An investigation of the ways in which school mathematics leaders support primary teachers' professional learning.* Monash University. Melbourne, Australia.

Faragher, R., & Clarke, D. (2014). Teaching mathematics effectively. In M. Gaffney & R. Faragher (Eds.), *Leading improvement in student numeracy* (pp. 47–65). Melbourne, Australia: ACER Press.

Fullan, M. (2001). *Leading in a culture of change*. Jossey-Bass.

Gibbons, L., & Cobb, P. (2017). Focusing on teacher learning opportunities to identify potentially productive coaching activities. *Journal of Teacher Education, 68*(4), 411–425.

Grootenboer, P., Edwards-Groves, C., & Rönnerman, K. (2015). The practice of "middle leading". In M. Marshman, V. Geiger, & A. Bennison (Eds.). *Mathematics education in the margins. Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 277–284). MERGA.

Hollingsworth, H. & Clarke, D. (2017). Video as a tool for focusing teachers self-reflection: Supporting and provoking teacher learning. *Journal of Maths Teacher Education*, p. 457–475.

Ingvarson, L. (2005). Getting professional development right. [Conference symposium]. ACER https://research.acer.edu.au/professional_dev/4.

Liljedahl, P. (2020). *Building thinking classrooms in mathematics, grades K–12: 14 teaching practices for enhancing learning.* Corwin Press.

Males, L., Otten, S., & Herbel-Eisenmann, B. (2010). Challenges of critical colleagueship: Examining and reflecting on mathematics teacher study group interactions. *Journal of Mathematics Teacher Education, 13*, p. 459–471. https://doi.org/10.1007/s10857-010-9156-6

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Stein, M. K., Smith, M., Henningsen, M., & Silver, E. (2009). *Implementing standards-based mathematics instruction* (2nd ed.). Teachers College Press and NCTM.

Tharp, R. & Gallimore, R. (1989). Rousing schools to life. *American Educator, 13*(2), 20–5.

Australian Junior Secondary Students' Approaches to Solving Ratio Problems Prior to Formal Instruction and Their Misconceptions

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Progressing from additive to multiplicative thinking is a key outcome of school mathematics, making ratios an essential topic of study in junior secondary. In this study, 15 Australian Year 8 students were administered a ratio test followed by semi-structured interviews to explore their conceptions of ratio prior to formal instruction. In this paper, students' responses to one of the ratio questions are analysed in detail. Analysis of incorrect responses was conducted using a modified version of Radatz's (1979) framework. Analysis of correct responses revealed that some students worked proficiently with ratio without formal instruction.

Ratio is termed a "big idea" in mathematics, and appreciation of the multiplicative relationship between quantities is fundamental to developing proportional reasoning (Siemon et al., 2012). The capacity to reason proportionally and work with ratios is a key outcome of high school mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2022). Ratios are an important topic of study in the secondary years since they unify the content strands of number, algebra, measurement, geometry, and data analysis and probability (Siemon, 2013). Despite the importance of ratios, researchers have considered the topic "the most protracted in terms of development, the most difficult to teach, the most mathematically complex, [and] the most cognitively challenging" (Lamon, 2007, p. 629). Analysis of student performance on the 2019 Trends in International Mathematics and Science Study (TIMSS) test confirmed students' difficulties with ratios. Only 35% of students internationally and 40% of Australian students were able to solve a ratio problem involving the enlargement of a figure (Mullis et al., 2020).

Given the importance of ratios and the noted difficulties that students experience when learning ratios, this study aimed to investigate Australian students' approaches to solving ratio problems prior to the formal teaching of the topic. Analysing student responses before explicit instruction allowed for the observation of students' natural approaches to ratio problems. This provides teachers with insight into the challenges and misconceptions students may experience when first introduced to ratios. The research question was "What approaches do junior secondary students use when solving ratio problems prior to instruction, and what misconceptions do they hold?"

Ratio Misconceptions

Although ratios have been identified as an area of challenge for both students and teachers, research into students' misconceptions in this area is limited. International research on ratios has mostly focused on misconceptions and difficulties held by primary school students and those with mathematical difficulties (e.g., Dougherty et al., 2016). One study analysed secondary students' misconceptions and difficulties with ratios in South Africa (Mahlabela, 2012), but similar research has not yet been conducted in Australia. On a national level, research has focused on diagnostic approaches to assess primary school students' proportional reasoning

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(Hilton et al., 2013) and interventionist approaches to promote students' emerging proportional reasoning (Fielding-Wells et al., 2014). Siemon and colleagues have also extensively researched the development of students' multiplicative thinking, but their work was not focused on misconceptions (Siemon et al., 2006; Siemon, 2019). Key themes pertaining to students' misconceptions and difficulties with ratios included difficulties transitioning from additive to multiplicative thinking, and confusion between fraction and ratio representations when analysing the findings from this research.

Proficiency with ratios is dependent on students' ability to think multiplicatively as proportional reasoning is the most sophisticated form of multiplicative thinking (Callingham & Siemon, 2021). During the transition from primary to high school, students' progression from additive to multiplicative thinking is one of the major barriers to learning mathematics, including the topic of ratios (Siemon, 2019). Research has shown that 30–55% of Year 8 students do not think multiplicatively, and differences between students' overall mathematics achievement can be attributed to an inadequate understanding of multiplication, division, fractions, decimals, and proportion (Siemon et al., 2006).

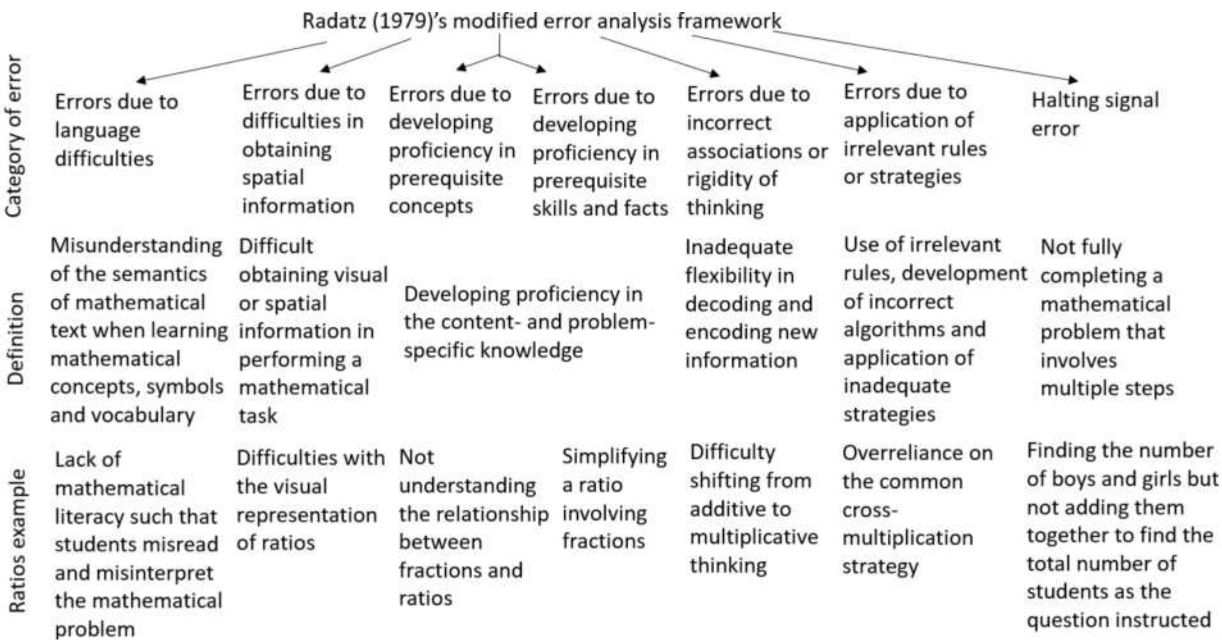
Misconceptions surrounding ratios have also been attributed to difficulties with prerequisite knowledge related to fractions (Dougherty et al., 2016). Students can have difficulties differentiating the part-part relationship of ratios from the part-whole relationship of fractions (Clark et al., 2003). Moseley (2005) also found that students' conceptual understanding of ratios is lacking as they focus on the numbers rather than the relations the numbers represent, as ratios are a multiplicative comparison.

Theoretical Framework

Categories of ratio errors are currently not well-established in the literature given the limited research on students' misconceptions and difficulties with ratios. This study drew from Radatz's (1979) error analysis framework that described five error categories: errors due to language difficulties; errors due to difficulties in obtaining spatial information; errors due to developing proficiency in prerequisite skills, facts, and concepts; errors due to incorrect associations or rigidity of thinking; and errors due to application of irrelevant rules or strategies.

Figure 1

Modified Version of Radatz's (1979) Error Analysis Framework



This framework uses an information-processing approach to classify errors because the "mechanisms used in obtaining, processing, retaining and reproducing the information in mathematical tasks" are examined (Radatz, 1979, p. 164). This study modified the original Radatz framework (Figure 1) through small language changes (removing reference to 'deficient' skills and knowledge to encourage a move away from a deficit view), adding the halting signal error (a partially complete response: Brodie & Bergie, 2010), and splitting the error category of 'errors due to the developing proficiency in prerequisite skills, facts, and concepts' into two for clarity (separating concepts and skills/facts).

Methodology

Participants were 15 Year 8 students from one class in an Australian independent school. The school's demographic was representative of a higher socioeconomic background, with a school ICSEA value of 1087 given the 1000 average. The study was conducted prior to the class formally learning ratios. A mixed methods approach was used, drawing on both qualitative and quantitative analysis of student error types on an 8-item ratio test. The test was developed from an analysis of the local mathematics syllabus (Stage 4 New South Wales Syllabus) which allowed for the identification of key ratio concepts and skills covered in Year 8. The test was completed in 20 minutes under exam conditions without calculators. In this paper, the findings from one of the test items are reported as they allow for a deep analysis of all students' attempts. Following the test, semi-structured interviews were conducted to clarify and confirm the researcher's interpretation of students' test responses. Interviews enhanced the validity of findings, since the different datasets elaborated, enhanced, and clarified each other (Greene et al., 1989). When coding students' incorrect responses, error analysis was conducted. All incorrect responses were collaboratively analysed and coded using the modified version of Radatz's framework (Figure 1).

Findings

This paper focuses on analysing students' correct and incorrect responses to one test question, "Can 10 people be divided into two groups with a ratio of 1:2? Explain your answer." Overall, the 15 students reported a 26.7% success rate on answering this item; 10 students answered incorrectly, and one student did not attempt the question. Students' challenges in answering the test question were expected since they had not yet been taught ratios. There was a diverse range of correct and incorrect answers provided by students.

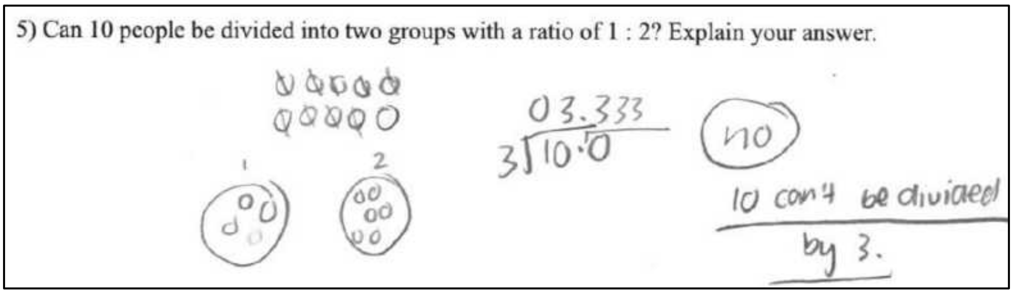
Correct Responses

Students' correct responses reflected varying levels of developing proficiency with ratios, although the semi-structured interviews after the ratios test revealed that some students still held some misconceptions on the topic.

In Figure 2, the student added the antecedent (i.e. 1) and consequent (i.e. 2) of the ratio together to reach 3.

Figure 2

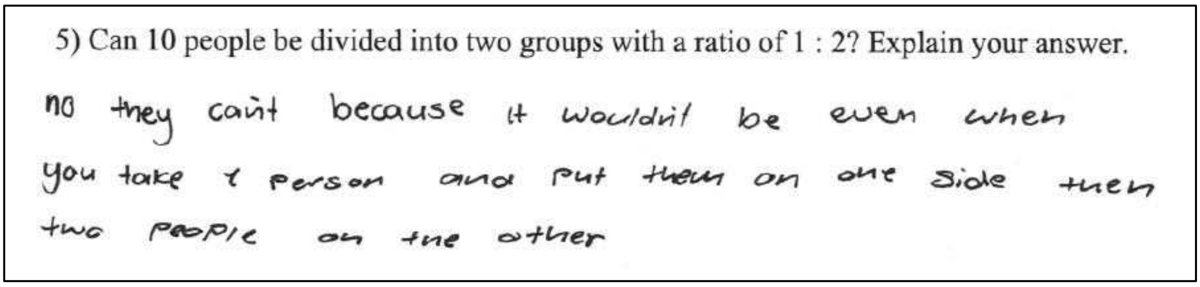
Example of Student's Correct Response Using Concepts of Division, Factors, and Multiples



To check if 10 people can be divided into two groups with a ratio of 1:2, this sum of the ratio parts (i.e., 3) must be a factor of the total number of people (i.e., 10), or conversely, the total number of people must be a multiple of the sum. The student also provided an accompanying diagram showing the first and second group having three and six people respectively, and their last uncrossed circle of 10 suggests the leftover person. This strategy for solving the question was also encapsulated in the working out of the student in Figure 3. Both students drew on key prerequisite knowledge of division and the concept of equal groups, which are foundational concepts of ratios.

Figure 3

Example of Student's Correct Response Using an Evenness Argument

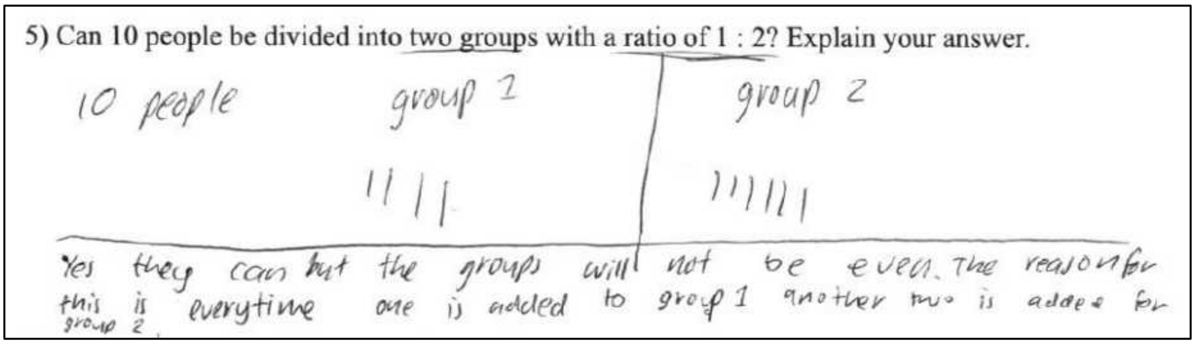

5) Can 10 people be divided into two groups with a ratio of 1 : 2? Explain your answer.

no they can't because it wouldn't be even when you take 1 person and put them on one side then two people on the other

Two students provided an alternative explanation that demonstrated a developing conceptual understanding of ratios. Instead of providing an explanation involving visuals or relating the problem to division, factors and multiples, their answer was that the group division would not be possible because the two groups would not be even in number. Although even groups were not a criterion of the question, these students identified that when ten is grouped in a ratio of 1:2, there would be one person leftover who has not joined a group. The response from the first of these two students, shown in Figure 4, shows emerging proficiency with ratios but some misconceptions are still held. Although their explanation aligns with the student's explanation in Figure 3, their diagram shows that they have not recognised that 4:6 = 2:3 is not equal to the desired 1:2 ratio.

Figure 4

Example of Student's Correct Response Using an Evenness Argument but With Misconceptions

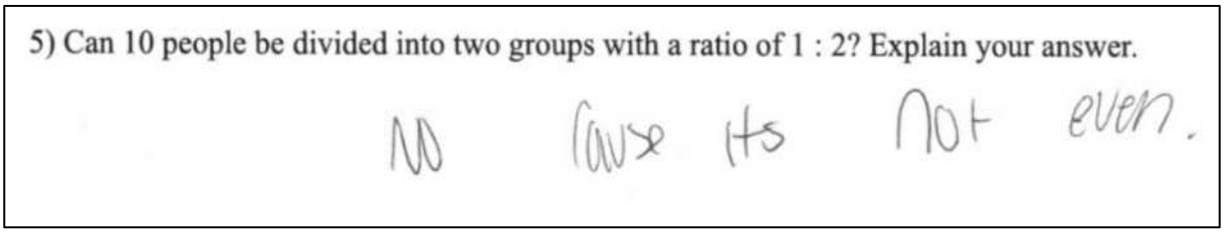

5) Can 10 people be divided into two groups with a ratio of 1 : 2? Explain your answer.

Yes they can but the groups will not be even. The reason for this is everytime one is added to group 1 another two is added for group 2.

The second student similarly identified that the groups would not be even (Figure 5), and when questioned on what they meant by 'even', they stated that "if there's one person on one side, for example, and there are two on the other", the number of people in each group would not be even. When the student was probed further, their misconceptions on ratios were revealed. After asking them their next steps in their working to ensure no halting signal error "Would you have added more people on either side or would you have left it as one person on one side and two people on the other?", the student responded "I would add an extra one to the other side" to make it even. Their thought process behind this was that "[the question] says 'to be divided into two groups' so it has to probably be even".

Figure 5

Example of Student's Correct Response but Their Interview Reveals Misconceptions

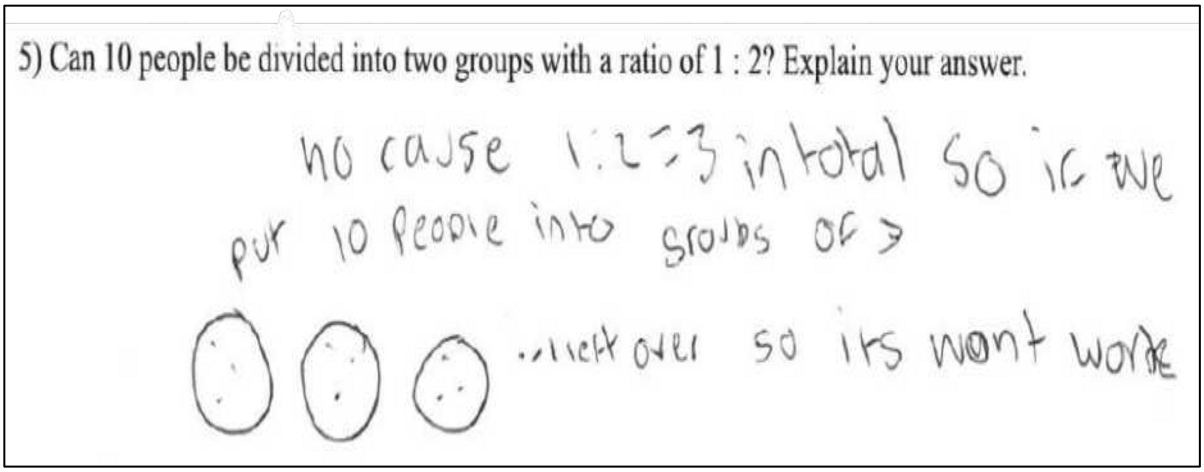


Incorrect Responses

While some students correctly solved the ratio problems, some students were not as successful. Like the student in Figure 2, the student response in Figure 6 demonstrated that they added the ratio parts together to help them solve this problem.

Figure 6

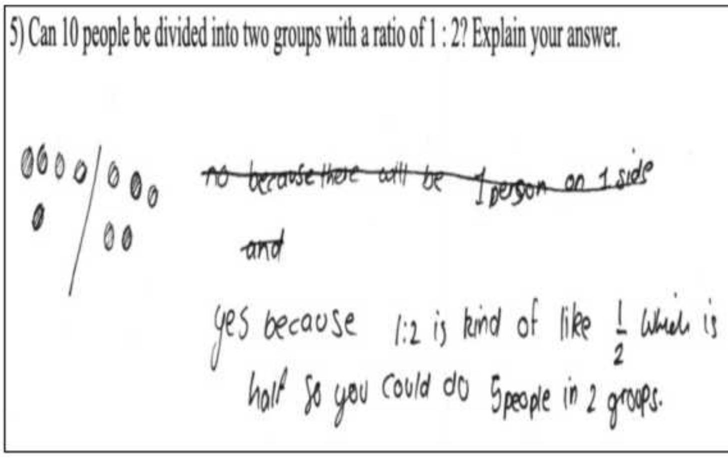
Example of Student's Correct Response but Their Working Reveals Misconceptions



However, instead of using this sum of three to conceptualise putting one person in one group and two people in the other group every round, they misinterpreted this as putting 10 people into groups of three. They were, however, able to identify that one person would be leftover so this division into three groups would be unsuccessful. Although the working here does not reflect the expected understanding of splitting a quantity into a ratio, it does indicate some developing conceptual understanding of ratios, based on the idea of division into equal groups. The root error cause was coded to be the developing proficiency in concepts and language difficulties, leading to procedural errors.

Figure 7

Example of a Student Misinterpreting 1:2 to Mean $\frac{1}{2}$



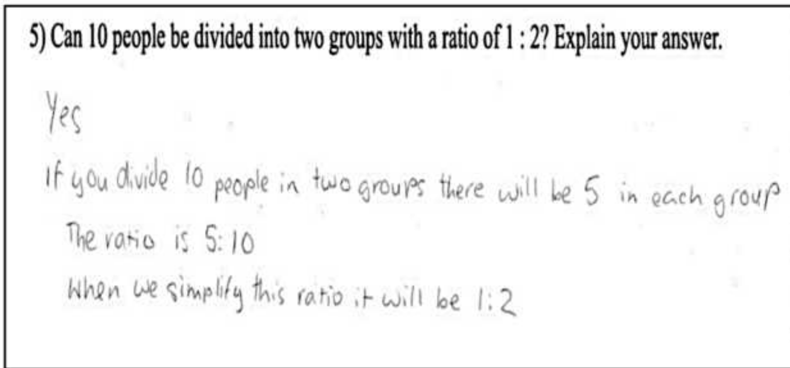
Error categorisation: Developing proficiency in ratio concepts, seen in students incorrectly associating ratios with fractions (occurrence: 5 out of 15 students).
Instead of interpreting 1:2 as 1 and 2 parts out of 3, five students interpreted it to mean $\frac{1}{2}$ or half, demonstrating difficulty with mathematical symbols (language difficulty). Hence, students split the group of 10 into two groups of 5, which would incorrectly form a ratio of 1:1.

When given the ratio 1:2, it is speculated that some students misinterpreted the colon sign of ratios to mean the same as the vinculum sign of fractions. This is because students lacked conceptual understanding of what a ratio is, which led to the language difficulty of misinterpreting the colon as a vinculum due to incorrect associations with fractions. An example of this misconception is shown in the student response in Figure 7.

Students also demonstrated conceptual difficulties with working with a fixed number of total people (i.e., 10) despite correctly identifying equivalent ratios (Figure 8).

Figure 8

Example of Student's Developing Proficiency in Concepts



Error categorisation: Developing proficiency in concepts leading to use of incorrect procedures (occurrence: 1 out of 15 students)
One student thought that 1:2 is equivalent to 5:10, which is correct. However, the new equivalent ratio involves 15 people instead of the required 10.

Students' responses also demonstrated that a flow-on error resulting from developing proficiency in concepts was the halting signal error, as students only partially completed the question (Figure 9). This can be attributed to students not having formally learnt ratios, although it reveals their approaches to solving ratio problems.

Figure 9

Examples of Two Students' Halting Signal Errors

Error categorisation: Developing proficiency in concepts, resulting in a halting signal error (occurrence: 2 out of 15 students)	
5) Can 10 people be divided into two groups with a ratio of 1 : 2? Explain your answer. 1:2 1:2 No because the ratio shows only six people	The student knew that 1:2 meant 3 people and how they should check whether 3 is a factor of 10. However, they did not continue up to the highest multiple of 9 and instead stopped at 6. It is possible that the student stopped at 6 because the question says, "two groups".
5) Can 10 people be divided into two groups with a ratio of 1 : 2? Explain your answer. no, because there would be 7 people left over.	The student only considered the first group of three people. This suggests that the student did not conceptually understand what the question was asking.

Discussion and Conclusion

This study aimed to investigate Australian students' approaches to solving ratio problems prior to the formal teaching of the topic and address the research question. For the analysed question, it was found that the main root error cause was developing proficiency in ratio concepts, leading to other error types including incorrect associations with fractions, application of irrelevant procedures and halting signal errors. This was not unexpected as the students had not learnt ratios yet. What was particularly interesting for teachers to observe was that several students successfully reasoned through the ratio problem despite not having learnt the topic. Students' prior experience with fractions (Dougherty et al., 2016) and emerging multiplicative thinking (Siemon et al., 2006) set the foundation for success when learning ratios. The implication of this finding is that it is beneficial for teachers to help students connect ratios with this prerequisite knowledge, rather than viewing it as a distinct and entirely new topic when first introducing it. Moreover, since students may resort to fraction strategies and concepts when first approaching ratios, another teaching recommendation is to clarify the similarities and differences between fractions and ratios in terms of vocabulary, concepts, and procedures.

The findings from this study have implications for research investigating student errors in mathematics. Through the error analysis conducted in this study, the modified version of Radatz's (1979) framework was shown to be a viable and useful way of coding student errors. Potential directions for future research include ascertaining whether the same errors exist before and after formal instruction on ratios, and testing whether Radatz's (1979) error analysis framework can be used to hierarchically classify errors for other mathematical topics.

Although it was anticipated that one class of students would provide adequate data to gain insight into students' understanding of ratios and corresponding errors, generalisability may be limited beyond this study because of its dependence on only one class of students. Despite this, the demographics of the school and participants provide context for the findings and support the reader to make judgements concerning the generalisability of the study to other contexts.

Acknowledgments

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References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *F–10 mathematics curriculum (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/

Brodie, K., & Berger, M. (2010). Toward a discursive framework for learner errors in mathematics. In V. Mudaly (Ed.), *Proceedings of the eighteenth annual meeting of the Southern African Association for Research in Mathematics, Science and Technology Education* (pp. 169–181). Durban: University of KwaZulu-Natal.

Callingham, R., & Siemon, D. (2021). Connecting multiplicative thinking and mathematical reasoning in the middle years. *The Journal of Mathematical Behaviour, 61*, 1–12. https://doi.org/10.1016/j.jmathb.2020.100837

Clark, M. R., Berenson, S. B., & Cavey, L. O. (2003). A comparison of ratios and fractions and their roles as tools in proportional reasoning. *The Journal of Mathematical Behaviour*, *22*(3), 297–317. https://doi.org/10.1016/S0732-3123(03)00023-3

Dougherty, B., Bryant, D. P., Bryant, B. R., & Shin, M. (2016). Helping students with mathematics difficulties understand ratios and proportions. *Teaching Exceptional Children*, *49*(2), 96–105. https://doi.org/10.1177/0040059916674897

Fielding-Wells, J., Dole, S., & Makar, K. (2014). Inquiry pedagogy to promote emerging proportional reasoning in primary students. *Mathematics Education Research Journal*, *26*(1), 47–77. https://doi.org/10.1007/s13394-013-0111-6

Greene, J. C., Caracelli, V. J., & Graham, W. F. (1989). Towards a conceptual framework for mixed-method evaluation designs. *Educational Evaluation and Policy Analysis*, *11*(3), 255–274. https://doi.org/10.3102/01623737011003255

Hilton, A., Hilton, G., Dole, S., & Goos, M. (2013). Development and application of a two-tier diagnostic instrument to assess middle-years students' proportional reasoning. *Mathematics Education Research Journal*, *25*(4), 523–545. https://doi.org/10.1007/s13394-013-0083-6

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. J. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–668). Information Age Publishing.

Mahlabela, P. T. (2012). *Learner errors and misconceptions in ratio and proportion: A case study of grade 9 learners from a rural KwaZulu-Natal school* [Unpublished Master's thesis, University of KwaZulu-Natal, South Africa].

Moseley, B. (2005). Students' early mathematical representation knowledge: The effects of emphasising single or multiple perspectives of the rational number domain in problem solving. *Educational Studies in Mathematics*, *60*(1), 37–69. https://doi.org/10.1007/s10649-005-5031-2

Mullis, I. V., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020). *TIMSS 2019 international results in mathematics and science*. IEA TIMSS & PIRLS. https://timssandpirls.bc.edu/timss2019/international-results

Radatz, H. (1979). Error analysis in mathematics education. *Journal for Research in Mathematics Education*, *10*(3), 163–172. https://doi.org/10.5951/jresematheduc.10.3.0163

Siemon, D. (2019). Knowing and building on what students know: The case of multiplicative thinking. In D. Siemon, T. Barkatsas, & R. Seah (Eds.), *Researching and using progressions (trajectories) in mathematics education* (pp. 6–31). Brill.

Siemon, D. (2013). Launching mathematical futures: The key role of multiplicative thinking. In S. Hernert, J. Tillyer & T. Spencer. (Eds.), *Mathematics: Launching futures. Proceedings of the 24th biennial conference of the Australian Association of Mathematics Teachers* (pp. 36–52). AAMT.

Siemon, D., Bleckly, J., & Neal, D. (2012). Working with the big ideas in number and the Australian curriculum: Mathematics. In B. Atweh, M. Goos, R. Jorgensen, & D. Siemon (Eds.), *Engaging the Australian National Curriculum: Mathematics—Perspectives from the field* (pp. 19–45). MERGA. https://merga.net.au/common/Uploaded%20files/Publications/Engaging%20the%20Australian%20Curriculum%20Mathematics.pdf

Siemon, D., Breed, M., Izard, J. & Virgona, J. (2006). The derivation of a learning and assessment framework for multiplicative thinking. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (pp. 113–120). PME.

Developing Complex Unfamiliar Mathematics Questions: A Perspective

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This paper reports on the development of complex unfamiliar mathematics questions using conventional mathematics questions. The autoethnographic research data included retrospective reflections of two educators and review of literature. Data analysis resulted in a three-phase development process: identifying conventional questions and subject matter to be modified to enhance levels of understandings and skills, modifying conventional questions and subject matter to complex unfamiliar questions, and identifying enhanced understandings and skills in modified questions. The paper then discusses the educators' insights developing complex unfamiliar mathematics questions for the practice of teaching and learning.

In Australia, the focus on developing students' mathematics proficiencies of understanding, fluency, reasoning, and problem-solving skills is meant to help them develop the capacity to solve complex unfamiliar problems (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2023). Likewise, in states like Queensland and South Australia students are expected to engage with complex unfamiliar or non-routine problems as part of the mathematics assessments. Particularly in Queensland, it is expected that about 20% of the marks in any general mathematics subject examination at senior secondary level should be complex unfamiliar. The subjects are General mathematics, Mathematical Methods and Specialist Mathematics and they provide a pathway for different tertiary courses. However consistent curriculum and assessment reports (Queensland Curriculum and Assessment Authority [QCAA], 2021; 2022; 2023; 2024; South Australia Council of Education [SACE], 2021) have identified lack of consistency by senior secondary mathematics teachers in Queensland and South Australia when it comes to developing complex unfamiliar questions for internal examinations. The reports noted that teachers still need to improve in assessing students under this level of difficulty. As a result, there has been limited use of complex unfamiliar problems in mathematics teaching and learning (Lee & Kim, 2005; Nguyen et al., 2020). Senior secondary mathematics teachers in Queensland are expected to include complex unfamiliar problems when they set internal assessments that are moderated by Queensland Curriculum and Assessment Authority (QCAA). Thus, providing resources that can help develop understanding and consistency in developing complex unfamiliar questions is central in promoting quality mathematics teaching and learning. This paper employed an auto-ethnographic method to report on how complex unfamiliar mathematics questions can be developed using conventional (routine) mathematical questions.

Complex Unfamiliar Mathematics Questions

There is a general agreement that complex unfamiliar or non-routine mathematics problems require deeper understanding of concepts, analytical thinking, interpreting and creativity beyond standard algorithm strategies as there is no clear path to the solution (Asman & Markovits, 2009; Mullis et al., 2009; Polya, 1966; QCAA, 2018; Schoenfeld, 1992). Complex unfamiliar or non-routine questions can take different definitions and forms depending on the nature of how they are presented and are solved. Some researchers (see for example Robinson, 2016) view non-routine problems as open-ended and involving multiple solutions. Similarly, other researchers (e.g., Asman & Markovits, 2009; Kloosterman, 1992) identify complex unfamiliar problems as having a unique solution but the procedures to solve and the concept

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 159–165). Gold Coast: MERGA.

the problem is derived from are not readily available. Importantly, "a non-routine problem appears when an individual encounters a given situation, intends to reach a required situation, but does not know a direct way of accessing or fulfilling his or her goal" (Elia et al., 2009, p. 606). Similarly, complex unfamiliar problems require students to demonstrate knowledge and understanding of mathematics and application of skills in a situation where:

- Relationships and interactions have several elements, such that connections are made with subject matter within and/or across the domains of mathematics;
- All the information to solve the problem is not immediately identifiable; that is, the required procedure is not clear from the way the problem is posed, and in a context in which students have had limited prior experience (QCAA, 2018 p. 44).

Student engagement with complex unfamiliar or non-routine problems in mathematics is said to enhance their development of mathematical knowledge (see Russo, 2019; Sullivan et al., 2015). Complex unfamiliar problems are credited for the promotion of strategic thinking capacity, integration of concepts and motivating students towards learning (Lampert, 2001). Calls to include complex unfamiliar questions in mathematics assessments have been going for a long-time (see Clarke, 2011; Ergen, 2020). It is fundamentally important that any mathematics activity should provide students with the opportunity to engage with content at different levels of sophistication (Clarke, 2011). Clarke went further to posit that mathematics curricula across the world is now advocating for inclusion of non-routine (complex unfamiliar) mathematics problems as a key goal of school mathematics (Clarke, 2011).

According to QCAA (2018), complex unfamiliar questions that require more levels of cognitive skills should not be equated with elaborate problem-solving tasks and modelling questions only. A single-answer, conventional question such as an example from Goos (2014): Find the equation of the line passing through the points (2,1) and (1,3); can be adapted to a more open-ended question such as: Write the equations of at least five lines passing through the point (2,1). This revised question targets the identical subject matter and assesses more and advanced cognitive understanding and skills. This means that, development of complex unfamiliar questions can start with: (1) identifying conventional mathematical questions and subject matter that can be modified to enhance levels of cognitive understandings and skills, (2) modifying the conventional questions and subject matter to complex unfamiliar questions, and (3) identifying the diverse and enhanced cognitive understandings and skills in the modified questions, leading to the development of assessment rubrics or guides to making judgements. This study is guided by the process of modifying conventional questions to complex unfamiliar questions as highlighted above by QCAA (2018). The paper reports on two educators' perspectives on how complex unfamiliar questions can be developed using conventional questions.

Method

The paper employed an autoethnographic approach to examine perspectives of two mathematics educators on how complex unfamiliar mathematics questions can be developed using conventional mathematics questions. According to Adams and colleagues (2017) autoethnography is a research method that uses personal experience ("auto") to describe and interpret ("graphy") cultural texts, experiences, beliefs, and practices ("ethno"). Chang and colleagues (2016) highlight that some auto ethnographers focus more on self, while others adopt a more analytical stance, focused on understanding and interpreting events and experiences involving self. This investigation takes the form of an analytical approach to autoethnography. The two educators met fortnightly for an hour over two school terms to articulate how complex unfamiliar mathematics questions can be developed using conventional mathematics questions. The meetings were reflective interpretations of experience, practice, and review of literature, which resulted in a three-phase analysis of the development of complex unfamiliar mathematics questions (Adams et al., 2017). Phase one centred on conventional mathematics questions and subject matter that can be modified to enhance levels of cognitive understanding and skills. Phase two centred on modification of conventional questions and subject matter to complex unfamiliar questions. Phase three centred on diverse and enhanced cognitive understandings and skills in modified questions and the development of assessment rubrics or guides to making judgements. The next section highlights a summary of the three-phase analysis and retrospective reflections of the educators.

Reflections and Analysis

This section reports on phases one, two and three retrospective reflections and analyses of how complex unfamiliar questions can be developed using conventional mathematics questions. Our overall position was that for developed complex unfamiliar questions: students will be required to apply their knowledge of multiple concepts to solve each question; the method required to solve the question will not be obvious in its wording; the question will most likely be in a context that students have not come across before; no scaffolding (for example provision of a graph); and that interpretation, clarification, and analysis will be required to develop responses to the questions.

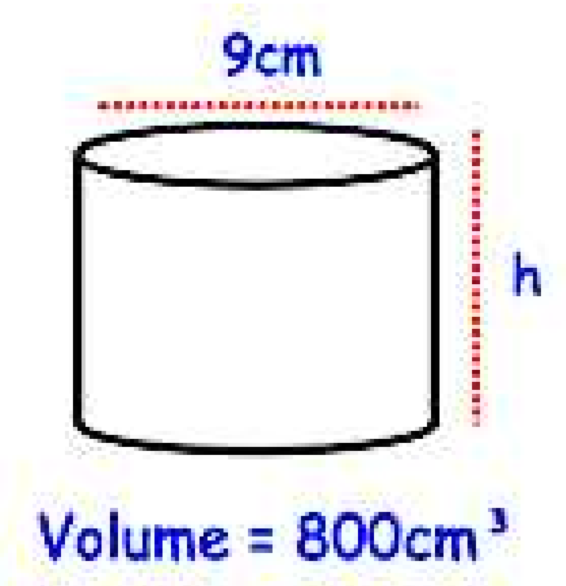
Phase one involved identifying the appropriate conventional mathematics questions and subject matter that can be modified to enhance levels of cognitive skills. We suggest starting with nominating a single-answer, conventional question; and then highlighting the subject matter, routine processes that may be involved, routine calculations that may be involved, and formulae and steps to aid the development of a solution that can be immediately applied to the solution of the given question.

For example, in a year 9 measurement question below, the original mathematics concept is on volume of a cylinder. The information provided includes the volume in cm^3, diameter in cm and the unknown height (h) as illustrated in the labelled diagram (Figure 1) below. The question also requires students' knowledge of one decimal place. This is a single-answer conventional question, and all students need to do is recall and evaluate the formula of volume of a cylinder:

> Conventional question: Figure 1 is a diagram of a cylinder with a volume of 800 cm^3, a diameter of 9 cm and a height (h). Calculate the height of the cylinder (to one decimal place).

Figure 1

Diagram of a Cylinder



Phase two involved modifying the conventional questions and subject matter to complex unfamiliar questions. We suggest starting with situating the conventional mathematics question in a real-world context/story or a context that the students have not seen before. The next step will be to remove any implied routine calculations and processes, then removing any labelled diagrams, formulae, and recognisable steps that aid the development of a solution. The aim is to de-identify routine procedures (no standard algorithm strategies, no clear path to solutions). The next step involves making connections of subject matter within and/or across the domains of mathematics. This makes the method and procedure to solve the question not immediately identifiable from the way the question is now posed. Using the previous year 9 measurement question stated above, it can be modified to the following:

> Complex unfamiliar question: A teacher proposes to the school administration to build a drum-shaped pool with no deep end to minimise the risk of drowning for prep to year 2 students. The pool must be bounded by the boundaries of a squared block measuring 3,600 m². How deep will be the pool (to two decimal places) for it to contain 2.5 million litres of water? Comment on the reasonableness of the teacher's plan.

The modified question is now more situated in a real-world context and the labelled cylinder diagram has been removed. The question is now multi-stepped and focuses on more than four aspects of mathematics, which include identifying 3-dimensional shapes, knowledge of calculating different areas of two-dimensional shapes (surface of the pool base) in m² (which can be any shape-from a circle, square, rectangle, etc.), different volumes of three-dimensional shapes in m³, conversion of volume units to capacity and knowledge of rounding off decimals. The question also seeks a contextual mathematical explanation that justifies the reasonableness of the answer.

Phase three involved identifying the diverse and enhanced cognitive understandings and skills in modified questions. We suggest identifying the multiple contexts, including diagrams and models, procedures and skills involved in solving the modified questions. Importantly, this phase hypothesises how diverse responses to the question can be developed. This process leads to the development of the assessment rubric or guide to making judgement. In the example above, this involves identifying the different areas and volumes of shapes and conversion of units. Additionally, having more than one correct answer means students have opportunities to justify the reasonableness of their answers and then have something unique to contribute to discussions with other students. The question can provide opportunities of generalisations as students explore a diverse of possible answers. Moreover, the question can be further modified for students to find just one answer, for example in an examination, the second last statement in the question can be: 'How deep can be the pool if it is to cover the maximum possible area of the available block?' Thus, depending on the objective and purpose of the assessment the modified complex unfamiliar question can adapt flexibly with very minor changes but still catering for the same content and cognitive demands.

Discussion

This section discusses our insights developing complex unfamiliar mathematics questions. During our reflective exchanges, a major challenge that we encountered was modifying complex unfamiliar questions that de-identify context and procedures and at the same time develop mathematics proficiencies of understanding, fluency, reasoning, and problem-solving skills, which are central to helping students to develop the capacity to solve complex unfamiliar problems (ACARA, 2023). Phase one agenda of identifying the conventional question that requires at least a three-step solution; highlighting the subject matter, context, diagram(s), routine processes that may be involved, routine calculations that may be involved, and formulae and steps to aid the development of a solution that can be immediately applied to the solution of the given question was the least challenging phase. We found it easy to reach consensus most of the times. Similarly, it was easy to reach consensus in phase three identifying the multiple contexts, including diagrams and models, procedures and skills involved in solving the modified questions. However, phase two agenda of modifying the conventional questions into real-world contexts/stories or unusual ways that students have very limited experience, removing implied routine calculations and processes, and making connections of subject matter within and/or across the domains of mathematics proved challenging and complex. Lampert (2001) suggests that this phase of modifying the conventional questions is central to promotion of students' strategic thinking capacity, integration of concepts and motivation towards learning. The challenges arose from finding the appropriate balance between removing appropriate levels of implied routine calculations and processes, making appropriate connections of subject matter within and/or across the domains of mathematics, and at the same time focusing on the mathematics proficiencies. As our reflective exchanges unfolded, it was evident that one purpose of our interactions which was to encourage collaboration and consensus on the modified products was proving elusive most of the times in phase two. To modify such conventional mathematics questions into complex unfamiliar questions requires collaborative effort among educators, as highlighted previously, curriculum reports (QCAA, 2021; 2022; 2023; 2024; SACE, 2021) have identified lack of consistency by mathematics teachers when it comes to developing complex unfamiliar questions for internal examinations. However, having such consistency and at the same time focus on the mathematics proficiencies proved to be very challenging.

The reflective exchanges and analyses of how complex unfamiliar questions can be developed using conventional mathematics questions, was particularly useful in helping us focus more on subject matter that can be modified to enhance levels of cognitive understandings and skills as well as making connections of subject matter within and/or across the domains of mathematics. The aim was to improve our theoretical and practical understanding of the nature of complex unfamiliar mathematics questions and the adaptive expertise developing them. In particular, developing deeper understanding of not only subject matter and concepts, but the required analytical thinking, interpreting and creativity beyond standard algorithm strategies (Asman & Markovits, 2009; Mullis et al., 2009; QCAA, 2018) was fundamental. During our reflective exchanges, we observed that the modification process might not be straight forward from the way the problem is posed and putting it in a context in which students have had limited prior experience. For the modification of complex unfamiliar questions, we tended to discuss which subject matter was better placed as the starting point for creating complex unfamiliar questions and then identifying routine procedures and standard algorithms that were evident in the question. We came to a position that a casualty in the whole process was making connections of subject matter within and/or across the domains of mathematics if the main focus is making the solution not immediately identifiable as suggested by Asman and Markovits (2009). Similarly, QCAA (2018) emphasise that the required procedure to solve the question should not be clear from the way the problem is posed, and it should be in a context in which students have had limited prior experience. Consequently, developing complex unfamiliar mathematics questions that connect subject matter within and/or across the domains of mathematics can be a challenging process.

The reflective exchanges and analyses played an important role in building our strategies on how to modify conventional mathematics questions to complex unfamiliar. We came to the view point that a good complex unfamiliar question tends to have a few of the following features but not necessarily all of them:

- Provides real world context/story; provides information in an unusual way (a context that the students haven't seen before).
- Can be worked backwards.
- Combines subject matter.

- Does not have a clear path to solutions (no implied routine calculations and processes, no labelled diagrams, formulae, and recognisable steps that aid the development of a solution).
- Is multi-stepped and creates different levels of sophistication.
- Depending on the assessment objectives, the question might have more than one correct answer.
- Asks to evaluate and justify the reasonableness of answers.

This perspective is not exhaustive; however, it can provide students with the opportunity to engage with content at different levels of sophistication, a position postulated by Clarke (2011) and Ergen (2020). Such a perspective can help develop understanding and consistency in developing complex unfamiliar questions and promote quality mathematics teaching and learning.

Conclusion

The autoethnography process has enabled us to reflect more deeply on how complex unfamiliar questions can be developed using conventional mathematics questions. The approach we report resulted in a three-phase analysis of the development of complex unfamiliar mathematics questions. Phase one focused on the identification of single-answer, conventional mathematics questions and the embedded subject matter that can be modified to enhance levels of cognitive understanding and skills. Phase two focused on modification of conventional questions and subject matter to complex unfamiliar questions. Phase three focused on the enhanced cognitive understandings and skills in modified questions, leading to the development of assessment rubrics and guides to making judgements. The autoethnographic process highlighted a perspective and the importance of consistency in teacher developed complex unfamiliar questions. Our hope is to encourage further research on developing complex unfamiliar mathematics questions using different types of mathematics questions to promote quality mathematics teaching and learning.

References

Adams, T. E., Ellis, C., & Jones, S. H. (2017). Autoethnography. *The International Encyclopedia of Communication Research Methods* (pp. 1–11). Hoboken, NJ: John Wiley & Sons. https://doi.org/https://doi.org/10.1002/9781118901731.iecrm0011

Asman, D., & Markovits, Z. (2009). Elementary school teachers' knowledge and beliefs regarding non-routine problems. *Asia Pacific Journal of Education*, *29*(2), 229–249. https://doi.org/10.1080/02188790902859012

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2023). *Australian Curriculum: Mathematics (v9.0)*. https://www.australiancurriculum.edu.au/resources/mathematics-proficiencies/

Chang, H., Ngunjiri F. W. & Hernandez, K. C. (2016). *Developing qualitative inquiry: Collaborative autoethnography*. Walnut Creek, US: Routledge.

Clarke, D. J. (2011). Open-ended tasks and assessment: The nettle or the rose. In B. Kaur, & K. Y. Wong (Eds.), *Assessment in the mathematics classroom* (pp. 131–163). Singapore: World Scientific.

Elia, I., van den Heuvel-Panhuizen, M., & Kolovou, A. (2009). Exploring strategy use and strategy flexibility in non-routine problem solving by primary school high achievers in mathematics. *ZDM Mathematics Education*, *41*(5), 605–618. https://doi.org/10.1007/s11858-009-0184-6

Ergen, Y. (2020). 'Does mathematics fool us?: ' A study on fourth grade students' non-routine maths problem solving skills. *Issues in Educational Research*, *30*(3), 845–865.

Goos, M. (2014). Mathematics classroom assessment. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 413–417). Springer, Dordrecht. https://doi.org/10.1007/978-94-007-4978-8_104

Kloosterman, P. (1992). Non-routine word problems: one part of a problem-solving program in the elementary school. *School Science and Mathematics*, *92*(1), 31–37. https://doi.org/10.1111/j.1949-8594.1992.tb12134.x

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven: Yale University Press.

Lee, M. & Kim, D. (2005). The effects of the collaborative representation supporting tool on problem-solving processes and outcomes in web-based collaborative problem-based learning (PBL) environments. *Journal of Interactive Learning Research*, *16*(3), 273–293. https://www.learntechlib.org/primary/p/5962/

Mullis, I. V. S., Martin, M. O., Ruddock, G. J., O'Sullivan, C. Y. & Preuschoff, C. (2009). *TIMSS 2011 assessment frameworks.* Boston: TIMSS & PIRLS International Study Center Lynch School of Education, Boston College. https://timssandpirls.bc.edu/timss2011/frameworks.html

Nguyen, H.A., Guo, Y., Stamper, J., McLaren, B.M. (2020). Improving students' problem-solving flexibility in non-routine mathematics. In I. Bittencourt, M. Cukurova, K. Muldner, R. Luckin, E. Millán, (Eds.), *Artificial intelligence in education. AIED 2020. Lecture notes in computer science* (Vol 12164). Springer, Cham. https://doi.org/10.1007/978-3-030-52240-7_74

Pólya, G. (1966). On teaching problem solving. *The role of axiomatics and problem solving in mathematics. Edward Griffith Conference Board of the Mathematical Sciences* (pp. 123–129). Boston, MA: Ginn.

Queensland Curriculum and Assessment Authority. (2018). *Senior general subjects: Mathematical methods*. QCAA. https://www.qcaa.qld.edu.au/downloads/senior-qce/syllabuses/snr_maths_methods_19_syll.pdf

Queensland Curriculum and Assessment Authority. (2021). *General mathematics subject report: 2020 cohort*. QCAA. https://www.qcaa.qld.edu.au/downloads/senior-qce/mathematics/snr_maths_general_20_subj_rpt.pdf

Queensland Curriculum and Assessment Authority. (2022). *Mathematical methods subject report: 2021 cohort*. QCAA. https://www.qcaa.qld.edu.au/downloads/senior-qce/mathematics/snr_maths_methods_21_subj_rpt.pdf

Queensland Curriculum and Assessment Authority. (2023). *Specialist mathematics subject report: 2022 cohort*. QCAA. https://www.qcaa.qld.edu.au/downloads/senior-qce/mathematics/snr_maths_specialist_22_subj_rpt.pdf

Queensland Curriculum and Assessment Authority. (2024). *Mathematical methods subject report: 2023 cohort.* QCAA. https://www.qcaa.qld.edu.au/downloads/senior-qce/mathematics/snr_maths_methods_23_subj_rpt.pdf

Robinson, L. M. (2016). *An exploratory study of the factors related to successful mathematical problem solving on non-routine unconstrained tasks* [Doctoral dissertation, Temple University, USA]. Digital Library. https://digital.library.temple.edu/digital/api/collection/p245801coll10/id/418552/download

Russo, J. A. (2019). Walking the line between order and chaos: A teacher-researcher's reflection on teaching mathematics with challenging tasks in primary classrooms. *International Journal of Innovation in Science and Mathematics Education*, *27*(3), 14–24. https://doi.org/10.30722/IJISME.27.03.002

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: Simon and Schuster. [reprint] https://doi.org/10.1177/002205741619600202

South Australia Council of Education (2021). Routine and complex calculations, processes, and questions in stage 1 mathematics. SACE. https://view.officeapps.live.com/op/view.aspx?src=https%3A%2F%2Fwww.sace.sa.edu.au%2Fdocuments%2F652891%2F3b5a5164-1661-49cb-8455-c463d8405f03

Sullivan, P., Askew, M., Cheeseman, J., Clarke, D., Mornane, A., Roche, A., & Walker, N. (2015). Supporting teachers in structuring mathematics lessons involving challenging tasks. *Journal of Mathematics Teacher Education*, *18*(2), 123–140. https://doi.org/10.1007/s10857-014-9279-2

Potential Fraction Concept Images Afforded in Textbooks: A Comparison of Northern Ireland and Singapore

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Fractions are among the most problematic concepts that children encounter in their primary school years because of the many different conceptions of fractions. Textbook analyses have tried to provide insights into how fractions are introduced, focusing on the different concepts and representations of fractions. In this paper, we contribute to these efforts by investigating the way fractions are first introduced using the notion of potential concept images as afforded by the textbooks. Analyses of two textbooks, one from Northern Ireland and the other from Singapore, will be presented to highlight these potential concept images and their implications for practice.

It is widely argued that fractions are among the most problematic concepts that students encounter in the primary school years (Charalambous & Pitta-Pantazi, 2006; Gabriel et al., 2013; Siegler & Lortie-Forgues, 2017; Smith, 2002). Despite numerous efforts by researchers, governments, mathematics educators, and textbook writers to reform the learning of fractions over the past few decades, students' performance on fraction arithmetic tasks has shown little improvement (Siegler & Lortie-Forgues, 2017). Fractions, as a "powerful mathematical idea", are difficult to learn because they can be used to "express so many different kinds of relationships" (Smith, 2002, p. 4). As Kieren (1976) had suggested, fractions are best conceptualised not as a single conception, but as a set of inter-related conceptions: part-whole, ratio, operator, quotient, and measure. However, fractions are often introduced only as a part-whole model in many countries. This is limiting and may have contributed to the difficulties faced by students when learning fractions (Simon et al., 2018). Furthermore, textbooks, which make up the main instructional or curriculum materials, do not introduce fractions using a *concept definition*; instead, they often present real-world scenarios, such as sharing pizzas, and other representations to support students in forming their own *concept images* (Tall & Vinner, 1981). Given that textbooks offer examples and representations that may facilitate or hinder learners' formation of useful concept images (Tall & Vinner, 1981; Zhang et al., 2014), it is important for mathematics educators to analyse how fraction concepts are presented in textbooks. Although there have been textbook analyses on the concept of fractions (e.g., see Lee et al., 2021), these analyses focused on the kind of fraction conceptions, tasks, and representations presented in the textbooks. In this paper, we contribute to this conversation by examining the *potential concept images* afforded by how fraction concepts are first introduced in mathematics textbooks from two relatively different education systems—Northern Ireland and Singapore. According to PISA 2022, Singapore remains the world leader in mathematics education, whereas pupils in Northern Ireland achieved a score of 475, not significantly different to the OECD average score of 472, but significantly lower than the score of 492 achieved in 2018. Besides international assessment data, the two countries have different education systems, which offer opportunities for interesting comparisons. The paper is framed by the following question:

> What are the potential concept images of fractions afforded by the textbooks from Northern Ireland and Singapore in the way they first introduce fractions?

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 167–174). Gold Coast: MERGA.

Review of Literature

We begin by reviewing the different conceptions of fractions and highlighting the challenges faced by students as they grapple with these various concepts. Furthermore, the way fractions are presented to students (manipulative, pictorial, verbal, real-world, and symbolic) may privilege certain conceptions over others. Hence, students' difficulties when learning fractions may be partially explained by the limited concept images they might have (Tall & Vinner, 1981; Zhang et al., 2014). As most students develop these mental images through their interactions with textbooks, we argue that it is important for us to investigate how fractions are introduced and represented in the textbooks. Finally, we suggest the notion of potential concept image to better understand how students may perceive fractions from textbooks.

Conceptualisations and Representations of Fractions

According to Smith (2002, p. 3), "no area of elementary mathematics is as mathematically rich, cognitively complicated, and difficult to teach as fractions, ratios and proportionality". Whilst several factors have been identified as contributing to this difficulty, the multifaceted conception of fraction is a key factor associated with the complexities of teaching and learning fractions (Kieren, 1976; Lamon, 2012; Simon et al., 2018). Fractions can be conceptualised in at least five different ways: part-whole, measure, operator, quotient, and ratio (Kieren, 1976; Lamon, 2012). For example, the fraction 2/3 is usually interpreted as two out of three equal parts (a part-whole conception). As a measure, 2/3 is perceived as the distance of two 1/3 units from zero on the number line, which presents fractions as numbers (Simon et al., 2018). Seeing fractions as numbers is also related to the idea that 2/3 is the answer to $2 \div 3$ (a quotient). As an operator construct, 2/3 of a quantity is found by multiplying the unit by two and dividing the result by 3 or dividing the quantity by 3 and multiplying the result by 2. Finally, 2/3 can also be seen as a ratio, which represents a multiplicative relationship between two quantities in a particular order. These different conceptions make fractions a "powerful mathematical idea" because they are used to "express so many different kinds of relationships" (Smith, 2002, p. 4).

However, these different conceptions are not all well understood by students. For instance, Simon et al. (2018) highlight that students often do not see fractions as numbers (measure concept); instead, they often perceive fractions as two numbers: the numerator denotes the number of shaded parts and the denominator denotes the total number of equal parts (a limited understanding of fraction as part-whole). This partial understanding of the various fraction conceptions may hinder students' ability to work with different fraction constructs (Lee & Hackenberg, 2014). As argued by Simon et al. (2018), introducing fractions as "part of a whole" is limiting and may hinder students in developing more "powerful conceptions of fractions" (p. 123). Hence, it is critical for students to encounter these different fraction ideas as part of their learning experiences.

To this end, Cramer et al. (2002) advocated providing opportunities for students to use multiple physical models of fractions and translate between and within the different modes of representation: pictorial, manipulative, verbal, real-world, and symbolic. Similarly, van de Walle et al. (2013) recommended the use of all three types of pedagogical models (area models (e.g., fraction discs), linear models (e.g., fraction strips, number lines, etc.), and set models (e.g., collection of discrete objects) to support students in developing a more robust understanding of fractions. Despite these recommendations, textbooks often favour the use of area models over the other representations (Lee et al., 2021; Simon et al., 2018; Watanabe, 2002), leading to a limited mental image of fractions, which could lead to an erroneous understanding of operations involving fractions (see Watanabe, 2002, p. 462 for an example).

Concept Images

Therefore, it is the networking of fraction conceptions and representations that enable students to perceive fractions in different ways that are useful to the tasks at hand. Specifically, it is crucial for students to have a rich network of mental images associated with the different fraction conceptions. These mental images, which could take the form of pictures, diagrams, graphs, symbols, or other representations, are called concept images. According to Tall and Vinner (1981), a concept image describes “all the cognitive structure in the individual’s mind that is associated with a given concept”, including “all the mental pictures and associated properties and processes” (pp. 151–152). The complication arises because learners’ concept images may not always be aligned to the corresponding formal concept definitions. Hershkowitz and Vinner (1980) suggested that students may develop concept images of a concept before they can make sense and use an appropriate concept definition. These images are developed through the students’ interactions with examples, diagrams, and other representations presented in the classrooms.

But for fractions, there seems to be an over-emphasis on the use of area models, and this results in a limited part-whole fraction conception. This limited conception “seemed to generate concept images of unit fractions which resulted in students learning much about drawing and shading circles, rectangles, and squares, but without developing sound conceptual understandings” of unit fractions (Zhang et al., 2014, p. 228). This issue is further exacerbated by the fact that most primary school textbooks do not introduce the concept of rational numbers, which has a clearly defined concept definition. These findings suggest the importance of textbook writers moving away from “area-model representations of unit fractions toward multiple embodiments” (p. 228) when writing textbooks.

We wonder, therefore, how textbooks can facilitate or hinder the development of concept images that can be productively evoked for solving fraction tasks. Studies that investigated students’ concept images often examined their evoked images when responding to carefully designed tasks (Hershkowitz & Vinner, 1980; Tall & Vinner, 1981; Zhang et al., 2014). But, if textbooks are seen as “objectively given structure” without considering how the textbooks might be used with students (Herbel-Eisenmann, 2007, p. 346), then there is a need to examine how textbooks might play a role in enriching or hindering students’ concept images. Most textbook analyses focus on characterising the concepts, representations, and tasks presented in the textbooks (Charalambous et al., 2010; Choy et al., 2020; Lee et al., 2021). In this paper, we want to analyse textbooks through the lens of concept images by using the notion of *potential* concept images. To be clear, we are not looking at evoked concept images; instead, we are trying to determine if various combinations of conceptions and representations might afford certain concept images, which give rise to opportunities for students to understand fractions.

Methods

To derive the potential concept images as afforded by textbooks, we first examined the textbooks with respect to the contextual variables of the two countries (Huntley, 2008) before we analysed the content and instructional variables (Charalambous et al., 2010; Lee et al., 2021). We focused on the five fraction conceptions (Kieren, 1976; Lamon, 2012) for the content variable and the fraction representations (Cramer et al., 2002; Watanabe, 2002) for the instructional variable.

Contexts of the Two Countries

In Northern Ireland (NI), it is compulsory for all children to have at least 12 years of education (seven years of primary education and five years of post-primary education) and the opportunity to obtain school leaving qualifications. The 12 years of compulsory education are divided into five Key Stages. Primary education comprises a Foundation Stage (Years 1 and 2), Key Stage 1 (Years 3 and 4) and Key Stage 2 (Years 5, 6 and 7). The NI Primary Curriculum (Council for the Curriculum Examinations & Assessment (CCEA), 2007) sets out the minimum requirement that should be taught at each of the three stages of primary education: Foundation Stage, Key Stage 1 and Key Stage 2. Teachers, however, have considerable flexibility to make decisions about how best to interpret and combine the requirements in order to prepare young people for the future. In the NI Curriculum for Mathematics and Numeracy, fractions are included within the 'Number' strand and are first introduced in Key Stage 1. In Key Stage 1, pupils "should be enabled to … recognise and use simple everyday fractions' (p. 62), and in Key Stage 2 they 'should be enabled to … understand and use vulgar fractions, decimal fractions and percentages and explore the relationships between them" (p. 65).

In Singapore, there is compulsory education for primary school education, which comprises levels from Primary One (students aged 7) to Primary Six (students aged 12). As a centralised education system, there is a standardised curriculum for all primary schools that specify the content learning outcomes, learning experiences, pedagogical approaches, and assessment requirements for each level (Ministry of Education-Singapore, 2019). Unlike in Northern Ireland, Singapore teachers generally follow the standards set out in the curriculum documents very closely. However, they have flexibility to decide on the teaching approaches to cater to the needs of their learners. Like NI, fractions are introduced as part of the 'Number and Algebra' content strand and are first introduced in Primary Two. Fraction concepts are progressively developed from Primary Two through Primary Six, covering key fraction concepts such as fractions as part-whole, equivalent factions, mixed numbers, and improper fractions, as well as fraction operations (see Figure1).

Figure 1

Development of Fraction Concepts in Singapore (Ministry of Education-Singapore, 2012, p. 36).

P2	P3	P4	P5	P6
1. Fraction of a whole 1.1 fraction as part of a whole 1.2 notation and representations of fractions 1.3 comparing and ordering fractions with denominators of given fractions not exceeding 12 • unit fractions • like fractions **2. Addition and subtraction** 2.1 adding and subtracting like fractions within one whole with denominators of given fractions not exceeding 12	**1. Equivalent fractions** 1.1 equivalent fractions 1.2 expressing a fraction in its simplest form 1.3 comparing and ordering unlike fractions with denominators of given fractions not exceeding 12 1.4 writing the equivalent fraction of a fraction given the denominator or the numerator **2. Addition and subtraction** 2.1 adding and subtracting two related fractions within one whole with denominators of given fractions not exceeding 12	**1. Mixed numbers and improper fractions** 1.1 mixed numbers, improper fractions and their relationships **2. Fraction of a set of objects** 2.1 fraction as part of a set of objects **3. Addition and subtraction** 3.1 adding and subtracting fractions with denominators of given fractions not exceeding 12 and not more than two different denominators 3.2 solving up to 2-step word problems involving addition and subtraction	**1. Fraction and division** 1.1 dividing a whole number by a whole number with quotient as a fraction 1.2 converting fractions to decimals **2. Four operations** 2.1 adding and subtracting mixed numbers 2.2 multiplying a proper/improper fraction and a whole number **without calculator** 2.3 multiplying a proper fraction and a proper/improper fractions **without calculator** 2.4 multiplying two improper fractions 2.5 multiplying a mixed number and a whole number 2.6 solving word problems involving addition, subtraction and multiplication	**1. Four operations** 1.1 dividing a proper fraction by a whole number **without calculator** 1.2 dividing a whole number/ proper fraction by a proper fraction **without calculator** 1.3 solving word problems involving the 4 operations

Selection of Textbooks

Schools in NI tend to rely on materials published in the United Kingdom to support their mathematics teaching. Anecdotal evidence suggests that, as in England, there is a vast array of mathematics curriculum resources in use, including textbook-schemes, online-schemes, and teacher-curated resources. For this study, we decided to focus on the New Heinemann Maths (NHM) series, a complete mathematics programme for children aged 4 to 11. Produced in England, NHM was first developed over 20 years ago and has been one of the more popular textbook-schemes in NI. In our analyses, we looked at NHM 3 (Scottish Primary Mathematics Group, 2000), where fractions were introduced for the first time in textbooks.

In contrast, there is only one textbook series, written by the Ministry of Education, to be used in Singapore primary schools for the current syllabus (Ministry of Education-Singapore, 2019), which was implemented in 2021. The former series of textbooks used for the 2013 syllabus will be progressively retired as the current syllabus progressively replaces the 2013 version from 2021 to 2026. As we are looking at how fractions are introduced, we will only analyse the current textbook for Primary Two, *Primary Mathematics Textbook 2B* (PMT 2B), published by Star Publications (Curriculum Planning & Development Division, 2022).

Data Analysis

With the aim of discerning the potential concept images as afforded by the textbooks, we followed Herbel-Eisenmann (2007) in seeing textbooks as "objectively given structure" (p. 346) without considering how the textbooks might be used with students. We wanted to examine how the examples, diagrams, and explanations might come together in presenting certain concept images to the readers. As we were interested in how fractions are first introduced in primary schools, we only analysed the relevant chapters that first present fractions as a mathematical concept (e.g., we did not analyse the chapter on time where terms such as "half past nine in the morning" might suggest some concepts of fractions). Although this may appear to be limited, it is important to consider how the foundations are established as this will impact how the learner's knowledge and understanding develops over time and how this knowledge and understanding will be used and applied in other contexts.

Analyses occurred at the item level where we saw each example, question, representation, paragraph of explanatory text, or task as an item. We first coded the items using the five conceptions of fractions—part-whole, measure, quotient, operator, and ratio (Kieren, 1976; Lamon, 2012)—depending on what was explained or needed to complete the item. We then coded the representations used (i.e., concrete, pictorial (area, length and set models), verbal, and symbolic) using the categorisation used by Lee et al. (2021). Coding was done independently in which the first author coded the Singapore textbook while the second author coded the NI textbook, before we came together to share and justify our analyses for our respective textbooks. Consensus was reached through discussion. We then proposed possible concept images based on our analyses of the conceptions and representations by connecting these two aspects of textbooks.

Findings and Discussions

In this section, we describe, compare, and contrast two key potential concept images afforded by the chosen textbooks from the two countries. They are "fractions as equal parts of a whole" and "fractions can have different sizes". As we describe these potential images, we also attempt to relate them to students' learning difficulties in extant literature before we conclude with some implications for practice.

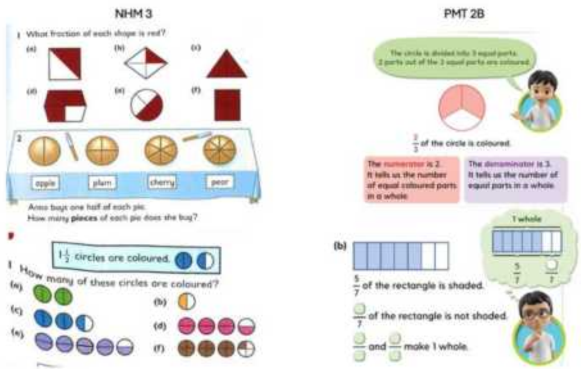
Fractions are Equal Parts of a Whole

We expected that "fractions are equal parts of a whole [shape]" to be one of the potential concept images afforded by the two textbooks. However, we made a nuanced distinction from the usual part-whole conception of fractions (Lamon, 2012) by highlighting what students might perceive about the equality of parts from the fraction representations. As seen in Figure 2, fractions are often represented in area models, comprising circles or rectangles, with equal parts that are *congruent* to each other. This image is also reinforced by explanations such as "The circle is divided into 3 equal parts. 2 parts out of three equal parts are coloured. So, 2/3 of the circle is coloured" [PMT 2B, p. 55].

This conception-pictorial-textual link might be wrongly perceived by students (and even teachers) to mean that there is no fraction when the parts are not equal (Choy, 2013). Seeing that the correct answers always involve equal parts with congruent parts, it might seem logical for students that fractions must always be equal and *congruent* parts of a whole (Simon et al., 2018), resulting in this flawed concept image. This flawed concept image may be potentially evoked by students when solving tasks which require students to identify pictures that show a given fraction or to identify the fraction in each picture [PMT 2B, p. 54; NHM 3, p. 70]. Therefore, our textbooks present at least two potential concept images associated with the part-whole conception. First, the mental image that a shaded part is 1/*n* of a given shape (area model) because its area is 1/*n* of the area of the given shape. Second, the flawed mental image that a shaded part is 1/*n* of a given shape because the shape is divided into *n* congruent parts.

Figure 2

Part-Whole Representations of Fractions (Curriculum Planning & Development Division, 2022, p. 55; Scottish Primary Mathematics Group, 2000, p. 70)

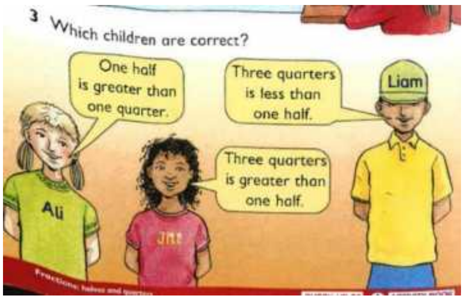


Fractions can Have Different "Sizes"

Several items in both the NHM 3 and PMT 2B afford the potential concept image of fractions having different sizes. Clearly, as a measure, fractions can be ordered *on a number line*. Here, the NHM 3 included a task that aims to encourage students to count in halves on a number line from 0 to 7. This task is followed by another one which provides students opportunities to determine which statement is true (see Figure 3). There are two possible answers to this task. First, if we take these fractions as a number (measure), then there is a fixed answer in that ¼ < ½ < ¾. Second, if we think of each fraction as part of a whole, then we will need to be clear about the referent before we can decide. Taken together, these tasks afford the potential image of fractions have different sizes because they can be *ordered* on a number line.

Figure 3

Task in NHM 3 on Comparing Fractions (Scottish Primary Mathematics Group, 2000, p. 71)

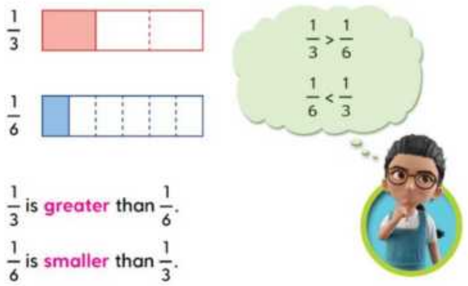


However, for the PMT 2B, there is no number line to indicate that fractions can be ordered independent of context. Instead, the potential concept image with respect to comparing and ordering fractions is associated with the *size of the shaded area* of the figure, which is consistent with the focus on the part-whole conception of fraction. Referring to Figure 4, we see that the basis for deciding which fraction is greater is the area of the shaded region—the bigger the area, the greater the fraction.

Although both textbooks offer potential concept images about the possibility of comparing and ordering fractions, they differ qualitatively in terms of what counts as size. The use of the number line in NHM 3 to illustrate the measure conception of fraction affords students a concept image that potentially relates the magnitude of a fraction to its location along the number line. Whereas in PMT 2B, the area model provides the basis for comparing and ordering fractions using the size (area) of the fraction pieces. This potential concept image may not be robust enough to deal with cases where the whole is of different sizes. The NHM 3, on the other hand, provides another potential image through the task in Figure 3.

Figure 4

Task in PMT 2B on Comparing Fractions (Curriculum Planning & Development Division, 2022, p. 57)



Concluding Remarks

Our analyses of the two textbooks through the lens of potential concept images is by no means comprehensive as there may be other potential concept images. However, they suggest a layer of complexity beyond coding for conceptions and representations in textbook analyses: the same representation in a textbook can potentially evoke different concept images in students. For instance, the heavy focus on the "part-whole" fraction concepts using the area models in PMT 2B can give rise to both an acceptable concept image and a flawed one. However, the surfacing of flawed potential concept images in a textbook does not mean that the textbook is inferior in any way. Nor would the number of potential concept images mean anything about the quality of a textbook. What we offer through the notion of potential concept images is to provide a way to connect conceptions, representations, and tasks in the textbooks for the purpose of highlighting different ways of understanding the same concept.

These potential concept images, as developed through our content and instructional analyses of the textbooks, can be seen as students' possible understanding of fraction ideas. As suggested by Smith (2002), it is useful to "work with students to understand fractions and ratios in their own terms first" for the purpose of making sense of the formal concepts of fractions later (p. 5). Hence, these potential concept images can serve as teachers' anticipation of students' ideas about fractions. They also sensitise teachers to the possibilities that students may perceive what was written in the textbooks differently. In addition, in the hands of a skilled teacher, an awareness of these flawed potential concept images can provide the locus of action for teachers to take when planning their instruction. Finally, given the powerful role that textbooks play in mathematics teaching and learning, it is vital that textbook writers carefully consider the potential concept images that may be evoked from the pedagogical approaches recommended.

References

Charalambous, C. Y., Delaney, S., Hsu, H.-Y., & Mesa, V. (2010). A Comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical Thinking and Learning*, *12*(2), 117–151. https://doi.org/10.1080/10986060903460070

Charalambous, C. Y., & Pitta-Pantazi, D. (2006). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics*, *64*(3), 293–316.

Choy, B. H. (2013). Productive mathematical noticing: What it is and why it matters. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Proceedings of 36th annual conference of Mathematics Education Research Group of Australasia.* (pp. 186–193). MERGA.

Choy, B. H., Lee, M. Y., & Mizzi, A. (2020). Insights into the teaching of gradient from an exploratory study of mathematics textbooks from Germany, Singapore, and South Korea. *International Electronic Journal of Mathematics Education*, *15*(3). https://doi.org/10.29333/iejme/8273

Council for the Curriculum Examinations & Assessment (CCEA). (2007). *The Northern Ireland curriculum: Primary*. CCEA.

Cramer, K. A., Post, T. R., & delMas, R. C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the rational number project curriculum. *Journal for Research in Mathematics Education*, *33*(2), 111–144. https://doi.org/10.2307/749646

Curriculum Planning & Development Division. (2022). *Primary mathematics textbook 2B*. Star Publishing Pte Ltd.

Gabriel, F., Coche, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2013). A componential view of children's difficulties in learning fractions. *Frontiers in Psychology*, *4*, 715. https://doi.org/10.3389/fpsyg.2013.00715

Herbel-Eisenmann, B. A. (2007). From intended curriculum to written curriculum: examining the "voice" of a mathematics textbook. *Journal for Research in Mathematics Education, 38*(4), 344–369.

Hershkowitz, R., & Vinner, S. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the 4th international conference for the Psychology of Mathematics Education* (Vol. 1, pp. 177–184). PME.

Huntley, M. A. (2008). A framework for analyzing differences across mathematics curricula. *Journal of Mathematics Education Leadership, 10*(2), 10–17.

Kieren, T. E. (1976). On the mathematical, cognitive and instructional foundations of rational numbers. In R. Lesh & D. Bradbard (Eds.), *Number and measurement: Papers from a research workshop* (pp. 101–144). Information Reference Center (ERIC/IRC), The Ohio State University.

Lamon, S. J. (2012). *Teaching Fractions and ratios for understanding* (3rd ed.). Routledge.

Lee, M. Y., Choy, B. H., & Mizzi, A. (2021). Exploring the introduction of fractions in Germany, Singapore, and South Korea mathematics textbooks. *Journal of the Korean Society of Mathematics Education Series D: Research in Mathematics Education*, *24*(2), 111–130. https://doi.org/10.7468/jksmed.2021.24.2.111

Lee, M. Y., & Hackenberg, A. J. (2014). Relationships between fractional knowledge and algebraic reasoning: The case of Willa. *International Journal of Science and Mathematics Education*, *12*(4), 975–1000. https://doi.org/10.1007/s10763-013-9442-8

Ministry of Education-Singapore. (2012). *Primary mathematics teaching and learning syllabus*. Curriculum Planning and Development Division.

Ministry of Education-Singapore. (2019). *Mathematics teaching and learning syllabus: Primary*. Curriculum Planning and Development Division.

Scottish Primary Mathematics Group. (2000). *New Heinemann maths* (Vol. 3). Heinemann.

Siegler, R. S., & Lortie-Forgues, H. (2017). Hard lessons: Why rational number arithmetic is so difficult for so many people. *Current Directions in Psychological Science*, *26*(4), 346–351. https://doi.org/10.1177/0963721417700129

Simon, M. A., Placa, N., Avitzur, A., & Kara, M. (2018). Promoting a concept of fraction-as-measure: A study of the Learning Through Activity research program. *The Journal of Mathematical Behavior*, *52*, 122–133. https://doi.org/10.1016/j.jmathb.2018.03.004

Smith, J. P. (2002). The development of students' knowledge of fractions and ratios. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 3–17). National Council of Teachers of Mathematics.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

van de Walle, J., Karp, K. S., & Bay-Williams, J. M. (2013). *Elementary and middle school mathematics teaching developmentally* (8th ed.). Pearson.

Watanabe, T. (2002). Representations in teaching and learning fractions. *Teaching Children Mathematics, 8*(8), 457–463.

Zhang, X., Clements, M. A., & Ellerton, N. F. (2014). Enriching student concept images: Teaching and learning fractions through a multiple-embodiment approach. *Mathematics Education Research Journal, 27*(2), 201–231. https://doi.org/10.1007/s13394-014-0137-4

Delving Deeply Into Interviews With Timeline Tools

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Semi-structured interviews are used to gain insights into participants' lived experiences and perspectives on issues, but they are open to subjectivity. To address this issue our study explored the combination of timeline graphic elicitation tools with semi-structured interviews as an approach to gain insights into teachers' experiences of mathematics teaching and professional learning. A qualitative study was conducted with ten participants from two schools who took part in professional learning activities for mathematics teaching. Findings indicated that combining these instruments can support researchers in gathering deeper insights into teachers' lived experiences.

Oral histories are personal memories and commentaries that hold historical significance and are commonly used for research. These memories offer insights from individuals deeply involved in the research subject matter (Galletta, 2013). Capturing participants' oral histories via semi-structured interviews offers valuable insights into lived experiences. Through a balance of flexible, open-ended questions and well-prepared structured interview questions, researchers can delve deeply into participants' histories (Kervin et al., 2016). Semi-structured interviews are a versatile method that allows researchers to explore lived experiences and investigate questions arising from existing theories (Galletta, 2013).

While semi-structured interviews are widely used in research, it can be argued that they have limitations. Charmaz (2014) pointed out a "common criticism of interviews is that they are tainted by the participants' subjectivity" (p. 80). Kervin et al. (2016) concur that participants' reflections during interviews may be influenced by limited, biased, or inaccurate recollections, as well as varying degrees of articulation. To ensure participants feel safe and comfortable, physically, mentally, and socially, it is critical to enhance our understanding of how "tools" might support emotions and reflections during interview design.

Incorporating graphic elicitation tools into the design of semi-structured interviews may assist mathematics education researchers in overcoming these limitations. Such tools are commonly used in the arts, psychology, and social sciences (Bravington & King, 2019), but are only emerging in mathematics education research. Lewis (2013) described them as a novel method of reflection while Galetta (2013) claimed they could help to elicit a narrative and shed light on participants' experiences. Providing tools to support mathematics teachers' reflections may deepen interview narratives and provide insight into significant events.

As part of the first author's PhD studies, a combination of timeline graphic elicitation tools and semi-structured interviews were used to collect data. This paper presents reflections on combining these instruments and methods to gain insights into teachers' experiences of mathematics teaching and professional learning (PL). The specific research question is:

- How might the combination of timeline graphic elicitation tools and semi-structured interviews support researchers to gain insights into teachers' experiences of mathematics teaching and professional learning?

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Graphic Elicitation Tools

This section presents the literature review with a focus on graphic elicitation and the use of timeline tools in mathematics education research.

Graphic elicitations are drawings or diagrams that may be produced by the researcher or participant as a contemporary addition to a conventional interview (Bagnoli, 2009; Crilly, 2006). Self-portraits, chain, hub and spoke, and network diagrams, photographs, timelines, and continuums are common variations (Bagnoli, 2009; Bravington, & King, 2019). From a researcher's perspective, combining graphic elicitation tools with a semi-structured interview "allows one to see things from different perspectives and to look at data in creative ways" (Bagnoli, 2009, p. 567) while gathering "rich and nuanced data" (Crilly, 2006, p. 342). From a participants' perspective, graphic elicitation tools may help participants to reflect on their experiences when conveying them to the researcher. For example, diagrams are effective tools to express one's thoughts and to communicate those thoughts to others (Crilly, 2006). In addition, graphic elicitation tools may provide a basis to elicit a deeper level of communication and reflection and encourage thinking in non-standard ways that may be difficult to obtain by other means (Bagnoli, 2009; Crilly, 2006). For example, Bagnoli (2009) described the use of self-portraits as a useful tool to *break the ice* in an interview while participants simultaneously expressed their identities through drawings.

Graphic elicitation tools have become increasingly popular among researchers (Bravington & King, 2019). Recently graphic elicitation tools in the form of timeline tools have been used to aid qualitative mathematics research. Timeline tools may be used to "collect the most important turning points and biographical events" (Bagnoli, 2009, p. 561) as seen from participants' perspectives. Lewis (2013) used a timeline tool to capture high school students' reflections of mathematics, whereas Bobis et al. (2021) used a similar timeline tool to explore teachers' development of mathematics identity. Combining a semi-structured interview with a timeline tool in this way yields insights into the impact of events that may not have been obvious to participants (Bobis et al., 2021). Furthermore, this combination of methods supports participants when explaining how and why experiences were considered impactful (Bobis et al., 2021.

As a "novel quantitative" method (Lewis, 2013, p. 76), the timeline tool allows teachers to reflect upon their perspectives and past experiences and consider what events were influential in developing their mathematics teaching practice. Corovic and Downton (2021) found the "inclusion of the timeline tool facilitated questioning to generate rich dialogue and valuable reflections" (p. 160). Semi-structured interviews allow for the conversation to be simultaneously flexible in design, moving within the flow of conversation, yet holding the ability to dig deeper into the research phenomena in a safe and trusting environment.

Method

To conduct the research, a qualitative method was used which involved combining a timeline graphic elicitation tool (see Figure 1) with semi-structured interviews.

Context and Participants

Ten participants were selected from two schools (based on specific characteristics required for the PhD study) using a purposive sampling method (Palinkas, 2015). Five teachers from each school agreed to participate in the study. They were all involved in a Foundation to Year 2 mathematics (students aged 5 to 8 years old) PL focused on teaching sequences of challenging tasks, facilitated by Emeritus Professor Peter Sullivan.

At School A (Melbourne Catholic school) teachers were selected by school leaders who perceived that the Year 1 team ($n = 2$) were resistant to changing their practice while the Year 2 team ($n = 3$) at the same school had embraced the PL practices. These five teachers had experienced the same PL, school context, and support, yet had different PL outcomes. While the PL method varied slightly for School B (Melbourne Government school) these teachers were teaching the same year levels as School A (Year 1 teachers $n = 2$, Year 2 teachers $n = 3$) and were also selected for the study.

Instruments

Two instruments were used in this study: a timeline graphic elicitation tool and an open-ended semi-structured interview. Both are explained next.

Semi-Structured Interview

An interview guide informed by Galletta's (2013) framework was developed. The guide had three parts. Part one was designed to discover participants' previous experience with mathematics education and to develop a positive and safe rapport between the interviewee and researcher. Part two focussed on open-ended questions to provide opportunities for teachers to provide commentary on their PL experiences. Part three was designed to provide time to clarify ideas and provide a final reflection. Questions throughout the guide aimed at gathering information relating to teachers' experiences including what the experiences were and their impact on teachers' practice.

Timeline Graphic Elicitation

Combining a timeline graphic elicitation tool with semi-structured interviews was inspired by the work of Bobis et al. (2021). After careful investigations and deliberation, revisions were made to the tool. These revisions were informed by Lewis' (2013) and Baglioni's (2009) versions and Murdoch and Wilson's (2004) student reflection timeline tool. Three sample tools were developed. After testing, discussing, and modifying, one tool was selected for the PhD study and used twice during each participant's interview, but with a different focus.

Data Collection and Analysis

For data collection, the first author conducted interviews with each participant using the timeline tool. School A's interviews were held online via Zoom due to COVID-19 in August 2021. School B's interviews were held in person in November 2023.

Prior to the interviews, participants completed the first timeline tool to elicit information about their previous experiences with mathematics education, starting from their preservice teacher education up until their recent participation in PL. Responses were brought to the interview by the teacher and used to help initiate discussions. Time was allocated during the interview (5 minutes approximately) to complete the second timeline tool. This tool focused on gathering information about teacher's experiences in recent mathematics PL. Each interview took approximately 45 minutes. Interviews were video recorded and then the audio was transcribed for data analysis. The completed timeline tools were also kept for analysis. The authors then watched the videos to record the frequency of teacher gestures, specifically pointing to the timeline tool. Data collected was used to respond to the research question.

An inductive thematic analysis was conducted using all forms of data collected: interview transcripts, non-verbal communication (pointing gestures), and timeline tools. Braun and Clarke's (2006) six-phase guide to performing thematic analysis was used. Following a phase of data familiarisation, initial codes were generated by the researchers. Codes were collected and sorted into potential themes such as: breaking the ice, alleviating anxiety, novel methods to collect nuanced data, capturing insights that were not immediately obvious, and insights into event influence. The themes were defined and named, and finally, extracts were selected to communicate the results. For this paper, we report on the use of instruments to gain insights into teachers' experiences of mathematics teaching and PL.

Results and Discussion

This section provides a brief overview of using the timeline tool and semi-structured interviews. Five themes will be discussed. Only one timeline tool example is included due to page limitations.

Breaking the Ice

Completing the timeline tool prior to the interview most likely helped participants prepare for the interview by reflecting on experiences that had influenced their mathematics teaching practice. This conclusion was confirmed because few prompts by the interviewer were required to instigate conversations. Teachers easily explained their recordings on each timeline tool, and they were able to take the lead in the conversation. This was evident as teachers pointed to and unpacked their responses. Table 1 shows the number of times each teacher participant pointed directly to their timeline tool.

Table 1

Frequency of Teacher Pointing to the Timeline Tool

Teacher	Timeline tool 1	Timeline tool 2
Andy	9	5
Nadia	17	9
Phoebe	2	3
Evelyn	2	2
Olivia	11	11
Ella	8	5
Ceilia	22	9
Celeste	25	6
Eliza	8	5
Natalie	8	12
Average	11.2	6.7

As shown in Table 1, teachers (pseudonyms used throughout) referred to, by pointing, the first timeline tool on average 11.2 times and the second timeline tool 6.7 times. Teachers were likely to point to the first tool more as there were more plot points drawn on them compared to the second tool. This is largely due to the first tool seeking reflections of teachers' career experiences, whereas the second tool was of a specific duration (two years for School A and one year for School B). For example, Ceilia (see Figure 1) drew seven plot points on her first timeline tool and three plot points on her second timeline tool. During the interview, participants frequently pointed to their recordings which helped them to 'open-up' and share reflections of their experiences. This suggests that the tool was effective in *breaking the ice* and encouraging conversations, as proposed by Bagnoli and King's (2009) notion that graphic elicitation can support interviews.

Alleviating Anxiety

Several teachers were noted to appear highly anxious at the commencement of the interview. This was evident in their body language, voice, and their expression of nerves. The combination of instruments appeared to help ease anxiety as the teachers drove the conversation and elaborated on various teaching events. Ceilia was an example of a teacher who arrived at her interview feeling very nervous. During the interview, the interviewer observed that Ceilia's voice was quavering; she laughed nervously and sat with her arms crossed l. However, as soon as the interviewer asked Ceilia to explain her first timeline tool (see Figure 1), she began to relax.

Figure 1

Ceilia's Timeline and Written Reflection of Teaching Experience Over Time (First Timeline Tool)

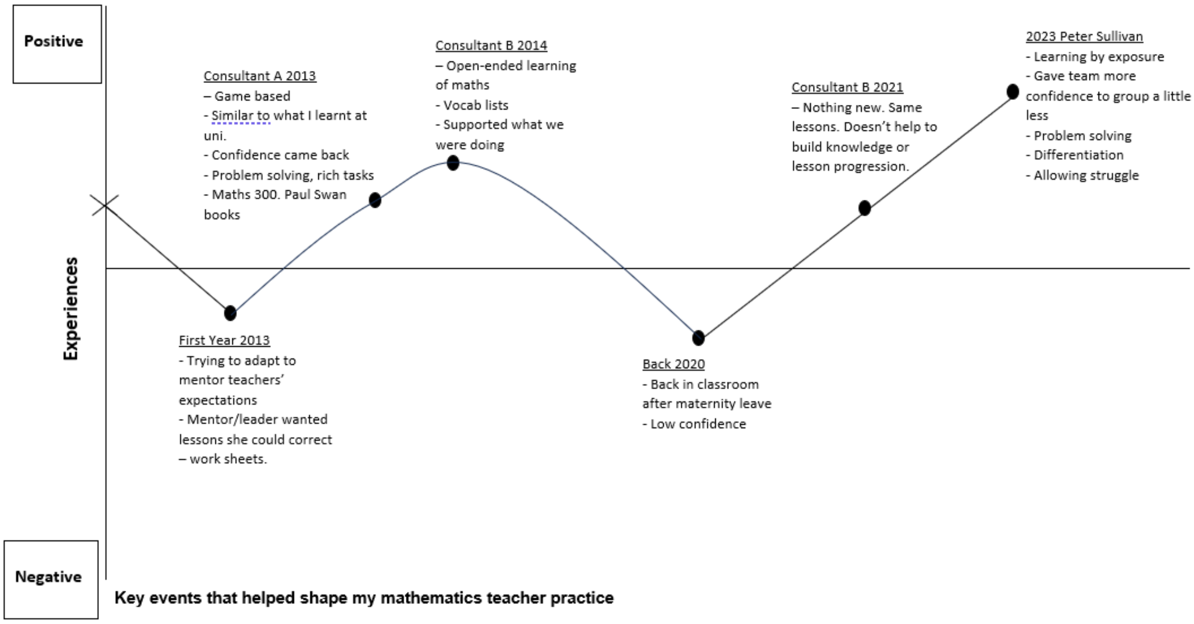

Figure 1 shows Ceilia's completed timeline tool of experiences that impacted her mathematics teaching practice. This response has captured background information related to the teaching experience of a particular teacher. It includes the duration of her teaching career, the schools she has worked at, the year levels she has taught, and any significant events that have influenced her mathematics teaching practice. While Ceilia was nervous to begin with, halfway through the first prompt she used her right hand to point to the timeline tool indicating the point she was referring to. Soon after she began to use her right hand to gesture. Ceilia began using two-handed gestures at 1.26 minutes into the interview. She spoke expressively while elaborating on her discussion points. The timeline tool provided an effective instrument for gathering Ceilia's historical data. Moreover, the instruments appeared to assist with alleviating anxiety by providing something for her to focus on stimulating the conversation.

Capturing Insights That Were Not Immediately Obvious

The combination of tools supported teacher reflections. This was evident as teachers were prepared to discuss experiences recorded on their timelines. In addition, most of the teachers added further reflections to their remembrances during the interview. At times the conversations deviated from a point on the timeline tool and revealed additional insights. Such was the case in Evelyn's interview. She alluded to her colleagues having an impact on her adoption of the PL practices. By probing into this additional information, the interviewer was able to capture further insights that weren't immediately obvious:

Interviewer: Then you've gone up again [pointing to planning on the timeline tool]. So, what have been some of the positives that have affected you?

Evelyn: So, I think that that planning, planning with someone. The observation was so helpful and the conversation we had afterwards. I know I got a lot out of it. That idea that it's not just scripted [the PL provided resource materials]. So, I think I was getting caught up with some of the other [team] talk which is 'oh no we have to go exactly by the book'

Interviewer: How was it helpful?

Evelyn: I think we've [team] kind of got to the point where we weren't reading the chapters [of the PL supplied resource book containing sequences of lessons], we were just looking at the activities. … I think

I lost a bit of track because the people around me had lost track. It was really good for me to be brought back into it and I think I just needed that refresher.

Interviewer: So, the team environment is having an impact on your participation in the learning and adoption, is that right?

Evelyn: I wish it was a bit more … I wish I could dive into it properly, but I can't [due to the teaching team environment]. So, it can be a bit demotivating.

This extract from Evelyn's interview highlights that while she began discussing an experience that had a positive impact on her, *planning*, she reflected on her needs as she spoke and revealed that her teaching team had a negative impact on her adoption of the PL practices. This had not been recorded on her timeline tool. Therefore, it was through a conversation about planning that she recognised that the team had had a significant impact on her. Due to the nature of a semi-structured interview, the interviewer was able to probe deeper into the experience to provide additional insights that were not immediately obvious to the participant. The combination of instruments supported reflection prior to, and deeper reflection during, the interview.

Rich Data

The combination of tools assisted in collecting rich data in a novel manner. Teachers' completed timeline tools provided an innovative way to collect historical data and the interview provided an opportunity to collect additional data of interest to the interviewer which had not yet been captured on the timeline tool. This was evident in Ella's interview:

Ella: In 2019 we did a walk-through where teachers had to do a maths lesson and parents and the principal watched.

Interviewer: What made this experience so impactful?

Ella: I think it was really hands on and we made it cross curricular. … We were doing maths, but it was linked to our [class] novel. The kids were really into it.

Interviewer: What grade were you teaching then?

Ella: Year 3

From this conversation with Ella, the data gathered included the following:

- Teaching background: In 2019 Ella was teaching Year 3;
- Significant events: A mathematics walk-through with parents and the principal;
- Impact of the event: It was hands-on, cross-curricula, and kids were into it;
- The degree of the impact: Very highly positively ranked on the timeline tool.

The extract illustrates how the timeline tool and interview supported data collection related to participant background information (teacher experiences) and insights into research phenomena (impact on teacher practice). This combined approach to data collection supported a rich understanding of the teacher's experiences.

Insights into Event Influence

Completed timeline tools provided an indication of the level of impact experiences had on teachers' practice. Conversations during interviews provided a greater depth of insight into the significance of these experiences. This was evident in Nadia's interview. Her timeline tool recordings indicated positive and negative experiences. She recorded the *new format* (pedagogical approach) as significantly low on the vertical axis and the *resource book* (support materials) as mildly low, indicating both were negative experiences for her. In contrast, Nadia recorded *colleagues* as mildly high on the positive side of her timeline tool, indicating a positive impact on her teaching practice. The extract below commences with Nadia explaining these first three points:

> So, I've got the new format to maths lessons being quite raw and uncertain about where I was headed … and the introduction of the resource book.

> Looking at those chapters and really breaking it apart, definitely helped.
>
> Working with colleagues and I'd say, working with and progressively even better as the year developed last year. I know [Maths Leader] was a big help in that way for me 'cause I really loved her modelling.

This extract shows how the data gathered in the interview complements the timeline tool by providing additional insights into Nadia's experiences. The timeline tool indicated some level of impact, however, the interview conversation allowed for further insights into the significance of the event. Without the timeline tool, the level of impact would be difficult to ascertain, or further questions in the interview would be required.

Conclusion

This study intended to explore the combination of timeline graphic elicitation tools with semi-structured interviews as a qualitative methodological research approach to gain insights into teachers' lived experiences. The findings demonstrated the effectiveness of this approach in facilitating conversations, alleviating perceived interviewee anxiety, and deepening reflections. The timeline tools emerged as valuable aids in prompting discussions about participants' teaching backgrounds and experiences with professional learning. Notably, participants frequently referenced the recordings they had made on their completed tools, underscoring their utility in shaping, and stimulating conversations.

This research contributes to the evolving landscape of qualitative research methodologies in mathematics education by building on Lewis's (2013) use of timeline tools as a novel method of reflection. Our findings suggest that in addition to supporting participants' reflections, the combined method may assist in fostering in-depth exploration of their experiences. As educational research increasingly recognises the importance of participants' perspectives, this study offers a valuable method for researchers seeking to employ innovative instruments that engage and empower participants. As Bobis et al. (2021) noted, the combination of timeline tools "encourages participants to provide rich descriptions of past experiences" (p. 137).

We acknowledge that a limitation of this study was focussing on one methodological approach, which may overlook potential insights gained from comparing or broadening approaches. However, the combined use of timeline graphic elicitation and semi-structured interviews emerge as an avenue for future investigations, offering researchers an innovative tool to capture nuanced data and deep insights into teachers' lived experiences of teaching, and professional learning in mathematics. Future studies that compared interview data captured with and without the timeline tool, and the use of other graphic elicitation tools would be of benefit.

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School B: Ethics approval [32570] was granted by Monash University, and teacher participants and parents/caregivers of the students provided informed consent.

References

Bagnoli, A. (2009). Beyond the standard interview: the use of graphic elicitation and arts-based methods. *Qualitative Research*, *9*(5), 547–570. http://doi.org/10.1177/1468794109343625

Bobis, J., Nguyen, J., & McMaster, H. (2021). Exploring the ‘high’ and ‘low’ points in primary preservice teachers’ mathematics identity. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo & S. L. Chin (Eds.), *Excellent in mathematics education: Foundations & pathways. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 131–146). MERGA.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Bravington, A., & King, N. (2019). Putting graphic elicitation into practice: Tools and typologies for the use of participant-led diagrams in qualitative research interviews. *Qualitative Research*, *19*(5), 506–523. http://doi.org/10.1177/1468794118781718

Charmaz, C. (2014). *Constructing Grounded Theory (2nd ed)*. SAGE Publications Ltd.

Corovic, E., & Downton, A. (2022). Resource materials as structured guidance in practice change. In. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 154–161). MERGA.

Crilly, N., Blackwell, A., & Clarkson, P. (2006). Graphic elicitation: Using research diagrams as interview stimuli. *Qualitative Research, 6*(3), 341–366. http://doi:10.1177/1468794106065007

Galletta, A. (2013). *Mastering the semi-structured interview and beyond: From research design to analysis and publication*. NYU Press.

Kervin, L., Vialle, W., Howard, S., Herrington, J., & Okely, T. (2016). *Research for educators (2nd ed)*. Cengage Learning Australia Pty Ltd.

Lewis, G. (2013). Emotion and disaffection with school mathematics. *Research in Mathematics Education, 15*(1), 70–86. http://doi:10.1080/14794802.2012.756636

Murdoch, K., & Wilson, J. (2004). *Learning links: Strategic teaching for the learning-centred classroom*. Curriculum Corporation.

Palinkas, L.A., Horwitz, S.M., Green, C.A., Wisdom, J.P., Duan, N., Hoagwood, K. (2015). Purposeful sampling for qualitative data collection and analysis in mixed method implementation research. *Adm Policy Mental Health*, *42*(5), 533–44. doi:10.1007/s10488-013-0528-y

Learning to Share Fairly: The Importance of Spatial Reasoning in Early Partitioning Experiences

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Young children often explore partitioning as the idea of fair sharing in contexts where equal parts are created and distributed based on spatial constructs of the objects, rather than enumerating parts or collections. However, the presence of spatial reasoning in children's early fraction experiences is implicit within much of the literature and has not been explored pedagogically in a range of early schooling contexts. This study reports on a selection of data from a larger Design Based Research study that demonstrates the power spatial reasoning plays in developing early partitioning. Implications for teaching and learning are discussed.

Partitioning is considered the foundational concept that enables children to develop the multiplicative foundations needed to work with an extended range of fractions (Confrey et al., 2014; Kieren, 1993; Lamon, 2007). Young children typically explore partitioning by creating fair shares and equal parts in everyday play experiences (such as sharing a set of counters equally or dividing lumps of playdough into equal shares). Siemon (2003) argued that partitioning is the missing link between young children's experiences with fractions in the early years of schooling and their performance and capabilities for multiplicative and proportional reasoning in later years. While Siemon's (2003) research reports on data from over 20 years ago, these issues still exist in primary and middle school children's understandings today in Australia and internationally (Callingham & Siemon, 2021; Vanluydt et al., 2022). This implies that there is still a great need to examine the ways in which young children develop flexible understandings about partitioning, and what pedagogical approaches may better support their learning.

Theoretical Framing

Regardless of the theoretical orientation of young children's development of fractions, partitioning is considered a critical foundation for fraction understanding because it is derived from division and reassembly of division (i.e., multiplication). This understanding is required to work with the various interpretations of fractions and the broader network of rational number concepts (Confrey et al., 2014; Kieren, 1993). The extensive body of research on fractions illustrates that it is a complex area of mathematics (Kieren, 1993) however, this paper does not have the scope to expand on these theoretical foundations and will focus specifically on young children's development of partitioning as the basis of this mathematical domain.

To establish a conceptual understanding of partitioning, it needs to be understood beyond the physical act of creating equal parts or shares from discrete or continuous objects. One of the critical ideas of this concept is understanding the relationship between how parts are formed and named; that is, the inverse relationship between the number of parts and their size (Lamon, 2007). Partitioning, therefore, develops closely in relation to the concepts of unitising (i.e., assigning a unit of measure to name a fraction in relation to how the quantity has been partitioned), and quantitative equivalence (i.e., fractions can be renamed infinitely) (Kieren, 1993; Lamon, 2007). Exploring 'half' is the typical introduction to fractions young children experience when fair sharing in everyday contexts (e.g., cutting a sandwich in half or filling a bucket half full of water). Halving is considered a critical entry point for exploring the inverse property of partitioning more broadly, as it introduces children to 'fraction families' (e.g.,

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halves, quarters, eighths, etc.), which integrates unitising and quantitative equivalence understandings (Confrey, 2012; Siemon, 2003). Even in the early years, teachers need to possess a deep understanding of how these concepts are connected so that the choice of pedagogical approach—including models and representations utilised—assists children in developing such connections. Examining current research on young children's early partitioning capabilities, in addition to the typical models, materials, and tasks teachers employ, provides an opportunity to re-examine early partitioning instruction.

Examining Children's Partitioning Experiences

Children are often provided with learning experiences that explore partitioning in continuous contexts, such as using 2D shapes and pattern blocks, and paper folding (Siemon, 2003; Clarke et al., 2011). However, research suggests that children will often be prompted to count the parts generated from partitioning, rather than examining their magnitude in relation to the parts generated, which masks the inverse property of fair sharing (Clarke, 2011; Confrey, 2012). While such literature emphasises the need for teachers to utilise different continuous models to explore the relationship between partitioning and fraction magnitude, the focus largely remains on the types of models used for a particular fraction task. Conversely, we must consider the underlying foundations of how the models support children in constructing the relationship between partitioning, unitising, and equivalence.

English (1997) advocates that by reframing the representation as the vehicle for mathematical reasoning rather than a tool to represent an end product, the opportunity to consider how we utilise various models is revealed. For example, taking a Cuisenaire rod or pattern block (2D shape) and asking, '*If this is … [1-half; 1-quarter; 1 and a half], what is 1?*' is a common task teachers currently employ in fraction instruction (Clarke et al., 2011). However, the power in such models is when they are used as a vehicle to discuss *how* and *why* different combinations of rods or blocks may represent equivalent fractions. From this perspective, the spatial attributes of the materials, such as how the orientation, symmetrical and proportional properties of the parts can be used to represent fraction magnitude, are of focus. When we consider using the materials in this way, it suggests spatial proportional reasoning is critical to how these materials can be used as vehicles to develop early partitioning understanding.

Spatial proportional reasoning is described as the ability to reason about non-symbolic, relative quantities (Möhring et al., 2015). Several studies have demonstrated children's capabilities to reason proportionally in activities that involve non-symbolic partitioning contexts-such as number line estimation tasks (see Gunderson & Hildebrand, 2021; Möhring et al., 2015). Spinillo and Bryant (1991) conducted an experiment where 6–7-year-old children were asked to compare a referent rectangle that was partly blue and white (typically half and half; three-quarters and quarter combinations of the colours) to two other larger rectangles, one of which had the same proportion of blue and white. The results revealed children's sensitivity to the 'half boundary', which children consistently used as a visual benchmark to reason proportionally about the size of the different coloured parts. Moreover, some of the rectangles were presented in different orientations to the referent image (e.g., horizontal versus vertical partitioning of the colours). This suggests that spatial visualisation was also a useful skill for children to draw on in this context. Spatial visualisation can be defined as the ability to mentally transform or manipulate spatial properties of an object or image (Lowrie et al., 2016). In this study, the results imply that the children also utilised spatial visualisation to imagine 'rearranging' the coloured parts to make comparisons between the rectangles. It is important to note that the children in that study demonstrated little understanding of symbolic notation relating to fractions. Thus, these findings suggest a strong connection between children's conceptualisation of what it means to partition and name a fraction as half through utilising spatial proportional reasoning and spatial visualisation in continuous contexts.

Discrete materials are also critical for the development of partitioning (Siemon, 2003) and can support children's early fraction understandings by emphasising their spatial attributes. Typical discrete materials include counters and small blocks to explore how sets can be shared fairly. Several studies illustrate that children create different structures and arrangements of objects to determine the equality of shares in a discrete context (Confrey, 2012; Wilson et al., 2012). Such arrangements included arrays or visual patterns (including symmetry and congruence of the arrangement of the objects) to justify the fair shares. Therefore, even in contexts that may imply a counting strategy is required, children will use a range of 'non-numerical' based strategies that help them reason about the magnitude and relationship of the shares created. Using arrays or patterns to create and distribute fair shares from a spatial reasoning lens implies that children use spatial structures in early partitioning experiences. Mulligan and Mitchelmore (2009) describe spatial structure as an awareness of mathematical relationships that are supported by spatial patterns and arrangements. Literature on spatial structure and early fraction understanding is limited. However, several studies have found that structure and pattern promote a multiplicative understanding of unit structures in measurement and whole number contexts (see Battista & Clements, 1996; Mulligan & Mitchelmore, 2009). Given the multiplicative relationship that exists between unitising and partitioning in the domain of fractions, spatial structuring appears to be an important construct to leverage in the teaching and learning of fractions—particularly in discrete contexts.

In summary, several spatial reasoning constructs have been identified as important in the development of children's early partitioning experiences. Yet, no studies to date have explored this connection through a pedagogical approach in the early years of school. The present study investigated this issue, and the following question frames this paper:

- How does spatial reasoning help young children develop a flexible understanding of partitioning?

Research Design

This doctoral study employed a Design-Based Research (DBR) methodology to iteratively explore a teaching intervention that employed a spatial reasoning approach to teaching and learning fractions in the early years of primary school. DBR is an important methodology to the mathematics education research community because it enables a dual focus on (a) designing innovative forms of instruction to explore children's processes of learning and (b) refining local instruction theories through iterative teaching experiments for wider theoretical and practical application (Prediger et al., 2015). This paper will report on a subset of data from two teaching experiments conducted as part of this study. Data from two junior primary classes from two public schools in regional South Australia (referred to as Class B and Class C). Forty-four children aged 6–7 years participated in this component of the study. Furthermore, the classroom teacher of each class acted as an additional researcher in this project.

Instruments

The intervention program comprised 13, 60-minute lessons taught by the researcher. The lessons were designed for children to explore continuous and discrete contexts, focusing on developing an understanding of the inverse property of partitioning through providing contexts that emphasised spatial reasoning strategies. Examples of activities from this intervention will be provided in the results and discussion section.

Data Collection and Analysis

The researcher collected and thematically analysed work samples, anecdotal notes, and observational data from each lesson during the two teaching experiments. Consistent with DBR, the classroom teacher acted as an additional researcher by collating their observations and anecdotal notes throughout each lesson for comparison with the researcher. Thematic analysis was employed to analyse children's thinking and behaviour throughout the intervention program. Thematic analysis provides researchers with a method for identifying patterns of meaning within the data that the researcher deems important concerning the research questions (Braun & Clarke, 2013).

Results and Discussion

The results of this paper are organised into three themes, which represent how spatial reasoning was integral to the development of the various partitioning ideas in discrete and continuous contexts.

Creating Fair Shares

The first theme that illustrates children's partitioning behaviours in this study is their ability to create fair shares. The first activity for discussion asked the children to collect 12 plastic counters that represented 'cookies' and explore how they could share them fairly between different groups of friends (as discrete collections). In the introduction of the activity, the researcher used six counters to model a 3 x 2 array to the class and asked how many counters there were and how they knew. Specifically, the children were asked to look at the way the counters were arranged and if that helped them describe how many counters were there. This was intended to elicit a discussion about how they might describe these parts and the share of cookies each person receives based on the structure and arrangement of the counters. Several children could articulate that they could see that '*six is three equal columns*; *three as a row is half of six*; *six can be shared fairly between three people, there are two equal rows (shares) of three cookies'*. The children were asked to think about how they might arrange their counters to help them describe and 'see' how many ways they could share 12 'cookies' fairly.

During this task, one group of children made three groups, distributing one counter to each group in a square-like arrangement (see Figure 1).

Figure 1

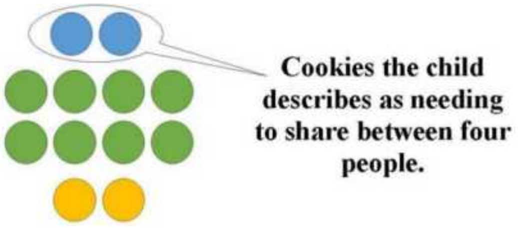
Children's Representation of Sharing 12 Cookies Between Three People

The children discussed how many counters there were in each group when sharing with three friends. Child 45 explained, "There are four, four and four because I can see a square in a group of four counters; my [12] counters make three squares!" This representation and description demonstrated how they utilised the spatial structure of the shares within the whole set to justify equal parts. That is, their creation of squares as a unit was created through the processes of partitioning, in this context, dealing one cookie to each 'square' at a time to exhaust the collection. However, their description of the 12 counters representing three shares of four indicates they unitised twelve as three–fours through the geometrical structure the child imposed on the shares. After this group experimented with sharing 12 cookies between four, five and six friends, Child 41 also stated that "when you make patterns with the shares, it's easier to tell if they are all the same or not, even when they get smaller because there's more people". This indicates that the spatial structure the children utilised within their created models was connected to their understanding of the inverse relationship generated through fair sharing.

In an extension of the cookie-sharing activity, the children were asked if they could share 12 cookies between 8 people fairly and how much each person would receive. A range of materials (paper circles, plastic counters) were provided for the children to work with. Although this was challenging for many children, three children worked together to make the following representation using the counters and presented it to the class (see Figure 2).

Figure 2

Child 57's Representation



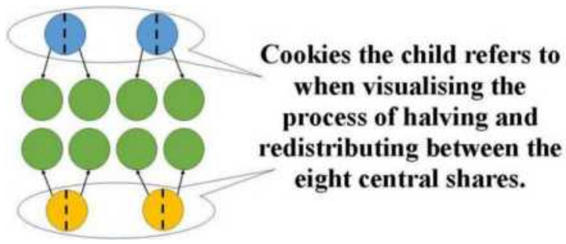
Child 57 explained their representation:

Child 57: We knew that there would be one cookie for each person, so we decided to line them up like this so you can see the four and four rows [gesturing horizontal movements with hand to indicate the central two rows of four]. This left four over, but to work it out it's easy because these cookies [referring to the two counters at the top of Figure 2] have to be shared between four people (Figure 3).

Child 57: This one cookie [referring to one of the two cookies at the bottom of Figure 3] also gets broken in half and shared to these two people and same with this one [gestures halving the two top cookies].

Figure 3

Child 57's Interpretation of Creating Fair Shares



This explanation highlights how the children imposed a spatial structure again to help them conceptualise the shares. However, they also indicated they used spatial visualisation to imagine partitioning the remaining four cookies in half to create equal shares of the collection, as the plastic counters were not able to be physically partitioned. This is a complex problem for children at this age, as mixed fractions are not typically considered appropriate for teaching in the early years and are not reflected in the current curriculum standards, at least in Australia.

Justifying Equal Shares

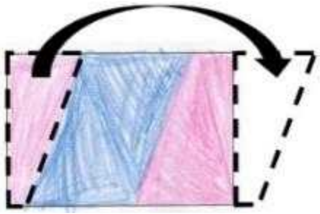
The second theme emerging from this study that highlights the connection between spatial reasoning and partitioning was children's ability to justify that they had created equal shares. The following activity was based on the context of designing and comparing tablecloths. The children were provided with blank paper rectangles and asked if they could design a tablecloth that met a specific brief (e.g., a design that had equal parts of blue and green, a design that was more purple than orange etc.) Some children chose to cut and fold their rectangles while others drew and coloured regions to represent their tablecloth designs.

Child 67 created the following design (Figure 4a) and stated that they had created equal parts of pink and purple on their tablecloth, that is, their tablecloth would be half pink and half purple. When asked how they knew the parts were equal, they stated that if they moved the left-hand pink section to join the pink region on the right-hand side, it would “look the same size as the purple part” (see Figure 4b).

Figure 4

Child 67’s Tablecloth and Interpretation of Their Visualisation Process

a) 

b) 

This child’s explanation demonstrated a complex visualisation process to justify that their parts were equal. The interpretation of this child engaging in spatial visualisation was supported by their use of spatial transformation vocabulary; that is, words indicating the child was mentally changing parts of the image in some way—such as move and slide—were captured in their descriptions. This child also indicated a sensitivity to the ‘half boundary’, consistent with Spinillo and Bryant’s (1991) study, by noticing the symmetrical relationship of the parts after (mentally) manipulating the tablecloth. Spatial visualisation was also evident in Child 30’s discussion about justifying their equal parts “You can have something that has lines all over it, and all different shapes, but it’s still a whole, and you can still make a half or a fourth if you look inside these patterns and move them in your head.”

These examples represent how many of the children were engaging in spatial visualisation, which allowed them to mentally move and transform parts of the tablecloth to make a judgement on the relative proportions of different colours. These findings represent early understandings of how fractions can be generated (partitioning), named (unitising), and renamed (quantitative equivalence), utilising spatial visualisation and spatial proportional reasoning.

Justifying Proportionally Equal Shares

The final theme exemplifying children’s partitioning development in this study was their use of spatial proportional reasoning when comparing and justifying fractions between unlike wholes. An activity that exemplified this theme was based on the context of fictional town maps (approximately 1.5m x 2m). The children were provided with a set of clues for where members of the public had seen dinosaurs (that had escaped their enclosure). The children had to use the clues to work out where the dinosaurs were hiding. The clues involved statements like, “The T-Rex was spotted halfway between the fountain and the bike path; …two-thirds along the train track, etc.). The children were not allowed to use any formal measurement tools as such, as the focus was on them considering the spatial elements of the map’s images.

The second part of the task required children to draw scaled representations of their pathways in their workbooks. Child 32 explained to the classroom teacher that the representation of their paths were different lengths, “so half is going to look different on each, but still in the middle [of each path]”. Teacher B also recorded that the child suggested that for the curved path, they had to take into account the impact of the curves in relation to the length of the whole path. In other words, the child could not just consider halfway between the destinations ‘as the crow flies’ but had to visualise the length of the entire pathway in relation to its orientation to determine the equal parts.

Figure 5

Child 45's Work Sample

Child 45 demonstrated a spatial proportional awareness of half in their explanation of "my lines [pathways] are all crazy, but there's still two same-size parts," referring to each half as proportionally equal to its relevant whole (Figure 5). With this explanation, the child used a gesture that suggested they were visualising straightening out the 2-halves of the curvy path at the top of their picture like it was a piece of string they were flattening out with the palms of their hands. This example also illustrates the close connection between spatial visualisation and spatial proportional reasoning in the creation and justification of equal parts. Finally, Child 32 concluded this activity with the following point in a whole class discussion:

> I could imagine that I can walk from here to far, far away, and its only half to where I'm going—like Adelaide or something. But I could walk from here to that table, and it's halfway of this room. You have to think about what the end is to know how big you've walked.

This is evidence that spatial visualisation and spatial proportional reasoning assisted the children in developing flexible understandings of partitioning. The ideas and understanding the children expressed in these statements are also consistent with Spinillo and Bryant's (1991) description of the 'half boundary' idea, which they argue young children utilise to judge the magnitude of parts within an object. Through emphasising spatial visualisation and spatial proportional reasoning, the children demonstrated the ability to justify the equality of the parts created, rather than seeing 'half' (or other fractions) as a quantity that is fixed to a particular object. This evidence demonstrates that utilising spatial reasoning is a powerful vehicle for developing early partitioning generalisations, that is, the inverse property of fractions. This is a critical finding as, according to the current literature, anticipating the outcome of partitioning and recognising the multiplicative foundations of fractions are ideas that many older children fail to comprehend (Callingham & Siemon, 2021).

Conclusion and Implications

This study demonstrates that the young children in this study were very capable of exploring early multiplicative and proportional foundations of partitioning through spatial contexts. That is, spatial reasoning enabled children to visualise the outcome of performing different partitions and determine the relationships between the part size and number of parts generated of like and unlike quantities. Furthermore, children's understanding of partitioning and the tightly connected concepts of unitising and equivalence was evident through focusing on structural and proportional comparisons of the objects and sets of objects they were partitioning. This paper is intended to be used as an opportunity for teachers and researchers to consider young children's capabilities for exploring the foundations for fractions, as the tasks presented in this study are not unlike activities and materials that are commonly used in primary classrooms for the teaching and learning of early fraction understanding. However, an implication of this study suggests that it is the way spatial reasoning was employed pedagogically that enabled children to make these connections about the foundations of partitioning, meaning teachers need to be supported on how to employ such an approach to provide rich opportunities for their students, whilst still meeting curriculum requirements. The evidence presented in this study is based on a modest sample, and therefore, more research is needed to explore the impact such an approach may have in the early years of schooling and beyond. Nevertheless, this study provides strong support for how spatial reasoning can be used as a powerful teaching and learning tool for developing flexible and sophisticated understandings of partitioning.

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References

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. (1998). Students' spatial structuring of 2 D arrays of squares. *Journal for Research in Mathematics Education, 29,* 503–532.

Braun, V. & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Callingham, R. & Siemon, D. (2021). Connecting multiplicative thinking and mathematical reasoning in the middle years. *The Journal of Mathematical Behavior, 61,* Article 100837.

Clarke, D., Roche, A., & Mitchell, A. (2011). One-to-one student interviews provide powerful insights and clear focus for the teaching of fractions in the middle years. *Australian Association of Mathematics Teachers*, 23–41.

Confrey, J. (2012). Better measurement of higher cognitive processes through learning trajectories and diagnostic assessments in mathematics: The challenge in adolescence. In V. F. Reyna, S. B. Chapman, M. R. Dougherty & J. Confrey (Eds.), *The adolescent brain: Learning, reasoning, and decision making* (pp. 155–182). American Psychological Association. https://doi.org/10.1037/13493-006

Confrey, J., Maloney, A. P., Nguyen, K. H. & Rupp, A. A. (2014). Equipartitioning, a foundation for rational number reasoning. In A. Maloney, J. Confrey & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. 61–96). Information Age Publishing.

English, L. (1997). Analogies, metaphors, and images: Vehicles for mathematical reasoning. In L. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images*. Lawrence Erlbaum Associates.

Gunderson, E. A., & Hildebrand, L. (2021). Relations among spatial skills, number line estimation, and exact and approximate calculation in young children. *Journal of Experimental Child Psychology, 212,* 105251.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Studies in mathematical thinking and learning. rational numbers: An integration of research* (pp. 49–84). Lawrence Erlbaum Associates.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–668). Information Age Publishing.

Lowrie, T., Logan, T., & Ramful., A. (2016). Spatial reasoning influences students' performance on mathematics tasks. In B. White, M. Chinnappan, & S. Trenholm (Eds.),*Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australasia,* (pp. 407–414). MERGA.

Möhring, W., Newcombe, N. S. & Frick, A. (2015). The relation between spatial thinking and proportional reasoning in pre-schoolers. *Journal of Experimental Child Psychology, 132,* 213–220. https://doi.org/10.1016/j.jecp.2015.01.005

Mulligan, J., & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal*, *21*(2), 33–49.

Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes—An overview on achievements and challenges. *ZDM Mathematics Education*, *47*(6), 877–891.

Siemon, D. (2003). Partitioning: the missing link in building fraction knowledge and confidence. *Australian Mathematics Teacher, 59*(3), 22–24. https://doi.org/10.1111/j.1467-8624.1991.tb01542.x

Spinillo, A. G. & Bryant, P. (1991). Children's proportional judgments: The importance of 'half'. *Child Development*, *62*(3), 427–440.

Vanluydt, E., Verschaffel, L., & Van Dooren, W. (2022). The early development of proportional reasoning: A longitudinal study of 5- to 8-year-olds. *Journal of Educational Psychology*, *114*(6), 1343.

Wilson, P. H., Myers, M., Edgington, C., & Confrey, J. (2012). Fair shares, matey, or walk the plank. *Teaching Children Mathematics*, *18*(8), 482–489.

Scripted Identities of the Mathematics Learner: Blurring Fiction and Fact in the Presentation of Research Data

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Research on mathematics learner identity often relies on interview data and presents case studies of individuals. This method allows an in-depth understanding of the person and their relationship with mathematics, but fails to generate a larger picture of identity with more breadth. In this paper we propose a method that blurs the boundaries of fiction and fact by creating fictional characters and a playscript derived from questionnaire response data of senior secondary mathematics students. We argue that this method allows for a wide representation of learner identity without sacrificing the individual voice that may resonate for various stakeholders within mathematics education.

Increasing student enrolment in tertiary mathematics (or STEM fields) remains a focus within mathematics education; requiring that senior secondary school students retain an interest in the subject and consider it to be something they might continue in the future. Mathematics learner identity is a conceptual lens that has often been applied to this retention problem, offering insights into why individual students may choose, or opt out of, continuing their mathematics study (Braathe & Solomon, 2015; Hernandez-Martinez & Williams, 2011; Solomon et al., 2016; Ward-Penny et al., 2011). These studies examine the stories students tell as they 'self-author' their relationship with mathematics (Braathe & Solomon, 2015). Identity provides the lens to consider aspects such as their experiences of transition to college/university (Hernandez-Martinez & Williams, 2011), reasons for leaving (Ward-Penny et al., 2011), or motivations for choosing further study (Black et al., 2010). Often the stories require some form of negotiation, e.g., the tension between liking and a fear of mathematics (Braathe & Solomon, 2015) or differing identities such as woman *and* mathematician (Solomon et al., 2016).

Research on mathematics learner identity seeking to understand students' decisions to engage with further mathematics, largely does so with individual cases. The studies mentioned above, for example, feature just one, two, or four participants in each case. Thus, whilst this body of research gives important and deep insights into how identity is implicated in students' affiliation with mathematics, it does not often gain a wider perspective from a broad range of students due to the methods typically employed in identity research (Darragh, 2016; Graven & Heyd-Metzuyanim, 2019). We consider it a worthwhile challenge to explore how we might create new methods for gaining the perspective of large groups of students without losing the richness of data that comes from looking at learner identity of individual cases.

Additionally, research exploring the reasons students may choose to (or not to) continue with further mathematics, may not reach beyond academic circles into practitioner groups. Careers advisors, mathematics teachers, and students may not have the interest nor access to read academic reports, especially those presented in a 'dry' manner. Developing innovative methods for the presentation of findings may go some way to extend the reach of the research.

In this paper we propose a method that aims to address such issues as described above. We draw from responses of a questionnaire sent to senior secondary students in schools across Aotearoa New Zealand. Following a more typical content analysis of the 641 respondents, we engaged in a methodological process that blurred the boundary between fiction and factual representation of data. Our aim was to present our data in a manner that *resonated* for the reader, yet also *represented* the broad range of responses, all the while maintaining a deeper sense of the person, as fitting a study on learner identity.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 191–198). Gold Coast: MERGA.

Our main purpose in this paper is to present the method we employed to synthesise the data and generate a playscript which we believe captured the main themes, albeit using a non-traditional presentation method. In the sections to follow we review literature that has similarly played with fiction in data; we detail our method; and we present an excerpt of the resultant playscript, together with a discussion of what might be read from these findings.

Research Using Fiction and Narrative

Fiction- or narrative-based approaches to qualitative data analysis have been popular for some time (Ollerenshaw & Creswell, 2002). According to Ollerenshaw and Creswell (2002), the stories that emerge as a product of narrative-based research methods tend to involve "a first-person oral telling or retelling of events related to the personal or social experiences of an individual" (p. 332). There are advantages to these methods, including that they enable us to convey the richness of the data through storytelling (Nardi, 2016), and permit learning to occur through the sharing of individuals' stories and experiences (Ollerenshaw & Creswell, 2002).

Narrative re-storying involves qualitative data, such as that from interviews or surveys, being analysed to identify potential story elements among responses, before being rearranged to form a new story (Ollerenshaw & Creswell, 2002). There are numerous ways in which re-storied data can be presented, affording researchers the flexibility to choose the shape they want their narrative to take based on their audience or purpose. In mathematics education, some fictional narratives produced through re-storying have taken the form of dialogues (da Costa et al., 2019; Nardi, 2016), inner monologues (de Freitas, 2004; Hannula, 2003), and movie synopses (Darragh & Radovic, 2019).

It is noted that re-storied data is ultimately shaped by the shared relationships and experiences of the participants and researchers (da Costa Neto et al., 2019). Nardi (2016) re-storied interviews verbatim to create a dialogue between a mathematician and mathematics education researcher, expressing that fictionalising the data in this way "assimilates the multiplicity of voices (researchers' and research participants' …) without suppressing or eliminating this multiplicity" (p. 364). Narrative-based approaches' ability to capture numerous perspectives was also commented on by da Costa Neto et al. (2019) who, like Nardi (2016), presented their re-storied interview data as a fictional dialogue between a researcher and retired mathematics lecturers. They highlighted that the re-storying method enabled them to amalgamate quotes from the interview transcripts of four separate participants into a single, coherent debate, whilst still showcasing similarities and differences among the interviewees' perspectives (da Costa Neto et al., 2019). Hence, fiction-based methods provide "a possible version of this story, through the lens of its actors" (da Costa Neto et al., 2019, p. 4) due to qualitative data analysis being "by definition a form of re-storying" (Nardi, 2016, p. 364).

Narrative formats also provide the opportunity for data to be situated in a context or setting that is relevant to the participants and their experiences (Ollerenshaw & Creswell, 2002). Hannula (2003) and de Freitas (2004) re-storied their respective student and teacher interview data into interior narratives set in mathematics classrooms to reflect the typical context in which mathematics teaching and learning takes place. Similarly, Darragh and Radovic (2019) situated their re-storied movie synopsis in Chile as it permitted them to "examine common societal discourses" (p. 518) that emerged from their interviews with Chilean primary school teachers and explore the identity of Chilean teachers more broadly. Because fiction-based methods act as "bridges between the different worlds of subjective experiences" (Hannula, 2003, p. 32), they enable readers and researchers to access, interpret, and learn from others' experiences and settings (Ollerenshaw & Creswell, 2002; Hannula, 2003). Thus, re-storying may be "a potent communicative tool" (Nardi, 2016, p. 364) in mathematics education research.

Methods

The questionnaire forming the data set elaborated in this paper was just one part of a larger project that aimed to understand mathematics learner identity, with a specific focus on the context of online instructional platforms for learning mathematics. Ethics were obtained for all aspects of the study. A mathematics learner identity can broadly be defined as:

> A socially produced way of being, as enacted and recognized in relation to learning mathematics. It involves stories, discourses and actions, decisions, and affiliations that people use to construct who they are in relation to mathematics, but also in interaction with multiple other simultaneously lived identities. This incorporates how they are treated and seen by others, how the local practice is defined and what social discourses are drawn upon regarding mathematics and the self. (Darragh & Radovic, 2018, para. 1)

For the purpose of this paper, we also note that identity is temporal: Student responses to a questionnaire prompt may be seen as one small identity act in that one moment of time. Following the definition of identity we use, stories or recounts of experience may be read in terms of the wider societal discourses that are drawn from, the affiliations made with the subject of mathematics, how mathematics learning is constructed in practice, and the ways in which others recognise the learner of mathematics.

The questionnaire was designed for senior secondary school students, for whom we assumed their identity would be somewhat consolidated, in order to gain a broad picture of common discourses or narratives about the learning of mathematics, both in-class and online, and an understanding of the various "local practices" in these contexts. The questionnaire included four opened-ended prompts; the final two questions were, "*Thinking back over all your experiences learning mathematics, which would be the highlight? Please explain why.*" and *"Thinking back over all your experiences learning mathematics, which would be the low point? Please explain why."* These prompts solicited stories from the respondents that could be read in terms of identity, albeit just one enactment of identity in a moment of time. Secondary schools around Aotearoa New Zealand were invited to participate, and seven schools agreed. The schools were diverse in terms of geographical location, demographic makeup, and socio-economic area. From these schools, 641 students responded to the questionnaire and most (98.3%) were taking a senior mathematics or statistics class.

First we completed a broad statistical analysis of closed questionnaire prompts, comparing differences in student ratings of their mathematics ability, their relationship with mathematics, and their perceived usefulness of mathematics by demographic groups and whether or not they planned a STEM career. Secondly, we conducted a content analysis of the open-ended responses. The first author worked with two research assistants to code every response following these steps: (1) The team read the first 50 responses together and identified codes; (2) Independently we coded the next 100 responses then met together to discuss and clarify any differences in our coding and decide if we needed to establish additional codes; (3) The remaining responses were divided amongst the three coders so that each response was coded by two people; and (4) We met a final time to discuss any conflicting codes and come to an agreement. This process was repeated with each of the four open ended responses.

The categories and counts we developed through these two analytical processes certainly summarised the data gained in the questionnaire, yet we felt the breadth of data came at a cost to depth. We had noted what we thought were heartfelt storied responses to the questionnaire prompts and we wanted to do better justice to the messages that were told and to bring this data to life in a way that might resonate for a broader audience.

Consequently, we completed a second wave analysis of the students' responses to the final two open-ended questions, using them to create characters and to form the basis of a play, as described below. Five researchers took part in the character development process. We formed a diverse group by ethnicity (Pākehā, Māori, and Korean), by gender, and by age (from recent high-school graduates to those with high-school aged children). From the whole data set of responses to these two questions, we independently each chose one or two quotes that *resonated* with us. We then imagined *who* the person who wrote that response might be, imagining them as a mathematics learner. Next, we found other excerpts from the corpus of responses that we felt fit with the character we had created. Finally, we returned to the questionnaire data and found the corresponding demographic information for each participant for whom we had identified a 'resonating quote'. We looked at their self-identified ethnicity, gender, age, current mathematics education status and career aspirations, and we gave each character a name.

After creating the characters, we reviewed the dataset again to identify other responses that we felt matched each character and 'assigned' to them further quotes. At this point we also identified any types of response that we felt could not connect to one of the characters we had created. We concluded that there was scope for three additional characters: one who had no strong feelings about mathematics, one who had disengaged with mathematics, and one who expressed feelings of anxiety toward mathematics. Again we began with the quote that expressed the character before looking to their demographic information. Finally, during the coding process, we also decided to merge characters together due to similarities in the types of responses we were allocating to their character description. This left us with eight student characters altogether. This cast was a little larger than desirable from a playwriting perspective, but we felt it was the minimum necessary to be able to allocate the full data set of 641 individual responses to the various characters in a coherent manner.

Before re-storying our responses (Nardi, 2016), we decided that the play's setting should be a school careers expo as we could imagine conversations about mathematics experiences taking place in this environment. Using the selected responses for each character, we arranged whole or part quotes to create a fictional dialogue between our characters derived from the student survey responses. It is important to note that while the dialogue was fictional, the content of each utterance in the play was directly derived from the real data collected in the questionnaire. Modifications were made to some responses, including tense corrections, spelling corrections, and adding words or linking phrases for clarity and flow. These changes are denoted in the play by square brackets. For example, two responses to the questionnaire prompt about highlights were, "A positive learning environment" and "Class, I have a very strong bong with them." These were re-presented in the play as "A positive learning environment [is a highlight in maths]. I have a very strong [bond] with [my class.]" The question context, a typo correction, and a grammatical change are shown in square brackets, but it can be seen the meaning of the two responses are retained. Finally, we also decided to include an additional student advisor character to guide the students' conversation. This character is entirely fictional—although it represents the university voice, such as the authors of this paper.

The draft playscript was given to a few mathematics educators to read and we asked them for feedback on the play. This feedback process allowed us to make informed adjustments to the play based on the educators' uncertainties, improving the study's trustworthiness (Stahl & King, 2020) and credibility (Shenton, 2004). One main adjustment we made was to give the advisor more of an agenda to match the rationale of the paper.

Findings: A Playscript of Mathematics Learner Identity

Whilst most respondents (97.7%) rated themselves as average or above average ability in mathematics, much fewer (64.1%) saw mathematics and statistics as useful for their future. Those who planned to go into a STEM career saw mathematics as more useful than those who did not, but these students accounted for only 35% of the respondents.

The content analysis of the open-ended prompts revealed 140 responses that mentioned achievement as a highlight, 116 mentioned understanding the content, 96 were coded as related to affect, and 79 mentioned assessments or end-of-year exams. Seventy-five responses described pedagogy as a highlight of learning mathematics, and 64 said the teacher had been the highlight. Similar themes were also prominent in the responses to the prompt asking for a description of the low point of mathematics learning experiences. Of these, 175 mentioned lacking understanding as a low point, 98 mentioned affect, 92 mentioned assessment, and 89 mentioned achievement. The low point pertained to pedagogy in 83 responses and the teacher in 69. As mentioned earlier, we felt a lot was lost in this categorisation and counting of the dataset. Whilst example quotes could be presented in a list format, we suggest that the playscript may help the responses to come alive for the reader.

The final play script on mathematics learner identity is lengthy and contains nine characters: eight students and one student advisor who is entirely fictitious. The length enabled us to incorporate a range of student responses from the original survey and weave several prominent themes from the survey responses into the discussion between our fictional characters. The student advisor character enabled our setting to reflect the rationale. The themes included in the playscript reflect those themes found in the content analysis of the questionnaire data set, yet also included statements about affiliating with mathematics.

The playscript length meant that we could not feature it here in its entirety. Instead, we present one excerpt during which the characters are mostly discussing the topics of competitions and peer support. To access the full play script, see Darragh and Smith (2024). When reading the playscript it is useful to remember that every comment by one of the characters, excluding the advisor, was in fact a written response to one of the two open-response prompts from the questionnaire. Topics of conversation in the full playscript include: affiliation with mathematics (or not), assessment, algebra and statistics learning, the usefulness of mathematics, teachers, competitions, peer support, and the classroom environment. Woven throughout the play are also a number of affective comments related to confidence, stress, and anxiety as well as a tension between struggle and understanding of mathematics content. Taken together, the play content gives some insight into societal discourses and local practices of learning mathematics that were drawn on in the students' responses to the questionnaire.

A group of students are at their school's careers evening. The students walk around the school hall together, surveying their potential career options as they wander. A student advisor from a nearby university waits at his stall for students to talk to him about STEM career pathways at his university

[The play begins with the students talking about their own feelings of mathematics learning, ability grouping, experiences learning algebra and statistics, before moving on to various teachers they have had]

Bao: Bad teachers don't really inspire me to try…

Advisor: *Sympathetically* [That's understandable.]

Manawa: Or they're mean and I just don't want to bother with their negativity.

Riley places the engineering pamphlet back on the table

Aisha: *Sounding frustrated* [Me neither! My] teachers forced me to do mathematics competitions.

Margaret: *Sounding surprised* [Oh…] I enjoyed participating in the Mathex competitions in years 7–10.

Riley: *Nodding* [So did I].

Aisha: *Looking at Advisor* The competitions slowly made me hate mathematics…

Advisor: *Sounding intrigued* [Oh…]

Aisha: And since they weren't optional, I would be punished if I hadn't participated in the 'personally paid' exams.

Advisor: *Sounding confused* [I thought competitions like that were meant to be fun?]

Riley: *Looking at Advisor* [Yeah! I always thought competitions] were a fun way to learn and compete with friends, classmates, and people from other schools [and] figure out where you are in terms of understanding maths.

Margaret: *Nodding* [Yeah,] the sense of accomplishment and the team relationships were very rewarding.

Aisha shrugs and starts having a conversation with Bao in the background about the neighbouring psychology careers stall.

Advisor: *Looking at Riley and Margaret* [Supportive environments like that must be beneficial for your learning.]

Margaret: *Looking at Advisor* Having a positive learning environment [is a highlight in maths]. I have a very strong [bond] with [my class.]

Manawa: *Nodding* [Me too!]

Aisha and Bao drift away from the group slightly to get a better look at the psychology stall.

Margaret: Discussing questions with peers throughout the year strengthened our relationships and [deepened] our understanding of math at the same time.

Manawa: *Looking at Margaret* [Absolutely! I've been] helping some of my friends that weren't fully interested in math to succeed and get good grades.

Margaret: *Looking at Manawa* [Same here! I was] able to help my peers in my class who were also struggling just like I was.

Manawa: Getting to help other people either with others struggling in class, tutoring younger years, or teaching younger siblings [is really cool.]

Riley: *Enthusiastically* [Being] able to explain [complex concepts] to other people gives [me] a feeling of achieving.

Manawa: I just think it's such a nice feeling being able to pass someone a set of information you know [and] help them with things they struggle with.

Margaret: *Nodding* [Totally! I could be] there for them when they felt like they were going to give up.

Advisor: *Looking at Margaret* [How awesome!] *Turns to Aisha and Bao who are still chatting in the background* [Do you agree that supportive class environments are important when you're learning maths?]

Aisha and Bao see the Advisor talking to them and stop their conversation. Margaret and Manawa continue chatting to each other in the background.

Aisha: *Nodding* [Definitely!] Not having a supportive teacher [can be] a big struggle.

Bao: *Looking at Aisha* [I agree.] When I am in a situation where I am unable to receive the assistance I need, I just [feel] lost and clueless [and] worse about my maths ability.

Sefo: *Sympathetically* It doesn't make you feel great [when you can't understand something].

Aisha: [Especially when there's an embedded] stereotype from an early age that [maths is] really hard.

Advisor: [That's very true!]

Aisha: *Looking at Advisor* [I like supportive classrooms because] it [often] takes me longer to understand things than others in my class…

Sefo: [Me too!]

Margaret and Manawa's conversation ends and they turn back to the group.

Aisha: [But] after assistance [from my teacher and friends, I'm] able to be back on track.

Jackson: *Shrugging* I [don't] care enough to ask for help.

Sefo: *Sounding surprised* [Really?]

Jackson: Yeah, I don't want to learn.

Manawa: *Looking at Advisor* [Well, I think it's nice having] good support in terms of friends who [can] help.

Sefo: [Yeah totally, and] after two years of consecutive not so great maths teaching and learning, [having a supportive teacher has made me] aware of how important maths is for my future.

Jackson nudges Margaret and suggests they go look at some of the other stalls.

Advisor: *Looking at Sefo* [That's great to hear.]

Margaret and Jackson slowly start walking away from the group. The Advisor spots them leaving and panics slightly.

Advisor: *Addressing the whole group but sounding slightly flustered* [It's been great chatting to you guys but before you go, would any of you be interested in signing up for one the STEM courses at our university next year?]

Margaret and Jackson continue walking away in the direction of the police stall.

Riley: *Enthusiastically* [Absolutely!]

The Advisor smiles with relief and hands Riley a sign-up sheet. Riley writes her name down and takes the engineering pamphlet she was looking at earlier as she leaves.

Advisor: *Looking optimistically at the remaining students* Anyone else?

[…]

Discussion

Our main purpose in writing this paper was to explore the potential of this fictional format for presentation of data, particularly the affordances and limitations. To that end, we wish to draw attention to a few noteworthy aspects we believe the method has demonstrated. Firstly, the play excerpt presented above explores the themes of competition and peer support/classroom environment, and in a more subtle manner we may also see the characters' planned careers coming through in their actions of picking up a pamphlet from the advisor's stall, versus their wandering off to other stalls, such as psychology or policing (these were desired careers stated in the demographic information for Bao, Aisha and Margaret's characters). The advisor's panic that very few people were signing up for STEM reflects the rationale mentioned at the beginning of this paper and highlights the affordance of creating a fictional setting that is relevant to this problem (Ollerenshaw & Creswell, 2002). In this manner the questionnaire data may be explicitly tied to the research purposes through its being pitched at an audience, beyond other researchers, who are stakeholders in the issue.

Secondly, the character development process (also described in Darragh et al., 2023), enabled a 'face' given to the anonymous data obtained in the questionnaire, perhaps lending it a certain authenticity. By choosing a *resonating* quote, we allowed student voice to surface after the content analysis reduced the data to categories and numbers. It could be suggested that choosing a quote in this way generated subjectivity in the data, however, we argue that this kind of subjectivity is always generated when selecting quotes to represent themes. We recognise the risks involved in giving our characters names that indicate ethnicity and gender. We did not randomly select the character demographics, these were responses connected to the initial 'resonating' quotes we picked to begin the process. We wish to emphasise that our characters are not intended to represent others in their ethnic or gender groups, however they do demonstrate the fact the questionnaire reached a diverse range of respondents. We wonder if the tension between resonating and representing data is worthy of further discussion—this is something likely present, yet hidden, in any data presentation that includes illustrative quotes. We also wonder if in assigning other quotes to the character 'type' we diminished the complexity one might see in a real person. These questions are worthy of further discussion.

In many ways, the presentation of data here differs little from typical qualitative research. Themes were found and quotes were selected to present those themes. However, we hope that the playscript presentation brings the questionnaire data 'to life' so that the implications of findings might be more apparent than otherwise, and may be more engaging for a wider audience. We suggest that issues of retention and students' choice to continue studying mathematics (or not) might be unpacked with senior mathematics teachers using a playscript such as ours. The success of this strategy would certainly be an area for future research.

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References

Black, L., Williams, J., Hernandez-Martinez, P., Davis, P., Pampaka, M., & Wake, G. (2010). Developing a 'leading identity': the relationship between students' mathematical identities and their career and higher education aspirations. *Educational Studies in Mathematics*, *73*(1), 55–72. https://doi.org/10.1007/s10649-009-9217-x

Braathe, H. J., & Solomon, Y. (2015). Choosing mathematics: The narrative of the self as a site of agency. *Educational Studies in Mathematics*, *89*, 151–166. https://doi.org/10.1007/s10649-014-9585-8

da Costa, N., Giraldo, C. D., & Nardi, E. (2019). Restoried narratives on the agency and disputes of mathematicians and mathematics educators in the education of mathematics teachers. *Eleventh Congress of the European Society for Research in Mathematics Education*. https://hal.science/hal-02422593/

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, *93*(1), 19–33. https://doi.org/10.1007/s10649-016-9696-5

Darragh, L., Amos, M., Kim, S. H., & Murrell, C. (2023). Character building—A new method for identity research in mathematics education? *Proceedings of the 12th International Conference of Mathematics Education and Society* (pp. 261–267).

Darragh, L., & Smith, A. (2024). *Scripted identities of the mathematics learner: Play script* [Online]. https://drive.google.com/file/d/1iT9Z4cEQjc-yeF3NWvGgd-Ut2cCvpoSU/view

Darragh, L., & Radovic, D. (2018). Mathematics learner identity. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer. https://doi.org/10.1007/978-3-319-77487-9_100017-1

Darragh, L., & Radovic, D. (2019). "To Tia with love": Chilean mathematics teacher identities after professional development. *ZDM Mathematics Education*, *51*, 517–527. https://doi.org/10.1007/s11858-018-01023-7

de Freitas, E. (2004). Plotting intersections along the political axis: The interior voice of dissenting mathematics teachers. *Educational Studies in Mathematics*, *55*(1), 259–274. http://www.jstor.org/stable/4150310

Graven, M., & Heyd-Metzuyanim, E. (2019). Mathematics identity research: The state of the art and future directions. *ZDM Mathematics Education*, *51*(3), 1–17. https://doi.org/10.1007/s11858-019-01050-y

Hannula, M. S. (2003). Fictionalising experiences—experiencing through fiction. *For the Learning of Mathematics*, *23*(3), 31–37.

Hernandez-Martinez, P., & Williams, J. (2011). Against the odds: resilience in mathematics students in transition. *British Educational Research Journal*, *iFirst Art*, 1–15. http://www.transmaths.org/iscar2011/PHM, Resilience in transition.pdf

Nardi, E. (2016). Where form and substance meet: Using the narrative approach of re-storying to generate research findings and community rapprochement in (university) mathematics education. *Educational Studies in Mathematics*, *92*(3), 361–377. https://doi.org/10.1007/s10649-015-9643-x

Ollerenshaw, J. A., & Creswell, J. W. (2002). Narrative research: A comparison of two restorying data analysis approaches. *Qualitative Inquiry*, *8*(3), 329–347. https://doi.org/10.1177/10778004008003008

Shenton, A. K. (2004). Strategies for ensuring trustworthiness in qualitative research projects. *Education for Information*, *22*(2), 63–75. https://doi.org/10.3233/efi-2004-22201

Solomon, Y., Radovic, D., & Black, L. (2016). "I can actually be very feminine here": Contradiction and hybridity in becoming a female mathematician. *Educational Studies in Mathematics*, *91*(1), 55–71. https://doi.org/10.1007/s10649-015-9649-4

Stahl, N. A., & King, J. R. (2020). Expanding approaches for research: Understanding and using trustworthiness in qualitative research. *Journal of Developmental Education*, *44*(1), 26–29.

Ward-Penny, R., Johnston-Wilder, S. U. E., & Lee, C. (2011). Exit interviews: Undergraduates who leave mathematics behind.*For the Learning of Mathematics*, *31*(2), 21–26.

Exploring Beliefs and Practices Towards Teaching Probability Using Games: A Case Study of one Fijian Secondary Mathematics Teacher

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Probability requires teaching approaches that allow children to predict, observe and experiment with concreate materials, games and simulations. In this study, we explored one teacher's beliefs and practices of teaching probability using game-based approaches. We employed a design-based research approach to explore how our case study teacher engaged with an hour-long professional learning on using a game-based probability teaching sequence. We found that our case study teacher's espoused beliefs and self-reported practices aligned with knowledge of teaching probability demonstrated during the professional learning activity.

In probability teaching, teacher beliefs, and knowledge are one area of concern (Batanero et al., 2004). Issues such as over emphasis on teaching procedural tricks, and lack of confidence in teaching probability exist, which lead to a lack of understanding and motivation for learners. (Sharma, 2015). According to Batanero et al. (2004), there are six main areas of teacher knowledge for teaching probability. Epistemic content represents knowledge of probability and statistics content, while cognitive and affective components broadly represent what Shulman (1987) called knowledge of student cognitions. The other three categories are media, interactional and ecological components. These components highlight some of the main pedagogical resources that are required by teachers. Unlike other branches of mathematics, probability requires relatively greater use of technological resources and an understanding of a wider range of applicability of the content in our everyday and professional lives (Estrada Roca & Batanero, 2020).

An important part of teacher knowledge includes having a productive disposition towards teaching probability. According to Estrada Roca and Batanero (2020), success of curriculum depends on teachers' interest in teaching probability. Estrada et al., (2020) provide a useful framework for understanding teachers' beliefs. Their framework explores beliefs about probability as well as the teaching of probability under three domains: affective (for example, teachers' emotions towards probability and its teaching), cognitive competence (for example, teacher's self-awareness of probability concepts and how to teach them) and behavioural (for example, is the teacher eager to teach probability). An added domain is value towards probability and its teaching (for example, does the teacher give importance to probability as a subject). Teacher knowledge and beliefs can be enhanced if they are exposed to a variety of teaching approaches (Koparan, 2022).

In this paper, we explored one Fijian secondary mathematics teacher's beliefs towards probability teaching before and during participation in professional development on a games-based teaching approach. Current approaches to teaching probability in the Fijian secondary classroom context generally rely on the traditional chalk-and-board style of teaching. In other words, probability teaching in the Fijian context does not take advantage of game-based teaching approaches that have connections with students' everyday experiences. This study is important because engaging teachers in games is likely to help develop their knowledge.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 199–206). Gold Coast: MERGA.

Theoretical Orientation of the Study

We used Vygotsky's socio-cultural theory which states that knowledge is socially situated and constructed through interactions with people (Vygotsky & Cole, 1978). In other words, knowledge construction requires tools such as language and cultural artefacts such as games. The socio-cultural views of learning see the role of the knowledgeable other such as peers as important. Also, viewed from a socio-cultural lens, teaching is a social activity that requires co-construction that often involves scaffolding from the knowledgeable other (Bell, 2010).

In other words, working together in collaborative practices and engaging in formative interactions with one another can enhance teachers' cognitive growth (Vygotsky & Cole, 1978). In our larger study, the teacher participants and the researchers provided the platform for our case study teacher to engage with a game-based probability teaching sequence and reflect on its usefulness. Our study involved an hour-long professional learning session and the socio-cultural view of learning gives us an opportunity to see how our case study participant contributed to knowledge construction of others. Given that probability teaching involves different dimensions of knowledge, including affective dimension, the socio-cultural theory provided us with a useful lens to explore our participants' beliefs towards probability as well, because human behaviour is highly influenced by cultural contexts and interactions (Moll, 2013, p. 16).

Literature Review

Two major interpretations of probability exist. Firstly, the theoretical viewpoint claims that all possible outcomes can be attributed to their own probabilities. On the other hand, the experimental probability acknowledges that the probability of an event happening can be determined by conducting experiments (Jones et al., 2007). Research literature provides some useful instances of how games can provide a useful context for exploring different interpretations and contexts suggested by Jones et al. (2007). One such example is provided by Batanero et al. (2004) who show how different probability teaching contexts can be explored using game-based teaching approaches. Having engaged a group of teachers in a game involving different coloured dice, they speculate that teachers do acquire knowledge that would be beneficial in their later professional work. Most of the research conducted with teachers, including prospective teachers, suggest that teachers find teaching probability and statistics difficult or challenging (Batanero et al., 2004; Leavy et al., 2013). Findings from a small sample study conducted by Leavy et al. (2013) in Ireland suggest that prospective secondary mathematics teachers perceive statistics as a challenge due to, among other factors, the need to think and reason statistically. Some studies also report similar findings with respect to both prospective and practicing teachers' understandings of probability (Estrada Roca & Batanero, 2020; Kazak & Pratt, 2017) as well as their understandings of teaching probability (Batanero & Álvarez-Arroyo, 2024; Batanero et al., 2010). Many studies have reported that children find probability to be a difficult topic and as a result show a number of misconceptions (Koparan, 2022; Sharma, 2015).

While the literature points out many challenges with respect to probability teaching and learning, it also points to some initiatives that can help overcome some of these challenges. Introducing teachers to activities that help uncover the relationship between theoretical and experimental aspects of probability is one such area. Typical activities proposed in the literature range from 'classical paradoxes that appeared in the history of probability' (Batanero et al., 2010; Estrada Roca & Batanero, 2020) to more recent innovations such as games (for example, tokens, cards, lotto, board, embodied, or cultural games) (Dayal & Sharma, 2021) or computer-generated games and simulations (Koparan, 2022). The use of a varied range of teaching activities such as games and computer-based simulations, models and experiments can provide excellent contexts for teaching probability (Dayal & Sharma, 2021), help teachers make links between statistics and probability (Estrada Roca & Batanero, 2020), and mediate theoretical and experimental probability (Kazak & Pratt, 2017). Estrada Roca and Batanero (2020) claim that many teachers have studied theoretical probability, which is one reason they may lack expertise in designing rich games and activities for the classroom. Exposure to rich activities has led to positive beliefs in teachers (Estrada Roca & Batanero, 2020). Other studies such as Batanero and Álvarez-Arroyo (2024), Batanero et al., (2010), Kazak and Pratt (2017) and Koparan (2022) confirm the many benefits of engaging teachers in such rich activities. Other studies, such as Veloo and Chairhany (2013) and Sullivan (2020), confirm the benefits of engaging children in such games.

Teaching probability and statistics is also a challenge for Pacific Island teachers. Dayal and Sharma (2020) reported that pre-service secondary mathematics teachers made incorrect predictions on a probability teaching sequence initially but were able to correct those misconceptions after conducting the experiments. Findings also revealed that pre-service teachers enjoyed the game-based teaching sequence because of the affective and cognitive learning challenges and opportunities it provided, as well as providing ideas for future teaching (Dayal & Sharma, 2021). This study hopes to add to our understanding of how in-service teachers can derive potential teaching ideas for both theoretical and experimental aspects of probability. The literature seems to suggest general prevalence of teaching challenges as well as an acknowledgement of the potential benefits of teaching using games. The current study also aims to add to our understanding of in-service teachers' perceptions of the degree of usefulness of games in teaching from a Pacific Islands context.

Methodology

For the purpose of this paper, we used a case study research methodology. Case study is the study of a unique stance in action (Cohen et al., 2007). In this study, a case study approach helped us provide a 'close-up reality' of the beliefs, knowledge and lived experiences of teaching probability of one secondary mathematics teacher.

The study reported in this paper involves only one teacher. Our participant, Jone, is a relatively inexperienced teacher with only two years of teaching experience at a single urban school. He recently graduated with a Bachelor of Science and Graduate Certificate in Education (BSCGCED) majoring in mathematics and physics. During the professional learning workshop, Jone partnered with another teacher from his school. The larger study consisted of 15 Fijian secondary mathematics teachers who went through three related phases: a pre-workshop one-to-one interview; an hour-long workshop; and a post workshop written reflection and interview. Details of the workshop can be seen in Table 1. A case study methodology was suitable because it allowed us to study one teacher's views and practices (Yin, 2009) in greater detail. The data collection was done in multiple ways that included a 15-minute-long interview, followed by an hour-long professional learning workshop. The final source of data reported in this study came from written reflections and a short interview at the end of the workshop. All interview data was audio recorded while the professional learning workshop was video recorded. Jone's interviews were analysed by deriving themes from different focus areas in the interviews. In describing Jone's workshop participation, we describe his actions based on the different parts of the workshop such as *posing a problem*, *playing the game in pairs* and *planning and exploring*. We purposively chose Jone as our case-study for this paper for two reasons: firstly, Jone was the least experienced of the 15 participants. Secondly, Jone had expressed his strong support for using games-based approaches in teaching during the initial interview, and as we report in the next section, Jone was highly influential during the workshop as well. The findings are presented next, in the same order as the research unfolded.

Table 1

A Game-Based Probability Teaching Sequence

Parts	Activities	Reflection and discussion
1: Posing a problem (Approximately 10 minutes)	Esha and Sarah decide to play a die rolling game. They take turns to roll two fair dice and calculate the difference (bigger number minus the smaller number) of the numbers shown. If the difference score is 0,1,2 Esha wins. If the difference score is 3, 4, 5, Sarah wins Is the game fair?	Why do you think the game is fair (or unfair). Explain your thinking
2: Playing the game in pairs (Approximately 20 minutes)	In pairs (or groups), teachers play the game with at least 20 trials	Is the game fair? Why or why not?
3: Planning and exploring (Approximately 30 minutes)	Pairs (or groups) make a plan to collect, record and analyse more data	Is the game fair? Why or why not? Think of the activity you just did. Can you share your views about this activity? Would you use this type of teaching sequence in your teaching? Where and how? What would be some of the benefits and challenges?

Findings

Jone's Views and Practices about Teaching Probability

During the interview, Jone began by acknowledging that he was new to the mathematics teaching profession. At the same time, he expressed having confidence in teaching probability. He stated that probability was a relevant topic for students because of its importance 'to our daily lives'. He argued that probability is closely linked to our day to day living and students tend to enjoy this topic in comparison to other mathematical topics. Jone also expressed that probability is one topic that involves both practical and theory. As such, he expressed strong views that probability be taught using practical activities along with normal pen-paper questions, "for me, it is something that we cannot just teach on paper … I have always used practical and demonstrative methods to teach students." He continued by sharing examples of using "certain coins, dice or spinner" to teach his students. He reiterated the need for teachers to "get the hands on experience" or using "real life examples of using weather or using the probability of an event they can relate to." Jone also showed greater understanding of the probability curriculum and mentioned that he has actually compared some aspects of the Fijian and NZ mathematics curriculum.

Upon further probing, Jone revealed that he has already conducted practical activities with his year 10 class. This activity involved a single die "we did play a few games and that involved a game of … we used to roll a die and students noted down the outcomes and we compared the outcomes and how frequently do we get a particular outcome."

When asked to share if he had used any other activity, Jone described the following:

> Each group was paper, pens to write and two dice. And the students were asked to roll the dice and to note down the outcome for each roll and for both dice and then we compared what is the difference in the outcome of both dice.

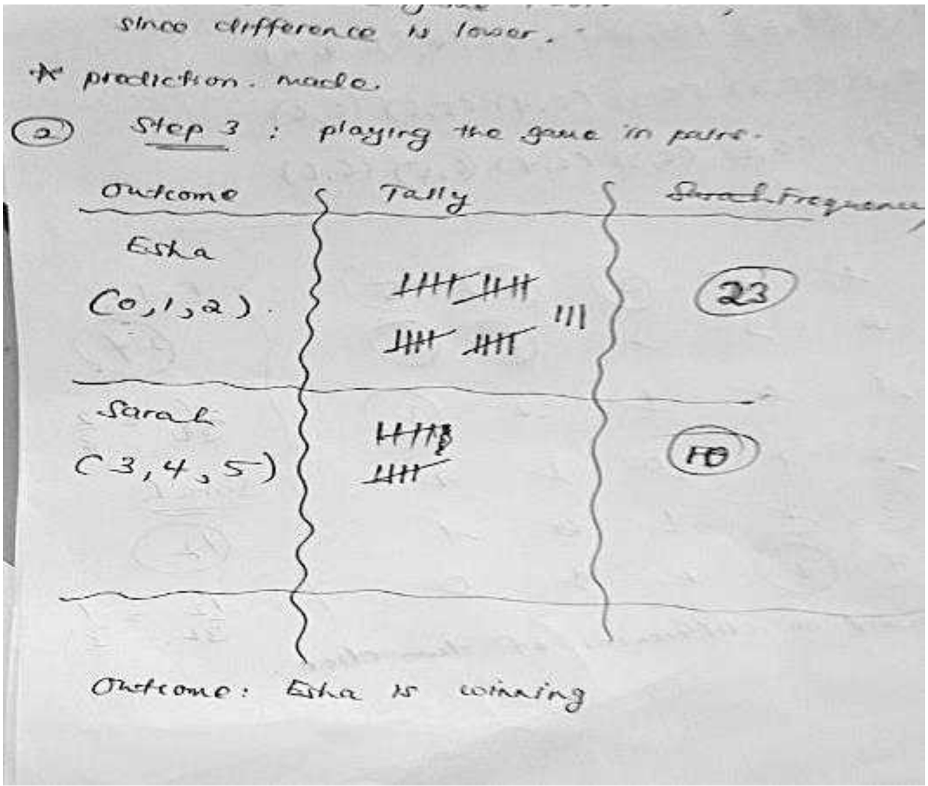
Jone reported that this activity was carried out with Year 10 students and they (the students) were able to find the sum and difference of two dice successfully. This was an interesting finding because the same idea of 'difference of two dice' formed the basis of our professional learning workshop. The workshop activity then provided us with an opportunity to learn more about Jone's self-reported teaching practices and validate his claim, especially the fact that he had some ideas about using the difference between the two die activities from his teaching experience with Year 10. How Jone participated in the workshop is described next.

Jone's Workshop Participation

During the workshop, we noticed that Jone became a resource not only for his group, but his initial predictions and his demonstrations during the workshop were found to be useful by other teachers that participated in the workshop

Figure 1

Jone's Trial Data



In 'posing the problem' part, Jone was the only teacher who was able to state that the game was not fair. He stated that "the game is not fair because they don't have an equal probability of winning. Most cases, the game favours Esha, since the difference of lower numbers is more". In part 2 of the activity, Jone conducted 33 trials with his team members and recorded the data, as shown in Figure 1. In order to convince his team member, Jone and his partner did some more trials until the team member was convinced that Esha was winning more often. Jone was able to offer a theoretical explanation by listing all the possible outcomes when two fair dice are tossed (see Figure 2). Next, he created a figure that showed the numerical difference between the outcomes of the two dice from Figure 2. The outcomes for Sarah were circled as shown in Figure 3 while the ones crossed were outcomes for Esha. The theoretical probabilities for both Esha and Sarah were counted out of the possible 36 outcomes and reduced to the simplest fraction.

Figure 2

Jone's Representation of Possible Outcomes When Tossing two Dice

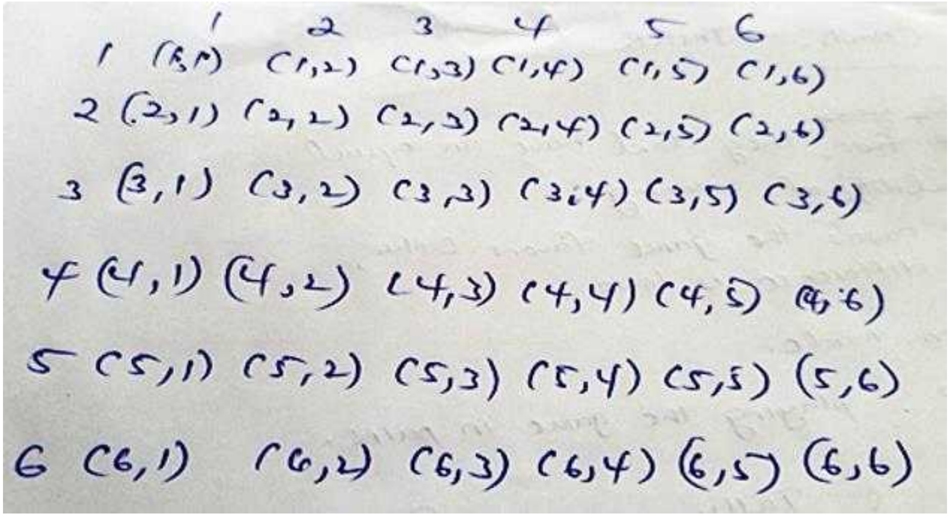
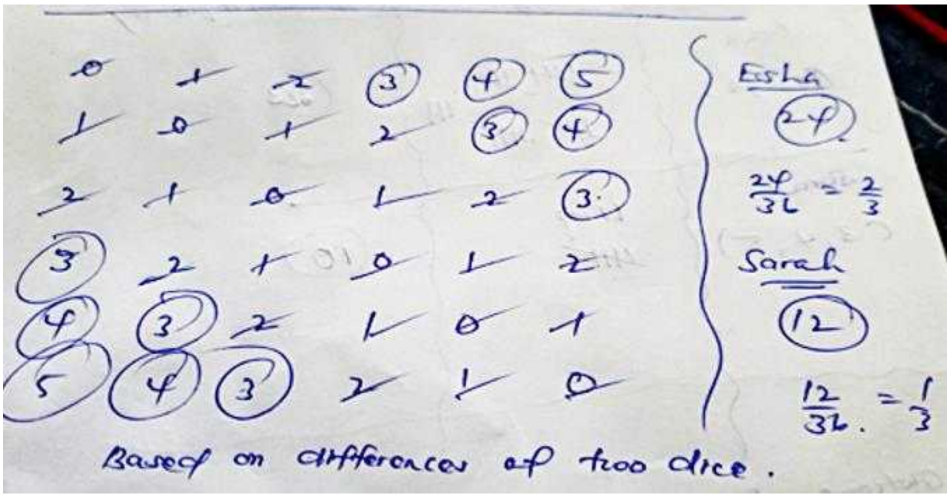


Figure 3

Jone's Understanding of Calculating Theoretical Probability When Tossing two Dice



Jone's Final Reflections

At the end of the workshop, we asked Jone to share his view on the probability teaching sequence, its applicability to classroom teaching, and associated challenges and benefits, and his intentions about using such techniques in future teaching. He expressed that the probability teaching sequence was a good opportunity for the teachers to learn to analyse the fairness of a game. He was able to suggest a number of activities that could be derived from this probability teaching sequence. For example, one of the interesting suggestions included using the same probability teaching sequence to teach about 'unfair gaming systems' and about outcomes that are 'not equally likely' outcomes. In addition, some common probability concepts listed were sample space, probability of an event, and probability distribution of each outcome. He was able to relate some of his ideas to actual classroom topics when he mentioned "Year 10-trials and experiments' and 'year 11-probability of an event". Jone stated several benefits such as "concrete learning, active learning, visual representation and engagement". In addition, he noted challenges such as large class sizes and varying levels of student understanding. He acknowledged that the probability experiments might be difficult for some students because they might find it difficult to "understand the results of the activity". Despite these challenges, Jone said that he will be using the probability teaching sequence in his teaching in the future and talked about sharing the idea with other department teachers at his school.

Discussion

This study examined one teacher's beliefs about teaching probability and his self-reported teaching practices, followed by an hour-long engagement with a game-based probability teaching sequence. The findings of this study reveal that Jone held strong views about the usefulness of probability and argued in favour of using games and demonstrations in teaching probability. He showed an awareness that probability demands a slightly different approach to teaching because it involves theoretical and experimental aspects. Jone, in reporting his style of teaching probability had revealed that he had already tried the probability teaching sequence with his Year 10 class.

The findings are interesting because previous studies done in Fiji and NZ using the same probability teaching sequence (Dayal & Sharma, 2021; Dayal & Sharma, 2020) albeit with pre-service teachers, show that pre-service teachers were not able to make the correct prediction at the start. They were, however, able to correct their responses after carrying out the trials. Our personal experiences of conducting professional learning workshops with secondary mathematics teachers in Fiji over the last couple of years reveal similar findings-i.e., practicing mathematics teachers see the game as fair and perceive the difference scores to be equally distributed. Another interesting aspect is that while pre-service teachers struggled to mention how they could use the probability teaching sequence in their actual teaching (Dayal & Sharma, 2020), Jone was able to state at least five quick ways of applying the lessons learnt from the workshop to his classroom teaching. Also, in his interviews, he stated that he was using similar activities for Year 10 students. In the Fijian mathematics teaching context, this is somewhat rare as previous studies in Fiji (Dayal & Sharma, 2020) and elsewhere (Batanero et al., 2004; Estrada Roca et al., 2020; Koparan, 2022) reveal that teachers themselves find such game-based probability teaching sequence quite challenging. Studies done with students also reveal that students find probability to be a challenging topic (Koparan, 2022). On the contrary, Jone reported that students in his class generally enjoyed probability. When asked to list some challenges when implementing this probability teaching sequence in Fijian classrooms, Jone acknowledged that some students might find the activity difficult. We speculate that teachers are unlikely to use game-like activities as teaching resources if they are uncomfortable in carrying out the activities themselves.

Conclusion

The research finding reported in this paper is based on a single case study. One limitation of the study is that its findings cannot be generalised. The findings provide us better insights into one teacher's beliefs, his self-reported practices and epistemic content knowledge as well as cognitive and affective components of knowledge for teaching probability (Batanero et al., 2010). The study's findings also support the need for teachers to have other dimensions of knowledge for teaching probability such as having a productive disposition towards teaching probability. Studies such as Estrada Roca and Batanero (2020) state the importance of factors such as teachers' interest in teaching probability. Previous research findings suggest that effective teaching of probability requires making greater use of resources in teaching probability such as using media, interactional and ecological components of teacher knowledge (Batanero et al., 2010). Our case study teacher, who held productive beliefs about teaching probability, and claimed to have made use of greater resources showed an advanced understanding of probability teaching through his participation in our probability teaching sequence. We speculate that our case study teacher would have interacted more with media, interactional and ecological components in order to build his knowledge of probability teaching.

We conclude that exposure to rich activities can lead to improved knowledge for teaching probability, including positive beliefs towards teaching.

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References

Batanero, C., & Álvarez-Arroyo, R. (2024). Teaching and learning of probability. *ZDM Mathematics Education*, *56,* 5–17. https://doi.org/10.1007/s11858-023-01511-5

Batanero, C., Contreras, J. M., Fernandes, J. A., & Ojeda, M. M. (2010). Paradoxical games as a didactic tool to train teachers in probability. In C. Reading (Ed.), *Data and context in statistics education: Towards an evidence-based society. Proceedings of the eighth international conference on teaching statistics (ICOTS8)*. International Statistical Institute.

Batanero, C., Godino, J. D., & Roa, R. (2004). Training teachers to teach probability. *Journal of statistics Education*, *12*(1). https://doi.org/10.1080/10691898.2004.11910715

Bell, B. (2010). Theorising teaching. *Waikato Journal of Education*, *15*(2). https://doi.org/10.15663/wje.v15i2.111

Dayal, H., & Sharma, S. (2021). Secondary pre-service teachers' views on using games in teaching probability: An international collaboration. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in mathematics education: Foundations and pathways*. *Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 187–194). Singapore: MERGA.

Dayal, H. C., & Sharma, S. (2020). Investigating probability concepts of secondary pre-service teachers in a game context. *Australian Journal of Teacher Education)*, *45*(5), 91–109. https://ro.ecu.edu.au/cgi/viewcontent.cgi?article=4630&context=ajte

Estrada, A., & Batanero, C. (2020). Prospective primary school teachers' attitudes towards probability and its teaching. *International Electronic Journal of Mathematics Education, 2020*, *15*(1), em0559. https://doi.org/10.29333/iejme/5941

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realities. *Second handbook of research on mathematics teaching and learning*, *2*, 909–955. Information Age Publishing and National Council of Teachers of Mathematics.

Kazak, S., & Pratt, D. (2017). Pre-service mathematics teachers' use of probability models in making informal inferences about a change game. *Statistics Education Research Journal*, *16*(2), 287–304. https://iase-web.org/documents/SERJ/SERJ16%282%29_Kazak.pdf

Koparan, T. (2022). The impact of a game and simulation-based probability learning environment on the achievement and attitudes of prospective teachers. *International Journal of Mathematical Education in Science and Technology*, *53*(9), 2319–2337. https://doi.org/10.1080/0020739X.2020.1868592

Leavy, A. M., Hannigan, A., & Fitzmaurice, O. (2013). If you're doubting yourself then, what's the fun in that? An exploration of why prospective secondary mathematics teachers perceive statistics as difficult. *Journal of statistics Education*, *21*(3). https://doi.org/10.1080/10691898.2013.11889684

Moll, L. C. (2013). *LS Vygotsky and education*. Routledge. https://doi.org/10.4324/9780203156773

Sharma, S. (2015). Teaching probability: a socio-constructivist perspective. *Teaching Statistics*, *37*(3), 78–84. https://doi.org/10.1111/test.12075

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–23. https://doi.org/10.17763/haer.57.1.j463w79r56455411

Sullivan, P. (2020). Is the last banana game fair? *Mathematics Teacher: Learning and Teaching PK–12*, *113*(1), 33–38. https://doi.org/10.5951/MTLT.2019.0131

Veloo, A., & Chairhany, S. (2013). Fostering students' attitudes and achievement in probability using teams-games-tournaments. *Procedia-Social and Behavioral Sciences*, *93,* 59–64. https://doi.org/10.1016/j.sbspro.2013.09.152

Vygotsky, L. S., Cole, M., Jolm-Steiner, V., Scribner, S., & Souberman, E. (1978). *Mind in society: Development of higher psychological processes*. Harvard University Press. https://doi.org/10.2307/j.ctvjf9vz4

Yin, R. K. (2009). *Case study research: Design and methods* (Vol. 5). Sage.

Secondary In-Service Mathematics Teachers' Self-Reported Teaching Practices and Their Views on Using Games in Teaching

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We report on findings from a study involving 15 secondary mathematics teachers from Fiji. The aim of the study is to describe teachers' current approaches to teaching probability and statistics; and to share their views on using game-based teaching approaches. We report briefly on the first stage of follow-up on one teacher who agreed to develop and implement a lesson. The findings suggest teachers use of concrete materials is limited to conducting simple demonstrations. In statistics, teachers reported using data derived from real life scenarios. Teachers also registered strong support for using game-based teaching. Lack of time was listed as the major inhibiting factor.

Probability and statistics is one of the strands in mathematics where students as well as teachers face a lot of difficulties (Batanero et al., 2004; Koparan, 2022; Leavy et al., 2013). One of the possible explanations for the challenges can be attributed to the need to think and reason statistically. This is because probability and statistics are areas of mathematics that rely heavily on our day-to-day activities, including making predictions about uncertain events related to our lives, such as predicting the weather or the winner of a sporting match. A related reason for such difficulty for students could be linked to the traditional chalk-board teaching approaches used to teach the topic (Koparan, 2022).

One of the ways to build a deeper understanding of probability and statistical literacy is through engaging learners in game-based teaching scenarios. As Koparan (2022) points out, games, including technology-based games, "has become one of the interests of the students" (p. 2333). Koparan (2022) adds that reducing the authoritarian teaching methods and offering more opportunities for active learning, where learners can discuss, visualise, estimate, control and experiment with probability and statistics situations can help with enhancing probability and statistics literacy. In light of this, there is an increasing need for additional research on teachers because most teachers teach probability and statistics like other mathematical strands with a heavy focus on procedures alone rather than on probabilistic or statistical reasoning (Koparan, 2022). There is also a need to investigate further into teachers' beliefs and attitudes as these can easily be transmitted into their actions (Batanero & Álvarez-Arroyo, 2024; Koparan, 2022).

In this study, we report findings on a small sample of Fijian secondary mathematics teachers' views and experiences about teaching probability and statistics. We also report on their perspectives about a game-based probability teaching sequence that they were exposed to during the study, including the perceived challenges and opportunities for including such games-based approaches in their actual teaching. We report on one of our participants' desire to plan and implement one such lesson in her classroom. The research questions addressed in this study are: What are Fijian secondary mathematics teachers' views and their self-reported practices about teaching probability and statistics? To what extent do they find the probability teaching sequence useful? An additional research question to which we offer partial answer is: How well does one participant plan a lesson on using a game-based strategy? After presenting the theoretical framework of the study, a literature review and the study's context and

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 207–214). Gold Coast: MERGA.

methodology are presented. This is followed by the main findings of the study. A short discussion and conclusion sum up this paper.

Theoretical Orientation of the Study

In this study, we used the socio-cultural learning theory. According to Cobb (2007), the socio-cultural theory seeks to "investigate the participation of the *individual-in-cultural-practice*" (p. 22, emphasis in original). According to this view, a learner learns based on his or her interactions, and using appropriate tools, with an adult or a more learned peer. This enables learning to move towards independent learning, a movement from the "plane of social interaction" towards the "plane of individual thought" (Cobb, 2007, p. 22). This notion of learning rests on the idea of the zone of proximal development (ZPD) which refers to the level of potential development which a learner is able to move to under the guidance of a more knowledgeable person (Heritage, 2013). Following Vygotsky and Cole (1978), the socio-cultural theorist sees cognition as "extending out into the world and as being inherently social" (Cobb, 2007, p. 23).

In this study, we focus on secondary mathematics teachers as learners. A socio-cultural view of adult learning is relevant because we propose a new design-based teaching approach to our participants after gaining an understanding of their current teaching practices. We then look at how or what our participants have learnt through our intervention. We also explore the limitations and opportunities our participants see in this new teaching approach. While socio-cultural theorists see learning as deeply rooted in participants' established classroom practices, these practices are subject to transformation and subject to potential new learnings that our participants might gather from our intervention. In our follow-up phase, we hope to continue working with some participants in future. In this way, the current study paves the way for establishing a community of practice by bringing individuals together to work on new instructional designs. According to Koparan (2022), it is important to support teachers to create different learning environments for their learners.

Literature Review

Research on teaching probability and statistics points out many unique challenges that teachers and students encounter. For example, findings from studies on pre-service mathematics teachers show that pre-service mathematics teachers find probability and statistics as difficult. A study by Leavy et al. (2013) for example noted how pre-service teachers found probability and statistics reasoning to be surrounded with uncertainties in comparison to other areas of mainstream mathematics. Similar findings have been noted in studies like Batanero et al., (2004) and Koparan (2019). Studies such as Koparan (2022) point out that students find probability as a difficult topic.

The literature also points out some of the effective ways in which teachers could overcome the challenges of teaching probability and statistics. One of the notable approaches seems to be the use of challenging teaching scenarios such as games (Batanero et al., 2004; Koparan, 2019, 2022; Sharma et al., 2021). In probability teaching, game-based learning involves situations where learners engage in a play-like scenario involving some aspects of probability. For example, the *stone-paper-scissors* is a common game that could be used to teach certain aspects of probability. Research shows multiple benefits of teaching using game-based strategies, such as improved student motivation and ability to work in groups (Koparan, 2022). Research on the use of such challenging game-based scenarios is generally focused on pre-service teachers. For example, one recent study involving 94 prospective mathematics teachers involving a quasi-experimental research design with pre-and-post tests showed that there was a significant difference in achievement and attitudes of prospective teachers in the experimental group who were exposed to a variety of materials such as worksheets, concrete materials, games and simulations. Koparan (2022) concluded probabilistic and statistical reasoning can be improved using activities “that push students to conduct research, make predictions, think, assess and to explain the logic they observe” (p. 2333). Similarly, a review on literature on teaching and learning probability noted that activities such as modelling can help teachers develop their content and pedagogical content knowledge (Batanero & Álvarez-Arroyo, 2024). In summary, there is tremendous support present in the existing research literature on using a variety of activities, including game-based scenarios, although most of these studies were focused on pre-service teachers (Dayal & Sharma, 2021; Dayal & Sharma, 2020; Kazak & Pratt, 2017).

In one study involving a small sample of six secondary in-service mathematics teachers, Sánchez (2002) reports findings about teachers’ views on a simulation software for developing probability concepts. The study’s participants reported having a better understanding of probability concepts, recognising simulations as a useful technique to solving probability problems. However, the participants could not suggest situations closer to students’ reality where simulations could be applied. From a Pacific Islands context, Dayal and Sharma (2020) report on how pre-service teachers were unable to make correct predictions on a game-based activity. The study noted that after doing the activity in full, pre-service teachers reported having a deeper understanding of the game-based teaching sequence. This study, while based in a Pacific Islands context, hopes to add to our understanding of how practicing mathematics teachers teach probability and statistics as well as what they feel about using game-based teaching approaches given that there are relatively fewer studies reported on in-service mathematics teachers.

Context and Methodology

Context

This study was part of a larger study carried out with mathematics teachers from five schools in the Western part of Fiji. The larger study unfolded in three phases, and involved 15 secondary mathematics teachers. Phase one involved one-to-one interviews with teachers. In phase two, the teachers took part in a two-hour workshop where they were exposed to the probability teaching sequence. Phase three involved post-workshop reflections. In the Fijian mathematics curriculum, the probability and statistics strand is known as Chance and Data. Chance and Data strand appears right from Year 1 curriculum. As early as Year 1, children are expected to discuss the likelihood or chances in an event, and draw and show simple data using a bar graph. Making predictions is one process that appears strongly from primary year levels. For example, in the Year 1 Numeracy curriculum, one of the activities requires students to throw stone into a circle made on the ground. Students first make predictions on how many stones will land inside the circle, then carry out the experiment and record data on their predicted and actual outcomes of the trials (Ministry of Education, 2017). In secondary schools, Chance and Data is offered from Years 10 to 13. Similar to primary school curriculum, teachers are expected to teach both content and processes such as statistical or probabilistic reasoning. For example, at Year 10, students are expected to demonstrate probability experiments by using a coin, die and pack of cards and list the respective outcome (Ministry of Education, 2014).

Methodology

We adopted a design-based research approach whereby our participant teachers were involved in a cyclic process involving action and reflection (Cobb & McClain, 2004). In this study, we report on phase one and phase three findings. In phase one, through one-to-one interviews, we explored participants' views about teaching and their preferred styles of teaching. We posed questions such as: *How do you normally teach this strand? Do you ever use any kind of game-based teaching?* In phase two, we conducted a two-hour professional learning workshop with the 15 participants. In this session, our participants were taken through the probability teaching sequence which involved a scenario where two players play a dice rolling game. Upon each throw, the difference between the two dice is noted. If the difference is 0, 1, or 2, player A wins. If the difference of the scores is 3, 4, or 5, player B wins. At the start of the probability teaching sequence, our participants were asked to predict whether the game was fair or not. Later they played the games in groups and calculated the experimental probabilities and reconciled these with the theoretical probabilities. After the probability teaching sequence workshop, all participants took part in written reflections that focused on questions: Can you share your views about this activity? Would you use this type of teaching sequence in your teaching? What would be some of the benefits and challenges? Consistent with design-based research, we included a follow-up phase where we requested our participants to develop lesson plans that could be used in their actual classrooms.

All the interviews were audio recorded and we conducted a thematic analysis on all interviews and written reflections (Braun, et al., 2022). For example, in analysing participants self-reported teaching practices we used categories such as 'no games used', "use of concrete materials or real-life data/activities' and 'use of games' and added participant's examples to the latter two categories. Similarly, in analysing their post-workshop reflections, we looked at teaching areas they had identified, as well as their anticipated challenges and opportunities. There were seven male and eight female participants in this study. Their educational qualifications ranged from a Diploma in Education (one participant), a bachelor of education with mathematics as one major (12 participants) and a postgraduate qualification in mathematics or education (two participants). On average, the participants had 11.5 years of teaching experience.

Findings

The first part of this section presents participants' views and experiences on teaching probability and statistics. The next sub-section presents participants' reflections and the teaching ideas they could derive from the probability teaching sequence. Finally, we present findings on how one participant developed a lesson plan based on a game scenario.

Participants' Views and Experiences on Teaching

All fifteen participants agreed that the teaching probability and statistics required teachers to cover both theoretical and practical dimensions of the subject, with all agreeing that using practical activities was better method because "when they do experiments on their own, they understand what is happening" (Hafiza's interview). Also, some participants strongly agreed that probability and statistics is one area of mathematics that rests heavily of daily activities. They mentioned daily activities such as sporting activities, weather patterns, or winning lotteries. Most of the participants said that they include practical aspects in their teaching although this was limited to certain activities, topics and year levels. Participants were able to share more examples of practical activities under the Statistics strand. Almost every participant shared data gathering activities which would lead to students creating different representations of the data. The examples teachers gave focused on students' real-life scenarios such as measuring height or weight (Anit's interview), tracking attendance and punctuality records of their class or school (Ratu's interview), or the modes of transport for students in a class (Sen's interview). However, all the participants agreed that the amount of time spent on such activities was small in comparison to the time used on traditional text book-based teaching methods that involved giving notes and examples and lots of questions to solve. Some participants, especially those teaching examination classes agreed to using lot of past year examination papers (Hafiza's interview). Three participants said they would use YouTube videos occasionally to demonstrate some concepts, while one participant said that she encouraged students to prepare PowerPoint presentations and engaged students in peer teaching.

Under the probability strand, all the participants said they used concrete objects such as coins, dice and playing cards, although teachers from two particular schools said they did not use these because of religious restrictions placed by their school on using any form of gaming materials in their teaching. Some teachers from the two schools said that they used coins, dice and cards only after seeking permission from their school principals. Almost every participant gave an example that made use of coins, dice or playing cards. For example, use of coins to conduct experiments or trials to find out the frequency of certain events (Ela, Rose, Ratu, Kaju's interview) or using a die to play a game of snakes and ladders (Kaju & Sheni's interview) or using a deck of cards (Navi's interview) were some of the examples given by the participants. While some of the participants could link the activities to student learning outcomes in probability such as finding the sample space when a coin or a die is tossed a given number of times, a few participants could not link the activities to any student learning outcomes. For instance, Sheni stated that she used coins and dice "just to give them a feel of it" when she began her lessons. Others such as Navi or Atish stated that coins or dice were used "for demonstrations". Only one participant explicitly said that she did not use any coins or dice for demonstrations simply because she was pressed for time to cover the syllabus (Hafiza's interview).

While all (except Hafiza) made use of simple activities involving concrete objects described above, 13 out of the 15 participants said they did not use any kind of games in their teaching. Reasons provided for not using games was often linked to the syllabi. In other words, the participants said that there were too many concepts to teach and thus a lack of time to cater for any kinds of games-based teaching. They also stated that the current curriculum did not support the use of games as only certain examples (such as conducting trials using two coins) are given in the curriculum and the textbooks. In other words, as the current curriculum did not support the use of games-based teaching approaches. As one participant stated, "the current curriculum is outdated, the text-book is way outdated" (Navi's interview). Another participant said that she "didn't see any games" in the curriculum (Hafiza's interview). For some other participants such as Jone and Mata, they said they developed their own games or activities. Jone and Mata were the only two participants who stated they had used some type of game-based scenarios in their teaching.

For example, Mata mentioned using crossword puzzles in her probability teaching but her description was not explicit. Similarly, she gave another example of involving students in throwing a pair of dice and adding the sum of the outcomes on each throw. According to Mata's description, the game would be played in groups of four and students roll a pair of die and add the outcomes on each throw. According to her, "if the group throws a "one", their total goes back to zero. The group that gets the highest total sum wins". When probed further on what learning outcomes this game would lead to, she explained that "they learn about the chances of getting a one". Jone stated that he used a similar game, but the students would calculate the difference of throwing a pair or die. He explained that "each group was given a paper and two die and they compared the outcomes for each roll of the two dice. We compared what is the difference of the two". When asked to link it to learning outcomes, Jone stated that "they would learn about trials and sample space". He argued that students would get a better understanding of these terminologies (such as a trial, event or sample space) if they were exposed to playing such games. All the participants took part in the probability teaching sequence after their initial interviews.

Participants' Reflections on the Probability Teaching Sequence

All the participants found the probability teaching sequence workshop a rewarding experience. It made participants reflect on their own probability knowledge as well as pedagogical knowledge. In other words, the participants reflected on how it made them aware of their own understanding of such games-based probability events. One of the reasons stated was that the workshop made participants 'actually see what happens' after making an initial prediction (Hafiza's reflection). For another participant, it was a 'good way to relate theory to practical' (Jone's reflection). One participant realised that the importance of understanding the context of the game as he mentioned that "catch was in the rule of the game itself (Navi's reflection). As another participant stated, 'predictions do not mean it will always be correct' (Sen's reflection). All the participants (except Jone) made predictions that were incorrect. In their reflections, some of the participants expressed how the activity them aware of the fairness of the game only after they had a chance to play the game.

Reflections on Probability Teaching Ideas

All the participants were able to list at least one teaching idea. Teaching ideas that appeared more frequently had to do with the use of coins or a deck of cards to conduct trials and to find out probabilities of simple events such as a head or tail. In addition to this, only a few participants noted other teaching areas such as sample space or checking on fair and bias games. Only one participant listed relating probability to statistics by using tally, frequency and other ways of recording data. However, two of the participants listed teaching areas that seemed incomplete. For example, one participant gave example of a volleyball game context and said students could calculate chances of hitting the volleyball (Sen's reflection). All the participants insisted the activity was useful because it will help students to make predictions, play games, and find the actual outcomes for themselves. It would help children get on hands on activities and develop their probability thinking. As one participant wrote, while both the dice are fair, the event was not fair. Carrying such experiments during class will "help students to learn that even though the die is fair and the probability of each throw is same, the chances of achieving each outcome was different." (Navi's reflection). Apart from these perceived advantages, one participant noted how the activity could involve everyone in the class (Rose's reflection). For example, some could play, some would record and all could discuss the outcomes. For two other participants, this activity would generate student interest in probability and statistics (Atish's and Mata's reflection).

When asked about some of the perceived challenges, lack of time came out as the main challenge. Many participants thought the activity was time consuming and may not be possible in one teaching period. Some participants saw large class size as a potential issue. Apart from time factor, two of the participants felt that the activity could be confusing to some students because it is a contradicting experiment about fairness (Kafi and Mata's reflection). Another teacher added that 'outcome is fair but difference in outcome is not fair' and this could be a contradictory point for some students to understand (Mata's reflection).

Developing and Implementing a Game-Based Lesson—A Follow-Up with One Participant

At the end of the workshop, we invited the participants to develop lesson plans on any teaching ideas that they derived from the workshop which they would likely use in their actual classroom teaching. At the time of writing, we had email correspondence from two teachers who are from the same school. Regrettably, one of the participants later resigned from teaching. The other participant, Rose, had emailed us her teaching ideas in the form of a lesson plan for a Year 10 probability lesson. While her initial lesson plan is only one page long resembling the typical lesson planning style in Fijian schools, the intended lesson goals are "Compute the probability from an experiment consisting of equally likely outcomes". Her description of student's activity contained only a short description on "Participate in group activity, throwing a pair of dice, and taking note of the results". We are currently working with her on how to add more details on student's activity section of her lesson plan.

Discussion and Conclusion

The aim of the study reported in this paper was to explore teachers' self-reported teaching strategies and their views on a game-based teaching activity. While our sample size was small, this study makes a useful contribution in terms of involving secondary practising teachers. As noted in the literature, there are limited studies involving secondary practising teachers (Batanero & Álvarez-Arroyo, 2024). With respect to our first research question, the study's findings suggest that all the participants agreed that probability and statistics require more activity-based teaching rather than chalk-and-board type teaching. In terms of their current teaching strategies and the extent of use of game-based activities, only two participants said they had used games in their teaching. The rest of the participants stated using common artefacts such as coin or die, but mainly for simple demonstrations. In teaching statistics, more participants could provide examples of using real-life examples and data derived from real-life scenarios. One of the possible reasons for the use of more real-life examples and data in the statistics strand could be related to the curriculum resources, such as textbooks having worked examples. The same cannot be said for the probability strand because, as many of our participants suggested, there were some concerns at the lack of support from the curriculum itself on the use of activity-based teaching approaches in probability. For example, some participants mentioned lack of examples or activities listed in the syllabi or the text book. We note that the Fijian Year 10 Probability Syllabi only mentions use of coins or dice for simple demonstrations (Ministry of Education, 2014). Also, teachers in Fiji generally have to cover the syllabus in preparation for external examinations. This reason was shared by some of the research participants. However, not all year levels have to take an external examination at the end of that academic year. There may be other reasons that prevent teachers from using game-based teaching strategies especially in the non-examination classes. For example, an added reason for a lack of involvement in game-based teaching could be attributed to a lack of understanding on how to integrate games in the lessons. Previous studies such as Dayal and Sharma (2020) and Koparan (2022) lend support for such arguments.

With respect to participants' views on the probability teaching sequence, there was overwhelming support about the value of using such game-based activities. Reasons provided by the participants included 'better student involvement' and 'improved student motivation'. These findings are consistent with the findings about the benefits of game-based teaching approaches in the literature (Dayal & Sharma, 2021; Koparan, 2022). In terms of the anticipated challenges, the findings indicated that lack of time would prevent the participants from using game-based teaching approaches. We speculate that teachers will need more support in creating and implementing lessons that make use of a wide range of activities such as games. This study has provided us with some initial insights on our first stage of follow-up phase where only two participants out of the 15 communicated to us, via email, their intentions of planning and using a game-based lesson.

We hope to work with one participant in the follow-up phase. Our participant has submitted her first lesson plan. While the proposed lesson's learning outcome seemed appropriate for Year 10, we have provided some feedback on how to include clearer steps for a deeper level of student engagement in the proposed lesson. From a socio-cultural perspective, this is encouraging since this collaboration gives some support to the claim that teachers can gradually adopt activity-based teaching when given appropriate support. Researchers need to continue to work together with teachers to find ways to help teachers develop probabilistic and statistical thinking.

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References

Batanero, C., & Álvarez-Arroyo, R. (2024). Teaching and learning of probability. *ZDM Mathematics Education 56*, 5–17. https://doi.org/10.1007/s11858-023-01511-5

Batanero, C., Godino, J. D., & Roa, R. (2004). Training teachers to teach probability. *Journal of Statistics Education*, *12*(1). https://doi.org/10.1080/10691898.2004.11910715

Braun, V., Clarke, V., & Hayfield, N. (2022). 'A starting point for your journey, not a map': Nikki Hayfield in conversation with Virginia Braun and Victoria Clarke about thematic analysis. *Qualitative Research in Psychology*, *19*(2), 424-445.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3–38). Information Age.

Cobb, P., & McClain, K. (2004). Principles of instructional design for supporting the development of students' statistical reasoning. In D. Ben-Zvi & J. Garfield (Eds), *The challenge of developing statistical literacy, reasoning, and thinking* (pp. 375–396). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Dayal, H., & Sharma, S. (2021). Secondary Pre-service teachers' views on using games in teaching probability: An international collaboration. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in mathematics education: foundations and pathways. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 187–194). Singapore: MERGA.

Dayal, H. C., & Sharma, S. (2020). Investigating probability concepts of secondary pre-service teachers in a game context. *Australian Journal of Teacher Education*, *45*(5), 91–109.

Heritage, M. (2013). *Formative assessment in practice: A process of inquiry and action*. Harvard Education Press.

Kazak, S., & Pratt, D. (2017). Pre-service mathematics teachers' use of probability models in making informal inferences about a change game. *Statistics Education Research Journal*, *16*(2), 287–304.

Koparan, T. (2019). Teaching game and simulation based probability. *International Journal of Assessment Tools in Education*, *6*(2), 235–258.

Koparan, T. (2022). The impact of a game and simulation-based probability learning environment on the achievement and attitudes of prospective teachers. *International Journal of Mathematical Education in Science and Technology*, *53*(9), 2319–2337.

Leavy, A. M., Hannigan, A., & Fitzmaurice, O. (2013). If you're doubting yourself then, what's the fun in that? An exploration of why prospective secondary mathematics teachers perceive statistics as difficult. *Journal of Statistics Education*, *21*(3). https://doi.org/10.1080/10691898.2013.11889684

Ministry of Education. (2014). *Mathematics syllabi years 9 &10*. Ministry of Education, Heritage and Arts.

Ministry of Education. (2017). *Maths guide for lower primary: Years 1 & 2*. Ministry of Education, Heritage and Arts.

Sánchez, E. (2002). Teachers beliefs about usefulness of simulations with the educational software Fathom for developing probability concepts in statistics classroom. In B. Phillips (Ed.), *Proceedings of the Sixth International Conference on Teaching Statistics: Developing a statistically literate society (pp*. 1–6). Cape Town, South Africa. https://iase-web.org/documents/papers/icots6/6e2_sanc.pdf

Sharma, S., Sharma, S., Doyle, P., Marcelo, L., & Kumar, D. (2021). Exploring probability concepts in a game context. *Teachers and Curriculum, 21*(1), 59–70.

Vygotsky, L. S., Cole, M., Jolm-Steiner, V., Scribner, S., & Souberman, E. (1978). *Mind in society: Development of higher psychological processes*. Harvard University Press. https://doi.org/10.2307/j.ctvjf9vz4

Mathematical Modelling for a Class Party: Challenges for Students in one Year 4 Classroom

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The revised Australian curriculum presents a new emphasis in primary schools on the process of Mathematical Modelling. A modelling focus brings close attention to confidence and capability with a potentially new problem-solving process, associated language, and pedagogical processes. This paper presents a Year 4 classroom modelling experience that arose as students planned their end-of-year party through Guided Mathematical Inquiry. Classroom video data captured two students working on vertical whiteboards as they formulated and solved a problem involving carrot sticks and dip. Findings reflect the students as doers of mathematics, engaged in productive struggle.

Mathematical modelling is presented in the revised Australian Curriculum: Mathematics as a new process that extends through all year levels from Prep to Year 6. The new emphasis supports students through problem solving to make connections between mathematics and real-life situations, between mathematical topics, and to strengthen interdisciplinary links (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2023). Previously in primary classrooms, such connections and links may have struggled to surface, especially when the focus in mathematics lessons remains on replicating taught procedures to correctly answer closed problem situations. The revised Curriculum presents an opportunity for students to ‘use mathematics to gain insights into, and make predictions about real-world phenomena’ to ‘inform judgements and make decisions in personal, civic and work life’ (ACARA, 2023). Prior to the release of the revised Curriculum, mathematical modelling as a problem-solving process in the primary years was not explicit. For example, previously in Foundation to Year 3, content descriptions included students *modelling* addition and sharing, *modelling* large numbers, and *modelling* unit fractions (ACARA, 2010). This content did not explicitly link to the proficiency of problem solving. The change to surface this mathematical process presents a significant shift for primary classroom teachers and brings competency with mathematical modelling to the forefront of professional development to support these experiences in the classroom. Although much literature offers excellent classroom modelling examples in secondary contexts (Geiger et al., 2022) and in STEM in the primary years (English, 2021), these examples do not address the current mathematical goals, nor specifically do they meet the needs of classroom teachers seeking classroom examples exploring the potential of modelling in Australian primary classrooms.

This paper offers insights into one Year 4 classroom as students problem solved through the inquiry, *How much does a class party really cost?*, addressing revised Curriculum requirements. The researcher and classroom teacher-researcher (first and second authors) were keen to make mathematical modelling explicit in this mathematics classroom, having never taught the process before, and felt that the class party context would offer a real-life situation in which to support the modelling process. We wondered, (RQ1) what modelling opportunities could exist for Year 4 students planning their class party through Guided Mathematical Inquiry, and (RQ2) how might Year 4 students in one classroom approach mathematical modelling for the first time.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 215–222). Gold Coast: MERGA.

Mathematical Modelling in the Primary Years

Mathematical modelling is a well-researched topic and many explanations of the modelling process exist. Blum's (2011) model is well accepted and presents 7 steps for structuring classroom mathematical modelling. Briefly, this involves starting with (1) a real situation to frame the construction of a problem which is then (2) simplified to create a *situation model*, and further structuring of the situation leads to a *real model* that can be (3) mathematised. A mathematical model of the problem (e.g., equations) allows the solver to (4) work mathematically to generate mathematical results. Problem solvers then return to the real-life context to (5) interpret their results to inform decision making. If results are (6) validated, the problem solver can decide whether further mathematising is needed to address other variables until the final solution is (7) exposed or communicated. A key tenet of modelling is the connection between mathematics and reality thereby reminding problem solvers of the usefulness of mathematics to their lives. This is especially important given the current school curricula focus on solving word problems that do not reflect the complexity of real-life problems (Maass et al., 2022). Real problems are messy and so in modelling, investigation progresses through multiple cycles of interpretation and explanation to inform decision making (Doerr & English, 2003). The design of modelling tasks attempts to capture real-life complexity through the inclusion of an open-ended approach (Geiger et al., 2022) and careful attention is paid to scaffolding, to 'unstick' blockages students may encounter when working to solve the problem (Park, 2023). To emphasise relevance, recent depictions of modelling processes integrate mathematics with socio-scientific issues and STEM knowledge, including the development of concrete materials that explore controversial issues, to progress the role of mathematics in STEM and to include a citizenship focus (English, 2021; Maass et al., 2019; Makar & Doerr, 2020). Student competence with mathematical modelling to address global challenge is important yet it can also be helpful for children with little modelling experience to initially explore and develop competence with closer contexts, as is the case in this paper.

Links made between mathematical modelling and inquiry-related processes articulate how both problem-solving approaches involve a collaborative approach between teacher and learner, that values creating questions, asking probing questions, using mathematical representations, iterations of refinement and improvement, and a sense of discovery (Maass et al., 2019; Maass et al., 2022; Makar & Doerr, 2020). Specific to Guided Mathematics Inquiry, is the use of an ill-structured question to support an extended nature, with opportunities for students to engage in productive struggle, and where the classroom culture aims to challenge and advance a student's conceptual development (Fielding & Makar, 2022). In the classroom depicted here, students were engaged in the inquiry *How much does a class party really cost?* to encourage thinking about and modelling the cost of foods, even when that food is made with products in the pantry. Solutions would present the amounts of money spent in each household of students who contribute a plate of food on party day. Due to the open nature of the question, many mathematical topics could be addressed, including measurement (e.g., cups of flour in one packet), additive and multiplicative operations (e.g., each carrot can be cut into 8–10 sticks for dipping. How many sticks can I get from 3–4 carrots?), and financial mathematics (e.g., the total cost of making chocolate brownies for 28 people). The modelling experience depicted in this paper centres focus on one aspect of the investigation; how much it might cost to provide carrot sticks and dip for the class to share. This was a simplified situation model that could be mathematised, where results could inform decision making. If dishes cost a family more than $10 to prepare, would it make better sense to order takeaway food for each person to be delivered to the classroom for the same price?

Key to both Guided Mathematics Inquiry and mathematical modelling, is a focus on problem solving. To clarify, the problem solving depicted in this paper illustrates students working mathematically on solutions where they did not know the solution process. The boundaries and constraints of this task offered students the freedom to explore how the problem could be approached, through solution pathways that appeared to be non-routine. For the students, the problem solving was challenging and, in this instance, a collaborative experience that promoted mathematical discussion and reasoning. The task outlined in this paper involved the use of Liljedahl and colleagues' (2021) vertical whiteboards approach to encourage students to discuss their thinking, to persist through mathematical challenge. The pedagogical aim was for students to learn *through* problem solving and to be engaged in thinking mathematically.

Research Design

A wider program of design research (Cobb et al., 2003) frames this case study, focused on understanding and improving the mathematical process of modelling in inquiry settings. The classroom teacher was experienced with Guided Mathematical Inquiry and a participant in the ongoing research. The aim of the larger study was to conceptualise and operationalise terms that contribute to ontological innovation (di Sessa & Cobb, 2004), with refinement and extension of developing theory. In this case specifically, we were interested in the modelling contexts that naturally arose from the inquiry and instead of teaching modelling, we wanted to know what the students could do when modelling, including the design of their solutions. Guided Mathematics Inquiry presented a suitable pedagogy in this instance to support the teacher with introducing mathematical content knowledge and processes using teaching *through* problem solving approach that is student-centred. The classroom episode reported on here as a case study, took place in the final term of the school year and the classroom teacher had already established a classroom culture that valued mathematical thinking and reasoning, including socio-mathematical norms of *building on the ideas of others* and *active listening.*

The Year 4 (8–9-year-olds) classroom depicted here was situated in a large Queensland metropolitan school. There were 30 students (co-educational) in the class, and only two of these students chose not to participate in the study. Consent was fluid in that students could opt-in or out to suit their level of comfortability once parent consent was received. The students were finding out through inquiry, how much having their end-of-year class party really cost, and to capture the messy nature of realistic classroom processes, the researcher used a video camera on a tripod to film lessons (6 sessions over 6 weeks, > 6 hours), accompanied by an iPad to capture student discussion in groupwork. An additional invite by the children meant the researcher also attended the party although this was not filmed.

Analysis

Videotape methodology (Powel et al., 2003) has been established as suitable for this kind of observational research in mathematics education. Analysis involves the research team frequently and flexibly viewing excerpts. To identify the case presented here, the researcher and teacher, who had been involved in all classroom lessons, reflected on and identified four instances in the inquiry that they felt were mathematical modelling experiences. These involved students formulating a problem from the real-life situation using number sentences and involving multiplicative thinking, to meet curriculum goals. To distinguish from the inquiry, student solutions to modelling problems were based on assumptions (that 3 serves of a menu item is a suitable amount for a student to eat, for instance) rather than being refined to suit the specific context (my data showed that one person wants 15 serves of this item so three won't be enough!). In an inquiry, assumptions often become the line of focus and interrogation. For example, rather than assume that three carrot sticks per person will cater for the class, a classroom inquiry involving statistics might find out more closely whether this is a reasonable serving size for Year 4 classmates.

Of the four instances identified, the problem involving carrot sticks and hummus was selected as a strong modelling example. The researcher, teacher, and the research assistant attentively viewed the selected video excerpt (from the third classroom session) independently, aided by a transcript of this section of the lesson, to identify critical events. The researcher and teacher then flexibly viewed the video again together, and narrowed focus to the critical event presented here, involving two girls working together on a vertical whiteboard surface, to work out how much it will cost to purchase enough carrots for the class to enjoy carrot sticks dipped in hummus at the end of year party. To respond to the second research question, coding focused on the content of the critical event, how Paris and Cristina approached modelling, related to the difficulties these students faced with mathematising the problem situation, and the feedback they received to support them towards a successful solution. The identified collection of events within the student-to-student discursive interaction offers an emerging narrative about the data and the storyline is presented in the following section.

Findings and Discussion

Modelling Task

By the third session of the inquiry, children had confirmed with their families and nominated a menu item that they would bring to the party. Each child was tasked with finding out the cost of making the dish that would be prepared at home. When asked for pricing for ingredients, Hayden revealed his research efforts: a bag of carrots cost $1.99, and he would only need 3 or 4 carrots to cut into sticks for his carrot-stick and hummus-dip dish. He also shared that he could cut 8–10 sticks from each carrot. In response, and in an impromptu manner, the teacher added that when she last bought a bag of carrots, there were 8 carrots in the bag. She rounded the $1.99 to $2 and posed the following questions:

(17m 8s) Teacher: So you guys now, can you work out how much it’s going to cost for the carrots that we need? Now think about if there are eight to ten carrot sticks from each carrot, consider how many you might want for the class (pauses). You’ve said three or four. Work out how many carrot sticks that’s gonna give us.

Whereas the inquiry was focused on finding out the costs and designs of dishes to share with 28 people at a party, based on their tastes and dietary requirements, this problem task reduced the focus on personal preference.

Table 1

Characteristics of Carrot Stick Modelling Task, Informed by Blum’s Model (2011)

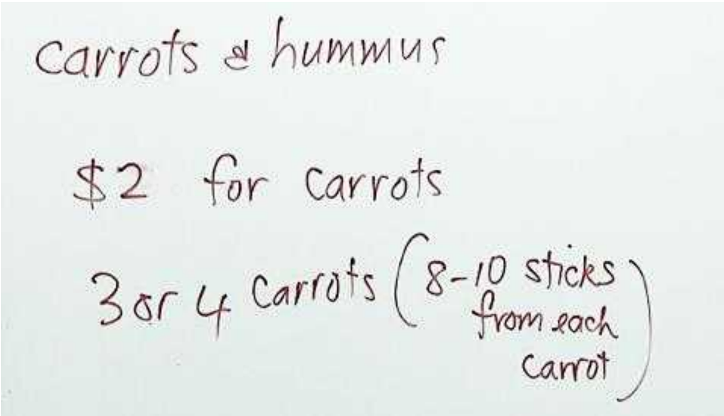
Modelling phase	Elements of carrot stick task
Real situation	Inquiry: How much does a class party really cost?
Situation model	Simplified to: number of sticks per carrot, number of carrots per bag, and cost of one bag of carrots
Real model	Mathematised as a series of equations involving multiplicative thinking
Working mathematically	Solving equations
Interpret results	Does the solution sound reasonable? What might this look like?
Validation	Through drawing, peer and teacher feedback, checking interpretations (5 iterations).
Communication	Solutions are displayed on vertical whiteboards and peers explain solution pathways.

To solve the carrot problem, students would work on the following assumptions: each carrot would be large enough in size to cut into 8–10 sticks, the carrot sticks would be large enough for dipping, everyone was going to choose to eat carrot sticks on party day and the total number of carrot sticks would be shared equally (the same number of sticks per person). Put simply, the problem now seemed to meet the characteristics of a modelling task (Table 1) that would align with curriculum goals. By the end of Year 4, students use mathematical modelling to solve financial and other practical problems, formulating the problem using number sentences, solving the problem choosing efficient strategies and interpreting results in terms of the situation (ACARA, 2022).

During analysis, focus turned to how the carrot stick problem was posed to the students as this framed the open nature of the task and how the students might approach the modelling situation. The transcript above outlines the moment the problem is identified and shared by the teacher, and when students are invited to help their peer, Hayden, work out a solution. Figure 1 is a photo taken of the board displaying the key information for students, recorded by the teachers as she posed the task. This is not displayed as a formal question.

Figure 1

Key Components of the Problem Recorded on the Board



Three key components are included in the excerpt (task proposal) and were recorded as key information for students to refer to (Figure 1). The first component is the first question posed by the teacher, *how much it's going to cost for the carrots that (the class) need*. To solve this part of the problem, two pieces of information are required: students would need to determine how many carrots are needed, then use this amount to determine a fraction of the cost of one bag. The second component is a consideration for the students, reminding them that they will need to determine *how many you might want (carrot sticks) for the class*. A solution might include: the number of carrots needed (3 or 4, as advised by Hayden), divided by 8 carrots (number of carrots in one bag), multiplied by the cost of one bag (rounded to $2). Such a problem is complex, and it is not obvious to students where to focus their attention first. To further complicate this task, the number of carrots would be determined by the third component; a question posed which is to work out *how many carrot sticks that's gonna give us*. This part of the problem opens the task in that one carrot may be cut into 8, 9, or 10 sticks (depending on the size of the carrot). Students could rely on Hayden's advice that 3 or 4 carrots would be needed, but it wasn't yet clear whether this would produce a reasonable number of carrot sticks for the class to share at a party (number of students = 28). How Paris and Cristina approach this problem is elaborated on in the following section.

Problem Solving

In less than one minute after introducing the task, students were moving to find a partner to problem solve with and a vertical surface on which to record their solutions. Paris and Cristina commenced by recording key information on their vertical surface, "each carrots $2, 8 carrots" followed by the equation, 2 x 8 = 16 (examples of students' written efforts on vertical surfaces includes student spelling and grammatical errors). Clarifying their progress, Paris notes to her partner that this should be sixteen *dollars* (the cost of 8 carrots) and they rewrite this as $2 x $8 = 16. Paris rubs the dollar sign off the board next to the 8, then rubs the entire equation of the board. Eventually they settle on the equation $2 x 8 = $16. In this brief start, we see the students are attempting to formulate the problem as an equation involving the two numbers they have already recorded on the board.

Following this first attempt, the two students check the board beside them where the teacher has recorded the components of the problem posed (Figure 1). The researcher with the camera is standing nearby and prompts the students by asking them to explain their thinking. They restate the number sentence they have recorded on the board, for the camera.

1 Researcher: You mean each carrot is $2?

2 Paris: I mean each carrot bag (rubs out $2 next to carrots and writes 'bag')

3 Cristina: So now we do 2 times 8

4 Researcher: So that's 2 times 8? (points to the board)

5 Cristina: Because there's 8 carrots.

6 Researcher: Yeah. So what are you finding out?

7 Paris: (Muffled. She looks to the key information written by the teacher which is beside them. She points 8 to each piece of information as she reads it.) Oh no wait! We did it wrong! We only need 3 or 4!

9 Researcher: Right!

10 Paris: (To Cristina) We need 3 or 4. (She rubs off the 8 on the board and replaces it with the number 4.)

11 Cristina: Ok.

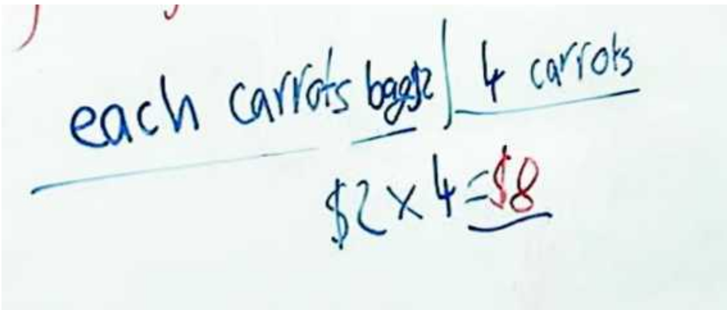
12 Paris: That means we're wrong. (Rubs off the answer '$16') Now it's (muffled-reads the equation)

13 Cristina: (records answer on the board, '$8')

Figure 2 displays the recording done on the vertical whiteboard by Paris and Cristina up to this point.

Figure 2

Vertical Surface Attempts by Paris and Cristina



In this next stage of problem solving, Paris has identified an error: $2 is the cost of a bag of carrots (Line 2). The students were prompted twice by the researcher to explain their efforts (Lines 1 and 4), and when the students checked the key components of the problem recorded on the board beside them (Lines 7 and 8, see Figure 1 also), they acknowledged another error in their efforts. The mistakes by Paris and Cristina take place during the very early stages of problem solving through modelling. In their initial attempts, we see the two students have devoted time to doing mathematics involving the quantities from the real-life situation. The students illustrate correcting each other when a mistake is made, using self-checking strategies such as checking key components of the problem written on the board, and respond when prompted by the researcher to interrogate their efforts by relating attempts to the real situation.

The students continued working on the problem for close to 14 minutes, exploring the different components of the situation, or variables, to solve the problem. Only the initial attempts have been presented here to illustrate what a teacher might expect to see in a Year 4 classroom when students approach mathematical modelling for the first time. Without direction on how to approach the problem, or teacher guidance on efficient strategies to use, we see two students at the end of Year 4 struggle with formulating the problem using number sentences. This highlights difficulties some students might face when simplifying and mathematising a real situation to create a situation model (Steps 1, 2, and 3 (Blum, 2011)) even when embedding the problem in an inquiry context. However, the inquiry framing the modelling experience (*How much does a class party really cost?*) may have supported students at this early stage of problem solving to interpret their results as invalid (steps 5 and 6 (Blum, 2011), generating feedback about how to progress with solving the problem of how much it would cost to buy the carrots they needed for the class.

Conclusion

In this classroom episode, we see all students working collaboratively on the same ill-structured problem, focussed on helping Hayden work out how much it will cost to purchase enough carrots for 28 students to enjoy at their end-of-year class party. The problem is framed by a Guided Mathematical Inquiry, and posed by the teacher in response to a real situation for one student. The teacher identified the carrot-stick problem as an opportunity to simplify a real situation as a modelling task for the class to solve that is aligned with Curriculum goals and is presented as a series of key components (variables, Figure 1). The task was open-ended (solutions would differ depending on whether each carrot was cut into 8, 9, or 10 sticks) and had open solution pathways (it was not clear to students which component needed to be addressed first). The problem involved a financial context, and the teacher has presented the key components of the problem in a way that challenges students to formulate the problem using number sentences, to choose their own strategies, and to interpret the results in terms of the situation (ACARA, 2022). The carrot-stick problem involved no 'correct' solution method and is combined with vertical surfaces to foster confidence through a collaborative approach. The task seems challenging to the two students presented in this paper, yet the struggle they face is productive when they employ their own ideas and strategies to mathematise a real-life situation and begin to identify themselves as *doers* of mathematics (Van de Walle et al., 2019).

We focus on the efforts of Paris and Cristina as they attempt to mathematise a real-life situation involving carrot sticks. Their efforts have been selected here to show the difficulties for students in completing this task, and how collaborative approaches can support students in the early stages of modelling. Blum and colleagues (2011) note *constructing*, as the first step in modelling, as difficult for students. Similarly, we see these students ignoring the context as they initially choose to multiply the values. Working on a vertical surface (Liljedahl et al., 2021) promoted collaboration between the girls as they sought to understand the problem through conversation. Researcher prompts for students to explain their efforts (So what are you finding out? (Line 6)) reminded students to check key components of the problem and progress through the modelling process. This prompt could be used by teachers when students construct situation models in their first attempts at modelling (mathematising the situation), to scaffold student progress.

We hope the problem is a useful example for teachers who are beginning to design their own modelling tasks to meet Curriculum requirements. We acknowledge that the context is not a universal issue, nor relevant to children everywhere. However, this example demonstrates how a teacher might capitalise on problems from students' lifeworlds that can connect to mathematics. We also hope that we have made a distinction through presentation of the carrot-stick problem, between inquiry in the mathematics classroom and mathematical modelling. In the inquiry, problem solving was complex and focused on the specific class of students with specific dietary requests and preferences. In the modelling task, assumptions generalised aspects of the problem-solving focus, simplifying the problem situation. However, inquiry and the process of modelling emphasise designing solution strategies that are purposeful rather than a focus solely on computation.

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References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2010). *Australian Curriculum: Mathematics*. ACARA. https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: Mathematics (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-4_year-5

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2023). *Understand this learning area: Mathematics*. ACARA. https://v9.australiancurriculum.edu.au/teacher-resources/understand-this-learning-area/mathematics

Blum, W. (2011). Can modelling be taught and learnt? Some answers from empirical research. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling* (pp. 15–30). Springer.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13.

diSessa A., Cobb P. (2004). Ontological innovation and the role of theory in design experiments. *Journal of the Learning Sciences*, *13*(1), 77–103. https://doi.org/10.1207/s15327809jls1301_4

Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education*, *34*(2), 110–136.

English, L. D. (2021). Mathematical and interdisciplinary modeling in optimizing young children's learning. *Exploring mathematical modeling with young learners* (pp. 3–23). Springer, Cham.

Fielding, J., & Makar, K. (2022). Challenging conceptual understanding in a complex system: supporting young students to address extended mathematical inquiry problems. *Instructional Science*, *50*(1), 35–61.

Geiger, V., Galbraith, P., Niss, M., & Delzoppo, C. (2022). Developing a task design and implementation framework for fostering mathematical modelling competencies. *Educational Studies in Mathematics*, *109*(2), 313–336.

Liljedahl, P., Zager, T., & Wheeler, L. (2021). *Building thinking classrooms in mathematics: 14 teaching practices for enhancing learning, grades K–12* (Ser. Corwin mathematics). Corwin.

Maass, K., Doorman, M., Jonker, V., & Wijers, M. (2019). Promoting active citizenship in mathematics teaching. *ZDM Mathematics Education*, *51*(6), 991–1003. https://doi.org/10.1007/s11858-019-01048-6

Maass, K., Artigue, M., Burkhardt, H., Doorman, M., English, L. D., Geiger, V., Krainer, K., Potari, D., & Schoenfeld, A. (2022). Mathematical modelling—A key to citizenship education. In N. Buchholtz, B. Schwarz, K. Vorhölter, (Eds.), *Initiationen mathematikdidaktischer Forschung*. Springer Spektrum, Wiesbaden. https://doi.org/10.1007/978-3-658-36766-4_2

Makar, K., & Doerr, H. M. (2020). Developing statistical modeling with paper helicopters. *Mathematics Teacher: Learning and Teaching PK–12*, *113*(2), 147–151.

Park, J. Y. (2023). A Case study of pre-service teachers' task design and implementation for a mathematical modelling lesson sequence in project-based instruction. *Advancing and consolidating mathematical modelling: Research from ICME-14* (pp. 275–289). Springer, Cham.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, *22*(4), 405–435.

Van de Walle, J., Karp, K., Bay-Williams, J., Brass, A., Bently, B., Ferguson, S., Goff, W., Livy, S., Marshman, M., Martin, D., Pearn, C., Prodromou, T., Symons, D., & Wilkie, K. (2019). *Primary and middle years mathematics: Teaching developmentally*. Pearson.

An Aesthetic Approach to Teaching Mathematics: A Proposed Framework Using Children's Picturebooks

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Research shows primary teachers report negative dispositions towards teaching mathematics, impacting their confidence and student outcomes. One approach to improve teacher and student enthusiasm is using picturebooks. While research supports the use of picturebooks in mathematics learning, specific guidance for teachers to implement this practice effectively is lacking. Using a methodology of metalogue, we developed a framework to support teachers in providing engaging mathematics activities through aesthetic perceptions. We reflect on the development of the theoretically informed framework and discuss how it might support teachers' work.

Research shows that teachers and preservice teachers often lack confidence to teach mathematics (Albay & Çetin, 2023), and struggle to teach mathematics engagingly (Pennell et al., 2021), in ways that highlight its aesthetic value. Consequently, this struggle impacts teacher self-efficacy, with many educators reporting negative dispositions towards teaching mathematics (Albay & Çetin, 2023). One approach offered in the literature to improve confidence in teaching mathematics is using quality children's picturebooks (Cox, 2016). Research explores the use of children's books with perceived, explicit, or embedded content (Marston, 2010). Perceived content is the unintentional and incidental inclusion of mathematical concepts in authentic texts. Explicit content is where books, such as mathematical picturebooks, are written explicitly to teach a mathematical concept. Embedded content occurs where books have been written to entertain but include purposefully embedded mathematics concepts. Further research explores frameworks to support teachers in selecting appropriate books (c.f. Marston, 2010). Frameworks can support teachers to implement the use of literature in the mathematics classroom by guiding them to design mathematical activities (c.f. van den Heuvel-Panhuizen & Elia, 2012) that support student learning. However, many teachers are still unsure about how to effectively use quality children's picturebooks in mathematics activities and lessons, highlighting the aesthetics of mathematics (Cox, 2016).

In his seminal theory of aesthetics, Dewey (1934) emphasises the vital importance of a teacher's role in preparing students to notice and appreciate aesthetic attributes. By recognising the inherent aesthetic qualities in mathematics, such as depth or symmetry, educators can foster a deeper connection between mathematical concepts and students' learning experiences. This conference paper begins with an analysis of research on the use of children's picturebooks in mathematics, and existing frameworks designed to support teachers' effective use of picturebooks in mathematics. Based on this literature and a process of metalogue, we answer the research question: How can we develop a framework that would be useful in guiding teachers to authentically incorporate children's literature into mathematics lessons, and to take an aesthetic stance for enhanced engagement and improved outcomes?

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 223–230). Gold Coast: MERGA.

Consequently, we designed an innovative *Beauty of Numeracy (BoN) framework* that teachers and preservice teachers can use as a guide when teaching mathematics engagingly in classroom contexts. Our framework is informed by Dewey's (1934) theory of aesthetics and the transformative potential of aesthetic education. It is also informed by the work of Krathwohl et al (1964) on a classification of educational objectives in the affective domain. This project specifically focuses on the aesthetic value of mathematics (Gadanidis et al., 2016). After reviewing the literature, we discuss the design process of our proposed BoN framework as experienced in its development and suggest how it might best support teachers' work.

Overview of Literature: Teaching Mathematics Through Picturebooks

Children's literature is a useful resource for mathematics instruction (van den Heuvel-Panhuizen & Elia, 2012) and can provide a diverse range of contexts to demonstrate mathematical concepts and applications, and foster connections to real-world scenarios. Selecting high-quality literature for mathematics tasks, designing practical activities, and challenging problem-solving tasks, leads to active learning processes (Cox, 2016). Active learning encourages students to explore mathematical concepts independently (Peter-Koop, 2005), and can be effective in helping students develop a deeper understanding of mathematical concepts (Cox, 2016). We examined research on the use of children's books to teach mathematics, identifying ways research supports teachers to use children's literature in the classroom. Research fell into three themes: (1) research that highlights storytelling to provide context for and support of students' understanding of mathematical concepts; (2) research that provides examples of activities and lessons for specific books; or (3) research on frameworks incorporating children's literature in mathematics instruction.

Research That Highlights Storytelling to Provide Context for and Support of Students' Understanding of Mathematical Concepts

Storytelling is a vital form of communication (Cox, 2016; Griffiths & Clyne, 1991). Mathematical stories can be used to focus student attention on mathematical ideas, although Cox (2016) argues that quality children's literature, without a mathematical focus, can be used to connect ideas for teaching mathematics in the classroom. Griffiths and Clyne (1991) state many concepts are embedded in stories, such as human relations, geography, and morality, but often, mathematics is not considered, making it difficult to embed literature in the mathematics classroom. Cox (2016) suggests teachers identify less obvious content within literature using Fermi problems (Peter-Koop, 2005) ─ non-traditional problems without numbers but with open beginnings and endings. Fermi problems encourage investigations and problem-solving amongst students. Cox (2016) argues they support teachers to integrate children's literature in mathematics, engaging students in meaningful, nonroutine mathematics explorations.

The power of story supports student engagement in the affective domain by learning mathematics through emotionalised and imaginative thinking (Griffiths & Clyne, 1991). In their work in Victorian schools (Australia), these researchers argued stories can: a) provide a context, b) provide a model, c) pose a problem, and d) stimulate an investigation or illustrate a concept. Stories can humanise mathematics, helping students see it as an integral part of everyday life and experience its aesthetic value. By using children's literature, teachers can vary concepts or topics, provide practice with different materials and methods, and provide opportunities for students to share ideas. This storied approach allows students to see how peers: solve problems, challenges students' thinking, and provides opportunity to reflect on learning. While providing a variety of examples, using specific books with explicit mathematics content, the research fails to provide a clear approach for teachers to see the mathematical and aesthetic potential of any children's book.

In a USA study (Barber & Neff, 2019), preservice teachers established *Math Night* events for students and their families. After completing studies in their university course on the use of picturebooks in mathematics, preservice teachers developed games and activities, potentially highlighting the aesthetic value of the chosen texts. While examples are provided of the mathematical goal of the chosen books, the process of developing activities is not described in this article. Research in this section demonstrates the value of children's literature in providing context and stimulus for mathematical thinking but does not provide clear guidance to teachers on how to implement the practice in the classroom.

Research That Provides Examples of Activities and Lessons From Specific Books

Some research describes mathematical tasks and lessons that use specific children's books, providing teachers with practical assistance in the mathematics classroom. For example, Larson and Rumsey (2018) used a picturebook and mathematics manipulatives, to integrate literacy skills into mathematics instruction. Their lesson vignette demonstrated how the book, combined with hands-on activities, supported learning goals and improved outcomes in both literacy and numeracy. The use of picturebooks and manipulatives progressed learning, "students [moved] beyond traditional, passive practices of learning by incorporating creativity, critical thinking, and the presentation of ideas" (Larson & Rumsey, 2018, p. 595). This approach supports teachers to "focus on mathematics worthy of attention, worthy of conversation, worthy of children's incredible minds (Gadanidis et al., 2016, p. 225) Further, Jenkins (2010) demonstrated the development of students' understanding of position, direction, and mapping skills using four picturebooks. Providing a range of activities suited to different year levels and linked with the NSW curriculum relevant at the time (Board of Studies NSW, 2002), Jenkins (2010) pointed out that the suggested texts "also lend themselves to mathematical discussions that cover a variety of other strands" (p. 31). Jenkins' work highlights the need for teachers to explore more deeply the mathematical content within children's literature ─ beyond what might be immediately obvious in the book.

Marston (2018) explored an *explicit* mathematical picturebook that was specifically developed and marketed to teach the concept of doubling. Marston analysed the mathematical content of the book using her framework (Marston, 2010) and found only 28% of the text related to the concept. Also identified in the text were: number, measurement, and geometry concepts. Three teachers in the UK and Australia used the book with three groups of 6–7-year-olds through a shared reading activity. Teachers were not given instructions on how to use the book. After the shared reading, the researcher asked students to "draw a picture or write about what comes into your mind when you think about the story" (Marston, 2018, p. 22). She then interviewed students and teachers. Teachers focused on the concept of doubling but had not considered the other concepts within the text. They used strategies such as pausing reading to explain mathematical content but provided limited opportunities for deep mathematical discussions arising from the content. Students' visual or written responses mostly reflected the doubling concept focused on by the teachers. While the explicit mathematical content in the text provided support for the teachers to integrate literature into mathematics, Marston (2018) found opportunities for deep mathematical exploration were missed, with students mostly engaging in teacher led questioning and discussion. While this research supports the value of children's literature to develop mathematical thinking in students, it also highlights the need for teacher support in effective implementation of picturebooks in mathematics instruction.

Research That Applies Frameworks to Incorporate Children's Literature in Mathematics

Recognising the need for implementation support, other research has developed frameworks to assist teachers. For example, Marston's (2010) framework guides teachers in selection of appropriate picturebooks to teach mathematics concepts. With an analysis of 122 award-winning children's books, Marston categorised mathematical content into perceived, explicit, or embedded categories to support teachers in evaluating the appropriateness of a text for teaching and learning in mathematics. The resultant framework includes sub-categories of mathematical content, problem-solving, reasoning, and pedagogical implementation. While Marston's framework provides a tool for teachers to consider texts for mathematics, limited detail on how to implement literature into the mathematics classroom is provided.

Interactive read alouds using picturebooks can enhance mathematical learning. Courtade et al. (2013) demonstrated the value of interactive read alouds using picturebooks in supporting students with moderate to severe disabilities to build mathematical knowledge. By selecting children's books that pose interesting problems and showcase solutions, the authors suggested that educators can integrate real-world connections into mathematical concepts. A special education teacher and a mainstream Year 3 teacher, worked with a Year 3 student with an intellectual disability who was assessed before, during and after an interactive read aloud activity within a mainstream classroom setting. The teachers collaboratively planned ways to support the student to access the mathematical content within their inclusive classroom. They developed four key steps to support teachers to incorporate children's literature into mathematics for students with intellectual disabilities: (1) choosing a book relevant to a mathematical concept; (2) adapting it to meet individual student needs; (3) employing concrete examples with systematic instruction; and (4) incorporating assessment to guide instructional decisions effectively. Using these four key steps, the teachers planned further experiences with other texts demonstrating the usefulness of the steps in guiding the incorporation of literature into inclusive educational settings for diverse students.

A problem-solving pedagogy utilising challenging and engaging tasks has shown success in recent mathematical research (cf. Stein et al., 2008). One example that has developed from this rich research is a recent study by Ingram et al. (2020) who demonstrated a framework to support 12 New Zealand primary teachers with implementation of challenging geometric reasoning tasks in inclusive classrooms as part of the Encouraging Persistence, Maintaining Challenge (EPMC) project. The research team provided professional learning sessions, sample problems and resources, and classroom anchor charts to support a problem-solving culture in the classroom. This framework proved successful in supporting teachers who stated they are now incorporating notions of challenge and struggle across other curriculum areas in their planning and teaching. Students reported how teachers focused instruction on the process of problem-solving and expected them to solve problems, to support them in their mathematics learning. This focus on problem-solving authenticates the development of critical thinking skills (Larson & Rumsey, 2018). This research focused on challenging tasks to provide an authentic context to develop critical thinking skills. How to provide the meaningful context to engage students in problem solving remains a challenge.

Using picturebooks provides meaningful contexts for mathematical learning and supports students' active engagement leading to better conceptual understanding. Forbringer et al. (2016) conducted a study in three classrooms; kindergarten, a multi age classroom, and a year 1 class, to develop a framework for teachers to use children's literature in a differentiated mathematics lesson. This framework comprises six steps: (1) select book that serves as an inspiration for mathematical problem-solving; (2) determine the mathematical focal point of the lesson and align it with the curriculum; (3) use the narrative to craft a mathematical problem or question; (4) list the concepts or skills that students must possess to successfully solve the problem; (5) consider additional questions to differentiate the task; and (6) plan how to facilitate students to engage in differentiated problem-solving tasks. Findings showed that the framework supported teachers to use children's literature to teach mathematical concepts, allowing differentiated tasks in an inclusive classroom. Research of frameworks demonstrates their potential value in mathematics instruction but also reveals some limitations.

Limitations of Existing Frameworks

While research demonstrates that using children's literature to support mathematical development is helpful (e.g., Cox, 2016; Marston, 2018), it provides scant ideas on how teachers might implement this practice in the classroom. Some literature provides advice for only specific titles (Larson & Rumsey, 2018). While Marston (2010) suggests mathematical content is perceived, explicit or embedded, she implies that not all children's literature could be useful in a mathematics classroom. Further, some frameworks provide guidance on the types of mathematical activities that support students' critical thinking skills (Forbringer et al., 2016), but explicit support for teacher implementation through children's literature is limited.

Research supports the benefits of teachers using picturebooks in mathematics to develop engaging and challenging tasks and resources. The aesthetic aspect of this approach, however, is largely missing in the research literature. Our study emphasises the significance of aesthetic appreciation in mathematics learning (Green et al., In press). We sought to develop a framework that progresses student learning in mathematics through perceptions of picturebooks, which enhance the aesthetic value of mathematics. This led to our research question: How can we develop a framework that would be useful in guiding teachers to authentically incorporate children's literature into mathematics lessons, and to take an aesthetic stance for enhanced engagement and improved outcomes?

Methods

To collaboratively bring together the different ideas and expertise of the authors, we used the methodology of metalogue. Metalogue involves conversation about a problematic matter, with subsequent discussion of the conversation in relation to the same matter (Bateson, 1972). Both the form and content of the discourse is analysed, and meaning made through reflexive practice (Willis et al., 2014). Metalogue is different from a conversation analysis (Toerien, 2014) in that researchers revisit the experience of an initial conversation through a reflexive lens, analysing the subsequent conversation. Emphasis is placed on the aesthetic dimension of the discussion (Yared & Davis, 2014). Metalogue provides space for participants to engage in generative dialogue, listen, and learn from each other (Willis et al., 2014). We conversed on the problematic matter of students' aesthetic engagement in picturebooks for mathematics learning, and then discussed the value of the conversational aesthetic experience (Green et al., in press). This reflexive exercise impelled the development of the BoN framework.

Developing the *Beauty of Numeracy (BoN) Framework*

The development of the BoN framework was informed by research on frameworks that support children's literature in mathematics instruction (e.g., Courtade et al., 2016; van den Heuvel-Panhuizen & Elia, 2012). This comprised frameworks designed to support teaching in mathematics (E.g., Ingram et al., 2020), promote the aesthetic value of mathematics (Gadanidis et al., 2016), and meet education objectives in the aesthetic domain (e.g., Krathwohl et al., 1964). Additionally, the BoN framework is underpinned by Dewey's (1934) aesthetic theory, often drawn on to advance mathematics instruction (e.g., Gadanidis et al., 2016) for its enculturating, long-term, transformative potential. Bloom's taxonomy of educational objectives in the affective domain (Krathwohl et al., 1964) also provides theoretical underpinnings.

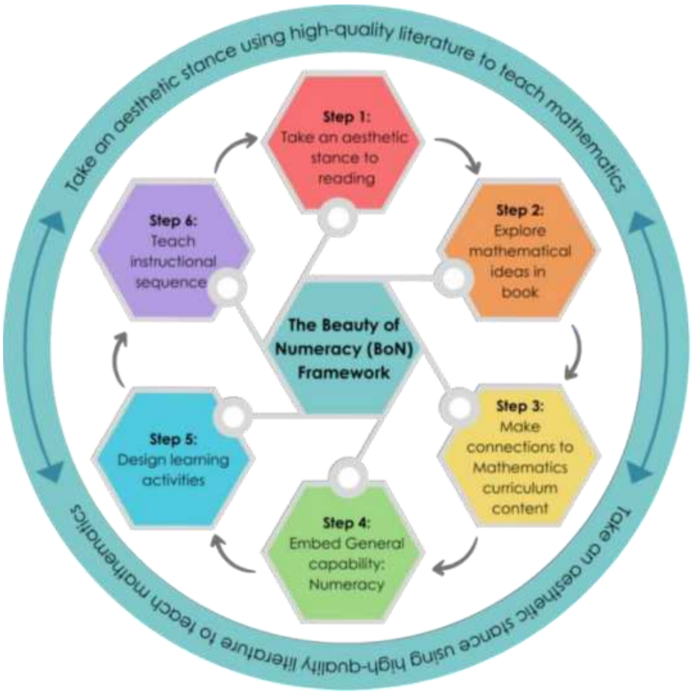
The Design Process

In the first conversation, the first two authors examined their own journeys towards planning and designing mathematics activities and challenging tasks. Following the process of metalogue (Green et al., in press), each author developed an initial framework. These frameworks were then discussed and analysed to identify key areas of innovation, agreement, and disagreement in a dynamic and interactive exchange, characterised by a rich diversity of perspectives, ideas, and insights. Through metalogue conflicts were resolved, and the initial six steps of the framework developed. Next, all five authors worked on different developmental steps of the framework, which were then brought back for further discussions and analysis by the first two authors. From the finalised design, the procedures for each step were fine-tuned by the first two authors. The final two authors incorporated the steps into an easy-to-follow diagram, displayed on an A3 placemat with the framework on one side, and explanations of each step on the other (see Figure 1). With the visual representation of the process in the centre of the placemat, brief points for each step were provided around the outside. A clear outline of the purpose was included at the top of the framework. The front page provides a quick reference guide for teachers. On the reverse side of the placemat, detailed explanations of each step were provided so that teachers could refer to this if they were unsure of their next steps.

The BoN framework was designed to provide in-service and pre-service teachers with a guide to authentically incorporate children's literature into mathematics lessons, through taking an aesthetic stance for enhanced engagement. The A3 size placemat was easy for teachers to keep on their desks to be readily available to support them in their planning and practice.

Figure 1

Beauty of Numeracy (BoN) Framework



With the visual representation of the process in the centre of the placemat, a clear outline of the purpose of taking an aesthetic approach to mathematics was stepped out into six clear principles. The aesthetic approach is the subject of a subsequent publication but briefly, it encourages teachers to appreciate and value:

1. An aware willing, open, and free exploration of mathematical concepts.
2. Satisfaction in responding to the sensory experience of mathematical inquiry.
3. Commitment to the development of perception associated with mathematical thinking.
4. Formation and organisation of ideas associated with mathematical beauty and reasoning.
5. Readiness to revise judgements associated with mathematical observations, evaluations, and reasoning.
6. Internalised, consistent belief in oneself as a highly numerate person who thoroughly enjoys mathematics.

The front side provides a quick reference guide for teachers with brief points for each step provided around the outside. With each step listed below, the key research literature that inspired and supported the importance of the inclusion of that step into the framework is listed:

1. Prepare the physical and mental space for taking an aesthetic stance to read (Gadanidis et al., 2016).
2. Explore high-quality children's literature with selective attention directed towards mathematical concepts (Cox ,2016).
3. Make connections between the mathematical concepts presented in the book and relevant mathematics curriculum content (Marston, 2018).
4. Identify any relevant General capability: Numeracy elements/sub-elements to broaden and extend learning (Forbringer et al., 2016).
5. Draw inspiration for the children's book, to develop a challenging learning activity/experience/inquiry/problem (Larson & Rumsey, 2018).
6. Inspired by Ingram et al. (2020):
    - *Prepare* for aesthetic engagement in reading and mathematics;
    - *Launch* the mathematical challenge;
    - *Explore* a range of possible responses;
    - *Discuss* ideas, potential to build on others' ideas, reasonableness of ideas and value the learning experience.

The detailed steps of the BoN framework, as shown on the reverse side of the placement, provide guidelines to teachers to explore quality children's literature, make connections to mathematical concepts and link that to the curriculum content for the context of their own class. Step 6, based on the problem-solving approach of Ingram et al. (2020), provides guidance for teachers in planning and implementing problem-based lessons with their students, with a reminder to consider an aesthetic stance. To support teachers to effectively implement the BoN framework in classroom planning and practice, a series of professional development sessions are currently being trialled. The trial is beyond the scope of this paper.

Discussion

The process of planning and developing a framework, through collaborative consultation and metalogue between all authors supported a progressive fine tuning of the BoN framework. The generative process led to a comprehensive and complete framework that is valuable for teachers' curriculum design and enactment. The framework offers support for in-service and preservice teachers in taking potentially any quality children's picturebook and using it as the basis for mathematical investigations. The aesthetic stance and problem-solving pedagogical approach offer support for both teachers and students to appreciate the aesthetic qualities of mathematical concepts, structures and content. This supports both teachers and students to see the beauty and value of mathematics in their world. The problem-solving, challenging task approach (Ingram et al., 2020) supports creativity and critical thinking as valued by Larson and Rumsey (2018), and Dewey (1934).

The next phase in the research project provides professional development (PD) sessions for in-service and pre-service teachers to trial and provide feedback on application of the BoN framework for improved engagement and outcomes. Initial feedback from the first two PD sessions, in which the BoN framework was introduced and explained, indicated 98% of teachers felt they had learnt something new and valuable. Further research will investigate teachers' implementation of the framework in classroom settings as well as impact on engagement and outcomes.

References

Albay, A., & Çetin, H. (2023). A mixed method research study on pre-service primary school teachers' mathematical disposition. *Journal on Mathematics Education, 14*(2), 311–338. https://doi.org/10.22342/jme.v14i2.pp311-338

Barber, K. A., & Neff, S. (2019). Through the eyes of a preservice teacher: Developing mathematics instruction with picture books during math night. *PDS Partners*, *14*(2), 18–22.

Bateson, G. (1972). Steps to an ecology of mind. Jason Aronson. Board of Studies NSW. (2002). *Mathematics K–6 Syllabus*. Board of Studies.

Courtade, G. R., Lingo, A. S., Karp, K. S., & Whitney, T. (2013). Shared story reading: Teaching mathematics to students with moderate and severe disabilities. *Teaching Exceptional Children*, *45*(3), 34–44. https://doi.org/10.1177/004005991304500304

Cox, T. (2016). Integrating literature into the teaching of mathematics. *Australian Mathematics Teacher*, *72*(1), 15–17.

Dewey, J. (1934). *Art as experience*. Minton, Balch & Company.

Forbringer, L., Hettinger, A., & Reichert, E. (2016). Using the picture book extra yarn to differentiate common core math instruction. *YC Young Children*, *71*(2), 22–29.

Gadanidis, G., Borba, M., Hughes, J., & Lacerda, H. D. (2016). Designing aesthetic experiences for young mathematicians: A model for mathematics education reform. *Revista Internacional de Pesquisa em Educação Matemática*, *6*(2), 225–244.

Green, M., Gaunt, L. & Barton, G. (in press). The aesthetic value of mathematics in children's picturebooks: A metalogue for improving teacher dispositions. *The Australian Educational Researcher*.

Griffiths, R., & Clyne, M. (1991). Once upon a time ... . *Australian Mathematics Teacher*, *47*(1), 10–13.

Ingram, N., Holmes, M., Linsell, C., Livy, S., McCormick, M., & Sullivan, P. (2020). Exploring an innovative approach to teaching mathematics through the use of challenging tasks: A New Zealand perspective. *Mathematics Education Research Journal*, *32*(3), 497–522. https://doi.org/10.1007/s13394-019-00266-1

Jenkins, K. (2010). New voices: Positioning picture books within the Mathematics Curriculum. *Australian Primary Mathematics Classroom*, *15*(2), 28–32.

Krathwohl, D., Bloom, B., & Masia, B. (1964). Taxonomy of educational objectives: The classification of educational goals: Handbook II affective domain. David McKay Company.

Larson, L. C., & Rumsey, C. (2018). Bringing stories to life: Integrating literature and math manipulatives. *The Reading Teacher*, *71*(5), 589–596. https://doi.org/10.1002/trtr.1652

Marston, J. (2010). Developing a framework for the selection of picture books to promote early mathematical development. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the Future of Mathematics Education. Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 383–390). MERGA.

Marston, J. (2018). Opportunities for rich teaching and learning in explicit mathematical picture books. *Primary Mathematics*, *22*(2), 21–24.

Pennell, S., Baker, K., & Fede, B. (2021). "The space between" literacy and mathematics: Interdisciplinary teaching through a graphic novel. *Voices From the Middle*, *29*(2), 64–67.

Peter-Koop, A. (2005). Fermi problems in primary mathematics classrooms: Fostering children's mathematical modelling processes. *Australian Primary Mathematics Classroom*, *10*(1), 4–8.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340. https://doi.org/10.1080/10986060802229675

Toerien, M. (2014). Conversations and conversation analysis. In E. Flick (Ed.), *The SAGE handbook of qualitative data analysis* (Chapter 22, pp. 327–340). Sage.

van den Heuvel-Panhuizen, M., & Elia, I. (2012). Developing a framework for the evaluation of picturebooks that support kindergartners' learning of mathematics. *Research in Mathematics Education*, *14*(1), 17–47.

Willis, L.-D., Kretschmann, G., Lewis, K., & Montes, C. (2014). An engaging pedagogy for social education: Co-teaching a teacher education program. *Social Educator*, *32*(3), 23–36. https://doi.org/10.3316/aeipt.206846

Yared, A., & Davis, H. (2014). Talking in circles: interview, conversation, metalogue. *Journal for Artistic Research*, *5*.

Harnessing the Expertise of Mathematics Intervention Teachers to Support Primary Teachers Through Co-Teaching Cycles

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This paper reports on a study that seeks new insight into how the expertise of specialist mathematics intervention teachers might be harnessed to support classroom teachers to assist students who experience difficulty learning mathematics. Findings show that the classroom teachers' confidence increased after engaging in co-teaching cycles led by the specialists. The most highly ranked helpful actions of the specialist intervention teachers were 'suggesting appropriate tasks for a given topic,' 'team teaching with me,' and 'suggestions about content for the next lesson.' The findings suggest that teachers' professional learning needs vary and require a personalised response.

Enabling all students to thrive with learning mathematics is an important goal for teachers. However, primary school teachers in Australia are generalists who do not have specific expertise or confidence in diagnosing or responding to the difficulties or diverse abilities of students. Hence, some schools employ a specialist mathematics teacher to implement a range of intervention approaches to support students (Bryant et al., 2008; Gervasoni, 2015; Sonnemann & Hunter, 2023). Although intensive intervention programs are effective (Gervasoni et al., 2021; Nickow et al., 2020), schools typically cannot resource intervention programs for all students who qualify, even if desired. A more strategic and sustainable approach may be to increase classroom teachers' expertise and confidence in providing high-quality inclusive mathematics teaching that enables all students to thrive. Our study explores how this goal might be advanced through professional learning involving co-teaching cycles (Sharratt & Fullan, 2012) that harness the expertise of specialist mathematics intervention teachers. The research questions addressed in this paper are: (1) How does the mathematics teaching confidence of primary teachers change after engaging in co-teaching cycles led by a specialist intervention teacher? and (2) What actions of the specialist intervention teachers in each phase of the co-teaching cycle do classroom teachers rank most highly to help them to support all students' mathematics learning? A particular focus for our study is considering the implications of the findings for designing approaches to assist students who are not currently thriving with mathematics learning.

Background Literature

Although intensive mathematics interventions are effective for increasing and accelerating students (Gervasoni et al., 2019; Nickow et al., 2020), not all students who may benefit are able to access these. Intervention approaches typically fall into three tiers (Bryant et al., 2008). Tier 1 approaches focus on high-quality classroom instruction to meet the needs of all students; Tier 2 approaches provide small group support for about 15% of students who fall behind; and Tier 3 approaches include intensive one-on-one support for students who make minimal progress in Tier 2 (Sonnemann & Hunter, 2023). With respect to Tier 1, teachers report that differentiating instruction to meet the needs of all students is one of the most difficult aspects of mathematics teaching (Downton et al., 2022; Gervasoni et al., 2021). Findings from the pilot phase of our study suggest that teachers find two aspects challenging when teaching students who are

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mathematically vulnerable: (1) differentiating instruction effectively; and (2) having sufficient time and opportunity in a lesson to work with students who experience difficulty (Gervasoni et al., 2023). Hence, building the capability and confidence of classroom teachers to provide high-quality mathematics instruction for all is vital.

Teacher Confidence

It has been long established that teachers' confidence about teaching mathematics is an important influence on their classroom practice (Munby et al., 2001). Indeed, Baxter et al. (2014) argue that there is a need to engage teachers in professional learning that supports the learning of content and pedagogy, while helping them to develop more confidence in their ability to teach mathematics. It is also established that large proportions of pre-service primary teachers, including in Australia, have low confidence in teaching mathematics (Norton, 2017) which is likely to influence their practice as graduate teachers. Teachers who lack confidence in teaching mathematics demonstrate this by avoiding teaching some aspects of mathematics, lacking variation in pedagogy, and relying on tightly scripted or unscripted approaches with minimal teacher input (Norton, 2017). It is likely that in situ professional learning that utilises co-teaching cycles for teachers led by a specialist mathematics teacher will be beneficial for building both the confidence and expertise of teachers.

Teacher Professional Learning and Co-Teaching Cycles

Previous studies have demonstrated the effectiveness of in-situ professional learning led by school mathematics leaders or coaches for enhancing mathematics teaching (Anstey & Clarke, 2010; Sexton & Downton, 2014), and for increasing student achievement (Bruce et al., 2010). Cobb et al. (2019) argued that modelling instruction, co-teaching, co-planning, and debriefing were potentially productive activities for mathematics coaches. Teachers have indicated that "modelling, observation, and debriefing were the most valuable components" of a professional learning model (Butler et al., 2004, p. 447). One model for leading classroom embedded professional learning that incorporates these approaches is the co-teaching cycle (Sharratt & Fullan, 2012) which comprises co-planning, co-teaching, co-debriefing, and co-reflecting. Professional learning based on co-teaching cycles aims to address aspects of practice that teachers wish to improve, so has potential for our study.

Given teachers have indicated that teaching mathematics for students with diverse abilities is difficult (Downton et al., 2022; Gervasoni et al., 2021), the use of co-teaching cycles for our study will need to address this aspect of practice. Previous research has highlighted that high-quality inclusive mathematics instruction promotes problem solving, collaboration, dialogue, and using tasks with enabling and extending prompts to enable all students to access the task (Russo et al., 2020). Further insight is needed about whether co-teaching cycles increase teachers' confidence to effectively differentiate teaching for diverse students, and which actions of the specialist intervention teacher are most helpful.

Context for the Study

Our study took place in primary schools situated within a System of 58 schools in Sydney that has focused on a constructivist aligned and problem-solving approach to mathematics education. In 2022 we conducted a pilot study, with a follow-up phase in 2023. The System approach to mathematics education has included using a task-based mathematics assessment (Clarke et al., 2002) enabling student progress to be monitored annually, and any students who were mathematically vulnerable to be identified, and supported through the Extending Mathematical Understanding (EMU) intervention program (Gervasoni et al., 2021; Gervasoni, 2015). The EMU program is taught by certified specialist teachers (ST) who complete a 6-day course and ongoing annual professional learning. More than 400 teachers in the system have qualified as EMU specialist teachers. The theoretical underpinnings, lesson structure, and teaching approach for the intervention program are described in detail in Gervasoni (2015).

Given that not all eligible students are able to access an EMU intervention program in Year 1, and that there are many students in Year 2 to Year 6 who are mathematically vulnerable also, our study seeks to investigate how the expertise of EMU Specialist Teachers (EMU ST) may be harnessed to enable classroom teachers to more intentionally support the mathematics learning of this group in their classrooms. Through experience with EMU intervention, the EMU STs had developed expertise in differentiating instruction for groups of three students, guided by diagnostic assessment and the ENRP growth point framework (Clarke, 2013). They were also experienced with designing lessons based on problem-solving and engagement with open tasks, and identifying concrete models to assist students' construction of knowledge; prompting students to visualise and explain their thinking and strategies for each other; and developing students' confidence and positive dispositions for mathematics. These practices are highly relevant for Tier 1 mathematics teaching.

Method

Qualitative methods were chosen as most relevant for our study. The research design involved EMU STs leading co-teaching cycles for classroom teachers for at least 10 weeks. At the conclusion of the co-teaching cycles, teachers were surveyed using an online platform (Qualtrics) and interviewed (via Zoom). The research followed the approved ethical guidelines, and pseudonyms used. Results and findings for classroom teachers are the focus for this paper.

Data Collection Instruments and Data Analysis

The teacher survey included items that invited teachers to rate their confidence in teaching mathematics and items that asked teachers to rank the helpfulness of a set of actions for each phase of the co-teaching cycle. Teachers were asked to rate their confidence at two time points: prior to the series of co-teaching cycles, and after. These time points allowed us to measure perceived changes in confidence. To gain insight about whether the teachers' perceived confidence was different when teaching mathematics for students who were struggling to learn, or highly capable, we asked them to rate their confidence for teaching these groups of students also. Open response items investigated what teachers considered most challenging about teaching students who were struggling with mathematics, and any other support from the EMU ST that they found valuable. The semi-structured interviews aimed to provide greater depth and clarity about the nature of the EMU ST support that teachers received, and their perceptions of the impact of the support. The survey responses were summarised. Open response items and the transcribed interview data were analysed and excerpts were used to further illustrate the survey findings.

Co-teaching Cycles

The co-teaching cycles in 2023 involved the EMU specialist teacher (EMU ST) co-planning with classroom teachers weekly across two terms, co-teaching a minimum of two mathematics lessons each week, and co-debriefing and co-reflecting after the co-teaching lessons. This process is known as EMU Level 2 (L2) intervention in our study. A series of professional learning sessions, facilitated by the System EMU Professional Learning Leaders, supported the EMU STs to lead the co-teaching cycles. Professional learning included role clarification, facilitating collaborative mathematics planning, professional readings, and observing and reflecting on an L2 co-teaching and co-debriefing session in one of the schools.

Participants

The six schools trialling EMU L2 in 2023 were invited to participate in the study. These schools had the support and commitment of the school's principal, were able to resource the intervention with appropriate staffing and time, and an experienced EMU ST on staff who was willing to participate. Three schools agreed to participate. School leaders selected the grade level for the EMU L2 support based on the proportion of students who were mathematically vulnerable, using data from the Mathematics Assessment Interview (Clarke et al., 2002). School A chose Grade 2, School B chose Grade 3, and School C chose Grade 4. The proportion of students who were vulnerable in at least one whole number domain in each grade was 75%, 96% and 66% respectively. The participants from the three schools were an EMU ST from each school, one Grade 2 teacher (School A), three Grade 3 teachers (School B), and two Grade 4 teachers (School C). The classroom teachers' experience ranged from 2 to 19 years, and EMU STs had 5 to 8 years of experience teaching EMU programs.

Results and Findings

Confidence in Teaching Mathematics

An aim of the study in 2023 was to gain insight about whether EMU L2 support was associated with increases in teacher confidence. At the end of the 2023 school year, classroom teachers were invited to rate their confidence as a mathematics teacher prior to the co-teaching cycles, and after. Similarly, they rated their confidence for teaching mathematics for students who are struggling to learn, and for students who are highly capable. Results are shown in Table 1. Note that Teacher 1 did not respond to this question.

Table 1

Classroom Teachers Confidence Ratings (Out of 10) for Teaching Mathematics, for Students who Struggle and Highly Capable Students, Prior to and Following EMU Level 2 Support

Teacher	Confidence rating as a mathematics teacher		Confidence rating for teaching students who struggle		Confidence rating for teaching highly capable students	
	Prior to L2	After L2	Prior to L2	After L2	Prior to L2	After L2
2	5	7	6	7	4	7
3	5	7	6	8	4	7
4	4	7	4	7	3	5
5	6	8	5	7	6	8
6	6	9	3	7	5	8
Mean	5.2	7.6	4.8	7.2	4.4	7.0

Prior to the EMU L2 support, no teacher rated their confidence highly, with ratings ranging from 3–6 for teaching students who struggle, or who are highly capable, and from 4–6 for confidence as a mathematics teacher. Following the L2 support, the confidence ratings were mostly in the range of 7–8 for all categories, which suggests a positive increase in confidence. The mean ratings suggest that the teachers were slightly more confident teaching students who struggle than teaching highly capable students. The greatest variation in teachers' increase in confidence was for teaching mathematics for students who struggle (1–4 points).

Valued Actions of EMU Specialist Teachers

In the 2023 study, we used analyses of survey items and interviews transcripts to gain deeper insight into which EMU ST actions were most helpful in each phase of the co-teaching cycle. In the survey, teachers ranked of a set of actions for each phase. In the interviews, teachers described the role of the EMU ST during the co-teaching cycles. The results are described below.

Co-Planning

For the co-planning phase, the classroom teachers ranked seven actions of the EMU ST, according to helpfulness (Table 2). The most highly ranked actions were 'suggesting appropriate tasks for a given topic' and 'supporting me to anticipate students' responses, solutions, and misconceptions.' The least helpful were 'providing advice about choosing and using concrete materials' and 'assisting me with understanding the growth points.' Considerable variation in rankings for most statements suggests that the teachers' needs differed.

Table 2

Frequency of Classroom Teacher's Ranking of EMU ST's Actions During Co-Planning

EMU ST actions during co-planning	Ranking for EMU ST actions						
	1	**2**	**3**	**4**	**5**	**6**	**7**
Assisting me to design enabling and extending prompts	0	2	0	1	2	0	1
Suggesting appropriate tasks for a given topic	4	1	0	0	1	0	0
Supporting me to anticipate student responses/solutions/misconceptions	1	1	3	1	0	0	0
Supporting me to understand the mathematics related to the lesson	0	2	0	2	0	1	1
Providing advice about choosing and using concrete materials	0	0	1	1	0	2	2
Providing advice about questions to elicit students' thinking	1	0	1	1	2	1	0
Assisting me with understanding the growth points	0	0	1	0	1	2	2

Data from the interview transcripts provided further insight into how the EMU ST helped teachers with suggesting appropriate tasks. For example: We'd be looking at tasks … we'd think about the area that we're … focusing on … then it was about that progression … and to have the enabling prompts and the extension prompts … ready to go. (Teacher 6)

So how do we modify the tasks to allow it to be accessible to other students? How do we engage them … so that they are willing to have a go … have the topics of the tasks be something that they're familiar with or …interested in … things like that that Kay was able to facilitate for us. (Teacher 5)

Teacher 1 described how planning discussions assisted her teaching to be more purposeful:

There's more anticipation now about student responses, …the reflection phase [of the lesson] is more purposeful. … And rather than us not being prepared for the task, we know perhaps student A has thought this …because he doesn't have the foundational understanding of this concept.

During Co-Teaching

The classroom teachers ranked six co-teaching actions by the EMU ST, according to their helpfulness. Table 3 shows the results. Note that the final two actions are not typical of co-teaching actions, but were described by teachers as helpful in the pilot study (Gervasoni et al., 2023). We included these to gain deeper insight into how teachers ranked these actions.

All teachers ranked 'Team teaching with me' as the most helpful of the six actions. 'Modelling the discussion at the end of a task or lesson' was highly ranked by most teachers. 'Modelling a full lesson' had the lowest overall ranking.

Excerpts from the interview transcripts provide further insight about the helpfulness of *team teaching*:

She'd roam across the whole space and she'd come and suggest, "Oh, this work sample is really good," if I needed help, or I'd go to her and ask for advice to see if I'm on the right track. (Teacher 5)

And I feel like the co-teaching was actually really helpful for me. She [EMU ST] would help with selecting work samples to put on the screen, helping make it visual. (Teacher 4)

Table 3

Frequency of Classroom Teacher's Ranking of EMU ST's Actions During Co-Teaching

EMU ST actions during co-teaching	Ranking for EMU ST actions					
	1	2	3	4	5	6
Modelling a full lesson	0	0	0	2	1	3
Modelling the discussion at the end of the task or lesson	0	4	0	2	0	0
Team teaching with me	6	0	0	0	0	0
Observing my teaching and providing feedback	0	0	4	1	0	1
Withdraw small group, focusing on moving to the next growth point	0	1	1	0	4	0
Reaching the students I can't get to in a lesson	0	1	1	1	1	2

The interview transcript data also illustrated how the EMU ST helped teachers by modelling or co-teaching during the discussion at the end of a task or lesson:

> Reflections in maths have always been a struggle for me because I never really know what to focus on. Kay [EMU ST] just showed me, … and that's what we reflect on. So I feel a lot more confident in my own teaching because of this. (Teacher 5)

> So on the spot, if we were looking at something … in a student's book that we wanted to reflect on, instead of us giving the game away, [the EMU ST] would often say, 'How can you prompt the students? What can you make them think?' I think it was the verbal cues. (Teacher 1)

During Co-Debriefing and Co-Reflecting

The classroom teachers ranked five actions by the EMU ST when co-debriefing and co-reflecting, according to their helpfulness. Table 4 shows the results.

Table 4

Frequency of Classroom Teacher's Ranking of EMU ST's Actions During Co-Debriefing

EMU ST actions during co-planning	Ranking for EMU ST actions				
	1	**2**	**3**	**4**	**5**
Feedback about what did not go well in the lesson.	1	0	0	1	4
Feedback about the students who were struggling.	1	0	4	1	0
Feedback about what worked well in the lesson.	0	1	1	3	1
Suggestions about content for the next lesson.	1	4	1	0	0
Suggestions pedagogies that may assist student learning in next lesson	3	1	0	1	1

The most highly ranked actions were 'Suggestions about content for the next lesson' and 'Suggestions about pedagogies that may assist students' learning in the next lesson.' Although four teachers ranked 'Feedback about what did not go well in the lesson' as least helpful, one teacher ranked it as most helpful. This result highlights the variation in rankings for most statements, suggesting that the needs of the teachers during debriefing differed.

The following interview excerpts further illustrate the helpfulness of debriefing actions:

> I also think the importance of our co-debriefing sessions during the lesson were crucial. After roaming, and then the three of us, 'let's have a reflection. What's our next step?' (Teacher 1)

> Kate [EMU ST] would always make sure that a couple of times a week … we'd be having the conversations with her, and she'll be seeing how we're going, what's working well, what can we adjust. (Teacher 2)

Discussion and Conclusion

The first research question sought insight into how the mathematics teaching confidence of primary teachers changed after engaging in co-teaching cycles led by an EMU specialist intervention teacher. The key finding is that engagement in the co-teaching cycles was associated with a positive increase in confidence for teaching mathematics for all five teachers, including for teaching students who struggle, and for highly capable students. This finding suggest that the EMU L2 support assisted teachers to be more confident in differentiating teaching for students with diverse knowledge. This finding is promising given the influence of confidence on mathematics teaching practice (Munby et al., 2001; Norton, 2017).

The second research question investigated which EMU ST actions classroom teachers ranked most highly, in each phase of the co-teaching cycle. The first key finding was the variation in rankings for actions, especially for the co-planning and co-de-briefing phases. This suggests that the professional learning needs of the teachers varied, and that a personalised approach for co-teaching cycles is warranted. This finding supports the view that co-teaching cycles need to address the specific aspects of practice that teachers wish to improve (Sharratt & Fullan, 2012). The second finding is the set of helpful EMU ST actions that teachers ranked most highly for each phase in the co-teaching cycle. These were: (i) suggesting appropriate tasks for a given topic, and supporting teachers to anticipate students' responses, solutions, and misconceptions (co-planning phase); (ii) team teaching, and modelling the discussion at the end of a task or lesson (co-teaching phase); and (iii) suggesting content for the next lesson, and suggesting pedagogies to assist students' learning in the next lesson (co-debriefing phase). It was clear from the interview data that these actions assisted teachers to become more confident, and to more effectively differentiate their teaching in response to the diverse range of student knowledge in a classroom. It is likely that these actions will be useful practices for other EMU STs to consider when leading co-teaching cycles. This would be a profitable area for further research. Also, it would be valuable to explore the impact of the EMU L2 support on the growth of students' mathematics learning and confidence.

Our study explored whether the expertise of specialist intervention teachers could be harnessed to increase the confidence and capability of classroom teachers to support students who struggle with mathematics learning. The findings provide new insight about the specific actions that teachers find most helpful in supporting their professional growth during co-teaching cycles. Although the overall goal of our research is to design approaches to assist students who are not thriving with mathematics learning, a surprising outcome of our study is that the EMU L2 approach also increased teachers' confidence and expertise in teaching highly capable students. This broader impact highlights the value of teachers learning in situ alongside a trusted and knowledgeable expert with whom they can engage in dialogue about what they are noticing about students' learning, and how to respond.

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References

Anstey, L., & Clarke, B. (2010). Perceived professional learning needs of numeracy coaches. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education. Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 45–52). MERGA.

Baxter, J.A., Ruzicka, A., Beghetto, R.A. and Livelybrooks, D. (2014). Professional development strategically connecting mathematics and science: The impact on teachers' confidence and practice. *School Science and Mathematics*, *114*(3) 102–113. https://doi.org/10.1111/ssm.12060

Bruce, C. D., Esmonde, I., Ross, J., Dookie, L., & Beatty, R. (2010). The effects of sustained classroom embedded teacher professional learning on teacher efficacy and related student achievement. *Teacher and Teacher Education: An International Journal of Research and Studies*, *26(*8), 1598–1608.

Bryant, D. P., Bryant, B. R., Gersten, R. M., Scammacca, N. N., Funk, C., Winter, A., Shih, M., & Pool, C. (2008). The effects of tier 2 intervention on the mathematics performance of first-grade students who are at risk for mathematics difficulties. *Learning Disabilities Quarterly*, *31*(2), 47–63.

Butler, D. L., Lauscher, H. N., Jarvis-Selinger, S., & Beckingham, B. (2004). Collaboration and self-regulation in teachers' professional development. *Teaching and Teacher Education, 20*, 435–455

Clarke, D. (2013). Understanding, assessing and developing children's mathematical thinking: task-based interviews as powerful tools for teacher professional learning. In: A. M. Lindmeier & A. Heinze (Eds.), *Mathematics learning across the life span. Proceedings of the 37th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17–30). PME.

Clarke, D. M., Cheeseman, J., Gervasoni, A., Gronn, D., … Rowley, G. (2002, February). *Early numeracy research Project final report*. Australian Catholic University.

Cobb, P., & The MIST Team. (2019). Investigating what it takes to improve the quality of mathematics teaching and learning on a large scale. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 1–14). MERGA.

Downton, A., Cheeseman, J., & Roche, A. (2022). Goals and challenges of school mathematics leaders. *Mathematics Teacher Education and Development*, *24*(1), 96–115.

Gervasoni, A. (2015*). Extending mathematical understanding: Intervention*. Ballarat, Australia: BHS Publishing.

Gervasoni, A., Roche, A., & Downton, A. (2021). Differentiating instruction for students who fail to thrive in mathematics: The impact of a constructivist-based intervention approach. *Mathematics Teacher Education and Development*, *23*(3), 207–233.

Gervasoni, A., Roche, A., Giumelli, K., & McHugh, B. (2019). Insights about the progress of grade 1 children who are mathematically vulnerable and participate in a mathematics intervention program. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 300–307). MERGA.

Gervasoni, A., Giumelli, K., Flanagan, L., Downton, A., Roche, A., & Wallis, O. (2023). Utilising the expertise of specialist intervention teachers in primary mathematics classrooms. In B. Reid-O'Connor, E.˜ Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives*. *Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 211–219). Newcastle: MERGA.

Munby, H., Russell, T., & Martin, A. K. (2001). Teachers' knowledge and how it develops. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 877–904). Washington, DC: American Educational Research Association.

Nickow, A., Oreopoulos, P. and Quan, V. (2020). *The impressive effects of tutoring on PreK–12 learning: A systematic review and meta-analysis of the experimental evidence*. Working paper 27476. National Bureau of Economic Research. http://www.nber.org/papers/w27476.

Norton, S. J. (2017). Primary mathematics trainee teacher confidence and its relationship to mathematical knowledge. *Australian Journal of Teacher Education*, *42*(2). https://doi.org/10.14221/ajte.2017v42n2.4

Roche, A. & Gervasoni, A. (2017). Perceived changes in teachers' knowledge and practice: the impact on classroom teachers from leader participation in whole-school reform of mathematics teaching and learning. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 442–449). MERGA.

Russo, J., Minas, M., Hewish, T., & McCosh, J. (2020). Using prompts to empower learners: Exploring primary students' attitudes towards enabling and extending prompts when learning mathematics through problem solving. *Mathematics Teacher Education Development*, *22*(1), 48–67.

Sexton, M. & Downton, A. (2014). School mathematics leaders' perceptions of successes and challenges of their leadership role within a mathematics improvement project. In J. Anderson, M. Cavanagh, & A. Prescott (Eds.), *Curriculum in focus: Research guided practice. Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia* (pp. 581–588). MERGA.

Sharratt, L., & Fullan, M. (2012*). Putting FACES on the data: What great leaders do!* Corwin Press.

Sonnemann, J., & Hunter, J. (2023). *Tackling under-achievement. Why Australia should embed high-quality small-group tuition in schools.* Grattan Institute.

Relationship Between Pre-Service Teachers' Early Mathematics Experiences and Their Current Self-Perception on Mathematics

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This study explores the relationship between pre-service teachers' (PSTs) early experiences and their current views on mathematics. The data were collected through an online survey from 107 PSTs and were analysed using descriptive statistics and Pearson's Chi-square test. Study results suggest that PSTs' early mathematics experiences relate to their current self-perception of mathematics. The majority of participants who loved mathematics as children are confident in using mathematics, whereas those who dreaded it at school lack current mathematical confidence. This highlights the importance of early mathematics education in shaping future perceptions.

In education, the influence of early experiences cannot be overstated. For future educators, pre-service teachers (PSTs), these early experiences are not only connected with their views of education but also their future teaching practices in the classroom (e.g., Sayers, 2013; Yang et al., 2020). This study investigates the relationship between PSTs' early mathematics learning experiences and their current view of mathematics in the Australian context. As guided by the Australian Curriculum, Assessment, and Reporting Authority (ACARA, 2019), mathematics education in Australia emphasises the importance of early numeracy skills as the foundation for later mathematical proficiency. The pivotal role of mathematics in students' academic development and the nation's future workforce underscores the significance understanding of how early mathematics experiences influence those prepared to become mathematics educators. While the literature on mathematics education is vast, with research on both early mathematics education (Björklund et al., 2020) and teacher beliefs and practices, there is a noticeable gap in exploring the relationship between early mathematics experiences during one's own schooling years and current view and future teaching in the Australian context. As a result, this study seeks to address the following research question: How did PST's early personal mathematics experiences relate to their current view of mathematics?

Understanding how early mathematics experiences connect with the beliefs and views of PSTs has the potential to inform teacher education programs, curriculum development, and classroom teaching strategies. It can provide valuable insights into how to nurture confident and capable mathematics educators who, in turn, inspire the next generation of mathematics learners in Australian schools.

Background

Early mathematics education plays a pivotal role in influencing the beliefs and practices of future teachers. In Australian schools and elsewhere, the significance of mathematics education in the early years cannot be understated, as it lays the foundation for not only students' mathematical proficiency but also informs the pedagogical approaches employed by teachers (Björklund et al., 2020). This literature review explores the intricate relationship between early mathematics experiences and the development of PSTs' beliefs and views of mathematics.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 239–246). Gold Coast: MERGA.

Early Mathematics Experiences and Beliefs

Bandura's Social Cognitive Theory (1986) highlights the concept of self-efficacy as the belief or personal judgement about one's competence in a subject matter relative to a specified standard. This concept leads to individuals' perceptions and behaviour and connects with prior experiences, in this case, early mathematics experience. Bandura emphasised the central role of self-efficacy in human agency and its influence on the choices of activities, persistence and efforts, and flexibility to adversity. For example, heightened self-perception or self-efficacy guides individual choices in making decisions, especially in effort expenditure and persistence, which may influence PSTs' beliefs, teaching strategy, and confidence. In contrast, low self-efficacy could result in unstable behaviour, discourages endurance and effort, and may cause total disengagement from learning. Hence, there is a need for continuing research on self-efficacy, self-perception and early mathematics experiences.

Early mathematics experiences are the impressions created by individuals about mathematics after engaging in learning mathematics during primary education levels. In mathematics education, early mathematics experiences can influence individuals' views about the subject. Various studies indicate that the quality and nature of early mathematics experiences significantly influence individuals' beliefs about mathematics (e.g., Sayers, 2013; Yang et al., 2020). For example, early positive experiences profoundly reinforce individuals' self-efficacy beliefs, leading to more confident and enthusiastic mathematics learners (Yang et al., 2020). Another study by Claessens and Engel (2013) found that early mathematics experiences, including exposure to engaging and interactive learning environments or traditional pedagogical approaches (Güler et al., 2023), are linked to changes in attitudes towards mathematics. Furthermore, Sayers (2013), through multiple case studies, revealed that these beliefs, seeded in early mathematical encounters, continue, and reappear in later beliefs and practices. Similarly, Ngan et al. (2003), in their study within early year centres, discovered that such experiences engender consistent beliefs and practices among kindergarten and primary grade teachers, illustrating the enduring influence of these formative encounters. An earlier study by Raymond (1993) explored the role of prior school experiences and teacher education programs, highlighting the relationship between these early factors and serving teachers' beliefs. Collectively, these studies underline the pivotal role of early mathematics experiences in laying the foundation for individuals' beliefs about mathematics.

Early Mathematics Experiences and Beliefs on the Current View/Practice

Relating the early mathematics experience of individuals to their current perception of mathematics, various studies consistently showed that the influence of early mathematics experiences and beliefs extends into one's current view and practice (e.g., Philipp et al., 2007; Yang et al., 2020). A study conducted by Philipp et al. (2007) demonstrated that individuals who actively engaged with children's mathematical thinking while learning mathematics developed more sophisticated beliefs about teaching and learning the subject. This underscores how early experiences influence not only beliefs but also teaching approaches. Also, Ampadu (2014) showed that teachers' beliefs and practices are not static but are continually shaped by background knowledge, professional development, and school culture, with early experiences serving as foundational elements. Moreover, Yang et al. (2020) provided empirical evidence of the enduring impact of early beliefs. They revealed that teachers' knowledge and beliefs, which often originate from early experiences, directly and indirectly, influence their current views and practices, demonstrating that early experiences continue to influence teaching throughout one's career. On the contrary, Purnomo et al. (2016) found that inconsistencies can emerge between beliefs formed in early experiences and actual classroom practices, suggesting that these beliefs do not always translate directly into current views or teaching actions.

While there is a substantial body of research exploring the relationship between early mathematics experiences and teacher beliefs and practices, there is a need for more specific investigation within the Australian context. The existing literature predominantly focuses on broader international perspectives, with limited attention paid to the unique challenges and opportunities the Australian education system presents. Recognising and understanding this relationship is crucial for the continuous improvement of mathematics education in the country. This study aims to contribute to the existing body of knowledge by exploring this relationship in the Australian context.

Method

Participants

This study involved 107 PSTs enrolled in a teacher education program at a regional university in Australia. These PSTs were in their third-year initial teacher education program to become primary school teachers in Australia. Data for this research was collected from two distinct cohorts of PSTs, comprising 68 PSTs from the 2021 cohort and 39 from the 2022 cohort, representing 59.8% of the total 179 enrolled students. The authors did not collect demographic information from the participants.

Data Collection

Data were collected through an anonymous online survey comprising three closed-ended questions. The three questions were: (1) How did you view mathematics in primary school; (2) How do you remember being taught mathematics when you were at school? Select the most common ways you were taught; and (3) How do you view yourself as a mathematician/educator today?

Data Analysis

The initial analysis stage focused on descriptive statistics, including frequency and percentage, offering an overview of PSTs' early experiences and views of mathematics. We used cross-tabulation techniques to explore relationships within the data and presented the findings using stacked bar charts. These techniques examined the relationships between various aspects, such as their current self-view of mathematics and their early mathematics experiences and views. Pearson's Chi-squared test was conducted to discern significant associations between primary school and current views and teaching methods across cohorts.

Results

Early and Current View of Mathematics

Analysis of the survey responses revealed several key themes related to the connection between early mathematics learning experiences and PSTs' current mathematics beliefs. Table 1 shows the early views of PSTs toward mathematics.

Table 1

PST's View of Mathematics in Primary School

Item	Frequency	Percent
Loved it	40	37.4
Could do it but I did not look forward to mathematics lessons	37	34.6
Just dreaded it!	30	28.0
Total	107	100

As shown in Table 1, many PSTs (37.4%) expressed positive views, indicating that they loved mathematics during their primary school years. On the other hand, 34.6% of the PSTs had the ability but did not particularly look forward to mathematics lessons, reflecting a neutral stance. Notably, 28.0% of the PSTs admitted having dreaded mathematics in primary school. These results underscore a range of perceptions among the PSTs, with a substantial portion having positive views about mathematics in their early education. In contrast, others held mixed or negative views.

The other theme investigated in this study was the PSTs' current view of themselves as a mathematician/educator. Table 2 summarises how the PSTs perceive themselves as mathematicians today.

Table 2

PSTs Self-View as a Mathematician Today

Item	Frequency	Percent
Love using mathematics to work out problems and enjoy learning about new mathematical ideas	21	19.6
Confident to apply mathematics in my everyday life	33	30.8
Ok, but I often feel like I need to double-check my calculations.	42	39.3
Not confident. I often avoid having to work things out if mathematics is involved.	11	10.3
Total	107	100

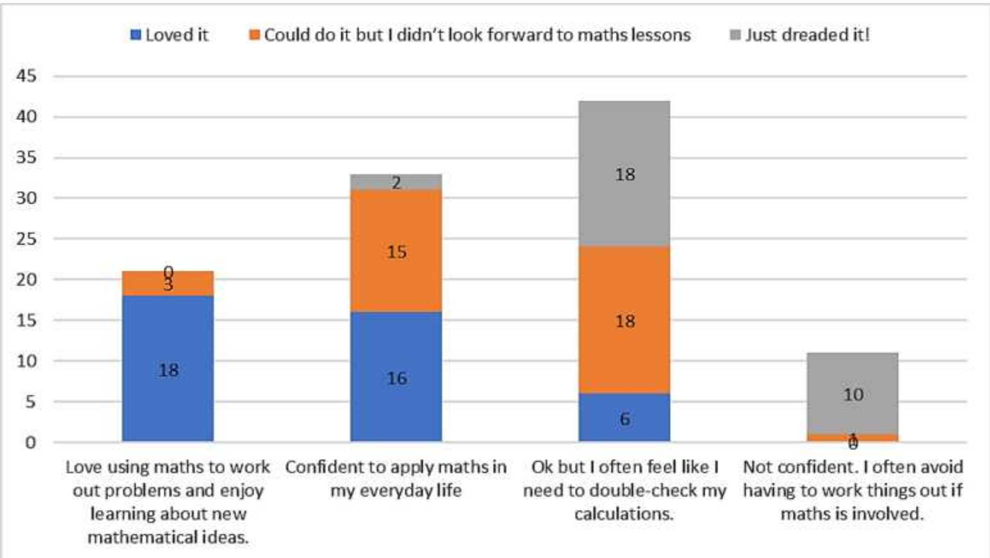
As shown in Table 2, 30.8% of the PSTs notably expressed confidence in applying mathematics in their daily lives. About 39.3% valued precision, often double-checking calculations. Additionally, 19.6% of the PSTs exhibited a strong love for mathematics, enjoying problem-solving and exploring new concepts. However, 10.3% of the PSTs lacked confidence and often avoided mathematics-related tasks.

Relationship Between PSTs' Early View of Mathematics and Their Current View as a Mathematician

Figure 2 reveals the relationship between how PSTs viewed mathematics in primary school and their current self-perception as mathematicians.

Figure 2

Link Between PSTs' Early View of Mathematics and Their Current View as a Mathematician



PSTs who loved mathematics during their primary school years predominantly fall into the category of "Love using mathematics to work out problems and enjoy learning about new mathematical ideas" today, with 18 out of 21 respondents falling into this group. On the other hand, those PSTs who dreaded mathematics in primary school tend to fall into the "Not confident, I often avoid having to work things out if mathematics is involved" category today, comprising 10 out of 11 the PSTs in this group. Interestingly, there is a relatively even distribution among the PSTs who could do mathematics but did not look forward to lessons in primary school, as they are spread across various categories of self-perception as mathematicians today. In addition, the Pearson Chi-Square value is statistically significant, $\chi^2(6, 107) = 61.21$, $p<0.001$. These findings illustrate a connection between early perceptions of mathematics and current self-identifications as mathematicians among PSTs, highlighting the importance of primary school experiences.

PSTs Early Mathematics Learning Experiences

Table 3 provides insights into how PSTs remember being taught mathematics during their own school years.

Table 3

PSTs Taught Mathematics at Primary School

Early learning/teaching strategies	**Responses**		**Percent of cases**
	n	**Percent**	
Copying sums from the board to complete after a demonstration of how to work an example by the teacher	103	30.0	96.3
Using real life materials and situations to learn a mathematical concept or apply understanding	20	5.8	18.7
Using concrete materials or other manipulatives to understand or work through math ideas	33	9.6%	30.8
Working on exercises in a textbook	96	28.0	89.7
Working in groups with other students to learn new math concepts	24	7.0	22.4
Using a calculator to work out answers	55	16.0	51.4
Going on math excursions to see real life examples of mathematical learning	2	0.6	1.9
Visits from experts to demonstrate how they use mathematics	2	0.6	1.9
Using computer software to make learning more interactive	8	2.3	7.5
Total	343	100	320.6

As shown in Table 3, the most prevalent teaching method recalled by the PSTs is "Copying sums from the board to complete after a demonstration of how to work an example by the teacher," representing 96.3% of responses. This traditional approach seems deeply deep-seated in their memory. Additionally, 89.7% of the PSTs reported "Working on exercises in a textbook," indicating a substantial reliance on textbook-based learning. Interestingly, only a smaller percentage, 30.8% of the PSTs, recall using "concrete materials or other manipulatives" to understand math concepts. In comparison, 51.4% of the PSTs recall using calculators, suggesting a shift toward technology in teaching methods. These findings underscore the dominance of traditional teaching approaches in the PSTs' memories of their own mathematics education, with some indications of evolving methods involving technology and hands-on learning experiences.

PSTs Early Mathematics Experience and Their Current View as Mathematician

Table 4 shows the connection between the PSTs' memories of their primary school mathematics education and their current self-perception as mathematicians.

Table 4

How PSTs Remember Being Taught Mathematics When They Were at School and how They View Themselves as a Mathematician Today

Early learning/teaching strategies	Self-view as a mathematician/educator today				Total
	1^{α}	2^{β}	3^{γ}	4^{δ}	
Copying sums from the board to complete after a demonstration of how to work an example by the teacher	19 (17.8%)	33 (30.8%)	40 (37.4%)	11 (10.3%)	103 (96.3%)
Using real life materials and situations to learn a mathematical concept or apply understanding	7 (6.5%)	3 (2.8%)	9 (8.4%)	1 (0.9%)	20 (18.7%)
Using concrete materials or other manipulatives to understand or work through math ideas	9 (8.4%)	9 (8.4%)	15 (14.0%)	0 (0.0%)	33 (30.8%)
Working on exercises in a textbook	16 (15.0%)	32 (29.9%)	38 (35.5%)	10 (9.3%)	96 (89.7%)
Working in groups with other students to learn new math concepts	4 (3.7%)	6 (5.6%)	13 (12.1%)	1 (0.9%)	24 (22.4%)
Using a calculator to work out answers	11 (10.3%)	19 (17.8%)	20 (18.7%)	5 (4.7%)	55 (51.4%)
Going on math excursions to see real-life examples of mathematical learning	1 (0.9%)	0 (0.0%)	1 (0.9%)	0 (0.0%)	2 (1.9%)
Visits from experts to demonstrate how they use mathematics	0 (0.0%)	2 (1.9%)	0 (0.0%)	0 (0.0%)	2 (1.9%)
Using computer software to make learning more interactive	1 (0.9%)	4 (3.7%)	2 (1.9%)	1 (0.9%)	8 (7.5%)
Total	21 (19.6%)	33 (30.8%)	42 (39.3%)	11 (10.3%)	107 (100%)

$^{\alpha}$Love using mathematics to work out problems and enjoy learning about new mathematical ideas; $^{\beta}$Confident to apply mathematics in my everyday life; $^{\gamma}$OK, but I often feel like I need to double-check my calculations; and $^{\delta}$Not confident. I often avoid having to work things out if mathematics is involved.

The PSTs who recall being taught through traditional methods, such as “Copying sums from the board to complete after a demonstration by the teacher” (96.3%), related with two self-perception categories, “Confident to apply mathematics in my everyday life” (30.8%) and “Ok but I often feel like I need to double-check my calculations” (37.4%). This underscores the association between traditional teaching techniques and confidence. Similarly, “Working on exercises in a textbook” (89.7%) is linked with varied self-perception categories, including “Not confident”. I often avoid having to work things out if mathematics is involved” (35.5%). Mainly, those taught with calculators for answers (51.4%) display mixed self-perception, but a significant portion lean toward “Confident to apply mathematics in my everyday life”. On the other hand, hands-on approaches like “Using concrete materials or other manipulatives” and “Working in groups with other students” yield more balanced distributions among self-perception categories of the PSTs. However, the Pearson Chi-Square value between their early teaching and learning mathematics experience and their current view as a mathematician is not statistically significant, $\chi^2(24, 107) = 35.14$, $p > 0.001$. These findings illustrate the lasting influence of primary school teaching methods on the PSTs' current self-perception as mathematicians, with traditional methods influencing confidence levels but not significant.

Discussion

This study investigates the relationship between PSTs' early mathematics experiences and their current self-perceptions as mathematicians/educators. By addressing our research question, we have uncovered findings that provide valuable insights into how early mathematics experiences connect with future mathematics educators.

The dominance of traditional teaching methods, as indicated by the substantial recall of "Copying sums from the board" and "Working on exercises in a textbook," underscores the deeply in-built nature of these methods in the PSTs' memories. This alignment with the existing literature suggests that traditional approaches have been a consistent feature of mathematics education (Güler et al., 2023). This emphasises how early exposure to teaching methods influences the self-perceptions of future educators.

The diverse range of perceptions expressed by the PSTs toward mathematics during their primary school years echoes previous research findings (Leder & Forgasz, 2002). While a considerable portion recalled "loving" mathematics, a significant percentage exhibited neutrality or even disliked the subject. This heterogeneity of perception aligns with the established understanding that students' experiences and views about mathematics during their formative years vary widely (Elçi, 2017). Moreover, the study emphasises a distinct link between pre-service teachers' (PSTs') early experiences in mathematics and their present self-view as mathematicians. This finding aligns with Bandura's Social Cognitive Theory (1986), which suggests that early experiences influence one's current self-perception, with a significant relationship from the Pearson Chi-Square value, $\chi^2(6, 107) = 61.21$, $p < 0.001$. Those who remembered positive primary school mathematics experiences, such as "loving" math, often exhibited confident and enthusiastic self-perceptions as mathematicians today, which resonates with the notion that early positive experiences can contribute to a stronger identity as a mathematics educator or learner (Sayers, 2013).

The indication of a shift toward technology in teaching methods, as evidenced by the recall of using calculators, reflects the evolving landscape of mathematics education. This suggests that technology is increasingly integrated into mathematics classroom teaching, influencing PSTs' perceptions and self-identifications as mathematicians in contemporary contexts.

This study emphasises the continuing influence of early mathematics experiences on PSTs' attitudes and self-perceptions as mathematicians. Traditional teaching methods play a significant role in influencing these perceptions, with some indication of evolving methods involving technology. Recognising these associations is vital for designing effective mathematics education programs that cater to the diverse needs and attitudes of future educators. Moreover, fostering positive early mathematics experiences is crucial for cultivating confident and passionate mathematicians among PSTs, thus contributing to improving mathematics education in schools.

Conclusion

The study reveals critical insights into mathematics education and teacher preparation, highlighting the need for customised support in teacher education programs for PSTs with varied perceptions of mathematics. Many PSTs recall either loving or dreading mathematics in primary school, which connects with their current self-perceptions as mathematicians. Those with positive early experiences tend to enjoy mathematical problem-solving, while those with negative experiences often lack confidence. The study emphasises the lasting relationship between primary school experiences and PSTs' perceptions and identities in mathematics.

Limitations

While this study provides valuable insights into the connection between early mathematics experiences and the current views of the PSTs, it has certain limitations. Self-reported data collected through surveys are subject to response and retrospective recall biases, potentially impacting data accuracy. In addition, the study results provide a snapshot rather than a longitudinal view of beliefs' evolution over time. The fixed-response survey format may not capture the full complexity of the PSTs' views and experiences, and the study does not explore contextual factors or causality. Additionally, the study's scope focuses primarily on views and experiences without a deep investigation of the specific teaching strategy effectiveness or interventions. Addressing these limitations in future research would contribute to a more comprehensive understanding of this intricate relationship and generalisations.

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References

Ampadu, E. (2014). Implementing a new mathematics curriculum: Mathematics teachers' beliefs and practices. *International Journal of Research Studies in Education*, *3*(1), 79–90.

Australian Curriculum and Assessment Reporting Authority (ACARA). (2019). *The Australian Curriculum*. https://www.australiancurriculum.edu.au/

Bandura, A. (1986). *Social foundation of thought and action: A social-cognitive view*. Prentice-Hall.

Björklund, C., van den Heuvel-Panhuizen, M. & Kullberg, A. (2020). Research on early childhood mathematics teaching and learning. *ZDM Mathematics Education*, *52*, 607–619. https://doi.org/10.1007/s11858-020-01177-3

Claessens, A., & Engel, M. (2013). How important is where you start? Early mathematics knowledge and later school success. *Teachers College Record*, *115*(6), 1–29. https://doi.org/10.1177/016146811311500603

Elçi, A. (2017). Students' attitudes towards mathematics and the impacts of mathematics teachers' approaches on it. *Acta Didactica Napocensia, 10*(2), 99–108. https://doi.org/10.24193/adn.10.2.8

Güler, M., Kokoç, M., & Önder Bütüner, S. (2023). Does a flipped classroom model work in mathematics education? A meta-analysis. *Education and Information Technologies*, *28*(1), 57–79. https://doi.org/10.1007/s10639-022-11143-z

Ngan Ng, S., Lopez-Real, F., & Rao, N. (2003). Early mathematics teaching: the relationship between teacher's belief and classroom practices. In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education* (pp. 213–220). Honolulu: PME.

Philipp, R. A., Ambrose, R., Lamb, L. L., Sowder, J. T., Schappelle, B. P., Sowder, L., & Chauvot, J. (2007). Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education, 38*(5), 438–476. https://doi.org/10.2307/30034961

Purnomo, Y. W., Suryadi, D., & Darwish, S. (2016). Examining pre-service elementary school teacher beliefs and instructional practices in mathematics class. *International Electronic Journal of Elementary Education*, *8*(4), 629–642. https://iejee.com/index.php/IEJEE/article/view/137

Raymond, A. M. (1993). Unraveling the relationships between beginning elementary teachers' mathematics beliefs and teaching practices. *Paper presented at the 15th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Monterey, CA. https://files.eric.ed.gov/fulltext/ED390694.pdf

Sayers, J. (2013). The influence of early childhood mathematical experiences on teachers' beliefs and practice. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.) *Proceedings of the eighth congress of European Research in Mathematics Education* (CERME 8), Antalya, Turkey: ERME. http://nectar.northampton.ac.uk/5204/

Yang, X., Kaiser, G., König, J. (2020). Relationship between pre-service mathematics teachers' knowledge, beliefs and instructional practices in China. *ZDM Mathematics Education*, *52*, 281–294. https://doi.org/10.1007/s11858-020-01145-x

Numeracy Across the Australian Curriculum: Opportunities from F to 6

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This study examined how opportunities to incorporate numeracy across different primary curriculum areas and grade levels are highlighted in the Australian Curriculum v.9. A document analysis approach was used to track the numeracy opportunities in the document across different curriculum areas and grade levels. The results showed that Science has the most identified numeracy opportunities across all grade levels, with 43 occurrences. Findings also showed Statistics as the most prominent mathematics content strand suggested as numeracy opportunities, followed by Space. The study provides examples of missed numeracy opportunities in underrepresented curriculum areas.

The need for numeracy in students as an important skill for effective participation in society is widely recognised. As a result, fostering numeracy has become a priority in school systems internationally, including Australia (Australian Curriculum and Assessment Reporting Authority [ACARA], 2022). The definition of numeracy as the ability to apply mathematics in diverse life situations has gained widespread acceptance and influenced research and curriculum development (Department of Employment, Education, Training and Youth Affairs [DEETYA], 1997). As it is situated within the Australian Curriculum: Mathematics, numeracy encompasses the knowledge, skills, behaviours, and dispositions that students need to use mathematics in a wide range of situations. It involves students recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully (Geiger et al., 2015).

Numeracy develops as students learn to confidently apply mathematics across different curriculum areas and in diverse life contexts (ACARA, 2022; Bennison, 2015; Ford, 2018). Recognising its cross-curricular importance, countries including the UK, the USA, New Zealand, and Australia have integrated numeracy into various curriculum areas. In Australia, it is a key component of the general curriculum capabilities, in addition to, for example, literacy and social skills. However, integrating numeracy across curriculum areas and various mathematical concepts can be challenging for teachers (Geiger et al., 2015; Gough, 2007; Getenet, 2023). For example, Geiger et al. (2015) noted limited numeracy integration in English, while Gough (2007) highlighted its prevalence in Science. Additionally, Getenet (2023) found that pre-service teachers are more interested in incorporating statistical concepts than other mathematical concepts into other curriculum areas. To support teachers, the Australian Curriculum v9.0 (and previous iterations) includes a numeracy icon that highlights numeracy opportunities across the curriculum content (ACARA, 2022). There is, however, a lack of comprehensive research on how these opportunities are distributed across different curriculum areas, year levels, and the type of mathematical concepts included.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 247–254). Gold Coast: MERGA.

This study aims to address this gap through a document analysis of v9.0 of the Australian Curriculum, addressing the following research questions:

- How does the representation of numeracy in the Australian Curriculum v9.0 vary across different year levels and curriculum areas?
- Which mathematical concepts or strands are most prominently represented in the numeracy aspects of the Australian Curriculum v9.0?

This research will also identify potential gaps, illustrate examples of missed opportunities in numeracy through lack of identification in curriculum documents across key learning areas, and provide insights into implications for teachers' practices. Furthermore, the results will assist teachers by highlighting the presence of extra numeracy opportunities throughout all curriculum areas, beyond those specified in the curriculum. This will encourage teachers to incorporate these missed opportunities into their lesson planning and teaching strategies, thereby addressing the issues associated with teachers identifying numeracy opportunities. Missed numeracy opportunities in this study refer to those mathematical knowledge, skills and dispositions that exist within curriculum areas but are not explicitly denoted with the numeracy icon.

Background

Numeracy Across Curriculum Areas

To develop students' skills in numeracy, teachers are required to provide opportunities for students to use their mathematical knowledge and understanding in multiple contexts. Various strategies and approaches to support this development include cross-curriculum area approaches and teaching numeracy as a separate discipline. When teachers are encouraged to identify and use the numeracy learning opportunities in various curriculum areas, students' numeracy capabilities and learning in each area are likely to be enhanced and help students learn to think critically (Bennison, 2015; Brown et al., 2002). This support allows students to transfer their mathematical knowledge and skills to contexts outside the mathematics classroom and across the curriculum areas (Bennison, 2015; Mathieson & Homer, 2021). In Australia, teachers are encouraged to develop students' numeracy skills by using mathematics confidently across all curriculum areas (ACARA, 2022). These curriculum areas are English, Humanities and Social Science (HASS), Health and Physical Education (HPE), Languages, Science, Technologies (digital and design), and the Arts. By including numeracy as a general capability in the Australian Curriculum v9.0, teachers are directed to opportunities to support students in developing their numeracy capabilities across curriculum areas (ACARA, 2022; Forgasz et al., 2017).

General capabilities, including numeracy and cross-curriculum priorities tagging, have been included in the Australian curriculum documents with revisions made in 2011 (ACARA, 2011). In 2020, the numeracy icons were linked with the Numeracy Progression version 3.0. The numeracy progressions supplement and underpin the Australian Curriculum v9.0 and describe the observable indicators of increasing complexity in the understanding of and skills in key numeracy concepts (ACARA, 2020). In version 9 of the Australian Curriculum, the numeracy icons are again linked with the numeracy progressions as they were previously, in relation to the elements of Number and Algebra, Measurement and Geometry, and Statistics and probability, rather than the updated six content areas in v. 9 of the national curriculum. While the inclusion of the numeracy icons highlights numeracy opportunities across different content areas and is designed for teachers to integrate numeracy into various curriculum areas, the links are not comprehensive and could inadvertently lead to teachers overlooking valuable numeracy opportunities. Studies looking into these opportunities, including those not explicitly indicated in the curriculum, could provide teachers with a broader range of options for incorporating numeracy, rather than being restricted to the opportunities specifically shown in the national curriculum.

Method

This study used a document analysis method to examine the numeracy presentations in the Australian Curriculum v9.0 from Foundation to Year 6. The focus was on how numeracy is represented across various curriculum areas and year levels and identifying missed opportunities. Document analysis, as defined by various researchers such as Bowen (2009) and Morgan (2022), involves categorising information from diverse sources like books, national guides, articles, and reports to address the research questions. This approach also considers the document's purpose, target audience, authorship, and original sources.

The Australian Curriculum v9.0: Mathematics is divided into six strands: Number, Algebra, Measurement, Space, Statistics, and Probability. Each strand has content descriptors and elaborations detailing student learning objectives and teaching examples. In addition, numeracy opportunities are made explicit throughout the other curriculum areas through the use of an icon linked to numeracy learning progressions. The curriculum incorporates references to *Relevant Content Descriptors* (outlined in Table 2), highlighting areas in mathematics where opportunities to develop numeracy skills can be identified.

Given that our team's expertise predominantly lies in primary education, our study's scope was strategically limited to Foundation to Year 6 levels. In addition, the Language curriculum, was not included in the analysis to further limit the scope of this study. This decision ensured a focused and in-depth analysis within our areas of expertise. To conduct the document analysis, we followed a structured three-step process as suggested by Bowen (2009). We began by skimming the curriculum documents for a preliminary understanding, followed by a thorough reading to quantify and characterise the numeracy representations, and concluding with an interpretation phase where the data was categorised based on curriculum areas, year levels, and mathematical strands. Our analysis was informed by the *Numeracy for the 21st Century Model* (Goos et al., 2012). The model was developed to support teachers in improving their students' numeracy capabilities in the range of curriculum areas, evaluating activities related to numeracy, as well as identifying opportunities for numeracy engagement. The model consists of four core dimensions: attention to contexts, application of mathematical knowledge, use of physical and digital tools, and promotion of positive dispositions. Of particular relevance to our analysis was the consideration of context which the curriculum areas logically provided.

In addition, to enhance the rigour and reliability of our findings, each curriculum area was independently analysed by two authors. This dual-analysis approach ensured a comprehensive review and fostered consistency across our analysis. Any inconsistencies uncovered during this stage were addressed through further collaborative discussions. The specific activities and steps involved in our analytical process are shown in Table 1.

Table 1

Steps of the Document Analysis in the Study

Step	Description
Skimming	A preliminary review of the curriculum documents by each author to identify initial numeracy opportunities for further analysis
Reading	Detailed examination of the curriculum to quantify how often numeracy opportunities occur within each curriculum area
Interpretations	Categorisation of the numeracy opportunities identified, based on the specific curriculum areas, year levels, and the mathematical concepts or strands emphasised

As shown in Table 1, the authors initially reviewed the document, followed by quantifying identified numeracy opportunities across curricular areas. Finally, in the interpretations step, the findings of the numeracy opportunities were categorised based on curriculum areas, year levels, and mathematical concepts or strands. In addition to these steps, the authors also identified and illustrated examples of missed numeracy opportunities in each curriculum area through a thorough analysis of the content descriptors of each curriculum area.

Results

The results of this study are presented to correspond with the research questions. These questions examine how numeracy is represented in the Australian curriculum across various year levels and curriculum areas and identify which mathematical concepts or strands are most prominently featured in the numeracy opportunities of the Australian curriculum.

Numeracy Representations in the National Curriculum Across Various Year Levels and Curriculum Areas

We have identified the articulated numeracy opportunities across various curriculum areas through document analysis of the Australian curriculum. With the exception of the Foundation year level, the numeracy opportunities in the curriculum for HPE, Technologies, and the Arts are presented for combined year levels. Specifically, these opportunities are grouped as Years 1 and 2, Years 3 and 4, and Years 5 and 6, as shown in Table 2 and further in Table 3.

Table 2 outlines numeracy opportunities across various curriculum areas and year levels. As shown in Table 2, in the English curriculum, there is a complete absence of numeracy opportunities from Foundation to Year 6, both directly and through related content descriptors. On the other hand, Science consistently provides numeracy opportunities across all year levels, peaking at 10 in Year 1 and varying between 3 to 9 in subsequent years. The HASS curriculum shows an increase in related content descriptors numeracy opportunities, ranging from three in Foundation to eight in Years 4 and 5, and finally leading to one direct opportunity in Year 6. In Table 2, the first number refers to the frequency of numeracy opportunities denoted with icons in the curriculum, whereas the number in brackets refers to the frequency of the related mathematics content links in each key learning area.

Table 2

Frequencies of Identified Numeracy Opportunities (Related Content Descriptors) in Various Curriculum Areas by Year Level

Curriculum area		**Year level**						
		F	**1**	**2**	**3**	**4**	**5**	**6**
English		0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
HASS		0 (3)	0 (6)	0 (6)	0 (8)	0 (8)	0 (7)	1 (7)
Science		9 (11)	10 (9)	5 (5)	6 (7)	3 (3)	4 (4)	6 (6)
HPE		1 (3)	1 (3)		0 (5)		0 (2)	
Technologies	Digital	1 (1)	2 (5)		3 (9)		3 (6)	
	Design	0 (2)	1 (2)		1 (5)		0 (4)	
The arts	Dance	0 (0)	1 (0)		0 (0)		0 (0)	
	Drama	0 (0)	0 (0)		0 (0)		0 (0)	
	Media	0 (0)	0 (0)		0 (0)		0 (0)	
	Music	0 (0)	0 (0)		0 (0)		0 (0)	
	Visual	0 (0)	1 (0)		0 (0)		0 (0)	

The HPE curriculum presents numeracy opportunities in Foundation (n = 1), Year 1 (n = 1), Year 2 (n = 1), and in Years 3, 4, Year 5 and Year 6 with none in the related content descriptors opportunities. The Technologies curriculum, particularly in Digital Technologies, shows limited numeracy opportunities ranging from one in Foundation to three in Years 3 and 4, while design technologies present fewer opportunities than digital technologies. However, more numeracy opportunities were found in the related content descriptors in both design and digital technology curricula. In the Arts curricula, numeracy opportunities are extremely limited, with only a few instances including in related content descriptors, such as one opportunity in Dance in Year 1 and one in Visual Arts in Year 1, with no opportunities in Drama, Media, and Music across all years. These results illustrate the varying degree of emphasis on numeracy in different curriculum areas and year levels, indicating areas where the inclusion of numeracy could be enhanced.

We further aligned the identified numeracy opportunities with the corresponding strands of the mathematics curricula. This alignment includes both opportunities directly suggested in the general numeracy capability and those related to mathematics content descriptors. This helps to determine which specific strands of Mathematics are given emphasis in various curriculum areas. These findings are presented in Table 3. In Table 3, the lead number is the number of strands linked to the numeracy button in general numeracy capability, and the number in brackets is those mathematics content descriptors linked.

Table 3

Frequencies of the Focus on Mathematical Concepts/Strands Across the Curriculum Areas

Strand	Curriculum area										
	English	HASS	Science	HPE	Technologies		Arts				
					Design	Digital	Dance	Drama	Media	Music	Visual
Number	0 (0)	1 (1)	6 (3)	0 (2)	0 (1)	1 (5)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
Algebra	0 (0)	0 (0)	1 (1)	0 (0)	0 (0)	2 (2)	0 (0)	0 (0)	0 (0)	0 (0)	1 (0)
Space	0 (0)	0 (7)	5 (4)	2 (6)	0 (0)	1 (4)	0 (1)	0 (0)	0 (0)	0 (0)	1 (1)
Measurement	0 (0)	0 (5)	13 (14)	0 (1)	2 (1)	2 (0)	1 (0)	0 (0)	0 (0)	0 (0)	0 (0)
Probability	0 (0)	0 (0)	1 (0)	0 (0)	0 (0)	1 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
Statistics	0 (0)	0 (32)	17 (23)	0 (4)	0 (0)	5 (7)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
Total	0 (0)	1 (51)	43 (45)	2 (13)	2 (2)	12 (18)	1 (1)	0 (0)	0 (0)	1 (0)	2 (1)

As shown in Table 3, HASS has a strong focus on Statistics, with 32 related opportunities. Science predominantly highlights Statistics and Measurement, with 17 and 13 direct suggested opportunities, respectively. The HPE curriculum focuses on Space ($n = 2$ direct suggested opportunities) and Statistics ($n = 4$ in related content descriptor opportunities). In Technologies, particularly in Digital technologies, there is a notable emphasis on the Statistics strands ($n = 7$) in the related content descriptor numeracy opportunities. These results showed that Statistics and Measurement are the most prominently featured strands in curriculum areas such as HASS, Science, and Technologies.

Numeracy Missed Opportunities Across Various Curriculum Areas and Year Levels

Although there are multiple potentially missed opportunities across all the curriculum areas, in this section, we provide examples of missed numeracy opportunities where the opportunities were found to be particularly scarce (see Table 2), such as English, Arts, HPE, and HASS. Table 4 provides an overview of some examples of missed opportunities, categorised by curriculum area and year level.

Table 4

Examples of Missed Opportunities for Numeracy Across Different Curriculum Areas

Curriculum area (year level)	Content Description	Description of demonstrated example	Maths content descriptor
English (1)	AC9E1LE0: Discuss how language and images are used to create characters, settings and events in literature by First Nations Australian, and wide-ranging Australian and world authors and illustrators	Map events in the literature on timelines; make timelines for own/families' lives	AC9M1M03: Describe the duration and sequence of events using years, months, weeks, days and hours Measuring Time: P1 Sequencing Time P2 Units of Time
HASS (2)	AC9HS3K02: Significant events, symbols and emblems that are important to Australia's identity and diversity	Discuss the importance of symbolism—religious/cultural/historical etc. Discuss the ... positioning, dominance, colouring etc, in the Australian and First Nations' flags. Design a new flag of relevance to the students with reference to unit fractions	AC9M3N02: Recognise and represent unit fractions including 1/2, 1/3, 1/4, 1/5, and 1/10 and their multiples in different ways; combine fractions with the same denominator to complete the whole Interpreting Fractions: P2: Repeated halving P3 Repeating Fractional Parts
Dance (1 & 2)	AC9ADA2E01: Recognising and describing patterns of movement in dances, using their own words and learnt dance terminology	Combine the dance concept of recognising and describing patterns of movement with mathematics as a "Dance Pattern Sequence Game"	AC9M1A01: Recognise, continue and create pattern sequences, with numbers, symbols, shapes and objects, formed by skip counting, initially by twos, fives and tens Number Patterns and Algebraic Thinking P2: Identifying and creating patterns P3: Identifying and generalising patterns
Media (1 & 2)	AC9AMA2E02: Exploring when, where, how and why First Nations Australians use media arts works to share knowledge about their cultures	Explore the concept of symmetry in Indigenous art, the use of geometric shapes, or the proportions in digital layouts	AC9M1SP01: Make, compare and classify familiar shapes; recognise familiar shapes and objects in the environment, identifying the similarities and differences between them Understanding geometric properties

As shown in Table 2, although some curriculum areas show fewer or no numeracy opportunities suggested directly or in the related content descriptors, it is important for teachers to recognise the potential and the availability for numeracy opportunities within various content descriptors across all curriculum areas (see Table 4). These opportunities are in addition to officially indicated opportunities in the curriculum document. For example, while English does not have explicit links to numeracy opportunities, we have identified several content descriptors within the English curriculum that offer opportunities for numeracy engagement. An example is the descriptor, "Discuss how language and images are used to create characters, settings, and events in literature by First Nations Australians, and wide-ranging Australian and world authors and illustrators". Under this content descriptor, children can engage in activities such as mapping events in literature on timelines, recording event durations, creating timelines of their own or their families' lives, and making journals or timetables for the week that involve noting times and calculating durations. Such activities draw upon and develop numeracy skills, specifically the ability to describe the duration and sequence of events using years, months, weeks, days, and hours, as outlined in the mathematics curriculum.

Discussion and Conclusion

The Australian Curriculum v9.0 has mandatory general Capabilities that are essential components of curriculum delivery. However, this content analysis has identified that there remain many numeracy opportunities that have not been identified to support primary teachers in developing mathematical concepts and mathematical literacy within the context of all key learning areas. Missed numeracy opportunities predominantly exist in key learning areas that are often categorised as 'Arts curriculum' and those in the domain of health and development. These curriculum areas did not capitalise on the relevant contexts highlighting numeracy opportunities which is a key dimension of the Numeracy for the 21st Century Model (Goos et al., 2012). Further support is required to support teachers in creating and delivering contextualised numeracy tasks to support conceptual development during mathematics lessons and identifying where mathematical literacy can be enhanced by primary teacher consideration and targeting of mathematical demands to enhance conceptual development in a range of key learning areas, and to assist growth in numeracy understandings.

There are too many numeracy opportunities that are possible to articulate in a curriculum document of this nature. Our findings highlight that teacher requires the identification of more numeracy links to support their planning, particularly in English and Arts, followed by HASS, HPE and Technologies that are currently available. Results from studies such as this one could assist teachers in identifying additional opportunities for numeracy across different curriculum areas beyond those specifically marked with a special symbol in the curriculum. These curriculum key learning areas were also highlighted as challenging for teachers to identify numeracy opportunities by previous studies (e.g., Geiger et al., 2015). Whilst the curriculum design lends itself to highlighting numeracy opportunities, these links have been identified in the curriculum at varied levels across the different key learning areas. The results of this study showed that the Science key learning area, which is often described as being hand-in-hand with Mathematics and an integral component of STEM studies, has the most identified links to the numeracy framework and links related to other key learning area content. This further hinders the numeracy connections that can be forged between subjects that are not considered in the STEM field. Gough's earlier study (2007) also underscored the prominent role of Science in this context.

Interestingly, whilst links are often made between numeracy and the number strand, the most emphasised strands in the Australian curriculum linked to the numeracy framework are Statistics, Measurement and Space. This result aligns with the findings of a recent study by Getenet (2023), which indicates that teachers tend to favour the integration of statistical concepts into various curriculum areas over other mathematical concepts. The potential exists to build on the numeracy connections included in the Australian curriculum to Probability, Algebra and Number, where there are many examples of numeracy demands that are relevant to societal interactions and functioning. The number of suggestions and content links is similar across the year levels, with the exception of the Foundation year in Science, which has high numeracy indicators for that year level. The numeracy opportunities refer teachers to concepts in the numeracy framework. However, examples of suggested numeracy tasks to support primary teachers' enhancement of numeracy development are not provided. To emphasise samples of missed numeracy opportunities in the national curriculum in English, HASS and Arts, the affordances of providing sample numeracy task suggestions have been highlighted.

This study has certain limitations that need to be recognised. First, to reduce bias and enhance credibility, documentary evidence can be supplemented with data from interviews and observations. However, in this case, the findings are solely derived from document analysis. Incorporating insights from school teachers regarding their application of the curriculum's numeracy indicators in designing classroom activities could have provided more robust results. Future research that addresses these limitations would offer a more substantial generalisation.

References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: Mathematics (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/

Australian Curriculum and Assessment Reporting Authority (ACARA). (2020). *National numeracy learning progression.* https://www.australiancurriculum.edu.au/media/3635/national-numeracy-learning-progression.pdf

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2011). *General capabilities consultation report.* ACARA. http://docs.acara.edu.au/resources/General_Capabilities_-_Consultation_Report_-_December__2011.pdf

Getenet, S. (2023). Investigating pre-service teachers' skills in designing numeracy activities across curriculum areas involving statistics. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 219–226). Newcastle: MERGA

Bennison, A. (2015). Supporting teachers to embed numeracy across the curriculum: A sociocultural approach. *International Journal on Mathematics Education*, *47*(4), 561–573. https://doi.org/10.1007/s11858-015-0706-3

Bowen, G. A. (2009). Document analysis as a qualitative research method. *Qualitative Research Journal*, *9*(2), 27–40. https://doi.org/10.3316/QRJ0902027

Brown, M., Askew, M., Baker, D., Denvir, H., & Millett, A. (2002). Is the national numeracy strategy research-based? *British Journal of Educational Studies*, *46*(4), 362–385. https://doi.org/10.1111/1467-8527.00090

Department of Employment, Education, Training and Youth Affairs DEETYA. (1997). Numeracy = Everyone's business. *The report of the numeracy education strategy development conference*, May 1997. DEETYA. Adelaide: Australian Association of Mathematics Teachers.

Ford, K. (2018). Persisting gaps: Labor market outcomes and numeracy skill levels of first-generation and multi-generation college graduates in the United States. *Research in Social Stratification and Mobility*, *56*(2018), 21–27. https://doi.org/10.1016/j.rssm.2018.06.003

Forgasz, H. Leder, G. & Hall, J. (2017). Numeracy across the curriculum in Australian schools: Teacher education students' and practising teachers' views and understandings of numeracy. *Numeracy*, *10*(2), 1–23. http://doi.org/10.5038/1936-4660.10.2.2

Geiger, V., Forgasz, H., & Goos, M. (2015). A critical orientation to numeracy across the curriculum. *International Journal on Mathematics Education*, *47*(4), 611–624. https://doi.org/10.1007/s11858-014-0648-1

Gough, J. (2007). Make your school's numeracy-across-the-curriculum policy. *Australian Mathematics Teacher*, *63*(3), 31–39. http://hdl.handle.net/10536/DRO/DU:30007800

Goos, M., Dole, S., & Geiger, V. (2012). Numeracy across the curriculum. *Australian Mathematics Teacher*, *68*(1), 3–7. https://search.informit.org/doi/10.3316/aeipt.197354

Mathieson, R., & Homer, M. (2021). "I was told it would help with my psychology": Do post-16 core maths qualifications in England support other subjects? *Research in Mathematics Education*, *24*(1) 1–19. https://doi.org/10.1080/14794802.2021.1959391

Morgan, H. (2022). Conducting a qualitative document analysis. *The Qualitative Report*, *27*(1),64–77. https://doi.org/10.46743/2160-3715/2022.5044

Student Engagement with Dynamic Digital Representations of Decimal Fractions to Prompt Conceptual Change

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Decimals and the related areas of ratio and proportion are recognised as one of the most challenging and complex areas of mathematics for young children to learn. In this study, four dynamic digital representations were used by Year 4 (9–10 years old) students in a set of video-recorded, individual task-based interviews to explore key concepts of decimals. Microgenetic analysis methods were used to detect conceptual changes through specific shifts in the learner's attention. The data collected revealed what features of each dynamic digital representation prompted attention shifts and how these translated to changes in conceptual understanding of decimal fractions.

Children have difficulty in understanding the abstract nature of decimals, and a plethora of misconceptions arise from incomplete understandings (Steinle & Stacey, 2004). Decimal fractions present complications as part of the number-relationship needed to make sense of the number, the denominator, is not visible. Developing conceptual understandings of decimal density, place value and relative magnitude of decimal fractions are considered key for consolidating number sense. The properties of density and place value unify all numbers and rational number magnitude knowledge is related to future mathematics achievement (Siegler et al., 2011). Minimal research has been conducted into what effective representations should be employed by mathematics educators to assist students in developing 'decimal sense' and managing 'cognitive conflict' in the context of decimal fractions (Swan, 1983). This study investigated a collection of dynamic digital representations that featured the three key concepts of decimals.

Representations are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning (Pape & Tchoshanov, 2001). An important feature of mathematics education is working with appropriate mathematical diagrammatic representations, but there is uncertainty around the best representations for decimals. Dynamic representations are potentially an effective support for student learning within mathematics because of their dynamic affordances. Unlike with static representations, students are able to actively interact with the mathematics content by making conjectures and testing them immediately to experience the concept from various perspectives and receive feedback on their thinking (Orrill & Polly, 2013). In particular, learner-centred computer tools that encourage interactive engagement with mathematics, allow students to 'see' notions that would otherwise be 'unseeable' (Orrill & Polly, 2013). Relatively little work has focused on the role of dynamic digital representations in teaching and learning decimals, although there is an abundance of research into the use of digital learning objects in mathematics education (Litster et al., 2018). The intention of this study was to establish a foundation for research in this topic by exploring how children's interactions with dynamic digital representations of decimal fractions may develop their conceptual understanding in this numerical area.

Theoretical Framework

In mathematics education, conceptual understanding involves being able to represent a variety of mathematical ideas in different ways, to see the connections between those representations, and to know how to use each representation for different purposes (Caldwell, 2020). The growth of students' conceptual understanding is synonymous with mathematics learning. Mason's (2008) theorisation of the notion of attention has been employed to account

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 255–262). Gold Coast: MERGA.

for students' learning of mathematics. The theory is underpinned by the idea that students must experience shifts of attention for learning to occur. As students interact with mathematical tasks and representations there are ongoing shifts in their focus and structure of attention. The learning of mathematics occurs through these attention shifts and students' developing awareness of them (Mason, 2008). At present, there is little conclusive research on how students are best supported to build a conceptual understanding of decimals. The theory of shifts of attention provides a scaffold for educators' understanding of teaching mathematics as well as proposing five observable attending stages that indicate what and how students are learning in mathematics lessons. Mason (2008) described the 'focus of attention' (what the learner attends to) and the form or 'structure of attention' (how the learner attends) whereby several shifts of attention occur in the process of thinking mathematically related to an object. Structure of attention refers to "holding wholes, discerning details, recognising relationships, perceiving properties, and reasoning on the basis of agreed properties" (Mason, 2008, p. 35). See Table 1 for examples. As attention shifts from one concept to another, its structure alters and connections are made, relationships are perceived, and new ideas are appreciated. Applying the theory of shifts of attention may provide a scaffold to plot a progression of understanding that students may take when learning about key concepts related to decimal fractions.

The focus of this paper is to explore how children in middle-primary grades engaged with dynamic digital representations of decimal fractions to develop their conceptual understanding. The overarching research question communicates the aim of the study:

- What features of dynamic digital representation do children attend to that prompt conceptual changes related to decimal fractions?

Method

The methodology considered most appropriate for the study was structured, task-based interview because it complemented the theoretical framework from a constructivist, learner-centred view. Constructivist learning theory has been linked with structured task-based interviews in the work of Goldin (2003) who found the research method to be an effective means of understanding conceptual knowledge. In this study, task-based interviews were used to prompt students' interactions with mathematical tasks and representations and record ongoing shifts in their focus of attention and the structure of attention.

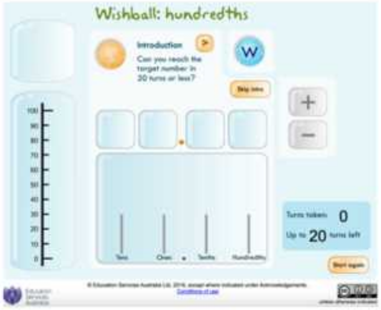
Data collection for this study consisted of four task-based interviews each focussing on a different dynamic digital representation: *Wishball-hundredths*, *Decimal Strips, Zooming in on Decimals,* and *Zoomable Number Line* (Figure 1). Four digital representations were selected from resources readily available to teachers on the internet, following an analysis of the affordances and constraints of each potential digital tool, and the focus on at least one of the three key decimal concepts. The number-line based, place-value symbolic, and area-model dynamic digital representations of decimals used were in the form of applets which are embedded into a webpage and displayed using a web-browser.

Four participants were from one Year 4 (9–10 years old) class situated in a school from the outer suburbs of Sydney, Australia. Although the students were not intended to be representative of the whole class, the researcher selected four that spanned a range of general mathematical achievement levels, based on results from a decimal comparison pre-test. This was in case there were differences in the ways in which students of varying abilities interacted with the dynamic digital representations. This grade level was selected because, according to the NSW syllabus (Board of Studies NSW, 2012), students are introduced to decimal notation for tenths and hundredths in Year 4. The decimal number concepts and skills were therefore still in the early stages of development at the time of the study.

Figure 1

Screenshots and Basic Information About the Digital Tools

Wishball

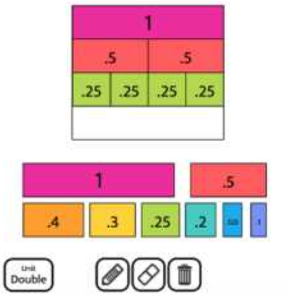


Focus on place value.

Involves adding or subtracting to reach a target decimal fraction number.

(https://www.scootle.edu.au/ec/viewing/L8456/index.html)

Decimal Strips

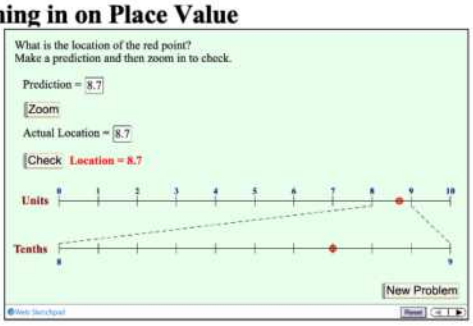


Focus on number magnitude.

Involves dragging coloured tiles representing different decimal values.

(https://toytheater.com/decimal-strips/)

Zooming in on Place Value

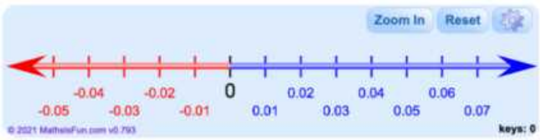


Focus on number magnitude.

Involves predicting the location of a decimal fraction on a number line, then self-checking with the embedded Zoom function.

(http://www.sineofthetimes.org/zooming-in-on-place-value/)

Zoomable Number Line



Focus on number density.

Involves using Zoom-in/out capability to respond to teacher-designed tasks.

(https://www.mathsisfun.com/numbers/number-line-zoom.html)

The researcher worked individually with each participant in four 30-minute sessions, over a period of four weeks. During the interviews, each digital resource (Figure 1) was introduced by inviting the student to freely investigate the tool for 5-minutes and them asking any clarifying questions. The tasks were then initiated by the researcher and completed by the student. In addition, the researcher, who was a qualified teacher, regularly asked probing questions to encourage further explanation by the student, for example ‘Can you tell me what you are thinking?’ but remained neutral regarding ‘correctness’ of responses. Video-recording the interviews was critical to collecting quality data as it was necessary to capture what was said and done by the child simultaneously, including gestures, expressions and emotional responses.

The data collected from the task-based interviews were analysed using a microgenetic approach that involved closely examining intensive data of what learners do and say over several task-based interviews to capture a record of moment-by-moment learning processes. This close examination allows researchers to draw conclusions about what prompts knowledge change, how learning occurs, and other aspects of the processes of acquiring conceptual understanding (Voutsina et al., 2019). Data analysis began with repeated viewings of the video-audio recordings of each task-based interview to deduce what new understandings the participants had gathered during the task-based interview. The initial analysis period occurred over several weeks as the researcher transcribed and documented all observed changes in decimal understanding. Four evidence excerpts were selected for this paper from the larger data-set. These examples showed key tasks that provoked the most change in conceptual thinking as a result of the student's interaction with a dynamic digital representation of decimals. A framework to support analysis was designed by the researcher as a reference tool when analysing the verbal reasoning and manipulative actions made by the participants (Table 1). The 'Shifting Attention with Decimal Concepts Framework' correlated Mason's five components of structures for attending with the developmental progression of decimal fraction concepts outlined in the National Numeracy Learning Progression (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2020). The framework was developed prior to data collection by anticipating the specific decimal concepts students should attend to when developing their conceptual understanding of decimal place value, magnitude and density. This was then modified after the interviews based on observations of student understandings. The value of this extended theory of shifts of attention enabled the researcher to understand what each participant experienced in the learning process and identify what elements of the dynamic digital representations prompted shifts of attention and changes in understanding. This analysis tool was used alongside a microgenetic approach to data analysis when the researcher discerned observable shifts in how and what the participants were attending to as they engaged with the four dynamic digital representations of decimal fractions.

Table 1

Shifting Attention with Decimal Concepts Framework

Structure and microqualities of attention (Mason, 2008)	Connecting Mason's structures with decimal fraction learning progression (ACARA, 2020)	Key concepts of decimal fractions
Holding wholes	Names decimal fractions without an understanding of place value	Place value
Discerning details	Discern mathematical details of decimal fractions with an understanding of place value and reforming decimal numbers with value	Place value Decimal magnitude
Recognising relationships	Understands the relationship between decimal fractions and the division operation, as the decimal is parts of a unit resulting from dividing the whole into equal parts. Understands the positional value of decimals as well as the scaling effect as digits move to the right and left in place value	Place value Decimal density
Perceiving properties	Perceives properties of decimal fractions as they demonstrate knowledge of the relative size of decimals when comparing and applies this knowledge to the relationship with measurement and position when plotting values on a number line	Decimal magnitude
Reasoning on the basis of agreed properties	Relates their understanding of positional value number knowledge to perform operations with decimals.	Place value Decimal density

Findings

Four scenarios were selected to illustrate how each of the students interacted with one of the four dynamic representations. The evidence and accompanying interpretation reveal the specific dynamic affordances that facilitated shifts in attention which indicated changes in conceptual understanding of decimals. Pseudonyms are used in replacement of students' names throughout.

Example One: Annelise Wishball-Hundredths

In the interview examining the affordances of Wishball-hundredths, Annelise experienced shifts in attention where she was able to generalise mathematical principles pertinent to adding and subtracting decimals, including the concept of place value when reforming decimals. During the task-based interview, Annelise needed to reach the target number by adding or subtracting in fewer than 20 moves, the current starting number was 29.67 and the spinner displayed 9. Annelise attempted to calculate the subtraction $29.67 - 0.09$ before the screen displayed the answer, to decide whether it was an ideal move within the game. This was a key moment of conceptual learning as she exclaimed "which will actually make it closer to the target number!" Through her engagement with the dynamic affordances embedded in Wishball-hundredths, namely the dynamic counting frame, as well as the feedback she received when the starting number changed based on her own carefully considered mathematical moves, the student's thinking progressed to *reasoning on the basis of agreed properties* (Table 1) as she related her understanding of positional place value knowledge to perform a subtraction operation with decimals. However, her approximated answer suggested that she was consolidating her knowledge across multiple structures of attention. Annelise acknowledged her change in conceptual understanding was the result of new learnings that occurred during Wishball-hundredths when she explained, "I knew that one of the tenths would need to be traded to the hundredths to subtract from, because I had seen the counting frame move before." The powerful partnering of observed dynamic affordances and Annelise's knowledge of theoretical concepts related to decimals allowed a transition of attention to generalising principles and therefore being able to *reason on the basis of agreed properties* when calculating a subtraction problem with decimal numbers.

Example Two: Valerie Decimal Strips

Valerie was asked to compare a series of two decimal numbers to determine the larger decimal value whilst completing the Decimal Strips task-based interview. One comparison challenge required the student to decide whether 0.2 or 0.25 was a larger decimal. Valerie acknowledged her insufficient decimal knowledge to answer the comparison question. Without hesitation she stated, "I'm not sure I'll have to check." Valerie was aware of the affordances of the dynamic digital representation to display the relative size of decimal fractions as was needed in this decimal comparison challenge. She dragged the 0.2 and 0.25 coloured tiles onto the blank fraction wall. The researcher used a probe question, "why do you think they are so close in value?" to elicit further student thinking. Valerie considered the question before responding, "maybe the zero-point-two has something else on it you just can't see it." A second probe question was asked, "what would be on the end?" Valerie replied, "it could be twenty-something and then this one [pointed to yellow 0.3 strip] could be thirty-something."

The experience of Valerie unpacking this particular decimal comparison question is an example of how the abstract nature of decimals present significant computation difficulties for students. Connecting her pre-existing decimal knowledge with how she manipulated the decimal strips to compare relative sizes, Valerie posited that 0.2 and 0.3 were actually "twenty-something" and "thirty-something". She independently uncovered the concept of decimal place value and accurately described 0.2 and 0.3 as having a 'hidden' value of tenths. Valerie's existing misconception thinking "longer-is-larger" (Steinle & Stacey, 2004) was being challenged through her engagement with the virtual manipulative as she dynamically compared the decimal lengths in this area-model representation. A significant first step to overcoming her erroneous thinking.

Example Three: Yehali Zooming in on Place Value

Yehali's default numerical thinking was focused on whole numbers only. She did not hold an understanding of decimal density and therefore initially couldn't predict decimals located between whole numbers in the Zooming in on Place Value interview. The interactive activity displayed the decimal 0.9 with the question, "what is the location of the red point? Can you make a prediction?" Yehali's first guess was "one", so she typed the predicted location in as 1.0 and watched the 'zooming in' animation reveal the location between zero and one with tenths integers marked as dashes on the number line. The researcher probed, "what did you see happen just then?" Yehali replied, "the numbers stretched out." The screen displayed a second question, "what is the actual location of the red point?" The student answered "nine" and typed her second predicted location in as 9.0 and then saw the actual location revealed as 0.9. The researcher checked for understanding, "what is the actual location written as?" Yehali read, "zero-and-nine." A follow-up question was posed, "why is that the actual answer for the location of the red point on the number line?" Yehali reasoned, "maybe because there is no number here it's just zero [pointed to zero integer marking on number line] and it hasn't gone to one yet [pointed to one integer marking on number line] so it's in the middle somewhere."

Yehali's current thinking involved *holding wholes* as her understanding was constrained by the whole number details presented in the first-step of the interactive activity. When Yehali observed the animated number line dynamically zoom-in to show ten-tenths between zero and one, she used her whole-number thinking to change her prediction to "nine" as she was viewing the integer markings having ones value rather than tenths. The feedback Yehali received from the interactive activity prompted her to reconsider her dominant whole-number thinking. When she noticed that her second prediction of "nine" was incorrect she needed to produce a mathematical reason why the actual location was zero-point-nine. By closely viewing the dynamic number line she grasped a more accurate understanding that included some decimal knowledge, "maybe because there is no number here it's just zero and it hasn't gone to one yet so it's in the middle somewhere." Yehali demonstrated a shift in attention to *recognising relationships* when she understood that between two whole numbers, decimal values are located, the concept of decimal density.

Example Four: Aarav Zoomable Number Line

Aarav displayed significant conceptual confusion around decimal fractions across a number of tasks in the Zoomable Number Line interview. He was asked to locate the decimal 0.9 on the zoomable number line and describe its location. Aarav immediately, zoomed-in to show hundredths on the number line and scrolled left from 0.12, he paused when 0.09 was shown on screen display exclaiming, "here is the number." The researcher questioned, "is that the same number as shown on the card?" The student considered and then answered, "no it's even littler" before he continued to scroll left from 0.09 and stopped when 0 was displayed. The researcher asked, "what are you thinking?" Aarav replied, "I zoomed in too much because the numbers are hundredths but this one [pointed to 0.9 decimal card] is tenths". He tapped the 'zoom out' button and adjusted the number line to show tenths only, he then scrolled back to the right along the number line and stopped upon 0.9 being shown, "I found it … zero-point-nine is between zero-point-eight and one-whole." Aarav's actions indicated that his understanding of tenths and hundredths place value in decimals was developed through his exploration of the zoomable number line. Initially, the student did not understand the concept of place value in decimals as suggested by his consideration of 0.9 to be the same value as 0.09. Using the 'zoom out' function of the zoomable number line helped Aarav correctly locate 0.9. Aarav was originally operating with a mathematical mindset of whole-number thinking and thus an incomplete ability to *discern details* related to decimal place value. However, through his interactions with the features of the zoomable number line, Aarav's attention shifted to *recognising relationships* by stating the place value of digits in the decimal number with an understanding of positional value.

Discussion

Representations are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning (Pape & Tchoshanov, 2001). The affordances of four dynamic digital representations were employed by the students in this study to develop conceptual understanding of decimal place value, magnitude and density. The facility of digital tools to adapt, change, and provide various interactive features and capabilities that traditional static representations may not offer were pivotal in enabling the students to successfully complete decimal fractions tasks such as the decimal location task in the Zoomable Number Line interview. By actively "scrolling backwards" Aarav could mathematically reason that the decimal 0.09 must be smaller in value than 0.9 as it was located closer to the whole number 0 on the number line. In addition, the dynamic digital representation Zooming in on Place Value helped Yehali 'see' notions that would otherwise be 'unseeable' through its unique 'zooming in' animation between two whole numbers to show a magnified view divided into tenths (Orrill & Polly, 2013). She initially had difficulty predicting what tenths decimal was located on the number line, however by engaging with the dynamic nature of the Zooming in on Place Value web-app, she successfully solved later questions. This showed that within the timeframe of the 30-minute interview, the student was comfortable and confident enough to use the highly effective learning tool to eliminate the abstractness of decimals.

All students involved in this study experienced moments of cognitive confusion generated by the dynamic affordances embedded within the digital tools which interrupted the students' flow of attention, to guide their conceptual learning journey of decimal fractions. Seminal work by Swan (1983) found that students who experienced 'cognitive conflict' when learning gained better outcomes overall than students who learnt through 'positive only teacher'. Decimal strips generated cognitive conflict as it challenged Valerie's current misconception thinking about decimals. It was not until she used the functions of the Decimal Strips web-app and dragged the 0.2 and 0.25 coloured tiles onto the blank fraction wall that she recognised the value of five-hundredths. Valerie was able to self-correct her thinking by interacting with this dynamic representation as it challenged her prior understanding of decimals and allowed her to examine her errors in thinking and misconceptions.

Dynamic representations of decimal fractions are capable of offering direction to learners' shifts of attention. The dynamic nature of digital manipulatives force shifts in the object of children's attention when they are actively interacting with the mathematics content as well as the structure of their attention in how they attend to features of the mathematics representation and the embedded mathematics concepts. Conceptual development occurred when the learners experienced a shift between attending to relationships within and between elements of current experience and perceiving relationships as properties that might be applicable in other situations (Mason, 2008). For example, Annelise acknowledged that her thinking had changed from her limited prior understanding of performing operations with decimal fractions. The shift from *discerning details* of the place values within a decimal to then being able to *recognise relationships, perceive properties,* and *reason on the basis of agreed properties* by performing operations with decimals with an understanding of positional value number knowledge, was the result of new learnings that occurred during the Wishball-hundredths game. These changes in decimal understanding were prompted by unique affordances embedded within the web-app including the dynamic counting frame. By being aware of shifts occurring, teachers can be more alert to the possibility that learners are not making appropriate shifts and therefore implement teaching strategies and learning tools such as dynamic digital representations, to prompt these conceptual changes.

Conclusion

Teachers need to use various strategies, including visual representations, manipulatives, real-world examples, and interactive activities to make decimals more tangible and understandable. Examining a group of students completing four task-based interviews revealed the full scope of conceptual understanding of decimal fractions inclusive of decimal place value, relative magnitude and decimal density, which emerged from the students' interactions with four dynamic digital tools. At times, the students' experienced productive cognitive conflict, as the dynamic representation of decimal fractions challenged their existing understandings and misconceptions. Outside of the research-interview situation, these would have been ideal 'teaching moments'. While keeping in mind the inferential limitations of a small-scale study, the findings clearly indicate the potential of dynamic, digital representations to be used as effective learning and teaching tools to develop conceptual understanding of decimal fractions, and to address previously established misconceptions.

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References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2020). *National numeracy learning progression*. ACARA. https://www.australiancurriculum.edu.au/media/3635/national-numeracy-learning-progression.pdf

Board of Studies NSW. (2012). *Mathematics K–10 syllabus*. Board of Studies NSW. https://educationstandards.nsw.edu.au/wps/portal/nesa/k-10/learning-areas/mathematics/mathematics-k-10

Caldwell, J. (2020). *Developing conceptual understanding: The case of place value*. Savvas Learning Company.

Goldin, G. A. (2003). Representation in school mathematics: A unifying research perspective. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 275–285). Reston, VA: National Council of Teachers of Mathematics.

Litster, K., Moyer-Packenham, P. S., & Reeder, R. (2019). Base-10 blocks: A study of iPad virtual manipulative affordances across primary-grade levels. *Mathematics Education Research Journal*, *31*(3), 349–365.

Mason, J. (2008). Being mathematical with and in front of learners. In B. Jaworski & T. Wood (Eds.), *The mathematics teacher educator as a developing professional* (pp. 31–55). Sense Publishers.

Orrill, C. H., & Polly, D. (2013). Supporting mathematical communication through technology. In D. Polly (Ed.), *Common core mathematics standards and implementing digital technologies* (pp. 22–36). IGI Global.

Pape, S. & Tchoshanov, M. (2001). The role of representation(s) in developing mathematical understanding. *Theory Into Practice*, *40*(2), 118–127.

Siegler, R. S., Thompson, C., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–296.

Steinle, V. & Stacey, K. (2004). A longitudinal study of students' understanding of decimal notation: An overview and refined results. In I. Putt, R. Faragher, & M. Mclean (Eds), *Mathematics education for the third millennium, towards 2010. Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia*, (Vol. 2, pp. 541–548). Sydney: MERGA.

Swan, M. (1983). *Teaching decimal place value: A comparative study of "conflict" and "positive only" approaches*. Shell Centre for Mathematical Education, Nottingham University.

Voutsina, C., George, L., & Jones, K. (2019). Microgenetic analysis of young children's shifts of attention in arithmetic tasks: underlying dynamics of change in phases of seemingly stable task performance. *Educational Studies in Mathematics*, *102*(1), 47–74.

Secondary Mathematics Teachers' Mathematical Competence

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Whilst educational goals in recent years for mathematics education are foregrounded the development of mathematical competencies, little is known about mathematics teachers' competencies. In this study, a group of practising teachers were asked to solve an algebra problem, and their solutions were analysed to determine the competencies apparent (or, equally importantly, absent) in these solutions. The most demonstrated competencies were *devising strategies* and *mathematising*, whilst *communication* and *reasoning* were mostly missed. The research provides a detailed methodology on how mathematical competencies can be assessed in the context of problem solving.

In recent decades, efforts in mathematics education reform and development have been directed to mathematical competencies. Many countries have introduced new standards influenced by mathematical competencies. Denmark, for example, has competency descriptions of its mathematics programmes all the way from primary school to tertiary level, and in teacher training (Jankvist et al., 2022). In Australia, the notion of mathematical competencies has become more explicit and established in the mathematics curriculum, by implementing a mathematical proficiencies framework (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2022). A shared understanding within the mathematics education community is that students' mathematical competencies are affected by their teachers' competencies (Jankvist et al., 2022). To be able to shed some light on this potential impact, we need to know what mathematical competencies teachers have. Extending on our earlier works with secondary mathematics teachers, in this paper, we present a study that examines mathematical competencies evident (or not) in a sample of secondary mathematics teachers' solutions to an open-ended algebra problem that is based on the idea of consecutive numbers. We aim to answer the research questions: (1) *What can be said about the teachers' mathematical competence for the given problem in terms of their ability to activate the mathematical competencies and arrive at a desirable answer?* and (2) *How (if at all) is the teachers' performance in the problem impacted by activation of the mathematical competencies?*

Mathematical Competencies

Broadly, mathematical competence is "someone's insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to a given situation" (Niss & Højgaard, 2019, p. 12). Mathematical competence consists of six distinct but mutually related mathematical competencies: *Communication*; *Devising strategies*; *Representation*; *Mathematising*; *Symbols and formalism*; *Reasoning and argument* (see Turner et al., 2015). The *communication* competency involves two aspects: *receptive* and *expressive*. The *receptive* aspect includes reading, deciphering, and understanding mathematical statements and data. The *expressive* aspect involves explaining, presenting, and debating mathematical concepts. *Devising strategies* is creating and executing a mathematical strategy to solve problems that arise from the task or context. *Representation* is creating or utilising representations of mathematical entities or relationships. These representations can be equations, formulas, graphs, tables, diagrams, or textual descriptions. *Mathematising* is converting real-world

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 263–270). Gold Coast: MERGA.

problems into mathematical ones. It also involves interpreting mathematical entities or data in the context of the modelled situation. *Symbols and formalism* is understanding, manipulating, and utilising symbolic expressions. It also involves using constructs based on definitions, rules, conventions and formal systems. Finally, *reasoning and argument* is logical thinking processes that explore and connect problem elements to draw conclusions from them. It also involves checking a given justification or providing a justification (Niss & Højgaard, 2019).

Previous Research on Mathematical Competencies

Most relevant to the current study, Büchele and Feudel (2023) carried out a study on data from a non-standardised mathematics entry test taken by 3076 economics students divided into different cohorts from 2012 to 2019. The authors used regression analyses to examine the changes in students' mathematical competencies. The students' ability to carry out symbolic calculations decreased over the years. However, their performance increased in some test questions focusing on other process competencies like *reasoning*, *mathematising*, or *using different representations*. The authors concluded that a curriculum reform emphasising these competencies was likely to have had the desired effect of improving students' abilities in the mathematical process. Albano and Pierri (2014) introduced a role-play activity designed to foster conceptual understanding of mathematics for first-year engineering students. The findings showed that the quality of the questions posed by the students highlighted a shift from the instrumental approach they were used to towards a relational one. This suggests that the role-play activity was successful in fostering a deeper, more conceptual understanding of mathematics among the students. Wess and Greefrath (2019) studied the professionalisation of preservice teachers through reflective practice when they were training for the mathematical modelling competency. The authors designed and implemented a teaching laboratory, which was a learning environment that simulated authentic classroom situations and allowed the participants to practice and reflect on their teaching skills. The teaching laboratory had a positive impact on the participants' task competency, as well as their self-efficacy and enthusiasm for teaching mathematical modelling. In none of these studies have the competencies been investigated in the context of practising schoolteachers, nor has it been shown what relationship exists between the various competencies and success in problem solving.

The Study

The data for this study comes from a workshop delivered by Hatisaru, in 2022, at an annual mathematics teacher conference in Western Australia that aimed to support lower secondary mathematics teachers' pedagogical content knowledge for teaching algebra. The teachers were teaching mathematics to students in years 7 to 12 (aged 12 to 18), and most were qualified mathematics teachers, with 16 out of 22 having at least a mathematics minor, and all others having completed some tertiary mathematics units. Each participant was provided with three Reflection Forms, each with an algebra word problem presented on it and the prompt: *Think of and explain as many different possible solutions to the problem as you can. Name the solutions as Solution A, Solution B, Solution C and so on*.

The participants completed this task in 20–25 minutes. Out of the 42 teachers, 22 of them gave consent for their responses to contribute to this research. They were assigned a code from P1 to P22 to protect their anonymity. In this paper, we focus on participants' solutions to one of these problems, referred to as *three consecutive numbers*: *Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two. Now do it with another three consecutive numbers. Can you explain it with numbers? Can you use algebra to explain it?* (Kieran, 1992). Here, we describe the mathematical content of this problem for each competency presented earlier.

Communication: The text is fairly straightforward to interpret, involving some reasonably low-level technical language (e.g., 'consecutive', 'square', 'subtract', 'product'). One must interpret the text (receptive component) as instructions to do various things—starting with deciding what numbers to work with, but also interpreting and dealing with the two somewhat ambiguous, or at least tentative 'can you…?' questions. The expressive or constructive component consists of explaining the process of their investigation, presenting a hypothesis, and arguing the validity of their findings. *Devising strategies*: Several steps are involved, starting with negotiating a string of instructions to be applied to three numbers chosen, applying the instructions again to another set of three numbers, before hopefully arriving at an 'explanation' of the observed result. The instructions are then to be applied again in a generalised setting using algebra, together with a generalised solution, and an explanation drawn/devised. One needs to keep the goal in mind, work out what meeting the goal might look like, monitor their progress, and move on from stage to stage. *Representation*: The heart of the problem lies in finding a way to express the context and text instructions in a mathematical form. The need to represent the task elements with numbers, arithmetic operations, and letters (algebraic entities) is fairly obvious, so only a very low-level representation demand is present. There is no given mathematical representation to interpret; the 'devise' aspect of representation fits better under mathematisation.

Mathematising: When one moves beyond the numeric exploration that falls directly out of the interpretation of the problem statement, to an algebraic representation, one needs to mathematise the phrase 'three consecutive numbers'. This can be done in a number of possible mathematical formulations, for example $a - 1, a, a + 1$; or $n, n + 1, n + 2$; or as $n - 2, n - 1, n$. Each of these leads to slightly different algebraic processing steps once the phrases 'square the middle number', 'subtract from it' and 'the product of the other two' have been mathematised. We decided to characterise these demands under mathematisation, because one must come up with a mathematical formulation of essentially real-world (extra-mathematical) context elements. Finally, interpreting the mathematical result ('Square − Product = 1') in the real-world context (i.e., in relation to any three consecutive numbers) is a 'de-mathematisation' step that is rather important. *Symbols and formalism*: One needs to apply their formal mathematical knowledge to carry out the arithmetic calculations (multiplication and subtraction), and then perform algebraic manipulation once that has been arranged (square an expression, multiply two expressions, and find the difference, in the specified direction), leading to the mathematical result 'Square − Product = 1'. *Reasoning and argument*: Reasoning and argumentation are needed to reflect on the numerical results found, link these to the generalised algebraic result, and to conceptualise an argument that expresses the finding that the difference found at the subtract step is always a number 1 less than the square.

Data Analysis Approach

Among 22 teachers, one teacher did not respond to the problem; 18 of the remaining teachers generated one solution and 3 of them gave two solutions. For teachers with more than one solution, we decided to only analyse their best solution. We first analysed these 21 solutions according to the marking scheme presented in Table 1. To find out what mathematical competencies were manifested in these solutions, next, we analysed the solutions according to the scheme presented in Table 2 (for examples, see Figure 1). The codes 'evident' and 'absent' were used where the relevant competency was apparent or not in the respective solution. As none of the participants attempted to explain the findings, *the constructive aspect of communication* was not coded. Finally, we compared differences in the observed mathematical competencies by teachers showing stronger performance in the problem and those showing weaker performance. A strong performance is defined as a solution receiving a score of 4, while a weaker performance is defined as a solution receiving a score of 3 or lower.

The data were analysed by two of the authors of this paper (Hatisaru and Richardson). To establish interrater reliability, they independently coded all the participant solutions. Their agreement for the coding of participants' solutions to the problem was approximately 86% and for the coding of competencies evident in these solutions was approximately 81%. Any differences or disagreements were resolved through discussion.

Table 1

Marking Scheme Used to Assess Participant Solutions to Three Consecutive Numbers

Score	Observed solution
5	Solution demonstrates correct and complete interpretation; correct mathematical formulation, result 'Square − Product = 1' is found together with explanation connecting the algebra to the original number context
4	Solution demonstrates correct and complete interpretation; correct mathematical formulation, result 'Square − Product = 1' is found but without a clear statement linking the result to the original number context
3	Solution demonstrates mostly correct interpretation; formulation correct, but with sense of difference reversed ('Product − Square = −1'); there is not a clear statement linking the result to the original number context
2	Solution demonstrates correct interpretation applied to numeric examples with the result of 'Square − Product = 1' without algebraic formulation being correctly found and/or correctly applied
1	Solution demonstrates mostly correct interpretation applied to numeric examples with the result of 'Product − Square= −1' without algebraic formulation being correctly found and/or correctly applied
0	Solution demonstrates no progress with mathematical formulation of problem situation

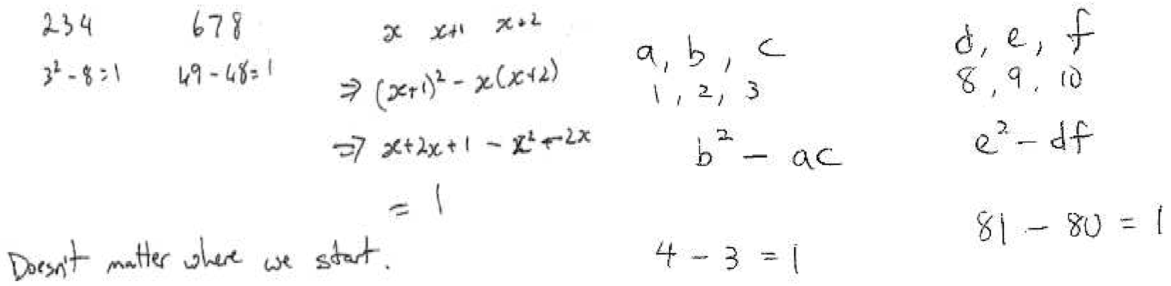
Table 2

Competencies for Solving Three Consecutive Numbers

Competency	Observed behaviour
Communication (COM)	The text is interpreted correctly; the terms 'consecutive', 'square', 'subtract' and 'product' are understood
Devising strategies (STR)	There is a strategy involving adhering to the steps outlined in the question statement. At least two steps are involved: applying the situation to numbers and to a generalised setting
Representation (REP)	The numerical and algebraic expressions are correct representations of the intent expressed in the problem statement
Mathematising (MAT)	The phrases are mathematised and the formulation of the problem situation 'Square–Product' is found
Symbols and formalism (SYF)	The formal knowledge needed to process both the numerical and the algebraic steps is evident. Algebraic manipulations are carried out correctly
Reasoning and argument (RES)	Numerical results are linked to the generalised algebraic result, a conclusion is made, and the mathematical result (i.e., 'Square − Product = 1') is connected to the 'real-word' result (i.e., any three consecutive numbers)

Figure 1

Example Solutions for P5 (Left) and P10 (Right) Coded Against the Marking Schema and Competencies



P5's response was scored as '4' with the codes COM, STR, REP, MAT, SYF, and RES.

P10's response was scored as '2' with the codes COM, STR, and REP.

Findings

Out of the 21 solutions to three consecutive numbers, none of the solutions were rated at score '5' because none of them included an explanation connecting the algebra to the original number context that was given. Two solutions were incorrect and were scored as '0' (P11 and P15). Two solutions were scored as '1' or '2' because, although these participants were able to interpret and correctly explore the problem with specific numerical examples, they did (or could) not formulate the problem algebraically (P10 and P16) or represent three consecutive numbers algebraically. Four were scored as '3' since they were able to formulate and devise the desired result algebraically, other than getting the order of Square − Product the wrong way around (P4, P6, P8, and P22). The remaining 14 solutions were scored as '4'; these responses presented the correct mathematical formulation (i.e., Square − Product = 1), but without an explanation. These findings suggest that many of the participants got to the essence of the problem, achieving differing levels of performance from scores '4' to '0'. Consideration of possible alternative approaches to the problem was happened in only three cases.

Figure 2 captures the frequency of mathematical competencies, apparent or not, in the solutions. Although participants varied in their use or activation of these competencies, patterns seemed to emerge. The mostly missed competency was *reasoning and argument*, followed by the *communication* and *symbols and formalism* competencies. In three solutions, all *devising strategies*, *representation*, and *mathematising* were absent. For example, in P4's solution in Figure 3 (left), a lack in the *communication* and *symbols and formalism* competencies is evident because P4 did not follow the instruction of implementing the procedure on numerical examples. Doing this might have given them a hint as to what they were looking to see from the algebraic consideration. Also, they did not order the 'Square–Product' correctly. Moreover, P4 lacked the algebra skills to simplify their final expression to -1.

The mathematical competencies in the participants' solutions who showed stronger (n = 13) and who showed weaker (n = 8) performance was compared (Figure 4). This comparison has revealed that all 13 participants who performed at 'score 4' level activated almost all of the first five competencies in solving the given problem, with one not activating the *symbols and formalism* competency. Conversely, the 8 weaker participants, i.e., those who performed at 'score 3' or lower, did not activate one (P6, P8, P22), three (P4, P15), five (P10), or all six (P11, P16) of the relevant competencies. This finding shows that activation of these competencies impacts on one's performance in solving a mathematics problem.

Figure 2

Distribution of Mathematical Competencies by the two Codes

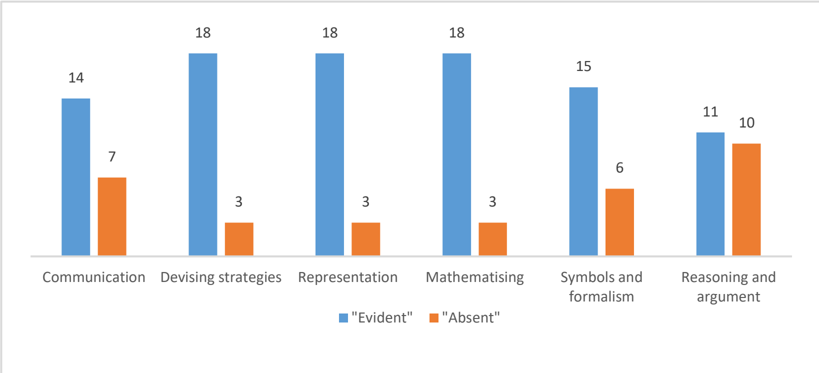
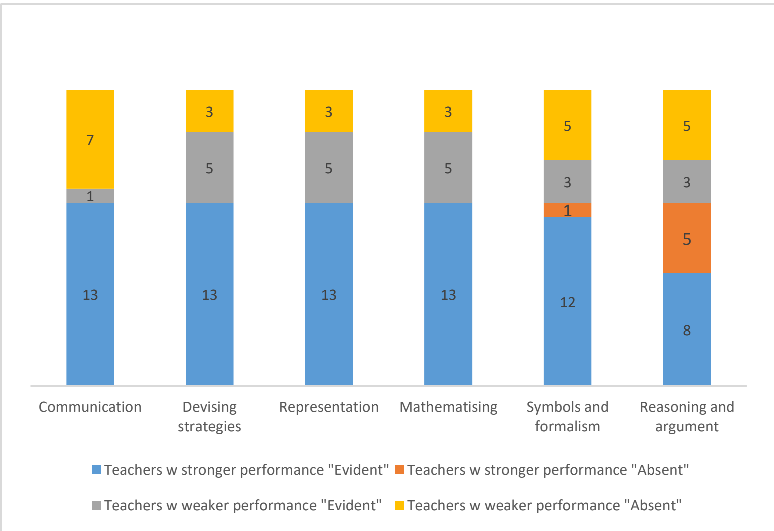


Figure 3

Solutions of P4 (Left) and P11 (Right) to Three Consecutive Numbers

Figure 4

*Distribution of Participants with Stronger (*n *= 13) or Weaker (*n *= 8) Performance by the two Codes*



Although the mathematical competencies are distinct, they are utilised in an interconnected way when solving problems. As a result, it is not always possible to accurately assess a specific mathematical competency using a presented solution, since a deficiency in one competency may eliminate the opportunity for another competency to be demonstrated. For example, P10 whose solution is presented in Figure 1 (right), only understood the question up to the numeric example level. The fact that they could not mathematise the question in algebraic form meant that we do not know if they could have simplified the expression (i.e., if they could have demonstrated *symbols and formalism*). In this solution, *mathematising* and *devising strategies* seem linked, as one cannot devise a strategy without being able to mathematise the problem. P11 (Figure 3) knew to square the middle number and multiply the others but was not able to interpret the subtraction part. They were unable to represent three consecutive numbers mathematically (i.e., *mathematising*) which meant that the remaining competencies could not really be assessed. That said, they did correctly expand (a–1) × (a+1) which is an element of *symbols and formalism*, although not part of a coherent solution.

Discussion and Suggestions for Future Research

In this study, we have provided a perspective for understanding and assessing mathematical competencies in the school education context. By using the case of *three consecutive numbers*, we have developed the process of applying the six mathematical competencies to an algebraic problem and assessing individual solutions to that problem, deconstructing the skills which need to be demonstrated for each competency to be verified. Our associated marking schemes (Tables 1 and 2) can be adapted to different problems, providing the basis for initial teacher education and professional development initiatives for preservice and practising teachers to assess and develop their mathematical competencies. They can also be used for teachers to assess the competencies of their students in the classroom, or for individuals to self-reflect on their work and consider whether they have demonstrated all six competencies in presenting their solutions. The conceptualisation presented here can be used in future investigations to understand the extent to which teacher competencies impact on student competencies, and thereby to gain a deeper understanding of any skills gaps being shown in national tests.

We analysed the mathematical competencies apparent and absent in a selection of practising teacher responses to an algebra problem. The most conspicuous competency absent in the solutions of our participants was that of *expressive communication*, which none of the solutions was deemed to contain. The weaker solutions also almost exclusively lacked *receptive communication* as well—that is, the ability to successfully interpret the instructions in the question. We consider teachers to be expert communicators: their main role is to explain complex concepts to beginners. The participants' solutions, however, showed almost no explanation, no commentary, conclusions, or interpretations. It is possible that the reason for this was contextual; that in the setting of a workshop teachers did not feel it was important to communicate their solutions as thoroughly as they would do in a classroom with students. Further investigation is required here to determine if mathematical *communication* skills are lacking in practising teachers, or if they are underused except in particular contexts. Connected to the competency of *expressive communication* is *reasoning and argument*, a skill that was the main competency lacking among both strong and weak solutions. This shows an inability to reflect on the results that were found, to generalise a result to a broader context, and to justify a result that was found. One could argue that the skill of *reasoning* is the most important competency, defining the very essence of mathematics: the ability to notice patterns and to question and justify the contexts in which they hold true. However, we see that even some successful solutions were missing this skill. As with the lack of *communication* skills, further investigation is required to determine the reasons for this within the context of the workshop. Mathematical competencies are not necessarily being learnt implicitly in initial teacher education. The research of Albano and Pierri (2014) and Wess and Greefrath (2019) show that these mathematical competencies can be developed in professional development sessions, both in terms of making the competencies explicit and emphasising their importance, and also in terms of improving teachers' skills and self-efficacy in these areas. We believe teachers of mathematics need to be supported to develop, use, and teach these competencies.

Like Büchele and Feudel's (2023) German participants, our Australian participants struggled more with *symbolic manipulation and formalism* than they did with *strategy* and *mathematising*. These latter skills were evident in all but three solutions, while *symbolic manipulation* was problematic in six. Further data on participants' educational backgrounds and current teaching focus would be useful here to determine the reasons for this weakness. For example, a teacher may not have taught algebra for some time or may have done their teacher training a long time ago. However, it could also be argued that *three consecutive numbers* required only low-level algebraic skills which we would expect that all secondary school mathematics teachers would have, regardless of their teaching level or educational background. As Australian students' algebra abilities are sometimes weaker than in other areas, our study is likely to representative of a wider problem. While it is good to see more of the higher-level competencies present in our participants (*devising strategies*, *mathematising*), we showed that it is important to focus teacher training initiatives on accurately applying formal mathematical knowledge to close this skills gap. This is likely to aid in the competency of *reasoning and argument*, as arguments cannot progress and conclude unless the intermediate steps are correct.

We acknowledge that our study had a relatively small sample size, with only 22 teachers participating. While all were teaching at the secondary level, we do not know more specifically what teaching experience the participants had. For example, someone teaching Specialist Mathematics at Year 12 would be expected to have stronger mathematical competencies than someone teaching mainly Years 7–9. It would also be interesting to investigate whether the degree program studied had an effect on the strength of the participants' solutions; for example, whether having a mathematics major as compared with a minor resulted in stronger competencies. Mathematical competencies should be investigated in a wider variety of problems and contexts, investigating the extent to which the wording of the problem influences the competencies being demonstrated and if factors such as the time given to solve a problem also have an impact. Finally, *three consecutive numbers* explicitly asked participants to first investigate with numbers and then to explain with algebra. A future investigation might omit these specific instructions to find out if teachers would naturally start from specific cases and generalise to all numbers and investigate how best we can cultivate and motivate this skill among teachers.

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References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: Mathematics (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/

Albano, G., & Pierri, A. (2014). Mathematical competencies in a role-play activity. In C. Nicol, P. Liljedahl, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36* (Vol. 2, pp. 17–24). Vancouver, Canada: PME.

Büchele, S., & Feudel, F. (2023). Changes in students' mathematical competencies at the beginning of higher education within the last decade at a German university. *International Journal of Science and Mathematics Education*, *21*, 2325–2347. https://doi.org/10.1007/s10763-022-10350-x

Jankvist, U. T., Geraniou, E., Pedersen, M. K., Bach, C. C., & Gregersen, R. M. (2022). Mathematical competencies in the digital era: An introduction. In U. T. Jankvist, & E. Geraniou (Eds), *Mathematics education in the digital era*, (Vol 20, p. 1). Springer, Cham.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan Publishing.

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28.

Turner, R., Blum, W., & Niss, M. (2015). Using competencies to explain mathematical item demand: A work in progress. In K. Stacey, & R. Turner (Eds), *Assessing Mathematical Literacy* (pp. 85–115) Springer.

Wess, R., & Greefrath, G. (2019). Professional competencies for teaching mathematical modelling: Supporting the modelling-specific task competency of prospective teachers in the teaching laboratory. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Eleventh congress of the European Society for Research in Mathematics Education (ERME)*, No. 27. Utrecht.

Teacher Expectations of Student Strategies for Algebra Problems

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This paper investigates the expectations that a group of preservice secondary mathematics teachers had for how students would approach four particular algebra problems. Their responses to a set of open-ended items were content analysed. Findings show that the teachers ranged in the expectations that they held for their students. Some teachers expected their students would approach the problems with more sophisticated problem-solving strategies, but many of the teachers expected that the students would only use less desirable "guess-and-check" strategies. We believe teachers' expectations for student strategies for problem-solving is a topic that warrants further investigation.

For about a decade, we have worked with teachers (both preservice and practising) to understand the knowledge bases that are needed to teach their subject and to support their professional knowledge and practices. When analysing data about preservice secondary mathematics teachers' own problem-solving strategies, an investigation captured a thought-provoking issue. The strategies that teachers most frequently expected students to use were "guess-and-check" or "trial and error", despite these approaches being less desirable than more sophisticated problem-solving strategies such as systematic numerical approaches or algebraic methods. Possible strategies to the relevant problems were articulated in a manner that underestimated the students' potential success and creativity in mathematical problem-solving. This stimulated the following research question:

- What expectations for student strategies to algebra problems are evident in preservice secondary mathematics teacher responses?

Every student can solve mathematical problems, so mathematics should be taught with high teacher expectations (National Council of Teachers of Mathematics [NCTM], 2016). The Australian Institute for Teaching and School Leadership (AITSL) standards stipulate that teachers know their students, including the suitable expectations for task difficulty (AITSL, 2017). Knowledge frameworks for mathematical problem-solving provide the foundation for developing teacher education programs and guiding teacher learning about how students think and approach mathematical problem-solving. Some research has investigated teacher expectations for student mathematical capabilities, but little is known about teachers' expectations for student approaches to problem-solving. The following literature shows that research about teachers' expectations of students' strategies for algebraic problem-solving can be important because of the well-established association between teachers' expectations and student academic attainment.

Teachers' Expectations of Students

Broadly, teachers' expectations of students are the beliefs that teachers hold about if, when, and how much students will accomplish academically at school (see Johnston et al., 2024). The expectations that teachers hold for their students can be short term, such as how they will perform on a given learning task, or long term, such as which post-secondary pathways they will take when they graduate (Wang et al., 2019). The association between teachers' expectations and student outcomes is called the 'teacher expectation effect' (Szumski &

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 271–278). Gold Coast: MERGA.

Karwowski, 2019). Research on teachers' expectations shows that high teacher expectations correspond with high student achievement, so students benefit from teachers who expect a lot from them academically (Rubie-Davies et al., 2015). Expectations can be accurate, but inaccurate low expectations hinder student results, while inaccurate high expectations can improve student results (Szumski & Karwowski, 2019) as students' responses to the teachers' differential treatment shapes their consequential educational attainment (Johnston et al., 2024).

Teachers' Expectations of Mathematical Problem-Solving

There has been very little research about teachers' expectations for students' mathematical problem-solving. Research in this area has usually been general to mathematics as a learning area, showing that teachers' mathematical expectations can be biased and based on student backgrounds, such as their ethnicity (Lorenz et al., 2016) or gender (Copur-Gencturk et al., 2023). Some teachers have higher expectations for boys' mathematical achievement than girls, but this is not always the case (Soto-Ardila et al., 2022). Classroom and school contextual factors, such as ability grouping and community engagement, can also influence teachers' expectations of students' mathematics abilities (Wang et al., 2019), as can a teacher's perception of their own ability (Copur-Gencturk et al., 2023).

Very few studies have investigated teachers' expectations for students' problem-solving in mathematics. Previous studies of teacher expectations of student mathematics achievement have asked teachers questions about their beliefs about students' potential, showing that teachers communicate these beliefs in ways that shape student achievement in mathematics problem-solving (Fyfe & Brown, 2020; Szumski & Karwowski, 2019). For example, Soto-Ardila et al. (2022) asked prospective teachers whether their students would be able to solve an elementary arithmetic problem while they were practicing in schools. Their findings revealed high teacher expectations of students in private schools, which corresponded to students' eventual high performance. This study did not control for prior achievement or consider how teachers' expectations were developed. However, a controlled experimental study from Fyfe and Brown (2020) found that teachers' expectations for mathematical problem-solving shape students' ability to generalise learning from one problem to another after feedback. Negative expectations caused students to perform significantly worse on these transfer problems, while positive expectations helped students to do better in the face of negative feedback. Copur-Gencturk et al. (2023) determined that when a teacher views a student as having a low ability (as opposed to a low effort) a cause-and-effect feedback loop occurs. For example, the teachers' views of students' helplessness are communicated to the student as a perception that they have a lower ability. Tohir et al. (2020) also investigated beginning teachers' projected expectations for their future students' mathematical thinking processes. Their findings were that the teachers' own approaches to answering the question were not related to their expectations of students, but poor written articulation of the findings made their important work unclear and unconvincing. Copur-Gencturk et al. (2023) found links between a teacher's own mathematical disposition and how they perceived student ability. For example, if a teacher viewed their own difficulties as a lack of ability, rather than as a problem to be solved, it could have a negative effect on what was expected of students. To our knowledge, these are the only studies that have investigated teachers' expectations for students' mathematical thinking or problem-solving specifically.

Interventions designed to raise teacher expectations can improve mathematical achievement of students from migrant and low-SES backgrounds. When teachers with low expectations for student mathematical achievement are encouraged to develop higher specific performance goals for mathematics, student achievement can improve (Ritzema et al., 2016). Thus, further exploration of teachers' expectations about their students' approaches to mathematical problem-solving could be a useful starting point for addressing any deficit or strength-based views of students' mathematical abilities. This paper seeks to establish what the participant teachers' expectations of their students are because these views can shape their future students' academic outcomes when they are enacted.

Methods

The study used to underpin this paper was conducted by the first author with the voluntary participation of Bachelor of Education and Master of Teaching preservice teachers. Both groups of participants were specialising in junior secondary mathematics teaching (Years 7 to 10; aged 12 to 16), and all were enrolled at a Western Australian university. Of the participants, 11 were enrolled in the Bachelor of Education program—a four-year degree at the undergraduate level, and 12 were enrolled in the Master of Teaching degree—a two-year degree at postgraduate level. The participant group was comprised of 9 females and 14 males.

In the two units where data were generated, preservice teachers were supported to enhance their mathematics pedagogy knowledge for Number and Algebra and to develop an understanding of the Australian curriculum structure. Across four weeks there was an explicit focus on the proficiency strand: problem-solving, as a pedagogical technique that would enhance student learning. Each week one of the problems introduced below was explored. Participants were asked to solve the relevant problem and then explain the strategies they would use to solve it in teaching. They were provided with three prompts: the first prompt was to answer the problem in as many ways as possible. In the second prompt, the participants were asked to compare their preferred solution strategies, the strategies that they anticipated students would use, and the strategies they hoped their students would use. In the final prompt, the participants were asked to describe why these problems would be useful to use in teaching and which of the problems they would use. In this paper, we focus on participant responses to the question about how students might solve these problems.

At the time of data collection, all participants had completed at least five weeks' professional experience, during which they planned, taught, and assessed student learning in mathematics. This is of contextual importance as the participants had some practical experience teaching mathematics, using pedagogical approaches to enhance mathematics, and witnessing learning within the students.

Mathematical Problems Used in the Study

The four problems used in this study—adapted from published literature (see Hatisaru et al., 2024)—are referred to in this paper as *any to numbers*, *farmer*, *dice*, and *books*. These problems are typically used (or can be used) in everyday junior secondary mathematics classrooms, and the participants had experience with the style of problems:

- *Any two numbers*: If you are given the sum and difference of any two numbers, show that you can always find out what the numbers are;
- *Farmer*: A farmer had 19 animals on his farm—some chickens and some cows. He also knew that there was a total of 62 legs on the animals on the farm. How many of each kind of animal did he have?
- *Dice*: Die A and Die B have twelve sides each. Suppose that you roll die A and die B at the same time. When do the dice satisfy the following two conditions? The sum of 2 times A plus B equals 15. 3 times A minus B equals 5;
- *Books*: You have some teen and young adult books. You gave one-half of the books plus one to a friend, one-half of the remaining books plus one to another friend, and one-half of the remaining books plus one to another friend. If you have one book left for you, how many books did you have at the start?

Each of these problems can be solved in multiple ways, which reveals not only the extent to which the participants were able to generate different strategies, but also the strategies they anticipated that students would use. Algebraic approaches are suitable for solving these problems, but they can also be solved by using numerical approaches that follow a systematic or logical reasoning process. Table 1 presents each of these approaches along with their descriptions, and Figures 1 and 2 capture example participant solutions where some of these strategies were used.

The last group of strategies in Table 1 refers to those that are less desirable because they are unsystematic or random numerical trials. Whilst systematic numerical approaches can provide a sufficient basis for solving these problems, strategies such as random guess-and-check or similar concrete methods (e.g., rolling dice and checking combinations) are not as desirable as systematic numerical approaches or algebraic strategies because they demonstrate no insight into the question. To truly demonstrate understanding of a problem, teachers and students should use a range of mathematical thinking processes and choose the most viable strategy, or combination of strategies, for the problem to be solved. The more logical and integrated the mathematical processes, the more sophisticated the thinking.

Table 1

Example Strategies for Solving 'Any Two Numbers', 'Farmer', 'Dice', and 'Books' (Reported in Hatisaru et al., 2024)

Strategy	Description
Equations	
Symbolic solving	Write algebraic equations and solve using standard algebraic method
Numerical solving	Write algebraic equations and solve numerically
Graphical solving	Write algebraic equations and plot to find the intersection point.
Using parameters, symbolic solving	Write algebraic equations with two unknowns and with two parameters and solve using standard algebraic methods
Using parameters, numerical solving	Write algebraic equations with two unknowns and with two parameters and solve for a specific example
No parameters, symbolic solving	Write algebraic equations with two unknowns but with two specific numbers and solve using standard algebraic methods
Pattern	Write algebraic equations with two unknowns and with selected specific sums and differences, solve using any method and look for a pattern linking solutions to sum and difference.
Numerical	
Systematic	Use a numerical path in a systematic way such as guess-check-and-improve or guess-and-check with tables
Logical arithmetic reasoning	Think about the relations between the numbers/quantities involved and work from known numbers towards the solution
Other less desirable strategies	Guess-and-check; concrete approaches such as rolling dice or counting

Data Analysis Approach

There were four problems and 22 participants; a few participants were absent in the class when the relevant problem was posed. So, in total, there were 70 responses. Unsurprisingly, most participants generated more than one solution to the problems—because they were asked to solve the problem in different ways—and that means in total we had 128 solutions for the four problems to analyse. Solutions could be 'complete', 'incomplete', 'erroneous', or 'suggested' without implementation.

The data were analysed by the first author of this paper who has extensive experience in content analysis (e.g., Hatisaru et al., 2024). She first determined the success rate; that is, if at least one of the solutions to the problem was complete and a correct answer was given. Next, she examined these solutions to find out what strategies the participants used in solving the problems. She classified each of the 128 solutions into solution strategies presented in Table 1. To illustrate this categorisation, example solutions are presented in Figures 1 and 2; strategy types mapped to them are noted. Participants were assigned codes: P1, P2, P3, P4, and so on to protect their anonymity.

To uncover the participants' expectations for student strategies to these problems, the author analysed participant responses to the relevant prompt in the same fashion. She coded the strategies addressed by the participants as possible strategies that student may apply to each problem. Some participants named more than one strategy, and each was counted. All authors reviewed and agreed on the data analysis.

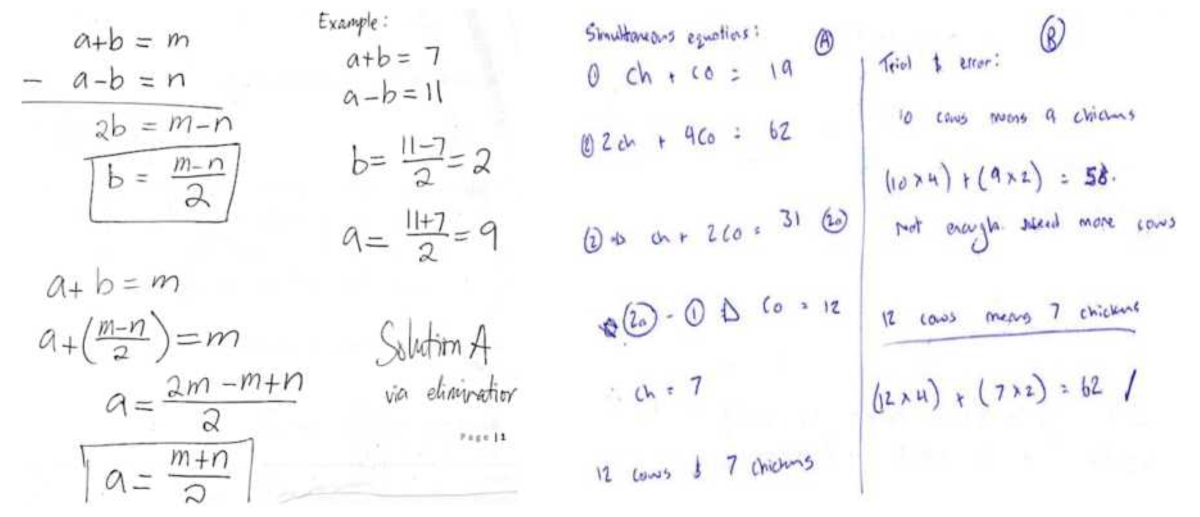
Results

Out of 70 solutions, 61 of them included a correct answer by at least one solution strategy (see Figures 1 and 2 for examples) and 4 responses were incorrect. Partially correct was only relevant for *any two numbers* when 5 participants solved the problem based on examples without generalising the solution to any sum and difference.

As noted earlier, a total of 128 solutions were identified for the four problems. Out of these, 13 were suggested solutions without showing any implementation, while 11 were erroneous responses where the participant had made incorrect assumptions about either the problem or the solution strategy (or both). Of the remaining 104 solutions, 7 were incomplete, while the remaining 97 were implemented correctly (76%). This means that most participants were able to solve the problems and also identify other possible ways to solve them.

Figure 1

Solutions of P16 and P12 to the 'Any Two Numbers' and 'Farmer' Problems



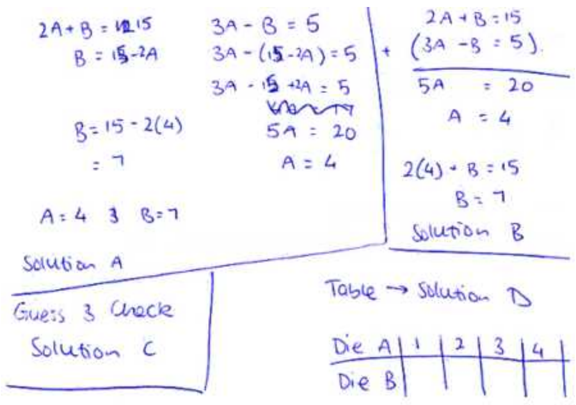
P16's solutions to any two numbers, including the equations, parameters, symbolic solving strategy.

P12's solutions to farmer, including the equations, symbolic solving (left) and numerical, systematic (right) strategies.

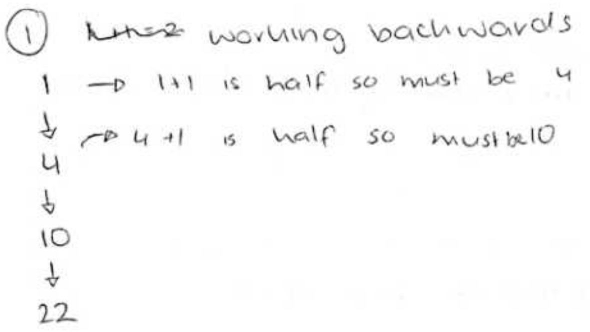
Focusing on participant responses where at least one of the identified—implemented or suggested—strategies to the relevant problem was correct (total of 114), we aimed to uncover the expectations of these participants with regard to how students would approach these problems. We excluded the other 14 responses as it was unlikely that the participants could anticipate sound student strategies where their own solution strategies were incorrect or erroneous. Table 2 summarises the number of strategies identified in the participants' solutions across all four problems, and in their responses where they anticipated student solutions.

Figure 2

Solutions of P9 and P3 to the 'Dice' and 'Books' Problems



P9's solutions to *dice*, including the *equations, symbolic solving* strategy.



P3's solution to *books*, including the *logical arithmetic reasoning* strategy.

Table 2

Distribution of Strategies in Participants' own and Anticipated Student Solutions

Strategy	In participant own solutions (114)	In anticipated student solutions (99)
Equations		
Symbolic solving	Farmer (20); Dice (19); Books (10)	Farmer (9); Dice (6); Books (6)
Numerical solving	Farmer (1); Dice (7)	Farmer (1); Dice (2);
Graphical solving	Dice (1)	Dice (1)
Using parameters, symbolic solving	Any two numbers (17)	Any two numbers (1)
Using parameters, numerical solving; no parameters, symbolic solving; pattern	Any two numbers (8)	-
Numerical		
Systematic	Farmer (9); Dice (4); Books (1)	Any two numbers (3); Farmer (7); Dice (1)
Logical arithmetic reasoning	Farmer (3); Books (4)	Books (3)
Other less desirable strategies	Any two numbers (2); Farmer (1); Dice (6); Books (1)	Any two numbers (18); Farmer (18); Dice (12); Books (11)

Out of 114 identified strategies for solving these four problems, many of them are viable or desirable strategies in the sense that they would give the correct answer, and only 10 of them (9%) classified as "other less desirable strategies". As opposed to this, out of 99 participant anticipations of student approaches, the majority (59/99 or 60%) are less desirable strategies for these problems, while only 26 are viable strategies.

Our analysis shows that for each of the four problems, and across the distribution of strategies, the participants repeatedly anticipated that the students would use less viable strategies. This highlights that the participants expected students to use strategies which would give less effective, complete, or correct solutions.

Discussion and Conclusion

Our findings from this study demonstrate that many of the participants held low expectations for their future students' mathematical strategies for algebra problem-solving. Low expectations of students are problematic, because previous research has shown that teachers who have low expectations of students' mathematical ability manifest low achievement through the teacher expectation effect (Szumski & Karwowski, 2019). Educators working in initial teacher education might consider strategies to raise teachers' expectations for students' mathematical problem-solving.

Previous research has shown that teachers with low expectations for students can treat their students in ways that shape the students' respective achievements in mathematics (Ritzema et al., 2016). This literature highlights how teachers frame mathematical problems for students in the ways that reflect their expectations, describing some problems as achievable and others as less likely to be achieved by students (Fyfe & Brown, 2020). Teachers' expectations can be conceptualised in terms of lower or higher order anticipated strategies that students will use to solve prospective mathematical problems (Tohir et al., 2020). Using this conceptualisation of students' problem-solving approaches, we determined that the expectations of the participants in this study were low. The strategies that they anticipated the students would use were categorised according to their sophistication and suitability. We found that many of the participants expected less desirable strategies to be employed by their students. These expectations about less desirable approaches to problem-solving need to be addressed so that students are set up for success in problem-solving.

Assessing the expectations that preservice (and practising) mathematics teachers hold for their own and their students' approaches to mathematical problem-solving could be an important first step in addressing any deficit views these teachers may hold. Previous literature about teachers' mathematical expectations has suggested that high teacher expectations can improve students' academic outcomes and encourage successful approaches to problem-solving (Ritzema et al., 2016). Our findings add to this literature by showing that teachers may bring pre-existing ideas about students' low mathematical problem-solving abilities to initial teacher education courses. If academics working in initial teacher education institutions know that their preservice teachers do not have high expectations for their future students, intervention strategies might be developed and employed to address this problem. Existing literature highlights the capacity to raise teacher expectations through interventions that create awareness about expectations and the practices of high expectation teachers (de Boer et al., 2018).

The limitations of this study should be noted, including that exploring the participants' expectations for student mathematical problem-solving in algebra was not the original intent of the research that informed data collection. The participants wrote responses on a written test and had not been interviewed, so rich data with explanation of their expectations was not available for analysis. Thus, questions about how these views were developed and more in-depth understanding of the expectations is not provided here. Interviews explaining the preservice teacher responses could be a potential avenue for further research. Interviews could also uncover whether participants' responses were based on students' actual performance in algebra problems while they were on placements, and if/how this was related to any characteristics of the schools and classes they had been based in.

Teachers' self-concept about their own mathematical ability and their mindset (either growth or fixed) can impact the level of the expectation they hold for their students (Copur-Gencturk et al., 2023). While the participants in this study had high mathematical knowledge themselves, they had limited teaching experience. We would be interested to know more about how these preservice teachers have developed varying expectations of their students, and how these varying expectations would shape their practice. Future research might also consider further exploration of methods for evaluating preservice teachers' expectations for their students' mathematical problem-solving in algebra and might consider other expectations that they hold of their future students. Such research might explore how these expectations can be raised through interventions, when necessary. Interventions that raise preservice teacher expectations could lead to students' benefitting from targeted high teacher expectations in mathematics teaching contexts.

Acknowledgements

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References

Australian Institute for Teaching and School Leadership. (2017). *Australian professional standards for teachers*. AITSL. https://www.aitsl.edu.au/teach/standards

Copur-Gencturk, Y., Thacker, I., & Cimpian, J.R. (2023). Teachers' race and gender biases and the moderating effects of their beliefs and dispositions. *International Journal of STEM Education*, *10*, 31.

de Boer, H., Timmermans, A. C., & van der Werf, M. P. C. (2018). The effects of teacher expectation interventions on teachers' expectations and student achievement: Narrative review and meta-analysis. *Educational Research and Evaluation*, *24*(3–5), 180–200. https://doi.org/10.1080/13803611.2018.1550834

Fyfe, E. R., & Brown, S. A. (2020). This is easy, you can do it! Feedback during mathematics problem-solving is more beneficial when students expect to succeed. *Instructional Science*, *48*(1), 23–44.

Hatisaru, V., Stacey, K., & Star, J. (2024). Mathematical connections in preservice teachers' solutions to algebra problems. *Avances de Investigacion en Educación Matemática (AEIM)*, *25*, 33–55. https://doi.org/10.35763/aiem25.6354

Johnston, O., Wildy, H., & Shand, J. (2024). A grounded theory about how teachers communicated high expectations to their secondary school students. *European Journal of Psychology Education*, *39*, 211–235.

Lorenz, G., Gentrup, S., Kristen, C., Stanat, P., & Kogan, I. (2016). Stereotype bei lehrkräften? Eine untersuchung systematisch verzerrter lehrererwartungen [Stereotypes among teachers? A study of systematic bias in teacher expectations]. *KZfSS Kölner Zeitschrift für Soziologie und Sozialpsychologie*, *68*(1), 89–111.

National Council of Teachers of Mathematics. (2016). *High expectations in mathematics education. A position of the National Council of Teachers of Mathematics*. NCTM.

Ritzema, E. S., Deunk, M. I., Bosker, R. J., & van Kuijk, M. F. (2016). The relation between teacher-set performance goals and students' mathematical achievement. *Studies in Educational Evaluation*, *51*, 17–28.

Rubie-Davies, C., Peterson, E., Sibley, C., & Rosenthal, R. (2015). A teacher expectation intervention: Modelling the practices of high expectation teachers. *Contemporary Educational Psychology*, *40*(C), 72–85.

Soto-Ardila, L. M., Caballero-Carrasco, A., & Casas-García, L. M. (2022). Teacher expectations and students' achievement in solving elementary arithmetic problems. *Heliyon*, *8*(5). https://doi.org/10.1016/j.heliyon.2022.e09447

Szumski, G., & Karwowski, M. (2019). Exploring the Pygmalion Effect: The role of teacher expectations, academic self-concept, and class context in students' math achievement. *Contemporary Educational Psychology*, *59*, 101787. https://doi.org/10.1016/j.cedpsych.2019.101787

Tohir, M., Maswar, M., Atikurrahman, M., Saiful, S., & Pradita, D.A.R. (2020). Prospective teachers' expectations of students' mathematical thinking processes in solving problems. *European Journal of Educational Research*, *9*(4), 1735–1748.

Wang, S., Rubie-Davies, C. M., & Meissel, K. (2019). Instructional practices and classroom interactions of high and low expectation teachers in China. *Social Psychology of Education*, *22*(4), 841–866.

Learning Mathematics Through Sequences of Connected, Cumulative, and Challenging Tasks: A Self-Determination Theory Perspective

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The current paper overviews a nine-month PhD study that investigated the impact of learning mathematics through sequences of challenging tasks on the mathematical competence and attitudes of Year 2 students (n = 59). Adopting a Self-Determination Theory lens, a pragmatist paradigm and a mixed-method design, the study found that at all levels of investigation and analysis, the experience of learning through sequences of connected, cumulative, and challenging tasks had positive benefits for Year 2 students.

Substantial evidence suggests that teaching mathematics through problem-solving approaches improves student outcomes (Schoenfeld, 2007; Sinha & Kapur, 2021). However, debate exists as to how early in their schooling students should be exposed to and engage in problem-solving approaches as part of their learning. In Australia, this issue has received recent attention in terms of the overall decline in both academic performance and student engagement in mathematics (Thomson et al., 2021). One of the key arguments in delaying exposure to problem-solving approaches is based on the premise that students should first develop and be able to demonstrate aptitude with basic skills, facts and procedures, taught through traditional methods, before being introduced to more complex mathematics problems (Kirschner et al., 2006). This stance is founded on an assumption that without established foundational knowledge, students' disengagement in mathematics could be exacerbated as a consequence of the difficulties encountered through less direct instruction (Westwood, 2011).

Alternatively, others advocate that it is through problem-solving approaches that students not only construct strong foundational mathematics knowledge but also develop greater mathematics proficiency (Baroody et al., 2007; Schoenfeld, 2007). This perspective aligns with the stance that simply replicating mathematics procedures does not automatically transfer to an effective application of knowledge (Schoenfeld, 2007). Posing appropriately challenging tasks, orientated through student-centred inquiry pedagogies, has been shown to increase student engagement and motivation to learn mathematics with students of secondary school age (Boaler, 2016; Gresalfi et al., 2009). The active instructional moves used in structured inquiry pedagogies, such as guiding questions and the facilitation of rich classroom discussions, offered older students opportunities to develop reasoning skills, strategize beyond traditional methods, and collaborate with peers (Chan & Clarke, 2017). Given these benefits, there is a need to investigate whether these findings are replicated when younger students are engaged in problem-solving approaches implemented through structured inquiry pedagogies.

The current study was conducted as part of a larger research project led by Emeritus Professor Peter Sullivan and colleagues titled Exploring Mathematical Sequences of Connected Cumulative and Challenging Tasks (EMC^3). This project encouraged the introduction and implementation of sequences of challenging tasks to students in Foundation to Year 2 (5 to 8 years old) by supporting teachers to utilise the EMC^3 materials within their mathematics programs (Sullivan et al., 2020). The use of challenging tasks (Sullivan et al., 2015), is one approach to teaching mathematics that incorporates active engagement in problem-solving, explored through a structured inquiry pedagogy. Often posed as non-routine questions within authentic contexts, challenging tasks provide opportunities for students to build on their prior

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 279–286). Gold Coast: MERGA.

knowledge, demonstrate persistence, and develop mathematics connections by thinking flexibly about concepts (Sullivan et al., 2020). To date, much of the research on challenging tasks has focused on teacher professional development (Ingram et al., 2020; Sullivan et al., 2015) and student responses from the middle to high school years (Russo & Minas, 2020; Sullivan & Mornane, 2014). Less is known about the ways students in the early years of schooling (aged 5 to 8 years old) respond to the experiences of learning mathematics through challenging tasks. While Russo & Hopkins (2017) found that Year 1 and 2 students respond positively when self-reflecting on the use of challenging tasks in mathematics, overall research studies reporting on challenging tasks with children of this age is limited.

The aim of the current PhD study was to investigate how ongoing exposure to learning through sequences of challenging tasks influenced the mathematical competence and attitudes of Year 2 students; whilst the goal of the current paper is to overview the study and share key findings with the MERGA community. Readers whose interest is piqued by the contents of this necessarily short overview are encouraged to read the full dissertation (see Hubbard, 2024).

One overarching research question guided the study, focused on the holistic nature of student learning: *How do sequences of connected, cumulative and challenging tasks shape Year 2 students' experience of learning mathematics?* However, recognising the distinctive attributes unique to each of mathematical competence and attitudes towards challenge, the following two additional subsidiary questions further supported the inquiry:

- How does learning through the EMC^3 project approach support the development of Year 2 students' competence in mathematics?
- How does learning through the EMC^3 project approach influence Year 2 students' attitudes towards challenging tasks?

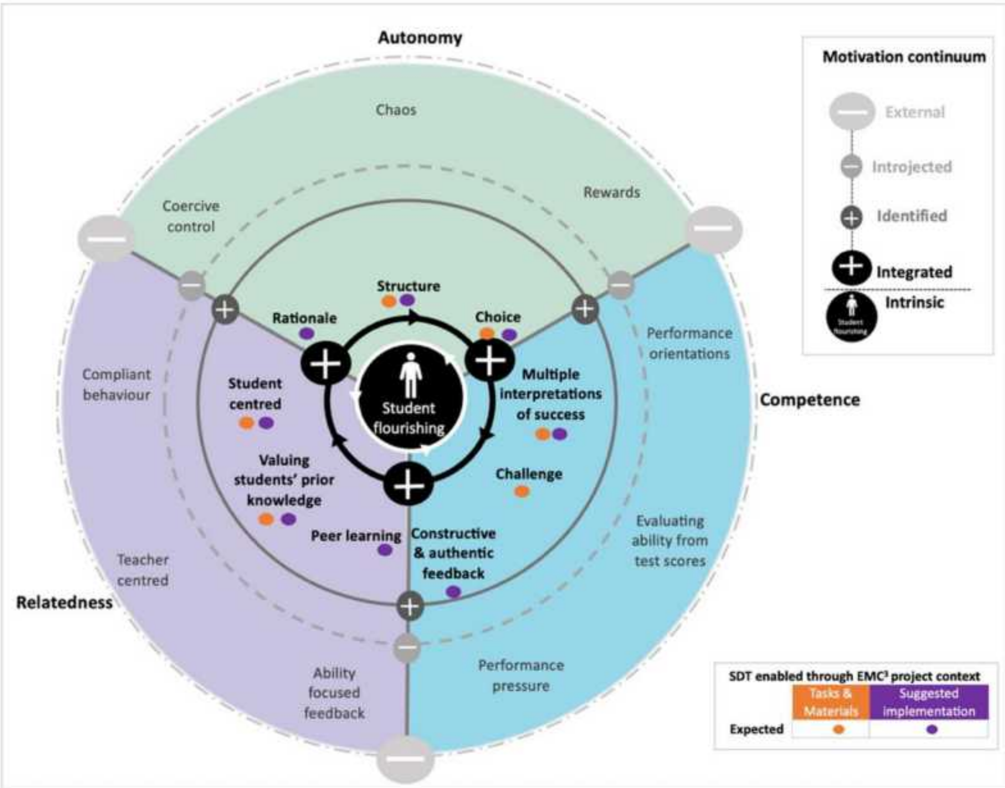
Theoretical Framework

The theoretical framework Self-Determination Theory (SDT) (Ryan & Deci, 2017, 2020) underpinned the current study. SDT is grounded in the belief that learning outcomes are likely to be positive when the basic psychological needs of autonomy, competence and relatedness are satisfied. These needs are dynamically interconnected whereby the satisfaction of one can in turn satisfy another. Autonomy is generally interpreted as having a desire to carry out particular actions with a sense of ownership. In a learning context, autonomy is satisfied when experiences align with interests or values and can be impeded when actions are externally controlled. Competence is satisfied through associated feelings of mastery, success, and growth. From this perspective, a sense of competence is framed as a process of ongoing development and effectiveness, rather than the attainment of a particular static outcome. Finally, the need for relatedness is underpinned by the desire to connect with others through feeling valued and accepted within social environments (Ryan & Deci, 2017, 2020).

The extent that these basic psychological needs are considered satisfied is dependent on a range of motivational tendencies that derive from positive or negative influences present within various learning conditions (Ryan & Deci, 2020). Interpreting student learning through this multifaceted lens aligns with key concerns of engagement and proficiency that are deemed critical in supporting mathematics development (Watt et al., 2017). The EMC^3 approach offered many prospective opportunities for students to develop a sense of competence, autonomy, and relatedness. In particular, the use of non-routine tasks where a solution is not immediately apparent challenged students to transfer prior knowledge to new situations, deepening conceptual understanding and supporting mathematical competence. Similarly, a sense of autonomy within the project approach manifested through, for example, students being offered choices in terms of how they would like to solve tasks and represent their thinking, while many of the features that constituted relatedness were addressed through the suggested lesson structure and proposed classroom norms. Therefore, aligning SDT and EMC^3 served the complementary purpose of developing holistic understandings of mathematics learning in order to better inform the implementation of challenging tasks in the early years of schooling. Drawing upon the SDT literature, Figure 1 was developed to show how the design features of the EMC^3 project are closely aligned to classroom conditions that are deemed integral for inducing positive motivational tendencies to enable student flourishing. It was proposed that using an SDT lens to investigate students' experiences as participants in the EMC^3 project creates an opportunity to develop greater insights into the ways learning through sequences of challenging tasks supports student flourishing in mathematics.

Figure 1

Explicit Theoretical Links Between the EMC^3 Project and SDT



Study Design

This PhD study was based on a pragmatist paradigm and utilised a parallel mixed-methods design to investigate the holistic experiences of Year 2 students as they learned mathematics through sequences of challenging tasks over a nine-month period. The quantitative component focused on the shifts in mathematical competence and attitudes towards challenge of the Year 2 cohort (n = 59) collected through a written mathematical assessment and attitude questionnaire. The process of analysing the written assessment responses resulted in the creation of a unique seven-point marking-key that not only identified different levels of content knowledge progression, but also showcased the problem-solving skills students utilised as they worked through the sequenced non-routine assessment items. The mathematical competence and attitude data from Phase 1 was combined to establish learning profile charts, representative of the overall influences that learning through sequences of challenging tasks had on the cohort.

Phase 2 comprised six focus students (n = 6) that were selected from the wider Year 2 cohort as participants for the qualitative component of this investigation. Monitoring the shifts in the focus students' mathematical competence throughout the study was determined in two ways. The first consisted of a one-on-one assessment context where observations about behavioural tendencies were documented noting the ways students worked through non-routine assessment items. The second interpretation of mathematical competence derived from classroom settings. Here, data collected from lesson observations, work samples, and interview responses were triangulated into learning artefacts that provided time bound insights into the dispositional behaviours students exhibited at different stages of challenging tasks lessons. Together, these insights enabled holistic interpretations of student progress that would not have been possible to ascertain from the analysis of single written responses or traditional assessment processes.

Throughout Phase 2, students were given a variety of opportunities to self-report their attitudes in relation to their experiences of learning mathematics within the EMC^3 context. The qualitative data collected included one-on-one response tasks, interviews, and lesson evaluation reflections, as well as behavioural observations documented throughout lessons and assessments. In adopting an iterative reflexive thematic analysis approach, it was possible to deduce clear interpretations about the ways students' attitudes towards challenging tasks were influenced over the nine-month study.

Further to this, additional in-depth case studies ($n = 2$) were investigated and reported based on the findings from the initial Phase 2 qualitative analysis. These students were selected according to their Phase 1 learning trajectory, developed by comparing their learning profiles to the Year 2 median. One of the affordances in compiling and reporting on these individual case studies in such depth was obtaining insights into the reflexive relationships that emerged between mathematical competence and attitudes that had not been possible to conclude through the other analysis processes conducted initially.

Summary of the Findings

Mathematical Competence

Overall, the reported findings concluded that at all levels of investigation and analysis, the experience of learning through sequences of connected, cumulative, and challenging tasks had positive benefits for Year 2 students. In Phase 1, the quantitative findings reported substantial evidence that students developed basic mathematical knowledge and facts. Furthermore, the majority of students demonstrated improved problem-solving skills, and were able to demonstrate flexible thinking and enhanced reasoning skills. The process of creating a written assessment instrument that simulated the actual learning experiences of students generated insights into broader interpretations of mathematical competence that would not have been possible to achieve through the use of traditional style assessments. The quantitative analysis that derived from the seven-point marking-key codes not only enabled specific learning progressions to be identified, but also offered clear interpretations about student efficacy, cumulative progress, as well as future learning needs relating to higher-order thinking skills (see Hubbard et al., 2022).

The Phase 2 qualitative analysis expanded on the Phase 1 findings, by including the dispositional behaviours students exhibited throughout the EMC^3 learning experiences. This provided critical insights into the development of students' problem-solving proficiency and reiterated there is much to interpret in-situ rather than relying on the evaluation of written output. Observing these behaviours throughout both individual assessment settings and lesson contexts generated holistic interpretations of mathematical competence that better explained how students utilised their mathematical knowledge and skills, rather than simply what they knew. The most informative of these processes occurred as a result of establishing learning artefacts which reflected different elements of mathematical competence that students demonstrated at various stages of learning. Such findings showed that when learning through structured inquiry approaches, students were cognisant of using particular strategies and skills at different times, demonstrating how their thinking shifted as the lesson progressed. It became clear that there were distinct patterns between the dispositional behaviours students demonstrated and the comprehensiveness of their written responses when problem-solving. Accounting for these behaviours has not traditionally formed part of previous mathematics assessment processes (Hubbard, 2023).

Self-Reported Attitudes Towards Challenging Tasks

Through both phases of the study, the Year 2 students' self-reported mostly positive attitudes towards challenging mathematics experiences. The quantitative analysis showed there was some evidence of improved attitudes towards challenging tasks, with the majority of students consistently reporting positive attitudes throughout each data cycle. However, the limited nature of the data collection process (student questionnaire) in this phase prohibited further conclusions to be drawn.

The qualitative analysis revealed that while the focus students' attitudes towards challenging tasks remained positive throughout the study, there were distinct shifts in the motivational origins that influenced these changes. Students reported satisfaction in learning to identify and overcome mistakes and could articulate how such experiences improved their mathematical understanding over time. The learning conditions created through ongoing and consistent implementation of the EMC^3 project approach supported students to broaden their parameters as to what worthwhile and positive learning experiences entailed. As such, by the end of the study the focus students recognised that positive experiences for learning encapsulated more than superficial interpretations of success and fun.

The Interconnectedness of Learning Pathways When Problem-Solving

Synthesising the findings through the lens of SDT provided a means to authentically combine mathematical competence with student attitudes enabling the learning pathways of students to be interpreted holistically. Using this framework, it was possible to identify specific turning points in students' learning experiences that explained the trajectory of their learning over the nine-month study. This process reinforced the notion that there were multiple pathways for students' mathematical improvement which were contingent on more than the evaluation of mastery of content knowledge.

The in-depth case studies reported in Phase 2 of the study demonstrated that even when students presented as having similar learning needs according to written assessments, the conditions that proved influential in their eventual progress varied significantly. Using the SDT framework it was possible to track not only the ways students' basic psychological needs were being satisfied through their learning experiences, but also to identify the motivational tendencies that influenced students' initial engagement. Especially noteworthy was that for both of the in-depth cases, the catalyst for overall mathematical improvement emphasised the enabling conditions that acknowledged and validated their unique preferences as learners. These insights are invaluable in better understanding how mathematics instruction can be accurately tailored to effectively support all learners to improve their capacity as problem solvers in ways that are holistic and student-centred. Collectively the findings summarised from this research raise several implications for future research and practice that are presented next.

Conclusions, Future Research and Implication for Practice

This current paper has served to overview a nine-month PhD study that investigated the experiences of Year 2 students as they learned mathematics through sequences of challenging tasks (see Hubbard, 2024). While the conclusions derived from this dissertation contribute to an understanding of the ways problem-solving approaches can be effectively implemented within the early years of schooling, it is also recognised they are representative of only a single study conducted within a single context. As such, suggestions for further research that expand on the findings from this study are offered.

Aligning the SDT literature (see Ryan & Deci, 2017, 2020) with the EMC^3 project (Sullivan et al., 2020) to explicitly highlight theoretical links (see Figure 1) offered a unique means of tracking student learning pathways that accounted for their internalised motivation, attitudes towards challenge, and mathematical competence. There is considerable scope to conduct further research using this proposed framework by attending to students that present with alternative learning profiles to the ones presented in this study. Such analysis would enable greater insights into the learning conditions that support students' self-determination satisfaction and enable teachers to more appropriately meet the diverse needs students present when they are learning mathematics through problem-solving approaches.

Similarly, the proposed framework enabled students' attitudes towards challenging tasks to be interpreted through the theoretical lens of the motivation continuum. Identifying if students' internalised motivation for learning mathematics was positively or negatively orientated provided an additional perspective through which their experiences within the EMC^3 project could be contextualised. Given this was reported in-depth only for the two student case studies, further research is recommended to verify:

- Whether such analysis processes are feasible at a larger scale and applicable in other contexts (e.g., upper primary school or secondary school);
- The extent that this process offers insights into students' attitudes towards mathematics that would otherwise remain undiscernible.

There were many findings that became apparent in only the latter stages of the study, such as the emergence of students' higher order thinking skills and the noticeable demonstrations of their productive behaviours for problem solving. Replicating this study but extending the timeframe to include longitudinal data would offer the potential to gain greater insights into these two aspects of student learning through problem-solving approaches. Additional advantages of designing a longitudinal version of this study would be in the inclusion of system wide assessment data for comparative purposes.

Finally, the assessment processes described in this thesis were constructed and refined as part of this particular study design and analysis approach. Therefore, they are considered novel and emergent in terms of their overall utility beyond the context of this investigation. Further research focusing on the assessment processes and structures generated through this study should endeavour to:

- Determine the suitability of adapting the written assessment instrument and marking-key codes beyond a Year 2 context;
- Evaluate the utility of these instruments and processes beyond the EMC^3 context;
- Verify and refine the dispositional competence elements that are derived from the classroom context in order to develop practical observation guides with the aim to support teachers in developing and using holistic formative assessment practices.

This research also raises important implications for both policy and practice when considering the affordances and constraints in introducing and sustaining effective problem-solving approaches in the early years of schooling. Although it is beyond the scope of the current paper to detail all such recommendations, two key sets of implications relevant for education systems, schools, and teachers are outlined below. The first set pertains to the nature of student learning experiences in mathematics, while the second set concerns corresponding assessment processes.

First, to establish and maintain supportive environments conducive to meeting students' needs at various points throughout their learning trajectory, it is recommended that an ongoing commitment is made to regularly and consistently provide opportunities for students to experience authentic problem-solving tasks as part of a balanced mathematics instruction. It is suggested that these problem-solving tasks are implemented through structured inquiry pedagogies that enable teachers to scaffold the learning experience in response to students' prior knowledge, responses to feedback, and collaborative class discussions. When interacting with students within such structures, teachers should strive to be constructive rather than evaluative when responding to students' thinking. Moreover, given that it was the ongoing and consistent exposure to learning through the EMC^3 approach that appeared to underpin the recalibration of students' attitudes towards challenging tasks, it is recommended that students of all abilities are provided with appropriate levels of mathematical challenge. This should include all students being given adequate time and space to adjust to working through a challenging task without direct instruction from the teacher. Doing so facilitates authentic opportunities to develop persistence and resilience as part of mathematics learning. These first-hand experiences offer scope for students to develop a better appreciation of what concepts such as growth mindset actually entail.

Second, to move away from outcome focused measures of achievement and towards more holistic interpretations of student progress, it is recommended that tiered assessment approaches are adopted that better reflect the holistic nature of problem-solving learning processes. In particular, when written assessments are used, they should preference the use of open-ended and non-routine items inclusive of problem-solving strategies to ascertain an accurate interpretation of the broad skills students apply and demonstrate as part of comprehensive written responses. When designed in this way, the instruments offer increased longevity and relevance to student learning that expands beyond evaluations of content knowledge attainment. As an alternative to written assessments, teachers should adopt lesson observation protocols that detail the nuanced but important productive behavioural tendencies students are likely to exhibit throughout problem-solving experiences. Utilising these observation protocols regularly will attune teachers into better recognising the range of behaviours students demonstrate at different stages of their learning process. It is important that such observation protocols aim to represent the variety of ways students may behave when solving problems to include both individual and collaborative experiences that occur frequently throughout structured inquiry pedagogies. Having a greater sense of what behaviours are productive and when serves to better inform the instructional choices of teachers in-situ, resulting in teaching practise that is more responsive to students' learning needs.

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References

Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. Journal for Research in Mathematics Education, 38(2), 115–131. https://doi.org/10.2307/30034952

Boaler, J. (2016). *Mathematical mindsets: Unleashing students' potential through creative maths, inspiring messages and innovative teaching*. Jossey-Bass.

Chan, M. C. E., & Clarke. D. (2017). Structured affordances in the use of open-ended tasks to facilitate collaborative problem-solving. *ZDM Mathematics Education*, *49*, 951–963. https://doi.org/10.1007/s11858-017-0876-2

Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational Studies in Mathematics, 70,* 49–70. https://doi.org/10.1007/s10649-008-9141-5

Hubbard, J. (2024). *Year 2 students' self-determination when learning mathematics through sequences of connected, cumulative and challenging tasks* [Unpublished doctoral thesis]. Monash University.

Hubbard, J. (2023). Mathematical competence exhibited by Year 2 students when learning through sequences of challenging tasks. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 275–282). Newcastle: MERGA.

Hubbard, J., Russo, J., & Livy, S. (2022). Assessing mathematical competence through challenging tasks. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 282–289). MERGA.

Ingram, N., Holmes, M., Linsell, C., Livy, S., McCormick, M., & Sullivan, P. (2020). Exploring an innovative approach to teaching mathematics through the use of challenging tasks: A New Zealand perspective. *Mathematics Education Research Journal*, *32*(3), 487–522. https://doi.org/10.1007/s13394-019-00266-1

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist, 41*(2), 75–86.

Russo, J., & Hopkins, S. (2017). Student reflections on learning with challenging tasks: "I think the worksheets were just for practice, and the challenges were for maths". *Mathematics Education Research Journal, 29,* 283–311. https://doi.org/10.1007/s13394-017-0197-3

Russo, J., & Minas, M. (2020). Student attitudes towards learning mathematics through challenging, problem-solving tasks: "It's so hard-in a good way". *International Electronic Journal of Elementary Education*, *13*(2), 215–225. https://doi.org/10.26822/iejee.2021.185

Ryan, R. M., & Deci, E. L. (2017). *Self-determination theory: Basic psychological needs in motivation, development and wellness*. The Guilford Press.

Ryan, R. M., & Deci, E. L. (2020). Intrinsic and extrinsic motivation from a self-determination theory perspective: Definitions, theory, practices, and future directions. *Contemporary Educational Psychology*, *61*, 1–11. https://doi.org/10.1016/j.cedpsych.2020.101860

Schoenfeld, A. (2007). What is mathematical proficiency and how can it be assessed? In A. Schoenfeld (Ed.), *Assessing mathematical proficiency* (pp. 59–74). MSRI Publications & Cambridge University Press. https://doi.org/10.1017/CBO9780511755378

Sinha, T., & Kapur, M. (2021). When problem-solving followed by instruction works: Evidence for productive failure. *Review of Educational Research*, *91*(5), 1–38. http://doi.org/10.3102/00346543211019105

Sullivan, P., Askew, M., Cheeseman, J., Clarke, D., Mornane, A., Roche, A., & Walker, N. (2015). Supporting teachers in structuring mathematics lessons involving challenging tasks. *Journal of Mathematics Teacher Education*, *18*(2), 123–140. https://doi.org/10.1007/s10857-014-9279-2

Sullivan, P., Bobis, J., Downton, A., Hughes, S., Livy, S., McCormick, M., & Russo, J. (2020). Ways that relentless consistency and task variation contribute to teacher and student mathematics learning. In A. Coles (Ed.), *For the learning of mathematics: Proceedings of a symposium on learning in honour of Laurinda Brown: Monograph 1* (pp. 32–37). FLM Publishing Association.

Sullivan, P., & Mornane, A. (2014). Exploring teachers' use of, and students' reactions to challenging mathematics tasks. *Mathematics Education Research Journal*, *26*, 193–213. https://doi.org/10.1007/s13394-013-0089-0

Thomson, S., Wernert, N., Buckley, S., Rodrigues, S., O'Grady, E., & Schmid, M. (2021). *TIMSS 2019 Australia. Volume II: School and classroom contexts for learning*. Australian Council for Education Research. https://doi.org/10.37517/978-1-74286-615-4

Watt, H., Carmichael, C., Callingham, R. (2017). Students' engagement profiles in mathematics according to learning environment dimensions: Developing an evidence base for best practice in mathematics education. *School Psychology International*, *38*(2), 166–183. https://doi.org/10.1177/0143034316688373

Westwood, P. (2011). The problem with problems: Potential difficulties in implementing problem based learning as the core method in primary school mathematics. *Australian Journal of Learning Difficulties*, *16*(1), 5–18. https://doi.org/10.1080/19404158.2011.563475

Riding the Wave of COVID-19: The afterMATH

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In this paper, we present research on New Zealand secondary teachers' perspectives of teaching and learning mathematics during the COVID-19 pandemic of 2020–2023. We use narrative analyses of interviews to explore the perspectives of eight participant teachers. COVID-19, and its resulting restrictions, were found to impact student-teacher relationships, teachers' ability to provide mathematical explanations and support students' learning, and students' engagement and participation.

In this paper, we focus our attention on eight New Zealand secondary mathematics teachers' perspectives of the impact the COVID-19 pandemic, and the resulting restrictions, had on teaching and learning mathematics. Learning mathematics is a social practice and is unique in terms of its nature and importance, as well as students' relationships with the subject (Ingram, 2011). We describe how the unique features of school mathematics led to specific and unique challenges for mathematics teaching and learning during that period. Therefore, we shed light on why these features of mathematics are essential characteristics, and how teachers can best address them in the context of ongoing secondary mathematics teaching and learning.

With the COVID-19 pandemic raging in other parts of the world, and with the government pursuing an elimination strategy, New Zealand closed its borders in March 2020 and, soon after, a strict lockdown was announced (Ministry of Health, 2021). This noon announcement closed schools immediately and teachers and students were sent home to navigate a long, initial, 8-week lockdown, as well as subsequent lockdowns. For the next two years, despite the borders remaining closed until 2022, schools emerging from lockdown needed to operate through restrictions and increasing sickness in their school communities. During that period in New Zealand, an infected person had to self-isolate for 14 days with their household contacts. If someone else in the household contracted the virus, all had to start on day one again, meaning that teachers and students could be absent for weeks or even months (Ministry of Health, 2021). Waves of COVID-19, and resulting impacts within schools, continue at the time of writing.

It was within this COVID-19 pandemic context that our main research project was situated. In mid-2023, we constructed a narrative of teaching and learning during the COVID-19 pandemic from the perspectives of mathematics and drama teachers in New Zealand. We were interested in their perspectives on the ongoing educational cost of the pandemic, and the differences and similarities between drama and mathematics teaching and learning. All teachers described their work, their challenges, and accomplishments. They were flexible and creative in their practices and focused on their students' well-being and learning. All teachers were impacted professionally and personally by the pandemic. Here, we focus on the mathematics teachers' perspectives. We first explore the recent educational research related to COVID-19. We then outline the unique features of mathematics teaching and learning and what they mean for mathematics teaching practice. Then, we explain and justify our narrative inquiry research design. Our findings are discussed regarding the relevant literature, and conclusions reached.

The Impact of COVID-19 on Teaching and Learning

Despite the COVID-19 pandemic being so recent, there is a body of research that explored this period, emerging from around May 2020 (e.g., Capurso, 2020). These accounts offer a window into the beliefs, values, and attitudes of researchers and participants while situated within the pandemic context and provide insights into how school communities evolved.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 287–294). Gold Coast: MERGA.

When countries went into lockdown and schools needed to move to online teaching, a range of barriers to teaching and learning emerged. There was the visceral uncertainty and distress caused by the pandemic and people's individual lockdown situations. The uncertainties of the pandemic and the fear of contracting the virus took a toll on the well-being of teachers and students and added to an already challenging school context (d'Agnese, 2023). During lockdown many schools prioritised students' mental health, and when schools re-opened, this emotional support needed to be balanced with students' academic recovery. This added to teachers' responsibilities and, in turn, further impacted teachers' own well-being (Capurso et al., 2020). Furthermore, the initial lockdowns and the subsequent years of sickness and restrictions are considered to have had a profound impact on students' educational outcomes as well as teachers' professional lives (Chaaban et al., 2021; McCluskey et al., 2021). There has been a particularly negative educational impact on those students who were already socially isolated, traumatised, or educationally disadvantaged in some way (Mutch, 2021).

Most schools required their teachers to move online for their teaching. At least initially, many teachers did not have the technological knowledge and skills necessary for working online, and there was unequal access to digital devices and good quality internet connections. Furthermore, there was relatively low student engagement and participation, and a range of other issues such as students' lack of camera use (Joseph & Trinick, 2021). These barriers made connections difficult between teachers and students and impacted learning (Akojie et al., 2022). For example, Walters et al. (2022), surveying 462 Welsh secondary school students found that student engagement, concentration, and ability to learn were significantly lower.

The range of barriers to teaching and learning related to COVID-19 described in the general education literature was also evident in research that specifically explored mathematics. For example, Noviani (2021) described internet access, time management and eyestrain as general barriers to learning. However, they also noted that students' reluctance to ask for help was a particular barrier to learning in mathematics during the COVID-19 period. Similarly, in the research of Mukaka et al., (2023), as well as confirming the pragmatic barriers to teaching during COVID-19, they concluded that mathematics is best taught when there is face-to-face guidance from the teacher. These factors are both related to students' need for individual interaction and support from the teacher and are particularly relevant to existing research that describe mathematics as a social practice with a unique nature, which will be explored now.

Learning Mathematics

Learning school mathematics is a social practice that is situated in the mathematics classroom (Op 't Eynde et al., 2006). Students need opportunities to discuss mathematics, and seek help from others (Ingram, 2011). An effective learning environment and positive relationships within the classroom community are therefore important as well as the co-construction of classroom social norms and socio-mathematical norms specific to the mathematical activity. Some examples of these norms are what constitutes a mathematical explanation, how a problem should be solved, and how students and teachers should interact during the classroom activity (Yackel & Cobb, 1996). Given the barriers of the online teaching environment necessitated by the COVID-19 pandemic, it would be a difficult context to enable discussion of students' emerging mathematical ideas, and it would be difficult for students to seek and get help, as found in Noviani's (2021) and Mukaka et al.'s (2023) research.

Furthermore, mathematics is a unique school subject. According to longitudinal research of secondary students' relationships with mathematics (Ingram, 2011), mathematics is more 'thinky', more important, more difficult, and more disliked than other subjects. Furthermore, many students described their mathematical knowledge as 'a tower'. In other words, they saw mathematics knowledge as cumulative and their learning of mathematics as a process of construction building on their existing knowledge, skills and understandings (Hantano, 2013).

Effective mathematics teaching therefore includes the provision of strong explanations and a range of strategies to support students' learning (Anthony & Walshaw, 2009). Furthermore, because of the cumulative nature of mathematics, it is particularly difficult to catch up on when a student is disengaged or is absent and they have a gap in their knowledge (Ingram, 2011). Lastly, students' engagement in mathematics—their doing of it—is synonymous to learning (Op 't Eynde, 2006). Given these perceptions and understandings, students learning mathematics may not have been resilient to the barriers they faced during the pandemic.

In this research, we were therefore interested in exploring mathematics teachers' perceptions of teaching and learning during the pandemic. We were attentive to the unique nature of teaching mathematics, and the necessity to support students' learning and engagement in the subject through participation and discussion, which were impacted during the COVID-19 pandemic. Our research questions were:

- What were the mathematics teachers' recollections of teaching during the COVID-19 pandemic?
- How did mathematics teachers support students' learning during the COVID-19 pandemic?
- How were students' engagement and participation in mathematics impacted by the COVID-19 pandemic, from the mathematics teachers' perspectives?

Research Design

To capture mathematics teachers' experiences and perspectives of teaching during the pandemic, we were informed by a narrative inquiry approach (Clandinin & Caine, 2013). Narrative inquiry draws from oral history, drama, psychology, and folklore traditions. Individuals construct their reality through the telling of stories (Chaaban et al., 2021). We lived alongside our participants as they shared their stories of the COVID-19 period.

Within the main research project, there were 14 volunteer participants recruited by advertising through subject associations. Eight of these were mathematics teachers, who are the focus of this paper. The mathematics teachers' ages ranged between 33 and 58 and they came from a range of rural and urban schools throughout New Zealand and had a range of years of teaching experience from 8 to 35 years. Pseudonyms have been used to protect their identity.

The participants shared their stories in interviews using Zoom either as an individual or in a pair with a drama teacher. The interviews were conversations with a purpose, semi-structured and informed by the research questions. The mathematics teachers were asked about how they prepared to teach during the enforced lockdown due to COVID-19, and their perspectives on how the pandemic impacted their mathematics teaching, and their students' learning.

The analysis occurred in four stages. The interviews were transcribed using a transcription tool, Otter, and then listened to as the transcripts were checked for accuracy. These transcripts were uploaded into a qualitative software package, NVivo, and the data set was coded into categories. Codes were changed, consolidated, and reduced as understanding grew of the data. Through constant comparison, the analysis moved to and fro between the general and the specific. Once main categories were formed, these categories were developed further and gradually 'teacher-student relationships', 'providing explanations', 'supporting students' and 'engagement and participation' emerged. Informed by narrative inquiry methodology (Clandinin & Caine, 2013), the data were ordered into a chronological narrative, structured by the time periods, and informed by the categories. The findings begin with this narrative. This narrative then provides the context for the specific focus on how teachers supported their students' learning and their perspectives on students' engagement and participation.

Findings

When lockdown was announced and mathematics teachers were sent home from school, they needed to scramble to get ready for online learning. Fern "broke the rules, going back into a building to pick up a bunch of textbooks that [they] could start scanning". Alex was allowed back into his school and picked up workbooks, textbooks, paper, and a small easel whiteboard. Sam also "stole a whiteboard at the last minute". Frankie had no idea what to take home and ended up filling "half her car". Sam had no power cord so her colleague "bashed in and picked it up". These stories account for the stress the teachers were under as the country went into full lockdown. This stress was particularly apparent for teachers who had little or no experience with online learning. For example, Reese was "panicking—I didn't know this stuff".

The teachers began to adapt their teaching and programmes for the lockdown. Their students did not have equal access to devices or consistent internet connection and, therefore, materials were delivered. For the students able to be online, the teachers used a variety of online tools, including Zoom, software from the Google suite (e.g., Meet, Classroom) or the Microsoft suite (e.g., OneNote, TEAMS). Fern switched her junior mathematics classes to measurement so students could do measuring tasks around the home. For her senior programme, Frankie removed several assessments, and Riley changed her topic to algebra because she perceived it would be easier to teach, although she "watered down what [she] would normally do". The data set suggests, that due to the cumulative nature of mathematics, the students were not receiving the same mathematical content that they would normally get during this period, which would impact their mathematical knowledge and preparedness for later learning.

Returning to face-to-face teaching after the lockdown was stressful for students and teachers due to a combination of unprecedented complexities, in particular, the COVID-19 restrictions such as physical distancing and mask-wearing. Furthermore, as the sickness spread throughout New Zealand, and students needed to stay home due to restrictions, absences were prolific. "My department got sick, I got sick, the kids were always sick" (Reese). The teachers described being overwhelmed by the "messy" situation (Sam). For example, Fern described the teachers needing to provide "in-class tuition for everyone who was there ... also the online stuff for all the kids who weren't in the room, and … the staff who weren't there". Fern's stress as she described having to cope with multiple elements during this period was still evident three years later when interviewed for this project. Like many teachers during the COVID-19 pandemic, Fern had a very young child at the time and was herself concerned about bringing the sickness home, attested to her professionalism and hard work during this period, especially as it came immediately after needing to teach from home during the stressful lockdown period.

This brief narrative of the teachers' experience during the pandemic has similarities to the literature related to teaching online. The mathematics teachers found creative solutions to keep students connected despite the distance. They utilised resources in the students' homes (Yates et al., 2021), they delivered materials directly to students to help ameliorate unequal access to online learning (as in Akojie et al.'s research, 2022), and they used digital tools seen in other research (e.g., Joseph & Trinick, 2021). The literature is around online teaching during the lockdown period—this research further describes the teachers' preparation for lockdown and the sheer chaos of coming back to school as sickness, and absences, grew.

The teachers' descriptions of teaching and learning mathematics during the pandemic provide context for the findings related to specific aspects of mathematics teaching. These include teachers' relationships with students, providing mathematical explanations, supporting students' learning, and supporting students' engagement and participation in mathematics.

Teachers' Relationships With Students

Student-teacher relationships were under pressure during the COVID-19 pandemic. Already dealing with the difficult situation of the impending illness, relationships were further compromised due to the pressure of online learning during an emergency lockdown, and the unequal participation due to devices and internet connection. Furthermore, many schools did not require students to have their cameras on because of concerns about student privacy and the quality of the internet connection. When the students came back to school, the restrictions hindered the ability of teachers and students to rebuild positive face-to-face relationships. It was especially "hard to get to know the students" because of the physical distancing and mask-wearing and because the students had begun the year only a few weeks before lockdown.

The teachers in the study knew relationships were important for their students' well-being and mathematics learning. Frankie explained, that if a mathematics teacher has a good relationship with their students, they are "more open to learning harder concepts" and do not have "walls built up as much … and are more willing to be challenged and more willing to try new things because they trust that you can get them through it". They therefore worked hard to continue to build their relationships with the students during the pandemic. For example, during lockdown, Sam introduced a "weird, funny, Friday". Frankie put "on a great show for the kids … because we have to teach maths to kids who don't like maths". Alex used "little things to catch their attention such as figurines, stuffed toys—a comic relief sort of thing". Reese encouraged her students to arrive early for their online classes so they could chat socially on Zoom. Fern explained one of the most useful adaptions she made was non-content related sessions such as "a lot of silly, let's paint on our selfies kind of stuff that kept them connected".

A clear theme was that the participants in the study underlined the need for collaboration and interaction among mathematics teachers and students, to foster a more relational and engaging online learning environment (d'Agnese, 2023; Joseph & Trinick, 2021). The COVID-19 pandemic, both during the online period and after schools resumed, made relationship-building complex and difficult. Consequently, student's mathematical learning and teacher-student relationships were compromised. The teachers' perspectives reinforced that having positive student-teacher relationships make a difference to the learning environment, to the students' well-being and to the quality of the students' mathematical learning (Yackel & Cobb, 1986).

Providing Mathematical Explanations

The mathematics teachers sought ways to provide mathematical explanations for their students. All attempted to make short content videos by recording their working in real-time as they talked. Fern described the awkward first days of attempting to film herself writing on a whiteboard with her phone using two cameras, so her students were "able to see me write and talk at the same time" (Fern). Other teachers used a camera attached to a computer, a camera focussed on the now precious 'stolen' whiteboards or asked their partners to hover their phones over a piece of paper as they wrote. Sam found a Screen Casting app on a tablet useful. The teachers either shared these videos in real-time and/or uploaded the recordings for the students. The whiteboards were essential for these teachers, and they all attested to their creativity in trying to capture their explanations. Except for Sam, they did not have access to a screen-casting app, or perhaps did not know about these apps, which are useful for recording lessons, students' problem solving and mathematical working (Ingram, et al., 2018).

Furthermore, the analysis illustrates teachers and students co-constructed new norms for working in the online environment. In Alex's class they developed class norms for working together online. "[The students] developed their own, essentially sign language for answering questions, because we were doing algebra. They came up with things for factorise, bracket, x, and square. It was a bit easier … because … you can't have everyone talk at the same time on video". These findings highlight how challenging it was to teach mathematics effective if the students' cameras were turned off, and attest to the importance of co-constructing a learning environment and sociomathematical norms in the classroom (Yackel & Cobb, 1986).

Supporting Students' Mathematical Learning

New ways needed to be developed to ensure students could get feedback and support from their teachers and peers about their mathematical learning. Frankie's students were asked to submit or share work by taking screenshots and posting photos or engaging in online programmes. Fern asked the students to use their cameras to share their working by holding it up to the camera to get immediate help. She noted the students who needed the most help often were the ones most reluctant to hold their work up and was frustrated by this. "In a normal class I'd make sure I was side-by-side with these students to ensure they were getting the support they needed". This is an interesting comment. Often, we talk about face-to-face learning, but in Fern's case she needed more than that—she needed to be side-by-side.

Not all students were reluctant to ask for support. Jo's school, which only offered asynchronous teaching, encouraged students to be in frequent contact with their teacher. This was difficult because the students had an expectation "they could contact [her] any hour of the day and night, and [the teacher] should respond immediately" (Jo). Sam described her senior students as "needy" because they seemed to have learned helplessness in their constant questions and their need for support to work through mathematics.

Even when school-based mathematics classes resumed after lockdown, the requirement for students and teachers to wear masks, and the need to be operating at a distance from each other, impacted their face-to-face mathematical learning. Fern could still could not get side-by-side students to support their learning. Sam explained that the mask wearing directly affected the students' learning of the mathematical content. She noticed some students:

> Struggled to hear … and understand what's going on. I use lip reading … when I teach. It is possible that the students use lip reading without realising it, so it would be much harder for them to comprehend and connect to take in what we are doing while wearing a mask. (Fern)

Indeed, Sam was surprised at how much she needed to read students' facial expressions as she was teaching to support her formative assessment and decision-making about her teaching. "You can't read their expressions. I didn't realise I get so much from looking at a student. It's really hard … without seeing them". Fern commented that "trying to read a confused face on a kid wearing a mask" was problematic. Reese agreed. "You realise how much you rely on reading the room because they're all behind a mask, and you can only see their eyes … yeah, that was hard". Much of the literature related to teaching and learning during the COVID-19 pandemic was related to online learning during lockdown.

There is little research available about mask-wearing when teaching face-to-face. Mathematics teachers, are constantly needing to make decisions as they teach, based on their minute-to-minute formative assessment, the feedback they get from students, the questions they are asking and the difficulties they are experiencing. It is clear from our data analysis that the COVID-19 period made some commonly relied-on teaching techniques difficult. The complexities of learning mathematics while wearing masks may also have reinforced some students' perceptions of mathematics as a difficult and disliked subject (Ingram, 2011). The need for the mathematics teachers to be physically side-by-side their students was apparent.

Students' Engagement and Participation in Mathematical Learning

According to the participants in our study, students' engagement in mathematical learning and absences from class were a major problem throughout the years of the pandemic. Some students were able to get into a routine of online learning during lockdown, making the space and time work for them. Also, despite the restrictions, some students were able to connect with the school environment when schools re-opened. All teachers described a large proportion of their students, however, who did not engage fully. They did not engage in discussion; they did not ask for help, and they did not consistently submit work. Some were not able to participate at all due to power or internet connections. Some parents did not want their children to be online for extended periods. Sam's school was in a rural area and many of her students were expected to help on the farm. Alex's school did not expect teachers to hold absent students to account due to concerns about their well-being. When schools resumed, absenteeism became even more rife as communities got sick and people were expected to stay home by law.

The teachers believed the students' well-being contributed to students' absences and engagement, as they "watched students' mental health decline" (Fern). This had an impact on their learning and their ways of working. Jo thought that "kids seem to have lost the importance of education". They were "very slow … it's a real struggle to get them through what we need to cover … and to get them to have a sense of urgency about completing tasks" In Alex's view, his students struggled with self-motivation and retaining mathematical content knowledge.

Many students were behind in their schoolwork, which further impacted their anxiety levels. Missing content in mathematics is a "much bigger deal" (Fern) because of the nature of the subject. Alex found that, after lockdown, some students, who "had not engaged at all, … were so far behind … [and] they didn't want to admit they were behind". Alex explained it was additionally hard to maintain relationships with students who were so far behind and feeling anxious about it. This was further exacerbated because teachers were asked to be more lenient to students about absences and assignment deadlines and therefore the return to any expectations that they would complete work by deadline dates would have been difficult. This is consistent with literature that describes the issues around leniency for work output expectations during the COVID-19 pandemic (e.g., Joseph & Trinick, 2021; Yates et al., 2021).

Conclusion

Teachers needed to deliver mathematics as effectively as possible throughout the COVID-19 pandemic of 2020–2023, throughout the initial lockdown, subsequent lockdowns when schools resumed, and throughout the ensuing sickness, restrictions such as enforced absences, masks, social distancing, and sanitation requirements. The teachers in our research described the ways they were limited in their ability to deliver mathematics effectively during that period in terms of their relationships with their students, their ability to provide explanations, and their support of the students' learning. They also described how their students' engagement and participation in mathematics were impacted.

These are the perspectives of only eight mathematics teachers in New Zealand—a limited number necessarily pragmatic because of the pandemic. However, richness is provided through descriptions of the barriers to teaching mathematics at the beginning of lockdown, in the stressful days before, and coming back to school. The challenges caused by the pandemic serve to accent the importance of the essential elements of mathematics teaching—having strong student-teacher relationships, co-constructing a learning environment, having strong student engagement, and teachers being side-by-side with students to support their learning.

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References

Akojie, P., Laroche, I., & Schumacher, J. (2022). Moving from face-to-face instruction to virtual instruction in the COVID-19 pandemic: Narratives of K–12 teachers. *American Journal of Qualitative Research*, *6*, 59–72.

Anthony, G., & Walshaw, M. (2009). *Effective pedagogy in mathematics* (Vol. 19). Belley, France: International Academy of Education.

Capurso, M., Dennis, J. L., Salmi, L. P., Parrino, C., & Mazzeschi, C. (2020). Empowering children through school re-entry activities after the Covid-19 pandemic. *Continuity in Education*, *1*(1) 62–82. https://doi.org/10.5334/cie.17

Chaaban, Y., Al-Thani, H., & Du, X. (2021). A narrative inquiry of teacher educators' professional agency, identity renegotiations, and emotional responses amid educational disruption. *Teaching and Teacher Education*, *108*, 103522. https://doi.org/https://doi.org/10.1016/j.tate.2021.103522

Clandinin, D. J., & Caine, V. (2013). Narrative inquiry. In A. A. Trainor & E. Graue (Eds.), *Reviewing qualitative research in the social sciences* (pp. 166–179). Routledge.

d'Agnese, V. (2023). Fear, angst, and the "startling unexpected". Three figures of teaching during the Covid-19 pandemic. *Studies in Philosophy and Education*, *42*(4), 389–409. https://doi.org/10.1007/s11217-023-09873-9

Hatano, G. (2013). A conception of knowledge acquisition and its implications for mathematics education. In L. P. Steffe, P. Nesher, P. Cobb, B. Sriraman, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 209–230). Routledge.

Ingram, N. (2011). *Affect and identity: The mathematical journeys of adolescents* [Unpublished doctoral dissertation]. Dunedin: University of Otago.

Ingram, N., Pratt, K., & Williamson-Leadley, S. (2018). Using show and tell apps to engage students in problem-solving in the mathematics classroom. In N. Calder, K. Larkin & N. Sinclair (Eds.), *Using mobile technologies in the teaching and learning of mathematics*. (pp. 301–316). Cham, Switzerland: Springer. doi: 10.1007/978-3-319-90179-4_17

Joseph, D., & Trinick, R. (2021). 'Staying apart yet keeping together': Challenges and opportunities of teaching during COVID-19 across the Tasman. *New Zealand Journal of Educational Studies*, *56*(2), 209–226. https://doi.org/10.1007/s40841-021-00211-6

McCluskey, G., Fry, D., Hamilton, S., King, A., Laurie, M., McAra, L., & Stewart, T. M. (2021). School closures, exam cancellations and isolation: The impact of Covid-19 on young people's mental health. *Emotional and Behavioural Difficulties*, *26*(1), 46–59. https://doi.org/10.1080/13632752.2021.1903182

Ministry of Health. (2021). COVID-19: Legislation and orders. New Zealand: Ministry of Health. https://www.health.govt.nz/our-work/diseases-and-conditions/covid-19-novel-coronavirus/covid-19-response-planning/covid-19-legislation-and-orders

Mukuka, A., Shumba, O., & Mulenga, H. M. (2021). Students' experiences with remote learning during the COVID-19 school closure: Implications for mathematics education. *Heliyon*, *7*(7). https://doi.org/10.1016/j.heliyon.2021.e07523

Mutch, C. A. (2021). COVID-19 and the exacerbation of educational inequalities in New Zealand. *Perspectives in Education, 39*(1), 242–256. https://doi.org/10.18820/2519593X/pie.v39.i1.15

Noviani, J. (2021). Perceptions of mathematics education students toward e-learning during the Covid-19 pandemic. *Kalamatika: Jurnal Pendidikan Matematika*, *6*(1), 15–30.

Op 't Eynde, P., De Corte, E., & Verschaffel, L. (2006). "Accepting emotional complexity": A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics*, *63*, 193–207.

Olsher, S., Yerushalmy, M., & Chazan, D. (2016). How might the use of technology in formative assessment support changes in mathematics teaching? *For the Learning of Mathematics*, *36*(3), 11–18.

Walters, T., Simkiss, N. J., Snowden, R. J., & Gray, N. S. (2022). Secondary school students' perception of the online teaching experience during COVID-19: The impact on mental wellbeing and specific learning difficulties. *British Journal of Educational Psychology*, *92*(3), 843–860. https://doi.org/10.1111/bjep.12475

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458–477.

Yates, A., Starkey, L., Egerton, B., & Flueggen, F. (2021). High school students' experience of online learning during Covid-19: The influence of technology and pedagogy. *Technology, Pedagogy and Education*, *30*(1), 59–73. https://doi.org/10.1080/1475939X.2020.1854337

Comprehending and Applying the First Isomorphism Theorem

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This qualitative study aims to investigate novice undergraduate mathematics students' first encounter with the First Isomorphism Theorem, which is, more often than not, the pinnacle of a typical introductory course in Group Theory. Several studies have reported on the challenges that this mathematical result poses to inexperienced mathematicians, mostly due to the numerous prerequisite abstract concepts. For the analysis of student responses, there has been used the Commognitive Theoretical Framework. This study suggests that the major challenges are due to the comprehension of the notions of kernel, image and isomorphism, and the application of FIT as a routine in the context of proofs.

First Isomorphism Theorem (FIT) is usually the culminating point of an introductory course in Group Theory (Ioannou, 2012). FIT is a particularly important mathematical result in such educational context, since it is essential in Group Theory, and it is rather complex in relating numerous concepts in Abstract Algebra in general (Melhuish et al., 2023). This complexity is due to the difficulty novice undergraduate mathematics students have in comprehending concepts such as the quotient group, which is reportedly the most challenging concept to grasp in an introductory course in Group Theory (Ioannou, 2016), the cosets, which novice students are challenged due to their inability to visualise them (Ioannou & Iannone, 2011), and the group isomorphisms, together with the kernel and image (Ioannou, 2019). Therefore, this study aims to investigate undergraduate mathematics students' first encounter with FIT, and report on the major pedagogical challenges they face. For this purpose, there will be used the Commognitive Theoretical Framework (CTF), proposed by Sfard (2008). Presmeg (2016, p. 423) suggests it is a theoretical framework of unrealised potential, designed to consider not only issues of teaching and learning of mathematics per se, but to investigate "the entire fabric of human development and what it means to be human." It proves to be an astute tool for the comprehension of diverse aspects of mathematical learning, which although grounded on discrete foundational assumptions, can be integrated to give a more holistic view of the students' learning experience (Sfard, 2012).

Literature Review

Abstract Algebra in general, and Group Theory in particular, is one of the most challenging mathematical fields for novice undergraduate mathematics students. Nardi (2000) reports that the grasp of the newly introduced notion of group is often an arduous task for novice students and causes serious difficulties in the transition from the informal secondary education mathematics to the formalism of undergraduate mathematics. Ioannou (2018) reports that the notion of subgroup, with the application of the Subgroup Test is considered another laborious mission to accomplish for novice students. In fact, novice students face numerous difficulties regarding both the comprehension and use of the notion of subgroup in the context of proofs.

The reported challenges students face in comprehending and applying group theoretic concepts are partly grounded on historical and epistemological factors, "The problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (expected to understand) the concepts today" (Robert and Schwarzenberger, 1991). Nowadays, the presentation of the fundamental group theoretic concepts including cosets, quotient groups, is "historically decontextualized" (Nardi, 2000, p169), since historically the fundamental concepts of Group Theory were permutation and symmetry.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 295–302). Gold Coast: MERGA.

to students who are beginning to study (expected to understand) the concepts today" (Robert and Schwarzenberger, 1991). Nowadays, the presentation of the fundamental group theoretic concepts including cosets, quotient groups, is "historically decontextualized" (Nardi, 2000, p169), since historically the fundamental concepts of Group Theory were permutation and symmetry.

Furthermore, quotient groups and FIT are two of the most challenging topics in an introductory course in Group Theory (Melhuish, 2019). Mena-Lorca and Parraguez (2016) reported that students who used a more generalised theorem had more sophisticated understanding than students who relied exclusively on group theoretic notions. Rupnow (2021) suggests that mathematicians who have managed to link the meaning of homomorphism and quotient groups are much more effective in the application of FIT and the comprehension of proofs. Brenton and Edwards (2003) discovered that understanding quotient groups requires novice students to realise that a coset is both an element in a group, as well as a set by itself. In addition, Siebert and Williams (2003) further the significance of the notion of coset, by suggesting that there are three distinct interpretations, namely coset as a set, coset as element set combinations, and cosets as representative elements, that need to be embraced.

Finally, a significant aspect of mathematical learning and practice at university level is the production of rigorous, explicit and elegant proofs, especially in the context of pure mathematics. Weber (2001) associates student difficulty with Group Theory with the difficulty to construct proofs, "when left to their own devices, students usually fail to acquire optimal strategies for completing mathematical tasks and often acquire deficient ones" (p. 116). Alcock (2010) similarly points out that learning Group Theory is challenging because of the abstract nature of its concepts and because it involves reading and writing proofs involving various learning practices and beliefs.

Theoretical Framework

CTF is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis. It involves a number of different notions such as *metaphor*, *thinking*, *communication*, and *cognition* (Sfard, 2008). In mathematical discourse, objects are discursive constructs and form part of the discourse. Mathematics is an *autopoietic* system of discourse, namely "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p. 129). Moreover, CTF defines discursive characteristics of mathematics as the *word use*, *visual mediators*, *narratives*, and *routines*.

Mathematical discourse involves certain objects of different categories and characteristics. *Primary object* (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p. 169). *Simple discursive objects* (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization". *Compound discursive objects* (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary" (Sfard, 2008, p. 166). In this context, group, coset and quotient group are examples of compound d-objects.

Other notions within the CTF, important for this study, are the rules of discourse, namely the *object-level* and the *metalevel rules*. Object-level rules are defined as "narratives about the regularities in the behaviour of the objects of the discourse" (Sfard, 2008, p. 201). In other words, these are rules that are directly related to the definition of the various objects, e.g., group, subgroup, coset, etc. Metalevel rules "define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (Sfard, 2008, p. 201). In other words, metarules govern the process of proof of new (to novice students) mathematical results.

Sfard (2008) describes two distinct categories of learning, namely the *object-level* and the *metalevel learning*. "Object-level learning … expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse" (Sfard, 2008, p. 253). In addition, "metalevel learning, which involves changes in metarules of the discourse … is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008, p. 254).

Finally, an important concept for the purposes of this study is the *commognitive conflict*, which is defined as a "situation that arises when communication occurs across incommensurable discourses" (Sfard, 2008, p. 296). Commognitive conflict is considered "a gate to the new discourse rather than a barrier to communication, both the newcomer and the old-timers must be genuinely committed to overcoming the hurdle" (Sfard, 2008, p. 282).

Methodology

The present study is a ramification of a larger research project (Ioannou, 2012), which conducted a close examination of Year 2 undergraduate mathematics students' learning experience in their first encounter with Abstract Algebra. The course was taught in a research-intensive mathematics department in the United Kingdom. It was a mandatory course, and a total of 78 students attended it. The course was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a doctorate student in the mathematics department. All members of the teaching team were pure mathematicians. The course assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of the semester.

The gathered data included the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 members of staff's interviews (5 members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave 3 interviews each), student coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, which uses only the student solution artifacts, there have been analysed the data gathered from the thirteen volunteers. The interviews, which covered a wide spectrum of themes, were fully transcribed, and analysed with comments regarding the mood, voice tone, emotions and attitudes, or incidents of laughter, long pauses etc., following the principles of Grounded Theory, and leading to the "Annotated Interview Transcriptions", where the researcher highlighted certain phrases or even parts of the dialogues that were related to a particular theme. Furthermore, coursework and examination solutions were analysed in detail, after the data collection period, using the CTF, and mostly focusing on issues such as students' engagement with certain mathematical concepts, the use of mathematical vocabulary and symbolization, and the application of discursive rules.

Finally, all emerging ethical issues during the data collection and analysis, namely, issues of power, equal opportunities for participation, right to withdraw, procedures of complaint, confidentiality, anonymity, participant consent, sensitive issues in interviews, etc., were addressed accordingly.

Data Analysis

The First Isomorphism Theorem was introduced by the lecturer as follows: *Suppose G, H are groups, and $\varphi: G \longrightarrow H$ is a homomorphism. Then ${}^{G}/_{ker\varphi} \cong im\varphi$.* There were two mathematical tasks that required the use of the FIT, one in the coursework and one in the final examination, as shown below.

Figure 1

Coursework Exercise on the FIT

6. (i) Suppose H is a non-trivial subgroup of $\mathbb{Z}$, the group of integers under addition, and let d be the smallest natural number in H (- why is there such a thing?). Prove that $H = d\mathbb{Z}$.
(ii) Suppose $G = \langle g \rangle$ is a cyclic group (written multiplicatively). Define $\phi : \mathbb{Z} \to G$ by $\phi(n) = g^n$ (for $n \in \mathbb{Z}$). Prove that this is a homomorphism.
(iii) Using (i) and (ii) and the First Isomorphism Theorem, deduce that if G is a cyclic group then there exists an integer d such that $G \cong \mathbb{Z}/d\mathbb{Z}$.

Figure 2

Examination Exercise on the FIT

5. (i) Suppose G is a group.

(a) What does it mean to say that a subgroup N of G is a *normal* subgroup? If N is a normal subgroup of G, explain how to make the set G/N of left cosets of N in G into a group. [3 marks]

(b) State the First Isomorphism Theorem for groups, defining the terms *kernel* and *image* in your statement. [4 marks]

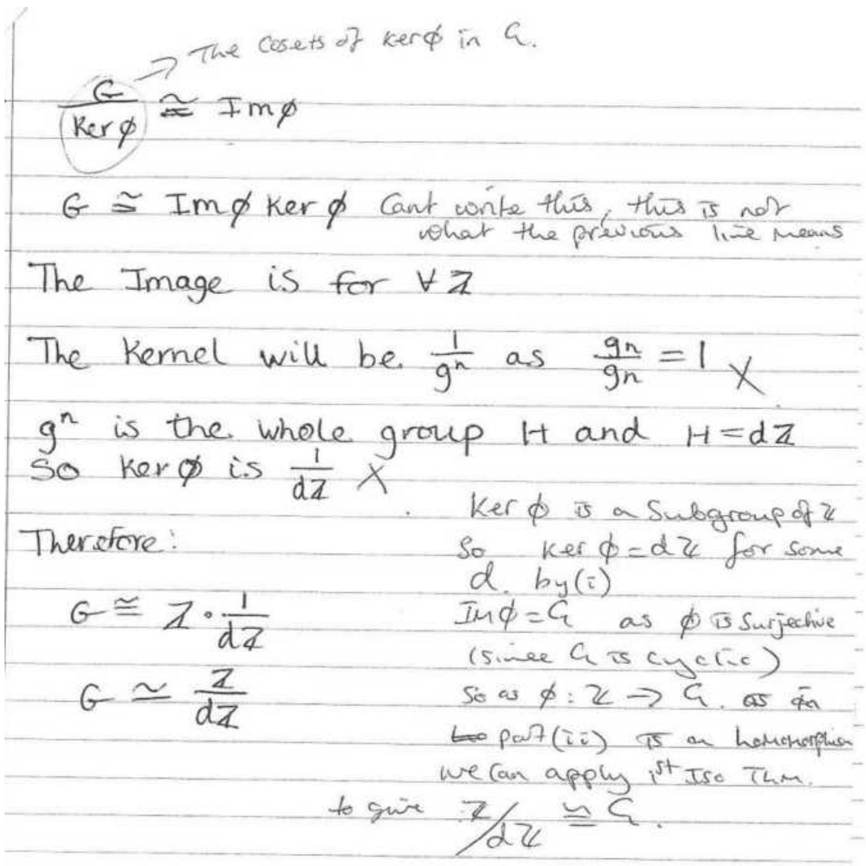
(c) Suppose H is a cyclic group. By defining a suitable homomorphism $\phi : (\mathbb{Z}, +) \to H$, or otherwise, prove that $H \cong \mathbb{Z}/m\mathbb{Z}$ for some $m \in \mathbb{Z}$. [3 marks]

Twelve out of thirteen (12/13) students' solutions indicated problematic engagement with FIT. There were strong indications of problematic application of the governing metarules of this routine by the majority of students. As mentioned before FIT is the pinnacle of an introductory course in Group Theory and students are required to overcome any object-level learning issues in order to be able to successfully apply the governing metarules that are required in the application of FIT as a routine. In what follows, we analyse a number of representative examples of the three types of errors that occurred in the student solutions.

Complete object-level learning of the involved d-objects, such as kernel, image and isomorphism, is of vital importance for successfully applying the FIT. For instance, as Figure 3 indicates, Student A's object-level learning of the notion of isomorphism appears to be incomplete and therefore he is still unable, to apply FIT successfully.

Figure 3

Student A's Coursework Solution

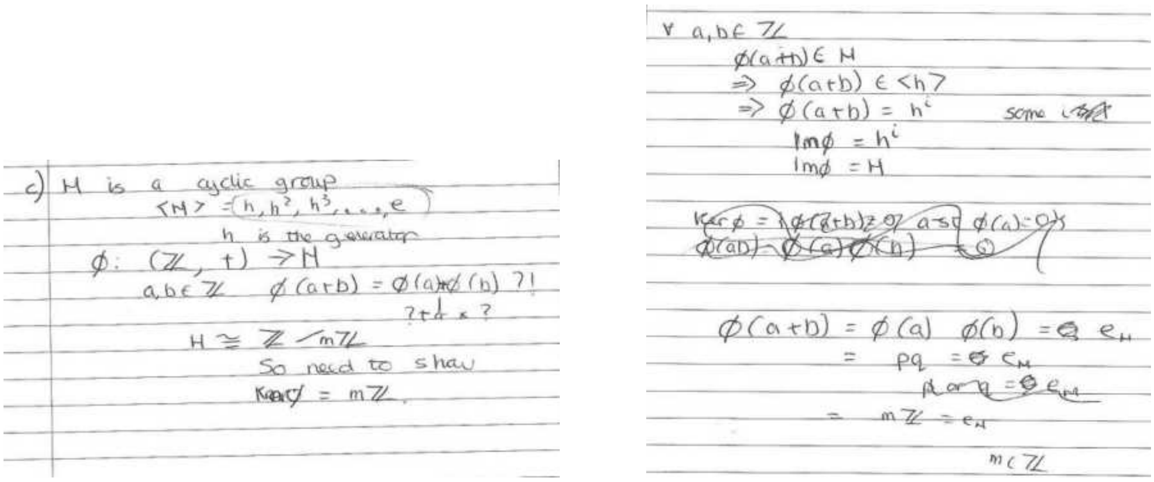


Here appears a typical example of a commognitive conflict. The notation $\frac{G}{ker\varphi} \cong Im\varphi$ is treated as an algebraic equation in which Student A has applied cross-multiplication, i.e., $G \cong Im\varphi Ker\varphi$. This is an erroneous metaphor from elementary algebra that indicates an incomplete object-level learning of the concepts of isomorphism, kernel and image, and therefore it reveals a typical situation which arises when communication occurs across incommensurable discourses, in this case elementary to abstract algebra. Student A has not realised that $G/ker\varphi$ is a mathematical structure and the symbol ≅ does not refer to equality relation but to bijective relation. His object-level and metalevel learning of the d-objects of kernel and image is also incomplete and consequently he is still not able to use FIT effectively. He has not realised that $im\varphi$ is a subgroup of H and that $ker\varphi$ is a normal subgroup of G.

Albeit asked to, Student B failed to properly state FIT in the final examination, since she did not provide the definitions for image and kernel. Possibly, failing to define kernel, image and stating FIT suggests that her object-level learning of kernel and image of a group is incomplete when applied in a more advanced context such as the FIT. The third part of her solution, involving the definition of a suitable homomorphism, reinforces the claim that she is still not able to effectively use the FIT. Her attempt to solve the last bit is fragmental and lacks linear reasoning and structure.

Figure 4

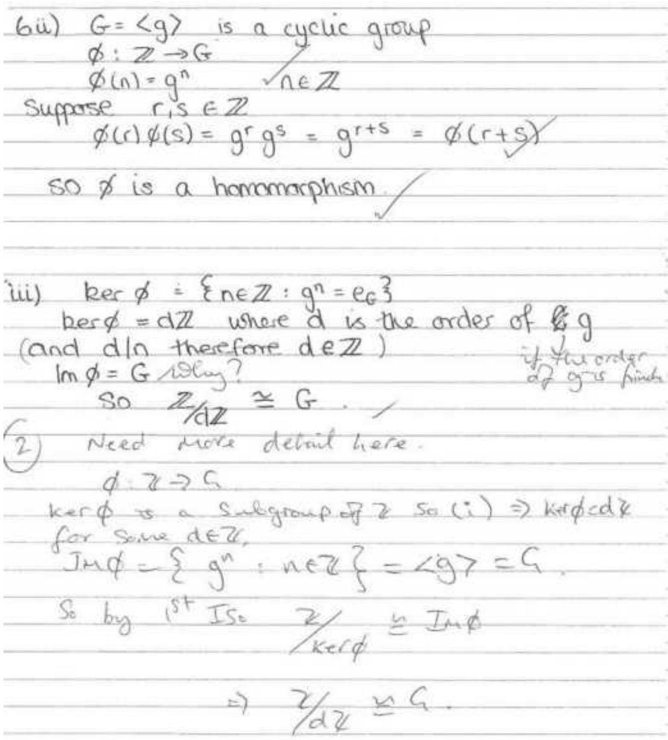
Student B's Examination Solution



A third type of errors regarding the application of FIT occurred Student C's coursework solution. Her attempt to solve this exercise has several problems, especially in the third part regarding the FIT. She correctly states that the image of the homomorphism is the group G itself, and therefore it is an isomorphism, but she does not mention anything about the kernel. This possibly suggests an incomplete object-level learning of the d-object of isomorphism, and in particular relating to the fact that one has to prove that a homomorphism needs to be both injective and surjective in order to be an isomorphism. Moreover, her solution indicates that she is not aware of the importance of the fact that the order of g is finite. Her narratives are not explicit, possibly indicating an incomplete metalevel learning of the involved routine as well as problematic application of the governing metarules. She does not seem to realise that the kernel in this case is $d\boldsymbol{Z}$, and the reasoning behind that.

Figure 5

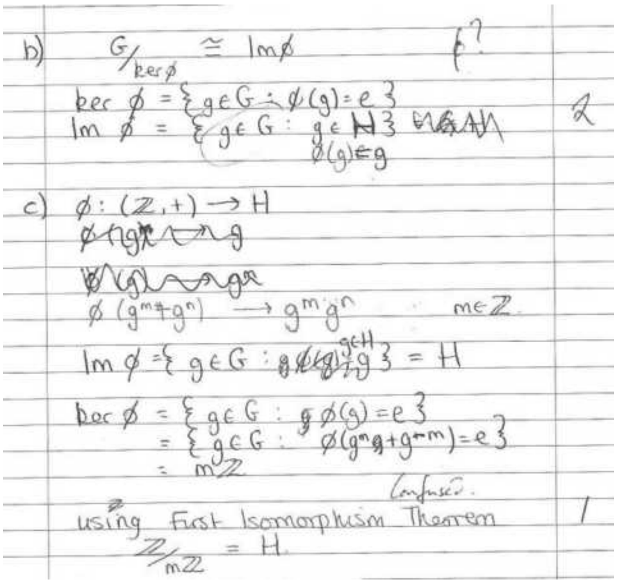
Student C's Coursework Solution



In Student C's examination solution, we can identify several problems. First of all, she has not properly stated FIT, but instead she wrote the mathematical expression $\frac{G}{Ker\varphi} \cong Im\varphi$ without any further explanation. The definition of image is problematic, since instead of writing $Im\varphi = \{\varphi(g): g \in G\}$ she stated $Im\varphi = \{g \in G: g \in H\}$. This probably indicates that she has an incomplete object-level learning of the definition of image and its elements. In addition, her narratives suggest that she is not fully aware yet of what she is writing mathematically. In part (c), her solution lacks explicitness, reflecting her problematic engagement with FIT, indicating incomplete metalevel learning of the particular routine, as well as imprecise application of 'norms' required for proving in this advanced mathematics course. The marker wrote the comment "Confused", which he rarely does in the examination solutions. Student C's performance in the exams indicates a decline regarding the application of FIT.

Figure 6

Student C's Examination Solution



Conclusion

First Isomorphism Theorem is, more often than not, the pinnacle of an introductory course in Group Theory. Its difficulty is due to several reasons. First, it is due to the fact that FIT relates numerous abstract theoretical concepts (Melhuish et al., 2023). These concepts include the quotient group (Author, 2016), which is the most challenging of all fundamental group theoretic concepts in an introductory level, the coset, which is often an obstacle due to students' difficulty to visualise it (Author & Iannone, 2011), and the group isomorphisms, together with the kernel and image (Author, 2019). The current study has analysed students' engagement with FIT.

The analysis has shown that the great majority of students have faced significant challenges with its application, with these challenges being of both object-level and metalevel nature. These challenges have been exemplified by using three representative examples which have repeatedly occurred in the solutions of many students. The first challenge was associated with the incomplete object-level learning of relevant mathematical concepts such as the kernel, image and group isomorphism. This challenge is relevant to the second one, which, according to the analysis, leads to metadiscursive level issues and the successful application of FIT, indicating that without full object-level learning of these notions it is almost impossible to apply the governing metarules successfully. Finally, the third challenge indicates a deeper difficulty with the notion of isomorphism, which emerges from the incomplete object-level learning of the notion of homomorphism, and the special characteristics that make homomorphism to be an isomorphism, namely being injective and surjective. These interesting results require further investigation in order to comprehend more deeply these challenges, and hopefully make Group Theory, and Abstract Algebra in general, more approachable to our students.

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References

Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. *Issues in Mathematics Education*, *16*, 63–91. https://doi.org/10.1090/cbmath/016/03

Brenton, L., & Edwards, T. G. (2003). Sets of sets: A cognitive obstacle. *The College Mathematics Journal*, *34*(1), 31–38. https://doi.org/10.1080/07468342.2003.11921982

Ioannou, M., & Iannone, P. (2011). Students' affective responses to the inability to visualise cosets. *Research in Mathematics Education*, *13*(1), 81–82. https://doi.org/10.1080/14794802.2011.550737

Ioannou, M. (2012). *Conceptual and learning issues in mathematics undergraduates' first encounter with group theory: A commognitive analysis* [Unpublished doctoral dissertation]. University of East Anglia.

Ioannou, M. (2016). Investigating undergraduate mathematics students' first encounter with quotient groups. *Mediterranean Journal for Research in Mathematics Education*, *15*, 51–67.

Ioannou, M. (2018). Commognitive analysis of undergraduate mathematics students' first encounter with the subgroup test. *Mathematics Education Research Journal*, *30*(2), 117–142. https://doi.org/10.1007/s13394-017-0222-6

Ioannou, M. (2019, July). On the learning of group isomorphisms. In M. Graven, H. Venkat, A. A. Essien, & P. Vale (Eds), *Proceedings of the 43rd annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 400–407). PME.

Melhuish, K., Guajardo, L., Dawkins, P. C., Zolt, H., & Lew, K. (2023). The role of the partitioning and coset algorithm quotient group partial meanings in comprehending the First Isomorphism Theorem and its proof. *Educational Studies in Mathematics*, *113*(3), 499–517. https://doi.org/10.1007/s10649-023-10207-2

Mena-Lorca, A., & Parraguez, A. M. M. (2016). Mental constructions for the group isomorphism theorem. *International Electronic Journal of Mathematics Education*, *11*(2), 377–393. https://doi.org/10.29333/iejme/340

Nardi, E. (2000). Mathematics undergraduates' responses to semantic abbreviations, 'geometric' images and multi-level abstractions in Group Theory. *Educational Studies in Mathematics*, *43*, 169–189.

Presmeg, N. (2016). Commognition as a lens for research. *Educational Studies in Mathematics*, *91*(3), 423–430. https://doi.org/10.1007/s10649-015-9676-1

Robert, A., & Schwarzenberger, R. (1991). Research in teaching and learning mathematics at an advanced lever. D. Tall. (Ed.), *Advanced mathematical thinking* (pp. 127–139). Kluwer Academic Publishers.

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *The Journal of Mathematical Behavior*, *62*, Article 100867. https://doi.org/10.1016/j.jmathb.2021.100867

Sfard, A. (2008). *Thinking as communicating: human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sfard, A. (2012). Introduction: Developing mathematical discourse—Some insights from communicational research. *International Journal of Educational Research*, *51–52*(3), 1–9. https://doi.org/10.1016/j.ijer.2011.12.013

Siebert, D., & Williams, S. R. (2003). Students' understanding of ZN. In N. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th International Conference for the Psychology of Mathematics Education* (Vol. 4, pp. 167–173). PME.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119.

Development of Items to Assess Big Ideas of Equivalence and Proportionality

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Big Ideas can be seen as overarching concepts that occur in various mathematical topics and strands within a syllabus. Within our project on Big Ideas in School Mathematics, we developed instruments to measure two Big Ideas: Equivalence and Proportionality. The instruments we developed seek to assess students' ability to see these Big Ideas as common ideas connecting within and across topics, and their ability to apply these Big Ideas in solving problems. In this paper, we discuss the development of items for these instruments.

Big Ideas can be described as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole (Charles, 2005, p. 10)". It would be reasonable to describe Big Ideas as overarching ideas that cut across topics. This follows from NCTM (2000) which highlighted the importance for teachers to "understand the Big Ideas of mathematics and be able to represent topics as a coherent and connected enterprise" (p. 17). In the most recent mathematics syllabus in Singapore (MOE, 2018a & 2018b), a significant addition is the explicit introduction of the need to teach towards Big Ideas. It defines Big Ideas as key ideas that "bring coherence and show connections across different topics, strands and levels" (MOE, 2018a). While the concept of Big Ideas in Mathematics has received increasing attention over recent years (NCTM, 2000; Charles, 2005; MOE, 2018a & 2018b), there is little research so far on the assessment of Big Ideas in Mathematics. One possible reason may be the varied definitions of what a Big Idea in Mathematics is. For our study, we adopt Charles' (2005) definition of a Big Idea as stated earlier. Knowing whether students can make connections between numerous understandings can help teachers improve their pedagogies and teaching approaches. However, there is currently little research done in developing instruments that can assess the ability to see connections using Big Ideas. Hence, the ability of students in Singapore to see the connectedness across and within topics using Big Ideas is a guiding principle in our study. This paper presents the development of an instrument to measure the Big Ideas of Equivalence and Proportionality.

While there are studies on the assessment of Big Ideas like *Equivalence* (Warren & Cooper, 2009; Fyfe et al., 2018; Niemi et al., 2006), and *Proportionality* (Carpenter et al., 1999; Izzatin, 2020), assessing students' ability to connect across or within topics was not evident. To fill this gap, our instrument is developed with the focus on assessing students' ability to apply Equivalence and Proportionality thinking to solve mathematical tasks and to elicit students' ability to 'see' connections across these tasks using Big Ideas.

Principles in Item Design

Our design of the items for the instruments is guided by two of the three characteristics of Big Ideas as detailed by Hsu et al. (2007): (a) connect different parts of the curriculum under the umbrella of Big Idea; and (b) be a basis for understanding other topics. To achieve a measurement of (a), a multi-part assessment item was developed. This multi-part structure is

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 303–310). Gold Coast: MERGA.

partly motivated by Niemi's (2006) *PowerSource* instrument which assesses students' ability to see Equivalence in algebra.

In this paper we focus on the assessment items measuring two Big Ideas: Equivalence and Proportionality. There are studies that explore and unpack Big Ideas in greater detail. For example, Fyfe et al. (2018), focused their study on the symbolic understanding of equivalence, i.e., the understanding of the equal sign, while Cook et al. (2022) provided three interpretations of equivalence, viz., Common Characteristics, Descriptive and Transformational. For Proportionality, Carpenter et al. (1999) identified four levels of proportional reasoning development, Özgün-Koca and Altay (2009) identified three types of proportional reasoning problems, and Izzatin (2020) identified five levels in the proportional reasoning process. Some studies went in depth into fine-grain details of Big Ideas such as proportional reasoning down to the detail of cross multiplication (Chin et al., 2022).

However, in the development of our items, we focus on tasks that require students to apply any forms or levels of Equivalence or Proportionality thinking while simultaneously teasing out their ability to see connections across tasks (Jahangeer et al., 2023). Thus, we define Equivalence as *a relationship that expresses the equality of two mathematical entities and realises the potential of an easier solution/understanding of one entity by converting it to the other entity*. We operationalise this definition to our Equivalence items by the statement: In this item, Entity 1 is equivalent to Entity 2, the equivalence of which usually enables the problem to be solved in an easier manner. (For example, 29 + 13 is equivalent to 30 + 12; the latter is easier to solve.)

We define Proportionality as "a relationship that expresses the direct variation of two mathematical entities and realises the potential of an easier calculation of one entity by knowing the value of the other entity". We operationalise this definition to our Proportionality items by the statement: In this item, Entity 1 is proportional to Entity 2, the proportionality of which usually enables one entity to be calculated from the other by multiplicative reasoning. (for example, if $y = 3x + 7$, then when x increases by 7, y will increase by $3 \times 7 = 21$).

We developed two separate instruments, one for each Big Idea. They are to be administered at two different levels: Primary (Grades 5 and 6); and Secondary (Grades 7 and 8). The items that we created to assess each Big Idea satisfy the operational definition of the Big Idea. Each item serves to assess students' ability to see the connections across the parts of the item through the Big Idea. The tasks of each item cover concepts either within or across topics in the Singapore school syllabus.

Example 1 shows a mathematics task that requires students to use Equivalence to solve:

> Example 1: Ali had \$220, and Colin had \$310. After each of them bought an identical pair of sneakers, Colin had thrice as much money as Ali had left. How much did the pair of sneakers cost?

Equivalence is established using the operational definition by stating that Entity 1 is "the difference in the money Ali and Colin had at the beginning" and Entity 2 is "the difference in the money Ali and Colin had at the end", and that Entity 1 is equivalent to Entity 2:

> Example 2 (Figure 1) is another task which shows how the operational definition ensures that it is valid for Equivalence. In this task, the shaded region (Entity 1) is equivalent to the numerical expression 1 + 2 + 3 + 4 + 5 + 6 + 7 (Entity 2). The shaded region is half of the rectangle, which can be easily calculated to be 7 × 8. Thus, the numerical expression is equivalently (7 × 8) ÷ 2.

For Proportionality, we consider Example 3 and Example 4 (Figure 2). In Example 3, the line shown is of the form $y = mx$, in which y varies directly with x, i.e., y is proportional to x. In Example 4, the line is of the general form $y = mx + c$. While y is not proportional to x, the **change in y** is proportional to the **change in x**. Hence, Example 4 still satisfies the operational definition of Proportionality and is included as one of the mathematical tasks for the instrument on Proportionality.

Figure 1

Example 2

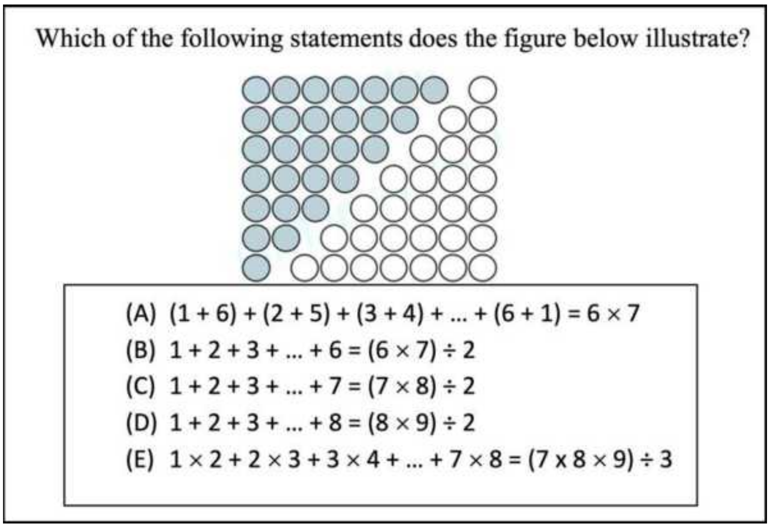
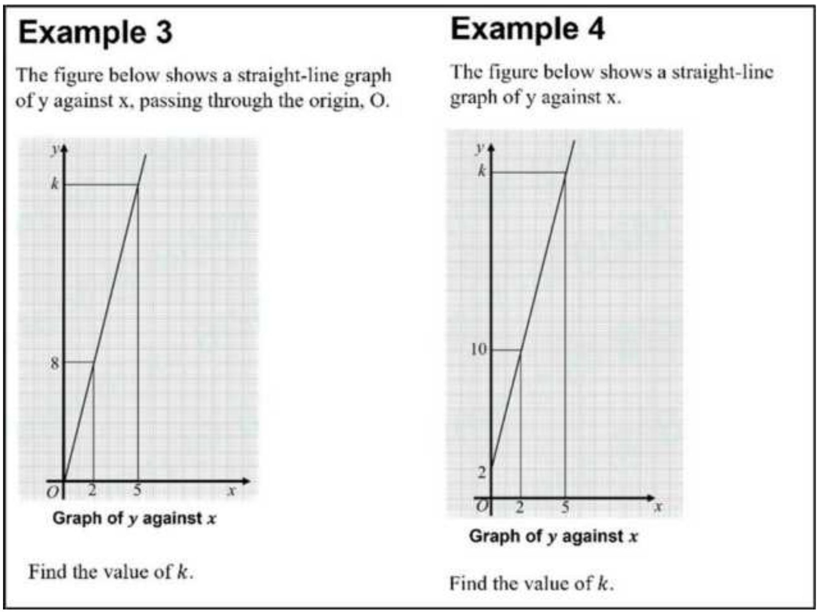


Figure 2

Example 3 and Example 4



Item Design

A typical item (Figure 3 is one such example measuring the Big Idea of Equivalence) consists of four tasks and two sets of reflection questions. The item parts are presented in the following order: Task 1, Task 2, Task 3, Reflection A-1, Reflection A-2, Task 4, and Reflection B.

Recall that we define a Big Idea as an idea that is central to the learning of mathematics, one that brings both coherence and connections 'within' and 'across' topics. The task shown in Figure 3 is an example which seeks to elicit students' ability to 'see' connections 'across' topics. The functionality of the task to assess Big Idea thinking across topics is as described in the proceeding paragraphs.

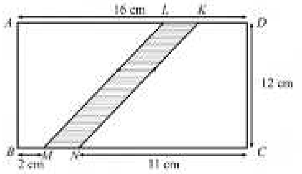
In Task 1 shown in Figure 3, students are supposed to find the shaded area, which is the area of the parallelogram. The shaded area can be found by finding the difference between the two unshaded areas and the larger rectangle. Note that the two unshaded areas can be put together to form a smaller rectangle. The equivalence here is the area of the two trapeziums (Entity 1) and the area of the smaller rectangle formed by joining the two together (Entity 2). Students at this stage have not been introduced to finding the area of trapeziums and parallelograms as it is not within the syllabus requirements for Grades 5 and 6 in Singapore.

Figure 3

A Complete Item for the 'Big Idea of Equivalence'

Task 1

The below shows a rectangle ABCD. The lines KN and LM are parallel.



Find the value of the shaded area.

Task 2

Charles has 248 Pokemon cards and David has 88 Pokemon cards. They each gave away ¼ of their original total number of cards. At the end, Charles has ____ Pokemon cards more than David.

Task 3

Alvin has $2023 and Betty has $1973. They were each given three times as much as their original total amount.

At the end, Alvin has $____ more than Betty.

Reflection A-1

What is common across Task 1, Task 2 and Task 3?

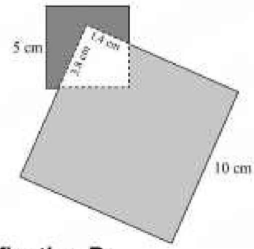
Reflection A-2

What is common across Task 1, Task 2 and Task 3?

- I used Diagrams for the tasks.
- I used Equivalence for the tasks.
- I used Guess and Check for the tasks.
- I used Proportionality for the tasks.
- Others (Please elaborate).

Task 4

The diagram below shows a lighter square with side 10cm placed on top of part of a darker square of length 5 cm. Their common region is unshaded. The difference between the area of the lighter region and the area of the darker region is _____cm2.



Reflection B

What is common across Task 1, Task 2, Task 3 and Task 4?

- In all these tasks, I used Diagrams.
- In all these tasks, I used Equivalence.
- In all these tasks, I used Guess and Check.
- In all these tasks, I used Proportionality.
- Others (Please elaborate)

In Task 2, students are asked to find the difference in the number of cards left between the two persons. It will be easy if the actual number of cards given away by each person is known. Since the amount at the start is known, the quantity at the end can be easily calculated and the difference found. However, the actual quantity given away is not mentioned. Since both persons give away an equal number of cards, the difference they have in the end (Entity 1) is equivalent to the difference they have at the beginning (Entity 2). It is easy to find the difference at the start with a known number of cards each person has.

Similar to Task 2, in Task 3, the difference in money they have in the end (Entity 1) is equivalent to the difference they have at the beginning (Entity 2). In this instance, both persons are now given the same amount of money.

Before attempting Task 4, students are asked Reflection A-1 and Reflection A-2. The two reflection questions require students to look for the common idea used to solve the first three tasks. In the first task, students could realise that it is easier to solve when they can find two entities that are equivalent. Thinking along the same idea of equivalence, the student could again look to see if the next two tasks can also be solved using equivalence. As Tasks 2 and 3 are in topics chosen from the syllabus that we know are familiar to students at that level, the students are expected to be able to see that the tasks could also be solved by comparing two entities that are equivalent. Once the connection (the common idea used for all three tasks) is established, students are then given Task 4 to solve.

Task 4 is generally unfamiliar to most students. Students who have made the connections carry over the same idea to try to find two entities that are equivalent. Here, the Big Idea of Equivalence is the bridge that connects the solution process to solve the previous three tasks with Task 4, i.e., looking for two entities that are equivalent. The more able (in that Big Idea) students will realise that the difference between the lighter and shaded regions (Entity 1) is equivalent to the difference between the larger and smaller squares (Entity 2), as the shaded regions are obtained by removing the same unshaded regions from both squares. It is easier to find the difference between two squares than the difference of two irregular shaded regions.

In explaining the example above, we make the assumption that students are encountering the item, consisting of the tasks and reflection questions, for the first time. When doing a second item, students may already start to look for connections of ideas between the first three tasks.

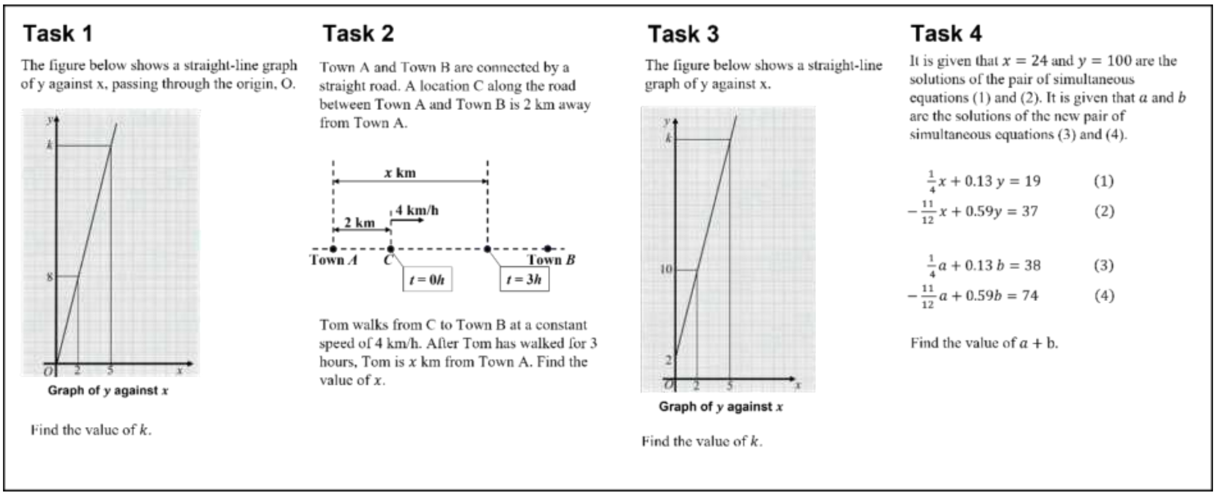
As a second example, we shall look at an item developed for assessing the Big Idea of Proportionality as shown in Figure 4. In Task 1, students are required to find the value of k by analysing the straight-line graph given. As the line cuts through the origin, students could conclude that the equation of the line is of the form $y = mx$, in which *y* (Entity 1) is proportional to x (Entity 2). Analysing the graph, students could notice that an x value of 2 results in a value of 8 for *y*, indicating that *y* is four times the value of x, i.e., $y = 4x$. With this observation, the value of k can be easily obtained by multiplying 5 by 4 to obtain the answer 20.

We shall proceed to elaborate how students are likely to answer Task 2. A secondary school student would be familiar with tasks of this nature as the topic of speed is part of the primary mathematics syllabus in Singapore. Recalling their prior knowledge, students will focus their attention on the speed given and know that the distance covered by Tom from Town C to Town B (Entity 1) is directly proportional to the time taken (Entity 2). The student could then obtain the distance of 12 km. However, to answer the problem posed, students will need to add the distance, 2 km, of Town C from Town A. We postulate that the more able (in the Big Idea of Proportionality) students would see the similarity between the two parts, in which the solution is based on the equation of a straight line, $y = mx + c$. For Task 1, $c = 0$ and for Task 2, $c = 2$.

Continuing to Task 3, we postulate that the more able (in the Big Idea of Proportionality) students would see that the value of y is related to the value of x by the equation $y = 4x + 2$, and that Task 3 is similar to both earlier tasks. While y is not directly proportional to x, the **change in y** (Entity 1) is directly proportional to the **change in x** (Entity 2). Arriving at Reflection A-1 and A-2, where they are prompted to look for the common ideas across all three parts, able students would be able to see that connecting across all these parts is the Big Idea of Proportionality. We postulate that the ability to see this connection will positively prime them with an efficient approach to solve the task in Task 4 later.

Figure 4

An Example of an Item (Without the Reflection Questions) to Assess the 'Big Idea of Proportionality' That Shows Connections Across Topics



When presented with Task 4, students could try to see if the Big Idea of Proportionality would be helpful in their solving process. Upon analysing the information given, a properly primed and able student should be able to see that the right-hand-side of equation 3 and the right-hand-side of equation 4 are twice those of equations 1 and 2 respectively. Students will use Proportionality to deduce that the values of a and b (Entity 1) will also be twice the solutions of equations 1 and 2 (Entity 2), resulting in the value of 48 for a and 200 for b.

Figure 5 is an example of an item meant for secondary school students under the Big Idea of Proportionality which elicits students' ability to see connections 'within' topics, in this example, the topic of rate. Tasks 1 and 2 involve direct proportion. Students use simple straightforward multiplicative reasoning to solve these tasks. Task 3 seems to require knowledge of inverse proportion. At these grade levels, the students are not yet introduced to inverse proportion. However, the task can be solved using direct proportion if the students can break down the task into parts. If we fix the number of days, then the number of workers (Entity 1) is proportional to the number of houses (Entity 2). We can obtain the fact that 36 workers will build $\frac{36}{60}$ of a house in 180 days. If we now fix the number of workers, then the number of days (Entity 1) is proportional to the number of houses (Entity 2). We can finally obtain the fact that 36 workers will build one house in $180 \times \frac{60}{36}$ days. Task 4 will be administered after the students have answered Reflections A-1 and A-2. Able students will see the common idea of proportionality and attempt to solve Task 4 by using proportion to find the number of chairs the five carpenters can make, and the number of chairs the other two carpenters can make separately and then combine them to obtain the answer.

Discussion

At this point in the research, we have created four sets of instruments, each containing eight items: two instruments for the primary level (Grades 5 and 6) and two instruments for the secondary levels (Grades 7 and 8). The two instruments for each level are for the Big Ideas of Equivalence and Proportionality separately. All tasks for each item have been carefully designed such that one has to typically use either Equivalence or Proportionality to solve them. Task 4 requires a higher cognitive demand as the task is typically unfamiliar to the students.

Figure 5

An Item (Without the Reflection Questions) for the 'Big Idea of Proportionality' That Shows Connections 'Within' Topics

Task 1

A car takes 120 min to travel 180 km.
At the same speed, how much time does the car take to travel 120 km?

(A) 20 min (B) 40 min (C) 60 min (D) 80 min (E) 180 min

Task 2

60 workers can make 180 chairs in one day. Each worker works at the same rate. How many chairs can 36 workers make in one day?

(A) 72 (B) 90 (C) 108 (D) 156 (E) 300

Task 3

A team of 60 workers takes 180 days to build one house. All workers work at the same rate. How many days does a team of 36 workers need to build the same house.

(A) 72 (B) 90 (C) 108 (D) 252 (E) 300

Task 4

10 carpenters take 5 hours to make 40 wooden chairs. All carpenters work at the same rate. In a particular week, 5 carpenters work for 40 hours, and 2 carpenters work for 35 hours. How many wooden chairs did the 7 carpenters make altogether in that week?

(A) 28 (B) 60 (C) 196 (D) 216 (E) 600

It may be possible for students to have answered correctly the first three tasks of each item based on their strong topical knowledge. However, we believe the first set of reflection questions will help students seek out the Big Idea among the first three tasks. We believe that the first set of reflection questions, Reflections A-1 and A-2, will trigger the students to 'see' that Task 4 can be solved using the same Big Idea. However, we acknowledge that (i) students may be able to obtain the correct answer by applying the correct Big Idea for the unfamiliar Task 4 but not see the connections between tasks (seen from their 'wrong' answers in the Reflection); and that (ii) students may be able to solve Task 4 using a different approach than that which we anticipated. Firstly, the proportion of students who can solve the task correctly but are not able to make the connections can give us valuable feedback on the current state of mathematics understanding. Such students have a good grasp of mathematical concepts but do not see mathematics as a connected whole. The findings can provide valuable feedback to teachers on how to modify their pedagogical approaches to help students see the connections. For the second case, we believe that such responses will be but a few and more of an exception rather than the norm. We shall explain this by referring to a typical mathematics assessment as an analogy. An item in a mathematics assessment is included to test a certain mathematics concept or process. The marking scheme for the item would be created based on typical responses expected. However, there will be responses that are atypical and the student's performance is scored using a different marking scheme without questioning the validity of the item. This is acceptable as the number of such atypical solutions are usually very small compared to the population. Similarly, while we acknowledge that there will be correct responses obtained by applying a different strategy or concept, the low occurrence of such cases can allow us to ignore its impact on the overall item analysis. In addition, our Task 4s have been designed such that a brute force computational approach would be time-consuming and inefficient.

The instrument, being 8-item long, requires a long duration for the students to complete. When we piloted two of the items to about 230 primary and secondary students, the mean time taken by the students was about 30 minutes. Thus, an eight-item instrument will require a duration of at least 120 mins. The time taken to administer the instruments developed will be a challenge as an assessment this long will take away precious curriculum time required for teaching and learning. At the same time, we risk reducing the validity of the data collected as students may experience fatigue completing the assessment or may decide not to perform to their ability due to the perceived fatigue of completing the lengthy assessment (Ackerman & Kanfer, 2009). A proposed solution will be to consider splitting the instrument into testlets and adopting a matrix design instead. We will be able to elaborate on this in a longer paper.

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References

Ackerman, & Kanfer, R. (2009). Test length and cognitive fatigue: An empirical examination of effects on performance and test-taker reactions. *Journal of Experimental Psychology: Applied*, *15*(2), 163–181. https://doi.org/10.1037/a0015719

Carpenter, T. P., et al. (1999, April 19–23). *An analysis of student construction of ratio and proportion understanding*. Paper presented at the American Educational Research Association, Montreal, Canada.

Chin, S. L., Choy, B. H., & Leong, Y. H. (2022). Considerations for teaching with multiple methods: A case study of missing-value problems in proportionality. In N. Fitzallen, C. Murphy, V. Hatisaru & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (pp. 112–129). Launceston: MERGA.

Cook, J. P., Reed, Z., & Lockwood, E. (2022). An initial framework for analyzing students' reasoning with equivalence across mathematical domains. *The Journal of Mathematical Behavior*, *66*, Article 100935. https://doi.org/10.1016/j.jmathb.2022.100935

Fyfe, Matthews, P. G., Amsel, E., McEldoon, K. L., & McNeil, N. M. (2018). Assessing formal knowledge of math equivalence among algebra and pre-algebra students. *Journal of Educational Psychology*, *110*(1), 87–101.

Izzatin, M (2020). Proportional reasoning in mathematics: What and how is the process? In W. Kusmaryani, Arifin, J. B. Darmayasa, & S. Wulandari, (Eds.), *Proceedings of the second international conference on innovation in education and pedagogy* (pp. 115–119). ICIEP. https://doi.org/10.2991/assehr.k.211219.022

Jahangeer, M. J., Toh, T. L., Tay, E. G., & Tong, C. L. (2023). Assessment of big ideas in school mathematics: Exploring an aggregated approach. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 44–47). Newcastle: MERGA.

Ministry of Education. (2018a). *2020 Secondary mathematics syllabuses* [Draft]. Singapore: Author.

Ministry of Education (2018b). *2021 Primary mathematics syllabuses* [Draft]. Singapore: Author.

National Council of Teachers of Mathematics (NCTM). (2000). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

Niemi, D., Vallone, J., & Vendlinski, T. (2006). *The power of big ideas in mathematics education: Development and pilot testing of POWERSOURCE assessments*. CSE Report 697, National Center for Research on Evaluation, Standards, and Student Testing (CRESST).

Özgün-Koca, S. A. & Altay, M. K. (2009). An investigation of proportional reasoning skills of middle school students. *Investigations in Mathematics Learning*, *2*(1), 26–48, DOI: 10.1080/24727466.2009.11790289

Warren, E., & Cooper, T. J. (2009). Developing mathematics understanding and abstraction: The case of equivalence in the elementary years. *Mathematics Education Research Journal*, *21*(2), 76–95. https://doi-org.libproxy.nie.edu.sg/10.1007/BF03217546

Yeo, J. B. W. (2019). Unpacking the big idea of proportionality: Connecting ratio, rate, proportion and variation. In T. L. Toh & J. B. W. Yeo (Eds.), *Big ideas in mathematics* (pp. 187–218). World Scientific. https://doi.org/10.1142/9789811205385_0012

Roles of Mathematical and Statistical Models in Data-Driven Predictions in an Integrated STEM Context

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This study elaborates on the pivotal roles of mathematical and statistical models in data-driven predictions in an integrated STEM context using the case of Year 4 students: (i) *a descriptive means* to describe the features of trends and variability of data and (ii) *an explanatory means* to explain causal relationships behind data. These roles are linked to models in other STEM subjects (i.e., prototypes and scientific models) and the application and development of STEM content knowledge. The results contribute to a better understanding of the role of mathematics/statistics in STEM education.

Predictions are found everywhere in life, society, and science. During the global COVID-19 pandemic, daily data on the number of positive cases, severe cases, and deaths were published in graphs and tables, and it became routine to keep track of the current infection situation and predict future waves of infection. *Data-driven prediction* by using data, mathematics, statistics, and interdisciplinary knowledge to predict and validate complex and uncertain events is indispensable for today's citizens and societies (e.g., Geiger et al., 2023). To provide a vehicle and platform for such data-driven predictions, modelling processes involving the generation, evaluation, and revision of *mathematical models* (deterministic representations) and *statistical models* (non-deterministic/stochastic representations) in data-rich interdisciplinary contexts are gaining attention in Science, Technology, Engineering and Mathematics (STEM) education from a mathematics education perspective (e.g., English, 2023). However, the literature does not clearly explain how and to what extent students use mathematical and statistical models for data-driven predictions in an integrated STEM context.

Unravelling the pivotal roles of mathematical and statistical models in STEM education offers two advantages for research and practice. First, it could elaborate on and advance the role of mathematics/statistics in integrated STEM education (English, 2016) as well as in mathematics curricula, such as ACARA (2022) and MEXT (2018), which emphasise mathematical modelling, statistical investigation, and STEM education. Second, it has implications on developing students' epistemic knowledge about the nature and role of models and representations in STEM disciplines, which are essential for STEM competencies (Tytler, 2020). Therefore, this study elaborates on the roles of mathematical and statistical models in data-driven predictions in an integrated STEM context using the case of Year 4 students.

Conceptual Framework

Data-Driven Prediction

The potential for introducing *data-driven prediction* from the primary school years has been identified by mathematics and statistics education research. *Informal statistical inference* (ISI), in which trends and variations in unknown data are predicted and generalised without adopting formal statistical procedures and methods, is actively studied since primary school years (Makar & Rubin, 2018). For instance, Oslington et al. (2023) highlighted primary school students' predictive reasoning as part of the ISI with representations of patterns such as seasonal trends and variability in data to predict temperature using tables, line graphs, and bar graphs.

Moreover, data-driven predictions are important in an integrated STEM context. Watson et al. (2023) conducted statistical investigations using ISI and technology with primary school

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 311–318). Gold Coast: MERGA.

students in a STEM context, revealing their representations, predictions, and understandings of variation. English (2023) implemented modelling involving data-driven predictions with mathematics and statistics for primary school students in an integrated STEM context. It explored how they applied multidisciplinary knowledge of mathematics, statistics, and science through predictions. Aridor et al. (2023) proposed a framework to describe the interactions between statistical reasoning, scientific reasoning, and the nature of scientific understanding using the case of citizen science.

Research (Aridor et al., 2023; English, 2023; Oslington et al., 2023) suggests that it is essential to use *deterministic* reasoning flexibly, based on mathematical models and *non-deterministic/stochastic* reasoning, through statistical model use, to consider data from multiple perspectives and make more reliable predictions for better decision-making. Non-deterministic/stochastic reasoning raises awareness of the limitations of human decision-making and provides an opportunity for critical reflection. Conversely, deterministic reasoning is required when predicting maximum certainty or controlling for uncertainty by seeking conditions that reduce variability. However, few research has demonstrated the roles of mathematical and statistical models in data-driven predictions in an integrated STEM context.

Interdisciplinary Data-Driven Modelling and Functions of Mathematical and Statistical Models

We adopted *interdisciplinary data-driven modelling* (IDDM) considering mathematical and statistical models in data-driven predictions in an integrated STEM context (Kawakami, 2023a, 2023b; Kawakami & Saeki, in press). The IDDM generates, validates, and revises mathematical and statistical models and models in other STEM subjects (science, technology, and engineering) based on data/context to make better predictions (Kawakami & Saeki, in press). Data have a structure comprising a deterministic aspect (*signal*) focused on exact numbers and causal explanations with certainty and a non-deterministic/stochastic aspect (*noise*) focused on uncertainty and variability (Innabi et al., 2023). A model refers to a representation of the structure of a given system and a reflection of the modeller's series of interpretations of an object (Hestenes, 2010). A *mathematical model* refers to a representation of the signal inherent in the data, reflecting the modeller's deterministic interpretation of the data and context (Kawakami, 2023b). A typical example is the linear model $y = ax+b$ (a and b are parameters), where the value of the variable y can be determined if the value of the variable x is determined. A *statistical model* refers to a representation of the noise inherent in the data, reflecting the modeller's non-deterministic/stochastic interpretation of the data and context (Kawakami, 2023b). A typical example is the linear model $y = ax+b+\varepsilon$ (a and b are parameters), where the value of the variable y cannot be determined even if the value of the variable x is determined and distributed by a random error ε. *Models in other STEM subjects* involve modellers' representations and interpretations of data, which are relevant to big ideas in STEM disciplines (Kawakami & Saeki, in press), such as scientific models (e.g., motion models of a falling body and structural models of seeds) and engineering models (e.g., scale model/prototypes).

Mathematical and statistical models describe phenomena and explain their prediction mechanisms. Ärlebäck and Doerr (2020) showed that models serve as *descriptive means* and *explanatory means*. The former is a function of understanding and describing the behaviour of events. The latter is a function of explaining the structure of events and the mechanisms of their structure in a unified and comprehensive way and elucidating why events behave as they do. They pointed out that a unified explanation of several events using the same model leads to the idea of generalisation, and it is necessary to combine several models with different perspectives to provide a comprehensive explanation of a single event.

Additionally, the descriptive and explanatory functions of mathematical and statistical models are essential in an integrated STEM context. To build a causal narrative about why a phenomenon occurs, Baptista et al. (2023), who identified the core features of a reasonable explanation for a STEM problem, required students to describe what was happening in a given scenario (by summarising patterns in the data), make connections between observations and mathematical or statistical models, and use scientific ideas. Thus, mathematical and statistical models are expected to contribute to data-driven predictions in an integrated STEM context through IDDM by describing and reading the signal and noise characteristics in the data and explaining them in a unified and comprehensive manner (Table 1).

Table 1

Functions of Mathematical and Statistical Models that Could Contribute to Data-Driven Prediction

Functions	Descriptions
Descriptive means	Mathematical and statistical models, such as graphs and statistics, provide an external representation of the signal and noise characteristics inherent in data, providing an understanding of how the data behaves
Explanatory means	Mathematical and statistical models, such as graphs and statistics, provide a unified and comprehensive explanation of the signal and noise characteristics inherent in data, providing an understanding of the behaviour of data and the mechanisms and causal relationship of events behind the data

Given the theoretical framework of the functions of the mathematical and statistical models in Table 1, we formulated and addressed the following research question:

- Considering the model functions of descriptive and explanatory means, how do students use mathematical and statistical models when making predictions through IDDM?

Research Design

Setting, Participants, and Context

To answer the research question, we used data from the IDDM practice implemented in Year 4, where students used mathematical and statistical models and models in other STEM subjects for data-based predictions. An overview of this practice is given in Kawakami and Saeki (in press); however, this study is substantially different from our previous work in that it analyses the role of models in data-driven predictions in practice.

The participants were students (n = 30) from a Year 4 class (aged 9–10 years) in a public primary school in Japan. They had learned about bar graphs, line graphs, and two-dimensional tables. However, they were unaware of representative values, dot plots, and histograms. The practice comprised nine 45-minute lessons in mathematics and cross-curricular enquiry classes and addressed the *Seed Dispersal Task* (Figure 1), incorporating data-driven predictions into Fitzallen et al.'s (2019) seed dispersal material for integrated STEM education.

Figure 1

Seed Dispersal Task (Partial)

> The buckleya lanceolata seeds (Figure 2a) come in various sizes. The speed at which they fall seems to vary depending on the seeds. Therefore, how can the flight time of the seeds be increased?
>
> **Sub-task:** After several experiments with the seed prototype (Figure 2b) to measure the flight time, predict what will happen to it if the slit length of the prototype is varied, as shown in Figure 2c. After making a prediction, validate it through experiments.

The goal of the task was to redesign the shape of the buckleya lanceolata seed (Figure 2a) to maximise flight time. After several experiments that measured the flight times of seed prototypes that behaved similarly to the seeds when they fell (Figures 2b and 2c), the sub-task involved predicting trends in flight times with the prototype shape as a variable and validating the predictions with real data. In the sub-task, the students generated mathematical and statistical models (e.g., line graphs) and models in other STEM subjects. For example, the students interpreted data trends deterministically and stochastically in relation to seed prototype size, weight, and shape as engineering models. Students also generated scientific models of the motion of a falling body under air resistance and the structure and function of the seeds.

Figure 2

Buckleya Lanceolata Seed and the Seed Prototypes (Photos Taken by Shohei Chiba)

To answer the research question, we focused on a sub-task involving prediction. In the lessons, before working on the sub-task, the students experienced dropping prototypes with vertical lengths of 15 cm and 20 cm and predicted the change in flight time as the vertical length increased, by drawing line graphs on the worksheets. After making predictions, they collected data on the flight times of the prototype with longer vertical lengths, plotted the datasets on a line graph, and compared the predicted graph with the actual data to validate their predictions.

In the sub-task, the students collected data by experimenting with the flight time of a prototype with a fixed vertical length and slit lengths of 3 cm, 6 cm, and 9 cm (Figure 2c). Then, they plotted these data online graphs and presented their predictions regarding the change in flight time when slit length increased. They drew line graphs on the worksheet and calculated the differences in the data (*Prediction*). Once the predictions were made, they collected data on the flight time of the prototype with a further increase in slit length, plotted the data on a line graph, and validated their predictions by comparing the predicted graph they made with the one containing the actual data (*Validation*).

Data Collection and Analysis

We analysed 30 students' worksheet excerpts in the *Prediction* and *Validation* activities and used a post-class interview protocol for complementary analysis. The prediction intention could also be written in the validation statement; therefore, it was included in the analysis. These data were analysed in three coding phases to answer the research question. The first author performed these coding phases, and the second author validated them. The differences in interpretation between the two authors were discussed until an agreement was reached. The analyses were revised as necessary.

In Phase 1, we identified and coded the mathematical and statistical models generated in the sub-task based on the framework of our study. In this analysis, the exact representation was not taken absolutely but relatively as a mathematical or statistical model depending on the student's intention to create and interpret the representation. For example, if a student interpreted a line graph deterministically, the model was taken as mathematical; if interpreted non-deterministically or stochastically, the model was taken as a statistical one.

In Phase 2, we examined whether these models' functions were descriptive or explanatory, based on Table 1, and coded them accordingly. In this analysis, the descriptions of the features of the trends and variability of the flight time data were judged as the emergence of the descriptive function. The use of the models to explain the causal relationship that changes flight time and its consistency with the results of previous experiments was judged as the emergence of the explanatory function.

In Phase 3, we disaggregated the interdisciplinary aspects of students' descriptive or explanatory use of mathematical and statistical models, focusing on the inclusion of models in other STEM subjects (i.e., engineering models such as prototypes and scientific models such as the air resistance model).

Results

To varying degrees, all participating students used a line graph as either a mathematical or statistical model to predict the flight times of seed prototypes. Considering the functions of the mathematical and statistical models (Table 1), we classified students' use of mathematical and statistical models into three types (Table 2).

Table 2

*Types of Mathematical and Statistical Model Use (*n *= 30)*

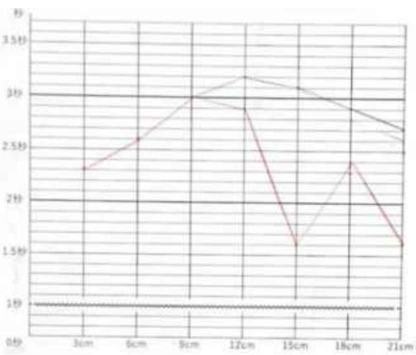
Types	Mathematical model	Statistical model	# (%)
1	Descriptive means	N/A	5 (17%)
2	Descriptive and explanatory means	N/A	20 (66%)
3	N/A	Descriptive and explanatory means	5 (17%)

Type 1 comprised students who used a mathematical model as a descriptive method. Students belonging to this type used the line graph only as a means of reading trends (signal) in the already collected data, but only superficially read the graphs (e.g., "As the flight time was rising, I thought it would continue to rise").

Type 2 comprised students who used a mathematical model for descriptive and explanatory purposes. This was the most common type. Students belonging to this type used line graphs as a means of reading trends (signals) in the already collected data and also as a means of asserting the reasonableness of predictions. They explained the causal relationship between the changing flight time and consistency with experimental results for different horizontal lengths using the different slopes of the graphs. Figure 3 shows an example of the Type 2 model use, demonstrated in a worksheet by Hata (student pseudonym), where on the left is a line graph of the real data and their prediction, and on the right is the reason for the prediction. They described the relationship between feather length and flight time from a line graph of the experimental results before the vertical length was stretched ("In the previous experiment of stretching the vertical length, if the length was stretched too far, the flight time became shorter"). Additionally, they explained why the slope of the flight time graph varied based on the prototype (e.g., "the weight of the prototype remained the same, so I thought the graph would not drop too steeply").

Figure 3

Descriptive and Explanatory Use of a Line Graph as a Mathematical Model From Hata's Worksheet



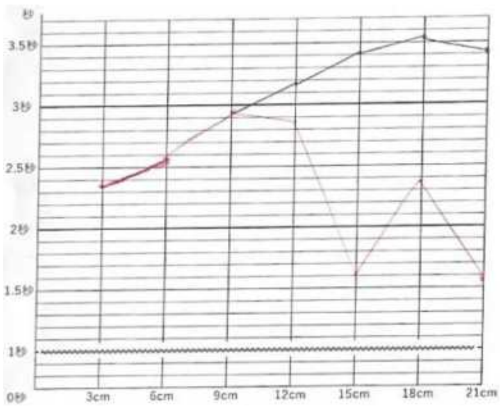
In the previous experiment of stretching the vertical length, if the length was stretched too far, the flight time became shorter, so I thought that if the slit was made too long, the flight time would become shorter. But the weight of the prototype remained the same, so I thought the graph would not drop too steeply. Moreover, I thought that the lower part of the prototype would become too short and the leaf (wing) part too long, so it wouldn't fly very well. …I thought that the graph would fall gradually, because the part of the prototype that is in contact with the wind would become larger.

Note. Black graph: Predictive data; Red graph: Real data; Horizontal axis: Prototype slit length; Vertical axis: Flight time.

Type 3 comprised students who used a statistical model as a descriptive and explanatory means. Students belonging to this type used line graphs as a means of reading variability (noise) in the data already collected and also as a means of asserting the reasonableness of predictions. Figure 4 shows an example of the Type 3 model used, demonstrated in a worksheet by Aki (student pseudonym). They described variability in the flight time from a line graph ("the flight time increased between 0.20 and 0.39 s") and utilised it to make predictions with an awareness of the range of values ("the flight time would also increase with a further increase in slit length, roughly between 0.20 and 0.39 s"). Referring to the geometry of the prototype, they also explained in a unified way the results of previous experiments with different vertical lengths and the results of the current experiments with different slit lengths ("The reason for lowering the graph at the end was the same as in the previous experiment with different vertical lengths: I thought that the longer the slit, the faster it would fall").

Figure 4

Descriptive and Explanatory Use of a Line Graph as a Statistical Model From Aki's Worksheet



Since the flight time increased between 0.20 and 0.39 s, I predicted that the flight time would also increase with a further increase in slit length, roughly between 0.20 and 0.39 s. The reason for lowering the graph at the end was the same as in the previous experiment with different vertical lengths: I thought that the longer the slit, the faster it would fall.

Note. Black graph: Predictive data; Red graph: Real data; Horizontal axis: Prototype slit length; Vertical axis: Flight time.

The interdisciplinary aspects (i.e., engineering and science relevance) in students' descriptive and explanatory use of mathematical and statistical models are shown in Table 3.

Table 3

Interdisciplinary Aspects in Students' Use of Mathematical and Statistical Models

Roles	Mathematical model	#	Statistical model	#
Descriptive means	Engineering relevance		N/A	
	Understanding trends (signal) in the data in relation to the prototypes	2		
Exploratory means	Engineering relevance		Engineering relevance	
	Explaining the data trends (signal) using information from the prototypes	20	Explaining reasons for varying flight times (noise) using information from the prototypes	5
	Science relevance		Science relevance	
	Explaining the data trends (signal) using informal scientific knowledge of air resistance	4	Explaining reasons for varying flight times (noise) using informal scientific knowledge of air resistance	1
	Explaining the data trends (signal) based on observations from the experiment	1		

Note. Statements that applied to more than one category were counted.

Twenty-five students used mathematical and/or statistical models to make engineering and/or science-related predictions. Regarding descriptive means, the students used the prototypical context to read trends (signals) in the graphs. They described the length of the wings and the weight of the prototype by replacing the line graph variable from the flight time with the weight of the prototype. As for the explanatory means, the students used information from prototypes (i.e., volume and weight remained the same, balance of form, and centre of gravity), informal scientific knowledge of air resistance, and observations from the experiment with prototypes to explain the data trends (signals) and reasons for varying flight times (noise). For instance, Hata explained data trends by relating a prototype's weight, shape, and informal scientific knowledge of air resistance to the slope of a graph (Figure 3). In both mathematical and statistical models, the explanatory means were more explicitly related to engineering (particularly prototypes) and science than the descriptive means.

Discussion and Concluding Remarks

This study addressed some aspects of Year 4 students' data-driven predictions in an integrated STEM context, taking the descriptive and explanatory functions of models as perspectives. We discuss two findings regarding this research question.

First, all the participating students consciously or unconsciously used mathematical or statistical models for descriptive and explanatory purposes (Table 2). In line with Oslington et al. (2023), mathematical and statistical models helped students recognise the patterns and structural features of data to support their predictive reasoning. However, more than half of the students tended to use mathematical models only for descriptive and explanatory purposes. This may be because the primary school students involved in this study were mainly exposed to mathematical models rather than statistical models in their everyday mathematics lessons (MEXT, 2018), and they have difficulty understanding complex variations (e.g., Watson et al., 2023). The fixed form of the line graph representation used for predictions might have further triggered the generation of a mathematical model (cf. Oslington et al., 2023; Watson et al., 2023), thereby indicating that more research is necessary to examine students' use of the statistical model in data-driven predictions.

Second, 80% of students connected mathematical or statistical models to other STEM subjects (Table 3). As seen in the case of Hata (Figure 3), mathematical or statistical models, when connected with models in other STEM subjects (i.e., prototypes and scientific models), describe the characteristics of the data and also provide explanatory power for the causal relationships behind the characteristics of the data (i.e., the reasons why flight time varied). Through these explanations with models, students' own scientific hypotheses and interdisciplinary knowledge—linking mathematics and engineering or science—are constructed (Figures 3 and 4), leading to the development of epistemic knowledge (Tytler, 2020). This finding provides evidence of the need for multiple models in demonstrating explanatory power (Ärlebäck & Doerr, 2020) and an example of a reasonable explanation in an integrated STEM context (Baptista et al., 2023).

The findings of this study that data-driven predictions in an integrated STEM context, including IDDM (Kawakami & Saeki, in press), could contribute to the development of STEM content knowledge in justifying predictions as well as their application. On the one hand, data-driven predictions contributed to other STEM subjects by encouraging the generation of students' scientific hypotheses and interdisciplinary knowledge through the process of using models as explanatory tools in prediction. On the other hand, other STEM subjects contributed to data-driven predictions by making sense of the data and models and strengthening the validity of predictions. These findings extend English's (2023) results, which revealed students' application of multidisciplinary knowledge, and advance the role of mathematics/statistics in STEM education as well as in the mathematics curriculum as more than just a service subject (e.g., ACARA, 2022; English, 2016; MEXT, 2018). These findings are based on a case study of the activity of a single prediction in the Seed Dispersal Task. A future step is to analyse in detail the teacher's role in promoting student predictions as well as the relationship between the development of students' predictions through the task and how they use mathematical and statistical models.

Acknowledgements

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References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *The Australian Curriculum (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/

Aridor, K., Dvir, M., Tsybulsky, D., & Ben-Zvi, D. (2023). Living the DReaM: The interrelations between statistical, scientific and nature of science uncertainty articulations through citizen science. *Instructional Science*, *51*(5), 729–762. https://doi.org/10.1007/s11251-023-09626-8

Ärlebäck, J. B., & Doerr, H. M. (2020). Moving beyond descriptive models: Research issues for design and implementation. *Advances in Research in Mathematics Education*, *17*, 5–20. https://doi.org/gjgsmz

Baptista, M., Jacinto, H., & Martins, I. (2023). What is a good explanation in integrated STEM education?. *ZDM Mathematics Education*, *55*(7), 1255–1268. https://doi.org/10.1007/s11858-023-01517-z

English, L. (2016). Advancing mathematics education research within a STEM environment. In K. Makar, S. Dole, J. Višňovská, M. Goos, A. Bennison, & K. Fry (Eds.), *Research in mathematics education in Australasia 2012–2015* (pp. 353–371). Springer.

English, L. (2023). Multidisciplinary modelling in a sixth-grade tsunami investigation. *International Journal of Science and Mathematics Education*, *21*(Suppl. 1), 41–65. https://doi.org/mbs5

Fitzallen, N., Wright, S., & Watson, J. (2019). Focusing on data: Year 5 students making STEM connections. *Journal of Research in STEM Education*, *5*(1), 1–19. https://doi.org/10.51355/jstem.2019.60

Geiger, V., Gal, I., & Graven, M. (2023). The connections between citizenship education and mathematics education. *ZDM Mathematics Education*, *55*(5), 923–940. https://doi.org/10.1007/s11858-023-01521-3

Hestenes, D. (2010). Modeling theory for math and science education. In R. Lesh, P Galbraith, C. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 13–41). Springer.

Innabi, H., Marton, F., & Emanuelsson, J. (2023). Sustainable learning of statistics. In G. F. Burrill, L. de Oliveria Souza, & E. Reston (Eds.), *Research on reasoning with data and statistical thinking: International perspectives* (pp. 279–302). Springer.

Kawakami, T. (2023a). *Research on the learning and teaching of data-driven modelling in school mathematics* [Doctoral dissertation, in Japanese]. Hyogo University of Teacher Education.

Kawakami, T. (2023b). A triplet of data/context, mathematical model, and statistical model: Conceptualising data-driven modelling in school mathematics. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.), *Proceedings of the 13th congress of the European Society for Research in Mathematics Education* (pp. 1243–1250). Budapest: Hungary, Alfréd Rényi Institute of Mathematics and ERME.

Kawakami, T., & Saeki, A. (in press). Extending data-driven modelling from school mathematics to school STEM education. In J. Anderson, & K. Makar (Eds.), *The contribution of mathematics to school STEM education: Current understandings*. Springer.

Makar, K., & Rubin, A. (2018). Learning about statistical inference. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 261–294). Springer.

Ministry of Education, Culture, Sports, Science and Technology (MEXT). (2018). *Elementary school teaching guide for the Japanese course of study (notified in 2017): Mathematics (Grade 1–6)* [in Japanese]. https://www.mext.go.jp/content/20211102-mxt_kyoiku02-100002607_04.pdf

Oslington, G., Mulligan, J., & Van Bergen, P. (2023). Shifts in students' predictive reasoning from data tables in years 3 and 4. *Mathematics Education Research Journal*. https://doi.org/mb63

Tytler, R. (2020). STEM education for the twenty-first century. In J. Anderson, & Y. Li (Eds.), *Integrated approaches to STEM education* (pp. 21–43). Springer.

Watson, J., Wright, S., Fitzallen, N., & Kelly, B. (2023). Consolidating understanding of variation as part of STEM: Experimenting with plant growth. *Mathematics Education Research Journal*, *35*(4), 961–999. https://doi.org/10.1007/s13394-022-00421-1

Reducing Mathematics and Examination Anxiety Using the Five Question Approach

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Mathematics anxiety and examination anxiety affect student well-being, engagement, and performance in the mathematics classroom. An increase in confidence in the mathematics classroom leads to a decrease in anxiety. This paper reports on the Five Question Approach (FQA) to teaching mathematics which provides teachers with a method of reducing these anxieties. This case study draws on data collected in three Australian secondary classrooms using the FQA as an intervention. Findings indicate that when added to normal classroom practice, the Five Question Approach alleviated these anxieties to some extent and increased engagement and academic performance.

Mathematics anxiety is a significant issue for many students and varies from other anxieties as it is specific to mathematics (Li et al., 2021). Much research has been conducted on mathematics anxiety in both teachers and students, with some research providing suggestions to assist in the alleviation of mathematics anxiety, including reinforcement activities, time for students to develop understanding, class discussions, formative assessment, timely feedback, and linking the level of difficulty of the questions to the students' level of understanding (Passolunghi et al., 2016; Pizzie et al., 2020; Russo et al., 2020). This paper proposes that the Five Question Approach (FQA) may be a pedagogical strategy that could not only reduce mathematics anxiety and examination anxiety but also increase engagement and academic performance in the mathematics classroom.

Literature Review

Mathematics Anxiety

Mathematics anxiety is a feeling or an emotional response to the challenge of completing a mathematical task, and it often takes the form of irregular breathing (Allen & Stambaugh, 2023). There are many definitions of mathematics anxiety, but it is accepted to be a feeling of fear, apprehension, and tension that negatively influences a student's ability to solve mathematical problems (Justicia-Galiano et al., 2016; Li et al., 2021). Mathematics Anxiety is specific to mathematics content, is separate from other anxieties, and can occur before the student commences the mathematical task (Moustafa et al., 2021). Students suffering from mathematics anxiety may not attend mathematics classes or avoid selecting classes that involve mathematics (Justicia-Galiano et al., 2016)

There is a discussion on which comes first, the mathematics anxiety or the poor performance; whatever the direction, an increase in performance through greater confidence could result in a reduction in mathematics anxiety (Fernandez-Blanco et al., 2023; Rossi et al., 2022). Mathematics anxiety reduces working memory by diverting cognitive resources away from procedural execution thus reducing the students' ability to solve mathematical problems (Girelli et al., 2000). Interventions like deep breathing exercises and positive affirmation can reduce mathematics anxiety and a change to teaching that allows students to correct conceptual misunderstandings, practice procedural skills with a view to automatisation, and solve problems in groups could also reduce mathematics anxiety (Allen & Stambaugh, 2023). Achievement, engagement, and enjoyment of mathematics are all impacted by mathematics anxiety, and improving students' mathematical skills is essential for a reduction in mathematics anxiety (Li et al., 2021). As the FQA is designed to improve achievement, engagement and the enjoyment

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 319–326). Gold Coast: MERGA.

of mathematics then this paper suggests that the FQA may then potentially reduce mathematics anxiety.

Five Question Approach (FQA)

Mathematics anxiety is heightened by many factors but crowded curriculum and restrictive time to develop understanding without personalised assistance are major issues (Ma & Xu, 2004; Mann & Walshaw, 2019; Putwain & Wood, 2023; Wang, 2024). Many current NSW mathematics classes use a traditional teaching approach with a prevalence of teaching mathematics with procedural questions, little opportunity for problem solving, a restrictive linear delivery of curriculum and not revisiting any student problem areas (Sullivan, 2011). The FQA was designed by the author to allow teachers to take small steps towards a change in practice without completely changing their entire approach to teaching. The FQA evolved as a means of not just consolidating procedural knowledge but also developing conceptual knowledge and increasing student confidence and engagement. The FQA does not replace the teacher's normal lesson but is meant to be one of the tools at the teacher's disposal to potentially increase positive learning outcomes and student confidence, thus potentially reducing mathematics anxiety. The FQA varies from the traditional teaching approach as multiple opportunities are provided over a period of time for students to consolidate their mathematical understanding of previously taught content areas and procedures, with a focus on building confidence, problem solving and conceptual understanding.

The FQA consists of four procedural questions and one conceptual question. The questions are presented at the commencement of the lesson and a new set of five questions are provided every lesson. The basis for the first four questions being of a procedural nature is threefold. Firstly, they are designed to develop procedural fluency, to decrease the load on working memory when problem solving potentially reducing mathematics anxiety (Mann & Walshaw, 2019). Secondly, they are developed from previously encountered material as direct revision and consolidation, aligning with the view that reinforcement activities reduce mathematics anxiety (Mehmet & Hulya, 2021). Thirdly, they are used to prepare students for future content areas by providing the procedural basis for future topics. The first four questions are usually based on one question from the previous lesson, two questions from previous content areas, and one question preparing for upcoming topics. The purpose is to provide time for the students to revise and develop understanding and confidence in all content areas. The procedural questions are not decontextualised right/wrong questions but the questions are carefully chosen based on student areas of difficulty as perceived by the teacher. Often, they are chosen from material that the students should have understood from previous classes or that they have not had time to fully understand. This aligns with the research by Mehmet & Hulya (2021) where a lack of understanding from a previous class was a significant factor in mathematics anxiety along with teachers not completing reinforcement activities. Putwain and Wood (2023) state that individualised learning matched to the student, while possibly difficult in the classroom can result in a decrease in mathematics anxiety. The first four questions are designed to address these issues through careful selection based on student needs. The complexity and degree of difficulty of the first four questions are progressively increased as student understanding increases using repetition with variation (Rohrer et al.,2020), thus providing reinforcement of the areas of student difficulty.

The fifth question is an open-ended conceptual problem-solving question on any topic that provides the opportunity for a deeper understanding of that concept which could result in a decrease in mathematics anxiety (Ma & Xu, 2004). By providing opportunities for students to discuss their solutions to question five with multiple strategies for solution may reduce mathematics anxiety (Mann & Walshaw, 2019). Students are required to complete the questions in order, from one to five, to ensure that they do not omit the questions that they were less confident in or, did not want to do. While the students are attempting the questions, the teacher moves about the room, providing assistance, if required, and giving positive diagnostic feedback. The one-on-one interaction between the teacher and the student is a factor that may reduce mathematics anxiety (Mehmet & Hulya, 2021). As the questions are based on student need every application of the FQA is a formative assessment opportunity with immediate diagnostic feedback, which may reduce mathematics anxiety (Wang, 2024).

The table shows an example of the FQA for the Year 8 class and includes the purpose of the question relating to the FQA description. The students only receive the question.

Table 1

Sample Five Questions

1	Write 35 : 20 as a ratio in simplest form: A revision question
2	Find 17 ½ % of $540: Preparation for the next topic
3	Solve 5 − 3x = 2x − 10: A revision question
4	Find the length of the hypotenuse in a right-angled triangle with sides of 8 cm and 11 cm: Current topic
5	Write 5 scores that have a mean of 4, a median of 3, and the mode is not 3. How many solutions are possible? Add another aspect that will reduce the number of solutions: Revision and consolidation

Given that students work and learn at different speeds, the first four questions are selected so that the majority of students can complete them successfully within the time allocated by the teacher. The purpose is to develop procedural fluency, assist in developing understanding, and increase student's level of confidence, thereby potentially reducing their mathematical anxiety (Mehmet & Hulya, 2021). Question five may be superficially answered by some students and deeply investigated by those completing questions one to four more quickly, allowing for differentiation in the classroom. While the teacher usually gives the answers to the first four questions, students are selected to present their solutions to question five to the class, providing the opportunity for students to learn from each other (Mann & Walshaw, 2019). Students with a superficial answer would be selected first and then students with a more extensive answer, allowing all students the option of presenting solutions over time.

The first four questions in the FQA have the questions increase in difficulty and complexity as students show that they understand the method of solution or solutions. The questions are not chosen to foster rote learning but to provide an opportunity to use memorisation, repetition with variation, and spacing to consolidate required mathematical skills, and to connect with other areas of knowledge (Rohrer et al., 2020). The teacher must make decisions about how often a question is repeated from lesson to lesson before the level of difficulty is increased. It is the level of knowledge of the individual student and the class group that must be considered by the teacher in the progress of question development (Pawley et al., 2005). By using this approach, there is an attempt to make the learning personalised within the classroom environment, which may assist in reducing mathematics anxiety (Putwain & Wood, 2023). For example, the equation questions in the FQA example would have begun as one-step equations, moved to two-step, and then continued as the student's competence increased to the equation given in question three.

While moving about the room the teacher will support any students that may still require assistance as the questions are being attempted so that all students are successful. Student success then builds confidence, and as a result, may increase student engagement and potentially reduce mathematical anxiety (Carey et al., 2016; Putwain & Wood, 2023; Skemp, 1989). The repetition of questions and the development of greater complexity at a rate tailored to student needs, through the FQA, provides students with success and positive feedback every lesson and as such provides an opportunity for both confidence and competence to grow. According to Skemp (1989), when students are working within their region of competence (domain), they feel confident and secure. When working outside of this region they may feel frustration and anxiety. The FQA allows students to gradually move to the limits of their region of competence and beyond by using a careful selection of questions to gradually extend students' mathematical knowledge and competence beyond their Zone of Proximal Development (ZPD) (Vygotsky, 1978).

Methodology

The FQA was introduced to the participating classes as an intervention designed to improve engagement, academic performance, enjoyment of mathematics and reduce mathematics anxiety. As the FQA requires time to be effective, data was collected from two schools over a school year. Focus group discussions with students, teacher interviews, and classroom observations were used to collect qualitative data on perceived academic performance, enjoyment of mathematics and levels of mathematics anxiety, at the end of terms 1, 2, 3, and 4. The examination performance of each student was analysed using data from the end of year examination in the previous academic year, compared with the half yearly and yearly examinations in the current academic year.

Contexts

Two Catholic Systemic schools both with seven graded mathematics classes in each year level 7 to 10 were chosen for data collection. Both schools graded their mathematics classes based on the students' results in the yearly examination of the previous academic year, with potential student movement based on half yearly examination performance.

School 1 allocated the second-highest graded Year 8 class, taught by two teachers who shared the class: one working three days and the other two days per week. While both teachers were mathematics trained, they both had less than five years of teaching experience. School 2 provided a Year 9 and a Year 10 class, which was the fifth-ranked graded class. Both teachers were mathematics trained with more than ten years of experience.

Data

All teachers received two days of training in the FQA at the beginning of the year. There were four follow-up half days during the year where fine-tuning of the writing of the questions and the application of the approach took place along with the teacher interviews. During the first three weeks, the teachers emailed the five questions daily for feedback. Quantitative and qualitative data on academic performance, engagement, anxiety level, and attitude to mathematics were collected. The quantitative data was collected by comparing the rankings of each student in comparison with all of the students writing the examination at that year's level compared with each student's ranking in the year level in the previous year's final examination. A student with a higher ranking would indicate academic improvement.

The qualitative data was collected through classroom observations, teacher interviews, and student focus group interviews which took place four times during the year. For each participating class the teacher selected a sample of six students from the class and they participated in the focus group discussions. The sample was stratified by selecting female and male students across a broad range of academic performance, engagement, and mathematics anxiety levels based on the teacher's knowledge of the students. All teachers had been at their current school for at least two years and had general knowledge of most students. The level of engagement and mathematics anxiety, at the commencement, was based on the classroom teachers' perception of the students. The Year 8 class engagement ranged from disengaged to highly engaged while the Year 9 and 10 classes ranged from highly disengaged to moderately engaged. The student focus group sample for each class along with their rankings in the final examination of the previous year is shown in Table 2.

Table 2

Focus Group Student Name and Rank Details

Year 8	Year 7 rank (out of 177)	Year 9	Year 8 rank (out of 180)	Year 10	Year 9 rank (out of 180)
Brenda	53	Jim	121	Allan	107
Sue	39	Allie	137	Joe	92
Corey	46	Shane	148	Sonia	136
Ella	37	Gina	102	Alison	115
Gemma	73	David	164	Sara	106
Lewis	25	Steven	133	Evan	139

There were four focus group meetings of one hour duration and the focus group questions were written beforehand. The questions centred around the students' perceptions of their mathematical ability, how the teaching and learning in their class was different from the previous year, how they felt about the five questions, any change in their level of engagement or enjoyment of mathematics, the level of stress or anxiety they felt about mathematics and the examinations and their perceived academic performance. The questions were used as a stimulus for discussion. The questions in the teacher interviews were similar and the discussion was extensive usually taking one hour for each teacher.

Findings and Discussion

The FQA allowed teachers that have not used problem solving strategies to slowly introduce these into their day-to-day teaching. The discussion around the fifth question was found to be the beginning of using student voice in the classroom. All teachers described how allowing students to explain their solution strategies for question five resulted in an improvement in understanding for some students and could reduce mathematics anxiety (Wang 2024).

As students are continually revising materials, through repetition with variation, providing reinforcement to reduce anxiety (Mehmet & Hulya, 2021), their level of confidence increases along with their success in class, and this has the effect of reducing their mathematical anxiety and particularly their examination anxiety (Mann & Walshaw, 2019). Most of the students in the focus groups for Year 9 and Year 10 classes initially stated that they had little confidence in mathematics, were not good at it, and found the classes stressful, but these aspects made a positive change over time. The students in the Year 8 focus group mostly felt confident about their mathematical abilities and related that to the fact they were in the second highest class in the year level.

Teacher Perceptions

All three teachers made comments about the increase in confidence and the reduction in stress and anxiety of the students as the school year progressed. The classroom observations, taken four times during the year confirmed the teachers' comments as the students were observed to be more interested and engaged in their learning and there was a significant positive change in the engagement of students overall. Many behavioural issues such as disruptive behaviour, students making negative comments about the material in class, and refusal to attempt questions decreased in the Year 9 and 10 classes. While the Year 8 class was always very compliant, there was a noticeable increase in engagement, which was evidenced by more students offering to provide answers and an increase in the number of students willing to present their solutions to the class. The opportunity for the teachers to spend additional time focussing on the problem areas as identified in class or at the end of a topic allowed additional time for the students to develop understanding, possibly reducing anxiety (Mehmet & Hulya, 2021). The level of engagement was directly observed to have increased through the classroom observations, and the student and teacher interviews provided additional confirmation. According to the Year 10 teacher, when discussing the yearly examination, “They built their confidence, and they were more prepared for the examination as they now had more confidence in mathematics.” The teacher meant, when stating they were more prepared, that the students were both academically and emotionally ready, and she felt, from the student feedback from the examination, that they were also less anxious.

Student Perceptions

The focus group students commented that the additional time spent on questions after the topic had officially been completed enabled them to have sufficient time to develop their understanding thereby increasing their confidence and reducing anxiety. The focus group students stated that the opportunity to continually revisit problem areas until they understood made them less anxious about mathematics and the examination (Mann & Walshaw, 2019; Mehmet & Hulya, 2021). The Year 8 focus group students felt that the FQA was significant in their improved results. Gemma said, “The five questions helped me get through the exams and achieve a higher mark.” Ella, “I answered all the questions and when unsure, thought back to a similar question in the five questions”

In the first Year 9 focus group interview, David expressed that he was not good at mathematics; it made him stressed, and he wasn’t interested. In the final focus group, he said, “It’s making [me] a lot more confident and more—what’s the word—engaged in it now. I can study easier, and with the five questions, it makes it a lot easier for me to be ready for the exam” By stating he was less stressed, David is indicating a decrease in anxiety.

The Year 10 final focus group had many comments about the examination as they had received their results and were quite pleased. Allen said, “The five questions helped me be less stressed about the exam because I knew I had a revision in the five questions.” Evan said, “It feels like an exam when you are doing them (FQA) so you’re not stressed when you are going into the exam.” The comments from both students indicated a reduction in mathematics and examination anxiety as a result of the FQA.

Academic Improvements

Many of the Year 9 and 10 focus group students indicated that they did not like mathematics and found the classes stressful, indicating a high level of anxiety. All of the classes in the study made significant academic improvements for many of the students, along with an increase in engagement and enjoyment, possibly indicating a reduction in any anxiety. The Year 9 class comprised the same students from the Year 8 class in the previous year and was taught by the same teacher. While there are many factors affecting student performance, the biggest change for this class and this teacher was the introduction of the FQA. When comparing the rankings from the Year 8 yearly with the Year 9 yearly examination thirteen students showed significant improvement, three of whom were ranked three classes higher. Three students improved slightly, with two staying the same and three performing below expectation. Considering that as a student moves up, then a student performing the same but previously ranked higher will move down one place, it is possible that all students except one showed improvement. The Year 10 class had a similar level of academic improvement.

The Year 8 class had nine students ranked in the top 20 in the yearly examination, indicating significant academic improvement. Twenty students performed above expectations, and eight performed below expectations.

Following the Year 8 half yearly examination two students, Gemma and Sue were moved up into a class taught traditionally, and one student, Zelda moved down into the class using the FQA. Zelda was added to the final focus group interview to provide the view of someone moving into the class using the FQA. Zelda made this statement about the FQA, “it (FQA) gives you confidence in the fact that you have already learnt it" and made this comment about the yearly examination, "I think it (FQA) made a huge difference … quality not quantity was important … I think it is very helpful." Both Sue and Gemma felt that the lack of FQA in their new class was a significant factor in their poorer performance in the yearly examination. Their rankings in the half-yearly and yearly examinations are displayed in Table 3

Table 3

Half-Yearly and Yearly Year 8 Rankings

Student	Year 7 yearly ranking	Year 8 half yearly ranking	Year 8 yearly ranking
Gemma	73	7	67
Sue	39	7	52
Zelda	29	63	5

Sue, who moved from the class using the FQA to one that didn't, made this comment that indicates a significant increase in anxiety when not in the class utilising the FQA:

> While I was in the exam, I skipped so many questions and ended up coming back to them, and I was freaking out because I thought I knew what I was doing. I just didn't have any of the confidence I had doing the last exam, it was just a lot of stress; but I never had any of that in the FQA class.

Conclusion

The FQA made a difference in the anxiety levels, academic performance, and engagement of many students. It is well documented that the traditional teaching approach as discussed earlier has some challenges with student engagement and enjoyment. Traditional teachers follow a linear scope and sequence that sets the order of topics and the time allowed to complete each topic usually with concentrated revision before examinations. The restricted time frame of the linear scope and sequence restricts the time students have to develop an understanding of each topic. This may exasperate the anxiety level of many students. The FQA could change how traditional teachers teach by revising topic areas after they have been completed in the linear scope and sequence allowing additional time for students to develop their understanding and introducing problem solving style questions into the classroom. This change in teaching approach may then result in a decrease in mathematics and examination anxiety for some students. Students in a classroom using the FQA have shown improvement in perceived academic performance, engagement, and enjoyment and the quantitative analysis shows an increase in academic performance.

Given the factors mentioned earlier concerning achievement, engagement, and enjoyment it has been shown that the FQA can enhance all three factors so it could be that the FQA may contribute to a reduction in mathematics and examination anxiety. Further research specifically targeting mathematics anxiety and the FQA may be beneficial.

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References

Allen, V., & Stambaugh, T. (2023). Math anxiety: Making room to breathe. *Sub-Stance, 52*(1), 217–225. https://doi.org/10.1353/sub.2023.a900557

Carey, E., Hill, F., Devine, A., & Szücs, D. (2016). The chicken or the egg? the direction of the relationship between mathematics anxiety and mathematics performance. *Frontiers in Psychology, 6*. https://doi.org/10.3389/fpsyg.2015.01987

Fernández-Blanco, A., Rojas-Barahona, C. A., Dib, M. N., & Orbach, L. (2023). Math anxiety assessment within the sate-trait anxiety model: Psychometric analysis of the "mathematics anxiety questionnaire" and "state-mathematics anxiety questionnaire" in Chilean school-aged children. *Current Psychology, 43*, 8812–8824. https://doi.org/10.1007/s12144-023-05032-y

Girelli, L., Lucangeli, D., & Butterworth, B. (2000). The development of automaticity in accessing number magnitude. *Journal of Experimental Child Psychology, 76*(2), 104–122. https://doi.org/10.1006/jecp.2000.2564

Justicia-Galiano, M.-J., Pelegrina, S., Lechuga, M.-T., Gutiérrez-Palma, N., Martín-Puga, E.-M., & Lendínez, C. (2015). Math anxiety and its relationship to inhibitory abilities and perceived emotional intelligence. *Anales de Psicología*, *32*(1), 125. https://doi.org/10.6018/analesps.32.1.194891

Li, Q., Cho, H., Cosso, J., & Maeda, Y. (2021). Relations between students' mathematics anxiety and motivation to learn mathematics: A meta-analysis. *Educational Psychology Review*. https://doi.org/10.1007/s10648-020-09589-z

Ma, X., & Xu, J. (2004). Academic achievement in adolescence: The role of parents' education. *Journal of Adolescence, 27*(2), 165–179.

Mann, L. C., & Walshaw, M. (2019). Making sense of mathematics: Students' perceptions of assessment feedback. *New Zealand Journal of Educational Studies, 54*(1), 101–120.

Mehmet, C., & Hulya, S. (2021). Factors that cause students to develop math anxiety and strategies to diminish. *Cypriot Journal of Educational Science, 16*(4), 1356–1367. https://doi.org/10.18844/cjes.v16i4.5984

Moustafa, A. A., Al-Emadi, A. A., & Megreya, A. M. (2021). The need to develop an individualized intervention for mathematics anxiety. *Frontiers in Psychology*, *12*. https://doi.org/10.3389/fpsyg.2021.723289

Pawley, D., Ayres, P., Cooper, M., & Sweller, J. (2005). Translating words into equations: a cognitive load theory approach. *Educational Psychology, 25*(1), 75–97. https://doi.org/10.1080/0144341042000294903

Passolunghi, M. C., Caviola, S., De Agostini, R., Perin, C., & Mammarella, I. C. (2016). Mathematics anxiety, working memory, and mathematics performance in secondary-school children. *Frontiers in Psychology, 7*(42). https://doi.org/10.3389/fpsyg.2016.00042

Pizzie, R. G., McDermott, C. L., Salem, T. G., & Kraemer, D. J. M. (2020). Neural evidence for cognitive reappraisal as a strategy to alleviate the effects of math anxiety. *Social Cognitive and Affective Neuroscience, 15*(12). https://doi.org/10.1093/scan/nsaa161

Putwain, D. W., & Wood, P. (2023). Personal and contextual influences on test anxiety: The mediating role of cognitive test anxiety beliefs. *ZDM Mathematics Education, 55*(2), 285–298. https://doi.org/10.1007/s11858-022-01390-2

Rohrer, D., Dedrick, R. F., Hartwig, M. K., & Cheung, C.-N. (2020). A randomized controlled trial of interleaved mathematics practice. *Journal of Educational Psychology, 112*(1), 40–52. https://doi.org/10.1037/edu0000367

Rossi, S., Xenidou-Dervou, I., Simsek, E., Artemenko, C., Daroczy, G., Nuerk, H., & Cipora, K. (2022). Mathematics–gender stereotype endorsement influences mathematics anxiety, self-concept, and performance differently in men and women. *Annals of the New York Academy of Sciences, 1513*(1). https://doi.org/10.1111/nyas.14779

Russo, J., Bobis, J., Sullivan, P., Downton, A., Livy, S., McCormick, M., & Hughes, S. (2020). Collaborative professional learning for teaching mathematics in primary school. *Teaching and Teacher Education, 88*, Article 102983.

Skemp, R. (1989). *Mathematics in the primary school*. London: Routledge.

Sullivan, P. (2011). *Teaching mathematics: Using research-informed strategies*. Acer Press. https://research.acer.edu.au/cgi/viewcontent.cgi?article=1022&context=aer

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes* (p. 86). Harvard University Press.

Wang, Y. (2024). The role of parental involvement and support in children's mathematics achievement: A longitudinal study. *European Journal of Psychology of Education, 39*(1), 431–453.

Incorporating Data Visualisation Into Teaching and Learning

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The profound advancements in technology have rendered novel forms of data and data visualisation increasingly accessible to individuals within society, thereby influencing daily decision-making processes. To address this change, this study sets out to review recent research on data-driven inquiries at the K–12 level from two perspectives: innovative data visualisation and non-traditional data sources. Our findings indicate that transnumeration of multiple data representations, along with data moves throughout the process of data visualisation, can potentially enhance the development of visual reasoning and data modelling skills.

Over the last two decades, there has been an increasing and ongoing demand for *data-literate* citizens who are capable of using data as sound evidence to make decisions on a wide range of topics (Engel, 2017). Data literacy, which can be broadly described as "learning to read and write the world with data" (National Academies of Sciences, Engineering and Medicine, 2023, p. 5), is not only a requirement for entering into the workforce but also a requisite skill for navigating everyday life. One approach to foster students' data literacy could be implementing a *data science* course that is designed for students to learn about the world through data (Gould, 2021). Defined as "the science of learning from data", data science is an interdisciplinary subject which focuses on using evidence-based approaches as well as technology to make sense of data (Donoho, 2017, p. 763). From this perspective, Donoho (2017) classified all the activities associated with data science into six divisions with *data visualisation* included in half of these divisions. In broad terms, *data visualisation* is both a *process* and a *product* that are embedded in data-based inquires rather than the equivalence of graphics (Arcavi, 2003). Moreover, the rapid development of technology has expanded the data that are available for classroom instructions (Gould & Çetinkaya-Rundel, 2013), and this change has fundamentally shaped data visualisations. This inclusion of visualisations (not just graphs) has started to appear in curricula. For example, in the latest Australian Curriculum Mathematics (AC:M), data visualisation is first introduced in Year 4 statistics as follows, "represent data using many-to-one pictographs, column graphs and other displays or visualisations" (AC9M4ST01).

As can be interpreted from this content descriptor, data visualisation is considered as supplementary graphical representations for numerical information in addition to conventional graphs such as pictographs and column graphs. Although the new version of the Australian curriculum vaguely mentions the use of "other display or visualisations", it fails to address data with "non-standard sources such as those with multiple characteristics or ambiguity (e.g., photographs, text, mobile apps)" (Makar et al., 2023, p. 976). The term *data visualisation* included in Grade 4 statistics and data display is used across Year 7 to Year 10 curriculum. This example indicates that more explicit guidelines for data visualisation need to be provided to classroom teachers and so they can incorporate this concept into everyday teaching practice. Therefore, the aim of this paper is to discuss opportunities for embedding data visualisation in the mathematics curriculum within the context of K–12 data science education. The research question guiding this literature review is:

- How can data visualisations be incorporated into classroom teaching and learning?

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 327–334). Gold Coast: MERGA.

What Constitutes Data

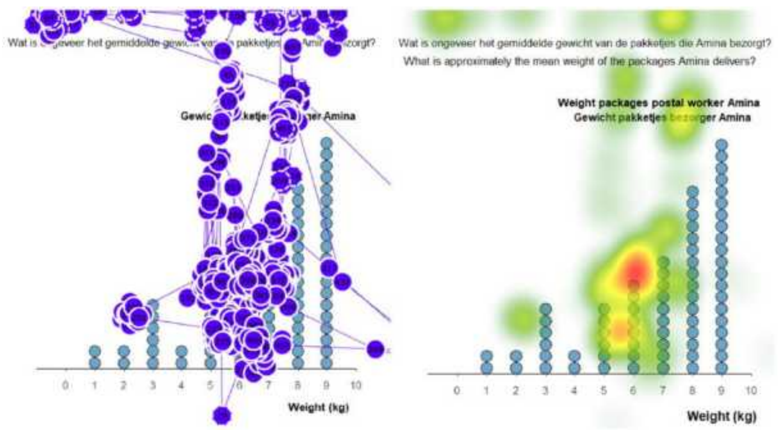
Back to the 17th century, data were exclusively produced, managed, and owned by scientists (Leonelli, 2019). During the last few decades, the development of technology has made large data sets accessible to the public. Consequently, people are immersed in statistical data within their daily experiences and are thus compelled to cultivate a discerning aptitude for data consumption (Watson, 2002). Moreover, these radical transformations have also shaped the conceptualisation of data. In other words, the categories of data considered by non-statisticians have been expanded from only *numerical* or *categorical* to a broader group which includes non-traditional data such as images, sounds videos, texts, and geospatial data. As such, data can be defined as any "information collected or generated from the world from which inferences about various phenomena can be made" (Wise, 2020, p. 165). This definition exhibits a breadth that encompasses not only the diverse forms of data but also the purpose for its manipulation and analysis. Stated differently, data serve as entities utilised by individuals to comprehend, explain, communicate, and even predict diverse phenomena within the physical world as well as our society. This calls for the need to incorporate modern data visualisations which focus on large, complex, and newer forms of data into teaching and learning practice (Nolan & Perrett, 2016).

Data Visualisation and Its Characteristics

In the realm of mathematical visualisation, data visualisation emerges as a distinct subset, characterised by its capacity to serve as both a procedural methodology (*process*) and an outcome (*product*). Prior to the work of Tukey (1977), who promoted the use of data visualisation for exploratory data analysis (EDA), the role of data visualisation was largely unknown. According to Tukey (1990), the use of data visualisation throughout EDA allows us to search, interpret, and analyse the data to explore phenomena that have occurred or that may occur. Likewise, Unwin (2020) pointed out that data visualisations for EDA were created to reveal new information. termed as "seeing the unseen" by Arcavi, (2003, p. 216). In light of this assertion, it is imperative to regard data visualisation as a facilitative instrument possessing inherent qualities conducive to discovery, rather than merely functioning as a graphical representation. The evolution of technology has substantially augmented the scale, accessibility, and modalities of data. This significant development has shifted data visualisation from conventional graphs for analysing quantitative information to instruments for reasoning about both traditional and non-traditional data.

Figure 1

The Gaze Plot (Left) and Heat Map (Right) (Boels, 2023)

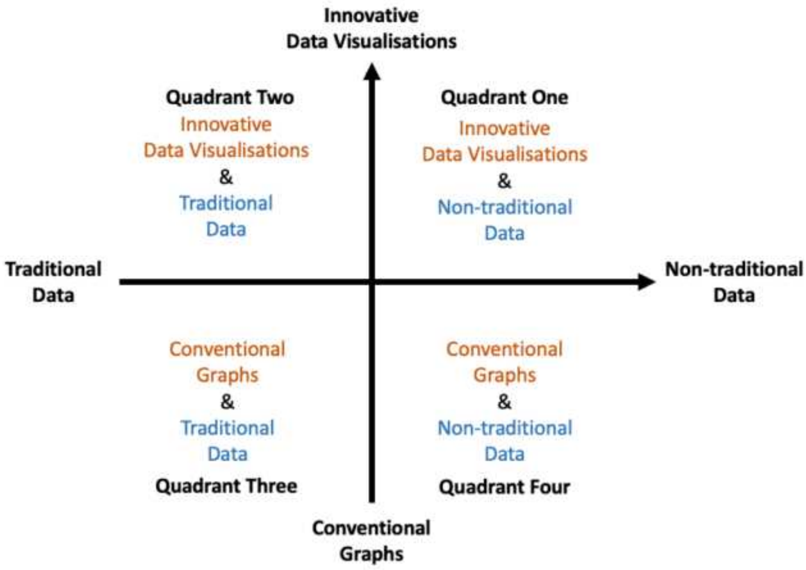


Although newer forms of data often contain valuable insights that might not be seen in traditional data, unstructured forms also make non-traditional data more challenging to visualise. That is, relying solely on conventional graphs may prove inadequate for comprehensively making sense of data, necessitating the utilization of innovative data visualisation techniques. This is evident in the study undertaken by Boels (2023), in which innovative data visualisations such as gaze plots and heat maps were used to investigate gaze data (eye-tracking data). The employment of these data visualisations facilitated researchers in understanding the methodologies that students applied into the interpretation of graphs, revealing strategies previously unnoticed by both educators and students. Additionally, novel data visualisations can be utilised to depict traditional data. A recent commentary by Lim et al. (2023) discussed the possibility of extending conventional graphs to innovative data visualisation. For example, adding a sensory dimension by attaching an audio clip to a time-series graph which depicted the sound level of New York City (numerical data) before and after the COVID lockdown. It is believed that the integration of different elements such as audio, video, interactive features as well as art is an efficient way of enhancing data storytelling. As posted by Clark and Paivio (1991), directing students to integrate events or objects into narratives will foster the establishment of mental imageries, thereby augmenting comprehension in learning.

Within the context of education, incorporating innovative forms of data and novel data visualisations may broaden the opportunities for teaching and learning of data science. The Cartesian plane shown below (see Figure 2) categorises different types of data and data visualisation into four quadrants. Too many classroom instructions and textbooks still focus on the combination of conventional graphs and traditional data (see Quadrant Three in Figure 2). As a result, further research is needed to investigate the remaining three quadrants, which encompass either innovative data, novel data visualizations, or both.

Figure 2

The Different Relationships Between Data and Data Visualisation



Review of the Literature on Innovative Data Visualisation and/or Data

The section below reviews the literature related to innovative data visualisation and/or newer forms of data. It commences with an exploration of studies involving innovative data visualisations (see Quadrant One and Two in Figure 2), subsequently progressing to an analysis of studies centred on non-traditional data (Quadrant Four).

Innovative Data Visualisations (Quadrant One and Two)

Within the context of K–12 education, there is a prevalent consensus that data visualisation ideally belongs within the discipline of probability and statistics. However, what data scientists actually do in the real world differs from what is taught in schools. That is, data visualisation is used as a tool to explore, analyse, and communicate data collected from different aspects of our life. This calls the need to design school-based learning experiences that resemble how data visualisation is used in practice. Some researchers have advocated for the integration of data science across different subjects (Jiang et al., 2022). In recent years, there has been an increasing interest in how to incorporate innovative forms of data visualisation into classroom teaching and learning across different disciplines. An example of this is the study conducted by Matuk et al. (2024) in which 15 to 30 middle-school students reasoned about both qualitative and quantitative data via *data-art inquiries*. In the *Photoessay* unit, students collected air quality data from a public map-based database, after making a scatterplot to investigate the association between air quality and life expectancy. Finally, students transcended traditional data by employing photography as a means to explain the diminished air quality within their neighbourhood area (see Figure 3). The use of photo-essay promoted "learners' existing relationship with data" and this facilitated students in gaining insights into the data (Wilkerson & Polman, 2020, p. 5).

Figure 3

A Student's Photoessay (Matuk et al., 2024)

Students in the preceding study utilised diverse forms of data representations, such as scatter plots and photoessays, across the duration of the instructional module. The process of "forming and changing data representations" is described as *transnumeration* by Wild and Pfannkuch (1999, p. 227). To be more specific, transnumeration is a data-visualisation process through which data are re-expressed and re-classified for the purpose of seeking new insights. In other words, it is a process as well as a technique that allows us to investigate the same data set via different perspectives so new meanings may be revealed. The shift from scatterplot to photoessay allowed students to visualise the potential impact of construction on environment in their local area (Matuk et al., 2024). This discovery aligns with the research by Roth and MacGinn (1997), who posited that the transition from *experience-distant* visual representations (e.g., scatterplot) to *experience-near* visual representations (e.g., photoessays) may enhance students' contextualization of data.

Another example of transnumeration in the process of data visualisation is the study undertaken by Makar (2016) who investigated the emergence of children's informal statistical inference. In this study, students used experience-near data visualisations (e.g., pictures) to record data about their classmates' shoe sizes (see Figure 4). Then, they constructed experience-distant visual representations with stacked and ordered white cards to model the shoe distribution of the class. The use of multiple types of data visualisations created opportunities for younger students to visualise the variability as well as the distribution associated with a data set. The change from *experience-near* to *experience-distant* data representation allowed young children to visualise abstract statistical concepts such as variability and distribution of data sets via the process of data visualisation.

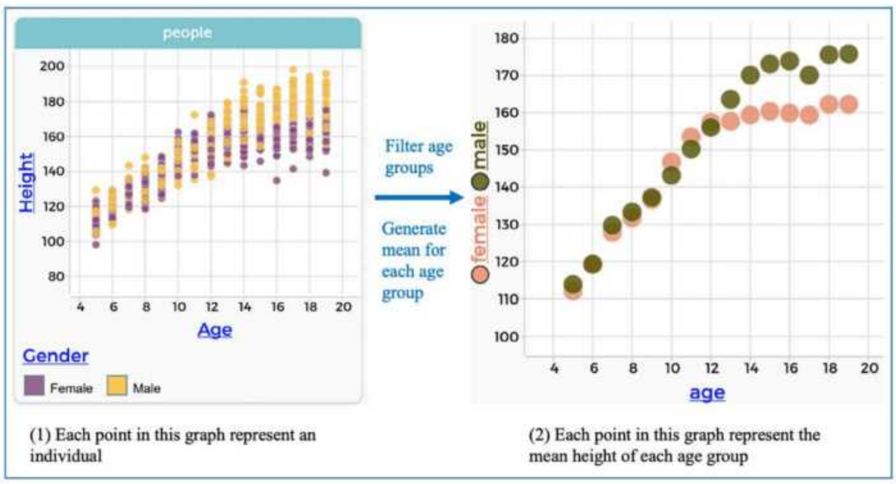
Figure 4

Transnumeration Between Different Types of Data Representations (Makar, 2016)

In addition to changing between different types of data representations, transnumeration can also occur within the same type of data visualisation via *data move*. For Erickson et al. (2019), a data move is "an action that alters a dataset's contents, structures, or values" (p. 5). The study by Erickson and Chen (2021) offers a comprehensive description of how data moves (e.g., filtering, grouping, and summarizing) can be used to investigate large and unruly data via technology-driven data visualisation tool (e.g., CODAP). As shown in Figure 5, students *filtered* each age group and then plotted *mean height* against all ages. The transformation of the scatterplot unveils the association between age and height, a relationship previously obscured within the original data representation. It should be emphasised that the adaptation of innovative data visualisation, exemplified by the dynamic visual display on CODAP, enhances the efficacy of the data visualisation process. Moreover, transnumeration does not happen spontaneously which means the pivotal role of guided discovery by educators during the process of data visualisation should not be underscored.

Figure 5

(1) The Data Representation Before Filtering; (2) The Data Representation After Filtering (Erickson & Chen, 2021)



The aforementioned study provides an illustrative instance of leveraging an innovative data visualisation tool such as CODAP to elucidate large and unstructured traditional data sets. Also, the integration of CODAP with additional online visualisation tools has the potential to streamline the decision-making process relating to machine learning. An example is the study undertaken by Erickson and Engel (2023) who proposed a data science project for high school students to construct and apply *classification trees*. In the outlined project, students explored how *the tree* worked in a real-life context of breast cancer diagnoses. Concurrently, supplementary data representations, including dot plots and scatterplots, are employed to aid students in identifying the threshold distinguishing positive from negative diagnoses. This procedure further supports the idea of using transnumeration as a way throughout the exploration and analysis of data. Although using *trees* as a mean to visualise data is new to most students, it is believed that such experience will provide students with a solid and profound understanding of how advanced statistics work in data science and machine learning (Erickson & Engel, 2023). Further, educators play a critical role in this data-based inquiry since guidance is needed for students to construct, understand, and apply *trees*.

Conventional Graphs and Non-traditional Data (Quarant Four)

Recently, a number of studies have begun to explore how to use driven visualisation to investigate non-traditional data. These studies may be broadly categorised into two groups: those employing primary data as data sources (Fergusson & Pfannkuch, 2022; Podworny et al., 2022) and those utilising secondary data from data sources that are open to the public (Jiang et al., 2022; 2023; Rao et al., 2023).

The acquisition and processing of primary innovative data frequently requires technology-driven methodologies and instruments. For example, the study by Podworny et al. (2022) offers the possibility of using sensors to collect real-time and authentic data from the local community to investigate environmental factors. In another example, the researchers designed a a step-by-step protocol to teach the participants how to use *R* code to collect and analyse data from a movie-rating website (Fergusson & Pfannkuch, 2022). In both investigations, code-driven tools (e.g., *R* code and *Python*) played essential roles in the transformation of raw data into conventional graphs such as linear regression graphs or line graphs. Put differently, students would not be able to visualise the data undergoing collection without the integration of coding. On one hand, these studies provided insights on how to "merge computational and statistical thinking" via coding in data science education (Gould, 2021, s17). Alternatively, it is uncertain whether computer programming imposes an excessive cognitive burden on learners as well as classroom teachers. Namely, students might allocate a significant portion of their time to acquiring coding skills rather than engaging in exploration and analysis throughout a data-centric inquiry. Additionally, classroom teachers may encounter challenges due to potential inexperience designing and implementing data science projects reliant on code-driven tools.

Figure 6

Using Multiple Data Representations to Evaluate Machine Learning Model (Jiang Et Al., 2022)

In contrast to data-based inquiries employing primary innovative data, the necessity of coding as a tool is less pronounced in investigations utilizing secondary innovative data. Instead, the focus of these data science projects is often on wrangling, analysis and evaluation of unstructured data. A recent study by Jiang et al., (2022) examined how high school students built machine learning models for classification of text data (e.g., positive and negative reviews of restaurants). In this study, multiple data visualisations such as dot plots and bar graphs were used for the evaluation of predictive features (e.g., keywords from the reviews) associated with the models (see Figure 6). Another example of this is a recent study undertaken by Rao et al. (2023) who designed a sequence of activities to guide learners how to use multiple data representations to make sense of real-world data. In the first activity, participants constructed network graphs to visualise the interactions between different characters in a movie. As can be seen in both studies, students constructed experience-distant representations (e.g., dot plots and network graphs) based on experience-near data (e.g., restaurant reviews and narratives of a movie). The transition from experience-distant to experience-near allowed learners to visually process innovative data and so built models to represent the data-driven phenomena.

Conclusion and Implications

The aim of this literature review was to explore how to incorporate data visualisations both as a process and a product into classroom teaching and learning. From the perspective of the four-quadrant relationship between data and data visualisation (see Figure 2), we examined recent studies focusing on data-driven inquiries. The literature view has identified that *transnumeration* of multiple data representations as well as *data moves* during the process of data visualisation is a way that may foster contextualisation, reasoning, and modelling with data. Moreover, the transition between experience-near and experience-distant, namely the data visualisation continuum could be used as a tool enhance students' visual reasoning in everyday teaching practice. However, most recent studies in this area are descriptive in nature and so more empirical studies are needed. Another uncertainty identified by this literature review is whether coding should be incorporated into data-based inquires involving innovative data. At last, there have been few attempts to investigate non-traditional data and innovative data visualisations (Quadrant One, Figure 2). Taken together, these results suggest that there is a need for researchers and educators to design and implement data-based inquiries which focus on using multiple data representations and data moves to investigate newer forms of data via the process of data visualisation.

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*(3), 215–241. http://www.jstor.org/stable/3483015

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Australian curriculum (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/

Boels, L. (2023). Reflections on gaze data in statistics education. *Teaching Statistics, 45*(1), S40-S51. http://doi.org/10.1111/test.12340

Clark, J. M., & Paivio, A. (1991). Dual coding theory and education. *Educational Psychology Review*, *3*(3), 149–210. https://doi.org/10.1007/BF01320076

Donoho, D. (2017). 50 years of data science. *Journal of Computational and Graphical Statistics*, *26*(4), 745–766. http://doi.org/10.1080/10618600.2017.1384734

Engel, J. (2017). Statistical literacy for active citizenship: A call for data science education. *Statistics Education Research Journal*, *16*(1), 44–49. http://doi.org/10.52041/serj.v16i1.213

Erickson, T., & Chen, E. (2021). Introducing data science with data moves and CODAP. *Teaching Statistics*, *43*(1), S124–S132. http://doi.org/10.1111/test.12240

Erickson, T., & Engel, J. (2023). What goes before the CART? Introducing classification trees with Arbor and CODAP. *Teaching Statistics, 45*(1), S104–S113. http://doi.org/10.1111/test.12347

Erickson, T., Wilkerson, M., Finzer, W., & Reichsman, F. (2019). Data moves. *Technology Innovations in Statistics Education*, *12*(1). http://doi.org/10.5070/T5121038001

Fergusson, A., & Pfannkuch, M. (2022). Introducing high school statistics teachers to predictive modelling and APIs using code-driven tools. *Statistics Education Research Journal*, *21*(2). http://doi.org/10.52041/serj.v21i2.49

Gould, R. (2021). Toward data-scientific thinking. *Teaching Statistics*, *43*(S1), S11–S22. http://doi.org/10.1111/test.12267

Gould, R., & Çetinkaya-Rundel, M. (2013). Teaching statistical thinking in the data deluge. In T. Wassong, D. Frischemeier, P. R. Fischer, R. Hochmuth, & P. Bender (Eds.), *Using tools for learning mathematics and statistics* (pp. 377–392). Springer.

Jiang, S., Lee, V. R., & Rosenberg, J. M. (2022). Data science education across the disciplines: Underexamined opportunities for K-12 innovation. *British Journal of Educational Technology*, *53*(5), 1073–1079. https://doi.org/http://doi.org/10.1111/bjet.13258

Jiang, S., Nocera, A., Tatar, C., Yoder, M. M., Chao, J., Wiedemann, K., . . . Rosé, C. P. (2022). An empirical analysis of high school students' practices of modelling with unstructured data. *British Journal of Educational Technology*, *53*(5), 1114–1133. http://doi.org/10.1111/bjet.13253

Jiang, S., Qian, Y., Tang, H., Yalcinkaya, R., Rosé, C. P., Chao, J., & Finzer, W. (2023). Examining computational thinking processes in modeling unstructured data. *Education and Information Technologies*, *28*(4), 4309–4333. http://doi.org/10.1007/s10639-022-11355-3

Leonelli, S. (2019). Data governance is key to interpretation: Reconceptualizing data in data science. *Harvard Data Science Review*, *1*(1). http://doi.org/10.1162/99608f92.17405bb6

Lim, V. Y., Peralta, L. M. M., Rubel, L. H., Jiang, S., Kahn, J. B., & Herbel-Eisenmann, B. (2023). Keeping pace with innovations in data visualizations: A commentary for mathematics education in times of crisis. *ZDM Mathematics Education*, *55*(1), 109–118. http://doi.org/10.1007/s11858-022-01449-0

Makar, K. (2016). Developing young children's emergent inferential practices in statistics. *Mathematical Thinking and Learning*, *18*(1), 1–24. http://doi.org/10.1080/10986065.2016.1107820

Makar, K., Fry, K., & English, L. (2023). Primary students' learning about citizenship through data science. *ZDM Mathematics Education*, *55*(5), 967–979. http://doi.org/10.1007/s11858-022-01450-7

Matuk, C., Vacca, R., Amato, A., Silander, M., DesPortes, K., Woods, P. J., & Tes, M. (2024). Promoting students' informal inferential reasoning through arts-integrated data literacy education. *Information and Learning Science*, *125*(3/4), 163–189. http://doi.org/10.1108/ILS-07-2023-0088

National Academies of Sciences, Engineering, & Medicine. (2023). *Foundations of data science for students in grades K–12: proceedings of a workshop.* National Academy Press.

Nolan, D., & Perrett, J. (2016). Teaching and learning data visualization: Ideas and assignments. *The American Statistician*, *70*(3), 260–269. http://doi.org/10.1080/00031305.2015.1123651

Podworny, S., Hüsing, S., & Schulte, C. (2022). A place for a data science introduction in school: Between statistics and programming. *Statistics Education Research Journal*, *21*(2), http://doi.org/10.52041/serj.v21i2.46

Rao, V. N. V, Legacy, C., Zieffler, A., & delMas, R. (2023). Designing a sequence of activities to build reasoning about data and visualization. *Teaching Statistics*, *45*(S1), S80–S92. https://doi.org/http://doi.org/10.1111/test.12341

Roth, W.-M., & MacGinn, M. K. (1997). Graphing: Cognitive ability or practice? *Science Education*, *81*(1), 91–106. https://doi.org/10.1002/(SICI)1098-237X(199701)81:1<91::AID-SCE5>3.0.CO2-X

Tukey, J. W. (1977). *Exploratory data analysis*. Addison-Wesley Pub. Co.

Tukey, J. W. (1990). Data-based graphics: Visual display in the decades to come. *Statistical Science*, *5*(3), 327–339. http://doi.org/10.1214/ss/1177012101

Unwin, A. (2020). Why is data visualization important? What is important in data visualization? *Harvard Data Science Review*, *2*(1). https://doi.org/10.1162/99608f92.8ae4d525

Wilkerson, M. H., & Polman, J. L. (2020). Situating data science: Exploring how relationships to data shape learning. *The Journal of the Learning Sciences*, *29*(1), 1–10. https://doi.org/10.1080/10508406.2019.1705664

Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, *67*(3), 223–248. https://doi.org/http://doi.org/10.1111/j.1751-5823.1999.tb00442.x

Wise, A. F. (2020). Educating data scientists and data literate citizens for a new generation of data. *Journal of the Learning Sciences*, *29*(1), 165–181. http://doi.org/10.1080/10508406.2019.1705678

The Assessment of Mathematical Proficiency in Written Exams: A Perspective From New South Wales (NSW)

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In this study, the evaluation of mathematical proficiency (MP) in one NSW Year 12 standardised exam paper was investigated. By considering the multi-strand model of MP, a literature-based criteria for scoring the assessment of MP in individual exam questions was first developed. The criteria were then applied to quantitatively evaluate the 2023 HSC Advanced Mathematics exam paper. The findings revealed a generally balanced assessment of all MP strands in the short-response exam questions. This study has methodological implications for evaluating the assessment of MP in external mathematics assessments, and the designing mathematics exams.

In the current education landscape, there exists continuing debates around the suitability and benefits of adopting 'teach to test' pedagogies. Critiques of such pedagogies suggest that they do not enhance student's skill domains holistically in mathematics (Robinson & Dervin, 2019), and result in students to neglect enhancing their MP skills while overly focusing on obtaining strong exam marks (Sullivan, 2011). However, Seeley (2006) pointed out that if the exams in mathematics were set in a way that assesses student's MP holistically, a 'teach to test' approach should be widely adopted and accepted. Drawing from Seely (2006), educators should re-evaluate how MP is assessed in exams to better understand why certain pedagogies are preferred and respond accordingly. The tension between the desire for comprehensive MP development and 'teach to test' approaches underscores the importance of evaluating how standardised exams, such as NAPLAN and HSC exams, assess student's MP. This evaluation can shed light on the alignment between current standardised mathematics exams and contemporary pedagogical approaches, enabling educators to refine their teaching methods, assessment strategies, and potentially adjust the curriculum to satisfy more stakeholders. This study will answer the research question of whether the 2023 NSW HSC Advanced Mathematics exam paper (NESA, 2023) assesses student's MP equally. Such analysis will allow this study to illuminate the current landscape of MP assessment in exams and elucidate potential implications for future research and educational practice.

Literature Review

Mathematical Proficiency

The MP model by Kilpatrick et al. (2001) serves as a comprehensive framework capturing the fundamental attributes conducive to successful mathematics learning. This model delineates five interdependent strands: conceptual understanding, procedural fluency, adaptive reasoning, strategic competence, and productive dispositions. Enhancement of overall MP hinges upon the concurrent development of all five strands, and each strand's enhancement is contingent on the advancement of others (Schoenfeld, 2007). Since its inception, the MP model has garnered widespread acceptance and recognition as a well-established framework, as evidenced by its widespread adoption and adaption in international mathematics curricula (e.g., DOE, 2014).

In recent research the MP model has evolved into an ultimate framework that educators are expected to comprehend and integrate into their teaching practices (Cavanagh, 2021; Sullivan, 2011). Despite its widespread acceptance among teachers, researchers, and curriculum developers, ongoing debates persist regarding the efficacy of translating the model into everyday interactions within mathematics classrooms. Questions linger about the extent to

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 335–342). Gold Coast: MERGA.

which educators successfully implement the model in their instructional approaches and whether it enhances student's MP in practical learning environments (Sullivan, 2011).

Mathematical Proficiency in the Australian and NSW Curriculum

In the Australian curriculum, MP is described as four strands (fluency, understanding, problem-solving, and reasoning), omitting the critical productive disposition strand described by Kilpatrick et al. (2001). The descriptions of the problem-solving and reasoning proficiencies in ACARA (2022) align with Kilpatrick's definitions of strategic competence and adaptive reasoning (Kilpatrick et al., 2001).

The NSW mathematics curriculum incorporates the content of the Australian curriculum into its own syllabuses, and its overarching key ideas align with the MP model. Departing from the terminology of MP, the NSW syllabus uses the term 'working mathematically' (NESA, n.d.). In the NSW senior mathematics syllabi, working mathematically encompasses fluency, understanding, justification, problem-solving, reasoning, and communication. The emphasis on fluency, understanding, and reasoning in both the NSW and Australian mathematics curriculum underscores their shared commitment to foundational aspects of MP (Cavanagh, 2021; NESA, n.d.). The three other working mathematically domains—justification, problem-solving, and communication—align with analogous strands in the MP model. Specifically, justification and problem-solving align with the strategic competence strand, emphasising the strategic application of existing knowledge to select appropriate and efficient methods for problem resolution (Schoenfeld, 2007). The communication skill domain is positioned as a prerequisite for every other domain, and involves using mathematical language effectively to articulate thoughts and solutions (Cavanagh, 2021). The various ways of conceptualising MP are consistent in demonstrating a movement away from pedagogy focused on rote memorisation and procedural mastery, to embracing cognitive processes such as critical thinking, reasoning, analysis, and problem-solving (Schoenfeld, 2007).

Tension Between Assessing Mathematical Proficiency and 'Teach to Test' Pedagogy

The shift in focus from rote learning to critical thinking and problem-solving in mathematics education has prompted a corresponding evolution in educators' perspectives on how and what to assess regarding student's MP (Corrêa & Haslam, 2021). Although the ideal scenario would involve the development of a mechanism capable of effectively and efficiently assessing all five strands of proficiency simultaneously, educators have encountered challenges in realising this goal. The primary challenge identified by educators is the realisation that not all MP strands lend themselves to assessment through traditional exams. Notably, the productive disposition strand, reflecting an individual's perception of mathematics as a meaningful pursuit in daily life, proves challenging to accurately assess within the confines of an exam (Corrêa & Haslam, 2021). Exams, often perceived as graduation prerequisites or progression milestones, may compel students to view mathematics as a compulsory rather than a personally rewarding activity. Consequently, assessing the productive disposition strand within exam settings becomes inherently problematic, as students may prioritise meeting societal expectations over cultivating an authentic appreciation for the subject (Sullivan, 2011).

In addition to the issue of assessing certain strands, Corrêa and Haslam (2021) highlighted the diversity of exams and their distinct purposes—ranging from diagnostic to formative, performance, and summative assessments—each serving specific roles in evaluating student learning. Different from 'well-balanced' mathematics programs, which can holistically enhance student's proficiency, mathematical thinking, and problem solving (Seeley, 2006), assessments can only assess a range of mathematical knowledge and skills (Schoenfeld, 2007). Due to stakeholders' attention on an incomplete image of student's mathematics learning revealed by formal assessments, teachers feel pressure to help students obtain satisfy results in assessments. This pressure transfers into ‘teach to test’ pedagogies, which focus solely on exam performance (Robinson & Dervin, 2019). While ‘teach to test’ pedagogies have received criticism from mathematics educators in both Australia and around the world (Sullivan, 2011; Robinson & Dervin, 2019; Seeley, 2006), this study posits that the issues created by ‘teach to test’ pedagogies are exacerbated by an unequal emphasis of MP strands in assessments. Therefore, it is necessary to evaluate how mathematics exams evaluates MP strands, and whether if there is an equal emphasis on each strand.

Methodology

The aim of this study was to determine whether exams assess all strands of MP, and measure the extent to which each strand is assessed. By doing this, this study aims to reveal whether MP strands are equally emphasised in the chosen exam paper. The 2023 NSW Advanced Mathematics exam paper is chosen for a document analysis as this subject is a prerequisite of a number of university degrees. Given that previous research had not established a reliable method to evaluate the assessment of MP in exams (Corrêa & Haslam, 2021), the first stage of this study involved the formulation of a set of criteria for determining which MP strands (and their sub-components) are assessed in an exam question. Establishing these criteria also supports reliability by providing transparency for subsequent analysis. The criteria were derived from a review of literature to establish a standardised foundation for the evaluation process (UoN, 2021; Schoenfeld, 2007). Following the development of criteria, the short response questions from the NSW Mathematics Advanced 2023 exam paper were analysed. The multiple-choice exam questions were omitted from the analysis due to the recognised limitations in assessing MP within such questions (Corrêa & Haslam, 2021). Each of the exam questions was individually analysed to determine which MP strands and their sub-components were assessed using the criteria established in the first stage of the study. Subsequently, each question was then provided with a score based on the scoring criteria, to reflect the extent to which the question assessed particular MP strand(s). Conducting this analysis will reveal whether the selected exam paper exhibits an equal emphasis on each of the MP strands in the exam. However, it is acknowledged that the evaluation of the questions is subjective in nature and open to interpretation. This subjective nature was unavoidable as there is little evidence of previous research establishing any accepted method to perform such analysis. To minimise the subjectivity when performing this evaluation in this novel situation, three trials were done to ensure the data obtained were consistent and valid.

Criteria for Evaluating the Assessment of Mathematical Proficiency in Exams

Given the limited research on the assessment of individual MP strands in standardised exams, there currently exists no rigorous framework to evaluate the assessment of mathematical proficiencies in exam questions. Therefore, establishing consistent criteria for each MP strand and assigning each question a score to reflect how they assess MP strands are necessary for this study. Thus, each exam question was analysed to determine (1) if they assess each individual proficiency strand, and (2) the depth to which the proficiency strand is assessed. To address the latter point, when scoring the depth to which each exam question assesses a particular MP strand, it will be scored at either a 1 (low level assessment of the strand), 2 (moderate assessment of the strand), or 3 (in-depth assessment of the strand). What is meant by 1, 2, and 3 specifically for each strand will be elaborated in the following subsections.

Conceptual Understanding (CU)

CU involves students developing both a robust and comprehensive knowledge of mathematics, and connections between different topics (Cavanagh, 2021). The Mathematics Assessment Resource Service (2017) illustrated that CU consists of six subcomponents: factual knowledge, comprehension, application, analysis, synthesis, and evaluation. The subcomponents have a hierarchical relationship, where factual knowledge was considered as the foundation and evaluation was considered to be the ultimate subcomponent within CU. Therefore, CU will receive a score of 1 if the question involves the application of basic factual knowledge and comprehension, a score of 2 if the question involves the application of learned knowledge in novel situations, a score of 3 if the question involves the evaluation of understanding through connecting multiple topics and synthesising new representations.

Procedural Fluency (PF)

PF captures student's ability to choose appropriate methods and carry them out efficiently, flexibly, and accurately (Schoenfeld, 2007). The three subcomponents of PF (efficiency, flexibility, and accuracy) all contribute to the overall enhancement of PF (UoN, 2021). Efficiency refers to the student's ability to select and execute a range of procedures, flexibility refers to the student's ability to select and use appropriate method(s) for solving the question, and accuracy refers to the student's ability to produce the correct answer (Schoenfeld, 2007). Therefore, PF will receive a score of 1 if the question involves the assessment of 1 identified subcomponent of PF, a score of 2 if the question involves the assessment of 2 identified subcomponents of PF, a score of 3 if the question involves the assessment of 3 identified subcomponents of PF.

Strategic Competence (SC)

SC emphasises the ability to apply knowledge in a range of situations and solve the problem accordingly (Cavanagh, 2021). SC also has three subcomponents (formulate, represent, and solve) which all contribute to the overall enhancement of SC (Schoenfeld, 2007). Formulate refers to the generation of relationships to solve the problem, represent refers to using mathematics to represent the given situation, and solve refers to evaluating results and calculating final results using appropriate methods in complex situations (Schoenfeld, 2007). Therefore, SC will receive a score of 1 if the question involves the assessment of 1 identified subcomponent of SC, a score of 2 if the question involves the assessment of 2 identified subcomponents of SC, a score of 3 if the question involves the assessment of 3 identified subcomponents of SC.

Adaptive Reasoning (AR)

AR refers to the ability to use mathematical language to communicate mathematical understanding (Cavanagh, 2021). In UoN's work (2017), AR involves three subcomponents: analysing, generalising, and justifying. Analysing refers to students exploring, comparing, and contrasting the problem and their knowledge. Generalising refers to students forming conjectures and identifying common patterns or properties. Justifying refers to students assessing the truth and making logical arguments. Therefore, AR will receive a score of 1 if the question involves assessment of 0 or 1 identified subcomponents of AR; a score of 2 if the question involves assessment of 2 identified subcomponents of AR; a score of 3 if the question involves assessment of 3 identified subcomponents of AR.

Productive Disposition (PD)

PD describes a student's ability to see mathematics as a worthwhile subject, and position themselves as capable mathematics learners (Cavanagh, 2021). However, there has been a strong emphasis on the connection between learning mathematics and daily life when discussing PD (Schoenfeld, 2007). Currently, research on how PD can be measured in school-based assessment is lacking. The development of criteria to assess PD is largely inferential and based on existing literature (Schoenfeld, 2007; Cavanagh, 2021). Therefore, the levels to which PD will be assessed in an exam question have been considered as follows: A score of 1 is given to questions that require students to solve a problem that is not related to daily life; a score of 2 is given to questions that require students to solve a problem that models a situation related to daily life; a score of 3 is given to questions that require students to solve a problem in a situation that is closely related to daily life.

Applying the Evaluative Criteria: Examples of Data Analysis Process

To provide transparency in how individual exam questions were analysed and the criteria outlined above applied, two detailed examples from 2023 NSW HSC Advanced Mathematics exam paper are discussed. It should be noted that the exam questions cannot be reproduced due to copyright restrictions, which means only descriptions of the questions can be provided. However, full questions can be accessed online (NESA, 2023).

For the first example, the question analysed is Question 15. This question assessed modelling financial mathematics concepts and involved two parts. Question 15 primarily assesses student’s ability to accurately and efficiently calculate the principal and future values using the formula $A = P(1 + r)^n$. Part (a) of the question asked students to calculate the principal using the formula. Therefore, part (a) required students to apply a basic understanding of the formula, and thus scored 1 for CU. Part (a) required only a simple substitution followed by solving an equation, thus it only assessed student’s accuracy in using the formula to answer the question, therefore scored 1 for PF. Part (b) of the question asked students to calculate the future value under the given condition. Therefore, part (b) required students to analyse the situation and apply their knowledge in a novel situation, thus it scored 2 for CU. Part (b) also required students to employ extra steps before substitution, thus it assessed student’s efficiency and accuracy. Therefore, it was coded Level 2 for PF. Both part (a) and (b) scored for SC, as both parts assessed the ability of formulating and representing, that both items required students to generate a relationship using the formula and given information and represent such relationship using mathematical language and equations. For AR, both parts (a) and (b) scored 2 since both questions required students to analyse a given situation and generalise the situation using the formula. Despite its focus on financial mathematical concepts, this question includes some connection with student’s everyday lives through the use of a problem context, thus both parts scored 3 for PD.

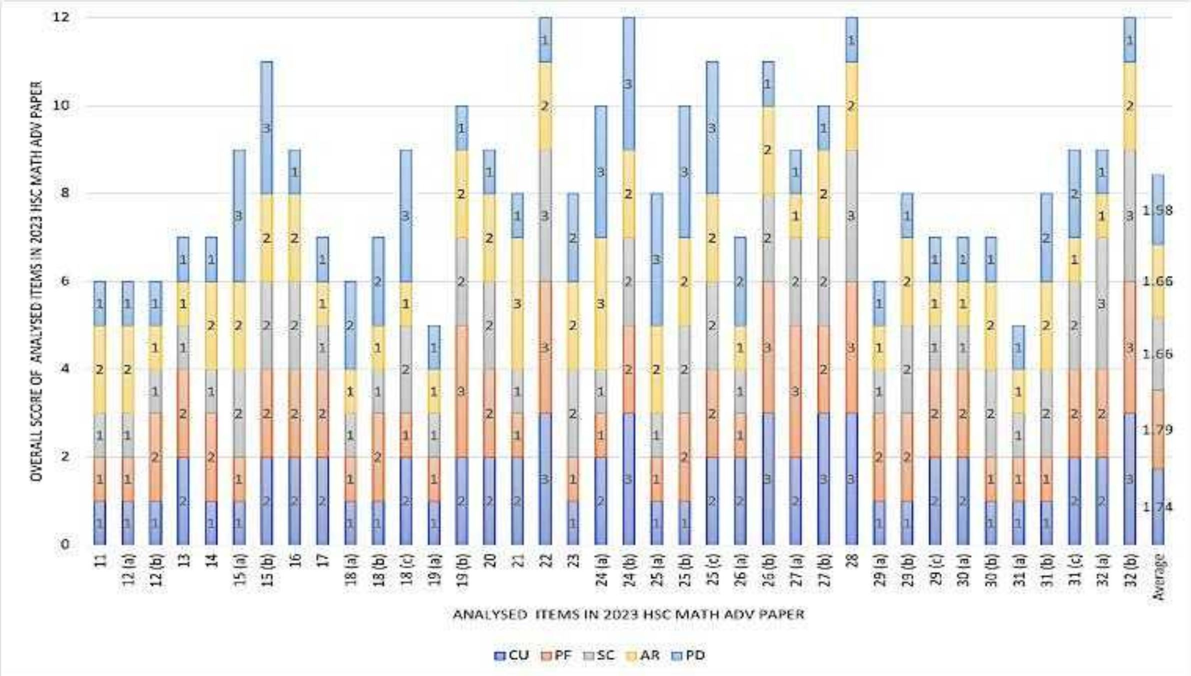
For the second example, the question analysed is Question 22. This question assessed 3D trigonometry. This item required students to utilise their knowledge of (3D) trigonometry within a complex, unfamiliar situation. Thus, it scored three for CU. This question was an excellent example of a question that assessed all three subcomponents of PF. Students were required to use different methods to perform a range of calculations accurately to get the final answer. Thus, this item scored three for PF. For SC, this question scored three. This question also tested student’s ability to formulate relationships based on the given information, represent the given information in an appropriate way to obtain further information about given situation, and calculate the final answer. For AR, this item scored two since it required students to analyse given situation and generalise the situation with the use of formulas. While this question excelled in assessing various MP strands, it is noteworthy that its inherent complexity and abstract nature may limit its direct connection to real-world contexts. Thus, this item scored one for PD.

Results

The 2023 HSC Mathematics Advanced exam paper contained 39 short-response questions (including each sub-question). The analysis and final score for each question is detailed in Figure 1. Observing the overall trends in average scores, the analysis reveals the extent to which MP strands are assessed in the 2023 HSC Mathematics Advanced exam paper.

Figure 1

Coding for Each Short Response Question in the 2023 Mathematics Advanced HSC



CU obtained an average score of 1.74 across the entire exam (*SD* = 0.7), signifying that questions predominantly evaluated students' capacity to apply fundamental factual knowledge and comprehension to novel situations. PF obtained an average score of 1.79 (*SD* = 0.7), with its subcomponents—accuracy, efficiency, and flexibility—scoring 0.9, 0.5, and 0.4, respectively. This suggests a focus on accurately solving questions, while relatively fewer questions necessitate efficient and flexible problem-solving approaches. For SC, the average score was 1.66 (*SD* = 0.7). The subcomponents—formulate, represent, and solve—achieved indices of 0.5, 0.5, and 0.6, respectively, indicating a similar emphasis on assessing student's ability to formulate equations or relationships, represent situations, and solve problems within the questions. AR obtained an average score of 1.66 (*SD* = 0.6). The subcomponents—analysing, generalising, and justifying—scoring 0.6, 0.6, and 0.4, respectively, indicating a higher emphasis on assessing the ability of analysing and generalising. PD obtained an average score of 1.58 (*SD* = 0.8), denoting a relatively lower prevalence of questions testing student's application of mathematics to everyday situations. This observation highlights a potential area for consideration and further exploration in terms of incorporating real-world contexts into mathematical assessments.

Overall, the findings illustrate a similar emphasis across the five examined strands, showcasing consistent averages without significant disparities, which further reveal an equal emphasis on MP strands within the 2023 HSC Mathematics Advanced exam paper. However, as elucidated through the analysis of specific examples, such as Questions 15 and 22, and observation of the standard deviations, it becomes apparent that the exam paper incorporates a range of questions which vary in the extent to which they assess the MP strands in a balanced way. This highlights the variability in the nature of questions, ranging from those emphasising daily life connections to those with a more abstract and disciplinary focus.

Discussion

The findings from this study provide a nuanced understanding of the distribution of how MP was examined in one exam paper, shedding light on the specific areas of MP that were emphasised. Overall, the 2023 HSC Mathematics Advanced exam paper assessed MP strands in a reasonably balanced manner. However, there were several exam questions where there was a stronger emphasis on one or two strands instead of all strands.

Despite the results demonstrating a relatively balanced relationship between the five strands, PF still scored highest among all strands, particularly on its accuracy subcomponent. This finding prompts a crucial question: is this emphasis expected and/or desirable? Examining the purpose of the HSC as a means of ranking students for graduation and university entrance, the relatively higher focus on PF, especially accuracy, is an interesting and noteworthy finding. Whether this emphasis is deemed appropriate is a subject of debate. That is, this study has found that obtaining a high score in the HSC for mathematics (Advanced) and increasing one's university entrance score is largely based on the ability to accurately replicate mathematical procedures. This is potentially an oversight in the type of students that are desirable in STEM degrees, given the importance of other skills like critical thinking and problem-solving. On the other hand, an emphasis on accuracy is not entirely unexpected in a mathematics exam. What is likely desirable, however, is emphasis on accuracy (PF) as well as other strands of MP. The need for comprehensive development of MP has been argued by academics, even in the context of selective exams like the HSC (Schoenfeld, 2007). Others contend that the selective nature of the standardised exams, for instance HSC, necessitates an objective measure of student's ability to perform tasks both accurately and effectively in the subject (Corrêa & Haslam, 2021). The purpose and focus of these HSC mathematics exams are important to consider given it influences university options in Australia.

Additionally, the findings of this study raise concerns about having an emphasis on CU or PF. While theoretically interconnected (Schoenfeld, 2007), the rise of teaching and learning practices focused on exam preparation prompts questions about whether students can enhance PF through rote learning without advancing in CU (Robinson & Dervin, 2019; Sullivan, 2011). The observed balance in the overall assessment of MP strands in this study contrasts with instances where questions assess lower CU and/or SC but higher PF. As mentioned earlier, this raises pivotal questions about the potential imbalance in the focus on proficiency strands particularly in senior years, and whether an emphasis on PF is desirable. Whether it is possible for a written format exam to more holistically assess MP is also debatable.

The study's data prompts reflection on whether the current emphasis in exams reflects an equal attention to proficiency strands and, if not, whether adjustments are needed. Furthermore, it prompts a broader exploration of assessment mechanisms to ensure students develop critical problem-solving skills and autonomy in various situations. These questions warrant further consideration and research to inform educational practices and policies.

Conclusion

This study commenced by acknowledging the recognition among mathematics educators regarding the importance of fostering comprehensive development of student's MP. However, as this study explored how the 2023 HSC Mathematics Advanced exam paper assesses MP strands, several noteworthy issues emerged. Despite the overall findings that there was a similar emphasis of each of the MP strands in the exam, it is noteworthy that there exists considerable difference between how different questions assess 5 MP strands. This observation prompts a critical inquiry into the suitability of questions in various types of assessment for effectively assessing MP. Ensuring that assessments align with the curriculum and pedagogical emphases advocated by policymakers, educators, and researchers is an important endeavour. By addressing these issues through rigorous research, the overarching goal of cultivating all MP strands can be more effectively realised in educational settings. The initial evaluation of one senior mathematics exam in this study highlight potential avenues for future research. It is potentially worthwhile to further explore how MP is assessed in exams at other stages of students' education. Also, it is worthwhile considering whether there are differences in how MP is assessed in school-based examinations compared to externally set exams such as the HSC. Such research could provide insights into whether the current emphasis on enhancing MP is identical across various educational levels and assessment types.

The assertion made by Schoenfeld (2007) regarding the varied interpretations of the MP model among educators holds significant relevance for this study. This contention becomes particularly pertinent in the context of this study, where the evaluation of questions to assess MP strands is a central focus. While there is widespread consensus on the importance of enhancing MP and its integration into teaching practices (Corrêa & Haslam, 2021; Sullivan, 2011), the absence of a universally accepted method for evaluating how MP strands are assessed in exam papers is a notable gap in the existing literature. In response to this gap, the current study has contributed to methodologies in this area by developing a set of criteria for analysis that is grounded in the existing understanding of MP strands. However, the acknowledgment that other researchers may hold different opinions regarding the details of the evaluation mechanism may influence the validity and reliability of this study. This acknowledgment underscores the need for continued dialogue and refinement in methodologies for evaluating MP in exam papers, with the aim of establishing more standardised and widely accepted assessment frameworks in the future.

References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: F–10 Mathematics Curriculum (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/

Cavanagh, M. (2021). Introduction: the learning and teaching of mathematics. In G. Hine, R. Reaburn, J. Anderson, L. Galligan, C. Carmichael, M. Cavanagh, B. Ngu, & B. White (Eds.), *Teaching secondary mathematics* (pp. 8–34). Cambridge University Press.

Corrêa, P., & Haslam, D. (2021). Mathematical proficiency as the basis for assessment: A literature review and its potentialities. *Mathematics Teaching Research Journal*, *12*(4), 3–20.

Department of Education (DOE). (2014). Mathematics programmes of study: Key stage 4 National curriculum in England. https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/331882/KS4_maths_PoS_FINAL_170714.pdf

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: The National Academies Press.

Mathematics Assessment Resource Service. (2017). Developing mathematical proficiency: The potential of different types of tasks for student learning. [Conference presentation]. University of Nottingham. https://www.mathnic.org/downloads/06_proficiency_slides.pdf

NSW Education Standards Authority [NESA]. (2023). *Mathematics advanced 2023 HSC exam pack*. NESA. https://www.educationstandards.nsw.edu.au/wps/portal/nesa/resource-finder/hsc-exam-papers/2023/mathematics-advanced-2023-hsc-exam-pack

NSW Education Standards Authority [NESA]. (n.d.). *Working mathematically*. https://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-standard-2017/working-mathematically

Robinson, B. K., & Dervin, A. (2019). Teach to the student, not the test. *Urban Education Research & Policy Annuals*, *6*(2).

Schoenfeld, A. (2007). What is mathematical proficiency and how can it be assessed? In A. Schoenfeld (Ed.), *Assessing mathematical proficiency* (pp. 59–74). MSRI Publications & Cambridge University Press. https://doi.org/10.1017/CBO9780511755378

Seeley, C. (2006). *Teaching to the test*. National Council of Teachers of Mathematics. https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Cathy-Seeley/Teaching-to-the-Test/

Sullivan, P. (2011). *Teaching mathematics: Using research-informed strategies*. Australian Council for Educational Research.

Conveyance Technology in Supporting the Teaching and Learning of Mathematics Through Student Reasoning and Problem Solving

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Technology can play a pivotal role in mathematics education. Qualitative methods were used to report on how a Year 1–2 teacher incorporated *Conveyance technology* (iPads), to support the teaching and learning of mathematics through student reasoning and problem-solving. The study was conducted in Australia. A single case was chosen, reporting a lesson where students engaged with a 12-cube task. The findings contribute to the broader discourse on the role of technology in early mathematics education, suggesting that conveyance technology supported by effective teaching practices and open-ended problems, fosters students' reasoning and problem-solving skills.

Technology is a powerful tool that can be incorporated into mathematics lessons. For meaningful learning, technologies can be used to advance students' mathematical thinking, reasoning, and problem-solving skills (Calder & Murphy, 2018; NCTM, 2014). Other benefits include supporting conceptual understanding, collaborative learning, exploration, assessment, and communication (Cullen & Hertel, 2023). However, some teachers are not convinced of the benefits of using technology as a teaching tool in mathematics classrooms, and others have been slow to adopt its use (Dick & Hollebrands, 2011). This is of concern because not allowing students to use technology could impede their mathematical learning experiences.

In addition to using technology, teachers are encouraged to foster students' mathematical understanding through problem-solving and reasoning as emphasised in the Australian Mathematics Curriculum (ACARA, 2022). When guiding problem-solving strategies, the teacher encourages students to model the problem, check for reasonableness, look for patterns or make conjectures (Van de Walle et al., 2024). Likewise, when supporting students in developing mathematical reasoning, teachers focus on guiding the processes of comparing, contrasting, conjecturing, generalising, explaining, and justifying (Herbert & Williams, 2021; Vale et al., 2017). If teaching with technology is difficult for some teachers, integrating problem-solving and reasoning most likely presents a further challenge.

Few studies have investigated ways technology might support the teaching and learning of mathematics, particularly through student reasoning and problem-solving in the early years. Further research is needed to identify how technology can be used productively, including in primary mathematics classrooms (Boon et al., 2021; Calder & Murphy, 2018; McCulloch et al., 2018). The purpose of this study was to gain a better understanding of the role of technology in supporting Years 1 and 2 students' reasoning and problem-solving skills.

Literature Review

Technology can offer teachers new ways for students to learn mathematics (Calder & Murphy, 2018). Examples of technologies that support teaching and student learning include computers, calculators, tablets, computer algebra systems, dynamic geometry software, online games, recording devices, spreadsheets, and a variety of online tools (NCTM, 2014; Van de Walle et al., 2024). Teachers might choose to use technology with their students in different ways, for instance, calculators when exploring larger numbers, online games to practice fluency of multiplication facts or a smart board to project a mathematics problem.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 343–350). Gold Coast: MERGA.

Theoretical Framework

A Framework for the Roles of Technology in Mathematics Education (Cullen & Hertel, 2023) draws on previous research and is used to present two uses of technology as a tool in mathematics classrooms (Table 1). Teachers and researchers can use this framework when considering how the role of technology as a tool supports mathematical teaching.

Table 1

Framework for the Roles of Technology in Mathematics Education (Cullen & Hertel, 2023, p. 317)

Roles of technology	Conveyance technology uses	Encouraging collaboration	Encouraging collaboration around mathematical problems
		Sequencing and sharing	Sequencing and sharing work on mathematical tasks
		Orchestrating discourse	Orchestrating mathematical discourse
		Monitoring and assessing	Monitoring and assessing mathematical learning
	Mathematical action technology uses	Serving as a tutee	Decomposing, abstracting, and encoding mathematical procedures and processes
		Promoting cycles of proof	Creating testing, revising and proving mathematical conjectures
		Supporting case-based reasoning	Generating, organising, and analysing data
		Presenting multiple representations	Presenting and connecting representations of the same mathematical object

As shown in Table 1 there are two roles or functions of technology (Cullen & Hertel, 2023). *Conveyancing technology* refers to the use of technology for conveying and transmitting information, such as a PowerPoint presentation. The four categories of Conveyance technology include Encouraging collaboration; Sequencing and sharing; Orchestrating discourse; and Monitoring and assessing, all of which can help to support effective teaching practices (Dick & Hollebrands, 2011). *Mathematical action technology* involves using technology to respond to students' actions, as in the case of GeoGebra and Desmos. For example, action technology, may assist students in generating, organising, and analysing data (Cullen & Hertel, 2023).

The Use of Tablets (iPads)

An affordable and common technology tool used for educational purposes in Australian schools is (iPads), equipped with applications (apps) software programs. Mathematical apps have been classified as those that develop a particular type of knowledge (conceptual or procedural) and mathematical content area (e.g., measurement and number skills) (Larkin & Milford, 2018). Similarly, depending on how students use iPad apps, they could be considered Conveyance technology or Mathematical action technology.

To assess the impact of iPads on student learning in middle-year classrooms, Boon et al. (2020) conducted a systematic review of the literature. The findings of the research studies were mixed. While iPads and mobile technology could motivate students' learning and dispositions, not all studies showed significant differences in student learning when comparing the use of iPads (or not) in mathematics lessons.

The purpose of the study reported in this paper is to extend what is already known by focusing on conveyance technology as a tool in a Year 1–2 classroom when considering how iPads might support students' reasoning and problem-solving. The study sought to answer the following research question:

- In what ways did a Year 1–2 teacher incorporate conveyance technology and support students' mathematical reasoning and problem-solving?

Method

Qualitative methods were used in this single-case study, involving a lesson observation of a Year 1–2 teacher. A single-case design was deemed appropriate for reporting cases that convey stories with messages, aiming to assist others by describing the observed activities within the case (Dumez, 2015). The single-case participant was selected because he had been teaching open-ended problems for several years. Open-ended tasks are characterised by having multiple solutions and/or different possible strategies (Sullivan et al., 2016), which were embedded into his teaching approach. As part of a larger project, (part research and part professional development) the author conducted school visits, each term for one year. These visits involved lesson observations and collaborating with Foundation to Year 2 (students 5 to 8 years old) teachers during the implementation and trialling of sequences of problem-solving lessons. The lesson reported here was the final lesson observation and was conducted in December 2018.

At the time of data collection, Jim (pseudonyms used throughout) was teaching the second lesson from a measurement and geometry sequence (Sullivan et al., 2023). The big idea of the lesson was that the same volume can be used to construct different prisms. The students were asked to solve the following problem:

A rectangular prism is made from 12 cubes. What might the prism look like?

Data Collection and Data Analysis

Data collection included video recordings, field notes, and students' work samples. Audio recordings were transcribed for subsequent coding and analysis. Lesson transcripts were thoroughly read and subjected to thematic analysis, utilising the Framework for the Roles of Technology in Mathematics Education (Cullen & Hertel, 2023). The results section includes vignettes from the lesson, including coding and evidence of *Encouraging collaboration, Sequencing and sharing, Orchestrating discourse, and Monitoring and assessing* (see Table 1). Additional coding showed evidence of the teacher supporting students' reasoning and problem-solving.

Results

During the lesson, students used the camera app on iPads to capture photographs of their responses to the 12-cube task. The primary purpose of the technology tool was to collect photographs of their constructions [coded: conveyance technology use]. The following vignettes report on the role of conveyance technology as a tool for supporting students' reasoning and problem-solving.

Vignettes of the Lesson

Lesson Launch and Using Technology

The lesson was launched to the class when sitting together on the floor. After reading the problem together, students moved to their tables to solve the 12-cube task (Figure 1).

Figure 1

Students Solving the 12-Cube Task

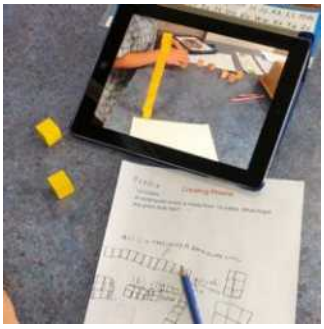
Each student had a set of 12 wooden cubes, the same colour, an iPad, a pencil, and a sheet of paper for recording their solutions (Figure 1). After the students used the 12 cubes to construct different prism, they took photographs, using their iPads. The iPad was useful as a tool for documenting the range of student responses to the task. The photographs were also used by the students to share their responses during the lesson with each other or their teacher [coded: encouraging collaboration; sequencing and sharing].

Teacher Observations

Students were also asked to draw their constructions. Many students had difficulties drawing three-dimensional prisms and instead drew two-dimensional diagrams (e.g., Figure 2).

Figure 2

Student Photograph of Their Cube Construction and Written Working Solution



As shown in Figure 2, a photograph and student's work sample offered insights for the teacher when monitoring and assessing mathematical learning [coded: monitoring and assessing]. Seeing the students' constructions provided immediate visual feedback for the teacher when considering how to respond to student learning. Interpreting only written solutions may have taken longer without the technology to support monitoring and assessing.

There were four solutions to the problem and a rotated prism was not an additional solution. When assessing student responses these were two key ideas that became the focus of the teacher and student one-on-one interactions [coded: orchestrating discourse] (Figure 3).

Figure 3

The Same Rectangular Prism Rotated

As shown in Figure 3, Jim used both the images and the blocks when modelling and exploring the problem with the students. Jim, “Now I am just going to go like this and tip over the rectangular prism.” Jim, “What can you tell me?” Jenny replied, “It is still the same, but it is tipped over.”

Jim’s questioning supported problem-solving by asking questions that helped Jenny to apply her knowledge and other questions focused on reasoning by encouraging Jenny to explain her thinking [coded: problem-solving and reasoning].

Lesson Conclusion

Near the end of the lesson, the teacher asked the students to come together on the floor and sit in a circle bringing their iPads and written solutions. Jim asked the students about the number of different solutions ($n = 4$). First, he chose to focus on students’ understanding and whether a prism rotated was a different or repeated solution [coded: assessing reasoning]. In the ensuing discussion, Peter, explaining his answer, talked about having to “turn it around” as he referred to the rectangular prism.

Jim: Put your hand up if you turned yours around or did what Peter did and popped it down?

Jim: Can I ask you if you were creating one that was the same or one that was different?

Girl: The same (*stating the correct response*).

To help students to visualise this concept Jim asked Peter to come and show the class his photographs on the iPad [coded: encouraging collaboration; sequencing and sharing; orchestrating discourse; monitoring and assessing; problem-solving and reasoning].

Jim: Peter, can you bring your iPad here and explain what you were telling me before? There was one you were looking at and I would like you to explain this … It was the one at the start … wasn’t it? It was that one …

After flicking through the photographs Jim then held up one solution (Figure 4: Image 1) to the students

Jim: Let’s start with this one … Put your hand up if you had this solution?

Figure 4

Peter’s Photographs of the Same Solution Rotated

After sharing the first image (Figure 5: Image 1) Jim flicked to another photograph (Figure 4: Image 2) and asked everyone:

Jim: Is this solution the same or a different solution? Now some people think it is a different solution, but Peter, can you explain why you think it is a different solution? (*Jim had spent time with Peter during the lesson to confirm his new learning before being selected during the whole class discussion to share his correct thinking*).

Peter: (*grinned and said*) It is the same because all you are doing is pushing it up.

Jim: (*repeats what Peter said and then confirms with the whole class*) Did the rectangular prism change?

Maddy: No, it just turned up.

Jim: Did any of the faces change?

Class: No (*all the children responded together*).

Jim: No, the faces all just stayed the same. Put up your hand if you thought it was a different solution and then you went back like Peter and said hang on it was just a different orientation?

By asking this question, Jim was highlighting to the students that they needed to check which ones were the same or different [coded: problem-solving and reasoning].

At the end of the lesson, Jim asked, "Who can convince me how many possible solutions there are?" [coded: reasoning]. Next, students constructed and shared the four possible solutions in the circle (Figure 5).

Figure 5

Constructing, Sharing, and Discussing all Possible Four Solutions

As shown in Figure 5, students used the data on their iPads to check if any other possible solutions were missing [coded: encouraging collaboration; problem-solving and reasoning]. Finally, students agreed that there were four possible solutions.

Discussion

During the lesson, the iPad served as an invaluable tool for enhancing both teaching and students' learning experiences. As reported in the results all codes of conveyance technology were demonstrated and the integration of technology was used effortlessly by students and the teacher. The teacher also used the conveyance technology to support students' mathematical reasoning and problem-solving skills. Next, these findings will be discussed.

The technology proved beneficial for documenting and sharing early years students' responses. Without this technology tool, some students might have encountered challenges when sharing answers with peers and the teacher, especially those facing difficulties when drawing three-dimensional prisms. In other words, the technology was an efficient tool for younger students to use when recording their solutions. The tool allowed students to capture solutions after constructing each prism, extending their mathematical understanding of the problem, and avoiding prolonged periods spent attempting to draw their answers. This finding suggests that using an iPad to capture images of their constructions was beneficial for Years 1 and 2 students when responding to the 12-cube task.

As reported in the results, the use of technology in Jim's lesson supported monitoring and assessment. The technology provided visual feedback for the teacher, assisting him when offering feedback to students in the movement of teaching. As Jim has a large cohort of students in his class, developing efficient strategies for supporting all students' learning is important. Similarly, the images of the students' constructions most likely provided feedback to students as they compared and analysed their solutions by flicking between responses. Creating opportunities for students to reflect on their learning aims to enrich their mathematical experiences.

Using iPads in the classroom offers the advantage of creating a visual record of students' achievements and learning. Images of students' mathematical responses could be added to a portfolio showcasing their progress across the year. In addition, at the end of the 12-cube lesson, Jim could have asked students to take a photograph of their written responses as a digital record of how they drew their prisms. Additionally, students could be asked to use the video app to record a reflection of their learning experiences, providing further evidence of student learning and the benefits of conveyance technology.

The use of conveyance technology played a crucial role in facilitating collaborations, enabling the sharing of student solutions, and fostering discussions related to the 12-cube task. Furthermore, the teacher's discussion and questioning approaches when using the images on the iPad during the lesson supported student reasoning and problem-solving. In particular, the students were encouraged to justify and explain their solutions and thinking and make a conjecture by explaining all four possible solutions when responding to the problem. The open-ended nature of the problem provided opportunities for students to record and discuss all possible solutions including developing mathematical vocabulary related to the properties of prisms such as faces and orientation. The images assisted students when pointing and justifying their thinking, especially during whole-class discussions at the end of the lesson.

Without technology, the teacher may have primarily relied on student work samples and the physical construction of three-dimensional prisms (12 cubes) to support students' mathematical reasoning and problem-solving skills. However, the combination of the technology tool and the quality of the teacher's pedagogical approaches in this study supported students' learning. The technology served as a tool for fostering collaboration, sharing solutions, promoting discourse, monitoring, and assessing. The technology was also useful in supporting the teacher when posing questions to extend students' reasoning and problem-solving skills. Although not explicitly addressed in this study, it is also plausible that incorporating technology could have further engaged students and fostered positive learning dispositions.

The findings of this study highlight one approach for using conveyance technology with early years students to support their reasoning and problem-solving skills. Most likely the learning in this lesson was attributed to the teachers' pedagogical actions, the task and technology. Calder and Murphy (2018) would agree that when teaching with iPads the quality of the teacher's pedagogy is influential in supporting student learning. In other words, a good teacher is needed to support learning with technology. In addition, Cullen and Hertel (2023) cautioned that when assessing technology as a tool we need to consider if student learning is attributed to the teacher or the technology. Further studies could explore this idea by interviewing teachers and students after using technology as well as teaching the same lesson with and without technology to compare how students learn.

Conclusion, Implications and Recommendations

This study investigated the use of conveyance technology, specifically iPads, to support mathematical reasoning and problem-solving through a single case study. The Year 1–2 lesson, centred around a 12-cube task, providing a platform for students to engage with technology. The incorporation of iPads was beneficial for sharing and recording student responses, supporting the teacher when monitoring and assessing, and providing opportunities to support students' reasoning and problem-solving. A short video capturing this lesson is available on the Monash University *TeachSpace* website (Livy, 2019).

Implications of this study suggest that integrating conveyance technology in early years' mathematics lessons when teaching with open-ended problems can support teaching practices for problem-solving and reasoning. Additionally, the study highlights the importance of teachers' pedagogical approaches when using technology to support student learning. Teachers, like Jim in this case, played a pivotal role in guiding students' mathematical reasoning, addressing misconceptions, and fostering mathematical discourse. Likewise, the use of iPads to document and share solutions supported peer learning and collaboration. Students benefited from seeing and discussing each other's approaches, leading to a deeper understanding and the development of conjectures and strategies. Overall, these findings contribute to the broader discourse on technology's role in the early years of mathematics education suggesting that the integration of conveyance technology when supported by effective teaching practices, and open-ended problems can foster students' reasoning and problem-solving skills.

This study was conducted before the onset of COVID-19. Conducting a follow-up study with the teacher could offer valuable insights into the evolution of the use of conveyance technology. Given the widespread shift to online teaching during the pandemic, exploring how teachers, including the one in this study, have adapted and grown in their use of technology would provide a timely and relevant perspective, contributing to further understanding of productive and unproductive uses of technology tools in the early years.

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References

Australian Curriculum Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: Mathematics (v9.0)*. ACARA. http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10

Boon, H. J., Boon, L., & Bartle, T. (2021). Does iPad use support learning in students aged 9–14 years? A systematic review. *The Australian Educational Researcher*, *48*(3), 525–541. https://doi.org/10.1007/s13384-020-00400-0

Calder, N., & Murphy, C. (2018). How might apps reshape the mathematical learning experience? In N. Calder, K. Larkin, & N. Sinclair (Eds.), *Using mobile technologies in the teaching and learning of mathematics* (pp. 31–50). Springer. https://doi.org/10.1007/978-3-319-90179-4_3

Cullen, C. J., & Hertel, J. T. (2023). Improving our criticism of technology in mathematics education. *Mathematics Teacher Learning and Teaching*, *116*(5), 314–321.

Dick, T. P., & Hollebrands, K. F. (2011). *Focus in high school mathematics: Technology to support reasoning and sense making*. National Council of Teachers of Mathematics.

Dumez, H. (2015). What is a case, and what is a case study? *BMS: Bulletin of Sociological Methodology/ Bulletin de Méthodologie Sociologique*, *127*, 43–57. http://www.jstor.org/stable/43761847

Herbert, S., & Williams, G. (2023). Eliciting mathematical reasoning during early primary problem solving. *Mathematics Education Research Journal*, *35*(1), 77–103. https://doi.org/10.1007/s13394-021-00376-9

Livy, S. (2019). *How to get your Grade 1 and 2 students to think like mathematicians*. Monash University. https://www.monash.edu/education/teachspace/articles/how-to-get-your-grade-1-and-2-students-to-think-like-mathematicians

Larkin, K., & Milford, T. (2018). Mathematics apps-stormy with the weather clearing: Using cluster analysis to enhance app use in mathematics classrooms. In N. Calder, K. Larkin, & N. Sinclair (Eds.), *Using mobile technologies in the teaching and learning of mathematics* (pp. 11–30). Springer International Publishing. https://doi.org/10.1007/978-3-319-90179-4_2

McCulloch, A. W., Hollebrands, K., Lee, H., Harrison, T., & Mutlu, A. (2018). Factors that influence secondary mathematics teachers' integration of technology in mathematics lessons. *Computers & Education*, *123*, 26–40. https://doi.org/https://doi.org/10.1016/j.compedu.2018.04.008

National Council of Teachers of Mathematics [NCTM]. (2014). *Principles to actions: Ensuring mathematical success for all*. NCTM.

Sullivan, P., Bobis, J., Downton, A., Livy, S., McCormick, M., & Russo, J. (2023). *Maths sequences for the early years F–2: Challenging children to reason mathematically*. Oxford University Press.

Sullivan, P., Borcek, C., Walker, N., & Rennie, M. (2016). Exploring a structure for mathematics lessons that initiate learning by activating cognition on challenging tasks. *The Journal of Mathematical Behavior*, *41*, 159–170. https://doi.org/https://doi.org/10.1016/j.jmathb.2015.12.002

Vale, C., Widjaja, W., Herbert, S., Bragg, L. A., & Loong, E. Y.-K. (2017). Mapping variation in children's mathematical reasoning: The case of 'what else belongs?'. *International Journal of Science and Mathematics Education*, *15*(5), 873–894. https://doi.org/10.1007/s10763-016-9725-y

Van de Walle, J. A., Karp, K. S., Bay-Williams, J. M., Brass, A., & Livy, S. (2024). *Primary and middle years mathematics teaching developmentally (2nd Australian ed.)*. Pearson.

Primary Students' Responses to a Cognitive Activation Lesson

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Cognitive activation is an approach to instruction associated with increased mathematical achievement and engagement. However, researchers have questioned its effectiveness for students with different learner characteristics. A Year 6 class participated in a lesson designed using cognitive activation practices. Interviews with ten case study participants, comprising varying achievement and engagement levels, revealed that practices such as collaborative learning and the use of cognitively demanding tasks evoked positive student responses. The findings highlight the instructional potential of cognitive activation across heterogeneous learner profiles.

Cognitive activation, an element of the Three Basic Dimensions framework posited by Klieme et al. (2009), has garnered attention for its relationship with improved mathematical performance (Le Donné et al., 2016; Lipowsky et al., 2009). Defined by Klieme et al. (2009) as "any observable pedagogical practice and pattern … of instruction that encourages students to engage in (co-)constructive and reflective high-level thinking and thus develop an elaborated, content-related knowledge base" (p. 140–141), cognitive activation is recognised by the Programme for International Student Assessment (PISA) as an instructional practice for enhancing mathematical literacy (OECD, 2013). In the context of primary mathematics education, there exists a broad consensus that sole reliance on a singular pedagogical approach is insufficient for a deep, holistic understanding of mathematical concepts. Instead, it is recommended that educators draw upon a diverse range of instructional practices to enrich their repertoire. As cognitive activation continues to be explored as one of the many instructional practices that teachers can employ, there is a growing need to enhance our understanding of how different students respond to its implementation.

Literature Review

Cognitive Activation

Although cognitive activation was conceptualised by Klieme et al. (2009), the construct was unpacked in greater detail by Lipowsky et al. (2009). Lipowsky and colleagues conceptualised cognitive activation as a multi-dimensional construct comprising three sub-elements: instruction for conceptual understanding, the cognitive level of students' activities, and the quality of interaction and participation in the classroom.

Instruction for conceptual understanding centres on the development of conceptual knowledge. Conceptual knowledge is defined as knowledge rich in relationships where teachers encourage students to develop meaningful connections between procedures and apply this understanding to novel situations (Brophy, 2000). Key features of conceptual understanding include explicit attention to concepts, the activation of prior knowledge and productive struggle (Hiebert & Grouws, 2007). The cognitive level of students' activities, the second sub-component, is closely linked to the first as facilitating productive struggle requires tasks demanding complex thinking. Lipowsky et al. (2009) found that cognitive activation is less likely when students solve mathematical problems in a standard manner. Therefore, tasks necessitating intricate, non-algorithm cognitive processing and decision-making are critical to

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this type of instruction. Such tasks require deep, sustained mathematical thinking and should be suitably challenging, falling within the learner's zone of proximal development (Vygotsky, 1931/1978). Vygotsky's research identified a zone just beyond what an individual can do independently where students can perform with the guidance and support of a more knowledgeable person, such as a teacher or peer. It includes scaffolding, challenge level, social interaction and cognitive development to support students to optimise their learning through challenge while providing the necessary support for growth. Finally, Lipowsky et al. (2009) also found the quality of interaction and student participation as an important factor of cognitive activation. They outlined practices such as questioning and student related discourse as facilitators for the construction of deeper meaning. Other research into student participation favours practices that encourage students to articulate ideas and engage in constructive dialogue, such as collaborative learning.

Table 1 provides a synthesis of the key practices/strategies outlined in Lipowsky et al.'s (2009) conceptualisation as well as integrating relevant theories.

Table 1

List of Cognitive Activation Pedagogical Strategies and Contributing Theories

Cognitive activation pedagogical strategies	Contributing theories
Activation of prior knowledge	Zone of proximal development
Explicit attention to mathematical concepts	Theories of student engagement
Productive struggle	
Cognitively demanding tasks	
Classroom discourse	
Collaborative learning	

Importantly, Lipowsky et al. (2009) acknowledged that "instructional features do not impact students' achievement in a direct manner; rather, the uptake of learning opportunities is thought to depend on various learner prerequisites" (p. 529). In their discussion, they argued the need for further examination of the interaction between cognitive activation strategies and learners' characteristics. This is of particular importance as "the classroom learning community is neither static nor linear" (Anthony & Walshaw, 2009, p. 149) and students arrive at any learning situation with a range of different learning characteristics. The focus characteristics selected for this paper include student achievement and engagement.

Achievement and Engagement Learner Characteristics

Student achievement is defined by Hattie (2013) to encompass their accomplishments. In this study, achievement refers to students' performance within the context of mathematics learning goals. Achievement has been an ongoing area of interest for mathematics educators. However, within student-centred and problem solving-based pedagogies, educators have expressed reservations about incorporating demanding or challenging tasks for students of low achievement backgrounds. A study by the National Foundation for Educational Research (NFER) in the UK (Burge et al., 2015) revealed that cognitive activation practices were less frequently observed among low-achieving students compared to their high-achieving counterparts. Notably, high-achieving students reported a 23% increase in situations where teachers presented problems where the solution was not immediately apparent. Although a nascent link exists between cognitive activation strategies and academic achievement, uncertainty persists regarding whether this discrepancy stems from high-achieving students witnessing more of these practices from their teachers than their lower-achieving peers. Furthermore, as evident from the updated New South Wales K–10 mathematics syllabus (New South Wales Education Standards Authority (NESA), 2020), students across all achievement levels should have opportunities to be creative and critical when solving complex tasks. There is a need to understand more deeply how students respond to moments of instruction that centre on students making connections, discussing key mathematical concepts and attempting cognitively demanding tasks; all key elements within a cognitive activation approach.

Student engagement has garnered significant attention due to its potential impact on positive educational outcomes (Fredericks et al., 2004). Engagement is well established as a multi-dimensional construct, encompassing behavioural, emotional and cognitive sub-constructs, all which play pivotal roles in fostering student involvement in their learning (Fredericks et al., 2004). Within their multidimensional framework, Fredericks et al., (2004) suggest authentic and challenging tasks can be associated with heightened behavioural, emotional and cognitive engagement. However, the authors acknowledge a gap in literature concerning how individual differences such as achievement levels, moderate the relationship between challenging tasks and engagement. Over the past two decades, researchers have continued to explore this gap. For instance, Bobis et al. (2021) explored instructional strategies to facilitate increased engagement in mathematics learning when challenging tasks were involved. Insights gleaned from teacher interviews highlighted the efficacy of using open-ended tasks as catalysts for enhancing student engagement. The present study aimed to capture student perspectives on teaching practices aligned with a cognitive activation approach that sustains or promotes engagement in their learning. Given the association between student engagement and achievement, understanding student responses to cognitive activation practices may be useful. Insights from this study can inform classroom practices and guide further investigations into the use of these teaching strategies across diverse student populations.

The present study aimed to address the following research question: How do Year 6 students with different engagement and achievement learner characteristics respond to components of a lesson designed using cognitive activation practices? For the context of this paper, 'respond' refers to students' reactions including their enjoyment and engagement.

Methodology

An exploratory case study methodology was used to investigate student responses to a lesson designed using cognitive activation strategies and contributing theories, hereafter referred to as a cognitive activation lesson. This paper is affiliated with a larger study comprising a series of six intervention lessons, all constructed around cognitive activation teaching strategies. The findings in this paper report on student responses of 10 subset case study participants gathered during the semi-structured interviews following the first cognitive activation lesson. Previous mathematics lessons were teacher-directed and students had minimal experience with cognitive activation practices prior to the intervention.

Intervention

A 60-minute lesson was led by the researcher, who was not the regular class teacher. The content focus of the lesson was on addition. A warmup activity was designed to activate prior knowledge where students were tasked to find multiple strategies to solve a three-digit additive number sentence. Students worked collaboratively with a peer using a mini-whiteboard to record their working. Solution strategies were shared and evaluated in a whole class discussion. The body of the lesson centred around a cognitively demanding 'Charity' task that posed the following question: Three schools raised $125,750 for charity. Each school raised a different amount. What amount might each school have raised? The open-ended nature of the task offered the possibility of stimulating higher-level thinking. It allowed students to generate their own strategies and apply their understanding of addition in a novel context. These key elements align with existing literature, which suggests that such tasks can enhance cognitive demands and challenge students intellectually (Dyer & Moyinhan, 2000; Sullivan, 2018). Support and extension prompts were pre-devised, adopting an approach recommended by Sullivan (2018) when providing students with challenging tasks. A support scaffold reduced the total to $25,750 while an extension scaffold challenged students to find multiple solutions where the total for each school was within $3,000 of each other. Students had a choice in the last few minutes of the task to use a calculator to help find more solutions. After completing the task, some student solutions were discussed. Students also had an opportunity to reflect on the strategies they used during the problem-solving process.

Participants and Data Collection

A heterogenous Year 6 class participated in the study. The school was an independent, co-educational school located in metropolitan Sydney. Students came from mid- to high-socioeconomic backgrounds. This age range was selected as it is a stage often linked with declining mathematical engagement (Skilling et al., 2020). Prior to the intervention, baseline data were collected to help select the subset of case study participants. Students' achievement and engagement characteristics were evaluated using Martin's (2007) Motivation and Engagement Scale (MES), Progressive Achievement Testing (PAT) and a researcher-made assessment on addition and subtraction. Results were ranked using a tripartite approach to determine low, moderate and high achievement and engagement. A subset of ten case participants, comprising five males and five females, represented a broad spectrum of achievement and engagement levels.

Each lesson was video recorded and researcher/teacher field notes were logged. Student artefacts including work samples and photographs of whiteboard work were collected. Individual student semi-structured interviews were conducted immediately following the first lesson. Interviews were video recorded and transcribed.

Data Analysis

A hybrid thematic approach was employed to discern patterns of significance within the semi-structured interview transcripts (Braun & Clarke, 2006). The initial phase of analysis involved a comprehensive review of the transcripts to obtain a holistic understanding of the students' reactions to the lesson. Deductive coding was primarily used to tag quotations when students alluded to a particular cognitive activation practice (refer to Table 1 for an overview of the cognitive activation practices). Inductive codes were also annotated when students mentioned other instructional components of significance. Following this, an inductive strategy was employed to detect patterns pertinent to the research question. These findings were further scrutinised, and categorised into sub-themes and major themes for this paper.

Results

Due to space constraints, a comprehensive coverage of all cognitive activation practices is unfeasible. Instead, the results in this paper hone in on four principles frequently underscoring students' responses: cognitively demanding tasks, zone of proximal development, productive struggle and collaborative learning. The findings are organised into two primary themes extracted from data analysis. Students' names are pseudonyms for confidentiality in the study.

Student Responses to Challenge

During the semi-structured interviews, students frequently discussed the cognitively demanding task or the associated challenge level. More than half of the students referenced the level of challenge, the majority in relation to the cognitively demanding 'Charity' task. Most students spoke negatively about the level of challenge because the cognitive demands of the task were excessively high. Oscar, a low achievement and engagement student, accounted for almost half of the negative responses, "I just don't know how to work out the question … I didn't get how to find the amount that each school raised. And I didn't get just what the question meant."

Similarly, Leah, a moderate achievement and engagement student, also found the main task too challenging, "I got confused on this task and I got quite stuck … I just didn't understand … With the other ones, I know how to do the strategies and I knew what to do."

Victor, a high achievement low engagement student, also found the challenge level for the 'Charity' task not suitable. He perceived part of the task too easy because the openness of the task meant answers could be derived by "coming up with random numbers". Victor did, however, show an increase in his cognitive engagement when afforded the opportunity to solve the extension prompt, "I think it was better when there was a bit less freedom … when it had to be a little closer together because you can't just do anything."

A similar sentiment was observed from Edward, a low achievement high engagement student, when referring to the opportunity to use the calculator towards the end of solving the main task, "well, when we started using the calculators, that wasn't as fun because it was giving us the answers. I like to work it out."

Both Edward and Victor reported diminished enjoyment when they perceived the task as straightforward, with answers readily available. Victor expressed greater enjoyment upon successfully completing the extended prompt, while Edward experienced a similar effect when he avoided using a calculator, as both instances required more effort in solving the cognitively demanding task.

Among the students who provided positive feedback regarding the cognitively demanding task, Maddie, a student with moderate achievement and low engagement, explicitly acknowledged the suitability of the challenge level, "it was pretty fun because the questions weren't too hard or too easy."

Almost all the students recognised the beneficial role of challenge for their learning. Sally, a low achievement and engagement student spoke positively about the role of productive struggle in the lesson, also noting the importance of scaffolding in responding to challenges:

> I think it can be sort of helpful because you get to challenge yourself and try. If it is something easy a lot of the time, I think you don't actually learn anything … I do like having that challenge but I also think it would be helpful … getting more help with challenging ones if it gets too challenging.

She also found interest in attempting struggle within the lesson and challenging herself not to only find easy solutions:

> It was fun to do different things and correct myself because I know I got a few wrong. I checked them and it was fun to redo it and do different examples instead of just having really easy ones the whole time.

Many high-achieving students exhibited favourable responses when confronted with challenges during the lesson. For example, George, a high achievement and engagement student, and Victor, both expressed enjoyment and a feeling of victory upon successfully solving complex problems:

> When I complete something hard, it just brings me joy. (George)

> When you're trying to figure out something and then at the very end or at the start of the next lesson, you finally figure it out, it gives you a sense of accomplishment and it feels good. (Victor)

Classroom Participation During the Lesson

Collaborative learning emerged as the predominant cognitive activation practise discussed in the student interviews. The majority of students spoke about their enjoyment engaging with a peer to solve the warmup task. For example:

> I really enjoyed it because we could work with new people and share our ideas and listen to each other. (Leah)

> I think my favourite part was when we were all sitting down on the carpet and we were working together with a partner because we could figure out who was going to do what and we could work more efficiently and we could understand what each other were doing. And we could also learn more ways to figure things out or be remembered. (Laura)

Sally, a low achievement and engagement student, expressed unfavourable sentiments regarding the collaborative learning activity. She highlighted a lack of collaboration with her partner and conveyed low enjoyment because she was not able to actively contribute to the shared learning experience, "I was with someone who understood math a lot better than I did and who knew lots of different strategies so it ended up being she did most of the work."

Similarly, when queried about her lowest engagement moments during the lesson, Sally attributed it to the collaborative activity, asserting "because my partner did most of the work." When prompted to identify a part of the lesson she would like to do again, she responded, "I'd probably do the warmup again and try and challenge myself to do some other things … instead of just letting my partner do it."

During the analysis of student responses, it became evident that collaborative learning was not the sole factor contributing to heightened classroom participation. Many students attributed their increased behavioural and cognitive engagement to the cognitively demanding task. Specifically, Laura, Edward and Connor reported that the open-ended task extended their involvement in the lesson, as they found the exploration of multiple solutions intellectually stimulating and intriguing:

> I enjoyed it because there were multiple answers … it was interesting to see how many answers you could come up with. (Laura)

> I enjoyed it because there wasn't one answer, because if there was one answer, it would just be like a boring lesson. We would figure out one and it would be over, but this there's multiple answers and multiple ways to figure out your answer. (Edward)

> I enjoyed it because it was a problem where you didn't just work it out and then you were done. You could have multiple answers, like lots of them, so you could sit there for a long time working out new answers. (Connor)

Discussion

The prominence of responses relating to the challenge level underscores the significance of Vygotsky's Zone of Proximal Development theory (1931/1978) in the context of cognitive activation instruction. Positive reactions, particularly in terms of engagement and enjoyment, were consistently observed when challenge levels aligned with students' academic capabilities. Conversely, excessively demanding challenges led to decreased engagement and enjoyment. When the challenge level was perceived as 'optimal' from the student's perspective, cognitive engagement appeared to increase. Although this study drew insights from a limited sample size of student voices, an interesting observation emerged: challenge level played a critical role in utilising Lipowsky et al.'s (2009) second sub-component of cognitive activation. The potential implications of this finding are promising, yet further investigation is warranted.

Furthermore, the observation of heightened negative reactions to challenging tasks among low-achieving students reinforced the importance of scaffolding. This finding aligns with Bobis et al.'s (2021) research, where teachers emphasised the effectiveness of a differentiated approach, including extending and enabling prompts, to engage a diverse student population. Student perspectives indicated that suboptimal challenge levels led to reduced engagement and enjoyment. It is acknowledged that this is one of many factors which may have influenced the way students engaged with the task, and further exploration of these factors is needed in larger-scale studies. Cognitive activation, which centres on the utilisation of cognitively demanding tasks, reveals the need for teachers to possess a deep understanding of their students. Student responses in this context highlight the role of teacher-student familiarity in optimising cognitively demanding activities. Specifically, awareness of students' achievement characteristics during the design and creation of cognitively demanding tasks could assist teachers to tailor differentiation effectively. By doing so, educators can aim to engage a wider student population. Further research should continue to explore this relationship and refine instructional strategies within cognitive activation teaching.

In the analysis, classroom participation emerged as a salient theme, with numerous students attributing heightened engagement to collaborative learning experiences and the solving of an open-ended cognitively demanding task. Sally's perspective illuminated the detrimental impact of ineffective collaboration on her learning. Her negative reactions stemmed from her lack of active contribution to the collaborative process, resulting in decreased task engagement and enjoyment. The large number of positive reactions, coupled with Sally's negative response, showed there may be a possible association between increased engagement and enjoyment with opportunities for collaborative learning, so long as the collaboration is shared equally and is productive. Finally, a potential link between cognitively demanding tasks and classroom participation was revealed. Notably, student perspectives demonstrated alignment with Frederick et al.'s (2004) findings, which associate challenging tasks with increased behavioural, emotional and cognitive engagement. Student responses to this lesson highlighted increased participation and heightened cognitive stimulation from both collaborative practices and open-ended, cognitively demanding activities as reasons for enhanced engagement and enjoyment in the lesson.

Conclusion

The objective of this study was to investigate how students with varying achievement and engagement profiles respond to different practices and aspects of a cognitive activation lesson. By analysing student feedback, we explored the potential of cognitive activation instruction. Notably, students reported heightened engagement and enjoyment when the challenge level aligned with their individual achievement characteristics. While these preliminary findings warrant further scrutiny, they hold implications for instructional practice. Student voices serve as a call for educators to incorporate opportunities for struggle and cognitively demanding tasks into teachers' teaching repertoires. A possible link between cognitively demanding tasks and increased cognitive engagement and participation emerged. Additionally, responses related to collaborative learning underscore heightened student participation, transcending engagement and achievement levels. In this lesson, successful collaboration not only facilitated peer learning but enhanced overall participation, with students appreciating the social dynamics inherent in this learning approach. This study sheds light on how learners with different characteristics engage within a cognitive activation framework. The hope is that these findings will encourage teachers to embrace this instructional practice, with the intent of elevating student engagement and the potential to foster a deeper enjoyment of mathematics learning.

Acknowledgments

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References

Anthony, G., & Walshaw, M. (2009). Characteristics of effective teaching of mathematics: A view from the west. *Journal of Mathematics Education, 2*(2), 147–164.

Bobis, J., Russo, J., Downton, A., Feng, M., Livy, S., McCormick, M., & Sullivan, P. (2021). Instructional moves that increase chances of engaging all students in learning mathematics. *Mathematics, 9*(6), 582. https://doi.org/10.3390/math9060582

Braun, & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Brophy, J. E. (2000). *Teaching: Educational practices (Series–1).* Brussels, Belgium: International Academy of Education (IAE)/UNESCO International Bureau of Education (IBE).

Burge, B., Lenkeit, J. and Sizmur, J. (2015). *PISA in practice: Cognitive activation in maths.* Slough: National Foundation for Educational Research (NFER). https://www.nfer.ac.uk/media/vmtj0jne/pquk04.pdf

Dyer, M., & Moynihan, C. (2000). *Open-ended questions in elementary mathematics: Instruction and assessment.* Larchmont: Eye on Education.

Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research*, *74*(1), 59–109. https://doi.org/10.3102/00346543074001059

Hattie, J., & Anderman, E. M. (Eds.). (2013). *International guide to student achievement*. Routledge. https://doi.org/10.4324/9780203850398

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In J. Frank & K. Lester (Eds.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Information Age.

Klieme, E., Pauli, C., & Reusser, K. (2009). The Pythagoras study: Investigating effects of teaching and learning in Swiss and German mathematics classrooms. In T. Janik & T. Seidel (Eds.), *The power of video studies in investigating teaching and learning in the classroom.* New York, NY: Waxmann Publishing Co.

Lipowsky, K., Pauli, C., Drollinger-Vetter, B., Klieme, E., & Reusser, K. (2009). Quality of geometry instruction and its short-term impact on students' understanding of the Pythagorean Theorem. *Learning and Instruction*, *19*(6), 527–537. https://doi.org/10.1016/j.learninstruc.2008.11.001

Le Donné, N., P. Fraser & G. Bousquet (2016). *Teaching strategies for instructional quality: Insights from the TALIS-PISA link data.* OECD Publishing, Paris. https://doi.org/10.1787/5jln1hlsr0lr-en.

Martin, A. J., (2007). Examining a multidimensional model of student motivation and engagement using a construct validation approach. *British Journal of Educational Psychology*, *77*, 413–440, http://dx.doi.org/10.1348/000709906X118036

New South Wales Education Standards Authority (NESA). (2020). *Nurturing wonder and igniting passion, designs for a new school curriculum: NSW curriculum review*. NSW Education Standards Authority. https://nsw.gov.au/sites/default/files/2023-04/NSW-Curriculum-Review-Final-Report.pdf

Organisation for Economic Co-operation and Development (OECD). (2013). *PISA 2012 assessment and analytical framework: Mathematics, reading, science, problem solving and financial literacy.* https://doi.org/10.1787/9789264190511-en

Skilling, K., Bobis, J., & Martin, A. J. (2020). The "ins and outs" of student engagement in mathematics: Shifts in engagement factors among high and low achievers. *Mathematics Education Research Journal*, *33*(3), 469–493. https://doi.org/10.1007/s13394-020-00313-2

Sullivan, P. (2018). *Challenging mathematical tasks: Unlocking the potential of all students*. Oxford University Press.

Vygotsky, L. S. (1931/1978). *Mind in society: The development of higher psychological processes*. In M. Cole, V. John-Steiner, S. Scribner & E. Souberman (Eds.), (A. R. Luria, M. Lopez-Morillas & M. Cole [with J. V. Wertsch], Trans.) Cambridge, Mass.: Harvard University Press. (Original manuscripts [ca. 1930–1934].

Pre-Service Teachers' Use of Jump Strategy on the Empty Number Line When Recording Micro-Teaching Videos

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This paper is part of a broader study which explores South African pre-service teachers' use of the jump strategy on the empty number line for enhancing their confidence to do and teach mental mathematics computation strategies. The focus of this paper is the use of micro-teaching in the form of video recordings by pre-service teachers. Forty videos were analysed thematically indicating that most pre-service teachers could implement the strategy with some level of fidelity. Three micro-teaching videos are shared in this paper to exemplify the range of strong, medium, and poor implementation fidelity.

The South African Department of Basic Education (South Africa. DBE, 2020) rolled out resources developed from the national Mental Starters Assessment Project (MSAP), intended for improving learners' mental mathematics and number sense skills (Graven et al., 2020). An offshoot of this project is the Mental Mathematics Work-Integrated Learning (MM-WiL) project, directed at preparing pre-service teachers (PSTs) for using the MSAP materials to effectively teach mental mathematics, with the additional benefit of improving their own confidence and competence in performing mental mathematics calculations. The broader study explores how South African PSTs use the jump strategy on the empty number line. This paper specifically aims to answer the research question, 'How do PSTs use the jump strategy on the empty number line when recording micro-teaching videos?' As mental calculations and number sense are vital for understanding mathematics (Bobis, 2007), I posit that supporting PSTs in learning how to teach strategies such as the jump strategy on the empty number line through the supportive and engaging context of micro-teaching videos is of interest to mathematics teacher educators internationally.

Background and Rationale

Literature indicates that South African teachers lack the necessary mathematical and pedagogical knowledge, support, and confidence, and are subsequently hindered by poor morale (for example, Graven & Heyd-Metzuyanim, 2014; Hoadley, 2012; Jojo, 2019; Spaull, 2019; Venkat & Graven, 2017). Studies on South African PSTs further show that their mathematical knowledge is lacking, and that there is little improvement during the four years of completing their BEd degree (Bowie et al., 2019; Fonesca, 2018). Fourth year PSTs in Bowie's study were unable to achieve a mean score over 50% for higher cognitive demand items across all mathematics topic areas, and she claims these results mirror the knowledge and proficiency of in-service teachers. Bowie (2019) therefore argues that mathematics teacher educators "cannot take it for granted that student teachers will arrive with a command of the mathematics they will be expected to teach" (p. 295). In the Australian context, Beswick and Goos (2012) similarly found that it could not be taken for granted that students entering as PSTs have fundamental mathematics knowledge and skills. PSTs are shown, across international studies, to have low number sense (e.g., Aktaş & Özdemir, 2017; Bowie et al., 2019; Courtney-Clarke & Wessels, 2014; Şengül, 2013). According to Bobis (2007), well-developed number sense is vital to efficient computation and understanding of mathematics. These factors indicate the rationale for projects such as MSAP and MM-WiL to support PSTs developing number sense and mental mathematics knowledge and skills. In this paper, I explore the opportunities for micro-teaching videos to support PSTs in learning how to implement mental calculation strategies, specifically the jump strategy on the ENL.

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Literature Review

The MSAP and MM-WIL Resources

The MM-WiL project uses the MSAP material (Graven et al., 2020) consisting of resources in a teacher guide such as scripted lessons, diagrams, video-recorded demonstrations (see for example, https://youtu.be/6RkP5bSpINQ), notes, pre- and post-tests and learner worksheets (available from https://www.education.gov.za/MSAP2022.aspx). Six mental mathematics calculation strategies are taught in MSAP: bridging through ten, jump strategy, doubling and halving, rounding and adjusting, re-ordering, and linking addition and subtraction. These strategies were carefully designed into tasks to prepare learners for fluent and flexible mental computation to move them away from over-reliance on unit counting (Graven et al., 2013; Graven & Venkat, 2021; Schollar, 2008). The current paper focuses on the jump strategy, which uses an empty number line (and eventually a mental image thereof) to gesture and visually represent addition or subtraction of a number. Mathematics teacher educators (MTEs) at eight South African universities are actively using the MSAP resources with their PSTs, followed by opportunities for them to implement teaching them during their practical teaching experience. This paper, however, reports on the use of micro-teaching as a way for PSTs to explore using and teaching the jump strategy for addition and subtraction. Rather than expecting these first-year PSTs to teach to a class of children, they are first given the opportunity to micro-teach to their peers in small groups of two to three members. PSTs worked together to record their micro-teaching similar to the demonstration videos provided in the teacher resources.

The Jump Strategy on the Empty Number Line

Şahin and Danaci (2020) suggest that one of the challenges learners face with mental computation is not visually showing their thinking. There is, however, a relationship between mental computation and number line representations, according to Ruiz and Balbi (2019). Vermeulen et al. (2020) further posit that the empty number line (ENL) has a "temporary value" (p. 233) in supporting learners with calculations by reducing their cognitive load in the process through visually showing what is happening mentally. The ENL is a number line with no predetermined numbers or unit markers, making it a useful tool to support addition and subtraction in any number range, and it can stimulate learners in explaining their thinking around calculation strategies (Bobis, 2007; Klein et al., 1998; van den Heuvel-Panhuizen, 2008). As learners develop understanding and fluency, they no longer need the visual representation, as it becomes a mental image supporting mental computation. The jump strategy on the ENL can assist learners in visually representing addition and subtraction on a linear representation, where one number is kept as a whole and the addend (or subtrahend) is jumped forwards or backwards in manageable parts, such as in tens (Blöte et al., 2000; Bobis & Bobis, 2005). The jump strategy should not be taught as a mandatory procedure, but rather as a meaningful, flexible strategy enabling learners to visualise and work with number relations (Bobis, 2007; van den Heuvel-Panhuizen, 2008; Vermeulen et al., 2020). Therefore, learning and practising how to use and teach the jump strategy on the ENL to support learners' mental computation can be a valuable strategy for PSTs, as encouraged in the MSAP and MM-WiL teacher resources.

Micro-Teaching for Enhancing Pre-Service Teachers' Skills

Micro-teaching is a widely used practice to prepare PSTs for professional classroom teaching and for education research (Cheng, 2017; O'Keeffe & White, 2022; Ünlü, 2018). O'Keeffe and White (2022) suggest that micro-teaching is an opportunity to move PSTs away from teaching the traditional way they were taught, which could result in negative views towards mathematics, to a more reflective and engaging way of teaching. Micro-teaching involves creating a controlled, supportive environment made up of smaller class sizes, normally of peers, and for a shorter duration of a standard lesson (Ünlü, 2018), allowing PSTs the opportunity to practise a specific teaching skill without the complexity of a typical classroom (Basturk & Tastepe, 2015). MTEs can use this opportunity to connect theory with practice, before sending PSTs out into schools for practicum. Literature reports benefits of micro-teaching for mathematics PSTs, including developing confidence with time and classroom management, planning, task design, use of questioning, self-efficacy, and precision (Cheng, 2017; Ünlü, 2018). Cheng (2017) highlights a "special kind of micro-teaching" (p. 281) where the university micro-teaching experience is linked to the mentor teacher experience at schools, which helped PSTs improve their precision in verbal (language use, reasoning) and non-verbal (accurate calculations, exact formulas) skills. Cheng (2017) however noted in their study that while the combined micro-teaching experience had value, some lack of precision was still observed in PSTs' lessons. Basturk and Tastepe (2015) caution against the "artificial nature" (p. 1) of the micro-teaching setting during a university course lecture. Nevertheless, probably the most beneficial of the micro-teaching experience is the opportunity for PSTs to be guided in reflecting on and self-evaluating their recorded lessons with peers and MTEs, stimulating collaborative discussion and development, thus improving their content and pedagogical knowledge (O'Keeffe & White, 2022). While an explicit focus on reflection was not guided by the MTE in the study reported on in this paper, it was clear that working in pairs, the PSTs used the opportunity to self- and peer-evaluate their micro-teaching of the jump strategy on the ENL. The next section describes the methodological process of the study.

Methodology

The MSAP resources and training were delivered by the author, also the MTE, to 190 first-year PSTs during their mathematics didactics course. One hundred and two PSTs agreed to participate in the broader study, including pre- and post-tests, questionnaires, and interviews. After completing the pre-test, the MTE shared the hardcopy and electronic MSAP resources with the PSTs. She also demonstrated the designed lesson starters and jump strategy on the ENL to the PSTs during the first 20 minutes of six lecture sessions spread across three weeks (with the intention of modelling how the PSTs would use the resources in their own classrooms). This was followed by the post-test. For the purpose of this paper, I focus on 40 PSTs' submitted micro-teaching videos for which consent to enter them into the data set was given. PSTs had the right to withdraw from the study at any stage; only their hands are shown in the videos, and pseudonyms were used to uphold their anonymity.

Powell et al.'s (2003) analytical model was used to guide the thematic analysis of the videos: the author (1) watched and re-watched the videos; (2) made detailed notes describing the data; (3) identified critical moments in each of the 40 videos; (4) transcribed them indicating both PSTs' actions/gestures and dialogue; (5) coded the data according to identified patterns; (6) constructed a storyline and (7) wrote a narrative description referring to screenshots of the critical moments' dialogue and gestures. This was a cyclical and iterative process. The author noticed four common key elements that contributed to the implementation fidelity of the MSAP resources in the PSTs' micro-teaching videos, i.e., use of (i) the ENL, (ii) jump strategy, (iii) gestures and (iv) key phrases. These four key elements were further used to analyse the 40 micro-teaching videos. It was noticed that for each of the four key elements the PSTs' videos either faithfully followed the scripted MSAP resources with strong implementation fidelity or they departed from the resources.

Findings

In Lovemore and Graven (in press) it was found that most PSTs followed the MSAP resources with some level of implementation fidelity in their micro-teaching videos. Over 70% of the PSTs used the ENL to teach the jump strategy in a highly similar way to the scripted resources in at least three of the four key elements (i)–(iv) listed above. Micro-teaching videos that had all four of the key elements present were categorised as having strong implementation fidelity. Videos with two or three key elements were categorised as medium implementation fidelity, and those with one or no key elements as poor implementation fidelity. In this paper, I move on from this global perspective of the broad data and focus on three examples of micro-teaching videos across the spectrum of implementation fidelity. This allows me to demonstrate specific examples of the PSTs' use of the jump strategy on the ENL when recording micro-teaching videos and to consider its implications for MTEs.

PSTs could select various calculations, taken from the MSAP resources, to demonstrate the use of the jump strategy on the ENL to create their micro-teaching video. For this paper, I selected the calculation 62 − 30 = __. This subtraction problem involves a known minuend and subtrahend, and the missing difference. It could be solved by placing the minuend (62) towards the end of the ENL and jumping backwards either by breaking the subtrahend (30) into three jumps of ten or by a whole jump of 30. Eight of the 40 PSTs selected this calculation, seemingly because it appears relatively straight-forward. Our analysis, however, shows that for this calculation two PSTs implemented the resources faithfully (all four key elements present); five PSTs implemented the strategy with medium implementation fidelity (three PSTs met three key elements, and two PSTs met two key elements); and one PST met only one key element. I selected this calculation to demonstrate the varying extent to which three PSTs faithfully implemented the key elements in their micro-teaching videos.

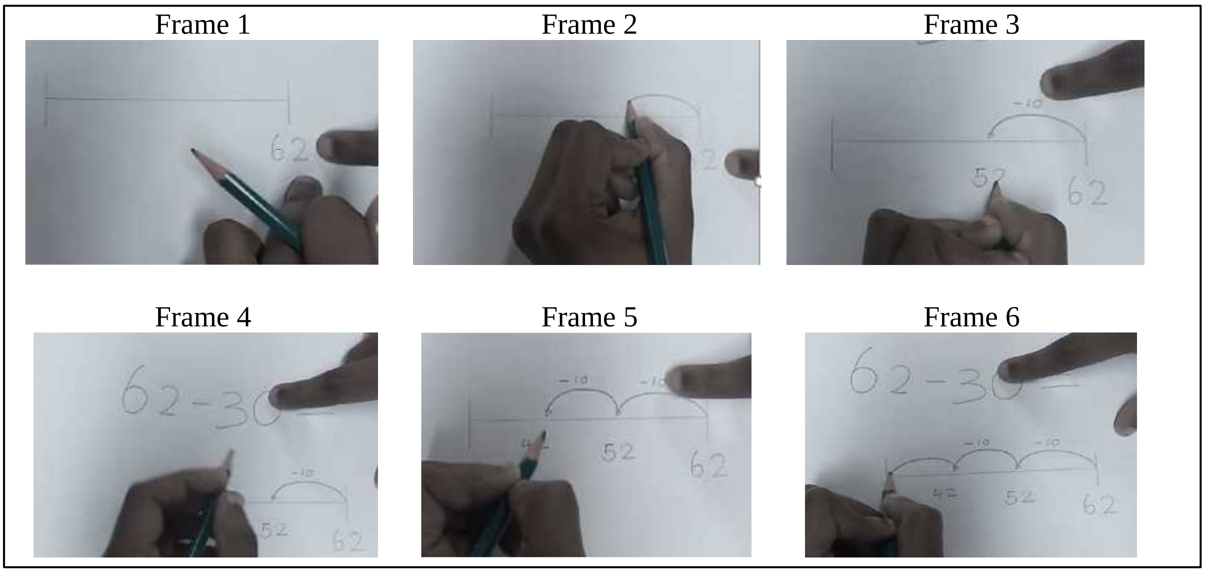
Strong Implementation Fidelity

Somila demonstrated an example of implementing the use of the jump strategy on the ENL in a way that is highly similar to the scripted MSAP lessons. An example from the MSAP scripted lesson for a subtraction calculation encourages the teacher to make the steps explicit to the learners, i.e., "Remind learners of the steps they learnt previously: **plot** [on the ENL], **break down** [into tens and ones], **jump** backwards, and **answer**" (Graven et al., 2020, p. 41). A demonstration video was also part of the teacher resource (see for example, https://youtu.be/JQq2zL6pwCM). Figure 1 below shows screenshots from key moments in her micro-teaching video, illustrate a transcript of her micro-lesson:

> You've got a sum of minusing 30 from 62, and we're going to be using the number strategy. The first step is to firstly draw your number line and place 62 on the end of your number line (*points to end of ENL–Frame 1*) since you're going backwards (*gestures backwards on the ENL*). So you need to take away three tens (*points to 30*) from 62 (*points to 62*). So you place your 62 at the end of the number line and you move backwards (*emphasises drawing jump arc – Frame 2*) by minusing 10. Then 62 minus 10 will give you 52 (*Frame 3 – points to 62 on ENL and then -10 jump*). Since you're not yet by 30 (*points – Frame 3*), you move back another 10 (*draws jump of 10*), and you minus 10 once again and you've got 42 (*keeps her righthand finger on the first ten jump while gesturing and drawing the next jump to 42*). So since we see that it's already two tens make up twenty (*points to jumps of ten – Frame 5*), that's why it takes one more ten to make 30, therefore we move back for the last time (*points with righthand finger on 30 in the calculation while drawing jumping the final 10 – Frame 6*), and you've got 32. These tens (*points to tens jumps*) make up 30, therefore you know you've moved back enough times.

Figure 1

Screenshots From Somila's Micro-Teaching Video



Somila first drew a straight line and then included markers for the first (minuend) and last (difference) numbers. The ENL is a straight continuous line with no unit markers other than the markers learners need and chooses to use to visually demonstrate their thinking (Bobis, 2007; van den Heuvel-Panhuizen, 2008; Vermeulen et al., 2020). She breaks 30 down into three groups of 10, which she uses for the jumps. As shown in Figure 1, her jumps are visually proportionate in size. While Bobis (2007) explained that learners are not expected to draw their jumps on the ENL proportionately, we suggest that the teacher should draw the jumps in a way that is visually proportionate, to indicate that groups of 10 are the same. Following the key phrases of the teacher resources, Somila indicates that 62 should be placed at the end of the line and gestures to demonstrate that 'minus' means we need to jump backwards. Her pointing shows the process linking the written calculation (specifically the subtrahend) to the jumps on the ENL, thus showing the fictitious learners exactly what is happening in each step. Other useful phrases include referring to "tens" and a strategy for learners to check if they are correct, "These tens make up 30, therefore you know you've moved back enough times". In this example, Somila effectively used the jump strategy on the ENL to solve the subtraction calculation, including key phrases from the teacher resources and gesturing to demonstrate the process. While Somila's micro-teaching video had all four key elements present, there are still areas where the MTE can guide her towards greater accuracy. This resonates with Cheng's (2017) study where micro-teaching was used to improve PSTs' precision in verbal and non-verbal skills. Somila, for example, referred to "a sum of minusing" and the "number strategy". On reflection of her micro-teaching video, her MTE could point out that 'sum' refers to addition, whereas 'difference' refers to a subtraction calculation. Similarly, the name of the strategy being taught is the *jump strategy* on the *ENL*. Somila also plotted her endpoint on the ENL, making it a line segment rather than a true *empty* number line. By practising teaching these strategies in a safe, conducive micro-teaching context, Somila has the opportunity to gain pedagogical and content knowledge as well as confidence before going to teach in a real school classroom.

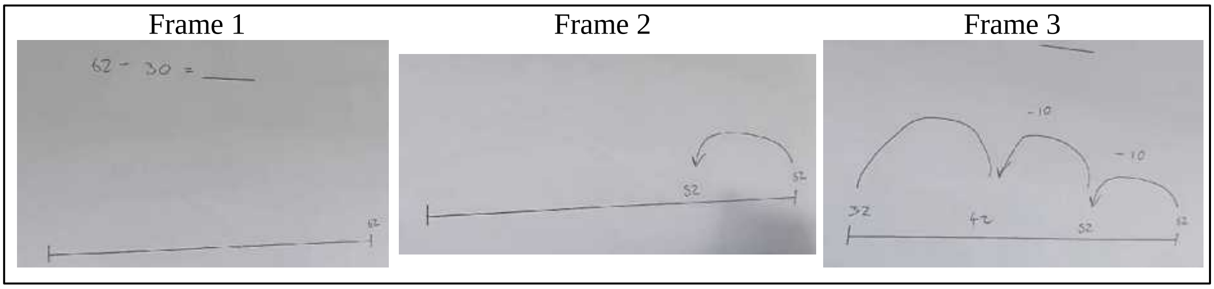
Medium Implementation Fidelity

Johannes demonstrated a medium similarity to the MSAP scripted teacher resources, by making use of the ENL, somewhat proportionate jumps and some of the key phrases. His micro-teaching video did not, however, include the gesturing encouraged in the teacher resources. His video transcript is below, followed by screenshots in Figure 2:

> 62 minus 30 is equal to (*62 plotted on ENL before starting – Frame 1*). To make it easier we are going to minus 10. 62 minus 10 is equal to 52 (*draws jump arc – Frame 2*). Minus 10 (*labels jump*). Minus another 10 gives 42 (*draws jump arc*). Minus another ten (*draws jump arc – Frame 3*) is your final answer, 32.

Figure 2

Screenshots from Johannes' Micro-Teaching Video



While Johannes made use of an ENL without unit markers, he had placed the minuend (62) at the end of the line without showing or explaining why the subtraction calculation would 'start' at the end of the line (see Frame 1). This omits an important pedagogical moment highlighted in the MSAP teacher resources, to give learners the opportunity to strategically represent and visualise the type of calculation (addition or subtraction). He proceeded to draw jumps, which are somewhat visually proportionate, but because he had placed the endpoints on the line, he had to ensure that his jumps landed on the market, thus resulting in his last jump being slightly longer than his first two. In this micro-teaching video, one key phrase from the MSAP resources is present, i.e., 'to make it easier', when indicating the breaking of 30 up into tens. He however does not refer to phrases such as 'at the end of the line' or 'jump backwards'. Instead, Johannes only uses the word 'minus' to indicate the subtraction of tens. Furthermore, his gestures are limited to the drawing of the jump arc. For the purpose of this study, I recognise the drawing of the jumps as a form of gesture, but also the emphasis of the jump through non-drawing hand movements, including pointing to the original calculation (such as in Somila's example), and indicating the jump direction (forwards or backwards) based on the calculation. While Johannes' micro-teaching video demonstrated the use of the jump strategy on the ENL to solve the calculation, there were some key elements missing that would enhance the pedagogical value of his demonstration. Using reflection on the micro-teaching video, the MTE could indicate the pedagogical value of using key phrases from the scripted lessons, and to scaffold learners with each step of the process.

Poor Implementation Fidelity

Wonga's micro-teaching video is an example of poorly implementing the MSAP teacher resources. In this video demonstration, he departed from the scripted teacher resources by breaking the subtrahend 30 into two groups of 15, rather than the suggested tens. The micro-teaching video is filmed at an angle which is difficult to clearly see, and as such, no screenshots are included. However, a transcript of his short video is below:

> (Draws ENL and plots 62). We have 62 minus (draws jump arc backwards) … 62 minus … 15 … this gives us … (is unsure and puts pencil down).

Wonga's 22-second micro-teaching video indicates that he is not yet confident with using the jump strategy on the ENL. He was able to draw the ENL, plot 62 and draw a jump arc backwards. However, he opted to break 30 up into two groups of 15, rather than using the suggested and scripted 'friendly numbers' such as groups of ten, in the MSAP resources. He only drew one jump arc and did not label it as '–15'. The delay indicated by the ellipsis in the transcript further indicates his uncertainty. When reaching the point of having to subtract 15 from 62 he stops the attempt. This suggests that he has not gained confidence in using the strategy for flexible and mental calculations. Wonga's micro-teaching video suggests that he needs further support in how to use the jump strategy on the ENL as well as explicit instruction on how to teach it.

Discussion and Conclusion

This paper sought to answer the research question, 'How do PSTs use the jump strategy on the empty number line when recording micro-teaching videos?'. From exploring the three examples of Somila, Johannes and Wonga, it is evident that PSTs are able to implement the jump strategy on the ENL with some (varying) level of fidelity. The micro-teaching videos were an opportunity for PSTs to work in pairs, to engage with the MSAP resources and find ways to enact them. The micro-teaching allowed opportunities for reflection (although not the focus of this paper) for PSTs, and their MTE, to establish where they are confident and competent, and which key elements needed further attention.

The three micro-teaching videos showed examples of how PSTs implemented the teacher resources with strong, medium, and poor implementation fidelity. This can be useful to researchers and MTEs to take note of the importance of explicit instruction and guidance for PSTs when learning how to teach the jump strategy on the ENL, and other mental calculation strategies. Just as Beswick and Goos (2012) and Bowie (2019) state that it cannot be taken for granted that PSTs arrive at the didactic courses with content knowledge, so too we cannot take it for granted that by sharing teacher resources and demonstrations on calculation strategies, PSTs have the knowledge, skills, and confidence to teach them. In this study, the MTE shared hardcopy and electronic scripted teacher guides, video-recorded examples, and demonstrated the jump strategy over a three-week period in six lectures. PSTs still did not have full implementation fidelity in their micro-teaching videos. It is promising though to note that PSTs were able to implement most of the key elements of appropriate jumps on the ENL, using some gestures and some key phrases. Creating micro-teaching videos in a safe space, and reflecting with peers, is a useful opportunity for PSTs to practise their teaching before going to teach in schools.

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References

Aktaş, M.C., & Özdemir, T.E. (2017). An examination of the number sense performances of preservice elementary school mathematics teachers. *European Journal of Education Studies*, *3*(12). https://doi.org/10.5281/zenodo.1117088

Basturk, S. & Tastepe, M. (2015). Examining primary pre-service researchers' difficulties of mathematics teaching with the micro-teaching method. *Acta Didactica Napocensia*, *8*(3), 1–10.

Blöte, A. W., Klein, A. S., & Beishuizen, M. (2000). Mental computation and conceptual understanding. *Learning and Instruction 10*(3), 221–247. https://doi.org/10.1016/S0959-4752(99)00028-6

Bobis, J. (2007). The empty number line: A useful tool or just another procedure? *Teaching Children Mathematics*, *13*(8), 410–413. http://www.jstor.org/stable/41198983

Bobis, J., & Bobis, E. (2005). The empty number line: Making children's thinking visible. In M. Coupland, J. Anderson, & T. Spencer (Eds.), *Making mathematics vital* (pp. 66–72), AAMT.

Bowie, L., Venkat, H., & Askew, M. (2019). Pre-service primary teachers' mathematical content knowledge: An exploratory study. *African Journal of Research in Mathematics, Science and Technology Education*, *23*(3), 286–297. https://doi.org/10.1080/18117295.2019.1682777

Cheng, J. (2017). Learning to attend to precision: The impact of micro-teaching guided by expert secondary mathematics teachers on pre-service teachers' teaching practice. *ZDM Mathematics Education*, *49*(2), 279–289. https://doi.org/10.1007/s11858-017-0839-7

Courtney-Clarke, M., & Wessels, H. (2014). Number sense of final year preservice primary school teachers. *Pythagoras*, *35*(1), a244. http://dx.doi.org/10.4102/pythagoras.v35i1.244

Fonseca, K., Maseko, J., & Roberts, N. (2018). Students' mathematical knowledge in a Bachelor of Education (foundation or intermediate phase) programme. In R. Govender, & K. Jonquière (Eds.), *Proceedings of the 24th annual national congress of the Association for Mathematics Education of South Africa* (pp. 124–139). AMESA.

Graven, M. & Heyd-Metzuyanim, E. (2014). Exploring the limitations and possibilities of researching mathematical dispositions of learners with low literacy levels. *Journal of Sciencia Educacione*, *5*, 20–35.

Graven, M., & Venkat, H. (2021). Piloting national diagnostic assessment for strategic calculation. *Mathematics Education Research Journal*, *33*(1), 23–42. https://doi.org/10.1007/s13394-019-00291-0

Graven, M., Venkat, H., Askew, M., Bowie, L., Morrison, S., & Vale, P. (2020*). Grade 3 mathematics mental starters assessment project (MSAP)*. Basic Education, Republic of South Africa. https://www.education.gov.za/MSAP2022.aspx

Graven, M., Venkat, H., Westaway, L., & Tshesane, H. (2013). Place value without number sense: Exploring the need for mental mathematical skills assessment within the Annual National Assessments. *South African Journal of Childhood Education*, *3*(2), 131–143.

Hoadley, U. (2012). What do we know about teaching and learning in South African Primary schools?. *Education as Change*, *16*(2), 187–202. https://doi.org/10.1080/ 16823206.2012.745725

Jojo, Z. (2019). Mathematics education systems in South Africa. In G. Porto (Ed.), *Education systems around the world*. IntechOpen. https://doi.org/10.5772/intechopen.85325

Klein, S., Beishuizen, M. & Treffers, A. (1998). The empty number line in Dutch second grades: Realistic versus gradual program design. *Journal for Research in Mathematics Education*, *29*(4), 443–464.

Lovemore, T.S., & Graven, M. (in press). Pre-service teachers' use of the jump strategy on the empty number line for efficient mental computation. *South African Journal of Childhood Education*.

O'Keeffe, L., & White, B. (2022). Supporting mathematics pre-service teachers reflection with 360 degree video and the knowledge quartet. *Australian Journal of Teacher Education*, *47*(3), 57–71. http://dx.doi.org/10.14221/ajte.2022v47n3.4

Ruiz, C., & Balbi, A., (2019). The effects of teaching mental calculation in the development of mathematical abilities. *Journal of Educational Research*, 112(3), 315–326. https://doi.org/10.1080/00220671.2018.1519689

Şahin, O., & Danaci, D. (2022). Investigating the effect of history-of mathematics activities on middle-grade students' mental computation and opinions: an action research. *International Journal of Mathematical Education in Science and Technology*, *53*(9), 2281–2318. https://doi.org/10.1080/0020739X.2020.1857859

Schollar, E. (2008). *Final report: The primary mathematics research project 2004–2007. Towards evidence based educational development in South Africa.* Johannesburg: Eric Schollar and Associates.

Şengül, S. (2013). Identification of number sense strategies used by pre-service elementary teachers. *Educational Sciences: Theory & Practice*, *13*(3), 1965–1974.

Spaull, N. (2019). *The education problem as it stands in January 2019* [Online]. https://nicspaull.com/2019/01/?blogsub=confirming#blog_subscription-3

Ünlü, M. (2018). Effect of micro-teaching practices with concrete models on pre-service mathematics teachers' self-efficacy beliefs about using concrete models. *Universal Journal of Educational Research*, *6*(1), 68–82. https://doi.org/10.13189/ujer.2018.060106

Van den Heuvel-Panhuizen, M. (2008). Learning from "Didactikids": An impetus for revisiting the empty number line. *Mathematics Education Research Journal*, *20*(3), 6–31. http://www.merga.net.au/node/41?volume=20&number=3

Venkat, H. & Graven, M. (2017). Changing teaching through a resources approach. In M. Graven & H. Venkat (Eds.), *Improving primary mathematics education, teaching and learning research for development in resource-constrained contexts* (pp. 163–178). Macmillan.

Vermeulen, J.A., Béguin, A., & Eggen, T.J.H.M. (2021). Task beliefs and the voluntary use of the empty number line in third-grade subtraction and addition. *Educational Studies in Mathematics*, *106*, 231–249, https://doi.org/10.1007/s10649-020-10016-x

Defining the Problem in a Changing Landscape: How Leaders Plan for and Address Mathematics Curriculum Change

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Curriculum change affords middle leaders opportunities for pedagogical and planning renewal. This paper reports case studies of two schools engaged in preparing for implementation of Version 9 of the Australian Curriculum: Mathematics. Drawing on interview data from a middle leader from each school, Bacchi's question, *What's the problem represented to be?* was used to explore factors underpinning how these leaders identified a *problem* to be addressed. Findings revealed that even when shown alternative interpretations of their perceived problem, these middle leaders were still *driven* by their own interpretation of the problem.

The Australian education landscape is marked by continual curriculum change, presenting both challenges and opportunities for educators. Remillard and Heck (2014) defined curriculum as "a plan for the experiences that learners will encounter, as well as the actual experiences they do encounter, that are designed to help them reach specified mathematics objectives" (p. 707, original emphasis). They presented a visual model of the curriculum policy, design, and enactment system that distinguished between the official curriculum, as specified by governing authorities, and the operational curriculum enacted in classrooms. How factors in the official and operational curriculum interact to exercise influence within a system is not well understood and need further exploration. Addressing this gap was one of the original aims of this research project, with a particular focus on how teachers could be supported to interpret and transform the revised official mathematics curriculum and bring it to life in the classroom.

Middle leaders (MLs) are focused on developing teaching and learning in classrooms (Grootenboer et al., 2023) and tend to lead collaboratively alongside their colleagues. MLs build teachers' capacity by organising how curriculum change will be implemented, providing opportunities for professional discussions, and negotiating instructional initiatives (Bryant et al., 2020). Recent changes to the F–10 *Australian Curriculum (AC): Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.), therefore, provide an avenue for MLs to embark on local curriculum change and reflect on pedagogical practices within their schools. The revised *AC: Mathematics* is due to be fully implemented in Queensland primary and secondary schools by the end of 2024 (Queensland Curriculum and Assessment Authority [QCAA], 2023). AC implementation follows curriculum change in the senior secondary curriculum to include external examinations, which represented the most significant reform to senior secondary schooling in Queensland for many years. Schools' processes to enact curriculum reform, in this case implementation of Version 9 of the *AC: Mathematics*, provide the context for the research reported in this paper.

The original aim of the project was to explore the ways in which MLs addressed a problem they identified in relation to implementation of Version 9 of the AC: Mathematics. However, initial findings revealed that, in some cases, assumptions about their own school context underpinning their perceived problem were not always a true reflection of the situation. Thus, there was a need to explore how MLs identified and addressed a problem within their own school context. Bacchi's (2009) approach to problem representation was used to explore factors

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 367–374). Gold Coast: MERGA.

underpinning how two MLs identified a problem within their own school context. Specifically, the following research question is addressed:

- How do MLs represent and address problems driven-by curriculum and pedagogy change?

Theoretical Framework

Investigating teacher professional learning in the context of curriculum reform provided the impetus for the project and led to the need to explore how MLs identify and represent problems within their own school context. The *official* curriculum sets out expectations for students' learning and has three components: curricular aims and objectives, the content of consequential assessments, and the designated curriculum (endorsed resources providing guidance to teachers) (Remillard & Heck, 2014). The *operational* curriculum includes the teacher-intended curriculum, the enacted curriculum, and student outcomes. Thus, the operational curriculum represents the transformation of the official curriculum into teachers' personal plans and how this plays out in the classroom.

Research has demonstrated that professional learning is enhanced when it occurs in school-based contexts so that teachers can try out and validate ideas in their own classrooms, and collectively discuss issues and problems with supportive colleagues (e.g., Loucks-Horsley et al., 2003; Mewborn, 2023). Garet et al. (2001) conducted a large-scale survey of more than 1000 mathematics and science teachers in the US to investigate characteristics of professional development that teachers found effective. Their analysis revealed that teacher knowledge, skills and classroom practice were positively influenced by three core features of professional development activity: a focus on content knowledge and how students learn this content; opportunities to engage in active learning, such as planning classroom implementation and sharing outcomes with colleagues; and coherent connections with local curriculum frameworks. Results also indicated that the structure of effective professional development was significant, with sustained and intensive learning being more effective than shorter duration activities such as one-off workshops. Each of these core and structural features of effective professional development was incorporated into the design of our project along with support for action research to implement aspects of the professional learning.

Identifying Problems

MLs and teachers usually rely on their own experiences and strategies to identify problems with curriculum implementation and pedagogical practices. Bacchi (2009) introduced a *What is the problem represented to be?* approach to policy analysis. Policy is the action plan designed to fix a specific issue (Parliamentary Education Office, 2023) and assumes that something needs fixing. Bacci's approach involves interrogating the problem from different perspectives and results in a problem-questioning paradigm rather than a problem-solving paradigm (Tawell & McCluskey, 2022). How problems are represented conveys different implications for how the issue is considered and, therefore, influences the solution methods trialled. Bacchi (2009) proposed using the following six interrelated questions to analyse problem representations:

- What's the "problem" represented to be?
- What presuppositions or assumptions underlie this representation of the "problem"?
- How has this representation of the "problem" come about?
- What is left unproblematic in this problem representation?
- What effects are produced by this representation of the "problem"?
- How/where has this representation of the "problem" been produced, disseminated and defended? How could it be questioned, disrupted and replaced? (p. 2)

The first question of problem representation is designed to clarify what the problem is and whether it is also connected to other problems. The second question considers presuppositions or assumptions, challenging policy makers to consider what knowledges are taken-for-granted and what meanings are drawn from different representations. Cultural premises and values should be considered. The third question is concerned with considering the history of the current representation and the conditions that facilitated this problem representation becoming the dominant one. With the fourth question it is necessary to consider the perspectives that have been silenced. The fifth question provides an opportunity to identify and assess critically the consequences of the problem representation using questions such as: what is likely to change/stay the same? who is likely to benefit/be harmed? The final question challenges policy makers to consider how representations become dominant and consider the production, dissemination, and defence of the problem (Bacchi, 2009). Although Bacchi's methodological approach is designed for analysing policy (including educational policy) rather than school-level or classroom-level contexts as in this project, her overarching question provides a means of capturing participants' own interpretation of the curriculum implementation problem they wish to solve; in other words, what they thought the "problem" was.

Research Design

This project was conducted over 2 years (2022–2023). The project supported 17 MLs, working in school-based teams in seven Queensland independent schools to develop small action research projects in which they identified a curriculum implementation problem, formulated their own questions to address the problem, and collected data to answer these questions (Loucks-Horsley et al., 2003). In this paper, we draw on data collected from participants in two schools, chosen because of the insights gained about how these MLs represented and enacted a curriculum implementation "problem" they identified.

Participants

The experiences of Sage and Tom, MLs from two independent schools in Brisbane, are reported in this paper. Sage was the Assistant Head of Primary at Coromandel College, a P-12 urban independent, faith-based school. She was a mid-career educator who was tasked with leading the curriculum implementation of mathematics, even though mathematics was not an area in which she felt confident. Tom was a very experienced educator and well-established as Head of Mathematics at Shaftston Hall, a P-12 independent girls' school with high academic expectations. He was seeking to develop students' problem solving and reasoning capabilities.

Data Collection and Analysis

The project team (the four authors of this paper) supported school teams through three face-to face professional learning workshops involving all participants and four Zoom mentoring sessions with individual school teams. The workshops were conducted at the beginning, middle, and end of the project and included opportunities for participants to share progress and receive feedback. Mentoring sessions were conducted between professional learning workshops and involved more individualised support for school teams from a member of the project team. Data sources included each school team's initial project plans and transcripts of the mentoring sessions and workshop sharing sessions. Additionally, two semi-structured interviews were conducted via Zoom with the leader of each school team which were recorded and transcribed.

Data analysis was conducted by considering the full data set for each case. Bacchi's (2009) questions were used to identify themes from the data (Tawell & McCluskey, 2022). Transcripts from workshop sharing sessions, mentoring sessions, and interviews were read by members of the project team. Comments made in relation to each of Bacchi's (2009) questions were identified. The project team met to discuss the analysis and resolved any differences in interpretation. The answers to Bacchi's questions for each case provide insights into how the MLs represented a problem in their school context. Two case studies follow.

Findings

Case Study 1

Coromandel College is relatively young school which has grown quickly in size in the last ten years. It is now a mid-sized, co-educational school with above average socio-educational advantage. Sage and four members of the teaching team participated in the project.

What's the "problem" represented to be? Initially Sage represented the problem as the implementation of a school mathematics program based on a textbook rather than the curriculum, "We need[ed] some programmes … because we didn't really have anything" (Sage, 1st mentoring). Now though, she thought the implementation of the pedagogical elements of the textbook program were patchy:

> Even within this team, you know, [there is] a variety of uptake of *Stepping Stones* [A commercial whole primary school mathematics program] as well. So, we've got some teachers who have implemented that as is and then others who haven't, for various reasons. (Sage, 1st mentoring)

What presuppostions or assumptions underlie this representation of the "problem"? Sage's representation of the problem developed from the presuppositions that fast growth of the school population had led to variable implementation of the *Stepping Stones* program (https://www.origoeducation.com.au/). Her aim was to build teacher capacity and confidence to teach mathematics and she saw the curriculum reform as an opportunity to build a more consistent whole school pedagogical approach and move away from the textbook:

> So, it was good timing to review what our program was, and to look forward to what's happening with the changes to the new curriculum … So, what we're trying to do is get that consistent approach and make sure that we've got everybody on the same page. (Sage, 1st mentoring)

How has this representation of the "problem" come about? The broader school team (four primary teachers from different year levels) agreed with Sage that decisions of pedagogy and resource selection were driven by lack of confidence among staff to teach mathematics. The school team expected that professional learning would result in increased confidence, "As we become more versed in the pedagogy, whatever it might be, and we get more trained and skilled in it, then without a doubt, our confidence will go up" (Lyn, 1st mentoring). Hence, project development at Coromandel College centred on boosting staff confidence to teach mathematics. The school team trialled pedagogical techniques for broader representations and modelling and shared their work in staff meetings routinely by unpacking the techniques, outcomes for student learning, and implementation. Teacher confidence data were collected pre- and post-intervention through online surveys.

What is left unproblematic in this problem representation? Sage acknowledged that she herself lacks confidence in her knowledge of mathematics and mathematics teaching practices:

> Maths is not my natural … Yeah, I have to work hard at maths. And what I've found, for my own schooling versus what I've done as a teacher, I've had to kind of reteach myself in order to teach students. (Sage, Interview 2)

She claimed that most of the school team, except for one teacher, would feel similarly about teaching mathematics. Consequently, it was assumed that teacher confidence was the issue.

Professional learning in mathematics was no longer a focus for Sage. Her recent professional learning has directly supported her leadership roles. She described a culture of support in providing professional learning opportunities for staff:

> We have we never really been denied access to professional development. You know, if people want to apply for something, nine times out of ten, they can go to it. But what's happened is it's been very ad hoc … it never really comes back to then bring everybody together or how that might align to that pedagogical framework of what we want. (Sage, Interview 1)

Thus, Sage assumed there had been opportunities for teachers at Coromandel College to undertake professional learning in mathematics. Although Sage acknowledged that the school is very well-resourced, she reflected that equitable resourcing is logistically challenging, particularly given the primary school size.

What effects are produced by this representation of the "problem"? This representation of the problem suggests that building teacher confidence in teaching mathematics will *fix* the problem. However, the representation of the problem was challenged by the findings from the pre-intervention survey. The findings showed that teachers overwhelmingly felt confident in teaching mathematics and creating tasks and rubrics to support the collection of student evidence of learning in mathematics. However, the teachers unanimously asked for more professional development to support teaching mathematics. Many commented that they could not remember the last professional development they attended with a focus on mathematics:

> That was good to see that people are feeling pretty confident with that [teaching mathematics]. But the area where they just jumped out at us is that people want more PD. That was the thing that they just, they couldn't remember when they'd last done any numeracy or maths PD in or … what they had done was pretty light on or they couldn't really remember what it was that they even did. (Sage, 2nd mentoring)

How/where has this representation of the "problem" been produced, disseminated and defended? How could it be questioned, disrupted and replaced? After considering the issue to be one of PD, Sage sought PD targeting mathematics pedagogical content knowledge. Finding PD addressing mathematics lesson planning and lesson structure, Sage, with one other member of the project team, attended the PD over several sessions. She then used this knowledge to develop a pedagogical framework for the teaching of mathematics in the primary school. In isolation of her team, Sage developed a framework that provided teachers with essential elements for lesson plans, unit overviews and mathematics resources kits. These became part of the "non-negotiables" of teaching mathematics. After developing the pedagogical framework and presenting this to staff, Sage still did not feel confident that this was the right direction. "I've made it up. I still don't know whether it's going to work, but we have to kind of test it out and see" (Sage, Interview 2).

Case Study 2

Shafston Hall is a well-resourced school with a long history of success in academic, sporting, and creative arts spheres. The school has a stable teaching staff who are attuned to the school's high academic expectations. Tom, the secondary school Head of Mathematics, has taught mathematics for 40 years, including nine years at this school. Over the last 20 years, he developed a reputation for innovative teaching of mathematical modelling using inquiry.

What's the "problem" represented to be? Tom came with a plan that aimed to develop problem-solving and modelling skills in Years 7–9 students. A further aim was to help students understand how they will be assessed in the senior secondary years when Problem Solving and Modelling Tasks (PSMTs) contribute 20% to the mandatory part of the high-stakes internal assessment of student's final grades. With the recent re-introduction of external examinations at the end of secondary school, Tom is also aware of time constraints for developing their mathematical understanding compared to the previous school-based assessment:

We tend to focus a lot more on just the knowledge and procedures and just getting through the course … And trying to develop some sort of understanding so that they can solve the complex problems. Now with the external exam, we have to cover everything, and we cover everything very superficially, really, each lesson is a new topic, and we don't have a lot of time to consolidate anything. (Tom, Interview 1)

Tom's action research project represented the problem in terms of lack of time and student characteristics (prior knowledge and attitudes to mathematical modelling) without appearing to consider the school culture that seems to be influencing student outcomes.

What presuppositions or assumptions underlie this representation of the "problem"? Tom wanted Years 7–9 students to learn to think mathematically so used the various aspects of the Instrument Specific Marking Guides to draw students' attention to the elements of the modelling processes: formulate a model, with appropriate assumptions and variables; solve the mathematical problem; evaluate and verify the results as well as the strengths and limitations of the model; and communicate the findings (QCAA, 2019). He thought this would allow these younger students an opportunity to learn the process, away from the pressures of the senior secondary assessment:

> And I guess some of the stuff that we're doing with the year 7s, 8s and 9s, we put it under the guise of preparing the kids for problem solving and modelling tasks at year 10, 11, and 12. But realistically, what I wanted to do is to get a foot in the door in terms of just giving the kids some time to be able to just work on problems. (Tom, Interview 1)

For several years at Shafston Hall, PSMTs had been assessed and were scheduled in Years 7–9 lessons for either two weeks of full-time work or four weeks part-time, with students spending lessons working on the problem and teachers helping them. However, there has been no noticeable improvement in PSMT performance in either the junior or senior secondary years. Some teachers were resistant to this approach: Tom explained that "I've got teachers who don't want to do it at all" and "it's a fight all the way". Other teachers feel obliged to "help" students because students are not accustomed to working on such tasks. Tom knows that this well-meaning approach only perpetuates a cycle of learned helplessness such that students expect to be told what to do when they get stuck (or even before they start). The Years 7–9 students also treat these tasks as high stakes assessment and are consumed with performance anxiety that reduced their willingness to take intellectual risks. Students focused only on the assessment grade rather than the knowledge and understanding associated with the task.

How has this representation of the "problem" come about? Exacerbating this problem is the school's cultural context, an elite institution steeped in conservative traditions that are difficult to change. Students have succeeded in mathematics without demonstrating advanced problem solving and modelling capabilities now demanded by the new senior secondary mathematics subjects. Teachers, likewise, have developed pedagogical repertoires that "work", but do not necessarily develop the skills and dispositions that students need to succeed in the new curriculum and assessment regime. Parents are also a part of this context. Tom reported that he often receives phone calls or emails from upset parents who want to know why he has changed the way mathematics is taught—that might potentially disadvantage their daughters:

> So, the parents have high expectations of their daughters doing well. And the kids have high expectations, and they equate the fact that just because they do lots of work, that they should get good results. And when they don't do that, then it becomes a bit of a bit of a problem. (Tom, Interview 1)

Tom explained that all his past efforts to begin developing PSMT-capacities in this school's junior secondary students have failed.

What is left unproblematic in this problem representation? The influences on the operational curriculum, including teachers' conservative beliefs and pedagogical practices, the high academic expectations of the school, and parental beliefs and expectations of what mathematics teaching should look like mean that instigating a cultural change will be challenging. The *silences* are that teachers and parents have a strong belief in how success in mathematics was achieved, albeit the previous school-based assessment system.

What effects are produced by this representation of the "problem"? Tom's *representation of the problem came about* because he understood the difficulty in bringing about cultural change and designed a project to work within these contextual constraints. To do this, he aimed to promote incremental change in teaching practices by raising teachers' awareness of students' unproductive attitudes and beliefs about mathematics. In this action research project, he wanted to try a slower and longer-term approach, introducing the separate parts of the modelling cycle over time in a developmental progression. He also planned to survey student attitudes at the beginning and end of the project, which will provide valuable data on one of the key influences on student outcomes.

How/where has this representation of the "problem" been produced, disseminated and defended? How could it be questioned, disrupted and replaced? Tom tried a slower and longer-term approach. He introduced the separate parts of the modelling cycle over time in a developmental progression and developed a teacher guide to support the implementation of the PSMT. He surveyed student attitudes at the beginning and end of the project to provide data on one of the key influences on student outcomes. The students worked collaboratively on problem-solving tasks with teacher input only as necessary:

> A think, pair share situation. So, we'd have a problem. ... We don't have any input to start with, we have the students sit in in groups of three, ... they had some time to think about the problem, what things they need to look up ... think about certain strategies that they might have themselves to approach it, then of course, they would have time together as a group to come up with these questions. ... we really wanted the teacher to step back and have more of a student-centred learning situation. ... the teacher would then be walking around looking at their groups, maybe having some input that would maybe steer them in the right direction. (Teacher at final PD Day)

Discussion and Conclusion

Sage wanted a pedagogical framework for teaching mathematics, believing it would build her teachers' confidence to teach mathematics and produce consistency across the school. However, the initial teacher survey showed most teachers were confident teaching mathematics and wanted mathematics-focussed professional development. Despite these survey results, Sage followed her initial assumption that all teachers lacked confidence in mathematics and chose for only herself and one other teacher to attend the professional development requested. Sage then developed a lesson structure to ensure a whole school approach to teaching mathematics.

Tom wanted to deepen students' understanding of problem solving and mathematical modelling to address student achievement concerns by developing an approach to teaching the elements of the PSMT task. Tom acknowledged the school's cultural pressures including teachers' conservative beliefs and pedagogical practices, the high academic expectations of the school, and parental beliefs and expectations of what mathematics teaching should look like. The high expectations of the school community drove him to remove the task from the assessment program while students developed their capacity to complete the PSMTs, rather than address the focus on assessment results and the strong belief in how success in mathematics was achieved, albeit the previous school-based assessment system.

This paper has focused on two schools with different contextual features and different approaches to identifying problems of curriculum implementation in mathematics at the primary and secondary levels. In both Coromandel College and Shafston Hall, the MLs were focused on developing teaching and learning in classrooms (Grootenboer et al., 2023) and the problems these MLs identified were based on assumptions and presuppositions that could be made visible and either challenged or affirmed through their participation in the project. Our interviews provided opportunities for these MLs to explain how they represented the curriculum implementation problem and reconsider or elaborate on their underlying assumptions about the problem and how it came about. We led them through a process to consider additional data which made them question the problem, but (after dabbling in the issue) they continued with their original issue/problem regardless. As a ML, Sage chose to organise how the curriculum change would be implemented in agreement with Bryant et al. (2020) through the lesson structure she developed but chose not to build teachers' capacity by providing opportunities for professional discussions and negotiating instructional initiatives. Tom too organised how the curriculum change would be implemented.

The MLs chose the problem to address with their action research project. Like Bacchi (2009), we suggest that what a school curriculum leader proposes to do or to change in response to curriculum reform reflects how they understand the "problem" within their context that needs to be solved. We argue it is important for researchers to understand and interrogate these problem representations to gain insights into their rationales and implications. This kind of analysis shifts our thinking from problem-solving towards exploring the dimensions of problems and how they are shaped, as well as assumptions that underlie problem representations and their possible effects.

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References

Australian Curriculum, Assessment and Reporting Authority [ACARA]. (n.d.). *What's changed in the new Australian Curriculum?* https://v9.australiancurriculum.edu.au/resources/stories/curriculum-changes

Bacchi, C. (2009). *Analysing policy: What's the problem represented to be?* Pearson Australia.

Bryant, D.A., Lun, W.Y. & Adames, A. (2020). How middle leaders support in-service teachers' on-site professional learning. *International Journal of Educational Research*, *100*, 1–17. https://doi.org/10.1016/j.ijer.2019.101530

Garet, M., Porter, A., Desimone, L., Birman, B., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal*, *38*(4), 915–945. https://doi.org/10.3102/00028312038004915

Grootenboer, P., Tindall-Ford, S.K., Edwards-Groves, C. & Attard, C. (2023). Establishing an evidence-base for supporting middle leadership practice development in schools. *School Leadership & Management*, *43*(5), 454–472. https://doi.org/10.1080/13632434.2023.2271957

Loucks-Horsley, S., Love, N., Stiles, K., Mundry, S., & Hewson, P. (2003). *Designing professional development for teachers of science and mathematics* (2nd ed.). Corwin Press.

Mewborn, D. (2003). Teaching, teachers' knowledge, and their professional development. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 45–52). National Council of Teachers of Mathematics.

Parliamentary Education Office. (2023). *What is the difference between policy, act, and legislation in Australia* [Online]*?* https://peo.gov.au/understand-our-parliament/your-questions-on-notice/questions/what-is-the-difference-between-policy-act-and-legislation-in-australia

Queensland Curriculum and Assessment Authority (QCAA). (2019). *Mathematical methods 2019 v1.2. General senior syllabus.* QCAA. www.qcaa.qld.edu.au/downloads/senior-qce/syllabuses/snr_maths_methods_19_syll.pdf

Queensland Curriculum and Assessment Authority (QCAA). (2023, July 11). *Memo: Australian Curriculum Version 9.0: Revised implementation schedules, professional learning and resources.* QCAA. http://www.qcaa.qld.edu.au/downloads/memos/23/043-23.pdf

Remillard, J., & Heck, D. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM Mathematics Education*, *46*, 705–718. https://doi.org/10.1007/s11858-014-0600-4

Tawell, A., & McCluskey, G. (2022). Utilising Bacchi's what's the problem represented to be? (WPR) approach to analyse national school exclusion policy in England and Scotland: A worked example. *International Journal of Research & Method in Education*, *45*(2), 137–149, https://doi.org/10.1080/1743727X.2021.1976750

Factors that Influence Primary Preservice Teachers' Self-Efficacy While Teaching Mathematics During Professional Practice

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This paper explores factors that contribute to increased self-efficacy in preservice teachers as they practise teaching mathematics in a primary classroom. Thirteen preservice teachers completing a Master of Primary Teaching contributed to the research, four of whom became the focus of intense examination during two professional practices conducted over a 12-month period. Findings revealed six factors. When conceptualised into a framework the factors contributed to a better understanding of ways to support preservice teachers' self-efficacy for teaching mathematics.

Learning to teach mathematics is challenging. Some preservice teachers (PSTs) struggle to connect education(al) theory and practice teaching in initial teacher education (ITE) programs, despite excellent support conferred by qualified supervising teachers (STs) during professional practice. Self-efficacy is essential for teaching mathematics, where teachers must teach competently and confidently. This paper examined changes to primary PSTs' self-efficacy during two professional practice placements. This study investigated the research question What factors influence primary preservice teachers' self-efficacy for teaching mathematics?

Literature Review

Self-efficacy has been studied extensively in educational contexts since the mid-1970s. While many researchers have contributed to the current understanding self-efficacy (for example, Arslan 2019; Segarra et al., 2021), Albert Bandura is widely recognised as the foremost author and the first to coin the term in his paper *Self-efficacy: Towards a unifying theory of behavioral change* (Bandura, 1977). Bandura defined self-efficacy as one's personal judgement about one's capability to complete a task to achieve a desired or required objective (Bandura, 1994). To comprehend self-efficacy more fully, one must first understand how self-efficacy beliefs are formed.

Sources of Self-Efficacy

Bandura (1994) posited that there were four main sources of self-efficacy beliefs. These are *mastery experiences, vicarious experiences, verbal and social persuasion, and emotional and physiological states*. This research also examined a fifth source which Bandura (1997) named *cognitive enactment.*

According to Bandura (1994), the most effective contributor to strong self-efficacy is mastery experiences. As PSTs experience successful mathematics lessons, there is a greater likelihood of developing a robust self-efficacy for teaching mathematics. Vicarious experiences are the second source of self-efficacy whereby learners observe successful social models with whom they identify, resulting in the belief that they too have the capabilities to succeed (Bandura, 1994). When a poor performance is observed, the learner's self-efficacy beliefs are less influenced (Bandura, 1994; Schunk & DiBenedetto, 2020).

Encouragement from others, known as verbal and social persuasion, can influence the learner's effort and motivation. This third source of self-efficacy is seen when people, such as teachers, parents, or peers, influence a learner's belief that they have the capabilities to master a challenging task. For verbal and social persuasion to succeed, there must exist a level of trust between the learner and the persuader (Bandura, 1994).

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 375–382). Gold Coast: MERGA.

Bandura (1994) argued that negative emotions and increased stress could impact negatively on self-efficacy, whereas positive emotions had the opposite effect. These emotional and physiological states is the fourth source of self-efficacy and occurs when an individual judges their capabilities in relation to how they feel (Bandura, 1994). Negative emotional and physiological states are less likely than other sources to impact self-efficacy (Bandura, 1994).

A fifth source of self-efficacy uses imagination to visually rehearse successful outcomes. Bandura (1997) called this cognitive self-modelling or cognitive enactment. For PSTs, imagining how their lesson could be structured when planning can be viewed as an attempt to pre-empt potential organisational, pedagogical, and behavioural problems.

Bandura and others maintain that increased self-efficacy is a higher predictor of performance, and is indicative of effort, actions, and motivation to persist with a task, whereas lower self-efficacy can negatively affect motivation and achievement (Bandura, 1994; Liljedahl & Oesterle, 2014). What is known is that self-efficacy beliefs for learning mathematics influence self-efficacy beliefs in teaching mathematics and the practices that teachers employ in the classroom (Hessen Bjerke & Solomon, 2020; Küçükalioğlu & Tuluk, 2021; Lo, 2021).

Self-Efficacy Beliefs for Learning Mathematics

Throughout many years of schooling, PSTs form a set of beliefs about teaching and learning in mathematics (Brady, 2012; Liljedahl et al., 2012). Prior learning experiences have been found to influence creativity and innovation for teaching mathematics, including the way PSTs plan their lessons and the instructional choices they make in the classroom (Chalkiadaki, 2018). Other studies revealed that primary PSTs typically lack confidence and have poor attitudes towards mathematics, before they come to ITE programs (Brady, 2012; Mapolelo & Akinsola, 2015). Hessen Bjerke and Solomon (2020) add that beliefs held about one's mathematics knowledge can influence PSTs' perceived self-efficacy for teaching mathematics.

Self-Efficacy Beliefs for Teaching Mathematics

Factors leading to improved self-efficacy for teaching mathematics continue to be debated. A growing body of evidence suggests that ITE programs positively influence PSTs' self-efficacy in mathematics. Some researchers found that positive and enthusiastic tertiary teachers played an important role in influencing PSTs' self-efficacy for teaching mathematics (Beswick & Goos, 2018; Küçükalioğlu & Tuluk, 2021). Others suggest that having opportunities to explore mathematical content in a supportive environment at university has a strong influence on PSTs' self-efficacy as learners and as prospective teachers of mathematics (Henderson & Rodrigues, 2008). Conversely, Brady (2012) claimed that when PSTs encounter the realities of teaching in the classroom and working with other teachers, it is then that they are more likely to adjust their beliefs about teaching mathematics and adopt practices that they observe. This is a similar view also held by Liljedahl et al. (2012) who state that experiences in the early years of teaching can positively influence primary PSTs' practice and beliefs for teaching mathematics.

Collectively, these studies provide important insights into some of the factors that influence self-efficacy beliefs held by PSTs for teaching mathematics. Overall, they highlight the need for further research into primary PSTs for self-efficacy for teaching mathematics. By examining changes to self-efficacy while PSTs are practicing their mathematics teaching during professional practice, additional features that contribute to positive self-efficacy for teaching mathematics can be examined.

Methodology

This study is situated in a qualitative epistemology that employed case study and narrative inquiry as merged methodologies. By choosing case study, the intention was to explore the real-life context (Stake, 2000) of the PSTs while they taught mathematics. The key features of the second methodology, narrative inquiry, are time and temporality (Clandinin & Connelly, 2000). Given that this research spanned 12 months, narrative inquiry allowed for observations of PSTs as they progressed along a continuum of learning. Their stories told in their own words revealed changes to self-efficacy over time and revealed the factors that contributed to those changes.

Data Collection and Analysis

Thirteen PSTs participated in this research, nine of whom had completed their first professional practice and four were yet to commence their first professional practice. Data was gathered from a range of sources consisting of two data collection phases spanning one year. Phase one commenced with participant identification surveys, a semi-structured interview before attending the school and another during the four-week professional practice. Phase two also held a semi-structured interview before attending the school and another during professional practice with a focus group interview at the end. Other data sources included at least four reflections, lesson plans and lesson reflections from each participant which were submitted at various points throughout both phases.

This research adopted a thematic analysis approach to analyse data across multiple case studies (Stake, 2000). The temporal nature of this yearlong study allowed for data collection and analysis to be conducted simultaneously. Interview transcripts, reflections, and lesson plans were maintained in digital form, and initial thoughts written in track changes in the margins. Preliminary themes were also colour coded which allowed for greater control when triangulating data across multiple data sources.

Chunks of data were transferred to large Post-it notes and arranged into rudimentary categories. The Post-it notes created a dynamic matrix where data could be classified into categories, descriptors, and themes. This iterative process revealed five descriptors, *history, motivation, self-efficacy and personal agency, and transformation*. Table 1 shows the process for the descriptor history.

Table 1

Codes, Categories, Descriptors, and Emerging Themes

Codes	Category	Descriptor	Themes	Theory
Lots of textbooks Rote learning Singing times tables Photo-copied pages of maths	Mathematics lessons	History	Historical experiences of mathematics	Influence of prior learning on goals for self-mastery
Hated maths Boring Bad teachers	Feelings about mathematics at school		Personal / historical evaluation of self	Self-efficacy belief about one's capabilities
Never any good at mathematics	Self-efficacy			

Key Findings

Analysis of the data revealed six factors that contributed to self-efficacy changes for teaching mathematics. These are *collaborative relationships*, *teacher feedback*, *resources, teaching practice, motivation and self-regulation* and *reflection*. These six factors operate in association with one another and if all six factors were apparent the PSTs were more likely to develop stronger self-efficacy for teaching mathematics. However, if any of the factors were absent, then the PSTs experienced an emotional reaction, and they felt their progress was constrained. While some of the factors influenced self-efficacy more strongly for certain individuals, there was no hierarchy considered when presenting them in this paper.

Collaborative Relationships

Relationships between the STs and PSTs are built on trust and rely on collaboration and reciprocal interactions. A recent study found that PSTs' greatest challenge during professional practice is to build relationships with their supervisors (Johnston & Dewhurst, 2021). Some relationships are more challenging than others where power and ill-chosen words can influence self-efficacy negatively.

John experienced a difficult relationship in his first professional practice. He said, "I became deflated when my supervising teacher said, how can we possibly get you to do 20 lessons given how hopeless you are?" and that "everything I was doing was wrong." It is important to remember that there is an inherent power differential in these relationships, which can make PSTs feel vulnerable and doubt their abilities (Waber et al., 2020).

Strongly linked with social persuasion, this type of interaction also connects with Bandura's social and emotional experiences. John's self-efficacy for teaching was profoundly impacted and he withdrew from professional practice. Contrastingly, John's second attempt at professional practice had a more positive impact on him. He said he had "developed a very strong rapport" and "feeling like I belong in the teaching team, has boosted my confidence." Even though John was already highly motivated, Land (2018) maintains that one's sense of belonging is important to self-efficacy beliefs.

Collaboration and planning as a team also had positive influences on Amy's self-efficacy and sense of belonging. She said, "The first day with Josh, I just observed, but later we planned a lesson for the next Friday, and then taught it together." Amy described this lesson as a "brilliant." She felt that she was "learning so many new things just because we are working together has given me so much confidence." The collaborative relationship between her and Josh had a strong influence on Amy's skill building and improved self-efficacy for teaching.

In contrast, Maria said she found it "difficult to engage with Peter" and "I'm not sure I trust him," and "he is still going to have to mark my report and I just don't want to rock the boat." Trust has been found to play a significant role in the supervisor, preservice teacher relationship (Tschannen-Moran, 2014). When collaborative relationships were weak or challenging, the emotion were *worry and self-doubt*.

Resources

Resources are associated with both the physical and non-physical resources required for teaching. For example, physical resources included lists of lessons the PSTs would be teaching, daily timetables and scope and sequences. Non-physical resources included information about students, such as student grouping, behaviour management or special needs, as well as opportunities to observe modelled teaching or seek advice from STs.

When non-physical resources such as guidance and support are provided, anxiety is reduced. For example, one of Emily's mathematics lessons did not go to plan. This greatly affected her emotional state and at that point her self-efficacy was significantly impacted. She said, "I burst into tears at lunch time … it brought back a flood of bad feelings because if I wasn't capable of teaching Kindy maths how on earth can I teach Year 6." After reflecting at length with her ST who modelled the lesson again, Emily retaught it. "I followed her instructions and remembered how she taught the kids so as soon as the next lesson was over the anxiety was gone again which was nice." Emily acquired the appropriate non-physical resources from her ST that assisted her to succeed.

It was apparent in this research that at the very least, PSTs needed the physical resources, such as lesson content and scope and sequences. However, it was not enough to provide these without clearly communicating their application in the context of the students, the school and the STs expectations. Ciara felt anxious because, "Neither of us knew what we had to do … I never got to see the scope and sequence for any subject never mind maths." Likewise, Yaz said, "When they sat down to do the actual Maths at their tables they couldn't trade. I didn't find out until after the lesson that the children hadn't been taught how to trade." Not having access to this non-physical resource left her feeling unsure and anxious and impacted her self-efficacy. She said, "I actually felt incompetent." When physical or non-physical resources are lacking, PSTs struggle to know what to do, and the emotional reaction is *anxiety*.

Feedback

The amount of and the perceived quality of lesson feedback was found to be a significant contributor to PSTs' self-efficacy for teaching mathematics. PSTs whose STs provided pre lesson and post lesson feedback developed good teaching strategies, were more confident, set goals and were better able to deal with challenges as they arose. However, when the PSTs felt that the feedback was lacking or that they needed more clarity, they felt uncertain about how they could improve their teaching. Amy said, "I need more information about how to improve." Maria maintained that, "He is more kind of focusing on my classroom management skills and it's really hard to know if I am improving or even where I can improve," and Cath alleged, "the feedback I received was fairly critical, and not really helpful … no feedback was ever provided prior to delivery of lessons, so I didn't know what to avoid or what to put in to improve."

The most effective feedback used a collaborative and reflective approach that included cognitive modelling and targeted goals. Emily's ST prompted her to "cognitively rehearse" (Bandura, 1989, p. 730) her lessons before teaching :

> She would ask me what are your goals? What if this doesn't happen, what will you do? If some of this blows up in your face, what are you going to do, if it goes smoothly and you have finished all the activities … what are you going to do? … She was good at making me think. (Emily)

Through this rehearsal process, Emily was able to anticipate possible scenarios and her ST could offer additional advice prior to teaching. However, when PSTs perceived the feedback to be ambiguous or limited, the resultant emotional reaction was one of *uncertainty* as they did not know what they needed to do, or how to improve their practice.

Teaching Practice

Practising teaching is fundamental to learning how to teach. In this study, opportunities to prepare mathematics lessons, practice teaching, and reflect on areas for improvement offered PSTs multiple ways to develop their skills. With practise, PSTs experienced mastery, resulting in increased self-efficacy for teaching mathematics.

Most of the PSTs had ample opportunities to teach mathematics during their professional practices. However, having only taught two mathematics lessons during the three-weeks of the first professional practice Amy expressed her frustration stating, "I've taught lots of [other] lessons, but I don't feel like I've done enough [mathematics lessons] to say I can teach maths well." While Amy's self-efficacy for teaching mathematics seemed unchanged to that point, her self-efficacy for teaching more broadly had improved. This appeared to be positively influencing her self-efficacy beliefs for mathematics. In a reflection, she said, "even though I didn't teach many [mathematics lessons] I felt really confident." If opportunities to teach mathematics were limited, the resulting emotional response was *frustration*. This was because PSTs were unable to determine areas of deficit, or to verify areas of growth that might contribute to improved self-efficacy for teaching mathematics.

Motivation and Self-Regulation

All PSTs participating in this study were highly motivated to improve their mathematics lessons. This research examined three motivational sources. These were *extrinsic*, *intrinsic* (Ryan & Deci, 2000) and a source I termed *influential* motivation. This construct contains many of the attributes of intrinsic motivation, that is, it is an internal source motivated by personal satisfaction. However, there is an additional element found in influential motivation which is related to the sense of responsibility that the PSTs felt towards their students. For example, Amy was motivated to ensure that she was "enthusiastic so the kids pick up on that. You don't want to go in there going, oh maths is horrible." Jen's responsibility was directly related to her fear of mathematics "it's quite scary thinking I am going to go into a classroom with my fear and trying to actually teach kids with confidence … it's a huge responsibility."

Influential motivation was a strong indicator of the behaviours and self-regulation choices PSTs made for teaching mathematics. Of note was the goals they set when preparing to teach mathematics. Maria stated that she struggled to connect the learning theories to her practice in the classroom. However, this did not dissuade her from setting specific pedagogical goals for her lessons such as cooperative learning "and not just doing mindless activities." She believed that "maths is really important to get it right, to teach it correctly". While it is true that all PSTs in this study were motivated at various levels, without motivation, it is anticipated that the emotional reaction may be *indifference*.

Reflection

In this study, reflecting-in practice, reflecting-on practice (Schön, 1991) and reflecting-for practice (Olteanu, 2017) were important to strengthening self-efficacy. For example, John reflected-for practice after he " … talked to Marcel [student] today and he said it [lesson on division] was much too easy, so now it's crucial to follow up with extension work at each skill level." He also reflected-in practice, when observing the whole class and offered to reteach the content in small groups to students who "were really struggling" with a concept.

While John's personal reflection contributed to improving his practice, joint reflection was found to strengthen self-efficacy for teaching mathematics more effectively. After bursting into tears following what she perceived to be a failed lesson, Emily reflected-on her practice with her ST. She said, "[Paula] was really good at going through the reflection with me [which] gave me an opportunity … to really brainstorm for myself what went wrong." In this study, lesson feedback from the ST acted as the conduit to reflection and when PSTs and their ST reflected jointly, self-efficacy for teaching was strengthened. However, when feedback lacked substance, the result was a shallow reflection and PSTs felt that their *progress was limited*.

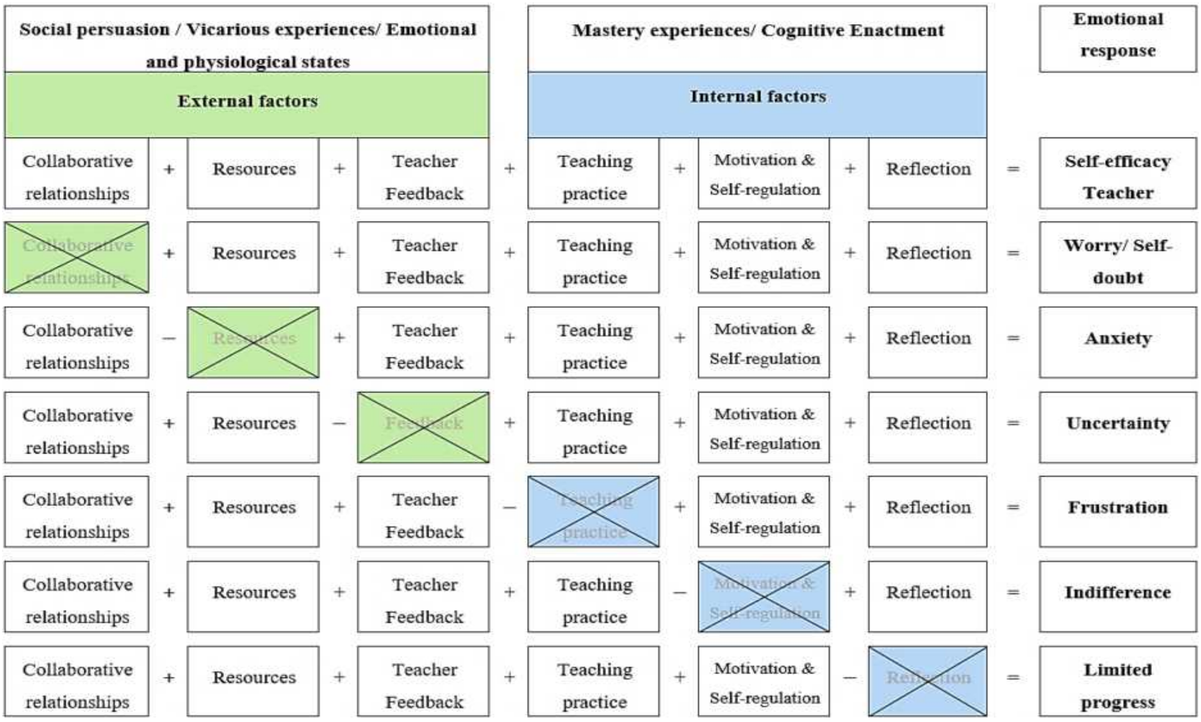
Implications for Practice

With any learning comes change and because there were observable self-efficacy changes over the two professional practices, the six factors collaborative relationships, resources, teacher feedback, teaching practices, motivation and self-regulation, and reflection contributed to this transformation. Drawing from Knoster's et al.'s (2000) Model for Leading and Managing Complex Change, I conceptualised the Building Teacher Self-Efficacy Framework (BTSE Framework, Figure 1). Although the BTSE Framework provides a graphical representation that appears organised, there is the human element that should be considered. It is true that neither PSTs nor their STs fit into neat rows or columns and learning to teach does not follow a linear progression (Khoshnevisan & Rashtchi, 2021). In fact, PSTs in this study repeatedly vacillated between high self-efficacy and self-doubt when teaching mathematics. Therefore, one should employ the framework knowing that some factors are dependent on the relationship that exists between PSTs and STs as well as the context in which they are working.

The six factors are divided into external and internal factors. Collaboration, resources and teacher feedback were factors that PSTs have limited control over and are therefore considered external to the PSTs. External factors were observed in three sources of self-efficacy. These were social persuasion, vicarious experiences, and emotional and physiological states. Factors that PSTs had greater personal control over included their teaching practice, motivation self-regulation and reflection. These are classified as internal factors and were analysed in relation to Bandura's mastery experiences and cognitive enactment.

Figure 1

Building Teacher Self-Efficacy Framework (BTSE Framework)



Limitations

While focusing on a small sample of participants (13) allowed for a deep analysis into factors that contributed to self-efficacy for teaching mathematics, including the ST's perspectives in future research may reveal additional insights. Furthermore, this study analysed qualitative data from a variety of sources. While these data revealed the narratives of the PSTs as they progressed through professional practices, by employing self-efficacy scales to evaluate their self-efficacy may have offered another dimension to the research.

Conclusion

This research has highlighted the complexities surrounding PSTs' self-efficacy for teaching primary mathematics. The findings of this study identified six factors that influence positive self-efficacy. Conceptualising these factors into the BTSE Framework, PSTs' specific learning needs can be identified to support self-efficacy for teaching mathematics during professional practice. It is clear that these factors are interconnected. I am not suggesting that PSTs' self-efficacy and professional development will only occur if all the factors in the BTSE Framework exist. However, this study has revealed that where weakness exists within one factor, it may add to deficits in other factors and therefore reduce positive influences on self-efficacy.

References

Bandura, A. (1977). Self-efficacy: Toward a unifying theory of behavioral change. *Psychological Review, 84*(2), 191–215.

Bandura, A. (1989). Regulations of cognitive processes through perceived self-efficacy. *Developmental Psychology, 25*(5), 729–735.

Bandura, A. (1994). Self-efficacy. In V. Ramachaudran (Ed.), *Encyclopedia of human behaviour*, (Vol. 4, pp. 71–81). New York: Academic Press.

Bandura, A. (1997). *Self-efficacy: The exercise of control*. W. H. Freeman and Company.

Beswick, K., & Goos, M. (2018). Mathematics teacher educator knowledge: What do we know and where to from here? *Journal of Mathematics Teacher Education*, *21*, 417–427. doi.org/10.1007/s10857-018-9416-4

Brady, K. (2012, June 2–6). Stories from the classroom: The developing beliefs and practices of beginning primary mathematics teachers. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Mathematics education: Expanding horizons*. *Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia* (pp. 122–129). Singapore: MERGA.

Clandinin, D. J., & Connelly, F. (2000). *Narrative inquiry: Experience and story in qualitative research*. Jossey-Bass.

Henderson, S., & Rodrigues, S. (2008). Scottish student primary teachers' levels of mathematics competence and confidence for teaching mathematics: Some implications for national qualifications and initial teacher education. *Journal of Education for Teaching*, *34*(2), 93–107.

Hessen Bjerke, A., & Solomon, Y. (2020). Developing self-efficacy in teaching mathematics: Pre-service teachers' perceptions of the role of subject knowledge. *Scandinavian Journal of Educational Research*, *64*(5), 692–705.

Johnston, D., & Dewhurst, Y. (2021). A study of student teachers' experiences of belonging on teaching practice. *International Journal of Educational Research*, *108*, 1–11.

Khoshnevisan, B., & Rashtchi, M. (2021). The first field experience: Perceptions of ESOL PST. *Advances in Language and Literary Studies*, *12*(5), 15–22.

Küçükalioğlu, T., & Tuluk, G. (2021). The effect of mathematics teachers' self-efficacy and leadership styles on students' mathematical achievement and attitudes. *Athens Journal of Education*, *8*(3), 221–238.

Land, C. (2018). Examples of critical coaching: An analysis of conversation between cooperating and PST. *Journal of Teacher Education*, *69*(5), 493–507. doi.org/10.1177/0022487118761347

Liljedahl, P., Oesterle, S., & Berneche, C. (2012). Stability of beliefs in mathematics education: A critical analysis. *Nordic Studies in Mathematics Education*, *17*(3–4), 101–118.

Liljedahl, P., & Oesterle, S. (2014). Teacher beliefs, attitudes, and self-efficacy in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer. doi.org/10.1007/978-3-030-15789-0_149

Lo, W. Y. (2021). Pre-service teachers' prior learning experiences of mathematics and the influences on their beliefs about mathematics teaching. *International Journal of Instruction*, *14*(1), 795–812.

Mapolelo, D., & Akinsola, M. (2015). Preparation of mathematics teachers: Lessons from review of literature on teachers' knowledge, beliefs, and teacher education. *American Journal of Educational Research*, *3*(4), 505–513.

Olteanu, C. (2017). Reflection-for-action and the choice or design of examples in the teaching of mathematics. *Mathematics Education Research Journal*, *29*, 349–367. doi.org/10.1007/s13394-017-0211-9

Ryan, R., & Deci, E. (2000). Intrinsic and extrinsic motivations: Classic definitions and new directions. *Contemporary Educational Psychology*, *25*, 54–67. doi/org/0.1006/ceps.1999.1020

Schön, D. A. (1991). *The reflective practitioner*. Ashgate Publishing Ltd.

Schunk, D., & DiBenedetto, M. (2020). Motivation and social cognitive theory. *Contemporary Educational Psychology*, 1–9, Article 101832. doi.org/10.1016/j.cedpsych.2019.101832

Segarra, J., Julià, C., & Valls, C. (2021). Pre-service teachers' belief about the efficacy of their mathematics teaching: A case study. *European Journal of Science & Mathematics Education*, *9*(4), 199–210.

Stake, R. (2000). The case study method in social inquiry. In R. Gomm, M. Hammersley, & P. Foster (Eds.), *Case study method*. Sage.

Tschannen-Moran, M. (2014). *Trust matters: Leadership for successful schools*. Jossey-Bass.

Waber, J., Hagenauer, G., & de Zordo, L. (2020). Student teachers' perceptions of trust during the team practicum. *European Journal of Teacher Education*, *45*(2), 213–229. doi.org/10.1080/02619768.2020.1803269

Mathematics Written Feedback for Pre-Service Teachers During Professional Experience

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In this study, the authors investigate the written feedback provided to primary and secondary pre-service teachers studying a specialisation in mathematics. The analysis focuses on 16 written reports completed by mentor teachers during the penultimate year placement of pre-service teachers. The evaluation employs a framework informed by the literature. Results indicate a scarcity of subject-specific written feedback concerning mathematics teaching and learning during pre-service teacher placements.

Feedback plays a pivotal role in the development of pre-service teachers (PST) during their professional experience placements, guiding them in shaping their future goals as they progress toward the completion of their courses (Grossman & McDonald, 2018). Constructive feedback not only provides valuable insights into strengths and areas for development but also guides PSTs in refining their teaching practices, building confidence, and fostering continuous professional development (Asregid et al., 2023). It creates a supportive learning environment, enhancing the overall effectiveness of their training and better preparing them for the challenges of the profession (Asregid et al., 2023). Feedback is an integral component of the broader practice of mentoring (Mullen et al., 2021), where effective embedded mentoring practice includes elements of feedback. Feedback can be in written or verbal forms, typically occurring within mentoring conversations or post lesson delivery between the mentor and mentee (Ellis & Loughland, 2017).

Mentors, also known as supervising teachers, both locally and internationally, are expected to provide crucial feedback to PSTs during their school based professional experience placements, fostering the professional development of PST growth. In the most recent Australian review, Strong Beginnings: Report of the Teacher Education Expert Panel 2023, there is mention of the need for PSTs to "receive regular observations, assessment and feedback to support their development" (Teacher Education Expert Panel, 2023, p. 62) during their professional experience placement.

Subject specific feedback is required for PSTs that are undertaking a course that requires explicit content and pedagogical knowledge. In Singapore, as part of professional development for experienced teachers, it is expected that all staff are involved in mentoring. The introduction of The Master Teacher program identifies experienced Singaporean teachers where mentoring is subject-specific, with a focus on content and pedagogical knowledge (Jensen et al., 2016). Internationally, it is common that feedback is provided post lesson observation to support ongoing improvement (e.g., in England, Shanghai, Finland, Jensen et al., 2016). For many Initial Teacher Education (ITE) programs, summative written feedback reports serve as a dual-purpose tool, fulfilling program accreditation requirements while also providing the platform for mentors to offer comprehensive written feedback to PSTs (White, 2007).

Presently in Australia, the workforce shortage has extended to impact the selection of mentors for PSTs during their professional experience placements. In Australia, teachers progress through four career stages: graduate, proficient, highly accomplished and lead. The current teacher demographic includes individuals in the early stages, often proficient, of their teaching career (up to 5 years), and occasionally, those teaching outside their subject area specialisation. Ideally, highly accomplished experienced teachers would be the optimal mentors for PSTs during their placements. However, among the 307,000 full-time equivalent teachers

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 383–390). Gold Coast: MERGA.

in the country, only 1211 hold certification at the highest accreditation levels, that is, Highly Accomplished or Lead (National Teacher Workforce Action Plan, 2022). Consequently, there is a need to consider how feedback can be effectively provided by mentors who are still developing professionally and may also be teaching outside their subject area.

This paper centres on examining a specific feedback approach, written feedback, for ITE primary PSTs specialising in mathematics and secondary mathematics PSTs. The focus extends to exploring subject-specific feedback. To streamline the information presented in this paper, only documents from the penultimate year PST reports will be included.

The Literature Informed Framework

Features of the theoretical framework can be categorised under five themes. They include: Australian Institute for Teaching and School Leadership (AITSL) governing standards; learner needs, feedback design, structure based on the written form/report and mentor provisions. For this paper, only five sub-themes are considered and further explained.

Sub-Theme 1: Use of AITSL Language in Written Feedback

In the Australian context, written feedback provided by mentors overseeing professional experience serves a dual purpose. It functions as a formative assessment, with a focus on continuous improvement, and as a summative assessment, serving accreditation objectives by determining if a PST meets the requirements for teacher accreditation (AITSL, 2019). The standards are structured into three domains of teaching: Professional Knowledge, Professional Practice, and Professional Engagement.

In the first domain, *Professional Knowledge*, teachers tailor instruction for diverse student needs, considering linguistic, cultural, and religious backgrounds. They adeptly integrate literacy, numeracy, and technology for enhanced learning (AITSL, 2019). In the second domain, *Professional Practice*, teachers create engaging and safe learning environments, using effective teaching strategies. They regularly evaluate their practice, interpret assessment data, and proficiently operate across all stages of the teaching and learning cycle (AITSL, 2019). In the final domain, *Professional Engagement*, teachers model effective learning and engage in ongoing professional development, demonstrating respect and professionalism in interactions with students, colleagues, and parents/carers, valuing connections between school, home, and community for students’ development (AITSL, 2019).

The three domains comprise seven teaching standards which encompass 38 focus areas offering additional descriptors into each teaching standard and domain. These focus areas exhibit a progression aligned with the four career stages (graduate, proficient, highly accomplished and lead). The Australian Professional Standards for Teachers provide a foundational theoretical framework guiding teacher development.

Sub-Theme 2: Subject Specificity

Several studies highlight a notable omission in the feedback and evaluation forms provided to PSTs, specifically the neglect of subject or discipline-specific aspects of teaching. Schwartz et al., (2018) and Brett and Parks (2022) have drawn attention to this oversight. Despite the crucial role that subject or discipline-specific elements play in shaping teacher identity and defining the parameters of teaching practice, these dimensions are frequently disregarded in the feedback and evaluation processes directed towards PSTs. An Australian study that surveyed 147 primary PSTs during their mathematics subject specialisation placement found that mathematics specific feedback was provided through verbal feedback, teaching feedback, written feedback and reviews of lessons (Hudson, 2009).

From a mentor’s perspective, factors which contribute to supporting PST subject-specialisation growth include: sufficient content knowledge; knowledge of the syllabus;

implementing feedback in subsequent lessons; positive outlook and attitude (Hudson, 2009). Msimango et al., (2020) extend this notion to include reciprocity in the mentoring process, where mentors and PSTs develop a shared understanding of mathematics-specific mentoring to enhance pedagogical and content knowledge. Therefore, mentors must offer subject-specific feedback to aid PSTs' development in their discipline and meet specialisation requirements.

Sub-Theme 3: Feedback is Actionable

The benefits of effective feedback in education have been well established in the research literature (Martínez, 2016; Moussaid & Zerhouni, 2017). For example, in a study of PST's responses to feedback from mentor teachers, Moussaid and Zerhouni (2017) found that "trainees preferred to receive plenty of feedback that is both written and oral, more performance-based than content-based, specific and helpful, honest and constructive, and encouraging and motivating" (p. 146). Similarly, in a study about Spanish student teachers, Martínez (2016) found that the student teachers expected detailed and constructive feedback focused on identifying areas of improvement.

Therefore, as PSTs seek feedback that aids their areas of improvement, this aspect will be included in the sub-theme focusing on actionable feedback.

Sub-Theme 4: Feedback Enhanced Knowledge and Sub-Theme 5: Feedback Improved Skills

Crucial to PSTs development are feedback mechanisms that enhance proficiencies in curriculum, teaching practices, and student management. Studies by Spear et al. (1997) and Akcan and Tatar (2010) revealed varying focuses on mentor feedback, with the former emphasising student engagement and behaviour and the latter concentrating on classroom activities and subject knowledge. These differences reflect diverse teaching identities and values across cultures (Kastberg et al., 2020). Despite these contrasts, mentors universally address practical concerns such as classroom strategies. Thus, feedback aimed at enriching PSTs' knowledge and skills in the teaching profession is integral to their development.

In order to ascertain the type of written feedback mentors are currently providing to PSTs, the study was underpinned by the following research question: What type of mathematics specific written feedback is found in written reports to PSTs?

Research Design

The data presented in this paper originates from a broader cross-university Australian study that primarily centred on non-subject-specific written feedback. Specifically, this paper delves into data obtained to gain a deeper understanding of the written feedback in mathematics education extended to both primary and secondary PSTs.

Participants

The study encompassed documentation from 13 primary and three secondary undergraduate PSTs in their third year from an Australian university. These PSTs completed their penultimate professional experience placements in schools spanning various systems and sectors (Catholic, Department, and Independent) in New South Wales. This placement was 15 days duration, with PSTs having already completed two, 15-day placements previously. A final 35-day placement followed in their final year. Professional Experience C (PEXC) Reports for each PST were checked to ensure mathematics was nominated as their primary subject specialisation or secondary teaching area. PEXC reports are used as a formative and summative assessment of a PST's achievement during placement.

Data Collection Methods

To investigate the type of subject-specific feedback found in PEXC Reports, document analysis was undertaken which involved examining and interpreting data in documents to extract meaning, achieve understanding, and generate empirical knowledge (Corbin & Strauss, 2008; Rapley & Rees, 2018). To make sense of the data, the analysis will be presented quantitively (Morgan, 2022). Qualitative data extracts or summaries from the researchers will be used to display illustrative examples to support the statistical data (Braun & Clarke, 2013).

Within the PEXC Reports, there are three main ways in which a supervising teacher/mentor can provide written feedback. The first is the identification of the grading the PST is working at against selected AITSL standards at the mid-point (formative) and end-point (summative) of the placement. The second are the written comments by the mentor at mid-point and end-point of the placement (Table 1). The Focus for Further Development provides mentors with an opportunity to provide specific guidance for the PST to improve practice during the placement and in the future. Additionally, both the mentors and PSTs are expected to add an overall reflective comment at the end of the report (Table 2). A review of the PEXC Report template was also conducted to identify how often and when prompts were provided to respond with subject specialisation feedback.

Table 1

Extract From the Professional Experience Report Written Feedback

Supervising teacher's comments re assessment of DOMAIN 1: Professional knowledge	
Mid-Point	End-Point
Evidence of development and achievement:	Evidence of development and achievement:
Focus for further development:	Focus for further development:

Table 2

Extract From the Professional Experience Report—Overall Comments From Supervising Teacher and PST

School supervising teacher's overall comments	**Pre-service teacher's reflective comments**
School supervising teacher signature: Date:	Pre-service teacher signature: Date:

The theoretical framework described earlier in the paper was used to guide data collection. Two coders reviewed mentor written feedback (Tables 1 and 2) independently. Using an excel worksheet, extracts or summaries from the report were aligned against the framework. Table 3 provides an example of the way in which the data was collected.

Table 3

Example of Data Analysis and Coding for PST 1

PST 1—Primary	**Coder A (Coder B completes a similar form for each placement report)**
Use of AITSL language in written feedback	☑ Mathematics Specific Comments ☐ General Comments *Example: "[PST] will be delivering a mathematics learning sequence on multiplicative relations in her final week." Mentor states that assessment and content knowledge will be enhanced. "This will provide her with an opportunity to demonstrate standard 2.3"*

PST 1—Primary	**Coder A (Coder B completes a similar form for each placement report)**
Subject specificity	☑ Mathematics Specific Comments ☐ General Comments *Example: Mentor has provided growth feedback from mid to end point for maths. Feedback focuses on the maths content, resources developed and assessment implemented. The mentor writes that student results indicate success in the PSTs teaching. No future goals set*
Feedback is actionable	☑ Mathematics Specific Comments ☐ General Comments *Example: Feedback can be actioned within the given time of the placement*
Feedback enhanced knowledge	☑ Mathematics Specific Comments ☐ General Comments *Example: Specific components of the primary maths syllabus are mentioned i.e. fractions, chance and multiplicative relations*
Feedback improved skills	☑ Mathematics Specific Comments ☐ General Comments *Example: Increase in selecting appropriate maths strategies and providing feedback*

Table 3 displays Coder A's collection of data from the PEXC Report for PST 1. PST 1 is a primary PST with a specialisation in mathematics.

Results

The analysis of the PEXC Report identified only one instance of the term "specialisation". PSTs were required to state their primary specialisation on the cover page. The PEXC Report includes a comment about the PST meeting a minimum requirement of 6 hours to observe, assist, teach in their named specialisation across the 15 days. The analysis of the PEXC Report completion rate revealed insights into the specific areas where mentors offered feedback. Table 4 provides an overview of what was evidenced in each report. It is noted that report completions varied across the components of the written report. The end-point focus for further development section (see Table 1) was not always completed. Further, within this section, 63% of reports had a comment that provided feedback on the professional engagement domain. The overall comment section (see Table 1) was completed by all mentor teachers and PSTs.

Table 4

*Completed Components Within the Reports Analysed (*n *= 16)*

	Evidence of development and achievement		**Focus for further development**	
	Mid-point	**End-point**	**Mid-point**	**End-point**
Professional knowledge	100%	100%	100%	88%
Professional practice	94%	100%	100%	81%
Professional engagement	100%	100%	88%	63%

Results were further categorised into the three AITSL teaching domains: (1) *Professional Knowledge* (PK); (2) *Professional Practice* (PP); and (3) *Professional Engagement* (PE).

This aligns with the university's structured feedback template, organised by these domains. The analysis of each professional experience report was then conducted against each sub-theme (use of AITSL language in written feedback; subject specificity; feedback is actionable; feedback enhanced knowledge; feedback improved skills). The findings are presented in Table 5.

Table 5

Analysis of the Reports by AITSL Teaching Domains and Sub-Themes

Sub-theme		**Primary (*n=13*)**			**Secondary (*n=3*)**		
		PK	**PP**	**PE**	**PK**	**PP**	**PE**
Sub-theme 1: use of AITSL language in written feedback	Mathematics Specific Feedback	**1**	**1**				
	Mathematics Specific and General Feedback	**2**			**1**		
	General Feedback	10	11	12	2	3	3
	No Response or Feedback Aligned with Sub-theme		1	1			
Sub-theme 2: subject specificity	Mathematics Specific Feedback						
	Mathematics Specific and General Feedback	**6**	**3**		**1**		
	General Feedback	7	7	4	2	3	
	No Response or Feedback Aligned with Sub-theme		3	9			3
Sub-theme 3: feedback is actionable	Mathematics Specific Feedback	**1**					
	Mathematics Specific and General Feedback	**1**			**1**		
	General Feedback	11	13	9	2	3	
	No Response or Feedback Aligned with Sub-theme			4			3
Sub-theme 4: feedback enhanced knowledge	Mathematics Specific Feedback	**1**	**1**				
	Mathematics Specific and General Feedback	**1**					
	General Feedback	3	1		1		
	No Response or Feedback Aligned with Sub-theme	8	11	13	2	3	3
Sub-theme 5: feedback improved skills	Mathematics Specific Feedback	**1**	**1**				
	Mathematics Specific and General Feedback						
	General Feedback	11	11	8	3	3	3
	No Response or Feedback Aligned with Sub-theme	1	1	5			

Professional Knowledge

In this domain, only a single written report contained specific feedback related to mathematics. Feedback within this report addressed sub-themes 1, 3, 4 and 5. In the mid-point report, sub-theme 2, focusing on subject specificity, was discussed generally. However, in the end-point report, the discussion was specifically on mathematics, “designed and implemented learning sequences for Stage 3 ... implemented open-ended, low-floor, high-ceiling tasks to allow for quality differentiation” [PST Primary Report 1]. The mentor also noted the PST’s growth from mid to end point in mathematics. The feedback specifically addressed mathematics content, resource development, and assessment implementation. The mentor acknowledged the success in the PST’s teaching based on student results.

Most feedback was of a general nature, with some instances where the mentor did not provide a response to that particular sub-theme. To illustrate, one report captured feedback that appears to be across multiple subjects, urging the PST to “continue to develop rich extension opportunities” and to “continue to consider assessment opportunities” [PST Primary Report 6]. Similarly, the feedback provided in another report was broad, suggesting that the PST “should explore and implement teaching strategies, e.g., pair/share” [PST Secondary Report 14]. These examples highlight a tendency for general feedback and instances where mentors did not provide subject specific comments.

Professional Practice

Examination revealed only one primary PST received specific feedback related to mathematics. No written reports from the secondary PSTs made reference to mathematics. This explicit reference to mathematics is likely to have been omitted as a subject area of the placement is identified as mathematics on the coversheet of the report. A comment by a secondary mentor "demonstrated a strong understanding of the content and effectively conveyed this knowledge to the students. … Lesson plans were well structured using a variety of techniques" [Report Secondary 16] has the reader assume the context is mathematics and was categorised as General Feedback. One primary PST received mathematics specific feedback within this domain. The mentor highlighted the PST's creation of a mathematics assessment task. In the same report, the mentor mentions the content area of patterns and algebra in connection to the PST's professional practice [PST Primary Report 1].

Professional Engagement

This domain had no instances of mathematics (or a subject area) specific feedback. The comments were either general in nature or focussed on school based professional learning opportunities. For example, "[PST] is encouraged to engage in professional learning and use this time as an opportunity to learn from his colleagues" [PST Primary Report 2].

Discussion

Addressing the research question, "What type of mathematics-specific written feedback is found in reports to PSTs?" reveals a scarcity of such feedback, raising concerns about potential oversights in addressing specific needs. The researchers agree that this study has limitations, for example, that it relies solely on written reports, potentially overlooking nuanced interactions or feedback that may occur during verbal communication between mentors and PSTs and that the study is restricted to a specific university, which may limit the generalisability of the findings to broader ITE programs.

The results prompt crucial discussions on the effectiveness of general feedback in fostering PST growth in mathematics, underscoring the need for clarity in the PEXC Report and specificity in mentoring practices. The finding that there is only one instance of a prompt to write feedback related to subject specificity suggests that the mentor may not have thought to include such feedback and greater signposting of the requirement is needed in the documentation. Instances of alignment with AITSL standards suggest the potential influence of professional standards in guiding mentor feedback, akin to findings by Schwartz et al. (2018) and Brett and Parks (2022). These emphasise the importance of examining PST preparation programs for subject-specific feedback and raising questions about mentors' qualifications and experience, particularly when mentoring outside their subject specialisation. The results obtained in this study will contribute to the broader research project, specifically addressing templates for written feedback, mentor professional learning, and subject-specific feedback strategies to enhance PST growth.

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References

Akcan, S., & Tatar, S. (2010). An investigation of the nature of feedback given to pre-service English teachers during their practice teaching experience. *Teacher Development*, *14*(2), 153–172.

Asregid, D., Mihiretie, D. M., & Kassa, S. A. (2023). Teacher educators use of feedback to facilitate reflective practice among pre-service teachers during microteaching. *Cogent Education*, *10*(2), 2257121. https://doi.org/10.1080/2331186X.2023.2257121

Australia. Department of Education. (2022). *The national teacher workforce action plan* [Online]. https://www.education.gov.au/teaching-and-school-leadership/consultations/national-teacher-workforce-action-plan

Australian Institute for Teaching and School Leadership (AITSL). (2019). A*ccreditation standards and procedures*. AITSL. https://www.aitsl.edu.au/deliver-ite-programs/standards-and-procedures

Braun, V., & Clarke, V. (2013). *Successful qualitative research: A practice guide for beginners.* Sage Publications.

Brett, P. D., & Parks, M. (2022). Demonstrating 'impact': Insights from the work of preservice teachers completing a graduate teacher performance assessment. *Australian Journal of Teacher Education*, *47*(1). https://doi.org/10.14221/ajte.2022v47n1.4

Corbin, J., & Strauss, A. (2008). *Qualitative research. Techniques and procedures for developing grounded theory* (3rd ed.). Sage Publications.

Ellis, N. J., & Loughland, T. (2017). 'Where to next?' Examining feedback received by teacher education students. *Issues in Educational Research*, *27*(1), 51–63.

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal*, *45*(1), 184–205. https://doi.org/10.3102/0002831207312906

Hudson, P. (2009). Mentoring preservice teachers in primary mathematics. *International Journal of Learning*, *16*(4), 119–132.

Jensen, B., Sonnemann, J., & Roberts-Hull, K. A. (2016). *Beyond PD: Teacher professional learning in high performing systems—appendices*. National Center on Education and the Economy (NCEE) Books and Reports.

Kastberg, S. E., Lischka, A. E., & Hillman, S. L. (2020). Characterizing mathematics teacher educators' written feedback to prospective teachers. *Journal of Mathematics Teacher Education*, *23*(2), 131–152.

Martínez Agudo, J. D. D. (2016). What type of feedback do student teachers expect from their school mentors during practicum experience? The case of Spanish EFL student teachers. *Australian Journal of Teacher Education*, *41*(5), 36–51.

Morgan, H. (2022). Conducting a qualitative document analysis. *The Qualitative Report*, *27*(1), 64–77. https://doi.org/10.46743/2160-3715/2022.5044

Moussaid, R., & Zerhouni, B. (2017). Problems of pre-service teachers during the practicum: An analysis of written reflections and mentor feedback. *Arab World English Journal*, *8*(3), 135–153.

Msimango, M. N., Fonseca, K., & Petersen, N. F. (2020). Mentoring and preparing primary school mathematics teachers. *Perspectives in Education*, *38*(2), 272–284. https://doi.org/10.18820/2519593X/pie.v38.i2.18

Mullen, C. A., & Klimaitis, C. C. (2021). Defining mentoring: A literature review of issues, types, and applications. *Annals of the New York Academy of Sciences*, *1483*(1), 19–35.

Rapley, T., & Rees, G. (2018). *Collecting documents as data*. FLICK, Uwe. The Sage Handbook of Qualitative Collection. Sage Publication.

Schwartz, C., Walkowiak, T. A., Poling, L., Richardson, K., & Polly, D. (2018). The nature of feedback given to elementary student teachers from university supervisors after observations of mathematics lessons. *Mathematics Teacher Education and Development*, *20*(1), 62–85.

Spear, M., Lock, N. D., & McCulloch, M. (1997). The written feedback mentors give to student teachers. *Teacher Development*, *1*(2), 269–280.

Teacher Education Expert Panel. (2023). Strong beginnings: Report of the Teacher Education Expert Panel 2023. Australian Government. https://www.education.gov.au/quality-initial-teacher-education-review/resources/strong-beginnings-report-teacher-education-expert-panel

White, S. (2007). Investigating effective feedback practices for pre-service teacher education students on practicum. *Teaching Education*, *18*(4), 299–311.

Implementing Dialogic Pedagogies in Early Years Mathematics Teaching

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In this paper, we present the findings from the first year of our study into the use of talk with young children (ages 6 and 7) in their mathematics lessons following four teachers' implementation of dialogic strategies. Based on Alexander's (2017) dialogic principles and notions of productive talk, we delivered a professional implementation through a series of four spaced workshops. Teachers were interviewed to reflect on their implementation at the end of the year. These interviews are analysed according to Lefstein's (2010) theoretical model of dialogic pedagogies to identify the factors that enabled and constrained the teachers in shifting their pedagogies.

Dialogic pedagogies have the potential to enhance learning in educational contexts (Alexander, 2017) through a focus on the value of talk and sharing of learner ideas and opinions (Lefstein & Snell, 2014). As teachers and students engage in dialogue, content and new understandings build cumulatively and purposefully (Harper et al., 2018). In terms of teaching approaches, traditional classroom roles are revised to place less emphasis on teacher-directed instruction, with the provision of open-ended tasks (i.e., those with more than one correct answer or method of determining an answer), and a commitment to the co-construction of knowledge between students and teacher (Alexander, 2017). Alexander's (2003) indicators of dialogic teaching include the structuring of questions that are likely to provoke thoughtful answers that then provoke further questions. Questions and answers build blocks of dialogue, rather than reaching a 'terminal point' (p. 37) and teacher-student and student-student exchanges are channelled into coherent lines of enquiry, rather than left 'stranded and disconnected' (p. 37). This is not to say that dialogic pedagogies that have no place for teacher-directed instruction, or that tasks are always entirely open, just that there is a greater emphasis on opening up opportunities for purposeful dialogue.

This paper reports on a project which aimed to develop a dialogic approach that enhances mathematical talk by young students, ages 6 to 7. Through engaging in school-based participatory research (Hennessy, 2014), we worked with four early years primary teachers to design, trial, and reflect upon the implementation of dialogical pedagogies and the development of appropriate, engaging tasks for student learning in groups. The aim was to develop teacher practice with a focus on the use of language (both teacher and student) and reflection on their professional development. Specifically, our research question for this paper was:

- What factors enabled and constrained early years' teachers' uptake of dialogic pedagogies in mathematics teaching?

Literature Review

Research into dialogic pedagogy has focused on the value of talk between students in early learning (e.g., Feez, 2023), primary (Edwards-Groves & Davidson, 2023), secondary (Myhill & Newman, 2023), and tertiary contexts (Simpson & Wang, 2023). Several researchers have sought to understand the value of dialogic pedagogies. A study by Howe et al. (2019), for example, which focused on teaching and learning in mathematics, literacy, and science, found

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 391–398). Gold Coast: MERGA.

that "classroom dialogue matters as regards student outcomes" (p. 462). More specifically in mathematics, Murphy (2013) explained how talk was crucial for speakers and listeners to orient each other and share their intentions in mathematics contexts. Despite its importance, a recent study by Moser et al. (2022) found that there is often a scarcity of dialogic classroom discussions in mathematics. Compounding the problem, Mercer and Sams (2006) found that the lack of language and reasoning skills is a major cause of underperformance in school education and also of behavioural problems. Clearly, it is important to gain a stronger understanding of the kinds and features of talk that are conducive to learning in mathematics.

While the research literature provides examples of teachers successfully implementing dialogic approaches in educational contexts (e.g., Clarke et al., 2016; Jay et al., 2017), and the use of prompts in supporting exploratory talk in collaborative group work in mathematics (Mercer & Sams, 2006), virtually no research has focused on the initial uptake of dialogic approaches by teachers who are used to traditional pedagogies. This paper addresses this gap and reports on the findings of teacher reflective conversations and interviews analysed through the lens of Lefstein's (2010) model of dialogic teaching. The findings are used to identify the factors that enable or constrain student learning through dialogue.

Conceptual Framework

Dialogic pedagogies move beyond monologic and teacher-centred dominance of Initial-Response-Evaluation (IRE) to the development of patterns of classroom interaction that open up the talk and, subsequently, the thinking of the learners (Alexander, 2003). Lefstein (2010) expanded Alexander's characterisation of dialogic content as purposeful and cumulative to include the criteria of critical and meaningful and so developed a model that "pedagogised" (p. 11) dialogue appropriate to contemporary school contexts. The model outlines four dimensions that can be considered when negotiating the various role demands of teaching through dialogue.

The first dimension is the meta-communicative, which involves teachers and learners giving attention to the role and features of the dialogue itself. Teachers can foster the meta-communicative dimension by establishing and maintaining communicative norms and encouraging and facilitating reflexivity (Lefstein, 2010). Second is the ideational dimension, which is concerned with the exchange of ideas, the building of knowledge and the pursuit of truth. Teachers foster the ideational when they open up the curriculum, relate students' contributions to each other, and probe and challenge students' ideas. The third is the interpersonal dimension, which focuses on the relationships between those engaging in dialogue. Teachers foster the interpersonal through building a classroom community and encouraging broad participation. The fourth is the aesthetic dimension, taken here to mean the enjoyment in and reasons for dialogue. While it can be related to all four dimensions, we see the design of mathematics tasks to be largely about the aesthetic, ensuring that tasks are appropriately challenging and engaging. For Lefstein, teachers promote the aesthetic when they "set the stage for engaging discussion" and "orchestrate pupil participation" (p. 16). Together, the four dimensions provide researchers with a useful tool for unpacking the complexities of dialogic teaching in educational contexts. We used this model to show what enabled and constrained learning for our focus teachers as they implemented dialogic pedagogies into their mathematics lessons.

Our project design was also informed by Clarke and Hollingsworth's Model of Professional Growth (2002). Through the workshops we were able to serve as an "external source of information or stimulus" and spaced workshops provided for teacher engagement in "professional experimentation" (Clarke & Hollingsworth, 2002, p. 951). We were seeking "salient outcomes" through uptake of dialogic teaching pedagogies and increased student participation in mathematical exploratory talk, and evidence of shifts in teachers' knowledge, beliefs and attitudes.

Methodology

The wider project involved a school-based participatory design (Hennessy, 2014), which began with the collection of baseline data through three lesson observations in each of the four classrooms, interviews with teachers about their approaches to teaching mathematics, and focus group and individual interviews with children about their mathematics learning and understandings. The first year of the project, which is the focus of this paper, involved four teacher researcher workshops, four lesson and small group task observations, and a reflective interview with each of the teachers about the implementation and evolution of the design. Interview questions focused on teachers' facilitation of mathematical talk, such as 'How are you facilitating student talk in your maths lessons?' and 'Can you describe instances where you have observed students engaging in productive mathematical talk?' The data collected from workshop reflections and teacher interviews are reported on in this paper. The participants were one Grade 1 and three Grade 1/2 teachers and their students.

In the first workshop, teachers were introduced to principles underpinning dialogic pedagogy that covered the four Lefstein dimensions. A key strategy based on the use of ground rules for productive talk (Mercer & Sams, 2006) was introduced, including prompts such as 'I agree/disagree because…' These rules were intended to promote metacommunicative and ideational dimensions. Subsequent workshops explored different ways of arranging students in small groups; how to promote safe, respectful learning environments (interpersonal dimension); and how to design small group tasks with appropriate timing, explicitness, and challenge (aesthetic dimension, Murphy, 2011).

Following a design cycle in line with the participatory nature of the project, the format of each workshop allowed for teachers to interpret and trial strategies that they could take into their classrooms ready for the video lesson observation visit. Each workshop started with an evaluation of the implementation of their strategies based on the video recording, followed by a discussion of strategies to further promote productive talk to take into their class for the next video lesson observation. Teachers were supported in resourcing and trialling challenging tasks that matched their planned teaching content. In this way, the workshops afforded the teachers opportunities to reflect on experiences in their classrooms and for the research team to listen, respond, and suggest ways forward based on previous research into dialogic practices. A key aspect of this approach was that different practices emerged for each teacher based on the four dimensions.

Results and Discussion

Following a summary of teachers' illustrative mathematics lesson structure and task examples, this section reports the findings of teacher reflections collected during the teacher workshops and reflective interviews. The findings are referenced to Lefstein's (2010) four dimensions of dialogic teaching.

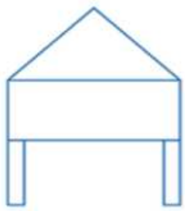
The four teacher participants engaged in the first year generally followed the gradual release of responsibility (GRR) model (Pearson & Gallagher, 1983) for their mathematics instruction. Henceforth, the GRR model seemed to leave a residue across much of the dialogue between teachers and researchers and in the teachers' decision making while implementing the design. This lesson structure involved three main stages: first, a setting up stage, in which the teacher would introduce the mathematics focus, introduce and talk children through a task or problem, and remind the children about the ground rules for productive talk about mathematics; second, an exploration stage, in which the students would split into small groups and talk through a mathematics task/problem, and the teacher would cycle around the groups offering assistance as required and keeping children on task; and third, a conclusion stage, in which the whole class would be brought back together to report their findings and for the teacher to facilitate a final discussion about the main learning outcomes for the experience. Importantly, this dialogic approach was not the only way students were taught mathematics; they still engaged in their usual GRR-based mathematics lessons, but the teachers committed to including at least one dialogic lesson per week and to increase the opportunities for students to talk and share ideas and opinions about mathematics in all lessons.

Based on Murphy's (2011) review of mathematical tasks, the research team supported the teachers to develop and implement small group tasks that were relatively short and with an explicit outcome that also included some ambiguity. The teachers avoided tasks where the students needed to take a long time discussing how to complete the task (e.g., what to cut out, where to stick it, etc.), as well as tasks that involved writing. The teachers were encouraged to use images (e.g., odd one out; sorting/pairing; true or false), and to create tasks where students would direct each other to the mathematics focus. Finally, teachers were encouraged to repeat given tasks several times, but with variations each time, such as using different numbers, shapes, and so on. Figure 1 provides an example of a task that was used.

Figure 1

Cubby House Task (Sullivan et al., 2023, p. 15)

Thirteen friends are playing in a cubby house.
Some of the friends play inside the cubby house, some play under the cubby house and some play outside the cubby house.
Draw a picture to show how many friends might be inside, how many might be outside and how many might be under the cubby house.



Metacommunicative Dimension

All four teachers talked about establishing and maintaining communicative norms and explicitly talked about rules for talking and listening. For example, James explained:

> I started to introduce some of the talking rules and some of the things like that. So then, when we were doing more stuff, where they're challenging and disagreeing with each other, they already had the kind of rules of communication in place. And I think that worked relatively well. I think it worked well.

In addition, James explained that his rules were loosely based off "established listening rules like the five L's anyway, which I assume are pretty universal. So that's eyes looking, ears listening, legs crossed, lips closed, hands on lap."

Similarly, Jill noted:

> I do feel like a lot of the work we've done has helped them to do that … and to cement their more foundational level of skills. It's given them more confidence to talk about the things that they've got going on.

The 'work' referred to by Jill includes the establishing and maintaining of communicative norms throughout the year's mathematics lessons, such as practicing established listening rules. In the reflective workshop sessions, we talked about moving beyond generic rules for listening and talking, and for looking for evidence that constructive listening was occurring. The teachers were reminded of the aim to engage students in productive talk, which required inviting criticism of participation in the direction of dialogue (Lefstein, 2010). This emphasis on the metacommunicative dimension was key to developing student confidence to share mathematics-related ideas and opinions during group and whole class learning, since the students had a shared understanding of what productive talk in mathematics lessons involved.

Ideational Dimension

A feature of the workshops was the opportunity for teachers to collaboratively view classroom footage of student talk collected throughout the project. These opportunities highlighted the importance of the teacher's role in maintaining conversational cohesion, which Lefstein (2010) described as an aspect of the ideational dimension. When teachers connect the contributions made by different students, this helps to draw together and summarise conversational threads, helping them see a developing logic in the talk. This appears to be particularly important when engaging in dialogic pedagogy with younger learners, since they often require more support in this area. In early primary contexts, dialogic pedagogy cannot be 'set and forget'; the teacher needs to scaffold conversations for students, threading their comments together when they do not have the language or thinking skills to do so themselves. If students are not drawn into a task quite quickly, they can lose interest and miss the learning experience. It appears that this dimension may have been the most challenging for teachers, and hence required more support to implement as the following quote illustrates:

> I think even just to debrief with you guys straight after and go, so this happened and this happened … . Is that normal? Like this is how the activity went. Should I try it again, with different numbers? Or should I just try and move on to something else? Yes, I think that's partially because I'm not as experienced as a teacher, but I think also just because what we're doing is a little bit different to what we've been trained to do and what I've done in the past. (James)

All four teachers actively encouraged students to use the terms 'I agree because' and 'I disagree because' as a means to foster respectful challenging of ideas. This strategy required explicit modelling and perseverance on behalf of the teachers before it became part of the talk routine. Anne said, "[We've been using] 'I agree', and 'I disagree, because' we need to use those, but they, you know, took a while to get to that language, [but now] they're actually using that language."

This way of teaching was new for these teachers and it was understandable that teachers lacked confidence with the implementation of dialogic pedagogies and the need for additional support will be noted in future iterations of the project.

The ideational dimension can also refer to opening the curriculum, which the teachers enacted through the provision of open-ended tasks and problems, such as the cubby house problem depicted in Figure 1:

> I introduced a few of those. And I think they changed some of the questions that I was asking, to more open-ended questions. And introducing some of those problems. As I said, with the cubby house problem, for example, having things like that a much more regular occurrence in my teaching planning. And yeah, the best listener was something I implemented a lot. (James)

Interpersonal Dimension

This dimension appeared to be particularly relevant, with all four teachers highlighting the importance of building a classroom environment which provided a safe space for talk, with one teacher, Jill, being particularly cognizant of modelling caring behaviours. She played lots of circle games with her students and used a puppet to model respectful listening:

> I definitely think those games were good for tuning everyone's thoughts around to you know, what, what you need to be when you're listening. And even just your volume when you're talking and taking turns to talk. I also introduced my little puppet, Bun-bun, and we had to teach Bun-bun how to listen. So it was looking at the person who was speaking, not interrupting and hands up and with Bun-bun, it was more about where his eyes were, where his body was; if he was paying attention, we could tell because he was looking at the person. So that was actually fun. They really loved that, and it was also us teaching him how to talk, which was good.

Teacher interview feedback received after the final workshop, indicated that the introduction of dialogic pedagogies had a positive impact on students' ability and willingness to talk:

> [The students] definitely did show a better ability to talk to everyone and to convey their own opinions. (Anne)

> It does feel like they have a better understanding of how to explain their thoughts … that could be because I now know them better, and it could be because, you know, I've seen them in different areas. (Sharon)

This feedback is related to the interpersonal dimension of dialogic pedagogy in that the teacher's efforts to build a classroom community and encourage student participation in many mathematical discussions with their peers developed their confidence to share their thinking about mathematics when working through problems.

Teachers also encouraged broad participation and structured discussions that maximised the chances that students would have something significant to say. The following quote demonstrates how the teacher orchestrated opportunities for all students to actively participate:

> I explicitly came back to how they needed to work together. There tended to be that dominant person still just doing the job and the others sort of waiting around, especially on the first attempt with things. So maybe on that first attempt, those people who weren't so sure, were watching and observing how to do it. And then when they had a second chance, and maybe a different group, it's like a fresh start, but they came with some knowledge, and they would then participate. I did find it wasn't always collaborative, especially as it was new learning. (Jill)

James was also aware of broadening participation and ensuring fairness, particularly in terms of roles:

> And then I'll have a listener for that. And then the listener will report back to the class on how they think their group was listening. And then I'll take their notes, and then I pick a different listener. And I did that so that everyone gets a chance to be a listener. But like, if I have one person to listen to for too long, other kids get a bit upset, because they wanted a turn to be the listener. That worked really well because there are times when I've been able to compare the listeners with each other like, when this student was a listener, he was doing this, so maybe think about how you can see what he did and copy that.

Aesthetic

As Lefstein (2010) described, teachers can attend to the aesthetic dimension through setting the stage for engaging discussions and orchestrating student participation by injecting good humour or a sense of drama into task design. The teachers recognised early on that they needed to model and facilitate respectful listening and talking generally in whole class and small group situations before they could focus on mathematical talk and ultimately exploratory talk, "I guess we spent two or three weeks just doing talking and listening sort of games and activities. So it wasn't really in maths it was just talking about, [just] in general yes."

Jill's use of a puppet as mentioned earlier in relation to the interpersonal dimension also provides an example of attending to aesthetic considerations. Lefstein (2010) acknowledges that the dimensions can overlap and that there can be tensions between them. For example, protecting students' social needs may mean not probing their thinking in public. There is also arguably a hierarchy of priorities as different dimensions may take priorities over others.

Setting up ground rules for talk and encouraging small group participation was not always enough to enable effective learning of mathematical concepts for these young students. Another teacher commented:

> I guess [it is a challenge] when you were applying it to maths and when the lesson's going on for a little while, especially if they're not engaged, say it's a little bit dry, or the content is hard. So if you're covering a new maths focus or something like that, I find that they're not as engaged as once they have some knowledge around it, and then they're happy to participate and talk and, you know, do that talking and listening stuff. Whereas otherwise, they seem to just tune out, which, you know, so they're not listening at all or talking. So, yes, I found there were just some difficult times with that. So, I often found the first time we did the activity, this is groups, there wasn't much really

> going on, especially when it was a bit challenging. But then if we repeated the activity, they were really good. (James)

Here, the teacher is aware of what can go wrong with dialogic pedagogy when the aesthetic dimension is not given enough attention. The young learners in these classrooms did not seem to cope well when the tasks were too open or too long, and care needed to be taken in scaffolding new mathematical concepts in a way that students could engage.

Conclusions

The research findings so far highlight that the teachers reported taking a dialogic approach to teaching mathematics had a positive impact on their students' ability to listen and participate in talk in mathematics lessons. In particular, teachers noted improvements in students' confidence to share their thinking during small group work, which other research has pointed to as a key enabling factor in older students' mathematical and science learning (e.g., Mercer et al., 2004). Giving attention to the metacommunicative and interpersonal dimensions was crucial here, as the teachers took time to establish and maintain ground rules for productive talk and to build positive classroom communities that valued the sharing of student ideas when solving mathematical problems.

At the same time, challenges were also apparent, especially in maintaining students' engagement in learning during difficult or unfamiliar mathematics tasks. Paying more attention to the aesthetic dimension was identified as a key factor in addressing this challenge, through meaningful task design personalised for the children in terms of the level of challenge. Additionally, aspects of the ideational dimension such as maintaining conversational cohesion would support the learners to grasp the developing logic in group discussions. Looking forward, Lefstein's (2010) framework has proven helpful in suggesting key areas that require further designing, trialling, and reflecting, as the teachers grapple with key questions such as how to shift from, what is essentially, co-operative listening and talking in completing a task together, to productive talk where the dialogue opens up and orients thinking.

Clarke and Hollingsworth's (2002) model proved useful in understanding how teachers' professional growth could be facilitated especially through attending to the first two elements of provision of an external stimulus and professional experimentation. Workshop design ensured that the teachers were provided with research-informed practices and resources to assist with the implementation of dialogic pedagogies. Spaced workshops and classroom visits allowed for trialling, professional experimentation, and reflection. Future directions for this project will include broadening participation, use of control groups to evaluate impact on students' learning and refinement of the project design to better support teachers with implementation.

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References

Alexander, R. J. (2003). *Talk for learning: The first year. First formative evaluation of North Yorkshire's talk for learning project, part of the education development plan for 2002–7*. North Yorkshire County Council. https://robinalexander.org.uk/wp-content/uploads/2019/12/North-Yorks-report-03.pdf

Alexander, R. J. (2017). *Towards dialogic teaching: Rethinking classroom talk* (5th ed.). York: Dialogos.

Clarke, B., Nelson, N., & Shanley, L. (2016). Mathematics fluency—More than the weekly timed test. In K. Cummings & Y. Petscher (Eds.), *The fluency construct* (pp. 67–89). Springer.

Clarke, D. & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, *18*(8), 947–967.

Edwards-Groves, C., & Davidson, C. (2023). Dialogicality as embodied multimodal communication in primary schools. In A. Thwaite, A. Simpson, & P. Jones (Eds.), *Dialogic pedagogy: Discourse in contexts from pre-school to university* (pp. 91–108). Routledge. https://www.doi.org/10.4324/9781003296744-9

Feez, S. (2023). Perceiving, labelling and knowing: Mediating educational meanings through multimodal dialogue in Montessori early years classrooms. In A. Thwaite, A. Simpson & P. Jones (Eds.), *Dialogic pedagogy: Discourse in contexts from pre-school to university* (pp. 35–51). Routledge. https://www.doi.org/10.4324/9781003296744-5

Harper, H., Lotherington, M., & Parkin, B. (2018). Carrying the conversation in our heads: Dialogue in a remote Aboriginal setting. In P. Jones, A. Simpson & A. Thwaite (Eds.), T*alking the talk: Snapshots from Australian classrooms* (pp. 75–87). Primary English Teaching Association Australia.

Hennessy, S. (2014). *Bridging between research and practice: Supporting professional development through collaborative studies of classroom teaching with technology*. Brill.

Howe, C., Hennessy, S., Mercer, N., Vrikki, M., & Wheatley, L. (2019). Teacher-student dialogue during classroom teaching: Does it really impact on student outcomes? *Journal of the Learning Sciences, 28*(4–5), 462–512. https://doi.org/10.1080/10508406.2019.1573730

Jay, T., Taylor, R., Moore, N., Burnett, C., Merchant, G., Thomas, P., Willis, B., & Stevens, A. (2017). *Dialogic teaching: Evaluation report and executive summary*. Education Endowment Foundation with Sheffield Hallam University. https://educationendowmentfoundation.org.uk/public/files/Projects/Evaluation_Reports/Dialogic_Teaching_Evaluation_Report.pdf

Lefstein, A. (2010). More helpful as problem than solution: Some implications of situating dialogue in classrooms. In K. Littleton & C. Howe (Eds.), *Educational dialogues: Understanding and promoting productive interaction* (pp. 170–191). London: Routledge.

Lefstein, A., & Snell, J. (2014). *Better than best practice: Developing teaching and learning through dialogue*. Routledge.

Littleton, K., & Mercer, N. (2013). *Interthinking: Putting talk to work*. Routledge.

Mercer, N. (2000). *Words and minds: How we use language to think together*. Routledge.

Mercer, N., & Sams, C. (2006). Teaching children how to use language to solve maths problems. *Language and Education*, *20*(6), 507–528.

Mercer, N., Dawes, L., Wegerif, R., & Sams, C. (2004). Reasoning as a scientist: Ways of helping children use language to learn science. *British Educational Research Journal*, *30*(3), 359–377. https://www.doi.org/ 10.1080/01411920410001689689

Moser, M., Zimmermann, M., Pauli, C., Reusser, K., & Wischgoll, A. (2022). Student's vocal participation trajectories in whole-class discussions during teacher professional development. *Learning, Culture and Social Interaction, 34,* Article 100633. https://doi.org/10.1016/j.lcsi.2022.100633

Murphy, C. (2011). Reviewing the effectiveness of mathematical tasks in encouraging collaborative talk with young children. In J. Clarke, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices. Proceedings of the 34th annual conference of the Mathematics Education Research Group of Australasia* (pp. 557–564. Alice Springs, NT: MERGA.

Murphy, C. (2013). Young children talking in mathematics: What is the point of 'That'? In V. Steinle, L. Ball & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow. Proceedings of the 36th annual conference of Mathematics Education Research Group of Australasia)* (pp. 538–545. Melbourne, VIC: MERGA.

Myhill, D., & Newman, R. (2023). Writing talk: Investigating metalinguistic dialogue about written texts. In A. Thwaite, A. Simpson & P. Jones (Eds.), *Dialogic pedagogy: Discourse in contexts from pre-school to university* (pp. 141–154). Routledge. https://www.doi.org/10.4324/9781003296744-13

Pearson, P. D., & Gallagher, M. C. (1983). The instruction of reading comprehension. *Contemporary Educational Psychology*, *8*, 317–344.

Simpson, A., & Wang, T. (2023). Dialogic pedagogy: Blended learning in initial teacher education. In A. Thwaite, A. Simpson & P. Jones (Eds.), *Dialogic pedagogy: Discourse in contexts from pre-school to university* (pp. 189–204). Routledge.

Sullivan, P., Bobis, J., Downton, A., Livy, S., McCormick, M., Russo, J. (2023). *Maths sequences for the early years, Challenging children to reason logically.* Oxford University Press.

Mathematics Teachers' Beliefs and Pedagogical Approaches Regarding Creativity Within a Novel STEM Creativity Framework

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Creativity is a task-specific construct that exhibits diverse characteristics depending on the context it operates in. Currently, there is little consensus on how creativity can be defined and taught within STEM education. In this study, mathematics teachers' beliefs about STEM creativity and their pedagogical approaches to teaching creativity within a novel STEM creativity framework have been explored. Results show that mathematics teachers have a variety of beliefs and practices about creativity that can be grouped into four pedagogical themes: exposure, exploration, experimentation, and execution.

Science, Technology, Engineering, and Mathematics (STEM) education has become widespread since the beginning of the 21st century. In STEM, creativity is identified as a key skill by various educational bodies including the Organisation for Economic Co-operation and Development (OECD), and the *Australian Curriculum: Mathematics (v9.0)* (Australian Curriculum Assessment and Reporting Authority [ACARA], 2022). http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-102022) addresses critical and creative thinking as a 'general capability' that should be developed across all learning areas. Despite the popularity of STEM programs, there is a scarcity of studies that examine the teaching of creativity in STEM learning. Additionally, the lack of consensus on what STEM education is and teachers' understandings of STEM and creativity within their teaching areas present further possibilities, creating the necessity of examining the characteristics of creativity in STEM. The aim of the study reported here was to explore mathematics teachers' beliefs and pedagogical approaches regarding creativity within the parameters of a novel STEM creativity framework developed specifically to improve creativity skills in STEM education.

Literature Review

STEM education is the integrated education of science, technology, engineering, and mathematics, in order to create a multidisciplinary curriculum with various objectives. There are multiple ways to define STEM education, and little consensus exists on what STEM education is and how it should be implemented (Falloon et al., 2020). STEM education can be considered as a loosely defined domain that can include studies of individual areas, a clearly defined domain that only includes cross-disciplinary study, or even only educational projects that include all four areas that make up STEM. Similar to STEM education, creativity is also a phenomenon that has several definitions with emphasis on its various aspects. Zabelina (2018) defined creativity as an ability of "producing work that is both novel and meaningful or useful" without specifying an end product. Smith and Henriksen (2016) see creativity as "developing ideas and/or objects that are novel (original) or interesting, effective (or useful), and have a certain aesthetic sensibility as a whole". A common theme among the various definitions is that creativity is regarded as a set of skills and processes, which may include developing ideas and systems that possess certain qualities that do not necessarily translate into a tangible product.

In the Australian Curriculum (ACARA, 2022), creativity is linked with critical thinking and seen as part of the seven 'General Capabilities' needed across all learning areas. It positions creativity in a matrix that contains individual learning areas and cross-curriculum priorities that are meant to be integrated in each learning area, which includes STEM learning areas. A

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learning continuum for critical and creative thinking is in place, indicating what critical and creative thinking may look like on different levels of development from Learning Continuum Level 1 (foundation) to Level 6 (grade 10). Critical and Creative Thinking is organised into four elements; inquiring, generating, analysing, and reflecting, each of which is further divided into sub-elements that specify the skills students develop within the individual elements. The implementation of the Critical and Creative Thinking general capability in the classroom is not consistent (Skourdoumbis, 2016).

In school classrooms, opportunities for creative expression are always present, which allows for the possibility of multiple definitions of creativity existing side by side. However, it is also necessary for creativity to have domain-specific definitions. Creativity in an artistic domain is different from creativity in a scientific domain, as well as the exact creative skills that constitute what makes a musician or scientist (Sternberg, 2020), a phenomenon that is also observed in STEM. Creativity is crucial for innovation in STEM; however, despite its significance, there is no definition of creativity specific to STEM education (Stretch & Roehrig, 2021), and education policies in several countries do not feature aspects of creativity in STEM learning (Stylianidou et al., 2018). Stretch & Roehrig (2021) argued that even though young people possess mathematics and science skills, their creativity skills remain lacking. This presents a need to define what creativity looks like in STEM education and how it can be taught.

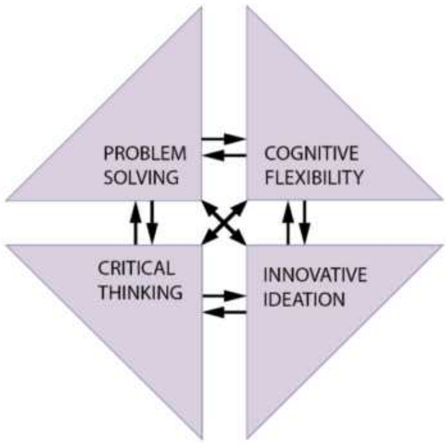
This paper reports part of a doctorate study on teachers' beliefs and pedagogical approaches regarding creativity in STEM education. The researcher draws on the literature to draft a STEM creativity framework. Acknowledging that some definitions of creativity may look different within the parameters of STEM education, this paper reports preliminary findings on the framework in understanding mathematics teachers' beliefs and pedagogical approaches regarding creativity.

A Novel Framework for STEM Creativity

In this study, a STEM creativity framework containing four specific elements is proposed, which, as a cluster, provides a working definition of STEM creativity: problem solving, critical thinking, innovative ideation, and cognitive flexibility (Figure 1); each element is drawn from aspects of creativity identified in the literature. The four elements are designed to be in interaction with each other in a STEM learning program and not to be implemented individually in a siloed style; all elements need to be present in a program in order for the framework to be implemented effectively. The framework was designed in light of existing creativity frameworks, current literature, the critical and creative general capability of the Australian curriculum, and the experiential learning theory.

Figure 1

STEM Creativity Framework



Problem solving is considered an essential element of STEM creativity; the positive impact of problem solving in STEM disciplines on student creativity and development of enquiry skills is acknowledged on policies and a survey of STEM instructors' perspectives across numerous European nations (Stylianidou et al., 2018. The contribution of problem solving to creativity in STEM education can also be linked to the requirement of integrating information from multiple sources during the problem-solving process (Wu & Koutstaal, 2020).

Various definitions have been proposed for critical thinking. Scriven and Paul (1987) defined critical thinking as a process of conceptualisation, application, analysis, synthesis, and evaluation of information. Critical thinking is a core educational goal whose importance has been widely acknowledged, specifically in STEM learning environments. The importance of critical thinking in creativity, have been examined by several scholars to date; some researchers recommended the integration of critical thinking and creativity, while others considered them as distinct processes (Wechsler et al., 2018).

Ideation is a key component of creativity and generating ideas is an indispensable process in STEM contexts (Hite et al., 2016) and hence, a significant aspect of STEM education is equipping students to be able to respond to problems. Ideation processes entail various aspects including the number of ideas that are generated, their variety and level of detail, and the originality of the ideas, that is, the level of innovation (Nijstad et al., 2010), all of which could have implications in the development of learning materials that target ideation skills. In this draft framework ideation is required to be innovative; in the process of ideation, many people refer to existing knowledge and concepts, and tend to favour generating accessible ideas; hence, it is important to distinguish between innovative and easily accessible ideation.

Cognitive flexibility is defined by Spiro (1988) as the ability of a person to restructure and adapt their knowledge in various ways in response to changing requirements or situations, which is a crucial ability specifically in STEM learning environments. Similarly, according to Nijstad et al. (2010), cognitive flexibility is "the ease with which people can switch to a different approach or consider a different perspective" (p. 42). Multiple studies found a positive relationship between creativity and cognitive flexibility. Therefore, it can be suggested that cognitive flexibility is a fundamental aspect of creativity and that it is present in the process of generating creative alternatives to problems.

Method

Potential participants who had experience teaching mathematics and at least one additional STEM discipline were invited to participate in a semi-structured interview. Participants, selected via snowball sampling, consisted of ten teachers; the majority of whom completed a qualification beyond initial teacher training. The ages of the participants ranged from 31 to 71. The participants' teaching experience ranged from 4 to 30+ years; the majority were experienced educators with over 20 years of teaching practice, and all participants taught student cohorts of every secondary grade. In line with the recruitment criterion, all participants had experience teaching one or more STEM disciplines in addition to mathematics: four participants taught mathematics and science, two mathematics, technology, and engineering, and a further two taught mathematics, science, and technology. Two participants taught all STEM disciplines.

Interviews lasted approximately 30–45 minutes; the interview questions ranged from introductory questions that seek to elicit information regarding teachers' attitudes and pedagogical approaches to creativity in STEM education to questions that aim to understand teachers' opinions on how to deliver the four STEM creativity skills. Interviews were transcribed using an online transcription tool and the transcripts were available to participants to review or edit their responses until the time interview data analysis commences. NVivo version 10 was used to code the qualitative interview data thematically. Data was coded according to categories. When data was recorded into NVivo, all interview participants were assigned a generic identifier and number.

Results

An evident difference between the participants' understanding of creativity is the distinction between participants with technology and/or engineering teaching experience (coded TE) and participants with science teaching experience (coded SI). Teachers with engineering and technology teaching experience had a more concrete and results-based approach to teaching creativity, whereas teachers with science experience did not consider a tangible end result or product as a requirement of creativity. This approach could be linked to the nature of the individual disciplines as well as the educational backgrounds of the respondents. The views of the two participants who had experience teaching all four STEM learning areas also aligned primarily with the participants with technology and/or engineering teaching experience. This distinction was evident in all responses to all questions.

For all TE respondents, a practical aspect is an essential element of what creativity is, and the creative process is meant to culminate in an end result in the form of a physical object or another product. This suggests that creativity on a purely theoretical level is not of the same level of sophistication, and a creative act can only be considered creative if it serves a purpose. Furthermore, according to TE teachers, creativity appears to be a tool to arrive at a better solution or response. Overall, TE teachers attributed many factors to what constitutes creativity that largely aligned with each other, and all of them stated specific criteria that required for a task or tasks to be considered creative. This contrasts with the views of SI teachers, who had a more open and fluid understanding of what creativity means and did not necessarily link it to the production of a visible product or end result. All SI respondents viewed the simple act of thinking in a different way or adapting a different approach when looking at things as creative acts. SI teachers viewed creativity as a general skill beneficial for the holistic development of a student, or a certain approach to academic and non-academic aspects of life, and indicated that creativity is personal; even if a creative act is insignificant in comparison to many other creative acts, it is still creativity for the specific student who undertook the creative act. Nevertheless, all SI teachers stated that while creativity in the mathematics and science domains do not require a product, there should still be a benefit for a student to undertake an act of creativity.

All respondents mentioned the importance of possessing base knowledge to practise creativity, and specified that in order to practise creativity, students need to acquire a knowledge and skill base, highlighting the task-specific nature of creativity. The process of teaching creativity commences with the introduction of new knowledge, skills, and perspectives to students. Base knowledge can be acquired through various methods such as direct teacher instruction and exposing students to examples of relevant projects. Respondents also identified potential barriers to students acquiring base knowledge and skills, including lack of teacher knowledge and technological literacy, availability of equipment in schools, and curricular limitations. Another significant factor that was identified in the interviewees' responses was creating an environment for creativity to be enabled and supported. This factor was elaborated by every respondent in largely aligning but individual ways. Pedagogical methods to achieve the right environment for creativity practice included collaborative work, peer feedback, and discussions in whole-class and small group settings, as well as developing open-ended learning experiences and assessment items that would allow learners to express and develop their creativity. Additionally, all respondents stated that creating a 'safe space' is essential via encouraging students to share their thoughts and perspectives, and ensuring every learner's voice is heard.

Pedagogical practices on how to teach creativity did not differ significantly across the two groups. All respondents stated that creativity can be taught, even though the techniques they used to teach creativity differed according to their teaching areas. One of the practices mentioned by all respondents was an experimental approach to learning, and the importance of students coming up with multiple options, approaches, looking from different perspectives, and trying out new things as factors in expressing and developing creativity. Approaches of trial and error, learning through making mistakes to develop new possibilities and ideas, and intellectual risk-taking are some of the factors in developing creativity in students. Experimental learning should also be supported through open-ended tasks and assessment items that give learners enough scope to try different possibilities, as mentioned by several respondents in both teaching domains. However, respondents also noted that scaffolding in both learning and assessment is also necessary to maintain the exploration in the right direction and ensure learning goals are met.

Respondents from both the TE and SI groups stated that contextualisation and application of learning is an essential pedagogical approach to teach creativity in STEM learning areas. This took multiple forms; however, in all cases, entailed some manner of connection between the newly acquired knowledge and skills and an aim that is to be responded to; an application area for the contextualisation of the new knowledge and skills. The implementation of this approach depended on individual learning areas, ranging from practical, hands-on tasks and assessment items in technology and engineering, student-created and guided experiments in science, and solving hypothetical problems in mathematics. All SI respondents also stated that creating scope for students to express themselves is important in the application of learning.

Each participant had unique responses regarding the four elements of STEM creativity, which suggests that despite their experiences throughout STEM learning areas, the exact definition of the elements and how they can be taught effectively may look different in each discipline. It was also noted that all participants referred, directly and indirectly, to all elements of the STEM creativity framework when they spoke about creativity even at the earlier stages of the interviews, before targeted questions about the elements of STEM creativity were directed to them. Furthermore, the respondents made individual connections between the four elements in many cases. These observations may signify that the participants teach the four elements, separately and concurrently, even when they do not do it consciously. A summary of the participants' beliefs and pedagogical approaches regarding each element is presented below.

Problem solving: All TE teachers had a clear outlook on what problem solving is and stated that problem solving is responding to a need or a situation. Conversely, SI respondents had a broader perspective on what problem solving can mean in mathematics and science. One respondent made a direct link to creativity and stated that problem solving occurs when there is a question that requires creativity, and another respondent noted that "pure creativity" comes into play when there is no precedent of a solution that would respond to a problem. In terms of pedagogical approaches, two TE and two SI respondents identified brainstorming and ideation tasks as precursors of problem-solving tasks, indicating a process of skill-building, and argued that problem solving as a creative practice is a skill that is learned and developed over time with practice, which allows learners to refer to prior learning as they build a skillset. Three TE respondents noted that teaching problem solving effectively entails giving students unguided, or minimally guided, challenges. Additionally, two SI respondents noted that teaching problem solving also requires the teacher to formulate the 'right' kind of problems.

Critical thinking: As opposed to problem solving, SI teachers elaborated on critical thinking more than TE teachers, and all participants offered distinct responses. One SI respondent stated that critical thinking is "a deeper level of problem solving" and added that it involves looking further into the question and using different approaches. Another stated that critical thinking means not taking anything at face value, and added that when learners receive new information, they should scrutinise its meanings beyond the surface. TE respondents' definitions of critical thinking entailed the analysis of the feasibility and producibility of an idea or product. In terms of pedagogical approaches, the respondents aligned critical thinking most closely with tasks that require students to explore new knowledge and skills. Two TE teachers stated that in technology education, critical thinking can be taught through evaluation and testing tasks, where learners examine their own designs according to set criteria. all SI respondents stated that discussions are essential to initiating and improving critical thinking.

Innovative ideation: Most TE participants defined ideation as developing multiple possible solutions to a problem or situation. As opposed to TE domain teachers, SI teachers approached ideation on a more abstract level and not necessarily as a process undertaken to directly respond to a problem or need. One respondent with experience every STEM learning area stated that ideation does not only involve generating options, but also comparing them to each other in order to produce the most optimum final option. Respondents identified brainstorming activities as effective idea generation processes, and recommended brainstorming to take place in groups to enable students to be inspired by each other. The majority of respondents across the two groups specified the need for a safe space for ideation to be practised in a group setting. However, each respondent had a different insight into how it can be taught best. Three TE respondents stated that learners should be offered a scaffold to engage with ideation, and two SI respondents highlighted the benefit for students considering other students' perspectives as an effective learning technique.

Cognitive flexibility: Most respondents were unaware of the term cognitive flexibility; however, they all referred to the individual elements of what constitutes it when asked about their understanding of and pedagogical approaches to creativity in previous questions. Out of all respondents, only one (TE) identified an explicit teaching of cognitive flexibility in the curriculum in the form of students developing 'contingency plans' for their projects, which indicated an underrepresentation of cognitive flexibility in the curriculum. TE respondents approached the teaching of cognitive flexibility in everyday life or hypothetical scenarios, and a further TE respondent argued that cognitive flexibility can only be acquired through practice. Three SI respondents interpreted the practice of scientific cognitive flexibility as learners using knowledge to make changes to their understanding of a theory or apply knowledge in a different way. A further SI respondent stated that this practice extends beyond science, and knowledge acquired in different subjects can also be used.

All participants interpreted the interactions between the elements of the STEM creativity framework in different ways. Half of the participants indicated a certain structure, linear or cyclical, to how the elements of STEM creativity would be implemented in a unit of study, which can involve one or two elements at a time. Four participants stated that the elements can occur simultaneously and without a particular linear or cyclical order throughout the implementation of a unit of study. Out of the four elements, problem-solving emerged as the element that was the most prioritised; half of the respondents indicated that they would initiate the implementation of the elements of STEM creativity with problem solving, which could be a consideration in their pedagogical approaches.

Discussion

The pedagogical approaches and practices stated by the respondents are categorised into four themes based on interview results: *exposure, exploration, experimentation, and execution*. These themes represent the overall categories of pedagogical beliefs and practices teachers expressed, with various methods and techniques to implement them. They were identified and defined through participants' responses to the interview questions. The names of all themes except for 'execution' have been used multiple times by all participants. The execution theme was named based on the participants' use of words such as 'application' and 'putting into practice' of the newly acquired skills and knowledge within a STEM learning program.

Exposure: This initial phase entails direct instruction from the teacher in various forms and is the base knowledge phase that the respondents referred to. According to the respondents' opinions, this phase is mostly teacher guided. However, in some cases, it can be scaffolded by a teacher, but the knowledge-finding and sharing can be undertaken by the learners.

Exploration: This phase is when learners begin to internalise newly acquired knowledge and skills, make connections with prior learning, and apply critical thinking and elaborate on the new knowledge and skills. It entails student-centred knowledge finding, exchange of knowledge, discussions, and other collaborative tasks that allow the learners to digest the knowledge and skills but not 'make' anything or produce new knowledge.

Experimentation: This phase is the precursor of execution and is inspired by the exposure and exploration phases. It involves brainstorming and ideation tasks, and often takes the form of learners coming up with their own problem-solving methods or experiments. In this phase, learners begin to use the newly acquired knowledge, skills, perspectives in some shape or form that may be the same or different from what they received in the exposure and exploration stages.

Execution: This phase takes the forms of contextualisation, connecting the newly acquired knowledge and skills to real life or a hypothetical scenario, and/or producing a tangible or intangible application of creativity, creating an end result, a final product, or solving a problem.

While it was observed that half of the participants in the TE group preferred to implement these themes in a rather rigid order of exposure, exploration, experimentation, and execution, they are nevertheless movable pedagogical themes that can be applied without a certain iterative, linear, or cyclical order. Most teachers identified that even when a certain structure is present in a project, students very often go back to a previous phase to fill gaps in knowledge, skills, or complete additional tasks. Furthermore, when a theme was identified in a response, another theme was often also mentioned, or one theme was implemented with the support of one or more other themes, which suggests that the four pedagogical themes are commonly applied in conjunction with each other, and in some cases, cannot be applied without the concurrent implementation of another theme. The validation of the framework was determined via the observation that all participants mentioned every element of the framework organically without being prompted and before the questions that identify and target the individual elements of the framework. However, since the participants' classroom practices were not observed, it was not possible to ascertain how they implemented the elements of the framework.

Concluding Remarks

In this study, a novel framework for STEM creativity has been designed and discussed. The researcher's primary objective was to explore mathematics teachers' beliefs and pedagogical approaches regarding creativity within the parameters of the framework. The findings indicate that mathematics teachers believe creativity is a learned skill and interpret creativity and the elements of the STEM Creativity Framework through a range of perspectives and four pedagogical themes: exposure, exploration, experimentation, and execution. Further research on the topic is needed to observe the teachers in their classroom practice in order to identify and examine consistencies and discrepancies between interview and observation data. Additionally, classroom observations could potentially lead to further amendments to the STEM Creativity Framework. The results of this study may inform future research regarding the instruction of creativity skills in STEM education and the design and development of future STEM creativity programs. The delivery of creativity skills within a framework designed specifically for STEM education that can operate within curricular boundaries can also lead to improvements in student outcomes and future STEM careers.

Acknowledgements

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References

Australian Curriculum., Assessment and Reporting Authority (ACARA). (2022). *Australian Curriculum: General capabilities (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/f-10-curriculum/f-10-curriculum-overview/general-capabilities

Falloon, G., Hatzigianni, M., Bower, M., Forbes, A., & Stevenson, M. (2020). Understanding K–12 STEM education: A framework for developing STEM literacy. *Journal of Science Education and Technology*, *29*(3), 369–385. https://doi.org/10.1007/s10956-020-09823-x

Hite, R., Jones, G., & Jur, J. S. (2016). Engineering imagination with ideation. *Journal of Interdisciplinary Teacher Leadership*, *1*(1), 9–24. https://doi.org/10.46767/kfp.2016-0002

Lubart, T. I. (1994). Creativity. In R. J. Sternberg (Ed.), *Handbook of perception and cognition, thinking and problem solving* (pp. 289–332). Academic Press.

Nijstad, B. A., De Dreu, C. K. W., Rietzschel, E. F., & Baas, M. (2010). The dual pathway to creativity model: Creative ideation as a function of flexibility and persistence. *European Review of Social Psychology*, *21*(1), 34–77. https://doi.org/10.1080/10463281003765323

Scriven, M., & Paul, R. (1987). Defining critical thinking. *Paper presented at the 8th annual international conference on critical thinking and education reform* (pp. 2–5). http://www.criticalthinking.org/pages/defining-critical-thinking/766

Skourdoumbis, A. (2016). Articulations of teaching practice: a case study of teachers and “general capabilities”. *Asia Pacific Education Review*, *17*(4), 545–554. https://doi.org/10.1007/s12564-016-9460-7

Smith, S., & Henriksen, D. (2016). Fail again, fail better: Embracing failure as a paradigm for creative learning in the arts. *Art Education (Reston)*, *69*(2), 6–11. https://doi.org/10.1080/00043125.2016.1141644

Spiro, R. J. (1988). *Cognitive flexibility theory: Advanced knowledge acquisition in ill-structured domains*. Technical Report No. 441.

Sternberg, R. J. (2020). Critical thinking in STEM disciplines. In R. J. Sternberg & D. F. Halpern (Eds.), *Critical thinking in psychology* (2nd ed., pp. 309–327). Cambridge University Press. https://doi.org/10.1017/9781108684354.014

Stretch, E., & Roehrig, G. (2021). Framing failure: Leveraging uncertainty to launch creativity in STEM education. *International Journal of Learning and Teaching*, *7*(2), 123–133. https://doi.org/10.18178/ijlt.7.2.123-133

Stylianidou, F., Glauert, E., Rossis, D., Compton, A., Cremin, T., Craft, A., & Havu-Nuutinen, S. (2018). Fostering inquiry and creativity in early years STEM education: Policy recommendations from the Creative Little Scientists Project. *European Journal of STEM Education*, *3*(3).

Ward, T. B. (1994). Structured imagination: The role of category structure in exemplar generation. *Cognitive Psychology*, *27*(1), 1–40.

Wechsler, S. M., Saiz, C., Rivas, S. F., Vendramini, C. M. M., Almeida, L. S., Mundim, M. C., & Franco, A. (2018). Creative and critical thinking: Independent or overlapping components? *Thinking Skills and Creativity*, *27*, 114–122. https://doi.org/10.1016/j.tsc.2017.12.003

Wu, Y., & Koutstaal, W. (2020). Charting the contributions of cognitive flexibility to creativity: Self-guided transitions as a process-based index of creativity-related adaptivity. *PloS One*, *15*(6), e0234473-e0234473. https://doi.org/10.1371/journal.pone.0234473

Zabelina, D. L. (2018). Attention and creativity. In O. Vartanian & R. E. Jung (Eds.), *The Cambridge handbook of the neuroscience of creativity* (pp. 161–179). Cambridge University Press. https://doi.org/10.1017/9781316556238.010

Pre-Service Primary School Teachers' Understanding of the Meaning of 'Capacity' in the Australian Curriculum: Mathematics

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Ambiguity of language can form a barrier to understanding mathematical concepts. The word 'capacity' is problematic as it holds a variety of meanings and might not be understood in school mathematics, to be an aspect of volume. We investigated pre-service primary school teachers' interpretations of 'capacity' when given the definitions of 'volume' and 'capacity' provided in the Australian Curriculum: Mathematics. Of the 72 participants, 20 appeared to hold the conception that the volume of a liquid is its capacity. We suggest reducing ambiguity by defining 'capacity' more clearly or by excluding this word from formal mathematics terminology.

In mathematics education, a distinction is commonly made between 'mathematical language' which is precise, explicit and averse to ambiguity, and 'everyday language' which is informal and has a largely implicit meaning based on the context in which it is used (Barwell, 2005). Alternative conceptions can arise when a term that students already understand informally, differs from the formal mathematical term defined in a mathematics curriculum (Avgerinos & Remoundou, 2021). The term 'capacity' is one such term.

Capacity has multiple meanings. It comes from an old French word, *capacité,* which means 'ability' in a legal, moral, or intellectual sense. In a physical context it means "ability to contain; size, extent" (Harper, 2024). *Capacity* can be understood as an ability to store electricity or an ability to hold either a volume, or a weight, or a number of people or items, depending on the context (Ho & McMaster, 2019). In the Australian Curriculum: Mathematics (AC:M), *capacity* is defined as "the amount a container will hold. It is often in relation to the volume of fluids. Units of capacity (volume of fluids or gases) include litres (L) and millilitres (mL)" (Australian Curriculum Assessment and Reporting Authority [ACARA], 2022). In this definition, the word 'amount' could mean a volume or a weight and the word 'often' could imply that *capacity* is not always a volume. However, the units of litres (L) and millilitres (mL) mentioned in the definition imply that *capacity* is an attribute of fluids. This alternative conception has significant implications for understandings in science where students learn about states of matter (McMaster et al., 2021). Most Australian primary school teachers teach both mathematics and science. If pre-service teachers (PSTs) are confused by the meaning of *capacity*, it follows that their students will be confused.

The exploratory research documented in this paper focuses on the question: What do pre-service primary school teachers understand to be the meaning of *capacity*, when given the definitions provided in the Australian Curriculum: Mathematics?

Literature Review

There is lack of prior research concerning teachers' understanding of the term capacity when it is formalised in a mathematics curriculum. There is however, a large body of research on alternative conceptions and how such conceptions develop (e.g., Vosniadou, 2013).

Resnick (1991) argues that all knowledge is situated, meaning that it reflects the particular conditions in which it was produced. Hence, to understand an alternative conception, the context in which that conception developed needs to be recognised. Hallden et al. (2013) present a theory of conceptual change which accounts for context. They propose a 'compounded model'

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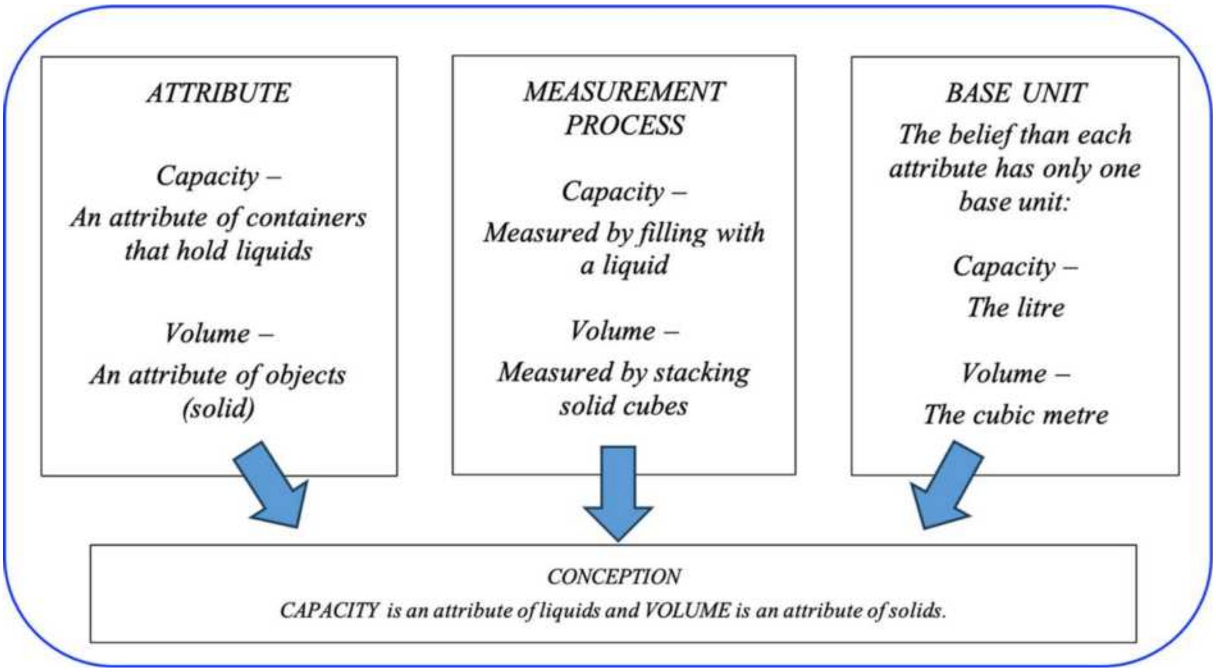
to describe the way in which individuals create a coherence of meaning from a collection of two different sources of information. Their model illustrates the piecing together of contrasting information to create a coherent whole that makes sense to the individual. Based on the theory of Hallden et al. (2013), we hypothesise a synthesis of the three sources of information and experiences that could account for the conception that capacity is an attribute of liquids while volume is an attribute of solids. These sources are: the definition of the attribute, the experience of measuring the attribute, and the base unit of the attribute. They are illustrated in Figure 1 and detailed in this literature review.

Figure 1

Information and Experiences Compounding to Form the Conception that Capacity is an Attribute of Liquids and Volume is an Attribute of Solids



When Harrison (1987) questioned young children (four to seven-year-olds) about the capacity of containers, they thought about how much a container *was* holding rather than what it *will* hold or *could* hold. Irrespective of the size of the container, children thought its *capacity* was how full it was. On reflection, Harrison (1987) thought their concentration on the liquid (the filled space) rather than the space inside the container, was logical. People naturally tend to notice first, the fullness of a container: how close it is to reaching its capacity.

The Measurement Process

In the AC:M, the process specified for measuring capacity is to fill containers with pourable materials such as water. In Year 1 it is suggested that they "pour sand/rice/water from one container to another to compare and order the capacity of 3 or more containers" (ACARA, 2022). This process is different from the process suggested in Year 7 for measuring the volume of a rectangular prism, namely stacking unit cubes in rows, columns and layers.

In Ho and McMaster's (2019) study, six 11- and 12-year-old students were asked during a task-based interview, how they would find the volume of a plastic box and then how they would find its capacity. Five of these students, having initially been confused about the difference between volume and capacity, became open to believing that the capacity of the container meant its volume in this context. However, the remaining student held firmly to the conception that if the volume inside the box was measured with unit cubes, this was the volume of the box and if the same space was measured with water, this was the capacity of the water. The conception that capacity is the volume of a liquid can be seen in a teacher's annotations of student work samples for Year 3 on the website housing the AC:M (ACARA, 2022).

The Base Unit

Another contributing factor could be the belief that in our metric system, each measurement attribute has only one base unit. In the International System of Units (known as the SI system), the metre is the only base unit for length, and the cubic metre is derived from this base unit. However, there are two official units that can be used to quantify a volume in Australia's metric system—the cubic metre and the litre (Australian Government, 2024). People who believe the cubic metre is the only metric unit for the volume of a solid, could believe that the litre is the only metric unit for the volume of a liquid. There are mathematics education textbooks for pre-service teachers that reinforce this conception, possibly due to the writers' interpretation of the AC:M. Tables in Cotton et al., (2023, p. 229) and Reys et al. (2023, p. 611) for example, do not list the litre as a metric unit of *volume*, but as a metric unit of *capacity*.

Method

The method was based on the premise of Hallden et al. (2013) that learners' conceptions can be understood by examining their responses to constructed contexts. The 'learners' in this study were primary school pre-service teachers (PSTs). For them, the context of responding to a student's statement and marking a question is particularly relevant to their future career. An online questionnaire was used because it enabled anonymity and a larger sample size than would be possible by interviewing participants. The research was conducted early in 2020.

The Participants

The study involved 72 PSTs at a metropolitan university which has a relatively high entry requirement into their initial teacher education programs for teaching in primary schools: a four-year Bachelor of Education (Primary) program (BEd) and a two-year post-graduate Master of Teaching (Primary) program (MTeach). The program and year level of the 72 participants is shown in Table 1. The BEd Year 4 cohort had attended a lecture and a tutorial class the previous semester, in which they were explicitly taught the meanings of *volume* and *capacity* in the mathematics curriculum. As part of the unit of study, they also experienced teaching the topic 'Volume and Capacity' to a small group of students at a local primary school. The other PSTs had not yet encountered this topic in their program, however four PSTs in BEd Year 3 (PSTs 47, 48, 49, 50) and four in MTeach Year 2 (PSTs 30, 31, 32, 33) had either taught the topic or seen it being taught, most likely during their profession experience placement.

Table 1

Participants in Each Cohort with Teaching Experience on the Topic of 'Volume and Capacity'

Program and year level	Number of participants	Number with teaching experience	Individual labels
MTeach (Primary) Year 1	25	0	PST1 to PST25
MTeach (Primary) Year 2	7	4	PST26 to PST32
BEd (Primary) Year 1	4	0	PST33 to PST36
BEd (Primary) Year 2	3	0	PST37 to PST39
BEd (Primary) Year 3	10	4	PST40 to PST49
BEd (Primary) Year 4	23	23	PST50 to PST72
Total	72	31	

The Questionnaire

The online questionnaire began by providing participants with the definitions of *volume* and *capacity* stated in the AC:M (version 8.4) and inviting them to read then state these definitions in their own words (Item 1). These definitions were the same as those in the current AC:M (version 9). *Volume* was defined as "a measure of the space enclosed by the solid" and *capacity* was defined as "a term that describes how much a container will hold. It is used in reference to the volume of fluids or gases and is measured in units such as litres or millilitres."

Item 2 was designed to obtain quantitative data concerning the number of PSTs' holding the conception that the volume of a liquid is its capacity, and qualitative data from those PSTs who gave an extended response. The PSTs were asked to state whether they would agree with a student who says, "Objects have volume and liquids have capacity" (Item 2a). It was expected that PSTs holding the alternative conception would agree with the student. To confirm whether they held this conception, they were given Item 2b about which capacity of a jug (Figure 2) they would accept as being correct. This question was sourced from a commercial test for Year 5 students and made a multiple-choice question. If a PST held the alternative conception, they were expected to select 1800 mL as the correct answer.

Figure 2

Item 2b) From the Questionnaire

Below is a question from a test given to Year 5 students. Which of the answers listed beside this test question, would you accept as being correct?

State the capacity of this container as a number of millilitres.

- 1800 mL
- 200 mL
- 2000 mL
- 2200 mL
- 400 mL
- other

If 'other', please state:

A final item (Item 3) asked PSTs if they had any concerns about the wording of the curriculum definitions. This item was to yield additional qualitative data and possibly clarify earlier responses.

Analysis of the Data

The data were analysed to determine the extent to which the PSTs held the conception that the volume of a liquid is its *capacity*. Data obtained from those students who gave extended responses to Item 2 and those who responded to the Items 1 and 3, were further analysed to determine possible reasons for their conception and the nature of their concerns about the curriculum definitions of *volume* and *capacity* in the AC:M.

Responses to the statement "Objects have volume and liquids have capacity" were sorted into three categories: agree (a response of 'yes', 'correct', or 'agree'), disagree (a response of 'no', 'incorrect' or 'disagree') and unsure (a response of 'I don't know' or something similar). Some responses raised ambiguities that were not envisaged. These were discussed by the two researchers until a common understanding was reached. For example, one PST responded by saying "Objects have volume, and they can hold a given capacity of liquid". It was decided that because they referred to 'capacity of liquid', they agree with the conception that liquids 'have capacity'. Responses were also coded as 'agree' if they said objects *also* 'have capacity' because this implies that they think liquids 'have capacity' e.g., "Objects can have capacity too" (PST39).

Responses in which PSTs said they 'partially agree' with the statement were coded as 'disagree', because it was assumed that by 'partially agree' they meant that the first half of the statement ("objects have volume") is correct and the second half ("liquids have capacity") is incorrect.

Regarding the capacity of the jug (Figure 2), if a PST judged 1800 mL to be correct, they were deemed to hold the conception that *capacity* means the volume of a liquid. Using the syllabus definition of *capacity*, every multiple-choice answer could be considered to be correct except an answer of 1800 mL. If the capacity of the jug means the volume to the top of the scale (its nominal capacity) the answer is 2000 mL. If what the jug 'will hold' is interpreted to mean the available space, 200 mL (2000 mL − 1800 mL) is another correct answer. If the capacity of the jug means its brimful capacity, the answers of 2200 mL and 400 mL (2200 mL − 1800 mL) are correct. Students choosing 'other' and stating an answer were classified as correct if they explained that the brimful capacity may not be exactly 2200 mL.

Results

Restating the Definitions

Of the 72 participants, 41 stated the definitions in their own words. Most explained how *volume* and *capacity* were different. A typical response was, "The volume of an object or substance is the amount of space it takes up, whereas capacity is the amount a container can hold". No-one referred to *capacity* as a volume, however one participant said "Volume and capacity are basically the same thing" (PST9) and another referred to both volume and capacity as space, "Volume is the space that something takes up, and capacity is the amount of space that is within a container" (PST71).

Three people defined *volume* to be an attribute of solids and *capacity* to be an attribute of liquids (PSTs 20, 21, 25), and three others thought *capacity* was how much a container "has in it" or "is holding" rather than how much it "will hold" (PSTs 5, 30, 48). These people also thought 1800 mL was the correct answer for the capacity of the jug in Figure 2, indicating that they held the conception that the volume of a liquid is its capacity.

Agreeing or Disagreeing with a Student Statement

Table 2 shows that a total of 30 participants (42%) agreed to the statement, "Objects have volume and liquids have capacity". Of the 72 participants, 17 gave an extended response, thereby providing insight as to why they and possibly other participants, chose to agree or disagree with the statement.

An extended response by a participant who agreed with the statement was "Objects have volume, and they can hold a given capacity of liquid" (PST23). One participant said they only agreed because they had just read the curriculum definitions, "I would have said the opposite until I read the description above" (PST20).

The six people who, in an earlier response had defined *capacity* as the volume of a liquid or as how much a container 'is holding', all agreed with the statement.

Three responses were classified as 'unsure'. One person wrote that they "would ask the student to elaborate about what they mean by each thing 'having' something" (PST64). Another wrote that "Objects measure volume and liquids measure capacity" (PST70) presumably confusing 'having' an attribute with measuring an attribute. The remaining 39 people disagreed with the statement. Some of their extended responses were:

- Objects can have a capacity. Liquids can be used to measure this capacity (PST43);
- Liquid is a means of finding the capacity (PST68);
- Liquids do not have capacity. It is the container that holds them that has capacity (PST71).

Marking Answers to a Question about the Capacity of a Jug

There are several correct answers to the question about the capacity of the jug (Item 2b). Fifteen people mentioned more than one answer. Almost half of all participants (35 people) said they would mark 1800 mL (the volume of the liquid) as correct, 25 would mark 2000 mL (the nominal capacity) as correct and 21 people would mark 2200 mL (the brimful capacity) as correct. Only one person included 200 mL or 400 mL as answers they thought were correct. This person also included 1800 mL as a correct answer. Altogether, three people included both 1800 mL and 2000 mL as correct answers, presumably thinking that both a container and a liquid can 'have capacity'.

Relationships Between Pre-Service Teacher Responses

The researchers envisaged that people who would mark 1800 mL as correct, held the alternative conception that *capacity* is the volume of a liquid, and would therefore agree with the statement that liquids 'have capacity'. The data in Table 2 shows that although they were twice as likely to agree that liquids have capacity compared to those who did not accept 1800 mL as correct (28% compared to 14%), they were still almost as likely to disagree or be unsure (23%) as to agree (28%) with the statement about liquids 'having capacity' (see Table 2).

Table 2

Responses to Item 2a) From the Questionnaire

	Agree (*n=72*)	**Disagree/unsure (*n=72*)**	**Total (*n=72*)**
1800 mL is not correct	10 (14%)	25 (34%)	35 (49%)
1800 mL is correct	20 (28%)	17 (23%)	37 (51%)
Total	30 (42%)	42 (58%)	72 (100%)

The Impact of Instruction

Only the BEd Year 4 cohort had received instruction on teaching students about 'Volume and Capacity' at the time when the survey was conducted. Table 3 shows a comparison between the responses of these PSTs and the remaining 49 PSTs.

Table 3

Responses of Pre-Service Teachers who had or had not yet Received Instruction About the Meaning of Capacity

Responses	**Before instruction (*n=49*)**	**After instruction (*n=23*)**
Agree that liquids have capacity	24 (49%)	6 (26%)
Would mark 1800 mL as correct	25 (51%)	12 (52%)
Both the above responses	11 (22%)	3 (13%)

Although students having received instruction were only about half as likely to agree that liquids 'have capacity', they were just as likely as others to see the level of water shown in the jug as its *capacity*.

Concerns About the Wording of the Curriculum Definitions

Only 31 PSTs responded to the final question of whether they had concerns regarding the curriculum definitions. Four people wrote 'no', three wrote 'not yet' and 17 wrote down their concerns. Most of those who expressed their concerns about the wording of the definitions of *capacity* and *volume* had had experience teaching the topic of 'Volume and Capacity'. Their main concern was the confusing nature of the terminology. Based on their experience, one wrote "The terminology can be confusing, so explicitly defining the difference between the two would be useful for younger students. Also emphasising that measuring capacity is more for real-world uses rather than volume." (PST46). A PST who had not yet taught the topic wrote "They are super confusing, I don't really understand the difference, and currently wouldn't feel comfortable teaching the difference." (PST45). Several PSTs thought the reason why the terms are so confusing, is that they are so similar. One wrote "I just generally don't understand why capacity exists" (PST69). Another thought that because they are so similar "it doesn't seem necessary to use both terms" (PST63). Further suggestions for making the topic less confusing were to begin the topic by teaching what *volume* is, using both liquid and solid contexts (PST66, PST69) and to only use *capacity* informally in real-world contexts (PST46, PST72).

Discussion and Conclusion

The conception that *capacity* is an attribute of liquids and *volume* is an attribute of solids, was prevalent amongst the PSTs who participated in the study, with 28% both agreeing with a statement that liquids 'have capacity' and believing that the liquid a container 'is holding' is the capacity (Items 2a and 2b), a conception reported by Harrison (1987).

It was surprising that about half of all the participants appeared to hold the view that the level of the water in the jug (Figure 2) was its capacity, including many who disagreed that liquids 'have capacity'. This could be because their eyes were immediately drawn to the visual reading of a scale rather than the wording of the question. By placing this item after inviting participants to read and comment on the curriculum definitions, and by underlining the word *capacity*, the researchers had hoped misreading of the question would be minimised.

Another reason for PTSs incorrectly answering Item 2b, while apparently knowing that liquids do not 'have capacity' (Item 2a) could be their misunderstanding of what 'having' means. Two PSTs mentioned the confusion this wording caused them. The statement could have also been misinterpreted by others who responded without giving a reason for agreeing or disagreeing. Clearer wording for Item 2a might be, "Volume is an attribute of solids and capacity is an attribute of liquids". It could have also been clearer if the first half of the statement was excluded because it makes the whole statement partially correct.

Another interesting finding was that some PSTs appeared to believe that the word *capacity* can be used in relation to both containers and liquids. The three participants who accepted both 1800 mL (the volume of water in the jug) and 2000 mL (the nominal capacity of the jug) to be correct answers, would fit in this category. One of the elaborations in the AC:M (Year 3) suggests that students "measure out different capacities of liquid" (ACARA, 2022).

Despite the limited number of participants and participants not being interviewed to clarify the meaning of their responses, this exploratory research suggests that there is considerable confusion amongst PSTs about the meaning of the word *capacity* in the AC:M. If these PSTs are confused, it follows that their future students will be confused. If *capacity* is only an attribute of containers, this needs to be stated clearly in the curriculum and emphasised by teachers. It could be helpful for teachers to use the term *internal volume* of a container as this is what is probably meant by *capacity*, rather than a container's nominal capacity or brimful capacity. Another possibility (one suggested by some of the PSTs) is to formally use the word *volume* and only use *capacity* informally when the context makes its meaning clear.

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Reference List

Australian Curriculum and Reporting Authority (ACARA). (2022). *The Australian Curriculum (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au

Australian Government. (2024). *Australian Government style manual*. Table 2: Examples of other commonly used legal units in Australia. https://www.stylemanual.gov.au/grammar-punctuation-and-conventions/numbers-and-measurements/measurement-and-units

Avgerinos, E. & Remoundou, D. (2021). The language of "rate of change" in Mathematics. *European Journal of Investigation in Health, Psychology and Education*, *2021*(11), 1599–1609. https://doi.org/10.3390/ejihpe11040113

Barwell, R. (2005). Ambiguity in the mathematics classroom, *Language and Education*, *19*(2), 117–125. https://doi.org/10.1080/09500780508668667

Cotton, T., Greenbaum, J. & Minas, M. (2023). *Understanding and teaching primary mathematics in Australia (4th ed)*. Routledge. https://doi.org/10.4324/9781003315155

Hallden, O., Scheja, M. & Haglun, L. (2013). The contextually of knowledge. An intentional approach to meaning making and conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (2nd ed. pp. 71–95). Routledge. https://www.taylorfrancis.com/books/edit/10.4324/9780203154472/international-handbook-research-conceptual-change-stella-vosniadou

Harper, D. (2024). *Etymology dictionary* [Online]. Entomology Dictionary. https://www.etymonline.com/word/capacity

Harrison, R. (1987). 'On fullness'. *Mathematics Teaching, 119*, 27–28.

Ho, A., & McMaster, H. (2019). Is 'capacity' volume? Understandings of 11 to 12-year-old children. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 356–363). Perth: MERGA.

McMaster, H., Preston, C., Wang, H, & Perivolarellis, M. (2021). The case for a sub-element 'measuring matter' within the Australian national numeracy learning progression. *Australian Journal of Education*, 65(3), 280–298. https://doi.org/10.1177/00049441211041855

Resnick, L. (1991). Shared cognition: Thinking as social practice. In L. B. Resnick, J. Levine, & S. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 1–20). Washington, DC: American Psychological Association.

Reys, R., Lindquist, M., Lamdin, D., Smith, N., Rogers, A., Bragg, L., Cooke, A., Fanshaw, M. and Gronow, M. (2022). *Helping children learn mathematics (4th Australian ed.)*. John Wiley & Sons.

Vosniadou, S. (Ed.). (2013). *International handbook of research on conceptual change* (2nd ed.). Routledge. https://www.taylorfrancis.com/books/edit/10.4324/9780203154472/international-handbook-research-conceptual-change-stella-vosniadou

Pre-Service Teachers' Struggles With Core Numeracy Concepts

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In Ireland, Initial Teacher Education [ITE] standards require all teachers to possess an adequate level of numeracy themselves in order to teach for numeracy learning across the school curriculum. This paper reports on a study that investigated 204 pre-service post-primary teachers' numeracy capabilities. Analysis of questionnaire data from participants in three universities showed that pre-service teachers tend to struggle to complete the numeracy tasks correctly, especially tasks related to ratio and proportional reasoning. If pre-service teachers of all disciplines are not capable of completing core numeracy concepts, then they will struggle to teach for numeracy learning.

Internationally, educators and government stakeholders are advocating that all citizens should have gained the necessary literacy and numeracy competencies in school, to live and work in today's world (Goos & O' Sullivan 2023; Department of Education and Skills [DES] 2011; Norwegian Directorate for Education and Training 2012). The Irish government developed a strategy for literacy and numeracy and in this strategy, have stressed the importance that all teachers should be teaching for numeracy learning across all subject disciplines (DES 2011, 2015). More recently, and in a bid to support this initiative of a numerate society, The Teaching Council of Ireland (2020), which is the regulatory body for teachers, have agreed that teachers need to develop their personal numeracy knowledge and also specified that all universities involved in preparing pre-service teachers must:

> Ensure that student teachers are afforded opportunities to enhance their own literacy and numeracy and are required to demonstrate an acceptable level of proficiency in literacy and numeracy. Students shall be required to demonstrate their competence in teaching and assessing literacy and numeracy appropriate to their curricular/ subject area(s). (CÉIM, 2020, p. 14)

Similarly, in Australia, Hall and Forgasz (2020) argue that pre-service teachers need to possess a certain level of numeracy skills themselves, prior to teaching for numeracy learning. Supporting teachers in gaining the appropriate knowledge required to teach for numeracy learning in an effective manner should be a priority for educators. Researchers in the field of numeracy argue that possessing mathematical knowledge is not enough to teach for numeracy, nevertheless they recognise that mathematical knowledge is a core aspect of numeracy and therefore is important for teachers to possess (Venkat & Winter, 2015). Forgasz and Hall (2019) argue that in order for teachers to improve the numeracy capabilities of their students, teachers must first equip themselves with the necessary skills to develop their own understanding of how mathematical concepts and numeracy affect their own lives and their subject area. The following paper presents results from a study conducted in Ireland which investigated pre-service secondary teachers' abilities to complete numeracy questions. For the purpose of this paper, responses to 2 numeracy questions were analysed to address the following research question: How well are pre-service post-primary teachers able to complete numeracy tasks?

Research Design and Method

Pre-service secondary teachers enrolled in the two-year Professional Master of Education (PME) programme in three different universities, were invited to take part in this research study. Pre-service teachers were asked to complete a questionnaire at one of their general education lectures at the beginning of their second year on the PME. There were 204 pre-service teachers who completed this questionnaire. The questionnaire consisted of three sections, Section A:

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 415–422). Gold Coast: MERGA.

Demographics, Section B:Pre-service Teachers perceptions of numeracy and preparedness to teach for numeracy, and Section C: Preservice Teachers assessment of numerate capabilities.

Section C of the questionnaire consisted of 7 numeracy tasks in total; however, the numeracy tasks were presented in 6 questions. The first two numeracy tasks were part of question 1. Table 1 presents a short explanation of each numeracy task in the questionnaire. Pre-service teachers were asked to display their workings for each task in a text box provided. Asking pre-service teachers to provide mathematical workings in the space provided enabled the authors to identify if the answer was correct or incorrect and furthermore allowed the authors to understand the pre-service teachers' mathematical thinking. Three of the numeracy tasks were published by the OECD as PISA test questions: Earthquake task (PISA, 2003), Car task (PISA, 2003) and the Salad dressing task (PISA, 2012). The other three numeracy tasks were developed specifically for this study. Pre-service teachers were allowed to use a calculator and they were asked to indicate at the end of the questionnaire if they had done so.

Table1

Explanation of Each Numeracy Task in the Questionnaire

Numeracy tasks	Explanation
Time task	Calculate the difference between two Olympic swimmers' finishing race times for a 100 metre butterfly race. The results were presented in a table and the pre-service teachers had to subtract one decimal number from another decimal number (51.14–50.39)
Distance task	Joseph Schooling had a result of 50.39 in the 100 metre race and if the race was 30 metres longer, given that he was travelling at the same average speed as he did in the first race, calculate the new time he would finish the 130 metre race
Earthquake task	A documentary about earthquakes and how often they occur is broadcast. A geologist stated "In the next twenty years, the chance that an earthquake will occur in Zed City is two out of three". Pre-service teachers were asked to use mathematical knowledge and understanding of statistics to predict an event occurring in this specific context. Pre-service teachers were provided with 4 different scenarios and asked to choose which one best reflected the meaning of the geologist's statement
Pie Chart task	Given a pie-chart, calculate the proportion of the pie chart (as a percentage) that represented the participants who chose biology as a subject for the Leaving Certificate
Best Car task	Calculate the score of the "Best Car" given an equation. The "Best Car" is evaluated based on scores for safety features (S), fuel efficiency (F), external appearance (E) and internal fittings (T) and these were the variables in the given equation (Ca) = (3 x S) + F + E + T. Pre-service teachers had to substitute values into the equation and work out the final answer for the Best Car
Salad Dressing task	A recipe for 100mls of salad dressing has three ingredients which are Salad Oil (60mls), Vinegar (30mls) and Soy sauce (10mls). Pre-service teachers were asked to calculate how much salad oil is required to make 175mls of salad dressing
Mobile Phone task	David uses 500 minutes per month and 15GB of data. Recommend the best mobile phone plan for David, given price tariffs for 3 mobile phone companies

Pre-Service Teachers' Numeracy Capabilities

The results of the overall numeracy tasks are presented in Table 2, which shows the number of pre-service teachers who answered each task correctly, incorrectly or left it blank.

Table 2

Breakdown of Correct, Incorrect and Blank Answers for Each Numeracy Task

	Correct	Incorrect	Blank
Time	112 (54.9%)	89 (43.6%)	3 (1.5%)
Distance	99 (48.5%)	68 (33.4%)	37 (18.1%)
Earthquake	165 (80.9%)	36 (17.6%)	3 (1.5%)
Pie chart	135 (66.2%)	53 (25.9%)	16 (7.8%)
Best Car	160 (78.4%)	22 (10.8%)	22 (10.8%)
Salad Dressing	118 (57.8%)	59 (29%)	27 (13.2%)
Mobile Phone	48 (23.5%)	94 (46%)	62 (30.5%)

The question with the highest number of correct answers was the Earthquake task with 165 (80.9%) pre-service teachers answering this task correctly. The Earthquake task was taken from PISA (2003) released sample items. It was the only question for which the pre-service teachers were given multiple choice options for their answer.

As can be seen in Table 2, the Time task had nearly as high a proportion of incorrect answers as it had of correct answers, which is interesting as I had considered this task one of the easier tasks as it involves only subtracting two decimal numbers. Also, it was noted that over 40% of pre-service teachers either answered incorrectly or left the task blank in four out of seven of the numeracy tasks. The Mobile Phone task was the question with the lowest number of correct answers. This question was also the only question for which there were more pre-service teachers who answered the question incorrectly than those who answered it correctly. The Mobile Phone task was also the question for which there was a substantial number of pre-service teachers who left the question blank. This may be due to the nature of the question and the fact that it involved much more time and more mathematical calculations such as working out how much extra data was needed per month and working out the cost of the extra data. Pre-service teachers needed to consider the total cost was over the course of 24 months and not just the price for one month when initially purchasing the phone. This question was also the last question on the questionnaire which may also have contributed to the low response rate.

From the initial analysis, 8 (3.9%) pre-service teachers were able to answer all 7 numeracy tasks correctly. There were 156 (76.5%) pre-service teachers who were able to answer 3, 4, 5 or 6 numeracy tasks correctly. However, there were still a considerable number of pre-service teachers (32, 15.7%), who were only able to answer either 1 or 2 numeracy tasks correctly. Finally, 8 (3.9%) pre-service teachers were unable to answer any numeracy task correctly and only one of these eight pre-service teachers left each answer blank, which means that 7 pre-service teachers attempted to answer the numeracy tasks but answered them all incorrectly.

Pre-Service Teachers' Struggles with Core Numeracy Concepts

This section presents the different types of answers pre-service teachers provided when completing the numeracy tasks. Initially, the numeracy tasks were coded as correct, incorrect or blank but then it became evident that many pre-service teachers had proceeded with answering the question incorrectly in the same way, thus suggesting there were many commonly held misconceptions. Therefore, I decided to code the types of answers which meant I was able to better understand the mathematical thinking and errors some pre-service teachers had made. For the purpose of this paper, I will focus on Question 1 which focussed on the Time and Distance tasks and I will also present errors in Question 5 which was the Salad Dressing Task. Both questions expect teachers to draw on their mathematical knowledge of ratio and proportional reasoning, which is fundamental and a core aspect of many numeracy tasks in the school curriculum and in the wider society that we live in.

The first numeracy task in Section C of the questionnaire asked pre-service teachers to calculate the winning margin in the 100 metre butterfly final in the 2016 Olympic Games, and then calculate the time to swim the race at this average speed over an additional 30 metres. The question and a model solution are shown in Figure 1.

Figure 1

Sample Correct Answer to Time and Distance Task

Section C: Numeracy problems

1. At the Olympics in Rio 2016, Joseph Schooling and Michael Phelps competed in the 100 metre Butterfly race. The results are shown in the table below:

Name	Result
Joseph Schooling	50.39
Michael Phelps	51.14
Chad Le Clos	51.14
Laszlo Cseh	51.14
Shuhao Li	51.26
Medhy Metella	51.58
Tom Shields	51.73
Aleksandr Sadovnikov	51.84

(i) By how much time did Joseph Schooling beat Michael Phelps?

51.14 – 50.39 = 0.75 of a second

(ii) How confident are you that your answer is correct?

Confident	Not Confident	Unsure

(iii) If the race was an extra 30 metres in distance, what would be the result time for Joseph Schooling given he was swimming at the same average speed that he swam in the 100m butterfly race?

100m ÷ 50.39s = 1.9845 m/s 130m ÷ 1.9845 m/s = 65.507s 65.5s
50.39s ÷ 100m = 0.5039 s/m 0.539s/m × 30m = 15.117s 50.39s + 15.11s = 65.5s

There were 89 pre-service teachers who answered this task incorrectly, and of this cohort, a considerable number of pre-service teachers arrived at the same incorrect answer. Different types of incorrect answers were observed and the mathematical thinking behind each error was also identified. As described in Table 3, there were three notable incorrect answers for the Time Task. There were 20 (22.5%) answers to this question which could not be allocated to either Type 1, Type 2 or Type 3 and the mathematical thinking of the error could not be drawn from the answer the pre-service teachers presented for these 20 answers. The incorrect answers for each type of answer are presented in the brackets in Table 3 along with a description of the error.

Table 3

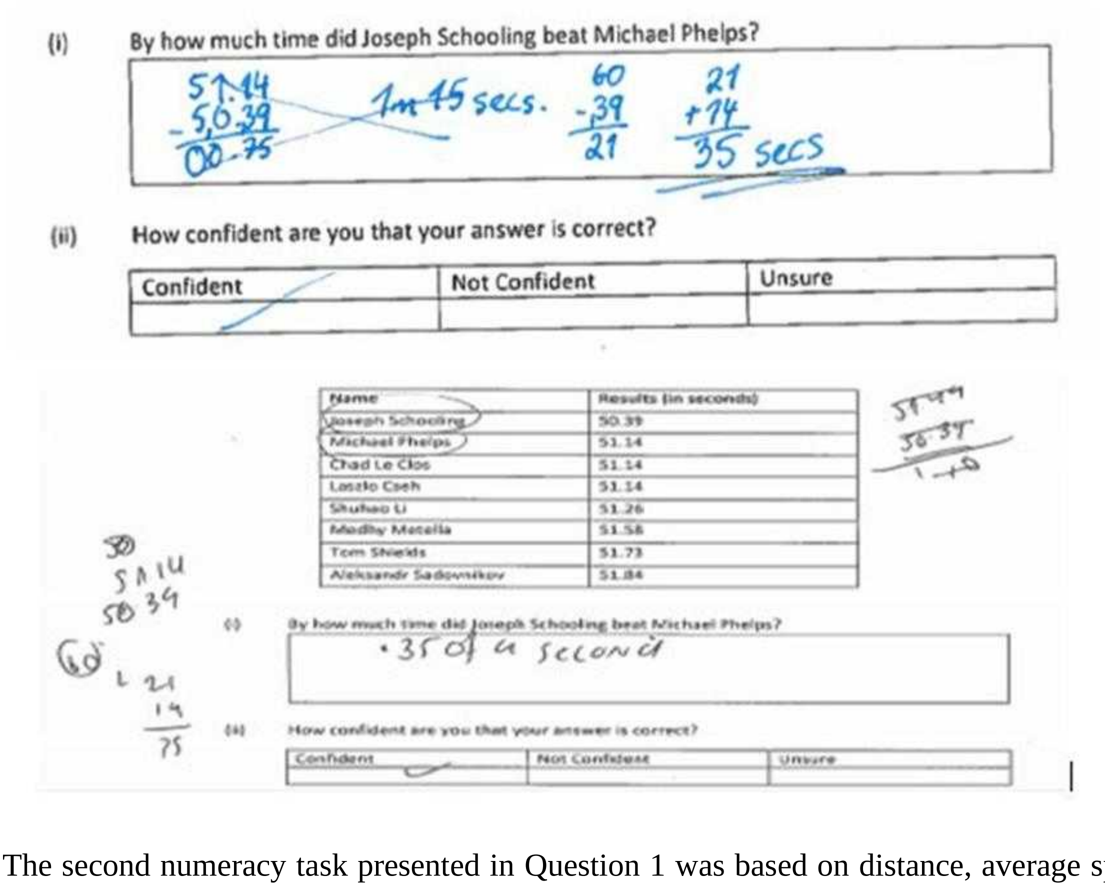
Types and Frequency of Occurrence of Incorrect Answers for Time Task

Types of answers	Description of misconception	*n* (%)
Type 1 (0.35 seconds or 35 seconds)	The number of centi-seconds in a second is the same as the number of seconds in a minute or number of minutes in an hour	52 (58.5%)
Type 2 (1 minute 15 seconds or 75 seconds)	Convert 0.75 to 1 minute and 15 seconds or 75 seconds whereby the respondents omitted the decimal place in their answer	9 (10.1%)
Type 3 (1.25 Seconds)	Subtract the smaller number from the larger number	8 (8.9%)

The main difficulties that led pre-service teachers to give a Type 1 answer (0.35 seconds or 35 seconds) involved the misconception that the number of centi-seconds in a second is the same as the number of seconds in a minute or number of minutes in an hour. Hence, 52 of the 89 pre-service teachers who gave a Type 1 answer incorrectly calculated the answer as either 0.35 seconds or 35 seconds. Examples of how some of these 52 participants calculated these answers are given in Figure 2. It seems that they have "carried over" 60 seconds after the decimal point to change the task from 51.14 − 50.39 to 50.74 − 50.39, which gives an answer of 0.35. However, some pre-service teachers who worked only with the numbers after the decimal point interpreted the answer as 35 seconds, which might suggest that students who answered 35 seconds do not understand the place of a decimal point.

Figure 2

Examples of Type 1 and 2 Incorrect Answers to Time Task

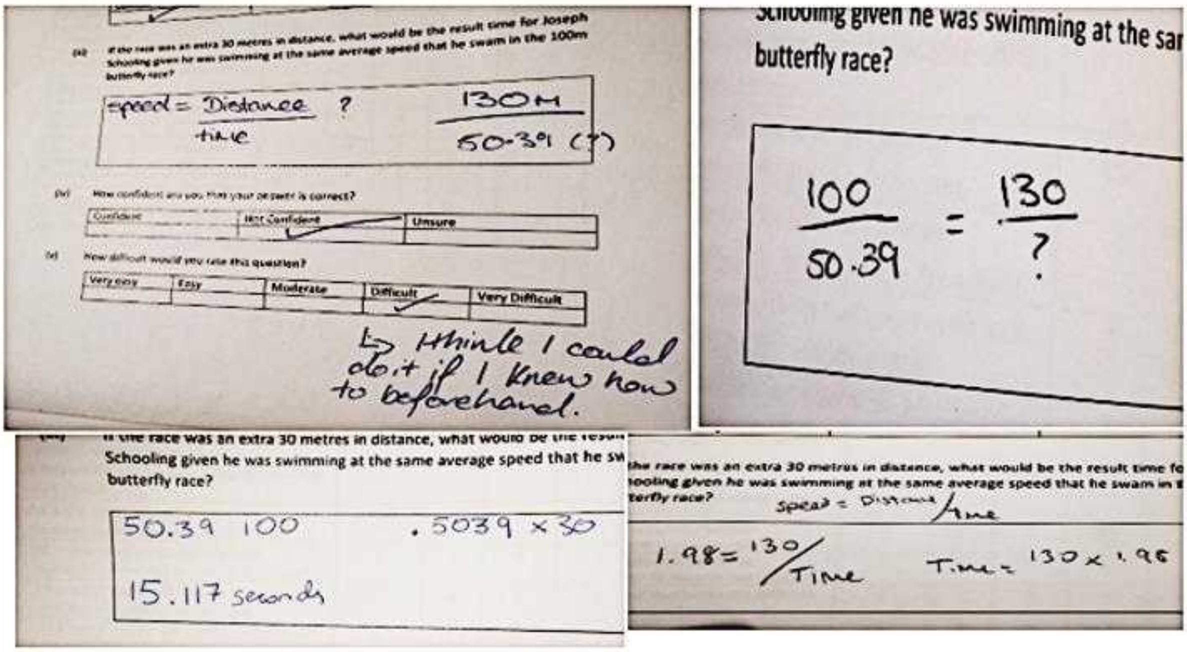


The second numeracy task presented in Question 1 was based on distance, average speed and time. This numeracy task asked pre-service teachers to calculate the result time of the butterfly race, if it was an extra 30 metres in distance. Here the pre-service teachers had to use the results table provided in the first part of the question to calculate the new finishing time for Joseph Schooling, given that he was swimming at the same average speed as when he completed the 100 metre butterfly race. Two possible correct responses to this numeracy task are presented in Figure 3.

Fewer pre-service teachers responded to this numeracy task in comparison to the first one, with only 167 (81.9%) completing this part. Of the 167 pre-service teachers who answered the Distance task, 101 (60.5%) answered it correctly. There were 10 (6%) pre-service teachers who had partially completed workings for this task. For example, some had calculated how many more seconds it would take for the extra 30 metres, but never added it to the time for 100 metres which can be seen in an example in Figure 3, while others recalled the formula to calculate the distance, average speed and time but didn't complete the task. One pre-service teacher stated that they would be able to complete the task if they knew how to beforehand.

Figure 3

Pre-Service Teachers Partial Working for the Distance Task



The next task was Salad Dressing and 177 (86.8%) pre-service teachers answered this task. This was a question adapted from PISA (2012) that involved applying their mathematical knowledge of ratio and proportion to make a salad dressing of 175millilitres, when given a recipe for 100 millilitres. Two-thirds of the pre-service teachers (118, 66.6%) who answered this question gave a correct response, answered correctly. Figure 4 shows the correct answer to this numeracy task.

Seven (4%) pre-service teachers stated that they understood what they were being asked to do but didn't understand how to answer the numeracy task. These pre-service teachers stated that the numeracy task was to do with "ratio", "fractions", and "working out the correct proportion", but stated they didn't know how to "get the answer" and also stated, "it would be easier if it was double the amount because you just double each portion". Additionally, there were 52 (29.4%) pre-service teachers who answered the task incorrectly. While there were two different types of incorrect answers, I was unable to decipher how they arrived at the answer provided. However, I did manage to categorise the answers and the different types are presented in Table 4, along with pre-service teachers' sample answers in Figure 5.

Figure 4

Sample Correct Answer to Salad Dressing Task

5. You are making your own salad dressing. Below is the recipe for 100 millilitres (mls) of salad dressing.

Salad oil	60ml
Vinegar	30ml
Soy sauce	10ml

(i) How many millilitres of salad oil do you need to make 175 mls of the salad dressing?

$$\frac{60mls}{100mls} \times 175mls = 105\,mls$$

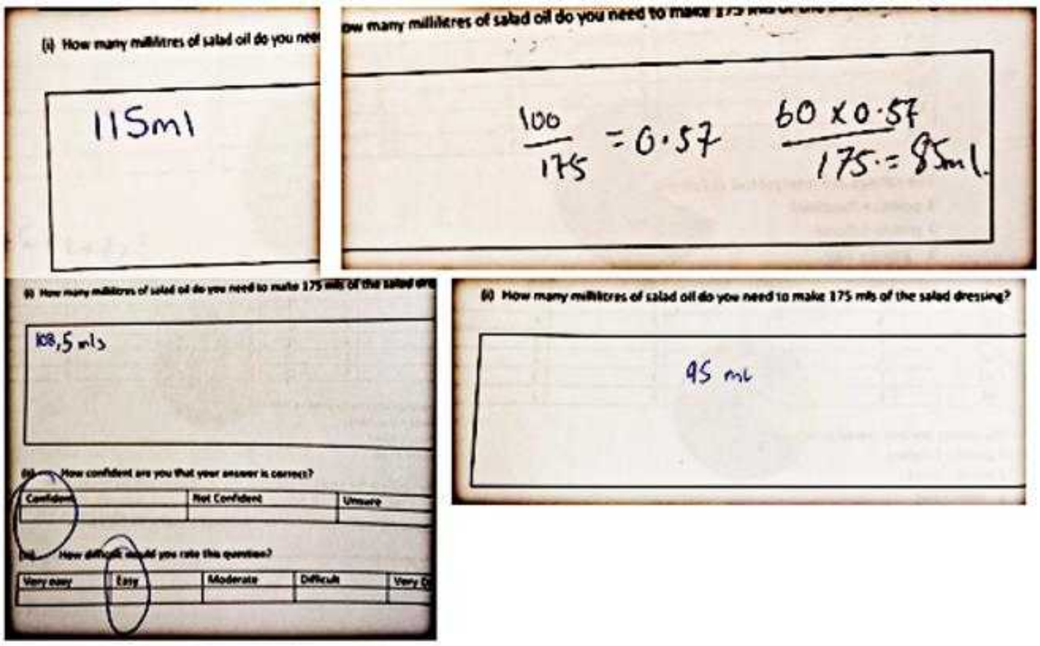
Table 4

Different Types of Incorrect Answers for the Salad Dressing Task

Types of answers	Description of misconception	*n* (%)
Type 1 (Random Number)	This type of answer saw pre-service teacher write down a random, incorrect number but they did not provide any evidence of mathematical workings	35 (19.8%)
Type 2 (Obscure working out)	This type of answer showed pre-service teachers demonstrate some mathematical workings but the mathematical reasoning did not make sense	17 (9.6%)

Figure 5

Examples of Type 1 and Type 2 Incorrect Answers to Numeracy Task



Discussion

Researchers in Australia have advocated that in order for teachers to teach for numeracy learning in the classroom, they must also be proficient in their own numeracy capabilities (Forgasz & Hall, 2020; Goos et al., 2019). The findings presented in this paper showed that overall, two-thirds of the pre-service teachers were able to answer four or more numeracy tasks correctly and approximately one-third of the pre-service teachers answered fewer than half of the numeracy tasks correctly. Nevertheless, it did not deter them from completing the tasks and it was obvious that the pre-service teachers demonstrated a high level of engagement by attempting the numeracy tasks. The error analysis revealing the struggles of pre-service teachers that are presented in this paper is informative, in that identifiable errors and common misconceptions in the numeracy tasks were ascertained. This is interesting to note as these pre-service teachers had come through an undergraduate degree but were presenting with common errors that students in primary school settings may also make.

Furthermore, these results also highlight the gaps in pre-service teachers' competency which need to be addressed. It is important for pre-service teachers to identify the common misconceptions students may have, with a specific focus on misconceptions relating to numeracy within their specific subject discipline. I argue that pre-service teachers could discuss and identify the different numeracy misconceptions that their students may encounter when teaching a certain topic. This would be in line with the current mandate from The Teaching Council of Ireland (2020), who have stated that pre-service teachers need to be given ample opportunity to develop their own personal numeracy skills which in turn will have an impact on the way in which they teach for numeracy learning.

Acknowledgements

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References

Department of Education and Skills [DES]. (2011). *Literacy and numeracy learning for life: The national strategy to improve literacy and numeracy among children and young people 2011–2020*. Dublin: DES. https://www.education.ie/en/Publications/Policy-Reports/lit_num_strategy_full.pdf

Department of Education and Skills [DES] (2015). *Framework for junior cycle 2015*. Dublin: DES. https://www.ncca.ie/media/3249/framework-for-junior-cycle-2015-en.pdf

Goos, M., & O'Sullivan, K. (2023). The evolution and uptake of numeracy and mathematical literacy as drivers for curriculum reform. In Y. Shimizu & R. Vithal (Eds.), *Mathematics curriculum reforms around the world: The 24th ICMI study* (pp. 345–358) Springer.

Forgasz, H., & Hall, J. (2019). Learning about numeracy: The impact of a compulsory unit on pre-service teachers' understandings and beliefs. *Australian Journal of Teacher Education, 44*(2). http://dx.doi.org/10.14221/ajte.2018v44n2.2

Hall, J., & Forgasz, H. (2020). Secondary pre-service teachers' experiences in a numeracy course. In N. Radakovic, & L. Jao (Eds.), *Borders in mathematics pre-service teacher education* (pp. 75–90). Springer. https://doi.org/10.1007/978-3-030-44292-7_4

Goos, M., Geiger, V., Dole, S., Forgasz, H., & Bennison, A. (2019). *Numeracy across the curriculum: Research-based strategies for enhancing teaching and learning*. Allen and Unwin.

Norwegian Directorate for Education and Training (2012). *Framework for basic skills* [Online]. Utdanningsdirektoratet (UDIR). https://www.udir.no/in-english/Framework-for-Basic-Skills/

Organisation for Economic Co-operation and Development. (2013). *PISA 2012 released mathematics items*. OECD. https://www.oecd.org/pisa/pisaproducts/pisa2012-2006-rel-items-maths-ENG.pdf

The Teaching Council of Ireland (2020). *Céim: Standards for initial teacher education*. https://www.teachingcouncil.ie/en/publications/ite-professional-accreditation/ceim-standards-for-initial-teacher-education.pdf

Venkat, H., & Winter, M. (2015). Boundary objects and boundary crossing for numeracy teaching. *ZDM Mathematics Education*, *47*, 575–586. https://doi.org/10.1007/s11858-015-0683-6

Examining Students' Mathematical Thinking: The Case of Porridge Words

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Mathematical thinking is a complex, multi-faceted process that has been described as messy and difficult but can also show growth and insights into conceptual understanding and knowing. This paper explores the application of Edward de Bono's practical thinking, in particular, the occurrence of porridge words to examine the mathematical thinking of primary school students. The research employed qualitative research techniques using participant's drawings, their written descriptions, and interviews about their drawings. Employing discourse analysis uncovered patterns in how students used porridge words to communicate their mathematical thinking.

In the realm of mathematics education, understanding and unravelling the intricacies of students' mathematical thinking is an enduring challenge. Mathematical thinking underpins students' abilities to problem solve in mathematics (Monteleone et al., 2023), and can lead to a more thorough understanding of mathematics (Stein et al., 1996). Further, mathematical thinking is a means to "describe mathematical growth" (Rasmussen et al., 2005, p. 52) and is a function of mathematical processes and operations (Burton, 1984). To achieve a deep level of understanding of mathematics, researchers suggest that students need to engage in the process of mathematical thinking (Schoenfeld, 1989; Stein et al., 1996). However, the thinking process is complex, and multi-faceted (Quane & Booth, 2023). Liljedahl (2021) describes mathematical thinking as "messy" requiring risk-taking (p. 72) is "difficult" (p. 87) and "is a necessary precursor to learning" (p. 5).

Despite the increased attention and acknowledgment of the important role mathematical thinking plays in mathematical understanding, there is little research examining how primary-aged children communicate their mathematical thinking and make their thinking visible to others. Research has investigated professional learning for teachers and pedagogical practices in the areas of mathematical thinking (Liljedahl, 2021; Sfard, 2008; Stein et al., 1996). Few studies have explored mathematical thinking from a student's perspective. No studies as far as the author is aware, have examined students' mathematical thinking and the use of porridge words.

Different scholars have theorised the thinking process. Edward de Bono, a proponent of teaching thinking and renowned for theorising thinking eloquently describes thinking as "simply a matter of moving from one idea to another" (de Bono, 1971, p. 55). The transition between ideas can be challenging, resulting in students using filler words phrases or emergent language to describe the transition. de Bono (1971) refers to these words or phrases as "porridge words" (p. 60) and uses the phrase in generalised contexts and not subject-specific contexts such as mathematics. This research investigated the research question:

- How do students use porridge words to communicate their mathematical thinking?

Conceptualisation of Porridge Words in Mathematics Education

Generalised language can inhibit and also enable thinking with de Bono (1971) using the term "porridge words" which, "allow us to make definite statements or ask definite questions when we do not really know what we are talking about". These vague blurred porridge words

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 423–430). Gold Coast: MERGA.

have an extremely important part to play in thinking (p. 61).

As such, “porridge words” help us make the connections to keep thinking moving and progressing “from one idea to another” (de Bono, 1971, p. 67). Without “porridge words” de Bono (1971) warns that thinking will cease providing no way to keep thinking moving. Further, “porridge words” can be a means to ask “vague questions” when a person is not yet familiar with a topic to help develop their thinking and understanding of a topic (de Bono, 1971, p. 67). Giving a name to different aspects of thinking allows for the identification of the different processes used when engaging in thinking. While de Bono (1971) used “porridge words” in generalised contexts, it can be argued that the process of using “porridge words” be applied to specific contexts and topics including mathematics education. In mathematics, the word ‘sum’ could be considered a porridge word, especially for children who use the word to mean ‘perform a mathematical operation’ such as multiplication. Here the word ‘sum’ is vague and does not accurately describe the processes involved in or the concept of multiplication.

However, de Bono (1971) warns that we often dismiss porridge words, especially in the thinking process. Yet, it is through such devices as “porridge words” that we can make a commitment to thinking, take action on our thinking, develop ideas through thinking, have parallel and intersecting thoughts, and abstract our thinking. de Bono (1971) posits that often we start with very general, non-specific thinking to establish more specific ideas and that there is a general attitude that we need to start with rigid, constrained, and specific thinking. The caveat of this attitude is that we are at risk of being “completely trapped by existing ideas” (p. 68). Porridge words in mathematics education, therefore, could be defined as *words or phrases that have an ambiguous meaning or may not describe the mathematical concept, process, or skill in a highly accurate manner using appropriate and accurate mathematical language*. That is, porridge words act as a device to facilitate mathematical thinking. In communicating mathematical thinking, even through the use of porridge words, students are engaging in mathematical discourse.

Mathematical Discourse and Discourse Analysis

Andreas (2011) argues that mathematical discourse differs from other forms of discourse due to the nature in which mathematics is communicated. Communicating mathematical thinking and understanding is multi-modal, involving various forms of communication and semiotic systems. These forms include but are not limited to words, symbols, graphs, drawings, and gestures (Andreas, 2011; Sfard, 2008). In communicating mathematics, andreas (2011) suggests students engage in both mathematical and generic discourses and explains that generic discourse may be relevant to the mathematical discourse that is occurring. In doing so, mathematical discourse is a dynamic and interactive process that can provide opportunities for students to explore, explain, and deepen their mathematical understanding.

To investigate the occurrence of porridge words in students’ mathematical thinking, discourse analysis was employed. Discourse analysis examines spoken or written texts to understand the ways words or phrases function in a particular context (Paltridge, 2008). Adopting a pragmatic approach, whereby understanding what students are saying rather than what the words or phrases mean in the “most literal sense” (Paltridge, 2008, p. 3) is used to understand the meaning behind what a student has said. In this way, the spoken text is analysed to identify porridge words and how they are used by students. In terms of mathematics, students may also have a “linguistic repertoire” (Paltridge, 2008, p. 29) that they use when communicating their mathematical understanding. A student’s linguistic repertoire then depends on the domain in which the mathematical language is used, such as the primary classroom and the interactions with others. Paltridge (2008) used the term “speech communities” to describe a group of people that interact with each other, using the same language that includes common geographical, cultural, age, and social factors (p. 28). In terms of social factors, Paltridge (2008) describes that the linguistic repertoire of a member will depend on who we are interacting with, the social context, the topic function and goal of the discussion or conversation as well as the formality of the setting and the status of each of the members. To understand the relationship between what is spoken and the meaning, pragmatics is employed as a means to study the context in which the spoken act has occurred. In this way, the context of the situation is crucial in interpreting and understanding what has been said.

Context

The study was conducted at a small inner-regional South Australian State School. The school had a total student population of 36 students separated into two classes. A junior class comprising Reception (first year of primary school), to year 2 inclusive. An upper class comprising of students in Years 3–6 inclusive. Flexible grouping was used for several learning areas including mathematics. The school is located in a small regional town, with an Index of Community Socio-Educational Advantage (ICSEA) value of 942 (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2023). ICSEA is calculated by adding a) the social education advantage, b) remoteness, and c) percentage of indigenous students. The social education advantage is determined by parental occupation, school education, and non-education levels. ACARA (2023) reports that the ICSEA measure provides the opportunity to compare Australian Schools where the median value is 1000 and the standard deviation is 100. The school has an ongoing strong community relationship, maintaining close connections with community groups and promoting the sharing of the school's facilities. The school population has experienced almost a 30% increase between 2020 to 2023 (ACARA, 2023).

Method

Children were withdrawn from class to complete a drawing and semi-structured interview. A drawing prompt was read to each child, outlining the requirement for children to "draw themselves doing mathematics" (Quane et al., 2021) with further instructions stating that children needed to include their face and that the focus of the drawing could be any aspect of mathematics. The drawing prompt guided students to show the mathematics that they were doing, thereby providing unique insights into how they communicate the mathematics depicted in their drawing. The same prompt was read to all children to ensure consistency. Children were prompted to write about their drawing and participated in a semi-structured interview that asked clarifying questions about what was drawn. To prompt students to communicate the mathematics that they had depicted in their drawing, students were asked "Can you explain the maths that you have drawn in your picture?". The semi-structured interviews were transcribed verbatim by the researcher including pauses, laughter, and filler words such as um, ah, er. The transcripts and audio recordings were reviewed numerous times for initial understanding to identify the occurrence of porridge words or phrases and the context in which the porridge words were used. Each interview transcript was uploaded to the NVivo 12 software, which provides "powerful processes of indexing, searching, and theorizing" (Creswell, 2012, p. 243). NVivo 12 was used to create nodes, cases, and case classifications to explore the data further in preparation for the coding process. The coding process identified words and phrases used by students that had an ambiguous meaning and therefore, constituted a porridge word or phrase.

Each drawing, written description, and transcribed interview were viewed initially as three separate pieces of data and then as a collection of work. Two main coding systems were then employed. First, a systematic analysis using the principle of atomism was used to examine each drawing. Once the drawing was examined at the atomic level, drawings were viewed holistically (Quane et al., 2019). The process of analysing at the atomic and holistic levels was repeated with the child's written description and interview responses. Second, the data generated from the three individual data collection techniques were analysed, they were combined to form a more holistic and comprehensive picture of children's mathematical thinking and, in particular, their use of porridge words. An inductive approach was used to identify porridge words within the generated data (Pezzica et al., 2016). To ensure that the data was consistently coded for the occurrence of porridge words, the researcher and an educator coded the data independently. Cohen's Kappa and the percentage of agreement were calculated using SPSS. Complete agreement between the two raters, that is 100% agreement (Fleiss et al., 2004), was achieved for the identification of porridge words.

Participants

An information letter and consent form were sent home via their child. The response rate 64%, with 13 male (57%) and 10 female (43%) students with all year levels except reception represented in the sample. Five students identified as Aboriginal or Torres Strait Islander. Table 1 shows the distribution of participating students by year level.

Table 1

Participant Numbers

Year level	1	2	3	4	5	6	Total
Male	3	3	0	1	2	4	13
Female	1	2	1	3	1	2	10

Findings

The first level of analysis was to identify the porridge words in the generated data. A total of 34 different porridge words or phrases were identified with 17 words being used once by different students (Table 2). D1 (Figure 1) used the most porridge words (n = 11), whereas there was a single child (Year 1, male) who did not use any porridge words. The most common porridge word was *math* or *maths* with 13 children ambiguously using the word, followed by the word plus or plusses (n = 6) and the phrase figured out (n = 4). At the surface level, some words classified as porridge words will appear to be used correctly. However, the examination of the intent of the use of these words, such as diameter, counting, and measure reveals that they are indeed used ambiguously.

Table 2

Occurrence and Frequency of Porridge Words

Porridge word	*n*	Porridge word	*n*	Porridge word	*n*
A blocks and B blocks	1	Figured out	4	Plus/plusses	6
Add	1	Getting answers	1	Practicing	1
Algebra	1	Groups	1	Problems	2
Big	1	Growing	1	Pop them/Put them/put some	2
Break	1	Make them	1	Shape	1
Break the numbers in Half	1	Making	1	Stuff	2
Building	1	Math/maths	13	Sums	3
Bunch/bunch of	2	Mental math	1	Take away	2
Counting	5	Measure	2	Times/timsing	2
Diameter	1	Numbers	4	Those/them	2
Equalling	2	Number maths	2	Thing	1
Find out	1			Working out	1

Note. *n* denotes the number of students.

The second level of analysis examined the intent and context in which the porridge words were used. Math or Maths were used as porridge words, particularly by younger students to provide a general description of what they had depicted in their drawing. For example, D5 (year 3) states “um, ah, I’m doing maths and I’d doing plusses” and similarly D7 (Figure 2) states she is “doing my maths” and later adds “I’m doing sums”. D7 uses the word “sums” as a porridge word to describe “the sums are two times eight and twelve times nine”.

“Figure out” was used to describe the process of solving a problem, deriving an answer, or identifying and classifying an object. For example, D21 (Figure 6) used the phrase “figure out” to describe her process of identifying and classifying the type of angle she depicted in her drawing, “Ah, just sitting, (pause), like just looking trying to figure out the best way, trying to concentrate.” D10 used the phrase “figure out” to describe the process of determining the next number in a number pattern increasing by fives “it’s um, where you, um (pause) try to figure out what’s the answer and not guessing”. D6 used the phrase multiple times to “figure out equations” and to describe an authentic situation of determining the number of carpet tiles required for a games room:

> Well we are finished now, but when we were doing the games room I was helping my dad built it and we had to figure out um cause we were putting carpet tiles down on the floor because I was getting my cast off and I was sad that I had to miss a day of school so he said I could do the maths and figure out cause they were a metre square each and I had to figure out the dia (sic), the diameter of the room and figure out how many boxes we needed of carpet tiles.

In all instances, students hesitated before using the phrase “figure out” by either pausing or using filler words such as ‘um’ or ‘ah’. As such the hesitancy may be an indication of students attempting to process their thoughts and communicate their understanding clearly but stumble, unable to find the most appropriate mathematical language. This is also the case in D6’s use of the word diameter to describe the dimensions of a rectangular room.

There were varying degrees of invisible thinking from students from not being able to articulate their thinking “I can’t really describe it” (D8, Figure 3) or “I don’t really know how to explain it” (D21), to students who gave an indication that they were thinking but providing little or no description or explanation. For example, “I did it in my head and I also know my two times tables pretty well” (D7). Porridge words were further categorised as either having a conceptual intention or a process intention. Invisible thinking also manifested in students’ partial descriptions of the mathematics they had shown in their drawing. For example, D17 (Figure 4) provided the following description of adding decimals:

> There was these, thousandths, hundredths, and tenths, and then there were these ones and tens. We had to pop them into (pause) add the numbers up and then after over 10 we had to put the one over to the next.

In D17’s description, we see how important language plays in describing and explaining mathematical thinking. In attempting to describe his thinking, D17 has used several porridge words (“pop them into” and “put the one over to the next”) to describe what he has done. The use of porridge words broadly describes the processes used, but D17 has yet to provide a clear conceptual explanation of the mathematics he has shown, rendering part of his thinking invisible. Continuing with porridge words to describe processes, D20 (Figure 5) uses several porridge words and phrases to describe her recollection of grouping and regrouping numbers:

> So, if the teacher said if you have 200 and 20 tens and twenty fives then you had to make them take away some and put them in the tens and put some of the tens in the tens, I mean ones into the tens.

In examining, D20’s use of porridge words, we can see that she is attempting to describe the place value of numbers as well as attempting to explain the process of regrouping. Here the intention of the phrases “make them”, “take away some”, “put them” and “put some” may be initially unclear. However, examining D20’s drawing we can see that she had depicted a range of Multiple Attribute Blocks, depicting these blocks as units (ones), longs (tens), and flats (hundreds). It appears that the porridge words used by D20 are evidence of early emergent informal mathematical language.

Figures 1–6

Student Drawings: Draw Yourself Doing Mathematics, Write About Your Drawing

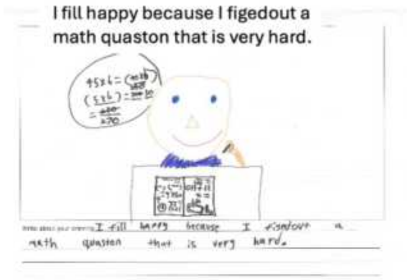

Figure 1: D1, Year 5, Male

Figure 2: D7, Year 4, Female

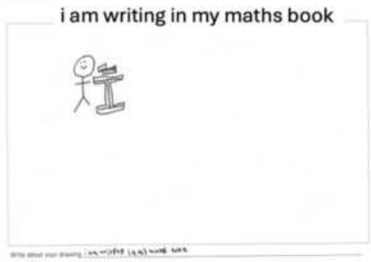

Figure 3: D8, Year 6, Male

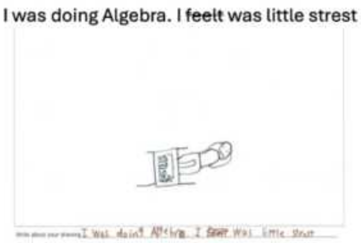

Figure 4: D17, Year 6, Male

Figure 5: D20, Year 4, Female

Figure 6: D21, Year 6, Female

D1's drawing (Figure 1), written description and interview responses show many examples of porridge words. D1 provides a partial explanation of the mathematics he has depicted.

Interviewer: Can you tell me about your picture

D1: My drawing is me working on a maths question that I didn't understand that was in the mental math book before but now that I have understood it. When I first got the question the first week that we had maths I got a question like 37 times 5 and I had no idea how to do that but then I figured out that I needed to break the numbers in half, I had to break the 30 away from the 7.

Interviewer: Yes

D1: So, I have 30 and I have 7. The number which would either be 5 times that number

Interviewer: Yes

D1: So then 5 times 30 would be, I would take off the 0 then it would be 5 times 3, 15 then I would add the 0 on which would be 150 I would write that down in the first box. Then I'd do the unit which would be 7 times 5 then I would add whatever that number is to the other number to get the answer

Interviewer: And how do you feel about that strategy?

D1: I feel it is a very good strategy to figure out big numbers times a little number

First, D1 was able to describe the mathematical procedure that he used to multiply a two-digit number by a single-digit number. In doing so, D1 used several porridge words including "break", "take off", "figured out", "box", "little numbers, and "big numbers". D1 used "break" to describe the procedure he has used in two different ways. Initially, D1 described that he needed to "break the numbers in half" and started to explain how he did this by adding "I had to break the 30 away from the 7". In attempting to explain how he partitioned the number 37, D1 used the word "half" to indicate partitioning the numerical value of the tens and units, which contradicts the true meaning of half. Further probing would have provided the opportunity to clarify D1's understanding of partitioning. D1 then launches into describing further how he multiplied the partitioned numbers by five, again using a series of porridge words including "I would take off the 0" and then returning to his description "I would add the 0 on". In this description, we gain insights into the procedure that D1 has applied, and we are starting to see a conceptual explanation of the mathematics in terms of D1 reference to "(t)hen I'd do the unit". It appears that the use of porridge words is helping D1 describe the procedure and providing an opportunity to develop the conceptual understanding to explain his thinking.

Discussion

Communicating mathematical thinking is multifaceted and the results from this study have shown that students are developing their mathematical linguistic repertoire via the use of porridge words. Most students (n = 22, 96%) engaged in invisible thinking either in their drawing, written description, or discussing their drawing as a means to attempt to communicate their mathematical thinking. Further analysis of the occurrence of invisible thinking revealed that students use invisible thinking in several ways. Returning to the work of de Bono (1971) helps us make sense of possible reasons why students communicate mathematical thinking that is invisible. First, thinking may be "intermediate impossible" which de Bono (1971, p. 139) describes as an idea that is not right but acts as a transitional point to another idea that is right. D17 could be considered an example of "intermediate impossible" where he describes his thinking using emergent language. It appears that many students knew that they were not using the most appropriate mathematical term but found familiar words to continue their descriptions or explanations of the mathematics that they depicted.

Second, ideation and conceptual understanding forms slowly over time as new knowledge is acquired. The data analysis suggests that the use of porridge words which may result in invisible thinking is a sign that students are hunting for mathematical clues to help them explain their mathematical thinking to others. de Bono (1971) classified clues into three types: (1) Clues that are obvious to everyone—but may still be misinterpreted; (2) Features that are obvious to everyone but do not become clues unless some significance is attached to them; and (3) Clues that are not at all obvious and have to be worked on (pp. 170–171).

de Bono (1971) describes clues have three distinct purposes. First, clues can be used "to suggest ideas" to help understand unfamiliar concepts by generating ideas, and engaging in noticing to focus on elements or features to understand the significance. Second, clues can "confirm ideas" to determine whether an idea fits with the schema that is developing and as such can be situational, or memory-based. Third, clues can be used to "exclude ideas" by eliminating possibilities that prove that thinking does not fit the situation, concept, or that thinking may need modifying (pp. 170–171). In this way porridge words "can serve as a starting point for a new line of thought" and in doing so, can provide insights into both conceptual understanding and possible alternate conceptions.

Conclusion

Communicating mathematical thinking is a core part of 'doing mathematics' providing insights into students' conceptual understanding of mathematics. However, the processes involved in communicating mathematical thinking can often be invisible. This paper reported a small-scale qualitative study examining primary students' mathematical thinking. The use of children's drawings, written descriptions, and interview responses provided illustrative case studies that moved beyond the identification of the aspects of mathematical thinking to understanding how mathematical thinking is enacted in the primary years. The study was situated in a small regional South Australian school which delimits the study to a very specific context. Studying students at a single school enriched the identification and classification of students' mathematical thinking across the primary years of schooling. The use of de Bono's (1971) porridge words may resonate with a wider context and prove to be invaluable in capturing students' mathematical thinking. An aim of applying porridge words to students' mathematical thinking was to make an ambiguous aspect of mathematical practice transparent and relatable. Further research into how why students use porridge words is warranted in a larger variety of contexts. Having participants review their explanations and their use of porridge words would yield further information about the use of porridge words. Additionally, research that explores how teachers use porridge words and how they address the occurrence of porridge words in a class context is recommended.

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References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2023). *My School*. ACARA. https://www.myschool.edu.au/

Andreas, R. (2011). Discourse research in mathematics education: A critical evaluation of 108 journal articles. *Journal for Research in Mathematics Education*, *42*(2), 167–199. https://doi.org/10.5951/jresematheduc.42.2.0167

Burton, L. (1984). Mathematical thinking: The struggle for meaning. *Journal for Research in Mathematics Education*, *15*(1), 35–49. https://doi.org/10.5951/jresematheduc.15.1.0035

Creswell, J. (2012). *Educational research planning, conducting, and evaluating quantitative and qualitative research* (fourth ed.). Pearson.

de Bono, E. (1971). *Practical thinking: 4 ways to be right; 5 ways to be wrong; 5 ways to understand.* Penguin Books Ltd.

Fleiss, J. L., Levin, B., & Paik, M. C. (2004). The measurement of interrater agreement. In W. A. Shewart, S. Wilks, J. Fleiss, B. Levin, & M. Paik (Eds.), *Statistical methods for rates and proportions* (pp. 598–626). https://doi.org/doi:10.1002/0471445428.ch18

Liljedahl, P. (2021). *Building thinking classrooms in mathematics: Grades K–12: 14 teaching practices for enhancing learning*. Corwin: A SAGE company.

Monteleone, C., Miller, J., & Warren, E. (2023). Conceptualising critical mathematical thinking in young students. *Mathematics Education Research Journal*, *35*(2), 339–359. https://doi.org/10.1007/s13394-023-00445-1

Paltridge, B. (2008). *Discourse analysis*. Continuum.

Pezzica, S., Pinto, G., Bigozzi, L., & Vezzani, C. (2016). Where is my attention? Children’s metaknowledge expressed through drawings. *Educational Psychology*, *36*(4), 616–637. https://doi.org/10.1080/01443410.2014.1003035

Quane, K., & Booth, H. (2023). Unpacking mathematical thinking: The case of describing and explaining thinking. *Australian Primary Mathematics Classroom*, *28*(2), 21–28. https://search.informit.org/doi/10.3316/informit.459869009869818

Quane, K., Chinnappan, M., & Trenholm, S. (2019). The nature of young children’s attitudes towards mathematics. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research: Impacting practice. Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia* (pp. 108–111), Perth, Australia.

Quane, K., Chinnappan, M., & Trenholm, S. (2021). Draw yourself doing mathematics: Developing an analytical tool to investigate the nature of young children’s attitudes towards mathematics. *Mathematics Education Research Journal*, *35*, 445–473. https://doi.org/10.1007/s13394-021-00399-2

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 51–73. https://doi.org/10.1207/s15327833mtl0701_4.

Schoenfeld, A. (1989). Mathematical thinking and problem solving. *SERBIULA (sistema Librum 2.0)*.

Sfard, A. (2008). *Thinking as communicating human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455–488. https://doi.org/10.3102/00028312033002455.

Flourishing Mathematics Teachers: The Effect of School-Based Placements on Preservice Secondary Mathematics Teachers Anticipated Job Enjoyment

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This paper reports on part of a longitudinal project investigating the formation of preservice secondary mathematics teachers' identities at one Australian university. Given that school-based placement experiences impact teacher identity development, the Flourishing Mathematics Teacher (FMT) project focuses on this experience and aims to identify features of placement experiences that enable beginning secondary mathematics teachers to flourish, and remain, in their teaching careers. In this paper, we report on preliminary data exploring changes to 23 preservice secondary mathematics teachers anticipated job enjoyment after their first teaching placement.

The recruitment and retention of teachers are pressing issues both nationally (Australian Institute for Teaching and School Leadership [AITSL], 2021) and internationally (Organisation for Economic Co-operation and Development [OECD], 2020), with shortages especially pronounced in secondary mathematics (Shah et al., 2022). Despite notable financial investment in teacher supply initiatives in Australia (e.g., NSW Government, 2021), teacher shortages remain. Currently, there is a noticeable lack of funding directed towards initiatives aimed at retaining beginning mathematics teachers, despite significant early career attrition (Wyatt & O'Neill, 2021). Retaining quality teachers is vital to Australia's ability to deliver quality mathematics education to all students. The Flourishing Mathematics Teacher (FMT) project addresses an important gap in beginning teacher retention interventions by investigating the conditions conducive to the success and 'flourishing' of beginning secondary mathematics teachers as they navigate through their initial teacher education and embark on their teaching career.

Teacher Identity

Teacher identity is multidimensional and encompasses cognitive and affective elements that influence how teachers view themselves and how they are viewed by others (Kaasila et al., 2012). Teacher identity is constantly formed and re-formed due to complex interactions between the personal, professional, and political (Mockler, 2011). It is shaped by career motivations and goals, and reflects the degree to which one believes they ascribe to being a member of the profession (Richardson & Watt, 2018). Given the fluid nature of identity, teacher identity can be influenced by the experiences they have during initial teacher education (Bobis et al., 2020), making this time crucial in setting future teachers up for success in their career. In this study, we consider teacher identity as "teachers' perceptions of their cognitive knowledge, sense of agency, self-awareness, voice, confidence, and relationship with colleagues, pupils and parents" (Izadinia, 2018, p. 109).

We coin the term 'flourishing mathematics teacher identity' and choose the verb 'flourish' to elicit the literal meaning of the word—"grow or develop in a healthy or vigorous way, especially as the result of a particularly congenial environment" (Oxford English Dictionary, 2023). To define a 'flourishing mathematics teacher identity', we consider motivational and contextual factors that enable the development of positive mathematics teacher identities. First, considering motivational factors, expectancy-value theory describes how career choices are informed by teachers' beliefs, expectancies for success, and values (Eccles & Wigfield, 2002;

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 431–438). Gold Coast: MERGA.

Richardson & Watt, 2018). The "FIT-choice" framework (Factors Influencing Teaching Choice) proposed that motivators to teach include intrinsic values, utility values, and attainment values (Watt & Richardson, 2024). Intrinsic value pertains to enjoyment or interest, utility value can relate to personal utility (e.g., job security) or social utility (e.g., a desire to make a meaningful difference), and attainment value concerns one's perceived teaching abilities (Wigfield & Eccles, 2000). Teacher self-efficacy and social-utility motivations have been empirically linked to job satisfaction (Zakariya & Wardat, 2023). In addition to these factors, innovativeness and creativity in teaching has also been linked to teacher job satisfaction (Blömeke et al., 2021; Uçar, 2022). Therefore, for preservice teachers, motivational factors that may impact their flourishing teacher identity include their self-efficacy, enjoyment, interest, and ability to be innovative and creative in their anticipated career. A growth-mindset is also pertinent to flourishing as a teacher, given that a growth mindset can support beginning teachers' resilience (Gero, 2013) and such mindsets have been positively linked to career motivation and engagement for teachers (Nalipay et al., 2021). Second, considering contextual factors and drawing on self-determination theory, a teachers' intrinsic career motivation can be supported if they feel a sense of autonomy, relatedness, and competence in their anticipated work environment (Ryan & Deci, 2017). Relatedness in particular aligns with the concept of the social identities of teachers, which pertain to interpersonal relationships and group membership (Richardson & Watt, 2018).

A core aim of the wider research project is aimed at the retention of early career mathematics teachers. Previous research indicated that job satisfaction is closely associated with early career teacher retention (Kelly et al., 2019). In addition to satisfaction, Sullivan et al. (2021) found that the enjoyment teachers got out of teaching others was vital to their commitment to the profession. Teachers who exhibit lower levels of satisfaction and commitment to teaching are more likely to leave the profession (Skaalvik & Skaalvik, 2016). The FIT-Choice project affirmed that intrinsic values predicted professional engagement (planned persistence) at the end of initial teacher education when transitioning to the workforce (Watt & Richardson, 2024).

The Flourishing Mathematics Teachers Project

The FMT project measured preservice secondary mathematics teachers' ratings for the flourishing identity factors enjoyment, interest, innovation, and autonomy, and will track how these change and reasons for changes throughout their initial teacher education program and their first years of teaching. The project will conclude with interventions to support preservice teacher flourishing, whether that be through interventions at the tertiary level or interventions during school-based placements. In this paper, preliminary data on preservice teachers anticipated job enjoyment is focused on and we report how school-based placement experiences influenced one cohort of preservice teachers' anticipated job enjoyment. Analysis of preservice teachers anticipated job enjoyment is important as research has identified that teachers who enjoy teaching mathematics are more likely to spend time teaching mathematics, have higher self-efficacy, and reduced teacher burnout (Russo et al., 2021). Though the concept of a flourishing identity has been theoretically explained, the FMT project also confirmed the extent to which preservice teachers value each of the examined factors, and this will also be discussed in this paper. The research questions that will be answered in this paper are:

- How does preservice secondary mathematics teachers' anticipated job enjoyment change as a result of placement?
- What experiences during placement influence preservice secondary mathematics teachers' anticipated job enjoyment?
- To what extent do preservice secondary mathematics teachers value each of the flourishing identity factors (i.e., self-efficacy, enjoyment, interest, and innovation)?

Methodology

Participants were undergraduate and postgraduate preservice secondary mathematics teachers from one Australian university. There were 54 students in the entire cohort, and 23 students form the final sample as they had completed their first school-based placement in secondary mathematics (the others completed a placement in their other teaching area and had not yet completed a placement in secondary mathematics).

Participants were administered a modified version of Frenzel et al.'s (2009) Teacher Enjoyment of Teaching Mathematics (TETM) questionnaire prior to and after their first school-based placement. The TETM was utilised to answer the first research question and ascertain how placement experiences influenced participants anticipated enjoyment in teaching mathematics. The questionnaire had 5-items, and items were modified to ask participants to reflect on their anticipated career enjoyment, given they were preservice teachers. Each of the TETM items was answered using a 6-point Likert scale with each being scored a value from 1 to 6 (strongly disagree = 1 to strongly agree = 6). Therefore, the minimum possible mean score across all items on the TETM was 1 and the maximum was 6.

Prior to placement, participants had completed one unit of study focused on teaching mathematics and had not yet undergone any formal professional experiences in schools. The school-based placements constituted a four-week teaching experience where participants engaged in observing and teaching secondary mathematics under the supervision of an in-service teacher. To answer research question two and understand reasons for any changes found on the TETM questionnaire after placement, qualitative questions were added to the post-placement survey. These questions were "what experiences on placement, if any, have changed your interest in teaching mathematics?" and "what did you enjoy the most on placement?"

In addition to measuring participants anticipated enjoyment, their values were explored. Using the framework for flourishing teacher identity, participants were asked pre- and post-placement to rate the importance of enjoying, having autonomy, being interested, and having flexibility to be innovative in their job. This allowed the third research question to be answered. Each of the values were rated on the same 6-point scale as the TETM and were scored in the same way.

Findings

Pre-Placement

Table 1 shows the descriptive statistics for each of the TETM items.

Table 1

*Descriptive Statistics for the Pre-Placement TETM Questionnaire (*n *= 23)*

Question	Mean	SD	σ^2
I anticipate that I will really enjoy teaching mathematics	4.83	1.07	1.10
I anticipate that I will look forward to mathematics lessons	4.88	0.92	0.81
I anticipate that teaching mathematics is so enjoyable that I will like preparing and planning my lessons	4.45	1.12	1.20
I anticipate that when teaching mathematics, I will be good-humored	4.83	0.88	0.75
I anticipate that teaching mathematics will give me many reasons to be pleased	4.73	1.05	1.06

Participants responses on the pre-placement TETM questionnaire had an overall mean score of 4.74 (SD = 1.02), demonstrating that the cohort anticipated that they would find enjoyment in teaching mathematics. This mean is indicative of high anticipated enjoyment when compared to participants in Frenzel et al.'s (2009) study with 71 German teachers (Mean = 3.69, SD = 0.68).

To explore the significance of the results outlined in Table 1, participants' ratings for the importance of each of the flourishing teacher identity factors in their prospective workplace illuminate the most important values for this cohort (Table 2). Of the explored factors, enjoyment was the highest rated followed by interest. Participants rated the importance of enjoying their job, on average, higher than their anticipated enjoyment.

Table 2

*Descriptive Statistics for Values Questionnaire Items (Prior to Placement),(*n *= 23)*

Question	Mean	SD	σ^2
Enjoying my job is important to me.	5.40	0.78	0.59
Having autonomy in my job is important to me.	5.00	0.90	0.78
Being interested in my job is important to me.	5.34	0.71	0.49
Being creative in my job is important to me.	5.04	0.93	0.83

Though the cohort's overall TETM mean indicated high anticipated enjoyment, analysis of individuals responses revealed variation among the cohort. Some participants, prior to embarking on a school-based placement, reported low anticipated enjoyment of their job on the TETM. To explore this further, participants with a mean overall score of 4 or below (n = 4) on the pre-placement TETM survey were identified and their responses to the career values importance questions were analysed. Of these four participants, two rated "enjoying my job is important to me" as strongly agree and one rated this question as agree. The combination of low anticipated enjoyment and agreeing that enjoyment is important categorises these four participants in an "at-risk" category for not flourishing in their career due to potentially low enjoyment, which could indicate potentially lower levels of satisfaction with a teaching career and a greater likelihood of leaving the profession (Skaalvik & Skaalvik, 2016).

Table 3

*Descriptive Statistics for the Post-Placement TETM Questionnaire (*n *= 23)*

Question	Mean	+/-	SD	+/-	Var	+/-
I anticipate that I will really enjoy teaching mathematics	4.86	+0.03	1.18	+0.11	1.33	+0.23
I anticipate that I will look forward to mathematics lessons	4.73	-0.14	0.96	+0.04	0.89	+0.08
I anticipate that teaching mathematics is so enjoyable that I will like preparing and planning my lessons	4.30	-0.15	1.18	+0.06	1.34	+0.14
I anticipate that when teaching mathematics, I will be good-humoured	4.79	-0.04	1.04	+0.16	1.05	+0.3
I anticipate that teaching mathematics will give me many reasons to be pleased	4.97	+0.24	0.98	-0.07	0.92	-0.14

Note. **+/-** denotes the difference from pre-placement questionnaire.

Post-Placement

Table 3 shows the descriptive statistics for the post-placement TETM, and the differences in the pre- and post-placement item means. The overall mean score of the post-placement TETM was 4.73 (SD = 1.07), demonstrating little overall change in anticipated enjoyment.

The question where participants were asked to assess whether teaching mathematics would give them many reasons to be pleased scored higher in the post-placement questionnaire. This higher score suggests that the placement experience had given participants more cause to anticipate that their future mathematics teaching careers would be satisfying. A further, qualitative question added to the post-placement questionnaire asked participants about the most enjoyable aspect of their placement. It was striking that 87% of responses (20 out of 23) mentioned an element of student interaction as the most enjoyable aspect of their placement, with individual participants saying they enjoyed "connecting with the students, and seeing them become more motivated" (participant 1), "seeing the students actually learned something" (participant 2), and "the interaction and relationships I got to build with the students" (participant 3). The ability to relate to and interact with actual students is a unique component of school-based placements, which cannot be easily replicated at university. The ability to interact with students seemed to be the feature of placement that preservice mathematics teachers found most enjoyable and rewarding.

Post-placement, participants were asked again to rate the flourishing teacher identity values. Each identity value mean score was the same pre- and post-placement apart from innovation, which scored lower post-placement (change in mean = −0.3). Of the flourishing identity factors, enjoyment was still rated highest, further affirming the need to scrutinise anticipated enjoyment for potential teacher flourishing.

When analysing individual responses on the TETM, eight students reported no difference in their anticipated enjoyment pre/post placement. Seven students reported increased anticipated enjoyment post-placement and cited reasons noted above about the rewarding nature of interacting with students. Concerningly, eight students reported decreased anticipated enjoyment post-placement. Qualitative responses from these students indicated that their placement experience had demonstrated to them that teaching was a more challenging profession than originally anticipated. Encapsulating this, one participant stated, "I've realised that the reality of teaching is a lot more difficult than I expected it to be" (participant 4). Supporting this finding was the quantitative data from the entire cohort, where a decline in anticipated enjoyment was observed for the TETM items relating to planning and teaching lessons. This suggests that the realities of planning and teaching impacted preservice teachers' anticipated job enjoyment. One student who reported decreases in anticipated enjoyment reported that difficulties with their supervising teacher made the placement experience less enjoyable than hoped, "my supervising teacher constantly correct me in front of students and peers, damaging my confidence … I now am unsure about a career in teaching" (participant 5).

Of the four students who were identified as "at-risk" pre-placement, two reported decreases in anticipated enjoyment, one reported no change, and one reported an increase but did not report an overall TETM score above 4 (classing them as "at risk" still). No students moved into the "at risk" category in the post-placement TETM. Given that all students who were originally identified as "at risk" remained so after placement potentially demonstrates that the TETM items can be a useful measure of determining students' risk status prior to teaching placements. It also indicates that a typical placement with no additional intervention is not effective in increasing their anticipated job enjoyment and can potentially be detrimental.

Discussion

The results of the TETM questionnaire indicate that the placement experience overall did not have a substantial effect on the cohort's anticipated enjoyment in teaching mathematics. Given that the pre-placement questionnaire indicated that many students anticipated that they would enjoy teaching mathematics, these results suggest that placements have overall not materially dampened or enhanced their predicted enjoyment in teaching mathematics. This may be a positive finding for a cohort who already reported high anticipated enjoyment, however it is somewhat surprising that there was no meaningful increase (even small) in anticipated enjoyment after a practical experience in a school. Whilst the cohort reported high anticipated enjoyment as a whole, variations among individuals are worth noting as some participants did report decreased anticipated enjoyment or consistently low anticipated enjoyment.

As evidenced in the minimal change in the TETM overall mean after placement, it seems plausible to suggest that most participants had a good understanding of what to expect on the placement experience. The small shifts seen in individual TETM items suggest that placement had shown them that teaching is a challenging profession that places significant demands on teachers. However, for this cohort, placement also affirmed that the profession is enjoyable and working with students is rewarding. The qualitative responses of participants suggest that the strongest influence on preservice teacher enjoyment was that that they were able to teach actual students. For these participants, this placement was their first opportunity to teach students in the classroom. The placement experience gave them a sense of relatedness with students, which has been showed to improve career motivation (Ryan & Deci, 2017). In addition, making connections with students should serve to help participants value the profession by improving the social utility of their work in schools, where they can see the fruits of their teaching (Wigfield & Eccles, 2000; Zakariya & Wardat, 2023).

Of the flourishing identity factors, it was clear that enjoying their job was important to this cohort. After enjoyment, interest was scored highest in the values questionnaire. People who do what they like enjoy their work and find it interesting (Benz & Frey, 2008). The data from this survey suggests that the participants want to do what they like. Therefore, for these preservice teachers to flourish it is important that they view mathematics teaching as what they want to do professionally. The data from the TETM suggests that most participants are doing what they want, but the variation in overall TETM scores among individuals indicates that some are not yet convinced that mathematics teaching is what they want to do. Therefore, where the TETM questionnaire appears to be most valuable in supporting beginning teachers to flourish is analysing their individual scores rather than only observing cohort means.

At an individual level, several students were identified as "at risk" prior to placement and this worsened or did not change after placement. These "at risk" students reported that they did not anticipate they would enjoy a career teaching despite reporting that it is important to them that they enjoy their work teaching. Potentially the TETM is a valuable tool to evaluate at risk status, as intervention is likely needed for these teachers as the typical placement experience was not sufficient to address their low anticipated enjoyment. For these at-risk students, frequent mention of "the realities of teaching" seem to indicate that some had mismatched expectations of the placement experience. A potential intervention to address this issue could occur at the university level, where we consider how well teacher training is preparing students for "the realities" of the classroom. Alternatively, intervention could occur at the placement level where supervising teachers consider strategies that better induct preservice teachers to classroom teaching, allowing for a smoother transition from university learning to professional practice. Comments from "at risk" students highlight the detrimental impact of negative interactions with supervising teachers, highlighting the importance of careful selection of supervising teachers. To further aid in design future interventions, the broader Flourishing Mathematics Teachers study will conduct case studies with preservice mathematics teacher's during their placement experiences to better understand where and why this mismatch in expectations occurs.

Conclusion

The data presented in this preliminary study on the what preservice teachers valued in their future work environment, and the impact of school-based placements showed that enjoyment and interest are highly valued factors for preservice secondary mathematics teachers. The participants anticipated that they would largely enjoy teaching mathematics, and this is a positive finding given that enjoyment has been proposed as crucial to flourishing as a mathematics teacher. Placement experiences did not affect the anticipated enjoyment of the cohort overall. Responses to qualitative survey questions indicated that placement had shown these preservice teachers that teaching was a challenging but rewarding career. This was evident in the enjoyment they derived from connecting with students and seeing their knowledge grow. The social identity (Richardson & Watt, 2018) that preservice teachers developed during placement due to interactions with students seemed to be most meaningful for developing a flourishing teacher identity. For participants who did report low anticipated job enjoyment, this finding was of concern given they rated enjoyment as important. Reasons for low anticipated enjoyment centred on a mismatch of expectations when faced with the realities of the classroom, and negative interactions with supervising teachers. It is worthwhile conducting further studies to ascertain whether "at risk" students remained so because they lacked the right type of experiences to develop the same positive social identity as others, or if those experiences were not sufficient to overcome the mismatch in expectations and negative experiences with supervising teachers.

The project from which this paper is drawn will conduct further research into preservice teacher placement experiences by carrying out multiple case studies to develop a deeper understanding of the factors that determine perceived enjoyment in teaching, along with the other flourishing teacher factors. These case studies will help to further illuminate the impact of initial placement experiences on the preservice teachers' perceptions of the teaching profession and how those perceptions change through engaging in classroom practice. Completing the remaining stages of the broader study will allow for deeper analysis and identification of supports that assist the flourishing of secondary mathematics teachers as they enter the teaching profession.

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References

Australian Institute for Teaching and School Leadership (AITSL). (2021). *Teaching futures: Background paper*. AITSL. https://www.aitsl.edu.au/teachingfutures

Benz, M., & Frey, B. S. (2008). The value of doing what you like: Evidence from the self-employed in 23 countries. *Journal of Economic Behavior & Organization*, *68(3–4),* 445–455. doi:https://doi.org/10.1016/j.jebo.2006.10.014

Blömeke, S., Nilsen, T., & Scherer, R. (2021). School innovativeness is associated with enhanced teacher collaboration, innovative classroom practices, and job satisfaction. *Journal of Educational Psychology*, *113(8),* 1645.

Bobis, J., Khosronejad, M., Way, J., & Anderson, J. (2020). "Sage on the stage" or "meddler in the middle": Shifting mathematics teachers' identities to support student engagement. *Journal of Mathematics Teacher Education*, *23(6),* 615–632.

Eccles, J. S., & Wigfield, A. (2002). Motivational beliefs, values, and goals. *Annual review of Psychology*, *53(1),* 109–132.

Frenzel, A. C., Goetz, T., Lüdtke, O., Pekrun, R., & Sutton, R. E. (2009). Emotional transmission in the classroom: Exploring the relationship between teacher and student enjoyment. *Journal of Educational Psychology*, *101*(3), 705–716. https://doi.org/10.1037/a0014695.

Gero, G. P. (2013). *What drives teachers to improve? The role of teacher mindset in professional learning* [Doctoral dissertation]. The Claremont Graduate University.

Izadinia, M. (2018). Mentor teachers. Contributions to the development of preservice teachers' identity. In P. Schutz, J. Hong, & D. Cross Francis (Eds.), *Research on teacher identity*. Springer, Cham. https://doi.org/10.1007/978-3-319-93836-3_10

Kaasila, R., Hannula, M.S., & Laine, A. (2012). "My personal relationship towards mathematics has necessarily not changed but …" Analysing preservice teachers' mathematical identity talk. *International Journal of Science and Mathematics Education*, *10,* 975–995.

Kelly, N., Cespedes, M., Clarà, M., & Danaher, P. A. (2019). Early career teachers' intentions to leave the profession: The complex relationships among preservice education, early career support, and job satisfaction. *Australian Journal of Teacher Education, 44*(3), 93–113. https://files.eric.ed.gov/fulltext/EJ1211673.pdf

Mockler, N. (2011). Beyond 'what works': Understanding teacher identity as a practical and political tool. *Teachers and Teaching*, *17*(5), 517–528.

Nalipay, M. J. N., King, R. B., Mordeno, I. G., Chai, C. S., & Jong, M. S. Y. (2021). Teachers with a growth mindset are motivated and engaged: The relationships among mindsets, motivation, and engagement in teaching. *Social Psychology of Education*, *24,* 1663–1684.

NSW Government. (2021). *NSW teacher supply strategy*. https://education.nsw.gov.au/about-us/strategies-and-reports/teacher-supply-strategy#:~:text=The%20NSW%20Teacher%20Supply%20Strategy,support%20and%20advance%20our%20students.

Organisation for Economic Co-operation and Development (OECD). (2020). *PISA 2018 results (volume V): Effective policies, successful schools*. PISA, OECD Publishing, Paris, https://doi.org/10.1787/ca768d40-en.

Oxford English dictionary. (2023). *Flourish (v.), sense I.1.b*. https://doi.org/10.1093/OED/3228912951.

Richardson, P. W., & Watt, H. M. G. (2018). Teacher professional identity and career motivation: a lifespan perspective. In P. Schutz, J. Hong, & D. Cross Francis (Eds.), *Research on teacher identity* (pp. 37–48). Springer International Publishing. https://doi.org/10.1007/978-3-319-93836-3_4

Russo, J., Bobis, J., Downton, A., Feng, M., Hughes, S., Livy, S., McCormick, M., & Sullivan, P. (2021). Characteristics of high enjoyment teachers of mathematics in primary schools. *Mathematics Education Research Journal*, *35*, 1–21.

Ryan, R. M., & Deci, E. L. (2017). *Self-determination theory: Basic psychological needs in motivation, development, and wellness.* Guilford Publications.

Shah, C., Richardson, P. W., Watt, H. M., & Rice, S. (2022). 'Out-of-field' teaching in mathematics: Australian evidence from PISA 2015. In L. Hobbs, & R. Porsch (Eds.), *Out-of-field teaching across teaching disciplines and contexts* (pp. 71–96). Springer, Singapore.

Skaalvik, E. M., & Skaalvik, S. (2016). Teacher stress and teacher self-efficacy as predictors of engagement, emotional exhaustion, and motivation to leave the teaching profession. *Creative Education*, *7*(13), Article 69852. https://doi.org/10.4236/ce.2016.713182

Sullivan, A., Johnson, B., Simons, M., & Tippett, N. (2021). When performativity meets agency: How early career teachers struggle to reconcile competing agendas to become 'quality' teachers. *Teachers and Teaching*, *27*(5), 388–403. https://doi.org/10.1080/13540602.2020.1806050

Uçar, R. (2022). The relationship between secondary school teachers' creativity and job satisfaction. *International Journal of Progressive Education*, *18*(4), 31–42.

Watt, H. M. G., & Richardson, P. W. (2024). Teachers' motivation to teach: A review through the lens of motivational theories. In G. Hagenauer, R. Lazarides, & H. Järvenoja (Eds.), *Motivation and emotion in learning and teaching across educational contexts: Theoretical and methodological perspectives and empirical insights* (pp. 128–142). Routledge. https://doi.org/10.4324/9781003303473-10

Wigfield, A., & Eccles, J. S. (2000). Expectancy–value theory of achievement motivation. *Contemporary Educational Psychology*, *25*(1), 68–81.

Wyatt, J. E., & O'Neill, M. (2021). Investigation of early career teacher attrition and the impact of induction programs in Western Australia. *International Journal of Educational Research*, *107*, 101754.

Zakariya, Y. F., & Wardat, Y. (2023). Job satisfaction of mathematics teachers: An empirical investigation to quantify the contributions of teacher self-efficacy and teacher motivation to teach. *Mathematics Education Research Journal*, 1–23. https://doi.org/10.1007/s13394-023-00475-9

Engaging Multilingual Students in Frequent and Supported Opportunities for Discourse to Strengthen Their Mathematical Thinking

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Determining the most effective ways to support student-to-student talk requires some negotiating from a responsive teacher. This paper reports on a case study of two emergent multilingual students in Grade 3 (seven to eight-year-olds) who explained their strategies to each other. The transcript of their conversation was analysed using Chan and Sfard's (2020) *participation profile framework*. Findings indicate that the two students learned from each other during the interaction because they revised their work to be more precise. One implication of this study is that strategic pairing may be a useful practice to eliminate inequitable power dynamics.

The value in having learners share their verbal or nonverbal mathematical ideas with each other is of increasing interest to mathematics educators around the globe. A growing body of research has emphasised the importance and many ways educators can create learning environments that support all students, specifically multilingual students, with engaging productively in rich mathematical activities and classroom discussions (Turner, Dominguez, Maldonado, & Empson, 2013; Moschkovich, 2007, 2013; Khisty & Chval, 2002). Participating in discourse is also important for developing conceptual understanding (Moschkovich 2015; Bailey 2007). When students share solutions publicly, they strengthen their own comprehension, learn how peers make sense of the mathematics, and present opportunities for teachers to better understand student reasoning and mathematical thinking.

The more students talk about mathematics, the more they see themselves as doers of mathematics, therefore dismantling and restructuring the current negative stereotypes about multilingual learners. Furthermore, students benefit from responsive teaching (Richards & Robertson, 2015), where teachers promote peer interactions by getting to know students as individual learners, differentiating instruction with enabling or extending prompts, and seating students in heterogeneous groups (Bobis, et al., 2021).

Finally, student learning through sharing can also be facilitated by Cognitively Guided Instruction (CGI, Carpenter & Fennema, 1992), which encourages exploration of student-centred thinking and problem solving over the provision of teacher-structured solution strategies. CGI is a foundational element of our work with teachers, specifically, Empson and Levi's (2011) framework for building students' conceptual understanding of fractions and decimals through discussing and solving word problems. The framework also guides our support of teachers with understanding the progression of student learning trajectories for solving fraction word problems to further guide instructional practice. In CGI classrooms, students learn mathematics by engaging in problem solving, explaining their problem-solving strategies to the teacher and their peers, and by listening to various ways of solving problems (Carpenter et al., 1999; Carpenter & Franke, 2004).

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 439–446). Gold Coast: MERGA.

Although we know that responsive teaching and CGI can help teachers provide valuable opportunities for interaction and discussion during mathematics instruction, additional research is needed on how to plan for, scaffold, and facilitate peer interactions effectively while also maintaining a high level of rigor and genuine inquiry, especially in the context of mathematics instruction with multilingual learners (Tai & Wei, 2020). Facilitating effective mathematical discussions, such as supporting students to produce conceptual explanations and connections between ideas, may be more difficult and take longer for teachers to develop (Hufferd-Ackles et al., 2009), and is more challenging in classrooms with linguistically diverse learners at various stages of English development (Walshaw & Anthony, 2008). Our study, part of a larger project on developing linguistic repertoires and mathematical conceptual knowledge in tandem in multilingual classrooms in the United States (US), sheds light on these important issues.

Our research seeks to understand:

- What does it look like when multilingual students have conversations with their peers about their solution strategies for equal sharing problems?

We examine here a conversation between two emergent multilingual students discussing their solutions to an equal-sharing fractions problem. We summarise key findings, including how CGI and responsive teaching can address inequitable power dynamics through strategic pairing while also encouraging collaborative revision when problem solving.

Relevant Literature

Most documented peer-to-peer interactions consist of either a pair or a small group of students collaborating on a problem together or one student tutoring another. Both types of peer-to-peer interactions can lead to asymmetric power relations which could be disadvantageous to learners. For instance, Chan and Sfard (2020) identified a phenomenon where pairs of students did not benefit from learning afforded by the participation structure because one student led the problem-solving effort and the other followed along without comprehending what the partner was doing. Given the possibility that partner work can be unproductive, we assert that partner and small group conversations need to be strategically organised to be effective.

One strategy teachers can use to address the power relationships during partner talk is to provide protocols for students to follow. In the UK, Amodia-Bidakowska et al., (2023) identified as one of their design principles that establishing various mathematics language routines may be beneficial to children as they interact with one another when sharing strategy explanations. The students in their study engaged in peer-to-peer dialogue that supported *elaboration* and *querying* of each other's ideas. Zwiers et al. (2014) suggest providing sentence frames that support development of certain conversation skills to help students build on each other's ideas. These skills include creating, clarifying, fortifying, and negotiating. To create an idea, students are provided with multiple opportunities to express original ideas about the content. Clarifying involves both partners figuring out ways to represent their idea through elaboration, paraphrasing and explanation. Fortifying ideas requires students to support or justify their ideas/problem solving with logic or models. Sometimes ideas are challenged with opposing ideas or strategies which requires the need to negotiate ideas. It may result in coming to a compromise, agreeing to disagree, or conceding to a new idea. All ideas are valued by both partners. Teaching students how to use conversation skills can prepare and support children for more effective interactions.

There are additional participation structures, tasks, and roles that teachers can leverage to facilitate productive interactions between students. In New Zealand, Hunter and Miller (2022) worked alongside a primary classroom teacher that used translanguaging and cultural artefacts in problem contexts to connect the mathematical situations to students' cultural backgrounds. Specifically, *waka* was a commonly used term for boat, *whanau* was a well-known term for family, *tamariki* for children, and *iwi* for tribe/community, etc. Discussing indigenous words and cultural situations during the launch of the problem supported students in making sense of the mathematical situation, identifying with classroom mathematics, and learning about the Indigenous and Pacific Island cultures in New Zealand. In Australia, Muir (2023) reported on a case study of a Grade 3–4 teacher who implemented some of Liljedahl's (2021) *Thinking Classroom Strategies*. The teacher noticed her students were more likely to engage in collaboration and persevere on tasks when they were given challenging tasks, when they believed they were grouped randomly, and when there was one person holding the pen for the group and that person was only allowed to write down other people's ideas. Researchers in the Netherlands found that groups of children, including multilingual students, discussed the meaning of unfamiliar words in story problems before they set out to solve the problems (Elbers & de Haan, 2005). They focused on the aspects of the words that were relevant to the mathematics at hand rather than the more general meaning of a word. For example, when asked what "rye bread" is, the children pointed to a picture in the textbook. Hence, peer-to-peer conversations have the potential to help or hinder student learning depending on how problems are launched and discussions are structured. Formal protocols such as sentence frames, random grouping, and one person holding the pen combined with rigorous, culturally-sustaining contexts are a few ways educators have eliminated inequitable power dynamics, leading to productive conversation between peers.

Theoretical Framework

We draw upon Chan and Sfard's (2020) *participation profile framework*, based on their commognitive framework, to evaluate the effectiveness of dyadic learning. They posited that conversations can be multi-modal, multi-channel, and not exclusively verbal. Chan and Sfard described one type of learning as a change in the command of mathematical discourse. They distinguished between mathematising talk and subjectifying talk. When learners use mathematising language they describe the mathematical features of the topic at hand (e.g., "intercept is negative 5"). Subjectifying occurs when learners navigate their moves out loud (e.g., "I was thinking", "oh then that makes sense to me"). Being able to assess what they've done so far and what their next steps are is the crux of learning. On the other hand, stating subjective evaluations of one's identity is not helpful for learning (e.g., "I'm not good at maths"). Chan and Sfard theorised that the proportion of mathematising and subjectifying utterances make up a learner's thematic profile and demonstrate the level of their command of mathematical discourse.

Methods

Context

This study was conducted in a large urban area in the western US from 2021 to 2024. Students came from linguistically, ethnically, and economically diverse backgrounds. Participants were involved in a design-based research project in which researchers met with teachers and coaches. Researchers met once a month with twelve participants during the first year and once every two to three months with the remaining eight participants during the second year. Teachers mostly taught Grades 3 to 5 students (seven to eleven-year-olds), with one teacher who switched to second grade and one who switched to transitional kindergarten during the second year. Each year of the study, participants identified four students who were classified as Emergent Bilingual because they spoke a language other than English at home. Following a CGI method of lesson planning during the integrated English Language Development mathematics period, the teachers asked pairs of students to share answers with each other after solving on their own. In addition, the pairs used a partner interview protocol (script) with Zwiers et al.'s (2014) conversation skills included as sentence frames. The pairs took turns explaining their strategies to each other using the protocol and then following up by asking each other questions about their individual strategies.

Data Collection and Analysis

Each month, the participants brought video, audio, written transcripts, and copies of students' solution strategies of one of the four focal students explaining to a partner how he/she/they independently solved an equal-sharing fraction problem. To code the transcripts, the research team used inductive and deductive codes to analyse the mathematical discourse between dyads. Following Chan and Sfard's (2020) framework, we differentiated between mathematising language and subjectifying language. When children described the mathematics that they or their peers did to solve the problem it was coded as *mathematising*. Comments about the mathematisers' actions were coded as *subjectifying*. We also noted intrapersonal and interpersonal talk between peers. Additionally, we used Amodia-Bidakowska et al.'s (2023) two core dialogue features, *elaboration* (including clarification/building) and *querying* (i.e. "doubting, full/partial disagreement, challenging, or rejecting a statement").

During deductive open-coding, we found that we also needed to include other codes such as *justification* (because reasoning is different than clarifying) and *equal-sharing language* ('what was your whole?' 'how many parts did you partition it into?') that were specific to the problems the students solved. Utterances were double-coded using both the umbrella codes, *mathematising* or *subjectifying*, and using the subcodes under each umbrella code. We used these codes to analyse the proportion of mathematising and subjectifying the pairs of students engaged in during each recorded session.

Findings

We highlight a case where two Grade 3 (seven and eight-year-olds) multilingual students individually solved an equal-sharing fraction problem using similar strategies. Conceptually, both student solutions were correct. The two peers, Student 1 (S1), male, and Student 2 (S2), female, started the conversation by each taking a turn explaining how they solved this problem, 'Four children share ten cookies. How much will they each get? Explain how you know your solution is correct'. Following the "partner interview protocol" (Zwiers et al., 2014), they each took turns sharing their explanations. Then, S1 probed his partner S2 for more justification. We found that the querying that occurred during the second part of the conversation fortified their understanding of unit fractions ("thirds").

Figure 1

Written Work from Student 1 Male (Left) and Student 2 Female (Right)

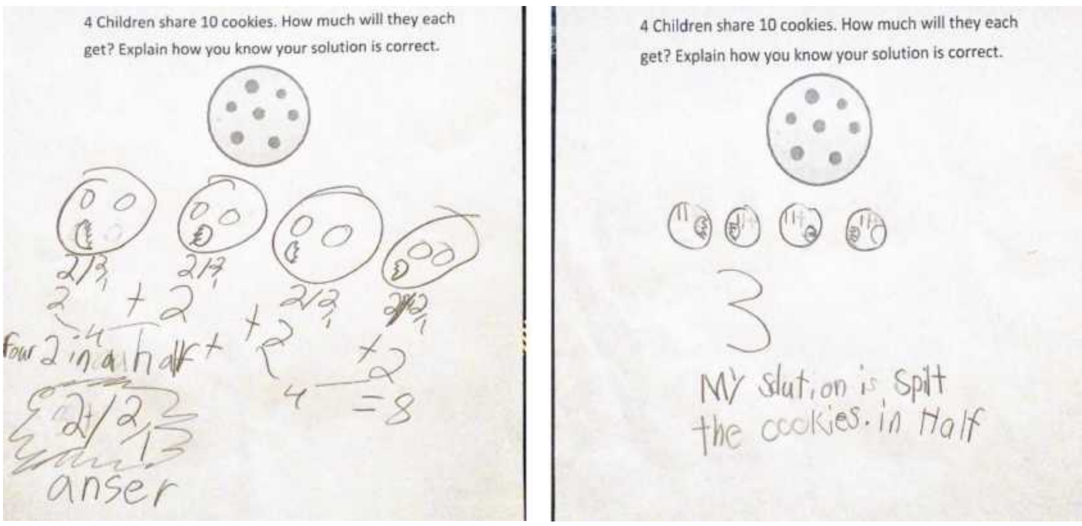

Table 1

Transcript (First 1 Minute, 30 Seconds) of a Dyad Conversation and Associated Codes

Transcript		Codes
S1:	How I solved it, S2, is that I wanted to skip count by twos. But when I did, I actually saw that if I add one here and another one here.	Subjectifying ("How I solved it … ") Subjectifying ("I actually saw … ")
S2:	Hmmm.	
S1:	I was just, it will be three, three, and two. So I decided to make little things like that they're cracked.	Subjectifying ("I decided … ")
S2:	Yeah.	
S1:	To put in each all of them. So then I put two and then half, two and half again, and two and a half like, and I put the same thing. So I put four and ah … four in … four twos and a half. So I did two plus two plus two plus two because I see one, two, three, four, so all of these equal to four. Then four plus four equals eight. So, the answer is, um, two and a half. So, and what did you do?	Mathematising/equal-sharing (distributing and halving cookies) Asked peer to share her explanation
S2:	First, I saw that there were four kids, right?	
S1:	Mmhmm.	
S2:	There was a partition ten cookies. And then what I did was do like line first and then like in each until I got to ten.	Mathematising/equal-sharing (distributing and halving cookies)
S1:	Mmhmm.	
S2:	But then I saw like two, these two didn't have, so, one I broke it in half, because there were two and there were supposed to be, so I broke it. And I just put that it's broken. And then like, I split them in half.	Subjectifying ("I saw") Mathematising/justifying ("because … ") Mathematising/equal-sharing ("I split them in half")

Table 1 displays the initial transcription where each student took turns explaining their individual strategies to each other. Both explanations resembled what could be described as simply reporting or recounting of procedural steps as neither engaged in elaborating or querying into each other's strategies. Both children used a similar strategy in which they distributed two whole cookies to each of the four children in the problem. Then S2 partitioned the last two cookies into halves to share equally between the four children. Although the method is conceptually accurate, S2 drew three similar-sized lines on their paper. She also wrote, "3," on her paper, implying that she knew the people in the problem each received three pieces of cookies without differentiating between wholes and halves. What occurred next is when the crux of the learning happened.

Table 2

Transcript (1:30–2:30) of a Dyad Conversation and Associated Codes

Transcript		Codes
S1:	But S2, how does this one have two and this one have three? I thought it was supposed to be some little lines to make it like half and half.	Querying
S2:	This one is half. And then like this one is half.	Mathematising/elaborating
S1:	Yeah but why are they all in three little rows? Like, for example like, this one, why would you put it in half? If you put it in half it would be two and a half, too. And this one would be two and a half too, also, because it's the same as this one. But this one would just be, um, two, and this one would be just three. So like I believe that that answer is not correct, so. But that does kind of make sense because you wanted to put them in half.	Querying Mathematising Subjectifying (correctness) Elaborating (clarifying)
S2:	Like, split them.	
S1:	Yeah, split them in half.	

In the second part of the conversation, the two children genuinely engaged in a productive dialogue as S1 argued that S2's written work did not align with her verbal explanation (Table 2). S1 stated that S2 initially drew "three little rows," even though S2 said aloud that the cookies were, "half." S2's written work (Figure 1) displays her revised solution after talking with S1. S2 erased the third tally mark and replaced it with a semicircle, similar to how S1 drew his diagram. S2 also wrote, "my solution is split the cookies in half", as if to clarify her solution strategy. Although S2 conceptually understood how to equally partition the cookies among the four people in the problem, her written and verbal explanations were fortified after engaging in a conversation with her peer. The student sample highlighted in this study, further demonstrates the importance of providing frequent and supported opportunities for students to interact. Engaging in mathematical discourse practices, such as explaining one's thinking to others can promote learning by encouraging learners to strengthen or restructure their own knowledge and understandings as well as acquire new strategies and knowledge (Webb et al., 2019; Erath 2017).

Discussion

In this case study, two students helped each other deepen their understanding of fractions via interpersonal dialogue. They both used similar strategies after they had solved it individually. The two students in this example were able to correctly partition the cookies even though they still needed to learn how to name the parts. This aligns with the learning trajectory put forth by Empson and Levi (2011). During the peer dialogue, S1 helped S2 clarify her thinking by asking why she drew three lines of equal length. After explaining her reasoning, S2 realised she needed to revise her picture to match her strategy. S1 also deepened his understanding of fractions when formulating his argument as to why S2's picture was inaccurate.

Unlike Chan and Sfard's (2020) observations, there was not an unequal power struggle between the two students because they had solved the problem individually and used similar strategies. S1's picture was more accurate than S2's, even though her verbal explanation implied that she "split the cookies in half". Rather than one student "tutoring" the other, S1 and S2 shared and compared their written and verbal explanations. They genuinely explored each other's reasoning and S1 helped S2 and himself accurately name the size of the parts. S1's querying helped S2 revise her work to reflect the mathematising she shared in her verbal description. This example demonstrates the ways S1 and S2 deepened their understanding of an equal-sharing fraction problem through dialogue.

The partner conversations that we are concerned with belong to a specific category of peer interaction that distinguish them from collaboration. Because CGI frameworks draw attention to students' development along a concrete to abstract trajectory, we do not advocate for students to collaborate in solving problems. Instead, we see value in independent problem solving followed by peer interaction. In this case, students explained their own approach to each other and the expectation was that they strove to understand what their partner had done.

Conclusion

This case study illustrates the importance of pairing students who have individually solved the problem with different, or in this case, slightly different strategies, and providing language scaffolds or protocols to extend student talk and meaning making. The two students in this example modelled conversation skills using a partner interview protocol. They also extended the script to query each other's ideas. Valuing students' solution strategies and asking their peers to listen to and respond to their explanations is not only beneficial for their mathematical and language development, it also elevates students' discipline-specific dispositions towards mathematics (Gresalfi & Cobb, 2006). This has implications for in-service and preservice teachers around the globe who understand the value of peer-to-peer discussions but need support with structuring those conversations, in-the-moment, in ways that lead to productive conversations between peers.

Acknowledgments

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References

Amodia-Bidakowska, A., Hennessy, S., & Warwick, P. (2023). Disciplinary dialogues: Exploring the association between classroom dialogue and learning outcomes within and between subjects in English primary schools. *Learning, Culture and Social Interaction*, *43*, Article 100742.

Bailey, A. L. (2007). *The language demands of school: putting academic English to the test*. Yale University Press.

Bobis J, Russo J, Downton A, Feng M, Livy S, McCormick M, Sullivan P. (2021). Instructional moves that increase chances of engaging all students in learning mathematics. *Mathematics*, *9*(6), 582. https://doi.org/10.3390/math9060582

Carpenter, T. P., & Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. *International Journal of Educational Research*, *17*(5), 457–470.

Carpenter, T.P.; Fennema, E., Franke, M.L., Levi, L., & Empson, S.B. (1999). *Children's mathematics: Cognitively guided instruction.* Portsmouth, NH: Heinemann.

Carpenter, T. P., & Franke, M. L. (2004). Cognitively guided Instruction: Challenging the core of educational practice. In T. K. Glennan, S. J. Bodilly, J. R. Galegher & K. A. Kerr (Eds.), *Expanding the reach of education reforms: Perspectives from leaders in the scale-up of educational interventions* (pp. 41–80). Santa Monica, CA: RAND Corporation.

Chan, M. C. E., & Sfard, A. (2020). On learning that could have happened: The same tale in two cities. *The Journal of Mathematical Behavior*, *60*, Article 100815. https://doi.org/10.1016/j.jmathb.2020.100815

Elbers, E., & de Haan, M. (2005). The construction of word meaning in a multicultural classroom. Mediational tools in peer collaboration during mathematics lessons. *European Journal of Psychology of Education*, *20*, 45–59.

Empson, S. & Levi, L. (2011). *Extending children's mathematical thinking: Fractions and decimals: Innovations in cognitively guided instruction*. Heinemann.

Erath, K. (2017). Implicit and explicit processes of establishing explaining practices: Ambivalent learning opportunities in classroom discourse. In T. Dooley, & G. Gueudet (Eds.), *Proceedings of the 10th congress of research in mathematics education* (pp. 1260–1267). Dublin: DCU Institute of Education and ERME.

Gresalfi, M. S., & Cobb, P. (2006). Cultivating students' discipline-specific dispositions as a critical goal for pedagogy and equity. *Pedagogies*, *1*(1), 49–57.

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education, 35*(2), 81–116.

Hunter, J., & Miller, J. (2022). The use of cultural contexts for patterning tasks: Supporting young diverse students to identify structures and generalise. *ZDM Mathematics Education*, *54*(6), 1349–1362. https://doi.org/10.1007/s11858-022-01386-y

Kazemi, E., & Stipek, D. (2009). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Journal of Education*, *189*(1–2), 123–137.

Khisty, L. L., & Chval, K. B. (2002). Pedagogic discourse and equity in mathematics: when teachers' talk matters. *Mathematics Education Research Journal, 14(*3), 154–168. https://doi.org/10.1007/BF03217360

Liljedahl, P. (2021). *Building thinking classrooms*. Corwin Press.

Moschkovich, J. N. (2007). Using two languages while learning mathematics. *Educational Studies in Mathematics, 64*(2), 121–144.

Moschkovich, J. (2013). Principles and Guidelines for Equitable Mathematics Teaching Practices and Materials for English Language Learners. *Journal of Urban Mathematics Education*, *6*(1).

Moschkovich, J. N. (2015). Scaffolding student participation in mathematical practices. *ZDM Mathematics Education*, *47*(7), 1067–1078. https://doi.org/10.1007/s11858-015-0730-3

Muir, T. (2023). Achieving teacher professional growth through a focus on making students' mathematical thinking visible. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 371–378). Newcastle: MERGA.

Richards, J., & Robertson, A. D. (2015). A review of the research on responsive teaching in science and mathematics. *Responsive Teaching in Science and Mathematics*, 36–55. https://doi.org/10.4324/9781315689302-2

Tai, K. W., & Wei, L. (2021). Constructing playful talk through translanguaging in English medium instruction mathematics classrooms. *Applied Linguistics, 42*(4), 607–640.

Turner, E., Dominguez, H., Maldonado, L., & Empson, S. (2013). English Learners' Participation in Mathematical Discussion: Shifting Positionings and Dynamic Identities. *Journal for Research in Mathematics Education*, *44*(1), 199–234. https://doi.org/10.5951/jresematheduc.44.1.0199

Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: A review of recent research into mathematics classrooms. *Review of educational research*, *78*(3), 516–551.

Webb, N. M., Franke, M. L., Ing, M., Turrou, A. C., Johnson, N. C., & Zimmerman, J. (2019). Teacher practices that promote productive dialogue and learning in mathematics classrooms. *International Journal of Educational Research*, *97*, 176–186.

Zwiers, J., O'Hara, S., & Pritchard, R. (2014). *Common core standards in diverse classrooms: Essential practices for developing academic language and disciplinary literacy*. Routledge. https://doi.org/10.4324/9781032680873

Catching the Translanguaging Wave: Considerations for Young Multilingual Learners' Mathematical Meaning-Making

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In this position paper we highlight language as a perennial factor contributing to compromised meaning-making in multilingual primary school mathematics classrooms. We note use of the term 'translanguaging' in discussions around mitigating this meaning-making challenge. The paper argues that, while much work remains to be done towards clarifying the pedagogical insights, skills, and resources needed to ensure that translanguaging practices achieve their intended goals, potentially important parallels may be found between horizontal and vertical forms of translanguaging and horizontal and vertical mathematisation.

This position paper explores some implications for primary-level school mathematics of the increasing use of the term 'translanguaging' in our professional literature on ways for enhancing bi- and multilingual learners' opportunities to make meaning of classroom mathematics. Our focus is principally upon ways for improving circumstances in mathematics classrooms in our own, South African, context, but we see our discussion as having relevance for many other contexts where mathematics learners may not yet have developed sufficient proficiency in the *de facto* dominant language of learning and teaching (LoLT).

As we discuss in our penultimate sub-section, the word translanguaging first entered educational parlance some two decades ago in relation to a particular pedagogical strategy for developing and strengthening learners' bilingual proficiencies. Perhaps the earliest use of the term in a publication in English was a 2000 journal article by Cen Williams. In the 1980s he coined, in Welsh, the term *trawsieithu* to describe a particular approach to bilingual (Welsh/English) language pedagogy. The term has subsequently taken on more of a social justice orientation; this in response to concerns about the educational prospects of bilingual learners whose full access to mainstream education may be compromised by the dominance of a particular LoLT. A leading figure in this more political facet of the term's (re-) emergence is Cuban-born Ofelia García, Professor Emerita (Urban Education and Latin American, Iberian, and Latino Cultures) for the City University of New York's Graduate Centre. Translanguaging theory poses significant challenges to 'established' theory around second-language teaching and learning and about how best teachers (mathematics teachers in this instance) might harness and mediate the linguistic and other meaning-making (semiotic) resources their learners bring into their classrooms. These challenges derive from the fact that, traditionally, as Makalela (2015) notes, the language teaching profession has tended to focus on languages as "separate and bounded entities" (p. 200). In other words, the teaching profession's focus is essentially 'monolingual' and 'purist'. By contrast, translanguaging advocates (see, e.g., Garcia & Wei, 2014), focus on bilingual speakers' languaging practices. They argue that these practices derive not from two or more separate language systems, but rather from a single, unified linguistic system or repertoire.

In honour of MERGA 46's surfing theme, our title speaks to the notion of 'catching' the translanguaging wave. Much more important, however, is the issue of successfully 'riding' this wave once caught. In this paper we argue that not being able to do so poses the real threat to equity, and that much work is still required towards ensuring that we clarify what pedagogical insights, skills, and resources are needed to ensure that our translanguaging practices do indeed help our learners get safely to their hoped-for shorelines.

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Our Trajectory Towards the Translanguaging Space

English remains the preferred LoLT in South African schools despite its being the native language of less than a tenth of our population. Reports on our learners' achievement in national and international assessments of mathematical proficiency directly implicate levels of proficiency in the LoLT as a significant factor influencing their performance. In light, for instance, of South Africa's disappointing TIMSS 2015 outcomes it was decided that, although TIMSS generally assesses learners at Grade 4 level, for 2019 we would instead test our learners at Grade 5 level using an easier version of the assessment (Reddy et al., 2020). These adjustments notwithstanding, only 5% of South Africa's participating Grade 5s attained the 'High and advanced' benchmark category (> 550); a distressing 63% scored in the 'Below low' benchmark category (< 399) (Gondwe, 2022). In relation to participating learners from no fee schools, only one in four spoke the language of the test at home (Reddy et al., 2022). "No silver bullet," these authors cautioned, "will fix low performance, remediate years of social imbalance throughout the system, and penetrate the indelible association between one's circumstances at birth and economic and social outcomes" (p. xvi). We regard the fact that the overwhelming majority of South African learners do not have sustained and systematic opportunities to take advantage of the mediating power of their strongest linguistic resource, their first language (L1), for making meaning of their school subjects, as central to addressing at least part of these inequitable circumstances. This requires that we clarify how best we might harness the meaning-making potential translanguaging offers.

Our pathway towards the translanguaging space began with our work at the literacy/ numeracy interface where we looked at the place of language in supporting young learners' mathematical development. Our focus was almost exclusively on classroom talk (or the virtual absence thereof from learners). Our analysis of the talk taking place in one Grade 4 mathematics classroom where English was the school's chosen official LoLT revealed that such talk as did take place was almost exclusively teacher talk. Observed learner talk was limited, largely monosyllabic and formulaic, often chorused, and frequently no more than a mirroring of small chunks of the teacher's talk. It gave scant evidence that the children were engaging in genuine mathematically-oriented verbal exploration of the ideas their teacher was putting before them (Robertson & Graven, 2019). Both the teacher and all of her learners were first language (L1) speakers of isiXhosa, the principal language for almost 80% of people living in the Eastern Cape province. Were it not for the school's stringent adherence to its straight-for-English policy and its discouragement of using isiXhosa anywhere other than in the isiXhosa language class, we believe that use of isiXhosa alongside English would have greatly aided learners in their mathematical meaning-making, encouraging also more participatory behaviour. Our analysis of the talk taking place in another Grade 4 mathematics classroom at a school where isiXhosa was the official LoLT for the first three years followed by a gradual transition across into English from Grade 4, showed a different picture. Although the teacher did most of the talking, her scaffolding of the learners' mathematical meaning-making through allowing access to isiXhosa, the L1 she shared with all her learners, appeared to give the learners a greater sense of confidence and agency in managing various mathematical tasks (Robertson & Graven, 2020a; Robertson & Graven, 2020b). To exemplify we share an excerpt from one of her Grade 4 mathematics lessons. It shows her blend of English and isiXhosa (in italics) usage, with some transliteration. English translations are in square brackets:

> Pointing to the test written up on the chalkboard, the teacher says,
>
> Nantsi itest ebhodini, Bethuna. Nantsi test ebhodini. Bendinixelele, mos? [Here is a test on the board, People. Here is the test on the board. I told you that you going to write, didn't I?]
>
> She quickly goes through various of the test item requirements. Pointing, for example, to one such item (to do with recognising number patterns), she says,

> Copy and complete. It's 93; 83; 73; -; - . What is next? 5, 9, 13; -; - . What is next? Siyevana? [Do you understand?] … Ukuba aku understandi uphakamise osandla. [If you don't understand, just put your hand up.]

Our next step towards the translanguaging space took us into multimodal territory. Here we focussed on the role non-linguistic semiotic resources can play alongside talk around mathematical ideas. This formed the substance of our presentations at the last three MERGA Conferences (Robertson & Graven, 2021; Robertson & Graven, 2022; Robertson & Graven, 2023) where we shared micro-ethnographic data illustrating some of the ways in which mathematical meaning was co-constructed in the course of an after-school mathematics club session with Grade 3 learners. The session was conducted almost exclusively in English, a language none of the children were at ease with even though it was the language of instruction in their classrooms. Most of the verbal input thus came from the club facilitator. With the additional semiotic input via gesturing, looking at and producing various images and inscriptions, and the use of concrete objects, the club members were able to work with the facilitator in reaching the solution to the mathematical challenge she had set them. This success highlighted the power of multi-modality for harnessing as much mathematical meaning-making potential as possible alongside the linguistic input, especially in contexts where understanding the linguistic input might be a challenge for learners. This could simply be because they have not yet become proficient users of a particular language. It might be a result of differences between the language registers of schooling and those experienced at home. In the case of the club members, both of these factors were at play. Not only were all of the children from isiXhosa- or Afrikaans-speaking homes, but all came also from socio-economically vulnerable backgrounds. Quite frequently in the latter circumstance, the oral and literacy patterns of the home differ substantially from such patterns at school. The seminal work of, for example, Bernstein (1971) bears testament to some of the challenges arising from such home/school linguistic divergence.

Most recently our edging towards the translanguaging space has involved exploring the growing importance of making multilingualism official. This is particularly important for educational contexts such as ours. Our colonial history has resulted in the imposition of a language from outside, with, as noted relative to our learners' TIMSS outcomes, potentially extremely negative consequences for academic and other success. We have frequently cited Setati's point (2008) about the perception that an English-medium education is the one most likely to provide learners their access to what she termed 'social goods' (p. 115). Such perception takes little account, however, of the additional burden learning through a second language (L2) often poses in relation to learners' epistemological access to, in the context of Setati's research, mathematics. We take the view that biliteracy is the obvious route through this impasse: academic proficiency in one's L1 plus academic proficiency in a global language (in this instance, English). Such biliteracy is more easily asserted than achieved, though, hence the almost worldwide ruing of the slowness of what has been called 'the multilingual turn' (after May, 2014). In two recent publications (Robertson & Graven, 2024a; 2024b) we explore reasons behind this slowness and highlight the importance that this turn be taken. Ironically our own country's Language in Education Policy strongly advocates multilingualism; and, in particular, the principle of additive bilingualism, yet with a 'purist' approach to the use of languages and translated mathematical terms in the classroom. The politics of language is preventing teachers and their learners from using their everyday home language for epistemological access, and, for a variety of reasons which we do not explore here, mere advocacy is proving an inadequate driver for ensuring its implementation. Increasingly, however, the idea of translanguaging is being identified as an important means of mitigating the linguistic hurdles faced by African learners (Essien et al., 2024).

'Translanguaging': Origins and Some ways Forward

Poza (2017) observes that "many questions remain about translanguaging pedagogies, especially regarding their implementation and outcomes" (p. 120). He cautions that inconsistencies in the way the term is conceived may dilute its 'social justice' implications. Heugh (2019) notes that while linguists agree that translanguaging "can benefit learning", there are some "contradictory understandings" of what this involves (para. 1). She distinguishes between two such understandings: one originating in the Welsh context in the 1990s; the other, a more recent, and increasingly popular understanding, of which Ofelia García (e.g., 2017) is amongst its leading proponents.

Vertical and Horizontal Translanguaging

In describing key elements of these contradictory understandings, Heugh makes the useful distinction between horizontal and vertical translanguaging. And, while she makes it clear that she accepts there is considerable merit in horizontal translanguaging relative to the initial stages of any meaning-making endeavour, she identifies the original (Welsh) practice of 'trawsieithu' (translanguaging) as coming closer to the kind of vertical translanguaging most conducive to learners' development of *higher levels* of academic proficiency in both their L1 and in whatever is their target L2. Williams (2000) explains of the Welsh bilingual education, that "where two languages are used equally in both oral and written contexts, there is room for 'translanguaging', i.e. reading in one language and writing in the other" (p. 144). We note the implicit emphasis on 'equally'. Such purposeful and systematic switching across discrete languages involves, as Heugh (2019) remarks, "highly complex metacognitive and metalinguistic processes and capabilities" (para. 3).

The emphasis in horizontal translanguaging foregrounded by linguists such as García is more on "fluid linguistic practices rather than deliberate alternation between two clearly demarcated standard languages" (Heugh, 2019, para. 4). Here the goal appears to be geared more towards political, social justice imperatives in education than towards pedagogical ones. This is not, of course, to imply that these imperatives are mutually exclusive goals. School language, García argues, "acts as the barrera that keeps the very few powerful" (2017, p. 257). She contends that "only those whose language practices can easily pass through the narrow linguistic passageway that schools construct, have then access to knowledge, knowledge of ciencia, historia, literature, matemáticas, and all other ways of understanding the world" (2017, p. 257). We note in these statements García's exercise of translanguaging. As she explained in an earlier article co-authored with Wei (2014), the 'trans-' in 'translanguaging' allows for this kind of transgression of traditional language boundaries of, for instance, the structures and practices both of language and of education systems. This transgression, achieved in transdisciplinary ways, aims at transforming conventional cognitive and social structures. Amongst the important social justice imperatives identified by García (2023a) is "ensuring that racialised bilingual speakers' lives and languaging are valued as meaning-making systems that are not only legitimate, but also academic [so flattening] … the hierarchies produced when named languages are attached to national or social groups that are always ranked on a social scale" (p. 7). In another recent publication she explains that "racialized bilinguals do language by assembling a repertoire of language and semiotic features and practices from all the different communities and individuals with whom they interact" (García, 2023b, pp. xxi–xxii).

Ways Forward

The above descriptions of the apparently contradictory interpretations of translanguaging derive from what Bonacina-Pugh et al. (2021) call the differences between the 'fixed language approach' and the 'fluid languaging approach'. Both, in our view, have a place in the mathematics classroom. This is particularly so for the multilingual classrooms that predominate in South Africa where, for too long, a majority of learners have had only diminished opportunity to use their dominant (L1) language as a genuine resource (after Ruiz, 1984), a genuine source of linguistic capital with which to invest (after Bourdieu, 1977) in their mathematical meaning-making. Heugh (2019), too, urges the need to focus on both the vertical and the horizontal forms of translanguaging, arguing that without such dual focus "we will neither reduce socio-economic and political inequalities nor achieve equitable opportunities for our students' futures" (Conclusion section). What we argue for here is that our mathematics education community comes to see the two forms as complementary rather than contradictory. Both have value, serving different purposes at different stages of the learning trajectory in the same way that movement from the more concrete (and familiar) towards the more abstract and symbolic (from the horizontal to the vertical) mathematisation occurs (after Freudenthal, 1973). A 'fluid' form of languaging using whatever linguistic and other semiotic resources are available is what happens in the initial phase of meaning-making. 'Fixed language' representations of the meanings are then built upon this, ideally in both the L1 and the L2, so consolidating and refining these meaning/s in more formal and mathematically-appropriate ways. Mathematics requires learners to (particularly in the assessment stage) express ideas from within the boundaries of a single language system. This facilitates maximal exploitation of that system's power of linguistic expression. In systemic functional linguistics Halliday and Matthiessen (2004) use the phrase 'cline of instantiation' to describe the movement towards increasingly sophisticated, precise, discipline-specific expressions of ideas. In the same way that using the language of, say, school geography to talk about school mathematics would inevitably compromise the precision with which mathematical ideas could be expressed, so too, we believe, could stepping too far beyond the boundaries of a particular language system compromise this precision.

A great deal of well-researched evidence around second language acquisition has demonstrated that, even in the case of so-called 'fixed language' views, it is impossible, and—indeed—undesirable to exclude learners' L1s from the classroom (see, e.g., Swain & Lapkin, 2013). We have elsewhere written (e.g., Robertson & Graven, 2024b) on the compelling evidence put forward by, amongst others, Skutnabb-Kangas (1981) and Cummins (2005) about the important role that learners' L1s play in the subsequent development of their L2 proficiency, and the longer-term benefit of working towards high levels of academic proficiency in both languages. In specific reference to the work of Cummins, and as Conteh (2018) noted, Cummins's "concepts of 'common underlying proficiency' and linguistic interdependence stress the positive benefits of transfer in language learning" (p. 445). Included amongst the benefits of bilingualism are the metacognitive and metalinguistic advantages that can accrue from becoming proficient users of more than a single language. Bialystok et al. (2012) reported on this. Included in their list of advantages are greater mental flexibility, increased executive and cognitive control, and, reassuringly, in the long term, a potential extension of the timeline towards the onset of age-related dementia. More immediately, in relation to young learners, are what might be termed the socio-emotional advantages of having classroom access to their L1 alongside an L2. These include making closer links between learners' home and classroom lives; developing a stronger sense of socio-cultural and personal identity and agency; and thereby increasing the likelihood of a greater willingness and motivation to participate actively in classroom activities and discussion.

Conteh (2018) notes that some question the need for the notion of 'translanguaging' "when the familiar concepts of code-switching and code mixing already provide a framework to understand multilingual language use" (p. 446). In the South African context, code-switching was frowned upon during the apartheid era as it was thought that mixing languages would interfere with learners' L2 acquisition. One isiXhosa-speaking science teacher captured this view 'confessing' to a researcher (Probyn, 2001) that he felt guilty of 'smuggling the vernacular into the classroom'. The liberation involved in increased acceptance of pedagogies employing translanguaging is captured in the third paper included in a recent symposium presentation. (See Tyler et al., 2024.) Paper 3's title began: 'No more smuggling the vernacular.'

A difficulty identified in relation to South Africa's constitutional commitment to 12 official languages (the recent 12th addition being sign language), and the Language in Education Policy ratification that, depending upon the decisions and circumstances of individual school's contexts, any one of these languages may be chosen as the official LoLT, is that standardisation of South Africa's indigenous African languages is still underway. Despite isiXhosa having a long history of lexicography dating back to the 18th century (Nkomo & Wababa, 2013), it is as yet not fully standardised, and disagreements about terminology and means of expression are common. Booi et al. (2024) note considerable variation depending on where speakers' come from creating problems both for teachers and for their learners. Translations of written texts have "tended to adopt a purist approach to language that results in the use of outdated or unfamiliar so-called 'standardised version' of isiXhosa that is unfamiliar" (Booi et al., 2024, p. 3). A further challenge these authors have identified is that of finding common terms across regional dialects, perhaps most particularly across the urban/ rural divide. These authors point out that the formal standardised isiXhosa used in school teaching and learning support materials is often at odds with the spoken home language—informal non-standardised isiXhosa used outside of school ('*lokshin*' isiXhosa). '*Lokshin*' here refers to a 'location' (English) or, in Afrikaans, '*lokasie*', a residential area set aside for black Africans during the apartheid era. Booi et al. (2024) suggest the use of translanguaging to mediate such differences (e.g., using the less formal term '*ukudabulisha*' for 'double', the '*uku-*' meaning 'to' and '*-dabulisha*' 'double') in place of the more formal and 'standardised', but less familiar phrase '*ukuphinda kabini*' (also meaning 'to repeat' or 'to multiply twice'). Evident in this example is some phonetic transliteration between English and isiXhosa in the less formal term ('*dabul*'/ 'double'). Such transliteration is useful in the sense that it potentially offers learners a 'double dip' at discerning the meanings of certain words. Another example would be using the word '*isikwere*' (or '*isqueri*') to refer to a 'square', a shape which one local isiXhosa-speaking person initially told us she would call '*ifourcorners*'. Working cross-lingually in this way, and asking children to consider the different equivalents for various mathematical terms would be one way of raising their metalinguistic awareness about how different languages work in expressing the same or similar ideas. More challenging than terminology, is mediating the learning of 'the language in-between' (Prediger, personal communication, January 12, 2023). This is the language required for exploring and reasoning through the demands of mathematical ideas and tasks, for justifying one's interpretations of what a task requires, or for responding constructively to the interpretations of others. It is perhaps in this largely oral domain that horizontal translanguaging can prove particularly powerful for helping learners' mathematical meaning-making, especially when augmented with other semiotic modes (e.g., gesture, image, inscriptions of various sorts).

Concluding Remarks

In this paper we have shared our concern that catching the translanguaging wave is all well and good, but that knowing how to ride it successfully is more important. We have outlined some of the differences in viewpoint reflected in the literature around translanguaging, and argued that such differences are perhaps more apparent than real. Heugh's distinction between horizontal and vertical forms of translanguaging (2019) is important. It helps towards the understanding that these forms can be seen as complementary rather than contradictory. We recognise complementarity also between these translanguaging axes and the axes of horizontal and vertical mathematisation. Just as horizontal mathematization starts with a focus on the everyday, so too does initial classroom discussion around mathematical ideas, concepts and procedures need a more everyday register, drawing on the widest possible repertoire of linguistic and other semiotic resources. Beyond this initial horizontal stage of communication more careful consideration is then needed in the vertical where specific terminology is key to faithfully capturing critical ideas in the concept. So, for example, the everyday, '*ifourcorners*' provides a mathematics teacher with a potentially valuable opportunity to start learners thinking through the reasons why there needs to be the specific name 'square' ('*isikwere*') in order to distinguish this shape from other four-cornered shapes. This is critical for vertical mathematisation of understanding squares as a special subset of quadrilaterals, parallelograms, and rectangles. The final point we want to reiterate is the meaning-making and conceptual value of what we called the 'double dip' aspect to the transliteration of English, and, frequently too, Afrikaans words into their isiXhosa form.

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References

Bernstein, B. (1971). *Class, codes and control (Volume 1): Theoretical studies towards a sociology of language*. Routledge.

Bialystok, E., Craik, F. I. M., & Luk, G. (2012). Bilingualism: Consequences for mind and brain. *Trends in Cognitive Sciences*, *16*(4), 240–250.

Booi, T., Vale, P., & Graven, M. (in press). Research-informed translation of comprehensible isiXhosa mental strategy teaching materials into isiXhosa: Doubling and halving as a mental strategy. *South African Journal of Early Childhood Education*.

Bourdieu, P. (1977). Cultural reproduction and social reproduction. In J. Karabel, & A. H. Halsey (Eds.), *Power and ideology in education* (pp. 487–511). Oxford University Press.

Bonacina-Pugh, F., da Costa Cabral, I., & Huang, J. (2021). Translanguaging in education. *Language Teaching*, *54*, 439–471.

Conteh, J. (2018). Translanguaging. *ELT Journal*, *72*(4), 445–447.

Cummins, J. (2005). Teaching for cross-language transfer in dual language education: Possibilities and pitfalls. In C. Alptekin, B. Seidlhofer, J. Cummins (Eds.), *TESOL symposium on dual language education: Teaching and learning two languages in the EFL setting* (pp. 1–18) Bogazici University, Istanbul, Turkey.

Essien, A. A., Sapire, I., & Moleko, M. M. (2024). The concept of language-as-resource/sources of meaning from an African perspective: Challenges and opportunities in mathematics education. In A. Essien (Ed.), *Multilingualism in the teaching and learning of mathematics in Africa: Issues for mathematics education research* (pp. 1–20). Bloomsbury.

Freudenthal, H. (1973). *Mathematics as an educational task*. D. Reidel Publishing.

García, O. (2017). Translanguaging in schools: Subiendo y Bajando, Bajando y Subiendo as afterword. *Journal of Language, Identity and Education*, *16*(4), 256–263.

García, O. (2023a). Translanguaged TESOL in Transit. *NYS TESOL Journal*, *10*(1), 5–18.

García, O. (2023b). Foreword: Doing translanguaging research/teaching/learning Juntos. In L. Shepard-Carey & Z. Tian (Eds.), *(Re)imagining translanguaging pedagogies through teacher–researcher collaboration* (pp. xvii–xxiv). Multilingual Matters.

García, O., & Wei, L. (2014). *Translanguaging: Language, bilingualism and education*. Palgrave Macmillan.

Gondwe, J. (2022). *Mathematics performance of South African primary school learners: Lessons from TIMSS 2019*. Department of Economics and the Bureau for Economic Research, Stellenbosch University.

Halliday, M. A. K., & Matthiessen, C. M. I. M. (2004). *An introduction to functional grammar*. Continuum.

Heugh, K. (2019). The place of translanguaging in multilingual education and assessment [Online]. *TIMSS SA Newsletter*. https://www.timss-sa.org/blog/the-place-of-translanguaging-in-multilingual-education-and-assessment

Makalela, L. (2015). Moving out of linguistic boxes: the effects of translanguaging strategies for multilingual classrooms. *Language and Education*, *29*(3), 200–217.

May. S. (2014). Disciplinary divides, knowledge construction and the multilingual turn. In S. May (Ed.), *The multilingual turn: Implications for SLA, TESOL and bilingual education* (pp. 7–31). Routledge.

Nkomo, D., & Wababa, Z. (2013). IsiXhosa lexicography: Past, present and future. *Lexikos*, *23*(1), 348–370.

Poza, L. (2017). Translanguaging: Definitions, implications, and further needs in burgeoning inquiry. *Berkeley Review of Education*, *6*(2), 101–128.

Probyn, M. J. (2001). Teachers' voices: Teachers' reflections on learning and teaching through the medium of English as a second language. *International Journal of Bilingual Education and Bilingualism*, *4*(4), 249–466.

Reddy, V., Winnaar, L., Juan, A., Arends, F., Harvey, J., Hannan, S., Namome, C., & Zulu, N. (2020). *TIMSS 2019 highlights of South African grade 5 results in mathematics and science: Achievement and achievement gaps*. Human Sciences Research Council.

Reddy, V., Winnaar, L., Harvey, J., Hannan, S., Isdale, K., Arends, F., & Juan, A. (2022). *The South African TIMSS 2019 grade 5 results: Building achievement and bridging achievement gaps*. Human Sciences Research Council.

Robertson, S-A., & Graven, M. (2019). Exploratory mathematics talk in a second language: A sociolinguistic perspective. *Educational Studies in Mathematics*, *101*(2), 215–232.

Robertson, S-A., & Graven, M. (2020a). A mathematics teacher's response to a dilemma: 'I'm supposed to teach them in English but they don't understand'. *South African Journal of Childhood Education*, *10*(1), 1–11.

Robertson, S-A., & Graven, M. (2020b). Language as an including or excluding factor in mathematics teaching and learning. *Mathematics Education Research Journal*, *32*(1), 77–101.

Robertson, S-A., & Graven, M. (2021). Tuning-in to non-linguistic resources during collective problem-solving in a second language context. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in mathematics education: Foundations and pathways. Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia* (p. 444). Singapore: MERGA.

Robertson, S-A., & Graven, M. (2022). Working on and with verbal, visual and gestured confluences in mathematical meaning-making. In N. Fitzallen, C. Murphy, V. Hatisaru & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia* (p. 610). Launceston: MERGA.

Robertson, S-A., & Graven, M. (2023). Orchestrating mediational means in solving a mathematical problem. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (p. 596). Newcastle: MERGA.

Robertson, S-A., & Graven, M. (2024a). Exploring the implications of the multilingual turn in mathematics education research for South African policy makers, researchers and educators. In A. Essien (Ed.), *Multilingualism in the teaching and learning of mathematics in Africa: Issues for mathematics education research* (pp. 23–45). Bloomsbury.

Robertson, S-A., & Graven, M. (2024b). Let us take that multilingual turn to make language a genuinely inclusive resource for mathematical meaning-making. In L. Westaway, C. H Stevenson-Milln, K. M. Ngcoza, & C. Simuja (Eds.), *Proceedings of the 32nd annual conference for research in mathematics, science and technology education* (pp. 183–196). SAARMSTE.

Ruiz, R. (1984). Orientations in language planning. *NABE Journal*, *8*(2), 15–34.

Setati, M. (2008). Access to mathematics versus access to the language of power: The struggle in multilingual mathematics classrooms. *South African Journal of Education*, *28*(1), 103–116.

Skutnabb-Kangas, T. (1981). *Bilingualism or not: The education of minorities*. Multilingual Matters.

Swain, M., & Lapkin, S. (2013). A Vygotskian sociocultural perspective on immersion education: The L1/L2 debate. *Journal of Immersion and Content-Based Language Education*, *1*(1), 101–129.

Tyler, R., Probyn, M., Set, B., Opanga, D., & Williams, V. (2024). The multilingual option: Pedagogical translanguaging in science classrooms (Symposium). In L. Westaway, C. H Stevenson-Milln, K. M. Ngcoza, & C. Simuja (Eds.), *Proceedings of the 32nd annual conference for research in mathematics, science and technology education* (pp. 306–310). SAARMSTE.

Williams, C. (2000). Bilingual teaching and language distribution at 16+. *International Journal of Bilingual Education and Bilingualism*, *3*(2), 129–148.

Teaching the Unexpected Mathematics: How Digital Technologies Unlocked Incidental Primary Mathematics Concepts

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Diminishing duplication and finding connections between the *Australian Curriculum: Mathematics* and *Digital Technologies* were a focus of the revision to version 9. This emphasis has provided enhanced opportunities for integration of these learning areas. In this exploratory multiple case study, interviews with teachers following teaching an integrated task identified connections to other mathematical concepts that had not been considered during planning. These findings indicate that integrating mathematics and digital technologies can potentially provide opportunities to deepen and consolidate learning in mathematics through connections beyond the initially intended concepts.

An emphasis on reducing duplication of content has meant that the Australian Curriculum (AC) version 9 was written to make the connections between learning areas more explicit (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2021). These connections emphasise the shared focus on data and computational thinking across *AC: Mathematics* and *AC: Digital Technologies* (ACARA, 2022a). Now, definitions and content in computational thinking and data collection, analysis, and representation are shared. Previously, there was insufficient alignment between the learning areas to develop synergies (Larkin & Miller, 2020). *AC: Technologies* is intended to be taught through integration with other learning areas, and this shared content provides clear opportunities for teachers to integrate.

In this project, professional development (PD) focused on integrating mathematics curriculum and technological content knowledge to demonstrate connections between version 9 of the *AC: Mathematics* and *AC: Digital Technologies* in the primary years. The effective integration of this content poses a challenge for generalist primary teachers (Larkin & Miller, 2020). The project aimed to support teachers in developing integrated tasks and learning experiences for their students through professional development and mentoring. The professional development focused on seamlessly integrating mathematics and digital technologies, teaching and assessing both curriculum areas in a single task. The tasks teachers developed were trialled in their classrooms, and their reflections were the focus of mentoring sessions. The mentoring sessions supported teachers' exploration of students' mathematics and digital technologies learning experiences during the task and determining the next steps for effective integration.

The project aimed to investigate the integration of the *AC: Mathematics* and *AC: Digital Technologies*, with a specific focus on diminishing duplication and uncovering connections between the two disciplines. This paper explored instances of unplanned teaching where connections emerge spontaneously when teachers leveraged their *knowledge of content and curriculum* to identify opportunities for establishing mathematical connections.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 455–462). Gold Coast: MERGA.

Literature

Curriculum Perspectives and Conceptual Threads

Curriculum integration allows teachers to weave learning experiences across subject boundaries to apply transdisciplinary skills for learning (Ross & Marshman, 2023). An integrated approach to teaching is often complex. Teachers can spend hours of planning time considering the curriculum to find threads that draw together learning areas (Ross & Marshman, 2023). Sometimes, these threads provide successful opportunities for learning. However, the challenge arises from developing these connections at age-appropriate curriculum levels.

While the teachers' plan describes the concepts and purposeful sequencing of learning experiences that they intend, the developed plan is not necessarily followed precisely in the enactment process (Ross, 2024). During the lesson, a teacher may take an unplanned detour to consider student questions or adjust the pitch of the content they are teaching. Further, there may be incidental topics that can be covered, i.e., topics covered that were not planned to be covered during that lesson or sequence. Srinivasa et al. (2022) described incidental learning as the unintended, unplanned, or additional learning that can occur because of other activities. Incidental learning may be additional content that needs to be taught to support a gap in student knowledge, or concepts that are beyond those intended for the lesson, or perhaps student-directed exploration or questioning beyond the scope of the planned lesson. In recognising these incidental teaching moments, the teacher needs deep knowledge to see the potential and capitalise on the learning opportunities (Nayler, 2014).

The Importance of Making Mathematical Connections

Ball et al. (2008) developed their six domains of mathematical knowledge for teaching by building on Shulman's (1987) content knowledge and pedagogical content knowledge (PCK). *Subject matter knowledge* is the mathematics knowledge that most people have. *Specialised content knowledge* is the mathematical knowledge and skills unique to teaching and may include "looking for patterns in student errors" (Ball et al., 2008, p. 400), identifying nonstandard approaches and unpacking mathematical ideas to make them more accessible to students. *Horizon content knowledge* is an awareness of where mathematical topics go in future learning. *Knowledge of content and students* includes anticipating students' thinking and responses to tasks, what they will find confusing, and the level of challenge, the language that students will use with "emerging and incomplete thinking" (p. 401) as well as common conceptions and misconceptions of the mathematical content. *Knowledge of content and teaching* includes knowledge of how to sequence the content and the usefulness of different representations, methods, and procedures. The final category, *knowledge of content and curriculum*, includes knowledge of both the mathematics and the pedagogical underpinnings of the curriculum including understanding the underpinning sequence of mathematical concepts.

Rowland et al. developed the knowledge quartet to describe the categories of knowledge needed to teach mathematics. *Foundation* is the knowledge, understanding, and beliefs acquired during initial teacher education; *transformation* is the ability to transform content knowledge so that it is accessible for students including the choice of resources and activities; *connections* include the sequencing of topics and the connections between different areas of mathematics; and *contingency*, sees a teacher responding to unexpected student responses. Contingency refers to the teachers' capacity to make *in-the-moment* decisions to decide whether to pursue their predetermined plan or to deviate to follow the student's introduced line of thinking. during teaching and learning (Rowland et al., 2009).

In this paper we explore teacher's *knowledge of content and curriculum* in *AC: Mathematics* and *AC: Digital Technologies.* We focussed on definitions and content in computational thinking and data collection, analysis, and representation. Teachers' capacity to select digital resources to teach mathematics using their pedagogical content knowledge can enhance students' mathematical learning (Loong & Herbert, 2018). Teachers are often provided with the technologies in primary mathematics classrooms and expected to integrate these technologies without the necessary PD support (Attard, 2013). Teachers can struggle to understand how the technology can be integrated (Perienen, 2020), and there is a need for further training to integrate digital technology into their mathematics teaching (Attard, 2013; Perienen, 2020) to enhance student learning opportunities.

Research literature has long emphasised that making mathematical connections between facts, procedures, and relationships is essential in constructing mathematical understanding (e.g., Eli et al., 2011) and between learning areas and the real world (e.g., National Council of Teachers of Mathematics, 2000). This is reinforced in one of the aims of the *AC: Mathematics*, "make connections between areas of mathematics and apply mathematics to model situations in various fields and disciplines" (ACARA, 2022b). The curriculum also identifies the role that integrated STEM learning can play in building these connections, "Interdisciplinary STEM learning can enhance students' scientific and mathematical literacy, design and computational thinking, problem-solving and collaboration skills." (ACARA, 2022b). The research presented here investigates how an integrated mathematics and digital technology task can create opportunities for making mathematical connections beyond the initial learning intention. The research question was: *How can teachers best use knowledge of content and curriculum when integrating mathematics and digital technologies?*

Research Design

Context

This 6-month study is part of a larger study. This smaller study focused on the initial experiences of teachers from two schools. Their data has been included in the analysis, and pseudonyms used. All teachers in the project attended a professional development (PD) day focused on developing an understanding of task design and integration. The PD event provided examples of integrated lessons using digital technologies to teach and consolidate mathematical concepts focused on statistics. After a trial period for the teachers with their integrated classroom task, they were offered a mentoring session to discuss the task and its enactment. An exploratory multiple-case study design (Yin, 2009) used mixed methods for collecting data, including surveys and a semi-structured interview protocol during the mentoring session.

Participants

This paper describes case studies of teachers in the study from two different schools in the Moreton Bay region in Queensland. Case study one shares the experiences of an early-career teacher from a designated special school, while Case study two reflects a team of two teachers (one early-career and one mid-career) from a government primary school. Both schools planned teaching and learning using the early primary levels of the *AC: Mathematics* integrated with *AC: Digital Technologies*.

Data Collection

The teachers completed a pre-PD survey to ascertain their starting beliefs and participated in a mentoring session once they had trailed a task in their classroom. The survey included questions from the Technology Beliefs and Barriers to Creating Technology-Enhanced, Learner-Centred Classrooms sections of An and Reigeluth's (2012) survey. Additional questions included beliefs about and barriers to integrating *AC: Mathematics* and *AC: Digital Technologies*.

The mentoring sessions were held through Teams and recorded. The discussion prompts used during the mentoring sessions were based on Rolfe et al.'s (2001) framework for reflective practice, asking the teachers to describe the integrated lesson the teacher had completed, including aspects that worked well and challenges (what); highlighted aspects that the teacher found interesting or surprising (so what); and outlined what they planned to do next (now what). Teachers were asked to recall the enactment of their integrated mathematics and digital technology tasks. They were asked to highlight what happened and encouraged to provide details about the nature of the task and the students' responses. The teachers were also asked to reflect on and analyse the event, including anything the students did that interested or surprised to them. These answers gave rise to several key examples of incidental mathematics learning.

Data Analysis

Data were analysed thematically using Braun and Clarke's (2006) six phases of thematic analysis. Researchers each analysed the data they had collected for individual schools, including survey responses and transcripts of mentoring sessions. Through a comparative discussion between the researchers presenting the data sets they were familiar with; emerging themes were used to code the transcripts of the mentoring sessions. This paper draws on data from one clear theme from mentoring sessions' data: the incidental mathematics the teachers recognised and taught that arose from the integrated mathematics and digital technologies task.

Findings

Case Study 1

David is formally an early career teacher; however, he has come from a career in teaching in another education-related field. He works in a special school where his students are developing their mathematical knowledge at a Preparatory (Foundation) level. In his task, David planned to explore mathematical concepts related to position and location in mathematics. To explore these concepts, he used the Bee-Bots and integrated the activity with digital technologies by considering data representation with his students. The students were tasked with moving a Bee-Bot with a small photo of them on it, through a maze representing the school to several pre-determined destinations (e.g., from the classroom to the library). While enacting the task, David noticed opportunities to explore other aspects of mathematics crucial to the students' capacity to complete the task:

> One thing that came up was that one-to-one correspondence is really helpful and that it wasn't planned. But the fact that they have to hit the button on the Bee-Bot and count as 1, 2, 3 is something that I sort of, I guess overlooked. And then [I] realised as they were doing it. Oh, yes, there is some underlying skill here that needs to be worked on. Yeah. So that was eye opening... [if] I had my time again, I would have just let them play with the Bee-Bot first. Yeah, just go nuts and push all the buttons that you want to push and make it do whatever you want it to do. And then I'll do the one-to-one correspondence.

David recognised the importance of the incidental mathematical concept of one-to-one correspondence as essential to the students being able to use the Bee-Bot buttons to program the robot. Without this knowledge and skill, David's students could not complete the task. Consequently, this aspect would need to be incorporated into future iterations of this lesson sequence. Further, David saw the opportunities in the incidental mathematics concept for one of his students:

> I was just like, basically counting breaking down the steps ... And he was struggling a lot with the one-to-one correspondence and the fact that he had to slow down and there was like a tangible goal, I think really helped him.

David describes a student racing through counting activities and other mathematical tasks. In the student's haste to finish, he was making errors. David felt that using the Bee-Bots meant the student had to slow down to ensure the correct entry of data, and this helped reinforce the one-to-one correspondence that the student was lacking. David also saw the students finding further opportunities to use this concept creatively in more complex coding patterns:

> This student did something that was shockingly creative. At first, I didn't know what he did ... they just installed a new spinner [at school]. So, he sent it [the Bee-Bot] to the spinner [on the map]. And then he made it spin. And at first when he did it, I didn't get it. And I was like what did you want it to do? Like I thought he made a mistake. And then he did it and I was like floored. (David)

The student programmed the Bee-Bot to leave the classroom on the map and head straight for the new spinner in the playground. Using the left turn button, he programmed the Bee-Bot to turn in a circle like the students would when using the spinner in the playground. David was excited by the student's capacity to quickly pick up the coding skills and the mathematical concepts that allowed for the creative expression of this mathematical knowledge.

Case Study 2

Vera was a young, early-career teacher and Sarah, an older, very experienced teacher from a government primary school. Together, they designed their integrated lesson to explore the mathematical concept of symmetry, which was shared with the other Year 3 teachers at their school. Their lesson used Pro-Bots to draw the mirror image of a symmetrical drawing, asking Year 3 students to "draw or follow instructions to draw the other side of the symmetrical item" (Vera). The task required students to follow and implement simple algorithms linked to learning in digital technologies. The teachers were aware that students had not done any programming previously, so gave the students a pre-lesson, "That I had like a pre-lesson on them just playing with the Pro-Bots before getting them to set a task because they've not done any sort of robotic stuff up into this point or any coding" (Sarah).

The teachers acknowledged that the activity included more mathematics than just the symmetry they planned to address. In their discussion of the activity, they explained the incidental mathematics that the students used in the lesson, which included the need to consider what angles to turn the Pro-Bots in the intended direction, directions (turning left or right), measuring how far the Pro-Bot travelled to set up grid references and ensure the Pro-Bot went the correct distance, patterning and mirroring, and inverse relationships, which they connected with addition and subtraction (*connections*). During mentoring, Sarah and Vera reported opportunities for developing directional language, including clockwise, anti-clockwise, quarter turn, and half turn:

Vera: It was not just good for their symmetry, but as well their directions, and actually realising which way was which and then, you know, we'd lift up the paper, and they go, "Oh, I actually went left instead of right. I didn't follow the instructions properly".

Vera: I think getting them to realise that it's not just forward because we had the Pro-Bots not Bee-Bots. So, actually, having the angle and making the angle first and then moving forward. That got them a few times. ... It was that directional language. It was measurement. It was symmetry, which the lesson was based around. Grids, as well, it had tons [of mathematics].

Sarah: Because the measuring was pretty tricky. ... they could work out that it had to be part of the grid system. You know, your turns that we talked about earlier in the year, where we're going clockwise, anti-clockwise, quarter turn, half turn, and then technology-wise, it being a specific pattern and code. Yeah, that had to be followed. Otherwise, it [Pro-Bot] wasn't going.

Vera: All that sequence. So, you're patterning that … And then mirroring.

Sarah: Mirroring. Yeah. With your symmetry. Yeah. So, then you had to reverse it. You had to think the opposite.

Sarah: Yeah. And so … you put that into the relationship of add and take away … they're the opposites of each other. So, you sort of make that connection. They're doing this and your turns are the opposite. Yeah. I mean, that's the same amount of steps, but just the direction is slightly different.

The teachers also identified that they could extend the task and incorporate a map and grid references as they moved into the next school term's learning and assessment:

Sarah: And I think it led into this term when the first maths assessment was looking at grid references in their assessment. So, I think they got really good at direction and giving directions from that

> [integrated task] as well...we could've maybe incorporated that with more of a map situation like we had on the day that we were playing. Yeah. With the grid and things like that. More of that.

Discussion

This study focused on the initial experiences of teachers from two schools and the incidental mathematics the teachers recognised and taught, which arose from the integrated tasks they designed. Analysis of the case studies provided insight into how integrating learning in mathematics and digital technologies supported the teaching of incidental mathematics and strengthened teachers' *knowledge of content and curriculum,* allowing *connections* to be made.

The activity in Case 1, focused on location and direction. Students programmed Bee-Bot using a series of arrows (forward, backward, left, and right) to visit areas of the school on a floor map and describe the robot's position relative to items depicted on the map. David recognised the incidental mathematical concept of one-to-one correspondence as essential to students being able to use the Bee-Bot buttons to program the robot before exploring concepts of position and location.

In Case 2, Vera and Sarah had planned their lesson to focus on symmetry using Pro-Bots. When teaching the lesson, they recognised that the use of the Pro-Bot provided an opportunity to consolidate previous learning in mathematics. This incidental mathematics included directions and directional language, angles, measurement, and patterning and mirroring. There was also an opportunity to discuss inverse relationships with the students, which they did by connecting with addition and subtraction.

The teachers, in both cases, leveraged their deep understanding of content, curriculum, and pedagogy to design and deliver an integrated mathematics and digital technologies task, demonstrating some *knowledge of content and curriculum* (Ball et al., 2008). They used their comprehensive knowledge of the subject matter to identify key mathematical concepts that could be effectively taught using robotics and computational thinking skills. Teachers were also willing to respond to students' needs and deviate from their original plans (*contingency*) and help students make *connections* (Rowland et al., 2005). The teachers chose technologies such as Bee-Bots for foundation years and Pro-Bots for primary years to enhance students' mathematics learning (Loong & Herbert, 2018). Thus, demonstrating an understanding of the pedagogical underpinnings of the curriculum and how both learning areas are connected.

Robotics provide students with tangible programming outcomes and an interactive context to apply mathematics and digital technologies concepts. Of interest to the study was the incidental mathematics that occurred due to the integration and use of these robotics in a task. In Case 1, David's knowledge of the concepts that underpin the understanding required to complete the task meant that he recognised that his students also needed to learn one-to-one correspondence, even though it is not listed in the developmental sequence in mathematics. One-to-one correspondence is an essential part of learning to count and is assumed knowledge at the Prep level. The need for one-to-one correspondence demonstrated how an integrated task using digital technologies can help bring up the students' level of mathematics and develop other mathematical concepts not in the original task design. David's ability to identify the connection between digital technology and mathematics enabled him to enhance the learning opportunities for his students. David realised during the lesson that he needed to draw on further *knowledge of content and curriculum* (Ball et al., 2008) that he had not considered whilst planning the lesson.

Teachers need a solid understanding of curriculum to integrate (Ross & Marshman, 2023). For Vera and Sarah (Case 2), it was only after they began teaching the lesson that they recognised the incidental mathematics underpinning the task. Programming robots to move involves understanding geometry and spatial reasoning. In Case 2, the teachers reported how concepts such as angles, distances and measurement were incidentally taught; for students to succeed in the task, they needed to use these concepts in a tangible manner to control the robot's movement. The integrated task using a Pro-Bot included more mathematics than just the symmetry they planned to address; it provided an opportunity to make mathematical connections between facts, procedures and relationships, an essential component in constructing mathematical understanding (Eli et al., 2011). Whilst teaching this integrated task Vera and Sarah realised that they too needed to draw on further *knowledge of content and curriculum* (Ball et al., 2008) not considered during lesson planning.

The incidental mathematics learning in the cases outlined was unplanned, underscoring the importance of teachers possessing understanding of both the mathematics and digital technologies curriculum content to design integrated tasks and optimal learning opportunities. Evident also in the cases was how integrated applications of mathematical skills and concepts required students to demonstrate a deeper understanding of mathematics than simple memory of formulas and procedures. In these examples, the integrated task allowed students to see connections between different mathematical concepts and the relationship between mathematics and the real world.

Conclusion

This paper provided a small snapshot of data analysed from a larger study. The paper serves to highlight the incidental mathematics that arose for teachers from two participating schools during teaching of integrated mathematics and digital technologies tasks. While the scope of the paper is confined to the experiences of these two schools, their insights serve to demonstrate the broader connections across mathematical concepts that can emanate from across diverse educational settings if the teachers see the conceptual connections.

The deliberate focus in version 9 of the Australian Curriculum to establish connections between mathematics and digital technologies has enriched the possibilities for seamless integration in these learning areas. As highlighted in the two case studies, the incorporation of digital technologies into mathematics education has unveiled opportunities for incidental mathematical learning. Making mathematical connections is an essential component in the construction of mathematical knowledge, and this study indicates that teachers need to possess *knowledge of content and curriculum* (Ball et al., 2008). Specifically, an understanding of both *AC: Mathematics* and *AC: Digital Technologies* curriculum content, and an understanding of the pedagogical underpinnings of the curriculum and knowledge of how both learning areas are connected, to create integrated learning experiences. Emphasising connections between the two curriculum areas and promoting a holistic view of mathematics lays the foundation for students to apply a range of mathematical concepts in real-world situations using digital technologies. Further, more sustained research across a broader range of contexts would be useful to further analyse the range of connections teachers are able to find from integrated contexts.

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References

An, Y.J. & Reigeluth, C. (2012). Creating technology-enhanced, learner-centered classrooms: K–12 teachers' beliefs, perceptions, barriers, and support needs. *Journal of Digital Learning in Teacher Education, 28*(2), 54–62.

Attard, C. (2013). Introducing iPads into primary mathematics pedagogies: an exploration of two teachers' experiences. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow: Proceedings of the Mathematics Education Research Group of Australasia* (pp. 58–65). Melbourne: MERGA.

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2021). *What has changed and why? Proposed revisions to the foundation—year 6 (F–6) Australian Curriculum.* https://www.australiancurriculum.edu.au/media/7157/ac_review_2021_primary_whats_changed_and_why.pdf

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022a). *Understanding this learning area—Technologies.* https://v9.australiancurriculum.edu.au/teacher-resources/understand-this-learning-area/technologies

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022b). *Understanding this learning area—Mathematics.* https://v9.australiancurriculum.edu.au/teacher-resources/understand-this-learning-area/mathematics

Ball, D.L., Thames, M. H. & Phelps, G. (2008). Content Knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Braun, V. & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Eli, J.A., Mohr-Schroeder, M.J. & Lee, C.W. (2011). Exploring mathematical connections of prospective middle-grades teachers through card-sorting tasks. *Mathematics Education Research Journal*, *23*, 297–319. https://doi.org/10.1007/s13394-011-0017-0

Larkin, K., & Miller, J. (2020). Digital technologies and numeracy-synergy or discord? In A. MacDonald, L. Danaia, & S. Murphy (Eds.), *STEM education across the learning continuum: Early childhood to senior secondary* [Online]. Springer. https://doi.org/10.1007/978-981-15-2821-7_8

Loong, E.YK. & Herbert, S. (2018). Primary school teachers' use of digital technology in mathematics: The complexities. *Mathematics Education Research Journal*, *30*, 474–498. https://doi.org/10.1007/s13394-018-0235-9

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. NCTM.

Nayler, J. (2014). *Enacting Australian Curriculum: Making connections for quality learning—An issues paper prepared for the Queensland Studies Authority*. QSA. http://www.qcaa.qld.edu.au/downloads/p_10/ac_enact_ac_paper.pdf

Perienen, A. (2020). Frameworks for ICT integration in mathematics education—A teacher's perspective. *Eurasia Journal of Mathematics, Science and Technology Education*, *16*(6), em1845. https://doi.org/10.29333/ejmste/7803

Rolfe, G., Freshwater, D. & Jasper, M. (2001). *Critical reflection for nursing and the helping professions: A user's guide. Basingstoke*, Palgrave Macmillan.

Ross, E. & Marshman, M. (2023). Barriers to integration: A case study of STEM-learning in mathematics and digital technology. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 427–434). Newcastle: MERGA.

Ross, E. (2024). Teachers' interpretation of curriculum as a window into 'curriculum potential'. *Curriculum Journal (London, England)*, 35(1), 38–55. https://doi.org/10.1002/curj.239

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*, 255–281. https://doi-org.ezproxy.usc.edu.au/10.1007/s10857-005-0853-5

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Srinivasa, K. G., Kurni, M. & Saritha, K. (2022). *Learning, teaching, and assessment methods for contemporary learners: pedagogy for the digital generation.* Springer. https://doi.org/10.1007/978-981-19-6734-4_4

Yin, R.K. (2009). *Case study research: Design and methods*. Sage.

Do Primary School Teachers Prefer Digital or Non-Digital Games to Support Mathematics Instruction?

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In this paper we explored primary school teachers preference for different game modes to support mathematics teaching and learning. Eighty-four teachers played digital and non-digital addition and subtraction games that were functionally equivalent during professional learning workshops. Most teachers indicated that they would be more likely to use the non-digital mode; despite more mixed views around perceived effectiveness for supporting learning and anticipated student preferences. Key reasons as to why teachers tended to prefer non-digital or digital games are examined.

The majority of primary school teachers use games in the majority of their mathematics lessons to support instruction (Russo et al., 2021). Accordingly, playing mathematical games has received substantial interest in the research and practitioner literature, partially because games are viewed as an effective pedagogy for making mathematics education more engaging for students, particularly in the primary school years (Bragg, 2012; Bright et al., 1985; Russo et al., 2021). In addition to facilitating engagement with mathematics, *non-digital games* have been associated with many positive cognitive and affective outcomes, including but not limited to: promoting mathematical reasoning (McFeeters & Palfy, 2018); supporting mathematics achievement (Kamii et al., 2005); reducing anxiety in learning mathematics (Alanazi, 2020); developing students' growth mindset (White & McCoy, 2019); and encouraging active learning, cooperation, and interactivity (Ernest, 1986). Moreover, studies focussed on using *digital games* to support mathematics instruction for primary-aged students have also revealed many cognitive and affective benefits, including mastery of number facts (Abdullah et al., 2012), improved attitudes towards learning mathematics (Miller & Robertson, 2011), and enhanced mathematical self-efficacy (Hung et al., 2014).

Despite evidence that they generate similar educational outcomes when used to support mathematics instruction, there are at least two differences between digital and non-digital games that are worth noting. First, non-digital games often incorporate physical activity, sometimes outside the mathematics classroom (e.g., Bahrami et al., 2012; Cichy et al., 2020), which tends to not be the case for digital games. Secondly, digital games involve students interfacing in an environment that has been principally "regarded as an entertainment medium" (Cojocariu & Boghian, 2014, p. 641), whereas non-digital games tend to involve repurposing educational resources and mathematical representations as game objects e.g., fraction walls (Clarke & Roche, 2010); and number lines, (Bofferding, 2014).

Generally, research into the effectiveness of mathematical games has focussed on one particular mode of game, either digital or non-digital, and compared playing games in this mode with non-game activities (e.g., Abdullah et al., 2012; Bragg, 2012; Kamii et al., 2005; Miller & Robertson, 2011). Indeed, we could not identify any studies that directly compared playing digital games to support mathematics learning with playing non-digital games. In addition, existing research into mathematical games has tended to focus on the impact of games on student outcomes, with fewer studies concerning themselves with how teachers use games in classrooms to support mathematics instruction (Russo et al., 2021). Consequently, substantial gaps exist in the literature, both with regards to research that aims to directly compare and contrast digital and non-digital games in mathematics education, as well as studies that examine the issue of how and why teachers choose to use particular games.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 463–470). Gold Coast: MERGA.

Background to the Current Study

We recently undertook a questionnaire asking primary school teachers if they preferred, and were more inclined to use, digital games or non-digital games to support mathematics instruction, and the reasons for these preferences (Russo et al., in press). It is important to note that the questionnaire asked about these teachers' general game mode preferences, rather than asking them to compare a specific non-digital and digital game. Teachers reported that they were three times more likely to use a non-digital game in a whole-class setting compared with a digital game, and five times more likely in a small-group setting. Moreover, most teachers indicated they preferred using non-digital games (59% whole class setting; 70% small group setting), with only a small number of teachers indicating a preference for digital games (6% whole class setting; 5% small group setting). The remaining participants had no general preference for a particular game mode (35% whole class setting; 25% small group setting).

When asked to elaborate on the reasons why they were more likely to use, and preferred, non-digital games, teachers tended to put forward pedagogical reasons first and foremost. This included elements such as non-digital games' enhanced capacity to promote collaboration and communication, that non-digital games could be more easily adapted by the teacher to suit the learning needs of students, and that non-digital games afforded the students opportunities to engage with manipulatives and concrete materials that would support their mathematics learning. Teachers also indicated that it was easier to monitor and assess students when they were engaged in a non-digital game, and that they valued the fact that choosing to use non-digital games rather than a digital game reduced the amount of screen time students were exposed to (Russo et al., in press).

However, in addition to these pedagogical and assessment related reasons for preferring non-digital games, some teacher participants indicated that this preference was driven by specific barriers to utilising digital games. Although some of these barriers were related to technology usage in the classroom more generally (e.g., a lack of expertise in technology, lack of access to reliable technology), others related to digital mathematical games in particular. For instance, some teachers described a lack of awareness of what they would deem to be high quality/suitable 'digital' games, which they juxtaposed with their deep knowledge and familiarity with non-digital games. As one teacher stated: "I found it is hard to find games online which match what you are trying to teach the kids" (Participant #120); whilst another indicated: "My own knowledge of good mathematical games tends towards non-digital" (Participant #190) (see also Russo et al., in press).

Reflecting on these findings, we speculated that it was at least possible that a lack of awareness of what teachers determined to be high quality digital games was the principal reason that teachers preferred non-digital games, effectively because they were not comparing 'like with like'. The contention is that this lack of awareness might be interacting with pedagogical factors to determine their preferences. For example, a teacher who states that "often the answer is required with little knowledge of strategy or understanding in online games" (Participant #12) is unlikely to view digital games as supporting effective collaboration and communication (Russo et al., in press). Consequently, we decided it would be valuable to explore the issue of game mode preference in a workshop context, where the barrier to being able to access high quality digital games was removed and the digital and non-digital modes of the game being compared were 'functionally equivalent'. The general research question guiding the current study can be stated as: Do primary school teachers prefer digital or non-digital games to support mathematics instruction? Our two sub-questions included:

- What preference, if any, do primary school teachers have for a particular game mode (digital versus non-digital) to support mathematics instruction, in the context of comparing two functionally equivalent games?

- To the extent that they exist, what are the reasons for these preferences?

Before proceeding, it is worth noting that, in this context, we use the term 'functionally equivalent' to refer to the fact that the digital and non-digital mode of the game were equivalent in important ways including: the specific mathematical concept(s) in focus; the game objective; the rules that governed game play; and the types of mathematical representations used within the game. This functional equivalence was a direct result of the two digital games being designed to be digitised versions of the non-digital games Get Out of My House and Part-Whole Triangles (Russo, 2020), as part of the ABC GOAT Maths Suite (Gervasoni & Russo, 2023).

Method

Participants and Procedure

Eighty-four primary school teachers from five Victorian schools participated in one of a series of school-based professional learning workshops with the first author, where they were given an opportunity to play three pairs of non-digital and digital games focussed on addition and subtraction concepts. For each pair of games, the digital game was explained and played (approx. 10 minutes), followed by the corresponding non-digital game (approx. 10 minutes).

After playing each pair of games, teachers were then invited to complete a pen and paper questionnaire to reflect on their experiences. For each game pair, participants were presented with three forced-choice items with a corresponding open-response item where they were asked to 'please explain their response'. The forced-choice items were each presented on a five-point scale. See Table 1, Table 2 and Table 3 for details about both the items and response options.

Following these workshops, the questionnaire data was subsequently entered into a spreadsheet program (Microsoft Excel v. 6.2.14) in preparation for analysis. Note that, although teachers played three pairs of games (six games in total) during these workshops, only two of these pairs (Pair 1: *Get Out of My House* and *Goat Crashers*; Pair 2: *Goat Squad* and *Part-Whole Triangles*) can be clearly described as 'functionally equivalent'. It is these comparisons that form the focus of the current study. Note that readers interested in learning about the precise mechanics of these digital games can read about (and play) these games through the ABC website: https://www.abc.net.au/education/topic-goat-maths/102180130.

Approach to Data Analysis

In order to address the first research question, quantitative data were imported into SPSS Statistics, v. 26 for analysis. This enabled differences between teachers' preferences and usage of functionally equivalent digital and non-digital games to be explored. In addition to the presentation of descriptive statistics, a series of one-sample t-tests were undertaken to compare the obtained mean values regarding teacher views about game mode with expected results if, on average, teachers did not have game mode preferences (reference mean score = 3.0).

By contrast, the second research question focussing on the reasons for these preferences involved analysing the open-response items qualitatively. Specifically, for each question (which one do students prefer, which one is more effective for learning, and which one are you more likely to use) the teacher's open responses were sorted into whether they had preferred (a lot or somewhat) Get Out of My House or Goat Crashers or whether they preferred each game equally. Note that only the qualitative analysis of the Get Out of My House and Goat Crashers comparison is included, as time constraints towards the end of three of the workshops meant that participants were not afforded adequate opportunity to respond to the final open-ended item inviting them to explain their preference for either Part-Whole Triangles or Goat Squad.

The written responses to the three questions were each analysed separately using qualitative line-by-line coding as outlined by Braun and Clarke (2012). The following stages were conducted: familiarisation with the data; generating initial themes; merging responses that can be accommodated into a single theme; defining and naming themes; and producing a table with illustrative quotes for each theme. For brevity, the data from each table that described teachers' explanations for choosing Goat Crashers or Get Out of My House were combined and are synthesised in Tables 4 and 5 respectively. Illustrative quotes that elaborate the definition for each theme are provided. Note that some responses were coded to more than one theme. Hence the sum of the *n* values may be greater than the number of responses (*n*). For example, when responding to: Which game would you be more likely to use with students in a primary school classroom, Participant A11 (who selected Get Out of My House) wrote "Easier to set up and more monitorable." This response was assigned to two themes *Easier to set up* and *Easier to monitor students* (see Table 4).

Results and Discussion

The first research question was focussed on the extent to which primary school teachers indicated a preference for a particular game mode (digital versus non-digital) to support mathematics instruction, in the context of comparing two functionally equivalent games. This involved teachers' comparing each pair of games along three dimensions: which mode they thought students would prefer playing; which mode they thought was more effective for supporting mathematics learning; and which mode they would be more inclined to use in the classroom with students. Results of these comparisons are detailed in Table 1, Table 2 and Table 3 respectively and are summarised below. Note that, within these tables, Pair 1 refers to Get Out of My House (non-digital) and Goat Crashers (digital), whilst Pair 2 refers to Part-Whole Triangles (non-digital) and Goat Squad (digital).

Table 1

Question: Which Game do you Think Students Would Prefer Playing?

Teachers' response option (percentage of respondents)	**Pair 1 (*n*=84)**	**Pair 2 (*n*=81)**
(1) Students would strongly prefer [Non-Digital Game]	12%	21%
(2) Students would somewhat prefer [Non-Digital Game]	15%	30%
(3) Students would like them the same	40%	28%
(4) Students would somewhat prefer [Digital Game]	23%	15%
(5) Students would strongly prefer [Digital Game]	10%	6%
Mean score (*SD*)	3.0 (1.1)	2.6 (1.2)

Overall, there was partial evidence that teachers anticipated students would prefer playing non-digital games. Specifically, while teachers, on average, anticipated that students would prefer playing Part-Whole Triangles to Goat Squad, $t(80) = -3.44$, $p < 0.01$, a one-way *t*-test using "Students would like them the same" (3.0) as the reference value revealed that, on average, teachers did not anticipate that students would prefer Get Out of My House to Goat Crashers, $t(83) = 0.19$, $p > 0.05$. Indeed, Table 1 indicates that around half of primary teachers thought that students would prefer Part-Whole Triangles (51%), compared with less than one-quarter anticipating that students would prefer the digital equivalent Goat Squad (21%). By contrast, there was comparatively little difference between the number of teachers who thought that students would prefer Get Out of My House (27%) compared with Goat Crashers (33%).

Similarly, there was only partial evidence that teachers believed that the non-digital mode of the game would be more effective for supporting mathematical learning than the digital mode. Again, while a one-way *t*-test using "They are both equally effective for supporting maths learning" (3.0) as the reference value indicated that, on average, teachers believed Part-Whole Triangles was more effective than Goat Squad $t(81) = -4.65$, $p<0.01$, there was no significant difference in perceived effectiveness when comparing Get Out of My House with Goat Crashers, $t(83) = -1.51$, $p > 0.05$. Whereas Table 2 indicates that most teachers (54%)

thought that Get Out of My House and Goat Crashers were equally effective for supporting student learning, only just over one-third of teachers (39%) held this view when comparing Part-Whole Triangles to Goat Squad.

Table 2

Question: Which Game do you Think is More Effective for Supporting Student Maths Learning?

Teachers' response option (percentage of respondents)	**Pair 1 (*n*=84)**	**Pair 2 (*n*=82)**
(1) [Non-Digital Game] is a lot more effective for supporting maths learning	10%	20%
(2) [Non-Digital Game] is somewhat more effective for supporting maths learning	19%	29%
(3) They are both equally effective for supporting maths learning	54%	39%
(4) [Digital Game] is somewhat more effective for supporting maths learning	13%	9%
(5) [Digital Game] is a lot more effective for supporting maths learning	5%	4%
Mean score (*SD*)	2.9 (0.9)	2.5 (1.0)

Contrasting somewhat with the data revealed in Tables 1 and 2, there was clear evidence that teachers believed they would be more likely to use the non-digital game mode of the game with students in the future. Specifically, using "I would be equally likely to use either game" (3.0) as the reference value, teachers indicated they were more likely to use both Part-Whole Triangles, $t(80) = -6.99$, $p < 0.01$, and Get Out of My House $t(82) = -3.19$, $p < 0.01$. Table 3 confirms that teachers were substantially more inclined to report having a strong preference for using the non-digital games (38% Part-Whole Triangles; 22% Get Out of My House) compared with their digital counterparts (2% Goat Squad; 6% Goat Crashers).

Table 3

Question: Which Would you be More Likely to use With Students in a Primary School Classroom?

Teachers' response option (percentage of respondents)	**Pair 1 (*n*=83)**	**Pair 2 (*n*=81)**
(1) I would be a lot more likely to use [Non-Digital Game]	22%	38%
(2) I would be somewhat more likely to use [Non-Digital Game]	16%	21%
(3) I would be equally likely to use either game	48%	31%
(4) I would be somewhat more likely to use [Digital Game]	8%	7%
(5) I would be a lot more likely to use [Digital Game]	6%	2%
Mean score (*SD*)	2.6 (1.1)	2.2 (1.1)

To summarise, teachers were far more likely to indicate interest in using the non-digital mode of a game with students back in their classrooms after playing both functionally equivalent modes in a workshop setting. This is despite the fact that their views about whether the non-digital game was more effective for supporting learning, or would be preferred by students, appeared to depend at least somewhat on the specific pair of games being compared. It is worth noting that such preferences for non-digital games are consistent with prior research inquiring into primary school teachers existing practice (Russo et al., in press).

In juxtaposing the two digital games, it was apparent that teachers viewed Goat Crashers more favourably than Goat Squad. Potential reasons for this might include the comparative richness of their respective digital landscapes in drawing students into the game world, and perceived opportunities to support mathematical thinking. Regarding this latter point, it is worth noting that Goat Crashers allowed students to choose an operation to construct an incomplete number sentence, and then provided immediate feedback to students about whether or not they are correct (e.g., an incorrect response brings up the prompt, "Think again: Is there another way to work out the problem?"). By contrast, Goat Squad only provided feedback once students had correctly placed three numbers that shared a part-part-whole relationship in the appropriate box (e.g., 3, 4, 7), and did not make connections back to number sentences.

The second research question was concerned with examining the reasons teachers provided for these preferences and was limited to comparing Goat Crashers with Get Out of My House. Results of the thematic analysis are presented in Tables 4 and 5. Note that explanations for neutral responses are not presented in this paper due to space constraints, however the most frequently provided explanation for a neutral response could be coded to the theme *Prefer both equally as they are similar in content or serve the same purpose* (e.g., "It is really the exact same game, so they are equally effective", Participant A42). Most remaining neutral responses could be coded to a second theme of *Prefer both equally but for different reasons* (e.g., "Students would like Get Out of My House as they could physically throw the opponents counter off the board. Likewise, the digital mode would appeal to students because of the animation, sound, as well as using a computer … Different formats appeal to different students to support their maths learning", Participant A39).

As indicated by Table 4, comfortably the most common reason provided for preferring Goat Crashers was the reported richness of the digital landscape to draw students into the game; that is, sound effects, graphics, movement, characters and storyline. This explanation resonates with the notion that digital games involve players interfacing in an environment that has been principally constructed to entertain (Cojocariu & Boghian, 2014).

Table 4

Teachers' Preferences for 'Goat Crashers' Including Their Beliefs Around Student Preferences, Effectiveness for Learning, and Their Likelihood to Use

Main themes	**Response (*n*=52)**	**Illustrative quotes**
Sound effects, graphics, move–ment, characters, & storyline	23	A3—Children would enjoy the sounds effects and graphics
More engaging "fun"	10	A8—It is more stimulating and engaging being an online game
Preference for digital games	8	A12—[I prefer] the tech aspect
Easier to set up	6	A32—Less preparation, easier access
Multiplayer/Group work	6	A36—Accessible for small groups to share and discuss
Feedback	5	A4—Goat Crashers gives instant feedback for students. If they're playing in pairs and get the answer wrong, in Get Out of My House they may not necessarily realise

By contrast, the most frequently offered reason for preferring Get Out of My House related to its hands-on nature and opportunity to utilise concrete materials (see Table 5). Other notable reasons included opportunities for interaction and discourse, and the potential to modify the game for different learners. Indeed, these three most frequently offered explanations for preferring Get Out of My House resonated with teachers' pedagogical reasons for preferring non-digital games in general as gleaned from previous research, namely: better for promoting collaboration, interaction, and communication; more 'hands-on' and opportunities to use manipulatives; and easily adapted/ differentiated (Russo et al., in press).

The notion that a comparative advantage of Get Out of My House is its 'modifiability' is particularly noteworthy and perhaps somewhat surprising, given that teachers were shown multiple versions of Goat Crashers in addition to the addition and subtraction mode that they played. This included versions to support multiplication and division fact practice with various target multiples, and a more cognitively demanding version that involved the choice of using any operation.

Table 5

Teachers' Preferences for 'Get Out of My House' Including Their Beliefs Around Student Preferences, Effectiveness for Learning, and Their Likelihood to Use

Main themes	**Response (*n*=83)**	**Illustrative quotes**
Use of materials/ Hands-on	20	A1—The rolling of the dice and the counters I see the students really engaging with
More interaction and discourse	14	B28—It provided opportunity for collaboration and exchange of emotions between players
Easier to modify, adapt or differentiate	12	A20—There are multiple ways you can play the game (i.e., more counters, multiplication, sight words)
More engaging "fun"	11	A30—Students would enjoy saying 'get out of my house' and having a laugh
Disadvantages of digital games	10	A14—Cognitive overload in the Goat Crashers game; noises detract from application of mathematical skill
Suitability for age group	10	A40—Easier for younger students. A29—[Suitable] for older students
Easier to set up	8	A15—Materials are on hand in the classroom. Quicker than getting out devices and logging on
Easier to monitor students	7	A5—Has ability for the teacher to monitor student progress better, so as to be able to scaffold students if needed
More opportunities for challenging maths work	6	A43—GOMH gives children more opportunity to think about their equations, decide on an operation, write it down if needed. More skills involved than the digital version

Consequently, it appears that teachers valued the capacity to adapt and modify Get Out of My House in highly specific ways to support students, rather than merely having access to pre-set variations. Interestingly, the idea that a given mode of the game was more engaging from a student perspective, or easier to setup, emerged as a reason for preferring both the digital and non-digital modes. Moreover, while some teachers preferred Goat Crashers specifically because they preferred using digital games, other teachers indicated a preference for Get Out of My House precisely because they perceived digital games to have specific disadvantages, such as the potential for cognitive overload.

Conclusions

Although research has revealed similar cognitive and affective benefits to using non-digital and digital games in the primary mathematics classroom, previous studies have tended to not consider teacher views concerning the types of games they prefer to utilise to support instruction. The current study supports our recent finding that most teachers prefer using non-digital games over digital games, principally for pedagogical reasons (Russo et al., in press). That these preferences were revealed even in a context where the non-digital and digital modes of the game were functionally equivalent is noteworthy. It suggests that teachers' tendency to prefer non-digital games is not principally driven by issues such as a lack of access to high-quality digital games. Future research should consider conducting classroom-based quasi-experimental and/or observational studies to examine whether the comparative pedagogical benefits of playing non-digital games anticipated by teachers are actually borne out in practice. Moreover, such research should examine the extent to which these benefits are moderated by factors such as the teacher's digital literacy, as well as their beliefs about digital technology.

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References

Abdullah, M. R. T. L., Abu Bakar, Z., Ali, R. M., Faye, I., & Hasan, M. H. (2012). The impact of video games in children's learning of mathematics. *World Academy of Science, Engineering and Technology*, *64*, 968–974.

Alanazi, H. M. N. (2020). The effects of active recreational maths games om maths anxiety and performance in primary school children: An experimental study. *Multidisciplinary Journal for Education, Social and Technological Sciences*, *7*(1), 89–112. https://doi.org/10.4995/muse.2020.12622

Bahrami, F., Chegini, Z. R., Kianzadeh, A., Emami, F., & Abdi, H. (2012). A comparison of the effectiveness of game-based and traditional teaching on learning and retention of first grade math concepts. *European Journal of Experimental Biology*, *2*(6), 2099–2102.

Bofferding, L. (2014). Order and value: Transitioning to integers. *Teaching Children Mathematics*, *20*(9), 546–554. https://doi.org/10.5951/teacchilmath.20.9.0546

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Bragg, L. A. (2012). The effect of mathematical games on on-task behaviours in the primary classroom. *Mathematics Education Research Journal*, *24*, 385–401. https://doi.org/10.1007/s13394-012-0045-4

Bright, G. W., Harvey, J. G., & Wheeler, M. M. (1985). Learning and mathematics games. *Journal for Research in Mathematics Education (Monograph)*, *1*, 1–189. NCTM. https://doi.org/10.2307/749987

Cichy, I., Kaczmarczyk, S., Kruszwicka, A., Przybyla, T., Klichowski, M., & Rokita, A. (2020). Participating in physical classes using eduball stimulates acquisition of mathematical knowledge and skills by primary school students. *Frontiers in Psychology, 11*. https://doi.org/10.3389/fpsyg.2020.02194

Cojocariu, V.-M., & Boghian, I. (2014). Teaching the relevance of game-based learning to preschool and primary teachers. *Procedia—Social and Behavioral Sciences, 142,* 640–646. https://doi.org/10.1016/j.sbspro.2014.07.679

Clarke, D., & Roche, A. (2010). The power of a single game to address a range of important ideas in fraction learning. *Australian Primary Mathematics Classroom, 15*(3), 18–23.

Ernest, P. (1986). Games: a rationale for their use in the teaching of mathematics in school. *Mathematics in School*, *15*(1), 2–5.

Gervasoni, A., & Russo, J. (2023, August 11). *These maths games are the greatest of all time—And students will love playing them*. Australian Broadcasting Commission. https://www.abc.net.au/education/these-maths-games-are-the-greatest-of-all-time/101946832

Hung, C.-M., Huang, I., & Hwang, G.-J. (2014). Effects of digital game-based learning on students' self-efficacy, motivation, anxiety, and achievements in learning mathematics. *Journal of Computers in Education*, *1*(2-3), 151–166. https://doi.org/10.1007/s40692-014-0008-8

Kamii, C., Rummelsburg, J., & Kari, A. (2005). Teaching arithmetic to low performing, low SES first graders. *Journal of Mathematical Behavior*, *24*, 39–50. https://doi.org/10.1016/j.jmathb.2004.12.004

McFeetors, P. J., & Palfy, K. (2018). Educative experiences in a games context: Supporting emerging reasoning in elementary school mathematics. *Journal of Mathematical Behavior*, *50*, 103–125. https://doi.org/10.1016/j.jmathb.2018.02.003

Miller, D. J., & Robertson, D. P. (2011). Educational benefits of using game consoles in a primary classroom: A randomised controlled trial. *British Journal of Educational Technology*, *42*(5), 850–864. https://doi.org/10.1111/j.1467-8535.2010.01114.x

Russo, J. (2020). Short activity: Part-whole triangles. *Australian Primary Mathematics Classroom*, *25*(3), 16–17.

Russo, J. Bragg, L. & Russo, T. (2021). How primary teachers use games to support their teaching of mathematics. *International Electronic Journal of Elementary Education*, *13*, 407–419. https://doi.org/10.26822/iejee.2021.200

Russo, J., Roche, A., Russo, T., & Kalogeropoulos, P. (in press). Examining primary school educators' preferences for using digital versus non-digital games to support mathematics instruction. *International Journal of Mathematical Education in Science and Technology*.

White, K., & McCoy, L. P. (2019). Effects of game-based learning on attitude and achievement in elementary mathematics. *Networks: An Online Journal for Teacher Research*, *21*(1), 1–17. https://doi.org/10.4148/2470-6353.1259

Mathematics Lecturer's Adaption to Online Teaching in Response to COVID-19

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Prior to the COVID-19 pandemic, the university educational system in Nigeria largely employed traditional, face-to-face classroom approaches for teaching. This study examines how mathematics lecturers adapted to online teaching in response to COVID-19 restrictions. A mixed methods approach was used to obtain both qualitative and quantitative data from ten mathematics and mathematics education lecturers, using questionnaires and semi-structured interviews. The results highlight mathematics and mathematics education lecturers' use of virtual boards, writing pads, and WhatsApp to improve interactions while teaching online via Zoom.

The online medium has created an entirely new landscape of educational opportunities (Sven & Julie, 2020). The globe is rapidly becoming more interconnected and online education is growing rapidly (Okyere et al., 2022). As a result, students can access a high-quality education at a reduced cost, regardless of their location or schedule (Xu & Xu, 2019). In Nigeria, online teaching and learning practices are at the developmental stage. Socioeconomic, sociocultural, and IT infrastructural factors are recognised challenges hindering the adoption of online learning in Nigeria (Abdulmajeed et al., 2020). The global outbreak of COVID-19 greatly impacted educational systems worldwide, including Nigeria, where institutions were compelled to close in response to the pandemic (Lawal, 2020; Oyedotun, 2020). This closure necessitated a shift to online delivery methods, utilizing various platforms such as radio, TV, *WhatsApp, Zoom, Google Meet, YouTube*, and other internet-based learning (Lawal, 2020). This transition was a significant move from traditional reliance on in-person teaching methods, prompting educational institutions to adapt online approaches for their content dissemination (Lawal, 2020). Prior to the pandemic, Nigeria had only 13 approved distance learning centres among its 221 universities and 123 affiliate campuses (National Universities Commission [NUC], 2023). Amidst the COVID-19 pandemic, lecturers and students adapted from a face-to-face instructional approach to teaching and learning remotely using online media (Lawal, 2020). Consequently, lecturers and students had to adapt to the new teaching and learning approaches (Sabitu et al., 2022) and utilise technology in the teaching and learning process (Sven & Julie, 2020). In addition, educational institutions had to upgrade their IT infrastructure to enhance students' synchronous and asynchronous learning experiences (Lawal, 2020).

According to Lawal (2020), educational institutions and providers have the primary responsibility for creating an enabling environment for online teaching, offering sufficient materials and resources to support the flexibility of online teaching. Adaptation involves making adjustments to teaching and learning methods, enabling teachers to devise strategies suitable for operating in the new teaching environment (Mardiana, 2020). The pandemic necessitated the widespread adoption of online teaching and learning as an alternative method of education.

Undergraduate mathematics courses necessitate students' comprehension of the taught concepts, which often entails engaging in various activities during lecture sessions. Mathematics is a subject that is often not preferred by many students (Sven & Julie, 2020). The technical nature of mathematics can make it difficult to teach online (Trenholm, 2013), although Borba et al. (2016) see online mathematics teaching as an opportunity to disrupt traditional teaching processes to the benefit of students and their learning. Teaching using online media

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 471–478). Gold Coast: MERGA.

and technology is not new, but online teaching approaches are thought to be more suited for non-mathematical courses (Hamdan & Amorri, 2022). Mathematics holds considerable importance in Nigerian tertiary education, serving as a fundamental unit across various undergraduate courses, particularly in STEM fields (National Universities Commission [NUC], 2015). All first-year undergraduate students are required to take a mathematics course as part of their general studies curriculum. This course is crucial as it equips students with essential mathematical and computational skills necessary for their studies. In Nigerian universities, mathematics lecturers typically belong to two categories: those teaching mathematics courses and those focusing on mathematics teaching methodology, covering primary and secondary mathematics content. The mathematics education curriculum includes both mathematics and methodology courses, with students being taught by both types of lecturers (NUC, 2015).

Onyema (2020) advocated for educators to incorporate and utilise emerging technologies in their teaching and learning. He observed that utilizing technology is now essential due to the need for flexibility in methods. The influence of technology on education shows the growing need for adaptable teaching methods to promote creativity and innovation in the teaching process (Lawal, 2020). The COVID-19 pandemic prompted a shift from traditional face-to-face teaching to online instruction. This transition led to modifications in the teaching approaches of lecturers, highlighting the importance of understanding their methods and actions during online interactions with students (Muir et al., 2022). The rapid switch from face-to-face to online teaching within a short period had its impact on lecturers (Tella & Obim, 2022). The impacts identified included lecturers' inability to communicate their intended learning content to students via the online mode, and the inability to effectively use the technological devices and platforms provided for effective teaching and learning (Lawal, 2020). Trenholm (2013) had already identified lack of student-teacher interactions as an issue for teaching and learning of mathematics online.

According to Trenholm, teaching mathematics online presents a challenging environment for instructors to effectively adapt their teaching methods to promote students' understanding of mathematics. To address this challenge, lecturers should utilise devices and tools that facilitate interaction during the teaching process (UNESCO UNICEF, 2020). Tella and Obim (2022) identified several barriers to online teaching and learning in Nigeria, including limited access to electronic devices, inadequate internet connectivity, power, challenges in time management, and financial constraints associated with hosting online lectures.

Therefore, this study investigates how lecturers in Nigeria adapted their teaching from face-to-face to online. The study explores mathematics and mathematics education lecturers' transition to online teaching as caused by responses to COVID-19. This study discusses how lecturers adopted online teaching tools for mathematics teaching, their perspectives about the nature of teaching mathematics online, and their knowledge and use of online teaching tools during the pandemic. This study seeks to answer the research question:

How did mathematics and mathematics education lecturers in a Nigerian university adapt their teaching approach to online teaching during the COVID-19 pandemic?

Methodology

This study, which comes from the first author's ongoing PhD research project, adopted a mixed-methods approach (Creswell & Clark, 2017) to obtain in-depth information about the lecturers' online teaching experiences. The study was exploratory and both qualitative and quantitative data were collected. A case study approach was adopted to explore the phenomenon of transition to online mathematics teaching among mathematics education lecturers (MEduL) as Case 1 and mathematics lecturers (ML) as Case 2. The research took place at a private university in southwestern Nigeria, involving five participants each from Case 1 and Case 2, who had prior experience or had recently transitioned to online mathematics teaching due to

COVID-19. The selection of these groups was based on the involvement of both mathematics and mathematics education lecturers in teaching the undergraduate mathematics education curriculum. Full ethical approval was obtained from the researcher's university and the participating institution.

A survey was conducted and followed up with a semi-structured interview; these were used to explore the experiences, interactions, perspectives, and technological knowledge in both cases. Semi-structured interviews allowed for tailored questions and prompts, facilitating in-depth discussions (Galletta, 2013) on topics such as previous teaching experiences, engagement processes, challenges and successes in online teaching, and perspectives on the transition to online mathematics education. Purposive sampling was used to select five participants from each case, and interviews were conducted at convenient times for the participants. Prior to the interviews, all ten lecturers indicated varying levels of agreement with specific survey questions. These questions covered statements such as "I find it easy teaching mathematics face-to-face", "I prefer to teach mathematics face-to-face than online", "I spend more time preparing to teach mathematics online than face-to-face", "My ability to use technology impacts my teaching abilities", and "Teaching online has helped to improve my overall teaching of mathematics".

Data analysis involved transcription of interviews, confirmation of transcribed data by participants, and thematic analysis using NVivo software. The iterative process included familiarization with the data, generating codes, identifying initial themes, reviewing and refining themes, and writing the final report (Clarke & Braun, 2021). The findings presented the experiences of mathematics education and mathematics lecturers during the transition to online teaching. Quotations from participants' interview transcripts were included with anonymization to protect participant identities.

Findings

Table 1 details participants' demographics, and their pre-COVID-19 online teaching experiences. Table 1 indicates that among mathematics education lecturers, three had previous experience using online tools for teaching, while two had no prior online teaching experience before COVID-19. Similarly, among mathematics lecturers, only two had previously utilised online tools for teaching. The semi-structured interview responses indicated that prior to COVID-19, all lecturers and students engaged with online tools or content asynchronously rather than in real-time (synchronously). This was due to the challenges associated with conducting real-time classes. However, students were able to access or search for online content at their own convenience, facilitating deeper understanding. The discussion below gives examples of participants' experiences with online tools prior to the pandemic, "I give students tasks and assignments from online material to encourage their internet use and search for materials on their own. To further motivate students, I conclude my classes by providing additional links to concepts taught" (ML2).

While not all lecturers explicitly provided links, some assigned students tasks that involved online research and assignments. Additionally, ML2 emphasised that these links exposed students to various learning materials related to the concepts taught. He believed this approach motivated students and accommodated different learning styles "Yes, sometimes I use online content to provide students [pre-service teachers] with additional explanations and examples on the approach to teaching children [in school] with high-level need" (MEduL3).

All lecturers attributed their limited usage of online tools for teaching before the pandemic to challenges such as poor internet infrastructure and power supply.

Table 1

Participants' Information

Participant information	Highest qualification	Teaching experiences (years)	Computer and internet skills usage	Using online platforms (years)	Used online tools for teaching prior to COVID
ML1	PhD	7	Proficient	5	Yes
ML2	PhD	8	Proficient	3	Yes
ML3	PhD	12	Proficient	5	No
ML4	PhD	6	Proficient	3	No
ML5	PhD	8	Intermediate	3	No
MEduL1	Masters	5	Proficient	3	Yes
MEduL2	PhD	6	Intermediate	2	No
MEduL3	PhD	27	Proficient	4	Yes
MEduL4	PhD	16	Proficient	4	Yes
MEduL5	PhD	8	Intermediate	1	No

Table 2 identifies the three key themes in the data along with the corresponding generated codes, illustrating how mathematics and mathematics education lecturers adapt to online teaching during the pandemic. The themes were developed through reflexive thematic analysis, derived from the dataset without any preconceptions (Braun & Clarke, 2019), and emerged from generated codes about the mathematics and mathematics education lecturers' adaption to online teaching. The themes represent the combined results from both Case 1 and Case 2, reflecting participants' responses regarding the themes explored in this study.

Table 2

Themes and Codes

Themes	Codes
Online teaching engagement	Approach to teaching, interactions with students, class control, and time management
Perspectives about online mathematics teaching	Nature of mathematics, beliefs, readiness, and capacity to teach
Knowledge and use of technology for teaching	Software used, use of internet to teach or learn, use for professional development

Online Teaching Engagement

The theme "Online teaching engagement" emerged from codes generated during the study, capturing lecturers' approaches to teaching, interactions with students, class control, and time management while teaching online during COVID-19. All participants acknowledged challenges during the initial transition to online teaching, including technical difficulties, poor internet reception, and adjusting to fully online teaching. For example, ML3 said "I transferred my usual face-to-face method to online, but less interaction made it one-sided." He added:

> Unlike the educational system in other parts of the world, our educational system is still developing. So, the experiences are new to most of our students since teaching and learning is usually in face-to-face. It was a fine experience but difficult during the initial lockdown.

Mathematics lecturers mentioned that they create recorded versions of their face-to-face lectures by teaching on the board in the university's ICT centre, with the sessions being recorded. These videos were accessible to students on the university portal before live lectures, aiding comprehension and allowing for questions on areas of difficulty and making the lecture interactive. One participant mentioned that his lack of technical skills, combined with poor internet and power at home, hindered his use of university-provided platforms for online teaching (MEduL2). Another participant expressed concerns about the educational infrastructure's limitations for online teaching, citing challenges such as power generation costs, internet access, and infrastructure expenses beyond the university's control (MEduL3). He further noted:

> As for me during the lockdown, my recorded lectures are online, I used demonstration method and during my presentation I will deliver a summarized lecture utilizing slides … Additionally, I will incorporate gestures to enhance communication, and occasionally, I will play video clips demonstrating responses and actions used for teaching pupils with special needs. Furthermore, students will have access to recorded Zoom lectures, allowing them to re-view the content at any time.

All participants expressed their concern about the level of interactions they had with students. They all expressed the importance of interaction in a mathematics class. ML1 stated:

> Considering the fact that mathematics science requires a lot of interactions during the teaching process. Lecturers' students, relationships on difficult aspects cannot be achieved due to timing in an online class. So, the direct observation of students' understanding through their facial gestures and interactions is missing.

MEduL2 shared similar sentiments regarding facial expressions, gestures, and interactions with students. In contrast, ML2 expressed that, "I feel online teaching has brought students closer to me in terms of interactions and at any time ... They are happy to communicate and resolve pressing problems outside lecture periods with me or their colleagues if need be". Additionally, MEduL1 noted how the breakout room enhanced students' interactions and collaboration while teaching and observed an increased confidence in students presenting the groups viewpoints despite being shy or hesitant in a regular face-to-face setting. MEduL3 mentioned challenges he observed and using *WhatsApp* to increase students' interactions despite students using the application to chat socially. He also said, "Teaching online also presents challenges as it consumes time, and I find it difficult to monitor or assist slow learners' progress". WhatsApp was used more by mathematics education lecturers due to better tolerance with poor internet connectivity.

Perspectives About Online Mathematics Teaching

The theme "Perspectives about online mathematics teaching" emerged from the generated codes, covering aspects such as the nature of mathematics, beliefs, readiness, and capacity to teach mathematics online. Both groups of lecturers expressed similar beliefs regarding online mathematics teaching, as reflected in the codes. Their attitudes align with the findings of Hamdan and Amorri (2022) and Trenholm (2013) who highlighted the technical nature of mathematics as a significant challenge in online teaching. MEDuL4 "With my background in mathematical science, I feel there are difficult branches of mathematics that are not suitable for online teaching". Both Case 1 and 2 participants talked about considering the nature of mathematics in the teaching and learning processes. Specifically, ML5 talked about "the content should be from simple to complex for students to understand". He noted that "Personally, I find teaching mathematics online particularly challenging due to the lack of appropriate teaching tools and the delayed response to and from students' questions".

The lecturers had varying views on how difficult they thought it was to teach mathematics content and mathematics pedagogical content online. ML4 noted that "online teaching platforms are more suitable for theoretical courses" and he relied on students' facial expressions and the questions asked to measure the level of understanding. The assertions of ML2 and MEduL2 supported ML4's view on mathematics teaching processes as they engaged with the contents. Additionally, ML3 noted:

> I cannot observe students' work in an online class, I cannot guide or give student hints when they are hooked [stuck] with steps while finding solution to the question. Unlike face-to-face where facial expressions, questions asked, collaborative solutions of problems on the board by students speak volume to their understanding of the concept taught".

In contrast, MEduL5 noted that "Teaching mathematics methodology online was easier compared to teaching mathematics content for secondary schools". Other participants made similar reference to the difficulty of teaching mathematics education content online, "My concern is how to equip undergraduate students with the practical skills and methods required for teaching mathematics to children [in schools]. These I feel are not exclusively available while teaching online" (MEduL4).

MEDuL3 commented on the significant role that interaction plays in the teaching/learning process. He further identified interactions between the lecturers and students as the major difference between teaching mathematics face-to-face or online.

Knowledge and Use of Technology for Teaching

All ten lecturers were familiar with the use of computers and the internet as shown in Table 1. The semi-structured interviews further revealed participants' use of the internet to search for information, use of emails, and use of online platforms such as *Zoom* and *Google Meet* for presentation at conferences, workshops, or seminars. Mathematics lecturers used additional tools like *Latex*, *Math Lab*, and equation editors for their research papers, and ML1 said he lectures students on how to use these tools and others to build software.

Three mathematics education lecturers and two mathematics lecturers identified using online teaching platforms prior to COVID-19. This was a result of their IT experience, and/or their use of asynchronous tools to teach or learn during their further studies. For example, ML2 said "I used most of these facilities during my studies in United States, therefore I found it much simpler" and MEduL1 said "I am familiar with the tools provided by the university. This is due to my experience with IT tools and some I learned with my current PhD colleagues."

Their passion for teaching, coupled with their experiences and the necessity for research, drove them to utilise online tools. However, this came at a personal expense, particularly regarding providing alternative power sources, and internet subscription fees for home and office use, especially during instances of university internet downtime or poor quality.

Generally, both mathematics and mathematics education lecturers shared similar opinions on their personal learning gains during lockdown in respect to teaching and learning. For example, MEduL5 said "The lockdown gave me more experience in teaching using online platforms. Because I never believed that mathematics and mathematics-related concepts could be taught online with our present IT infrastructure and skills". Similarly, ML1 noted "My experience with teaching online has encouraged the development of my understanding of online teaching processes and has boosted my confidence in utilizing online tools for teaching and research purposes."

Discussion and Conclusion

This study examined the perspectives of mathematics and mathematics education lecturers and their adaption of online mathematics teaching in response to COVID-19. Three themes were derived from the data analysis of ten interviews. The findings centred on the viewpoints of individual lecturers regarding their adoption and involvement in online mathematics teaching. The perspectives on online mathematics teaching and the level of technology utilization among both groups of lecturers were influenced by their prior experience with the internet and various online platforms. Personal and professional development with online tools played a key role during the transition to online teaching. Both groups of lecturers noted challenges in adapting face-to-face teaching methods to an online format, particularly among those with no prior online teaching experience, but there were some positive views on online mathematics teaching. Mathematics education lecturers were concerned about how to teach students (pre-service teachers) the practical approach required for teaching in schools. These findings are consistent with the observations of Hamdan and Amorri (2022) and Trenholm (2013) on the constraints of online courses in fostering students' understanding of mathematics.

The quality of lecturer-student interactions during the teaching process was identified a major concern while teaching mathematics online. That students had difficulty being able to give timely responses during lectures was a challenge. Both groups of lecturers emphasised the need for two-way communication between the students and the lecturer. Trenholm (2013) suggests the importance of synchronous interaction to enrich teaching and learning mathematics online. As a result of this, mathematics lecturers were provided with virtual board software and the use of tablets as a writing pad to improve illustrations of what the lecturers might write during online lectures. Others used the facilities at the university ICT centre. Similarly, mathematics education lecturers used video clips for further clarification. Also, both groups of lecturers noted the use of WhatsApp for synchronous voice and image interaction in conjunction with Zoom and Google Meet to enhance teaching. In situations when internet connectivity was poor, lecturers utilised WhatsApp to enhance interaction. Also, mathematics lecturers prepared pre-recorded lectures to mirror their face-to-face lectures. Additionally *Zoom*-recorded lectures, alongside the pre-recorded lectures, were uploaded on the university learning management system to enhance students understanding. This aligns with the findings of Muir et al. (2022), which examined the strategies and actions employed by lecturers to achieve success in an online mathematics course.

All the lecturers endorsed the blended approach to teaching mathematics, emphasising that online platforms ought to complement face-to-face instruction, which aligns with the conclusions of Adnan and Yaman (2015). Previous utilisation of online resources for teaching or learning, as reported by both mathematics and mathematics education lecturers, had a substantial impact on their teaching. The utilisation of online technologies, whether for their continued education or for academic endeavours such as conferences, workshops, and seminars, also had a substantial impact on their teaching. Enhancing professional development in the context of online teaching and learning will significantly contribute to the advancement of online teaching methodologies. Both mathematics and mathematics education lecturers expressed a strong desire to continue utilising online teaching methods, provided the necessary infrastructure is readily accessible. Given the lecturers' worried over conveying the nature of mathematics, it would be interesting to ascertain which facets of mathematics pose challenges in terms of instruction using online platforms.

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References

Abdulmajeed, K., Joyner, D. A., & McManus, C. (2020, August 12–14). Challenges of online learning in Nigeria (paper presentation). *Proceedings of the seventh ACM conference on learning@scale* (pp. 417–420). New York: Association for Computing Machinery. https://doi.org/10.1145/3386527.3405953

Adnan, M., & Yaman, B. B. (2015). Faculty members' perspectives on teaching mathematics online: Does prior online learning experience count? *Turkish Online Journal of Qualitative Inquiry*, *6*(1), 21–38.

Borba, M. C., Askar, P., Engelbrecht, J., Gadanidis, G., Llinares, S., & Aguilar, M. S. (2016). Blended learning, e-learning and mobile learning in mathematics education. *ZDM Mathematics Education*, *48*, 589–610. https://doi.org/10.1007/s11858-016-0798-4

Braun, V., & Clarke, V. (2019). Reflecting on reflexive thematic analysis. *Qualitative Research in Sport, Exercise and Health*, *11*(4), 589–597.

Clarke, V., & Braun, V. (2021). *Thematic analysis: A practical guide*. London: Sage. https://uwe-repository.worktribe.com/output/9004204/thematic-analysis-a-practical-guide

Creswell, J. W., & Clark, V. L. P. (2017). *Designing and conducting mixed methods research*. Sage Publications.

Galletta, A. (2013). *Mastering the semi-structured interview and beyond: From research design to analysis and publication* (Vol. 18). NYU Press.

Hamdan, K., & Amorri, A. (2022). The impact of online learning strategies on students' academic performance. In M. M. C. Shohel (Ed.), *E-learning and digital education in the twenty-first century* (pp. 1–19). IntechOpen.

Lawal, I. (2020). Why Nigeria must invest in open, distance learning by Prof Jegede. *The Guardian Nigeria News*.

Mardiana, H. (2020). Lecturers' adaptability to technological change and its impact on the teaching process. *Jurnal Pendidikan Indonesia*, *9*(2), 275–289.

Muir, T., Livy, S., Murphy, C., & Trimble, A. (2022). Making the transition from on-campus to online learning: Pre-service teachers' experiences of online learning as a result of COVID-19. *Journal of University Teaching & Learning Practice*, *19*(5), Article 03.

National Universities Commission (NUC). (2023). *The list of approved distance learning centers*. National Universities Commission. https://www.nuc.edu.ng/distance-learning-centers

National Universities Commission (NUC). (2015). *Benchmark minimum academic standards for undergraduate programs in Nigerian universities: Education.* National Universities Commission. https://nuc.edu.ng/wp-content/uploads/2015/09/Education%20Draft%20BMAS.pdf

Okyere, E., Salusalu, M., Goundar, R., & Marfoh, K. (2022). What do university students say about online learning and the COVID-19 pandemic in central Fiji? A qualitative study. *Plos one*, *17*(8), e0273187. https://doi.org/10.1371/journal.pone.0273187

Onyema, E. M. (2020). Integration of emerging technologies in teaching and learning process in Nigeria: The challenges. *Central Asian Journal of Mathematical Theory and Computer Sciences*, *1*(1), 35–39.

Oyedotun, T. D. (2020). Sudden change of pedagogy in education driven by COVID-19: Perspectives and evaluation from a developing country. *Research in Globalization*, *2*, Article 100029.

Sabitu, K. A., Saddiq, K., & Kehinde, M. A. (2022). Effect of e-learning on students' emotion and achievement in mathematics among senior secondary school students in Oyo State. *African Journal of Science, Technology and Mathematics Education*, *8*(1), 1–7. https://doi.org/https://www.ajstme.com.ng/admin/img/paper/1-7%20AJSTME8_1-003..pdf

Sven, T., & Julie, P. (2020). Teaching undergraduate mathematics fully online: A review from the perspective of communities of practice. *International Journal of Educational Technology in Higher Education*, *17*, Article 37. https://doi.org/10.1186/s41239-020-00215-0

Tella, A., & Obim, I. E. (2022). Teaching and learning online during COVID-19 lockdown, encouraging and discouraging? The perspectives of students and staff of library and Information science in Nigerian Universities. *Proceedings of the 50th annual conference of the International Association of School Librarianship and the 25th international forum on research on school librarianship* (pp. 1–16). Columbia, South Carolina. https://journals.library.ualberta.ca/slw/index.php/iasl/article/view/8530/5163

Trenholm, S. (2013). *Adaptation of tertiary mathematics instruction to the virtual medium: Approaches to assessment practice* [Unpublished doctoral dissertation]. Loughborough University.

UNESCO UNICEF. (2020). *Framework for reopening schools*. UNESCO Digital Library.

Xu, D., & Xu, Y. (2019). *The promises and limits of online higher education: Understanding how distance education affects access, cost, and quality*. American Enterprise Institute. https://files.eric.ed.gov/fulltext/ED596296.pdf

Raising Students' Awareness and Actions Through a Sustainability Project

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Little research is evident about integration of mathematics in sustainability tasks to raise students' awareness and actions to solve a sustainability challenge. In this paper, we explore how a teacher designed and implemented a sustainability project, which included mathematics to raise Year 10 students' (aged 15–16) awareness and actions about sustainability. Data, including classroom observations and interviews, were analysed based on one characteristic of the green mathematics framework: *projection*. We report on how students used mathematics to project future situations and how the projection may raise students' awareness and actions to address sustainability.

Greta Thunberg has been school-striking every Friday for years to campaign for climate justice; her campaign has inspired other people including students from all over the world to do the same (Sabherwal et al., 2021). To participate in public discussion and debate to convince people about the urgency of climate change, students need to understand how climate change eventuates and how to address the problem (Barwell & Hauge, 2021). School education can play a pivotal role in engaging students in the topic of sustainability and providing opportunities to become curious about and aware of current environmental challenges (UNESCO, 2020).

"To develop students' ability to tackle real-world problems and apply mathematical knowledge successfully, schools and education systems need to go beyond formal mathematics education" (OECD, 2023, p. 60). Formal mathematics education aims to help students understand mathematics concepts and use them to solve problems that relate to reality. However, such problems are often intentionally simplified for educational purposes (Vos, 2018). Meanwhile, problems that humanity faces, such as climate change, are very complex (Barwell & Hauge, 2021). If students are only taught mathematics as a means to solve contrived problems, it may be challenging for them to use mathematics in complex real-world problems (Tran & Dougherty, 2014). Therefore, students should be given the opportunity to investigate complex problems, including sustainability challenges, using mathematical ideas.

Sustainability has been increasingly included in school curricula and research has shown how teachers have taught sustainability either through learning areas (da Silva-Branco & Woods-McConney, 2021) or co-curricular projects (Anggraena et al., 2022). However, little research is evident about how their implementation used mathematics to raise students' awareness and actions. In this paper, we aim to explore how a high school teacher, Kartini (pseudonym), designed and implemented a sustainability project, in which students investigated a local environmental challenge, then proposed and took actions in addressing the problem. We used the green mathematics framework (Salim, 2023) to analyse the project design and implementation to understand how projects could raise students' awareness and actions.

Educational Policies and Sustainability: Raising Knowledge

The United Nations Educational, Scientific and Cultural Organization (UNESCO) has called on all countries to start teaching students about climate change and other sustainability challenges through Education for Sustainable Development (ESD) for 2030 (UNESCO, 2020).

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 479–486). Gold Coast: MERGA.

ESD for 2030 aims to “raise knowledge, awareness, and action” (p. 17). Some countries have already included sustainability into their curricula (e.g., Anggraena et al., 2022; da Silva-Branco & Woods-McConney, 2021). In Australian curriculum, for example, sustainability is one of cross-curriculum priorities, which is to be integrated across learning areas. In Indonesia’s new curriculum (known as *kurikulum merdeka* or *emancipated curriculum*), sustainability is one of seven co-curricular project themes, taught separately from intra-curricular subjects (learning areas in Australian curriculum) (Anggraena et al., 2022). Indonesian schools must allocate 25–30% of school time to thematic co-curricular projects.

Such educational policies aim to give opportunities for students to learn about sustainability challenges, but only teachers can provide students with a learning experience that is meaningful and potentially raises their awareness of sustainability challenges (UNESCO, 2020). Some Australian teachers self-reported that they were able to integrate sustainability across learning areas like science, digital technologies, and mathematics (da Silva-Branco & Woods-McConney, 2021). Likewise, some Indonesian schools have implemented a sustainability themed co-curricular project (Anggraena et al., 2022). However, little research is evident about how the integration of sustainability into learning areas (especially mathematics) can influence students’ awareness and actions. In the next section, we discuss how mathematics and sustainability education can be integrated to raise students’ awareness.

Integrating Mathematics and Sustainability Education: Raising Awareness

The focus in mathematics classrooms is often on concepts and procedures, which can be categorised as formal mathematics (OECD, 2023; Schmidt et al., 2022). Word problems, intended to show how mathematics can be used, are often inauthentic and lack complexity of realistic situations. Students may therefore focus only on prescribed mathematical procedures, disregarding the problem context (Lubienski, 2000; Palm, 2007). However, mathematics education could play a role in supporting students’ understanding and awareness of complex sustainability challenges (Barwell & Hauge, 2021). For example, by investigating variability in data collection, students could experience some of the complexity of a real-life system (Dierdorp et al., 2017). Using mathematics as a tool to analyse the system and interpret data within the system provides a path to building awareness of a sustainability phenomenon.

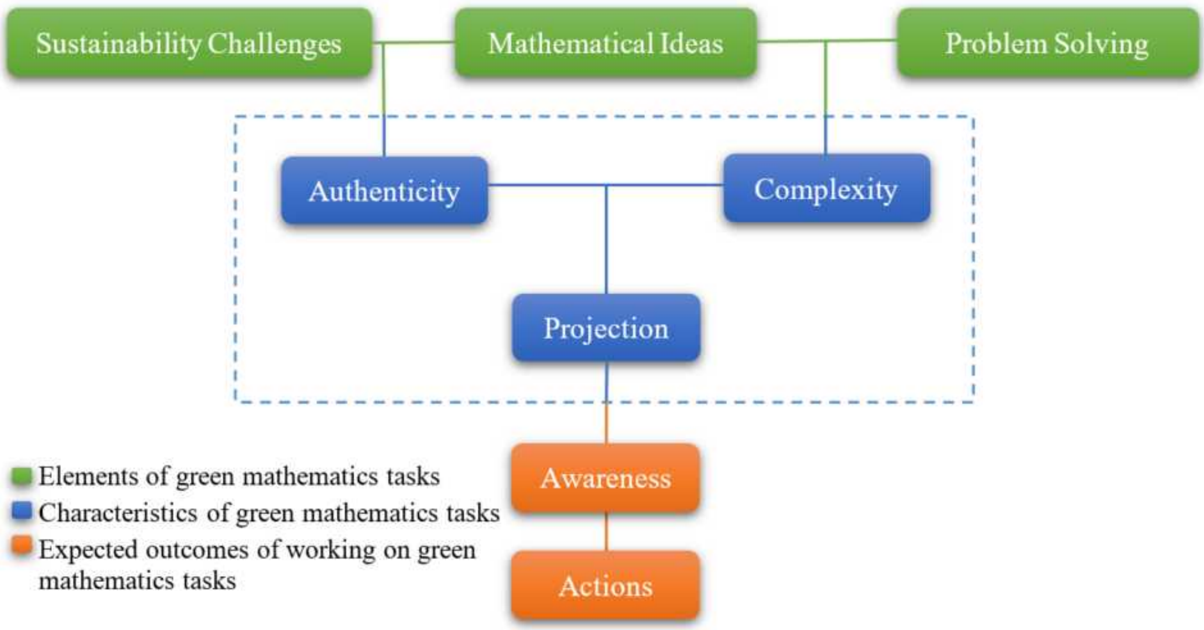
Another important aspect to consider is that students need to have purpose to use mathematical ideas (e.g., measurement and statistics) if they are to meaningfully analyse data to understand a problem (Ainley et al., 2006). Sustainability contexts that can be related to students’ lives may motivate them to use *authentic* data to investigate a *complex* problem thoroughly (Smith & Watson, 2023). The more the students consider the task as being authentic and understand the complexity of a sustainability challenge, the more likely they become aware of the problem that addressing this challenge presents (Endsley, 2017). In addition to importance of authenticity and complexity, Salim (2023) previously emphasised that integrating mathematics and sustainability education requires opportunities for *projection* (Figure 1). After outlining the green mathematics framework, we explore how opportunities for projection may strengthen sustainability projects by invoking students’ actions to address a sustainability challenge.

Projection in the Green Mathematics Framework: Raising Actions

Three basic elements need to be present in tasks or projects that allow to successfully integrate mathematics and sustainability education: a sustainability challenge(s), mathematical ideas, and problem solving. These elements are interdependent; missing one of them may make it difficult for students to understand a sustainability challenge (Salim, 2023). We explained above that to raise students’ awareness, the task also needs to be *authentic* and *complex*. By *authenticity*, we refer to adoption for classroom use of data/information, context and problem related to a sustainability challenge that has happened, is currently happening, or is predicted to happen, and where task designers expect that students would come to have a stake in addressing the challenge. *Complexity* refers to the extent to which the task is presented in a classroom as a real-life system, where mathematical ideas and activities (e.g., data collection and analysis) may need to be called upon when students engage in the task. To raise students' actions to address a sustainability challenge, the students need to encounter a need to project (predict or estimate) future possibilities of the sustainability challenge being investigated in the task (Endsley, 2017).

Figure 1

The Green Mathematics Framework (Adapted from Salim, 2023)



The term *projection* in this framework refers to predicting and visualising future implications based on current data and information (which is different from projection in geometry). Making predictions requires students to reason and make sense of patterns and "allows students to activate and refine their existing knowledge [and] … to increase students' level of engagement" (Lim et al., 2010, p. 598). Likewise, projection in the green mathematics framework is expected to support students in attending to patterns and connections between past, present, and future situations (Endsley, 2017), thus engage in activities that are overtly mathematical while working towards understanding a sustainability problem.

The complexity in the framework is expected to give students opportunities to investigate a sustainability challenge as if they were experts and scientists in the field who engage in a real-world problem solving activity (Schoenfeld, 2016). Projecting throughout a complex sustainability challenge may allow students to see a problem from different perspectives. The projection makes it possible for students to see themselves being impacted by or perhaps contributing to the problem, which can raise not only their awareness of the problem but also a need for action. By allowing students to see themselves in future situations, projection may motivate them to explore possible solutions and actions to avoid unwanted outcomes.

Research Methods

This paper is part of a research project exploring a case in which a high school teacher, Kartini, designed and implemented a sustainability project, following Indonesia's *emancipated curriculum*. We employed the green mathematics framework to analyse the design and implementation of Kartini's sustainability project. In this paper, we focus on how students used mathematics in making *projections* and how this raised their sustainability awareness and actions. The project was designed by a team of teachers, but only Kartini (the chair) participated in this study. The data were collected through interviews with Kartini, classroom observations, focus group discussion with students, a survey for students, and document artefacts.

School Context, Participants, and the Sustainability Project

The research was conducted in an Indonesian Islamic secondary boarding school. The school was established in a village surrounded by green mountains, located around 35 kilometres from the nearest city centre. The school employed approximately 50 teachers and accommodated around 150–200 students from Year 10 to 12 (15–18 years old). The school started implementing the *emancipated curriculum* in 2022. Kartini is a biology teacher with four years of teaching experience. She designed the sustainability project for Year 10 students (15–16 years old). Twenty-seven students who worked on the project participated in this study.

Kartini and the team designed the sustainability project to address a local environmental challenge in the school: *waste management*. Kartini was concerned about the waste being burnt around the school and not managed properly. Since the students spent most of their time at the school, they were familiar with the issue. Through this project, Kartini involved the students to investigate the problem in order to make them aware that they were contributing to the school waste. This project was implemented in five sessions (3–4 hours each). First, Kartini explained the project's aims and introduced the waste problem around the school (*introduction*). The students then collected data about the school waste in groups (*contextualisation*). Each group investigated different types of waste (e.g., single-use plastic bottles, food waste, and paper). In the next session, each group presented their data and findings to other students in a *mini exhibition*, then they reflected on their findings to discuss possible actions to address the waste problem. The students carried out proposed *actions and solutions*, and, in the last session, they *reflected* on the project.

Data Collection and Analysis

In this case study, four audio-recorded interviews (30–60 minutes) were conducted with Kartini: one interview before the project implementation and three interviews afterwards. Of five project sessions observed, only two could be video recorded due to the nature of the project that was mostly outdoors and required the students to interact with non-participants. After the project implementation, the students participated in a focus group discussion and filled in a short, anonymous survey to explore their experience in working on the project. Copies of documents related to the project (e.g., students' work, the project's plan) were also collected.

The data were analysed by identifying *critical* events, defined as episodes in which students use mathematics as part of their project work, and then coding the transcriptions based on the green mathematics framework (Powell et al., 2003). More specifically, we focused on analysing a set of questions requiring students to make predictions and estimations, which we term *projective questions*. The data were analysed to investigate how the *projective questions* in the sustainability project oriented the students to use mathematics to understand the local waste problem (raising knowledge). We also investigated how the projection could raise students' awareness and actions to address the problem.

Results

We report on the analysis of the projective questions in the sustainability project, students' projections, and their responses to the waste problem after the project implementation.

Projective Questions in the Sustainability Project

Kartini and her team designed what they called the 'Save The Earth Project', in which five structured questions were created for students (Figure 2). Questions 1, 2, and 3 required students to observe and collect data around the school related to waste production and management. Questions 4 and 5 required them to predict the future situation based the data collected.

Figure 2

The Questions Designed by Kartini's Team as Part of the Sustainability Project Design

Save The Earth Project

Bahasa Indonesia	English
1. Menurut Anda, sampah/limbah apa yang diproduksi di lingkungan sekitar Anda?	1. In your opinion, what kind of waste is produced around your (school) environment?
2. Bagaimana pengolahan sampah/limbah yang diproduksi di lingkungan sekitar Anda?	2. How is the waste managed around your (school) environment?
3. Berapa banyak produksi limbah yang dihasilakan tiap hari? (Dapat diperoleh melalui wawancara atau survey ke warga sekolah)	3. How much waste is produced every day? (You can find the data by interviewing or surveying the school community)
4. Buatlah prediksi berapa jumlah sampah yang dihasilkan dalam jangka 5 tahun dan 10 tahun ke depan!	4. Estimate how much waste will be produced in the next 5 and 10 years.
5. Buatlah prediksi apa yang terjadi jika permasalahan tersebut tidak dapat diselesaikan dengan baik!	5. Predict what will happen if the problem cannot be solved properly.

These questions make the sustainability project complex because students had to plan how they collected the data (research design), how they analysed the data (proportion, percentage, and statistics), and how they presented their findings (graphs, charts, and tables) in a mini exhibition. The aim of these questions was indeed to investigate the waste management, including the waste sources, as well as its current and future impact.

> The aim I made the questions is so that students become aware of the fact that the waste they produce, although it is small and often neglected, but if it is accumulated in a month, two months, or even in one year, that can be a lot. My aim for the questions … is to make it well-structured, so that there is a visualisation regarding the source of the waste, either in small or big amount, it is from us. Hence, later when doing the actions, certainly any small things that can be done need to start from us first. (Kartini, Interview 1, 12:09)

Kartini did not ask students about waste production in the abstract, but she sought to help students to envision their personal contributions to the waste production problem. Her questions encouraged the use of mathematics to facilitate students' predictions, and hence projection towards personal awareness and future action.

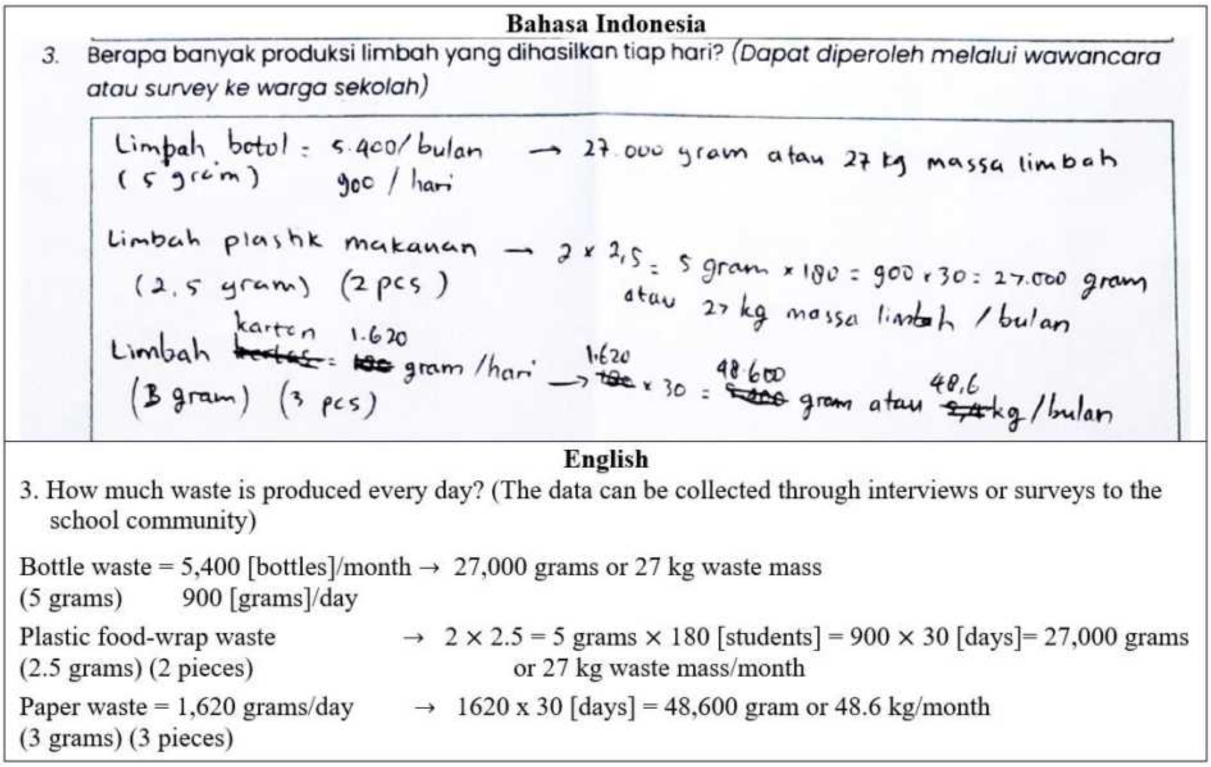
Projection Throughout the Project Implementation

In the introduction session, Kartini asked her students to work in groups on the questions. Figure 3 shows an estimation made by Nadya's group in response to question 3. The students estimated that in one day each student used one single-use plastic bottle, two pieces of plastic wraps, and three pieces of paper. They estimated the mass of each and calculated that if there were 180 students in the school, how much waste they would produce in one day and one month. The quantities increased significantly when the students responded to question 4, projecting that in five years students at the school would use 1.6 tons of single-use plastic bottles and food wraps respectively, and 2.9 tons of paper. In ten years, these numbers were doubled.

Other groups made different estimations than Nadya's group estimations. Instead of using 180 students, one group used 130 students considering the Year 12 students had graduated (Introduction Observation, 01:23:39). Another group used 148 people by including the school staff (Introduction Observation, 01:44:02). The evidence showed how students considered different variables that could influence their estimations.

Figure 3

A Group Students' Estimation of the School Waste Production


Bahasa Indonesia

3. *Berapa banyak produksi limbah yang dihasilkan tiap hari? (Dapat diperoleh melalui wawancara atau survey ke warga sekolah)*

Limpah botol = 5.400/bulan → 27.000 gram atau 27 kg massa limbah
(5 gram) 900 / hari

Limbah plastik makanan → 2 × 2,5 = 5 gram × 180 = 900 × 30 = 27.000 gram
(2,5 gram) (2 pcs) atau 27 kg massa limbah / bulan

Limbah karton = 1.620 gram /hari → 1.620 × 30 = 48.600 gram atau 48,6 kg/bulan
(3 gram) (3 pcs)

English

3. How much waste is produced every day? (The data can be collected through interviews or surveys to the school community)

Bottle waste = 5,400 [bottles]/month → 27,000 grams or 27 kg waste mass
(5 grams) 900 [grams]/day

Plastic food-wrap waste → 2 × 2.5 = 5 grams × 180 [students] = 900 × 30 [days]= 27,000 grams
(2.5 grams) (2 pieces) or 27 kg waste mass/month

Paper waste = 1,620 grams/day → 1620 x 30 [days] = 48,600 gram or 48.6 kg/month
(3 grams) (3 pieces)

In the contextualisation session, each group collected data about different types of waste. This time, students used collected data (rather than estimates) to make a projection. Nadya's group collected data about students' use of single-use plastic bottles. In the focus group discussion, I asked the students which part of the project helped them to understand the waste problem. The following was Nadya's response, "when collecting the data, we finally know that the students at the school produce, um since I am in the plastic bottle group, so I know that on average a student uses one to two bottles per day" (Focus Group Discussion, 09:47).

The students found that, on average, each student used one to two plastic bottles per day. This was higher than their estimation in the introduction session (one bottle per student per day). This means that their initial estimation in Figure 3 could be doubled. Most students agreed with Nadya that the data they found from interviewing the school community (students, teachers, and staff) helped them to understand the problem better.

Students' Awareness and Actions

While implementing the project, Kartini noticed some changes in the students' behaviours:

> After the mini exhibition, I saw that the students have been cautious to sort out their plastic waste. In their dormitory, I got information that in every level—there are three levels there—students have provided bins for plastic waste. (Interview 2, 03:54)

Only a couple of students seemed to believe that "the waste problem in the school is not very urgent" (Survey #1). Most students' survey responses show that the sustainability project made them aware of the waste problem and that they needed to solve the problem immediately. For example, one student wrote "I became aware that waste is a serious problem that if not addressed, will lead to a huge problem" (Survey #15). Another student responded, "the sustainability issue needs to be addressed properly to prevent negative impacts from the problem" (Survey #14). Based on these responses, using current data to project the future situation of the waste problem may have helped the students to become aware of how serious the problem was. They then proposed some solutions like sorting out recyclable materials, encouraging the use of re-usable tumblers, and urging the school to provide water dispensers. Ultimately, they used the data not only to justify their proposed solutions, but also to raise other students' awareness and actions during the mini exhibition.

Discussion and Conclusion

The analysis showed that as the students used mathematics to project future situations, they became more aware of the environmental challenge and proposed actions to address the problem. The results indicate that the projective questions in the design and implementation of the sustainability project had likely helped the students to become aware of a local sustainability challenge and motivated their actions to tackle the problem. In this part, we discuss two notions that could have contributed to raising students' awareness and actions: *purpose* and *projection*. First, because the context in the project was made local and relevant to their lives, the students had their own *purpose* to use mathematics when attempting to understand a real-world situation (Ainley et al., 2006). The students might have learned the mathematics (e.g., proportion, percentage, and statistics) previously and used it in a pseudo-realistic problem. An authentic, relevant context allowed the students to notice utility of mathematics in understanding a local sustainability challenge. Having used mathematics as an essential tool in analysing and interpreting environmental data, the students encountered mathematics as being important to addressing an environmental problem (Barwell & Hauge, 2021).

In Kartini's view, quantifying the school's waste production was essential to making students aware that the accumulation of individual production of waste could make a significant impact. This is where *projection* of the future situation based on the data that they collected could have helped students to clearly see their own impact. For example, the students predicted that in the next five years, their school will have produced around 1.6–3.2 tons of single-use plastic bottles if 180 students keep using 1–2 bottles each every day. This could be a great opportunity to further engage the students to think about sources of variability in their projection (e.g., whose waste production was included) to make the practical use of mathematics even more authentic and realistic (Dierdorp et al., 2017).

Questions 4 and 5 in Figure 2, which we refer to as projective questions, made it possible for students to envisage the future situation of the waste problem. The student projections were based on two variables: the number of students and the amount of waste produced by each student per day. As the number of students was not likely to decrease in the future, the students explored actions and solutions that could reduce the individual waste production, such as water dispenser and re-usable tumbler use in school. The findings from the projections were used to justify the proposed solutions and then to convince the school and other students (the mini exhibition visitors) to take actions.

To conclude, purposeful use of mathematics appears to be essential in integrating mathematics and sustainability education. Including the need for *projection* (e.g., from current to future data) in a task or a project design appears to afford mathematics use to become purposeful from students' perspectives. Additionally, choosing a local problem that can readily be brought to matter to students may play a pivotal role in whether students choose to engage in complex project activities (e.g., data collection, exhibition, etc.). As the problem matters, the students have their own *purpose* to work on the project and motivate their actions to use mathematical ideas to understand and address a local sustainability challenge.

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References

Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal*, *32*(1), 23–38. https://doi.org/10.1080/01411920500401971

Anggraena, Y., Felicia, N., Eprijum, D., Pratiwi, I., Utama, B., Alhapip, L., & Widiaswati, D. (2022). *Kajian akademik kurikulum untuk pemulihan pembelajaran* [Monograph]. Pusat Kurikulum dan Pembelajaran, Jakarta, Indonesia. http://repositori.kemdikbud.go.id/id/eprint/24972

Barwell, R., & Hauge, K. H. (2021). A critical mathematics education for climate change: A post-normal approach. In A. Andersson & R. Barwell (Eds.), *Applying critical mathematics education* (pp. 166–184). Brill.

da Silva-Branco, K., & Woods-McConney, A. (2021). Teachers' experiences of implementing sustainability as a cross-curriculum priority in Western Australian schools. *Teaching Science*, *67*(3), 39–50.

Dierdorp, A., Bakker, A., Ben-Zvi, D., & Makar, K. (2017). Secondary students'considerations of variability in measurement activities based on authentic practices. *Statistics Education Research Journal*, *16*(2), 397–418. https://doi.org/10.52041/serj.v16i2.198

Endsley, M. R. (2017). Toward a theory of situation awareness in dynamic systems. In E. Salas & A. S. Dietz (Eds.), *Situational awareness* (pp. 9–42). Routledge.

Lim, K. H., Buendía, G., Kim, O.-K., Cordero, F., & Kasmer, L. (2010). The role of prediction in the teaching and learning of mathematics. *International Journal of Mathematical Education in Science and Technology*, *41*(5), 595–608. https://doi.org/10.1080/00207391003605239

Lubienski, S. T. (2000). Problem solving as a means toward mathematics for all: An exploratory look through a class lens. *Journal for Research in Mathematics Education*, *31*(4), 454–482. https://doi.org/10.2307/749653

Organisation for Economic Development (OECD). (2023). *PISA 2022 results (Vol. I)*. OECD Publishing. https://doi.org/10.1787/53f23881-en

Palm, T. (2007). Features and impact of the authenticity of applied mathematical school tasks. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI study* (pp. 201–208). Springer US. https://doi.org/10.1007/978-0-387-29822-1_20

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, *22*(4), 405–435. https://doi.org/10.1016/j.jmathb.2003.09.002

Sabherwal, A., Ballew, M. T., van Der Linden, S., Gustafson, A., Goldberg, M. H., Maibach, E. W., Kotcher, J. E., Swim, J. K., Rosenthal, S. A., & Leiserowitz, A. (2021). The Greta Thunberg effect: Familiarity with Greta Thunberg predicts intentions to engage in climate activism in the United States. *Journal of Applied Social Psychology*, *51*(4), 321–333. https://doi.org/10.1111/jasp.12737

Salim, S. (2023). A framework for designing green mathematics tasks. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 451–458). Newcastle: MERGA.

Schmidt, W. H., Houang, R. T., Sullivan, W. F., & Cogan, L. S. (2022). *When practice meets policy in mathematics education: A 19 country/jurisdiction case study*. OECD Publishing. https://doi.org/10.1787/07d0eb7d-en

Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics (Reprint). *Journal of Education*, *196*(2), 1–38. https://doi.org/10.1177/002205741619600202

Smith, C., & Watson, J. (2023). Using the practice of statistics to enhance education through UN sustainable development goal 13, climate change. In K. Beasy, C. Smith, & J. Watson (Eds.), *Education and the UN sustainable development goals: Praxis within and beyond the classroom* (pp. 539–556). Springer.

Tran, D., & Dougherty, B. J. (2014). Authenticity of mathematical modeling. *The Mathematics Teacher*, *107*(9), 672–678. https://doi.org/10.5951/mathteacher.107.9.0672

UNESCO. (2020). *Education for sustainable development: A roadmap*. UNESCO Paris, France.

Vos, P. (2018). "How real people really need mathematics in the real world"—authenticity in mathematics education. *Education Sciences*, *8*(4), 195. https://doi.org/10.3390/educsci8040195

Mathematics Leaders as Agents of Project Sustainability

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This paper explores the complex problem of project sustainability, focusing on the leadership of three primary school mathematics leaders. Using cultural-historical activity theory (CHAT), the leaders' efforts are reported, highlighting their contribution to project sustainability. The CHAT-informed research design supported the generation of findings, revealing how the mathematics leaders enacted a form of resourceful practice. This paper contributes new knowledge about mathematics leaders, characterising how they acted as agents of project sustainability. Implications for mathematics education project design are also offered.

Project sustainability in mathematics education is defined as the continuous adaptation and integration of reform efforts that remain true to the project's intent and content in response to the ever-changing post-project contexts within which the efforts are enacted (Clements et al., 2012; Tirosh et al., 2015). The sustainability of project reforms is a complex problem because the development initiated during the life of the project tends to cease once participation in the project concludes (Tirosh et al., 201), and there is also a lack of research about how reform efforts continue beyond project participation (Bobis, 2011). It is recognised that school leadership acts as a crucial factor of project sustainability, but this tends to focus on principal leadership (Coburn et al.), neglecting the influence of middle leading practice enacted by mathematics leaders. Drawing on concepts from cultural-historical activity theory (CHAT), I report on the efforts of three mathematics leaders working as middle leaders in their schools (Grootenboer, 2018). I aim to present their contribution to project sustainability as a form of the CHAT-aligned activity known as *resourceful practice* (Edwards, 2010). By doing so, I characterise how the mathematics leaders acted as agents of project sustainability, providing a new perspective on the leadership activity of mathematics leaders enacted in primary schools.

Background Literature

Project sustainability involves the maintenance of the impact on student learning outcomes and the continued influence of the reform on mathematics teaching practice beyond project participation (Clements et al., 2015; Coburn et al., 2012). The sustainability of projects requires fidelity to the project intent and content (Clements et al., 2015), and it also involves engagement in processes of self-renewal where new routines and practices surface (Coburn et al., 2012; Zehetmeier, 2017). As such, there is a need for school staff to adapt and integrate project intent and content in response to the changed conditions that face schools beyond project participation (Tirosh et al., 2015). Fullan (2008) named school staff who engage in project sustainability as *change agents*, but this title was reserved for principals and executive leaders only.

Studies of project sustainability have focused on the factors that sustain practice reform, recognising their potential to enable and constrain sustainability efforts (Saito et al., 2012; Zehetmeier, 2017). Sustainability factors *within* schools are considered *internal* factors (Saito et al., 2012). Internal factors focus on the primacy of *school leadership* with other factors mediated by that school leadership factor including *staff turnover, school-based professional learning*, and *project resource use* (Bobis, 2011; Datnow et al., 2005; Kaur, 2015; Fishman et al., 2011; Pritchard & McDiarmid, 2006; Tirosh et al., 2015; Warren & Miller, 2016).

School leadership is critical in project sustainability, with principals featured predominately within the literature due to their authority in creating the conditions that maintain the project-

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 487–494). Gold Coast: MERGA.

initiated reform and development (Coburn et al., 2012; Datnow et al., 2005; Saito et al., 2012; Tirosh et al., 2015). Principals can make decisions, set expectations, and provide the resources for continued teacher professional learning (Bobis, 2011; Warren & Miller, 2016). A lack of principal leadership can constrain sustainability efforts, leading to practice regression that sees the resurfacing of pre-reform pedagogies (Tirosh et al., 2015). Rare cases in the literature have reported teacher leaders who have contributed to project sustainability, naming them as *school-based facilitators* (Bobis, 2011) and *reform coordinators* (Datnow et al., 2005).

Staff turnover is understood as the changes at the teacher (Pritchard & McDiarmid, 2006), principal (Saito et al., 2012), and district levels (Datnow et al., 2005). Staff turnover tends to constrain sustainability, disrupting the continuity of institutional knowledge and practice initiated through project participation (Pritchard & McDiarmid, 2006). Staff turnover requires professional learning for newly appointed staff which tends to revise project content rather than extend practice development (Saito et al., 2012). Continued professional learning is crucial for sustaining practice development, with principals positioned as the leaders who create the conditions and provide resources necessary for school-based professional learning (Bobis, 2011; Saito et al., 2012; Warren & Miller, 2016; Zehetmeier, 2017). It is vital that post-project professional learning maintains fidelity to the project's intent and content (Clements et al., 2015; Kaur, 2015). This fidelity extends to the continued use of project resources with leaders and teachers maintaining shared understanding of the pedagogical potential of resources following project participation (Fishman et al., 2011; Saito et al., 2012; Warren & Miller, 2016).

In recent years, mathematics leadership has received attention from mathematics education researchers as a form of school leadership (Driscoll, 2017; Grootenboer, 2018; Sexton, 2023). Mathematics leadership has been conceptualised as middle leading practice different from teacher leadership because mathematics leaders tend to hold formal school leadership positions and undertake teaching responsibilities (Grootenboer, 2018). Mathematics leaders practise leadership in the space *between* the principal and the teachers working in classrooms (Grootenboer, 2018; Sexton, 2023). As middle leaders, mathematics leaders engage in leadership that influences teaching practice due to their unique positioning and proximity to classrooms (Grootenboer, 2018). Their influence is realised in the ways they lead school-based professional learning (Jorgensen, 2016; Sexton, 2023) using co-teaching episodes (Driscoll, 2017), developing assessment practices (Jorgensen, 2016), leading staff meetings (Driscoll, 2017), and developing visions for mathematics teaching practice (Sexton, 2023). Jorgensen (2016) named mathematics leaders as *change agents* due to their influence on teaching practice in ways that principals may not because of their executive positioning.

Despite knowledge of its factors, project sustainability remains a complex problem for schools and mathematics education researchers (Bobis, 2011; Clements et al., 2015). This problem tends to exist because project-initiated reforms tend not to last once project participation ceases (Tirosh et al., 2015), and most research reports about project impact tend to focus only on the change that happens during the life of projects (Coburn et al., 2012; Fishman et al., 2011). While existing literature emphasises the pivotal role of principal leadership as a sustainability factor (e.g., Tirosh et al., 2015), a notable knowledge gap exists regarding how the efforts of mathematics leaders, as middle leaders within school leadership systems, contribute to project sustainability. To address the research problem, I pose the following question: *How do mathematics leaders contribute to project sustainability through their post-project leadership activity as middle leaders in schools?*

Research Design

This paper is drawn from my doctoral study (Sexton, 2023), that investigated mathematics leaders' contribution to project sustainability. Recognising mathematics leadership as a form of middle leading practice (Grootenboer, 2018), the practice-based theory of CHAT was chosen to investigate mathematics leadership as a form of activity (Engeström, 2015).

Theoretical Framework

CHAT understands activity as *object-oriented* and draws psychological and practical development forward in simultaneous ways. The CHAT concept of the *activity system* acts as the unit of analysis, providing ways of understanding activity by situating it within the context in which the activity occurs (Engeström, 2015). Within the activity system, the *subject* (individuals or a collective group) acts on the *motive-object(s)* in order to transform it, using *cultural tools* through a process known as *mediation* (Engeström, 2015). The subject directs their activity towards the motive-object to achieve a desired and valued *outcome* (Engeström, 2015). Motive-objects are interpreted as the driving force of activity (Leont'ev, 1978) and are understood as the *problem space* at which the subject directs their activity (Engeström, 2015).

Mediation within the activity system occurs through the influence of mediational means beyond cultural tools, which include *rules* that are the implicit and explicit norms and routines that govern interactions within the activity system (Engeström, 2015); *community*, which includes the other people involved in the activity as human activity does not exist outside of social relations (Leont'ev, 1978); and, the *division of labour* that includes the distribution of power and responsibility for tasks and actions enacted within the activity system (Engeström, 2015). In CHAT, the subject enacts a series of *actions* that mediate motive-objects (Leont'ev, 1978) whilst using the mediational means present and available within the activity system.

Resourceful Practice

Resourceful practice, a contemporary CHAT concept, has explanatory power to understand how the subject, when faced with *practice problems* (Edwards, 2010), uses resources creatively to resolve contradictions or tensions within the activity system (Edwards & Thompson, 2013). Resourceful practice recognises resources as the cultural tools, rules, and division of labour within and beyond the subject's activity system. It also highlights the subject's use of the transformative potential of resources (Edwards, 2010). Resourceful practice also theorises the agentic role of the subject in driving activity forward amid contradiction resolution.

Resourceful practice is characterised by several actions, including *reconfiguring motive-objects*, *adapting cultural tools*, *rule-bending*, and *accessing distributed expertise* (Edwards, 2010; Edwards & Thompson, 2013). Reconfiguration of motive-objects is realised when the subject objectifies *what matters* in new ways when faced with practice problems (Edwards & Thompson, 2013). Tool adaptation involves using resources in adaptive and creative ways to resolve contradictions by attributing new meaning to them to pursue reconfigured motive-objects (Edwards, 2010). Rule-bending entails adapting norms and routines by modifying or breaking historically followed rules (Edwards & Thompson, 2013). As a collective process, resourceful practice also emphasises engagement with others within and across activity systems. Accessing distributed expertise involves using resources from various practices and using expertise from neighbouring systems to drive activity forward (Edwards, 2010).

Context and Participants

Contemporary Teaching and Learning of Mathematics (CTLM) was a large-scale project that involved 82 Catholic primary schools in Victoria between 2008 to 2012 inclusive. Each CTLM school participated in a two-year program supported by Australian Catholic University (ACU) and Catholic education staff members. As a requirement, participating schools nominated at least one staff member to undertake the mathematics leadership role. During CTLM, mathematics leaders led teaching practice development in accordance with the project's intent and content. Their leadership was realised through the facilitation of teachers' planning meetings, co-teaching lessons with colleagues, the organisation of demonstration lessons undertaken by ACU staff, and the leadership of fortnightly professional learning meetings.

Three mathematics leaders, Penny, Cindy, and Rachel (pseudonyms), who worked in three schools that participated in the CTLM project in 2011 and 2012, were participants in my study. Each mathematics leader remained in the role for the entirety of the data generation period, where they engaged in leadership activity that saw them leading teachers' professional learning, completing management tasks associated with their schools' mathematics program, and undertaking mathematics teaching responsibilities in classrooms.

Data Generation and Analysis

The research design included a prolonged data generation period involving site visits to the mathematics leaders' schools from November 2014 to February 2018. The extended data generation period was enacted to investigate the lasting effect of project sustainability (Zehetmeier, 2017). Semi-structured interviews were coupled with observations of the mathematics leaders' practice, with interviews used prior to (~15-min interview) and after (~60-min interview) observations of professional learning opportunities (~70-min observations). This was done to mitigate methodological issues related to the reliance on self-reports of project sustainability efforts, which can impact validation of findings (Tirosh et al., 2015). Documents were collected as cultural tools used by the leaders. Each mathematics leader was visited at least five times during the data generation period. Interviews and observation records were transcribed and uploaded into NVivo™ with the retrieved documents for data analysis.

To analyse data, I used concepts from CHAT and resourceful practice as sensitising concepts to support a deductive thematic analysis (DTA) approach (Fereday & Muir-Cochrane, 2006). This allowed the opportunity to create and use a coding scheme that supported the generation of evidence of theoretical concepts within the dataset. The concepts within the scheme were also used as nodes in NVivo™, and data were tagged and captured within those nodes. The DTA approach also supported the development of themes from the deductively coded data. The analysis involved seeking evidence of concepts through reading, coding, and interrogating data with my doctoral supervisors, ensuring the saturation of themes. The themes supported naming the mathematics leaders' efforts as leadership actions, thus explaining the mathematics leaders' contribution to project sustainability.

Findings and Discussion

I report and discuss findings together in relation to the theoretical and background literature (Sutton & Austin, 2015). This is done to support achievement of my aim which is to present the mathematics leaders' contribution to project sustainability as a form of resourceful practice.

Before explaining the mathematics leaders' contribution, it is essential to state that they were afforded that contribution because of their principals' commitment to maintaining the mathematics leadership role after CTLM had ended. This supports previous evidence that, as a school leadership factor, principals play a critical role in engaging their authority that sets the direction for project sustainability (Bobis, 2011; Datnow et al., 2005; Warren & Miller, 2016). In the case of my study, this principal direction setting was concerned with the continuation of funding for the mathematics leadership role and maintaining Penny, Cindy, and Rachel in their leadership roles in the six years following CTLM participation.

Leadership Actions: Realising Resourceful Practice

I now focus on explaining the leadership actions enacted by the mathematics leaders that realised their contribution to project sustainability. The six leadership actions are evidenced through the discussion of data, supporting my explanation of the mathematics leaders' contribution to project sustainability as a form of resourceful practice.

Committing to Sustaining Project-Initiated Reforms

Despite facing the changed post-project school conditions (Tirosh et al., 2015), which included the practice problems of withdrawn district leadership and shifted principal support, teacher turnover, and diminished priority and frequency of mathematics professional learning, the mathematics leaders enacted a clear commitment to sustaining the CTLM reforms. Penny evidenced this when she explained what motivated her leadership in the years following CTLM participation:

> This is what I know to be right, and these are the things that a leader needs to be doing, but lack of time and maths priority is stopping me. But I keep going because I care for the students and the teachers. I care about maths and what we started in CTLM, so I try to be creative. (26.03.15)

The leaders' commitment to project sustainability focused on care for improving students' learning, maintaining the project-initiated practice development, and honouring the historical changes in practice initiated through CTLM participation. This focus on "what mattered to them" (Edwards & Thompson, 2013) was their way of reconfiguring the motive-object of their leadership activity in response to the post-project practice problems they faced.

Influencing Principals to Maintain Facilitated Mathematics Planning Meetings

Facilitated mathematics planning meetings were established as a routine in the mathematics leaders' schools during CTLM. Post-CTLM participation saw the mathematics leaders engage in leadership that sustained that project-initiated routine (Clements et al., 2015). This was realised in their efforts to persuade their principals to retain the planning meetings as a school routine. Rachel confirmed this when she explained the reason why facilitated planning meetings remained an enduring routine and why the leadership of them stayed part of her work activity:

> You've got to keep the planning meetings going for the changes we started to become part of the common practice and shared practice. I keep the principal informed about how the planning meetings are important because change takes time, and I want them to continue. (29.04.15)

This "influencing principals" action was also highlighted by Penny in 2016, "I spend time sharing, especially with the principal, that we should keep the facilitated planning meetings here. I tell him that we should keep them." Acknowledging their middle leadership role and their limited authority (Grootenboer, 2018), the mathematics leaders knew that principal endorsement was crucial for maintaining facilitated planning meetings as routines that sustained practice development. This is an example of how the SMLs leveraged the division of labour within their activity system (Engeström, 2015) and engaged in resourceful practice by accessing the distributed expertise of their principals within their activity system (Edwards, 2010). This action also mediated the leaders' focus on what mattered through their reconfigured motive-object realised in their commitment to project sustainability (Edwards & Thompson, 2013). Enactment of this action saw the mathematics leaders maintain facilitated planning meetings as a routine and co-opt those meetings as professional learning opportunities. This action resourcefully addressed the practice problem of diminished mathematics professional learning opportunities they faced following CTLM participation.

Co-Opting Facilitated Planning Meetings as Professional Learning Opportunities

All three leaders claimed that mathematics professional learning became scarce following CTLM participation because other curriculum areas claimed space in school improvement agendas. This was highlighted by Penny when she shared: "Our PLTs [professional learning team meetings] used to be numeracy and literacy, and that was a fortnightly seeing each team [of teachers], but now, I am very lucky to have two PLT meetings a term" (25.03.15).

The mathematics leaders creatively repurposed facilitated planning meetings as teachers' professional learning opportunities to resolve that practice problem. This move became a recurring leadership action observed during each school visit, emphasising its prevalence in their contribution to project sustainability. Cindy evidenced what I have interpreted as rule-bending (Edwards, 2010) through her co-option of facilitated planning meetings:

> I use the facilitated planning meetings as PD with teachers. I mean, we don't have the PD in maths like we used to in the "CTLM days", so the planning meetings are a way for me to get around that and so that we can keep going on with what we started in CTLM. (06.11.14)

The mathematics leaders repositioned facilitated planning meetings by attributing new meaning to them and adapting the meetings as a resource to mediate their reconfigured motive-object of what mattered for project sustainability (Edwards, 2010; Edwards & Thompson, 2013). They created spaces for continued professional learning, understood as a factor of project sustainability (Bobis, 2011; Kaur, 2015; Pritchard & McDiarmid, 2006). Their enactment of this action further realised resourceful practice (Edwards, 2010) as they adapted to the changed condition of reduced professional learning opportunities following CTLM participation.

Repurposing Project Resources as Sustainability Tools

The mathematics leaders also demonstrated resourceful practice by repurposing CTLM resources as sustainability tools. As the leaders co-opted facilitated planning meetings as professional learning opportunities through rule-bending (Edwards, 2010), they redefined the purposes of project resources. Those resources included mathematics tasks highlighted in CTLM workshops and planning documentation created during CTLM participation. As evidenced by Penny, the mathematics leaders preserved and extended the use of the project resources, acknowledging their historical significance and potential as sustainability tools:

> By using those tasks from CTLM, we are keeping CTLM going here and what we started in CTLM keeps going. It's important that those who went through CTLM use them so that teachers who didn't do CTLM get to know about them, and they use them in their teaching, too. (02.12.16)

This repurposing was not merely a continuation of resource use (Fishman et al., 2011). Instead, the CTLM resources strategically served as cultural tools that mediated the motive object of what mattered to the leaders' sustainability efforts (Edwards & Thompson, 2013). Furthermore, the mathematics leaders extended their resourceful practice to engage in tool adaptation and rule-bending (Edwards, 2010). This was done as they attributed new meanings to the CTLM resources as enduring cultural tools within their activity system. This leadership action highlights the importance of access to and use of project resources (Bobis, 2011; Fishman et al., 2012; Warren & Miller, 2016) and contributes new knowledge about how mathematics leaders resourcefully repurposed them as project sustainability tools.

Using Student Assessment Data as a Convincing Tool

Another leadership action concerned the innovative use of student assessment data. While the CTLM project developed practices for data use to inform teachers' planning decisions, the mathematics leaders redefined the significance of data following CTLM participation. As they facilitated planning meetings, the leaders used data as a cultural tool to persuade teachers to continue using the teaching practices developed during CTLM. This was exemplified by Rachel when she explained why she used NAPLAN data during facilitated planning meetings:

> It's good to use the NAPLAN data with teachers to show them that by continuing with what we started with CTLM, we have kept going, and we saw improvements in the NAPLAN data. That's why we have to keep going, too. The NAPLAN data is good for that. (19.11.15)

I interpret this as further evidence of resourceful practice by how the mathematics leaders engaged in tool adaptation and rule-bending (Edwards, 2010). Due to their commitment to what mattered, the leaders attributed new meaning to data as convincing tools by bending the rules about data use. Data were no longer only used to inform teachers' planning decisions; instead, data were used to persuade teachers to maintain the use of CTLM resources, including mathematical tasks, and to continue using teaching practices developed during the project.

Seeking Support From External Mathematics Educators

The final leadership action that I interpret as the mathematics leaders' resourceful practice concerned how they sought support from outside their schools. Faced with the contradiction of changed district and principal leadership, the leaders proactively sought relationships and assistance from mathematics educators beyond their school sites. This included mathematics leaders in other schools, mathematics consultants, and university mathematics educators. Cindy exemplified this leadership action as she explained how she dealt with diminished support from her principal and withdrawn assistance from Catholic office district staff:

> Having an outside person from the school who is into maths is so helpful for me as the maths leader. That person acts as a 'sounding board' because I know I cannot access the [central Catholic office] staff, and the principal support has really dropped off with maths. You need that outside person who 'gets it' for advice on ways to continue what we started. (23.10.18)

I interpret this action as a new routine for the mathematics leaders and as evidence of accessing distributed expertise beyond the mathematics leaders' activity system (Edwards, 2010). They acted in agentic ways as they sought advice from others' practice in neighbouring activity systems. The mathematics leaders recognised the expertise of others and engaged in volitional action as they accessed distributed expertise (Edwards, 2010).

I have focused on the efforts of the mathematics leaders, presenting them as leadership actions and interpreting them as their contribution to project sustainability. This provides a new perspective on how mathematics leadership, as a form of middle leading, acts as a school leadership factor neglected in previous project sustainability studies (e.g., Datnow et al., 2005; Saito et al., 2012). I evidenced that through the enactment of middle leading practice by focusing on what mattered, engaging in rule bending, adapting cultural tools, and accessing distributed expertise, the mathematics leaders surfaced new and adapted routines that contributed to sustained practice development (Coburn et al., 2012; Tirosh et al., 2012). The mathematics leaders were not only change agents who engaged in project sustainability and influenced teaching practice (Fullan, 2008; Jorgensen, 2016), but they sought to lead in agentic ways by driving practice development forward amid the post-project practice problems they faced (Edwards, 2010). Drawing together my interpretation of their efforts, I claim that the mathematics leaders' contribution to project sustainability was realised through a form of resourceful practice that saw them act as agents of project sustainability.

Conclusion and Implications

Addressing the complexity of project sustainability requires an expanded understanding of school leadership beyond principal leadership. While existing literature focuses on principals and, to some degree, teacher leaders, I drew on CHAT and the explanatory power of resourceful practice to claim that mathematics leaders acted as agents of project sustainability. This is a new contribution to mathematics education research about project sustainability that theorises mathematics leadership activity as a crucial element of the sustainability factor of school leadership. Recognising that CHAT interprets activity within the context in which it occurs, further investigations must take place into how other mathematics leaders act as agents of project sustainability in situations different from the one reported in this paper.

The impact of my study for mathematics education relates to project design. Projects can be costly endeavours with their impact fading once projects cease. One implication to mitigate this phenomenon concerns leveraging the role of mathematics leaders in project design. By building in intent and content that supports practice development for mathematics leaders focused on efforts that sustain reforms, the continued impact of projects could be mediated. This could happen by offering school leaders the concepts of resourceful practice during project participation and exploring how other mathematics leaders have enacted leadership of project sustainability. Findings about Rachel, Penny, and Cindy's leadership could be used as "stories of resourceful practice" to inform the design of school project sustainability plans that utilise the mathematics leader as an agent of project sustainability.

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References

Bobis, J. (2011). Mechanisms affecting the sustainability and scale-up of a system-wide numeracy reform. *Mathematics Teacher Education and Development*, *13*(1), 34–53.

Clements, D. H., Sarama, J., Wolfe, C. B., & Spitler, M. E. (2015). Sustainability of a scale-up intervention in early mathematics: A longitudinal evaluation of implementation fidelity. *Early Education and Development*, *26*(3), 427–499.

Coburn, C. E., Russell, J. L., Kaufman, J. H., & Stein, M. K. (2012). Supporting sustainability: Teachers' advice networks and ambitious instructional reform. *American Journal of Education*, *119*, 137–182.

Datnow, A., Foster, L., Kemper, E., Lasky, S., Rutherford, C., Schmidt, M., Stringfield, S., Sutherland, S., & Thomas, J. (2005). Five key factors in supporting comprehensive school reform. In N. Bascia, A. Cumming, A. Datnow, K. Leithwood, & E. Livingstone (Eds.), *International handbook of educational policy* (pp. 195–215). Springer.

Driscoll, K. (2017). Primary school mathematics leaders' views of their mathematics leadership role. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 213–220). MERGA.

Edwards. A. (2010). Being an expert professional practitioner: The relational turn in expertise. Springer.

Edwards, A., & Thompson, M. (2013). Resourceful leadership: Revealing the creativity of organizational leaders. In A. Sannino & V. Ellis (Eds.), *Learning and collective creativity: Activity-theoretical and sociocultural studies* (pp. 99–115). Routledge.

Fereday, J., & Muir-Cochrane, E. (2006). Demonstrating rigor using thematic analysis: A hybrid approach of inductive and deductive coding and theme development. *International Journal of Qualitative Methods*, *5*, 80–92.

Fishman, B. J., Penuel, W. R., Hegedus, S., & Roschelle, J. (2011). What happens when the research ends? Factors related to sustainability of a technology-infused mathematics curriculum. *Journal of Computers in Mathematics and Science Teaching*, *30*(4), 329–353.

Fullan, M. (2008). *The new meaning of educational change* (4th ed.). Teachers College Press.

Grootenboer. P. (2018). The practices of school middle leadership: Leading professional learning. Springer.

Jorgensen, R. (2016). Middle leadership: A key role of numeracy reform. *Australian Primary Mathematics Classroom*, *21*(3), 33–36.

Kaur, B. (2015). What matters? From a small scale to a school-wide intervention. *ZDM Mathematics Education*, *47*(1), 105–116. https://doi.org/10.1007/s11858-015-0665-8

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. Prentice-Hall.

Pritchard, R., & McDiarmid, F. (2006). Promoting change in teacher practices: Investigating factors which contribute to sustainability. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures, and learning spaces. Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia* (pp. 432–439). Canberra: MERGA.

Saito, E., Khong, T. D. H., & Tsukui, A. (2012). Why is school reform sustained even after a project? A case study of Bac Giang Province, Vietnam. *Journal of Educational Change*, *13*, 259–287.

Sexton, M. (2023). *The professional learning leadership of school mathematics leaders after participation in a large-scale primary school mathematics improvement project: An activity theory perspective* [Doctoral dissertation, Australian Catholic University]. SearchBank. https://doi.org/10.26199/acu.8z10q

Sutton, J., & Austin, Z. (2015). Qualitative research: Data collection, analysis, and management. *The Canadian Journal of Hospital Pharmacy*, *68*(3), 226–231. https://doi.org/10.4212/cjhp.v68i3.1456

Tirosh, D., Tsamir, P., & Levenson, E. (2015). Fundamental issues concerning the sustainment and scaling up of professional development programs. *ZDM Mathematics Education*, *47*(1), 153–159. https://doi.org/10.1007/s11858-015-0665-8

Warren, E., & Miller, J. (2016). *Mathematics at the margins.* Springer.

Zehetmeier, S. (2017). Researching the sustainable impact of professional development programmes. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the tenth congress of the European Society for Research in Mathematics Education* (CERME10, 1–5 February 2017, pp. 3019–3025). CERME.

Can a Short Online Test Diagnose Student Thinking?

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This paper provides evidence that a short, fully online, well-constructed diagnostic test based on research literature can give teachers information about their students' thinking and strategies that is sufficiently accurate to use for formative assessment purposes. The example is a test for students beginning to learn to solve equations. The main goal is to inform their teacher of the solving strategies each can use. Accuracy data from 3010 students is analysed to judge how well the overall performance of each group matches predictions. Alongside a previous analysis of misconceptions and common errors, we show this test gives teachers a good picture of most students to plan their lessons.

The purpose of this paper is to provide evidence that a short, fully online diagnostic test can give teachers information about their students' thinking and strategies that is sufficiently accurate to use for formative assessment purposes, i.e., to identify individual student's understandings and to plan future teaching that builds on their current knowledge. Of course, we do not claim that a computerised test is the best way to learn about an individual's mathematical thinking and knowledge, but our focus is on developing quick and easy tools to give teachers a 'good enough' guide for teaching an up-coming topic to all their students. Over many years, we have created *Specific Mathematics Assessments that Reveal Thinking* (SMART::tests) on 66 topics for middle years students.

In this paper, using a test of solving equations as an example, we illustrate one part of the evaluation of these tests from the large-scale data, here using the accuracy of student responses to assess its adequacy as a diagnostic tool and to identify refinements. To evaluate a SMART::test, we also conduct an in-depth analysis of students' common errors and misconceptions, as demonstrated for the present test by Steinle, Stacey, et al. (2022). Small-scale mixed methods studies have also been undertaken to validate SMART::tests (e.g., Klingbeil et al., 2024)

SMART::tests (www.smartvic.com) have been described elsewhere (e.g., Stacey et al., 2018). For this paper, the key points are that volunteer teachers assign the tests, which students can do at school or at home when needed. All aspects of the test are fully automated. The diagnosis is not just based on the correctness of items but also considers actual responses. The teacher receives, for each student, a 'stage of learning' and possibly some flags which indicate that there is evidence of a misconception, a missing conception or a common (procedural) error. Each test has a highly specific focus on a central aspect of a topic that we believe influences students' underlying understandings. The focus is on what mathematics education researchers have identified lies behind students' work and affects their success, not what is easy to see from a standard test. Professional learning is an important goal, by helping to deepen teachers' knowledge of hidden cognitive obstacles as they see how their own students think and see teaching suggestions for students at each stage.

In the next section, we describe the Australian curriculum expectations that are the focus of this test; briefly describe research which identifies an important learning challenge; give a mathematical analysis of the three strategies that students should learn, and explain the diagnosis made by the test. The method section briefly describes the test and its construction,

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and the data collection. We then look in turn at the performance of students in each of the diagnosed stages, examining whether their performance supports the diagnosis supplied.

Space limitations restrict this paper to reporting only on analyses of accuracy statistics—the numbers of students who are correct, incorrect or omit each item. This paper is supplemented by data from a detailed study of errors within the item responses (Steinle, Stacey et al., 2022). In this paper, it is valuable to see how much accuracy statistics can reveal.

Solving Linear Equations—Mathematical Analysis and Research

Curriculum Expectations

This paper focusses on the SMART::test *Solving Linear Equations*. Its target group is students beginning to learn (symbolic) algebra. The Australian Curriculum: Mathematics (Version 9) (Australian Curriculum, Assessment Reporting Authority [ACARA], 2022) has one main content descriptor for learning to solve linear equations, placed at Year 7 "AC9M7A03 "Solve one-variable linear equations with natural number solutions; verify the solution by substitution" (https://acara.edu.au/curriculum).

Supporting this, the (non-compulsory) elaborations suggest that students might use substitution to determine whether a given number is a solution, or solve using *backtracking* (called *unwinding* in this paper), or balance equations perhaps illustrated with a concrete model. The only given example of an equation is $3x + 7 = 19$ which has one occurrence of the pronumeral, and a natural number solution.

These elaborations for AC9M7A03 point to three different strategies for solving equations, all of which significantly contribute to mathematical progress. In order of cognitive complexity, these are substituting, unwinding then balancing. Both substituting and unwinding are foreshadowed in Year 6 by AC9M6A02 which is about finding unknown numbers in numerical equations (e.g., an equation where a square placeholder is written instead of the unknown number). At Year 8, AC9M8A01 highlights algebraic manipulation including rearranging, simplifying linear expressions, and using inverse properties (presumably to solve equations). These capabilities open up the range of equations that can be solved, especially by the balancing method.

Literature Review

The fundamental research on the teaching of equation solving was conducted several decades ago, although related research continues (e.g., Linsell 2010; Knuth et al., 2006). At that time, researchers were interested in characterising the essential elements of 'algebraic thinking' with a goal to move teachers' and researchers' conceptualisation of the school topic of algebra away from emphasis on the manipulation of symbols towards understanding the fundamental concepts that students must develop. This movement led to the formalisation of an algebra strand in primary school curriculum descriptions. Within the topic of solving linear equations, Filloy and Rojano (1989) observed there was a 'didactic cut' between the demands of solving 'arithmetic equations' that can be solved by successively unwinding operations on numbers (e.g., $\mathrm{A}x \pm \mathrm{B} = \mathrm{C}$, $x/\mathrm{A} = \mathrm{B}$) and other 'non-arithmetical equations' like $\mathrm{A}x \pm \mathrm{B} = \mathrm{C}x \pm \mathrm{D}$ (for A, B, C, D positive numbers). As other researchers have, they showed that this 'didactic cut' was hard for students to cross, not just because of new procedures to be followed but because the second required a different way of thinking about equations. Kieran (1992) described this distinction as being (p. 392) between (i) procedural operations with "ostensibly" algebraic items, i.e., "arithmetic operations carried out on numbers to yield numbers" and (ii) structural operations "that are carried out, not on numbers, but on algebraic expressions" which can have the result being an algebraic expression. Herscovics and Linchevski (1994) referred to a similar 'cognitive gap' between arithmetic and algebra, (p. 59) which is "characterised as the students' inability to spontaneously operate with or on the unknown". They observed that Year 7 students' strategies and success changed depending on the location of the unknown in the equation, especially for subtraction and division. For example, the most frequent strategy for solving $n - 13 = 24$ was to add 13 to 24 (using the inverse operation), but $37 - n = 18$ was solved by a variety of strategies such as "complementary subtrahend" $(37 - 18)$ using knowledge of subtraction. They concluded that the difference in success solving these two similar equations resulted from students "working around the unknown at a purely numerical level" (p. 70). This literature provides a justification for the importance of this test, as well as the fundamental concepts that illuminate the three strategies and enable the design of the *Solving Linear Equations* test. Note that there is no indication of the 'cognitive gap' in the curriculum statements, so helping teachers understand it is a major goal of this test.

Analysis of Strategies

Table 1 analyses the three strategies, their roles, and the characteristics of items that the test uses to identify the strategies that students are able to use. In practice, experts use all these strategies, often interchangeably in the case of unwinding and balancing, but for beginners they are separate skills drawing on separate concepts. The various statements about students in the early stages of learning algebra in Table 1 draw on evidence from the literature and our pilot studies (Steinle, Price, & Stacey, 2022).

Table 1

Characteristics of Three Equation Solving Strategies

Feature	Substituting	Unwinding$^{\alpha}$	Balancing$^{\beta}$
General statement	A guess and check strategy, sometimes with an 'improve' step.	Read notation as specifying a sequence of operations applied to an unknown number.	A series of transformed equations including simplifying and reorganising steps.
Example: Solve $2n + 5 = 20$	Guess $n = 3$ Substitute: $2 \times 3 + 5 = 11$. Compare result: 11 is too small, try larger *n*, etc.	Unknown was multiplied by 2, then 5 was added to give 20. To undo, first subtract 5 (gives 15) and divide by 2 (gives 7.5)	$2n + 5 = 20$ $2n + 5 - 5 = 20 - 5$ $2n = 15$ $2n \div 2 = 15 \div 2$ $n = 7.5$
Operations used	Given operations	Inverse operations	Inverse operations
Operations on and with	Numbers	Numbers	Numbers and pronumerals
Applicability	Can be used to solve any equation, but beginners only successful with small natural number solution.	Beginners only successful with one occurrence of the variable easily seen to be at the start of the 'story'.	Powerful for a wide variety of equations.
Main teaching purposes	Reading algebraic notation; understanding what a solution is; introducing tables of values.	Introducing inverse operations; connecting to 'function machine' diagrams.	Widely applicable principle for equation solving.

$^{\alpha}$Used to describe a process which is applied without any preliminary algebraic manipulation.

$^{\beta}$Often referred to as do the same to both sides and process often abbreviated to change side-change sign.

Table 1 includes a simple example and then specifies various salient characteristics from the literature. The example does not show operating on or with pronumerals and expressions, which is required when there is more than one occurrence of the pronumeral (e.g., to solve $5x + 9 = 2x$). Not noted is the fact that the logic of balancing additionally requires a conception of operating structurally on a whole equation to gradually transform it to a form where the solution is obvious. To stress the importance of all the strategies to progress in algebra, the table also briefly outlines their curriculum functions. The applicability row is the key to the diagnosis made by the test. We exploit the limitations of the methods, in particular for beginners, to identify what strategies students can use.

Method

The present test is an outcome of considerable preliminary investigation, pilot testing, and data analysis as recorded by Steinle, Price, and Stacey (2022). Each item requires the student to type the solution to the equation. Table 2 gives the 14 equations, noting features of equation structure and the solution, as well as a label which indicates which of the five groups each equation belongs to. Students are allocated to one of five stages according to accuracy of responses to items in Groups A to D, as described in Table 3. Items in Group E provide another part of the report to teachers. In specific terms, the research question for this paper is to determine the extent to which the performance of the students diagnosed in each stage matches the predictions based on the strategies.

Space does not allow a thorough discussion of the choice of items, but we note a few general points. The pilot work and literature highlighted how much beginning students are affected by apparently small changes of algebraic structure (e.g., changing $14 - 2x$ to $2x - 14$). Therefore, to be able to attribute changed success to just the identified features of structure (and hence strategy), the items in the diagnostic Groups A, B, C and D are as close to the canonical linear form as possible. Because teachers want to know about how students deal with various algebraic notations, the Group E items were included but not used for the main diagnosis. The number of equations were reduced to a minimum, to keep the average time of completion under 15 minutes. This test is Version 1; there is also a parallel test (Version 2) which can be used as a post-test, which has been similarly analysed. Results are similar to those reported here.

The data was collected from the 3010 anonymous students in the classes of volunteer teachers who assigned this test during the years 2016 to 2018. Most students were in Year 8 or 9. In general, the teachers seemed to have allocated the test to the intended target group, since the median test score was 7/14. We have no information about the conditions for taking the test, which may have varied significantly. Some students had trouble typing fraction and decimal answers (e.g., typing 1\5 instead of 1/5 or 1.1.625 instead of 1.625). Teachers can access any completed SMART::test to see why a student has received an unexpected result.

Results

In this section, we examine the extent to which the performance of the students diagnosed in each stage matches the predictions based on the strategies. Table 2 shows which items are expected to be within the capability of students who can use each strategy. The validity of the tests requires evidence that students in each diagnosed stage can use the strategy specified in Table 3. Of course, beginning algebra students make many slips, so that giving an incorrect response is not always indicative of a lack of knowledge of a strategy. Hence we have used the word ‘feasible’ in Table 2 for an appropriate strategy rather than ‘correct’. However, we expect that the predicted successes of Table 2 should give a good guide to the observed success on all items, including Group E items not used for diagnosis.

Table 2

Items Showing Equations, Details of Structure and Applicability of Strategies (Test Version 1)

Label	Equation	Equation Structure: Pronumeral Location: Left, Right	Equation Structure: Explicit Oper-ations	Solution	Strategy Applicability: Substitute	Strategy Applicability: Unwind	Strategy Applicability: Balance
A1	$3x + 8 = 23$	1, 0	+	5	F	F	F
A2	$4x + 9 = 37$	1, 0	+	7	F	F	F
B1	$5x + 7 = 15$	1, 0	+	1.6	A	F	F
B2	$8x + 3 = 16$	1, 0	+	1.625	A	F	F
C1	$8x + 5 = 3x + 14$	1, 1	+ +	1.8	A	N	F
C2	$12x + 2 = 8x + 15$	1, 1	+ +	3.25	A	N	F
D1	$7x - 11 = 2x - 4$	1, 1	− −	1.4	A	N	F
D2	$12 - 11x = 5 - x$	1, 1	− −	0.7	A	N	F
E1	$7x - 2 = 16$	1, 0	−	$^{18}/_{7}$	A	F	F
E2	$14 - 2x = 8$	1, 0	−	3	F	N	F
E3	$3x + 6 + 2x = 7$	2, 0	+ +	0.2	A	N	F
E4	$\frac{x+2}{5} = 3$	1, 0	+ ÷α	13	F	F	F
E5	$\frac{x}{3} + 1 = 5$	1, 0	÷α +	12	F	F	F
E6	$4(x - 3) = 21$	1, 0	− ×α	8.25	A	A	F

F: strategy is feasible, A: strategy is awkward, N: strategy not possible without initial algebra.

αFraction or brackets may or may not be seen as 'explicit' operations.

Notes. E1 is only item with solution longer than three decimal places. Pronumeral *a* was used in Version 1.

Table 3

Rubric for Allocating Stages to Students Based on Scores on Groups of Items

Stage	Description of stages	Score on group of items: A	B	C	D	E
0	Not yet at Stage 1	0	–	–	–	–
1	Students can solve simple linear equations that are easy to solve by repeat substituting;	1,2	0	–	–	–
2	…and can solve linear equations with more difficult solutions requiring systematic strategy (such as unwinding);	1,2	1,2	0	–	–
3	…and can solve linear equations involving addition only, with the pronumeral on both sides (balancing strategy needed);	1,2	1,2	1,2	0	–
4	…and can further solve linear equations involving subtraction with the pronumeral on both sides and non-integer solutions.	1,2	1,2	1,2	1,2	–

Note. "–" indicates that group is not used in the stage diagnosis.

Table 4 shows the accuracy for each item by stage, reporting two measures: Facility 1 is the percentage of all students who are correct on an item and Facility 3 is the percentage of the students who respond who are correct. If there are any omissions, Facility 3 is larger. When reading the tables, it is useful to note that algebraic manipulation shows the omission rate for an item is (Facility 3 − Facility 1) / Facility 3. If Facility 3 − Facility 1, there are no omissions; if Facility 3 = 1.5 × Facility 1, one third of students omitted; and if Facility 3 = 2 × Facility 1, the same number of students omitted the item as responded. Omissions generally increase later in the test although students also look through the test selecting items that look easy. (See Steinle, Stacey, et al. (2022) if you are curious about Facility 2.)

Table 4

Facility Range (Facility 1~Facility 3) of Test Items by Diagnosed Stage

Label	**Equation**	**Solution**	**Diagnosed stage**				
			Stage 0 (*n*=338)	**Stage 1 (*n*=575)**	**Stage 2 (*n*=762)**	**Stage 3 (*n*=287)**	**Stage 4 (*n*=1048)**
A1	$3x + 8 = 23$	5	rubric 0%	94%~94%	96%~96%	98%~98%	97%~97%
A2	$4x + 9 = 37$	7	rubric 0%	87%~89%	95%~96%	95%~95%	97%~97%
B1	$5x + 7 = 15$	1.6	3%~4%	rubric 0%	95%~95%	97%~97%	97%~97%
B2	$8x + 3 = 16$	1.625	3%~3%	rubric 0%	83%~88%	87%~90%	95%~95%
C1	$8x + 5 = 3x + 14$	1.8	6%~9%	12%~22%	rubric 0%	90%~91%	96%~96%
C2	$12x + 2 = 8x + 15$	3.25	5%~8%	7%~15%	rubric 0%	63%~76%	94%~94%
D1	$7x - 11 = 2x - 4$	1.4	4%~7%	6%~13%	3%~9%	rubric 0%	92%~93%
D2	$12 - 11x = 5 - x$	0.7	3%~5%	5%~10%	3%~9%	rubric 0%	69%~72%
E1	$7x - 2 = 16$	$^{18}/_{7}$	4%~6%	11%~20%	40%~61%	53%~70%	81%~83%
E2	$14 - 2x = 8$	3	7%~10%	27%~46%	33%~54%	39%~52%	77%~81%
E3	$3x + 6 + 2x = 7$	0.2	2%~4%	7%~15%	19%~44%	39%~61%	81%~87%
E4	$\frac{x+2}{5} = 3$	13	9%~16%	23%~46%	35%~68%	52%~78%	79%~87%
E5	$\frac{x}{3} + 1 = 5$	12	10%~19%	22%~46%	34%~66%	46%~70%	76%~84%
E6	$4(x - 3) = 21$	8.25	3%~6%	5%~11%	18%~39%	37%~59%	72%~80%

The shading (from Table 2) indicates the applicability (F, A, N) of the hypothesised strategy for each stage. 'Rubric 0%' indicates the item must be incorrect for student to be allocated to that Stage (as in Table 3).

Stages 3 and 4: Do Responses Match the Balancing Strategy?

Stage 3 and 4 students were predicted (Table 2) to do well on all items (except Stage 3 on Group D). They have very high performance on Group A, B and C and higher performance than other stages on all other items. After A1, Stage 4 students had a somewhat higher facility range than Stage 3 on every item. We see that Stage 4 students are more secure than Stage 3 in their balancing strategy (e.g., comparing Group C facilities) and the Group E facilities show they are better able to manage algebraic variations. Importantly, the facility ranges for Group D reveal the importance difference between subtraction of a number (D1) and subtraction of a pronumeral (D2) even for these relatively accomplished students (as noted by Herscovics & Linchevski, 1994). This difference is also apparent in the decrease in facility between E1 and E2 for students in Stages 2, 3 and 4. Only Stage 0 and Stage 1 students had higher facility on E2 than on E1, supporting the hypothesis that these students are substituting numbers rather than working with the algebra and hence found the item with a natural number solution easier.

Stage 3 students' responses generally fit the predicted pattern, with high success on items in Groups A, B, and C, and moderate success on Group E. We have no good explanation for the large drop in facility for C2 compared to C1. Overall, the Stage 3 pattern is consistent with students still developing the balancing strategy and the algebraic skills that are necessary to manage variations in algebraic structure (see, for example E3 and E6).

Stage 2: Do Responses Match the Unwinding Strategy?

The *a priori* analysis in Table 2 predicted the unwinding strategy would give success with eight items but, assuming no prior algebraic manipulation to put the equation in an amenable form, it would fail for six items (C1, C2, D1, D2, E2, E3) and that E6 would be difficult. The facility ranges for the Group A, B, C and D items match the prediction. Items E1, E4, E5 were predicted to be feasible with the unwinding strategy (Table 2), and around two thirds of those Stage 2 students who attempted these were successful. Item E6 can also be solved by unwinding, but success relies on reading the brackets notation appropriately; first 3 is subtracted from the unknown number then the answer is multiplied by 4. Perhaps this is why the facility was much lower. Items E2 and E3 cannot be solved by unwinding without preliminary algebraic manipulation. After collecting like terms, E3 belongs to Group B where this group of students have already shown very high facility, but only 44% of those who responded to E3 were successful and the omission rate was high. This is consistent with students learning unwinding strategies with clear stories of what happened to the unknown number, written in algebra (e.g., E4, E5, A, and B items), and not yet doing preliminary algebraic manipulation. Stage 2 students did much better on item E2 than predicted with nearly half being successful; we assume they turned to substitution (natural number answer), which may also have boosted facilities of E4 and E5. The omission rate was high for all E items, supporting the picture of these students just beginning to learn equation solving, still having little exposure to variations of algebraic form.

Stage 1: Do Responses Match the Substituting Strategy?

The *a priori* analysis in Table 2 predicted the substituting strategy would be successful when solutions were small natural numbers, but in other cases would be awkward for beginners. Locating a non-integer solution takes a relatively long time and is prone to error because more complex arithmetic is required. This is well supported by the data. All the items with Facility 3 over 40% have small natural number solutions. All other facilities are under 25%. The omission rate is high at around one third after Group A items, which is consistent with students taking a long time to complete the questions. Further support for use of substituting was presented by the error analysis (Steinle, Stacey et al., 2022). Stage 1 students often gave an answer of 1 more or less than the correct answer (so a numerical slip) or answered N.5 when the correct answer was between N and N + 1 (e.g., to give an answer 3.5 instead of 3.25). We hypothesise that this response was used to indicate that they know the answer is somewhere between 3 and 4.

Stage 0: What do These Students Know?

Students were put into Stage 0 if they answered neither A1 nor A2 correctly. It is to be expected that a small number of capable students made slips (mathematical or response entry) on both A1 and A2 and therefore are misdiagnosed by the strictly hierarchical rubric we have used (see Table 3). A small rate of misdiagnosis is supported by the observation that every item in Groups B, C and D was answered correctly by at least 3% of the Stage 0 students. The items with the highest facility for Stage 0 students are E2, E4, and E5, all of which have small natural number solutions, indicating some students were substituting. For E4 and E5, around 10% of students were correct. These are the only items that do not include implicit multiplication. We conclude that at least 10% of the students diagnosed as Stage 0 are using substitution (and we estimate about a half), but they do not reliably interpret implicit multiplication. In support of this, the error analysis (Steinle, Stacey et al., 2022) found that incorrect responses consistent with interpreting mx as $m + x$ were given by over one-third of Stage 0 students.

Conclusion

The purpose of this paper was to demonstrate that a short online test can diagnose students' thinking sufficiently well to help a teacher plan teaching. The constraints were that the test must be short, student responses must be machine-readable, and the diagnosis given to the teacher must be easily understood and important. The literature review and curriculum analysis demonstrated the importance of the strategies for solving linear equations. This paper illustrated the contribution of analysis of overall accuracy to validating the resulting diagnoses; error analysis reported elsewhere supplements this. The match between the theoretical predictions of item facility and the actual facilities provided evidence that most students in Stage 1 use substituting only, most students in Stage 2 can (also) use unwinding and that students in Stages 3 and 4 can use balancing, at least with basic algebraic structures. The distinction between Stage 3 and 4 students was not as clear-cut as expected, so could be refined. The test results also highlighted the importance of students frequently working with linear equations that are not written in the canonical ax + b form.

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References

Filloy, E. & Rojano, T. (1989). Solving equations, the transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 19–25.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, *27*(1), 59–78.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 390–419). New York: MacMillan Publishing.

Klingbeil, K., Rösken, F., Barzel, B., Schacht, F., Stacey, K., Steinle, V. & Thurm, D. (2024). Validity of multiple-choice digital formative assessment for assessing students' (mis)conceptions: Evidence from a mixed-methods study in algebra. *ZDM Mathematics Education*. https://doi.org/10.1007/s11858-024-01556-0

Knuth, E. J., Stephen, A. S., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equals sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, *37*, 297–312.

Linsell, C. (2010). Secondary numeracy project students' development of algebraic knowledge and strategies. *Findings from the New Zealand numeracy development projects 2009* (pp. 100–116). Wellington: Ministry of Education.

Stacey, K., Steinle, V., Price, B., & Gvozdenko, E. (2018). Specific mathematics assessments that reveal thinking: An online tool to build teachers' diagnostic competence and support teaching. In T. Leuders, J. Leuders, & K. Philipp (Eds.), *Diagnostic competence of mathematics teachers. Unpacking a complex construct in teacher education and teacher practice* (pp. 241–262). New York: Springer.

Steinle, V., Price, E., & Stacey, K. (2022). *SMART::test solving linear equations: development and student results* [Technical report]. University of Melbourne. https://doi.org/10.26188/20278920

Steinle, V., Stacey, K., & Price, B. (2022). Beyond accuracy: A process for analysis of constructed responses in large datasets and insights into students' equation solving. *Journal of Educational Research in Mathematics*, *32*(3), 1–28. https://doi.org/10.29275/jerm.2022.32.3.1

Promoting Mathematical Reasoning in the Early Years Through Dialogic Talk

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This article presents research focused on establishing ways that dialogic talk between teachers and students promotes mathematical reasoning in early years classrooms. Data are drawn from recorded and transcribed Year 1 mathematics lessons. Conversation analysis provides close examination of the talk-in-interaction practices of teachers and students and reveals how mathematical reasoning is co-produced through the turn-by-turn exchange structures between teachers and students. Analysis of selected transcripts illustrates how different teacher talk moves create a dialogic space that make student mathematical reasoning possible. Implications for classroom practices are made.

Lessons by their very nature are interactive events where teachers and students come together in interactions centred around promoting and eliciting student's learning (Edwards-Groves, 2024). In mathematics lessons, one of the central pedagogical goals is facilitating students' reasoning; for students, reasoning is generally experienced as they participate in whole class, small group or paired discussions (Mercer, 2008). Mathematical reasoning is well addressed in both the research literature and curriculum documents as being a critical process necessary for supporting young learners to identify patterns, organise ideas, solve problems, discern relationships between concepts, and draw logical well justified conclusions (Australian Curriculum, Assessment Reporting Authority [ACARA], 2022; Russell,1999). Interaction research also establishes that talk and interaction itself is a reasoning activity (Hutchby & Wooffitt, 2008) which requires interlocutors, like teachers and students in lessons, to display their reasoning in their turns where intersubjectively, sense is made as conversations unfold turn-by-turn (Schegloff, 2007). Therefore, to support early years teachers to develop deeper understandings about mathematical reasoning, there is a critical need to draw closer attention to the complexity that reasoning talk in mathematics lessons presents. For effective teaching, this requires accounting for the inextricable interrelationship between classroom dialogue, communication practices and mathematical reasoning, and what this means for early reasoning proficiency. To understand the nature and influence of classroom talk on young children's mathematical reasoning, this article examines some core (and often taken-for-granted) dialogic talk practices of teachers and students in their mathematics lessons.

Literature Review

In this section, two main bodies of literature relevant to mathematical reasoning and classroom talk are presented.

Mathematical Reasoning

Mathematical reasoning has a central role in learning mathematics and is critical for student's developing mathematical understandings, creative thinking, problem solving, and knowledge (Chapin et al., 2007; Vale et al., 2017). In classroom lessons, developing mathematics reasoning aims to strengthen student competencies for solving increasingly complex mathematical problems (Stein et al., 1996; Wusterberg et al., 2012). Mathematical

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 503–510). Gold Coast: MERGA.

reasoning is comprised of three main reasoning actions: analysing, generalising and justifying. These reasoning actions are necessary for engaging in tasks requiring students to examine and interpret concepts and problems, elicit generalisations and provide well-reasoned justifications. It is through participating in more extended discussions with and among students that account for these reasoning actions, that an interconnected web of understandings and knowledge is built (Zevenbergen et al., 2001). This provides a foundation for mathematical sensemaking and so create conditions for developing one's mathematical memory (Russell, 1999).

In the *Australian Curriculum: F–10 Mathematics* (ACARA, 2022), reasoning is one of four interrelated proficiencies students are expected to develop as part of their learning. Along with understanding, fluency and problem-solving, reasoning is described as a necessary process for working mathematically. In lessons, students are required to engage in reasoning by working on mathematical tasks that require them to 'discover' and make sense of mathematics concepts and ideas. According to the curriculum, the reasoning process and the actions which comprise it, requires Early Stage One and Stage One students to demonstrate: (1) an increasingly sophisticated capacity for logical thought and actions; (2) thinking through their explanations; (3) mathematical thinking that creates and validates mathematical ideas and new knowledge; and (4) meaning making. Further, an emphasis on mathematical reasoning incorporates the study of flawed or incorrect reasoning as an avenue towards a deeper development of mathematical knowledge (ACARA, 2022, pp. 1–2). "Greater discussion among students and the recognition that mathematics is not constrained by always having a single answer" (Zevenbergen et al., 2001, p. 9) is necessary for providing educative conditions for students to produce and display mathematical reasoning. One way to foster such discussion is by creating a dialogic talk space that forms a strong pedagogical basis for facilitating mathematical reasoning.

Dialogic Talk

A growing body of research has focused on understanding and promoting teachers and students' repertoires of talk moves that form a critical feature of pedagogical efficacy (Attard et al., 2018; Chapin et al., 2013). As explained by Edwards-Groves et al. (2014), understanding talk moves concerns the work of a turn in an interaction; for example, this might be in the form of a question that opens a discussion sequence, a response to a prior speaker's turn/question, or an assessment of a turn through verbal (a comment, a rephrasing) and/or nonverbal (a gesture, a nod) means. Each, and in combination, evoke a particular kind of next turn speaker action and signals the function of a speaker's turn (e.g., a *why* question solicits a reasoning response) (Edwards-Groves, 2024). Hinton et al. (2021) described *dialogic talk moves* as turns which deliberately and strategically move *student turns* to:

- *Sustain a point* where a particular line of thinking is pursued and to produce evidentiary talk by providing clarifications, exemplifications, reasons and justifications;
- *Extend and deepen thinking* about their ideas by orienting to and building on the thinking of others, to learn from the opinions, reasons and knowledge of others;
- *Challenge and question* the thinking of others by building argumentation through agreeing and disagreeing and promoting the acceptance and production of a range of alternative points of views;
- *Demonstrate active listening* by providing feedback to the turns of others, asking for clarification, and responding directly to the thinking of others (saying what they think about what they'd heard in the prior turn or across a sequence), repeating (word-for-word) and revoicing (rephrasing in their own words) what they have heard;
- *Reflect on and review* their learning (often through retelling, paraphrasing or summarising) at different points across a lesson.

It is argued that these moves are dialogic since they move the next turn speaker towards producing longer more sustained turns, and so have the potential to make discussions more academically productive, high leverage and intellectually rigorous (Chapin et al., 2007). Edwards-Groves (2014) identified two deliberately used *communicative strategies* that further extend opportunities for student to talk and engage with the thinking of others: i) *wait time* (a deliberate pause of more than three seconds after a teacher turn); and ii) *vacating the floor* (reflected in strategies such as turn-to-talk, walk-and-talk, think-pair-share). Allowing longer *wait time* provides more time for student thinking, formulating and rehearsing ideas, helping them produce a more considered, and often longer, response before they 'go public' with their contribution. When teachers *vacate the floor* they 'remove' themselves from the immediacy of the conversation to allow students time and opportunity for practicing, rehearsing and 'testing' ideas, and managing, controlling and taking responsibility for the conversation, their contributions, and ultimately their learning. Vacating the floor might take the form of the teacher withholding of turns to allow students take consecutive turns. Importantly as Stibbard et al. (2020) also showed, that student-student talk sequences focused on the task (in paired or small group interactions) are reflective of a dialogic classroom.

The Study

Participants and Contexts

Data for this article are drawn from a broader study conducted in five Early Stage One/Stage One classrooms in three primary schools in regional NSW, Australia, aiming to understand the nature and influence of classroom talk on young children's mathematical reasoning. This article examines some core (and often taken-for-granted) dialogic talk practices of teachers and students in their mathematics lessons, and sought to answer this specific research question: *What is the nature of teacher talk moves which promote mathematical reasoning in early years problem solving lessons?* To answer this question, recorded and transcribed mathematics lessons (15 in total) formed the corpus of data for analysis. Participants were recommended by the district mathematics consultant, then invited to participate. Informed consent was requested and given from teacher participants, along with the parents of students in each class. Class sizes ranged from 14–28 students. Selected transcript excerpts are from one Year 1 problem solving lesson where in the whole classroom discussion phase the teacher was focused on eliciting student's understandings about the differences between rows and columns.

Data Analysis

In this study, conversation analysis (Sacks et al., 1974) was employed to reveal the detail and intricacies of conversational exchanges in Year 1 mathematics lessons to show how reasoning is accomplished turn-by-turn and across teacher and student sequences. Reasoning, as a core process for working mathematically, emerged as a prominent analytic focus since accomplishing mathematical reasoning simultaneously requires interactive reasoning. Conversation analysis (CA) was applied to the data as a method for conducting a systematic analysis of the turn structure and organisation in the lesson talk-in-interaction (Sacks et al., 1974). As a premise, CA researchers study audio and/or video recordings of interactions, transcribing them into detailed transcripts, followed by the close analysis of interactional phenomena as it is produced in the talk (Davidson, 2009). Analysis focuses on the basic taxonomy (presented below) for understanding the speech-exchange system suggested by conversation analyst Emmanuel Schegloff (2007). This system is useful for identifying and understanding the talk-in-interaction produced by teachers and students in lessons (Edwards-Groves, 2024); specifically, attention is given to the:

- Length of the turns and exchanges;
- Nature of the contributions provided by each party to the talk (teachers and students);

- Types of interactive processes used in the task-setting, instructional or discussion parts of the talk (e.g., what is being asked of the students in terms of mental activity or what thinking is needed, verbal/literate activity or what language and discourses are required, material activity or what activities, tasks and resources will be spoken about, and relational activity or what grouping and interactive arrangements are required);
- Orientation, logical order, and relevance of the talk related to the:
    - *local*—what was just said or done then and there in the interaction;
    - *sequential*—what has been said & heard in and across a sequence or lesson;
    - *topical*—the relationship of the turn to the topic at hand;
    - *categorical*—the expectations or norms displayed by participants in and by their turns, like a teacher or student (as they talk in pairs, groups or whole class); and
    - *reasoning*—the specialised reasoning and/or linguistic practices of a discipline/ curriculum like mathematics (Schegloff, 2007).

These features are typical of how meanings are produced in ordinary talk-in-interaction, but also highlight the complex interactional demands required of students participating in lessons asking them to demonstrate reasoned sensemaking about what is happening at the same time produce mathematical reasoning.

Results and Discussion

The excerpts presented next are drawn from the 'launch' phase (Sullivan et al., 2016) in the final lesson of a five-lesson sequence focusing on multiplication and division. To begin, the 14 students were asked to organise themselves into rows of three, after which the teacher asked the framing question "*what's a row?*" The mathematical knowledge for the task included understanding arrays (specifically arrays with the dimensions of 4 by 3); knowledge of arrays being structured using equal rows and columns; and understanding remainders (items left over). The students were expected to make four rows of three students, with two students 'left over'. In this first excerpt, several teacher and student talk moves illustrate how reasoning is promoted across a series of turns. (Note: each excerpt accounts for less than two minutes of talk).

Excerpt 1: Producing and Co-Producing Reasoning

37	Tea:	Charlie↑ have you thought about how many rows we will have↑(0.5)
38	Cha:	four rows
39	Tea:	you'll end up with four rows↑(.) okay↑
40		how did you(.) why do you think four Charlie?
41	Cha:	because I knew four
42	Tea:	one row yep↑ tell me more
46	Cha:	and um Mischa and(.)and Cassie and (.)
47	Tea:	ye:ah↑=
48	Cha:	=and me and Fletcher=
49	Fle:	=but there's not enough three peoples
50	Tea:	hang on, let Charlie finish↑
51	Cha:	me, Noah and Fletcher then two (.) then we'd need one more
53	Tea:	we'd need one more to make our↑(.)
54	Fle:	fifth row
55	Mis:	it has to be one, to make a full row=
56	Tea:	=row Charlie(.) well done I think you did some <u>good</u> maths thinking
57		then↓ remember you are making <u>rows of three</u>

In this sequence, the teacher's use of wait time (the non-typical extended pause of 5 seconds (0.5) in line 37), the "why" question (line 40), the "tell me more" request (line 42), that more turns than is typical is allocated to Charlie to sustain his point so that he could continue his line of thinking where he could explain his reasons for his initial answer “four” (line 38), form a collection of turns that exemplify a more dialogic sequence of talk.

The “why” question (line 40) not only functioned as a pass on talk move which returned the floor to the Charlie after his first response (Willemsen et al., 2020), but signalled that a reasoning response was required. Why questions form a dialogic talk move as they typically provoke a reasoning response whereby student/s are asked to explain their thinking (Edwards-Groves, 2024). Here, reasoning is produced not as a single student response, but as evident in this excerpt, across a series of turns involving Charlie, other students, and the teacher. The teacher, in a deliberately orchestrated way, allowed Charlie more turns to provide a more complete explanation, then justification for his answer. That the teacher (line 50) made it clear that she wanted Charlie’s line of thinking to not be interrupted by another student created the space for Charlie to do so. Charlie extended his initial response by explaining the thinking he used to work out his answer of four; he did this by grouping students into rows. Contributions by other students Fletcher and Mischa provided additional evidence of how mathematical reasoning was co-produced in this instance.

As the lesson progressed, the teacher called for students to suggest other examples that answer the framing question, “what’s a row”? The next excerpt shows how the responsibility for producing examples and explanations is shared among the students as they work through the problem.

Excerpt 2: Sharing the Responsibility for Making Thinking Evident

121 Tea: ok so back to our question, what’s a row↑ (0.3) what’s a row↑ (0.3)
122 do you know what a row is↑ (0.2) what do you think Cleo↑
123 Cle: um(.) like a line↑ (0.2)
124 Tea: like a line↑ so do you think rows can go (.) like this? ((teacher draws
125 three 0’s horizontally on the whiteboard)) or this↑ ((draws three 0’s
126 vertically)) or this (.) or this↑ ((draws three 0’S diagonally))
127 Sts: no↓ ((no:o)) ((one student saying “they’re not rows when they go down”))
128 Lia: any way↑ they can go any way↑ rows can go any way↑
129 Tea: you think rows can go any way↑
130 Cle: look they’re going that way, going that way
131 Tea: I heard someone say no though↓ who said no↓ you don’t think they’re
132 rows↓ alright (.) Cassie you said they’re not rows when they go down↑
133 (0.3) tell us why you think that
134 Cas: rows are like(.) the movie theatre
135 Tea: can you show us (0.1) can you draw some rows for us↑ ((Cassie draws
136 straight lines horizontally on the whiteboard to demonstrate rows))
137 so Cassie(.) said rows are like when go to the movie theatre (0.5) thanks
138 Cassie↓ (0.4) *what do you think* about what Cassie has just shown us↑(.)
139 Lin: Cassie’s right, she said rows are like (.) when you go to the movie theatre
140 and you sit in the seats in rows …

In this sequence, producing mathematical reasoning is aligned with both articulating thinking and showing evidence (e.g., Cassie’s drawing her ideas on the board). This is made prominent in the teacher’s prompts and requests, such as “what do you think” (line 122), “so do you think?”(line 124), “you think” (line 129), “you don’t think” (line 131), “why you think that’ (line 133), “can you draw some rows…” (line 135), and “what do you think about...” (line 138). The sequence progresses with a range of student responses, that “rows are like lines”, that “rows can go any way”, and that “rows are not rows if they go down”. It is at line 131, that the teacher shifts the student responses to move from the reasoning action of generalising (rows are like lines) to exemplifying (drawing rows on the board) to justifying (tell us why you think that). Here the teacher turn "Can you show us…?" (line 136) functions as a request for producing evidence to support Cassie's earlier point (line 127) that "they're not rows when they go down". Lincoln's subsequent agreement with and rephrasing of Cassie's explanation (lines 139/140) is a demonstration of active listening by overtly indicating his orientation and interpretation of another person's ideas. This brief excerpt shows how the mathematical reasoning is co-produced in the talk-in-interaction that students are supported (through prompts and requests) to make their thinking 'public' through extended explanations and justifications.

In this final excerpt, alternative responses are offered as possible solutions to the framing "why" question. Teacher prompted the dialogue in strategically and sequentially relevant ways, through talk moves such as using "I wonder", inviting multiple ideas like "let's have thumbs up if you have something to share", or asking open questions that require students to produce analysing, generalising and justifying their reasoning.

Excerpt 3: Using 'I wonder' for Producing Alternative Responses

141	Tea:	I wonder what other ideas (.) let's have thumbs up (.) if you've
142		got something else to share (.)…((turns omitted)) what do you think Noah↑
143	Noa:	um when Cassie said rows at the civic theatre but they they
144		could not be equal because you don't know if people could make it↑
145	Tea:	[okay
146	Noa:	[there could be some empty seats↓
147	Tea:	there could be empty seats but can you see what Cassie has [drawn↑
148	Noa:	[yeah
149	Tea:	so let's look at what Cassie is saying that (0.1) lines like that are rows↑
150	Noa:	yes
151	Tea:	whereas before people said that lines could go anyway(.) and
152		they're rows so what do you think about that↑(.) I wonder if it's a row
153		or is it just a line? (0.3) which one do you think might be rows(.) Noah↑
154	Noa:	the top one ((Noah refers to teacher drawing horizontal 0s))
155	Tea:	you think that's rows(.) why↑
156	Noa:	because um it doesn't matter how long your row is because(.) they're rows
157		aren't different to most of the [rows

In the first turn of this excerpt, the teacher's use of "I wonder" (line 141) functions as a move designed to open up the conversation to elicit other ideas and thoughts. Here, the teacher moves provided students with extended opportunities to articulate their thinking and justify their ideas and form evidence of the dialogic nature of this exchange. The use of 'I wonder' frees the students to contribute further explanations and designed to enhance student participation. This is a direct dialogic move as it is an opening-up rather than closing-down move. According to Houen et al. (2016), the use of "I wonder" plays an important role in discussions as a move that intentionally elicits additional "thoughts, ideas and opinions rather than being required to recite facts" (p. 75).

In this example, the "I wonder" formulation recruits Noah to answer. He subsequently demonstrates the analysing reasoning action (lines 143–4, 146, 154, 156) as he orients to, engages with, and challenges, Cassies' earlier explanation. Noah challenges Cassie's point by proposing a problem with her idea (albeit not entirely correct in terms of what she was proposing). To address this flawed reasoning (Russell, 1999), the teacher reformulates her question "what do you think about that" (line 152) into a wondering, "I wonder if it's a row or is it just a line?" (lines 152–53). This wondering seeks clarification and solicits more reasoning to rectify Noah's partially correct answer. As a response, Noah pursued his thinking by adding "but they need to be parallel", and on his reasoning, rows in a movie theatre are not parallel.

Summary

Analysis of these brief excerpts from this Year 1 mathematics lesson showed ways dialogic talk in this problem-solving lesson supported these early years students to produce the mathematical reasoning actions of analysing, explaining, and justifying. Across the sequences, dialogic talk was evident in how teacher talk moves opened the communicative space for students to analyse the ideas and the turns of others, generalise concepts (like rows and lines) and justify their responses as they attempted to explore 'rows' at the local, sequential, and topical levels. Across the turns, through interpretations, explanations and justifications, students jointly produced mathematical reasoning as they oriented to, took up, and extended the thinking of others. When the teacher vacated the floor (to withhold a turn or wait longer with a pause), students directly built on and responded to the ideas of others. It was evident that students were provided with extended and additional opportunities to offer alternatives to other student's ideas, take up or present new ideas, challenge others, and/or contribute additional information, personal opinions and alternative reasons.

Findings shows ways that Schegloff's (2007) five interactional features for meaning making and coherence is evident across the turns *as* students created and maintained meaning at the local turn level, sequential (across the turns and sequences), topical (about rows), categorical (as student and teachers co-producing meaning) and reasoning (in terms of producing mathematical reasonings) as they displayed their meanings about the talk, the topic and the task at hand was displayed.

Implications

Conversation analysis, through its fine-grained attention to how talk unfolds turn by turn, provided a useful method for understanding the nature of interconnectedness of interactive and mathematic reasoning with implications for dialogic talk practices. Analysis of this Year 1 lesson showed evidence that mathematical reasoning is co-produced across a series of turns rather that found in a single student response. It is shown that through the strategic and deliberate teacher use of talk moves, mathematical reasoning is accomplished. For example, moves such as the use of "I wonder", the use of a "why" question to provoke a reasoning response, affording students the opportunity to sustain and/or extend a point, challenging ideas, and/or producing alternatives are notable. This has important implications for teachers' understandings about how mathematical reasoning is produced as a process, and that it is made evident by students as a lesson unfolds, turn by turn, and across sequences of talk-in-interaction. Further, it is necessary to understand that talk-in-interaction is a reasoning task requiring intersubjective meaning making, that at the same time produces mathematical reasoning.

Conclusion

This article examines one of the most fundamental, yet taken-for-granted, activities that goes in classrooms—the talk and interaction that happens between teachers and students in their lessons. It is through this talk and interaction that intersubjective meanings are made, that mathematical reasoning is promoted and demonstrated, and that mathematical knowledge is developed. Given what is at stake, how lesson talk is understood and practiced by teachers and students matters to student learning is necessary for improving mathematical reasoning.

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References

Attard, C., Edwards-Groves, C., & Grootenboer, P. (2018). Dialogic practices in the mathematics classroom. In J. Hunter, P. Perger, & L. Darragh (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia*, (pp. 122–129). MERGA.

Australian Curriculum, Assessment & Reporting Authority (ACARA). (2022). *Australian Curriculum: Mathematics*. https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Chapin, S., & O'Connor, C. (2007). Academically productive talk: Supporting student learning in mathematics. In W. G. Martin (Ed.), *The learning of mathematics, 69th NCTM yearbook* (pp. 113–128). National Council of Teachers of Mathematics.

Chapin, S., O'Connor, C., & Anderson, N.C. (2013). *Talk moves: A teacher's guide for using classroom discussions in math*. Math Solutions.

Davidson, C. (2009). Transcription: Imperatives for qualitative research. *International Journal of Qualitative Methods*, *8*(2), 35–52.

Edwards-Groves, C. (2014). *Talk moves: A repertoire of practices for productive classroom dialogue*. Primary English Teaching Association Australia.

Edwards-Groves, C. (2024). Dialogicality for literacy learning in the middle years: Moves, meaning-making and motivations. *Literacy Learning: The Middle Years*, *32*(1), 19–31.

Edwards-Groves, C., Anstey, M., & Bull, G. (2014). *Classroom talk: Understanding dialogue, pedagogy and practice*. Primary English Teaching Association Australia.

Edwards-Groves, C., & Davidson, C. (2017). *Becoming a meaning maker: Talk and interaction in the dialogic classroom*. Primary English Teaching Association Australia.

Hinton, L., Edwards-Groves, C. & Davidson, C. (2021). Facilitating sustained dialogic discussions in a year 5 classroom: A shared teacher and student practice. *Practical Literacy*, *26*(1), 42+. https://link.gale.com/apps/doc/A657592307/AONE?u=anon~e5e860d6&sid=googleScholar&xid=129cf2f3.

Houen, S., Danby, S., Farrell, A., & Thorpe, K. (2016). Creating spaces for children's agency: 'I wonder…' formulations in teacher-child interactions. *International Journal of Early Childhood*, *48*, 259–276.

Hutchby, I., & Wooffitt, R. (2008). *Conversation analysis*. Polity Press.

Mercer, N. (2008). Talk and the development of reasoning and understanding. *Human Development*, *51*(1), 90–100.

Russell, S.J. (1999). Mathematical reasoning in the elementary grades. In L. V. Stiff & F. R. Curcio (Eds.) *Developing mathematical reasoning in grades K–12*, (pp. 1–12). National Council of Teachers of Mathematics.

Sacks, H., Schegloff, E. A., & Jefferson, G. (1974). A simplest systematics for the organisation of turn taking for conversation. *Language*, *50*, 696–735.

Schegloff, E. A. (2007). *Sequence organisation in interaction: A primer in conversational analysis* (Vol.1). Cambridge University Press.

Stein, M.K., Grover, B.W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455–488.

Stibbard, A., Davidson, C., & Edwards-Groves, C. (2020). Changing talk and interaction for dialogic teaching in an early years' literacy classroom. *The Journal of Teacher Action Research*, *6*(3), 4–25

Vale, C., Bragg, L.A., Widjaja, W., Herbert, S., & Loong, E.Y. (2017). Children's mathematical reasoning: Opportunities for developing understanding and creative thinking. *Australian Primary Mathematics Classroom*, *22*(1), 3–8.

Willemsen, A., Gosen, M. N., Koole, T., & de Glopper, K. (2020). Teachers' pass-on practices in whole class discussions: How teachers return the floor to their students. *Classroom Discourse*, *11*(4), 297–315.

Wusterberg, S., Grieff, S., & Funke, J. (2012). Complex problem solving—More than reasoning. *Intelligence*, *40*, 1–14.

Zevenbergen, R., Mousley, J., & Sullivan, P. (2001). Using open ended tasks for teaching, learning and assessment. *Teaching Mathematics*, *26*(1), 7–10.

Understanding of the Equal Sign: A Case of Chinese Grade 5 Students

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An understanding of the equal sign is a fundamental concept for early algebra. While literature claimed that Chinese students commonly master the relational understanding of the equal sign in the elementary school, these claims are under-researched. This study used the Mathematics Equivalence Assessment with 237 Chinese Grade 5 students. The results showed that the majority of students possess a relational understanding of the equal sign, with some able to confidently apply the concept of structural equivalence. To complement the test results, six Grade 5 teachers were also interviewed to explore teaching approaches and contexts used to foster an understanding of the equal sign.

Students' misconceptions of the equal sign still remain widespread in many western countries nowadays (Stephens et al., 2022). Many students possess an operational understanding of the equal sign, considering the equal sign as a symbol of indicating calculation results, instead of perceiving the equal sign as an indication of an equivalent relationship of both sides (i.e., a relational understanding, McNeil et al., 2015). A relational understanding of the equal sign is fundamental during students' progression from arithmetic to algebra (e.g., Carpenter et al., 2003). A narrow operational conception of the equal sign can cause students' difficulties in algebra (Stephens et al., 2022). For instance, students without the relational understanding of the equal sign struggle to understand the number sentences such as '3 = 3', and '3 + 2 = 4 + 1', and these students will further experience difficulties in making sense of solving equations with unknowns on both sides (e.g., $3x + 2 = 2x + 1$) when learning algebra. There has been extensive research contributing to pedagogical approaches to develop students' relational understanding of the equal sign, such as the number sentence evaluation activity (Carpenter et al., 2003) and exposing students to non-conventional forms of arithmetic expressions, such as '__ = 1 + 2' and '3 = 1 + __'. (McNeil et al., 2015). These pedagogies aim to interrupt students' one-directional (left to right) conception of the equal sign and provide students with an opportunity to attend to the equivalence of both sides of the equal sign.

While the misconception of the equal sign is common in western countries, it is not universal. Researchers (e.g., Li et al., 2008; Jones et al., 2012) reported that by the end of primary school, students in China generally have developed a relational conception of the equal sign. For instance, a pioneer work by Li et al. (2008) revealed that the majority (98%) of Grade 6 tested students (n = 145) in China possess a robust relational understanding of the equal sign. Since Li et al. (2008), a growing body of literature is exploring Chinese students' conceptions of the equal sign, indicating their process of gaining the solid relational understanding is not without setbacks (e.g., Sun et al., 2023; Yang et al., 2014). This leaves a space for a further investigation on how Chinese students develop a relational understanding of the equal sign. The purpose of this report is to contribute further insights to this topic by examining Chinese Grade 5 students' understanding of the equal sign and exploring the possible factors influencing the development of their understanding.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 511–518). Gold Coast: MERGA.

Literature Review

As mentioned, Li et al. (2008) reported students in China generally hold a relational understanding of the equal sign. Through a text-book analysis, Li et al. (2008) found that in China the equal sign was introduced to students before their exposure to addition and subtraction. Students started learning the formal symbol of the equal sign in a context of comparing and describing quantities relationship (e.g., 'more than', 'the same as', 'less than') of concrete objects, with the formal symbol of the equal sign being introduced afterwards. (Sun & Gu, 2023) conducted a close examination of the pedagogical approach to introduce equal sign in China. Echoing Li et al. (2008), Sun and Gu (2023) showed that the concept of the equality and the formal symbol of the equal sign is first introduced to children in kindergarten (for the children aged 5) in a quantity comparison context before they begin learning arithmetic operations (addition and subtraction), so the interference of misconception that the equal sign is a symbol of displaying results of calculation possibly brought by traditional forms of arithmetic expressions (e.g., 1 + 2 = 3, 7 − 5 = 2) can be reduced. Furthermore, they showed that children in kindergarten were introduced to an activity of drawing an equivalent number of any objects they like to match the other side of the equal sign, and the objects to be matched were allocated on either right side or left side of the equal sign (see Figure 1 below).

Figure 1

Match the Quantity of Objects Activity (Sun & Gu, 2023)

This activity supported students in building a bi-directional view of the equal sign. Finally, Sun and Gu (2023) mentioned introducing the formal symbol of the equal sign, the official curriculum document requires teachers to highlight the way of drawing the equal sign: 'two short horizontal lines with the same length'. This step appeared to reinforce students' conception that the equal sign refers to the 'sameness'. Overall, Sun and Gu (2023) concluded that children's experiences with the equal sign in kindergarten provided them with a foundation of a relational view towards the equal sign before beginning primary school. However, Sun et al. (2023) reported that only about half of Grade 3 tested Chinese students (n = 501) held the robust relational understanding of the equal sign. This percentage is even lower for Grade 1 students tested (approximately 36%, n = 497). Similarly, Yang et al. (2014) showed about 30% of tested Grade 3 students (n = 110) considered the equal sign as 'show results'. These results tended to demonstrate that while Chinese students may have a foundation of relational understanding of the equal sign at pre-school level, their emerging relational view can revert to operational view after they enter early grades in primary school. Sun et al. (2023) speculated the extensive arithmetic operation drill in early primary grades could be a major factor leading to this failure. On the other hand, as mentioned above, others (e.g., Jones et al., 2012; Li et al., 2008) showed by Grade 6, students in China generally possessed a more robust relational understanding of the equal sign. While these studies were conducted in the different contexts (e.g., schools, regions) in China, the high degree of uniformity of national curriculum, nationally approved textbooks, and the consistency of teaching approaches in Chinese schools suggest that these findings from different studies are likely to be well-grounded to portray a landscape of students' conception of the equal sign in China. That is, while students in China generally have developed a relational understanding of the equal sign by end of primary school, the process that they gain this understanding is not without obstacles.

In this sense, there is a space to further explore how Chinese students' development of conception of the equal sign, in particular, to understand how their conceptions are further supported in progressing to relational level since Grade 3. This study contributes to this research gap by focusing on students who were at the start of Grade 5. The main part of this study reports on the use of the Mathematical Equivalence Assessment (MEA) to investigate students relational understanding of the equal sign. In addition, to complement the MEA test results, exploratory interviews with the students' teachers were conducted to identify possible factors that may help explain students' developing conceptions of the equal sign.

Methodology

The research was conducted in a primary school in Changchun, Jilin Province, China. The context of the participating school and students is similar to those in Sun et al. (2023), in terms of similar region, SES background and academic rankings. A total of 237 Grade 5 (aged 11–12) students participated in this study, 110 boys and 127 girls. Students took a diagnostic test that examined their understanding of the equal sign (elaborated below). Their responses were coded against three categories of understanding of the equal sign suggested by Stephens et al. (2013). Afterward, interviews were conducted with six mathematics teachers to gather their insights on the factors enhancing or impeding students' relational conception of the equal sign (four were the teachers of participating students when they were in Grade 4, the other two were not, but they also taught Grade 4). The study was conducted in September, when students had just started Grade 5.

Instrument

The Mathematics Equivalence Assessment (MEA) instrument is a well-established tool designed to measure students' understanding of the equal sign, and it has been proved effective in cross-cultural contexts, including its application in China (Simsek et al., 2021).

Table 1

Example Test Items for Number Sentence Evaluation and Solving

Number sentence type	Elaboration and example
$a + b = c$	Students evaluate true or false for number sentences such as 31 + 12 = 43 Students fill the missing number to the form __ + 35 = 91
$c = a + b$	Students evaluate true or false for number sentences such as 25 = 16 + 9 Students fill the missing number to the form 52 = 13 + __
$a = a$	Students evaluate true or false for number sentences such as 41 = 41 Students fill the missing number for the form 23 = __
$a + b = c + d$	Students evaluate true or false for number sentences such as 41 + 23 = 31 + 33 Students fill the missing number for the form 53 + 31= __ + 21

MEA consists of three types of problems: (1) structure evaluation, students to determine whether number sentences such as 10 = 3 + 7, 7 = 7 and 3 + 7 = 4 + 6, are true or false; (2) structure solving, students to fill the missing number in a number sentence such as 12 + 23 = __ + 25; and (3) definition of the equal sign, students to explain the meaning of the equal sign. Sun et al. (2023) further modified the items in MEA to the context in China (e.g., modifying the numbers to better align with students' grade levels) and applied them to measure students' understanding of the equal sign in the first three grades. Considering the similarity in school contexts between this research and Sun et al. (2023), this study used the same test items as Sun et al. (2023). In the test, students evaluated true/false for number sentences given first. They then filled the missing numbers in the number sentences. For number sentence evaluation and solving items, students were required to explain how they arrived at the answers. There are four types of number sentences for evaluation and solving, as shown in Table 1, which are conventional and non-conventional forms of number sentences that are commonly applied to examine students' understanding of equal sign. Finally, students wrote the definition of the equal sign.

Coding Process

Students' responses were coded according to three levels of understanding of the equal sign in Stephens et al. (2013). The first level is 'operational': students perceive the equal sign as 'show answer' symbol displaying the results of calculation carried out from the left side. The second level is 'relational-computational': students recognise the equal sign as indicating an equivalence of both sides, but they use full calculations to demonstrate this equivalence. The third level is 'relational-structural': students can apply relationships among the quantities to show the equivalence of both sides, with a minimum calculation. Both the second and third levels are evidence that students possess the relational understanding of the equal sign (Stephens et al., 2013). In this study, for the definition of equal sign, if a student stated that the equal sign meant "adding numbers", "answers", "results" or "totals", this response was coded as 'operational'; if a student explained equal sign means equivalence of both sides with a specific calculation example, it was coded as 'relational-computational'; if a student expressed equal sign meant equivalent quantities of both sides in general, it was coded as 'relational-structural'. For number sentence solving, for instance, when solving "7 + 3 = __ + 4", if a student filled in 10 or 14 for the missing number, it was coded as 'operational'; if a student calculated 7 + 3 = 10 and then 10 − 4 = 6 for the missing number, it was coded as relational-computational; if a student recognised that 4 is 1 more than 3 and so the missing number should be 1 less than 7 so it is 6, it was coded as 'relational-structural'. The coding procedure for number sentence evaluation items was similar.

Interview

After the test was coded, six mathematics teachers were interviewed. The experiences of these teachers range from five to twenty years. All teachers had taught lower and middle primary levels. In the interviews, they shared their experiences in developing students' conceptions of the equal sign and provided their opinion about factors that supported and/or impeded the relational understanding of the equal sign. The interviews were open-ended and prompted by questions such as "*In your everyday teaching, would you emphasise the conception of the equal sign? If so, how?*", and "*What do you think helps or hinders students to understand the equal sign as a symbol indicating the equivalent relationship of both sides?*"

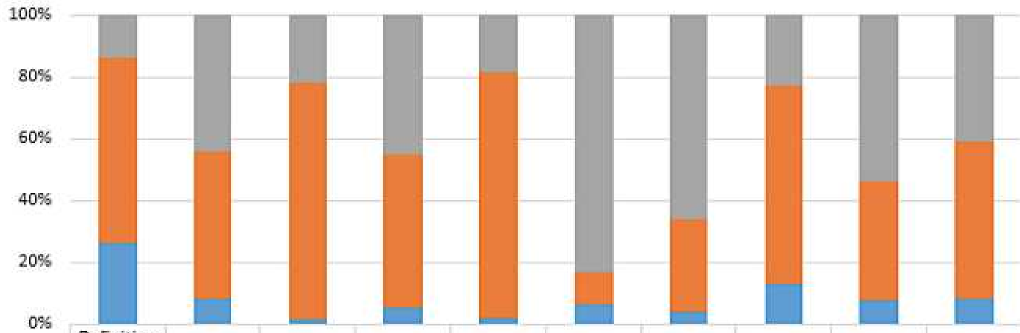
Results and Discussion

MEA Test

Figure 2 below shows the coded test results. There are ten result columns, with the first one for the definition of the equal sign. Subsequently, there are two columns for each type of number sentence (one for evaluation and one for solving). The final column shows the average results of the previous nine columns. For each column, the percentages of students' responses at three levels of understanding of the equal sign are categorised.

Figure 2

Distribution of Students' Responses to Test Items Against Level of Understanding (%)



	Definition of equal sign	"a+b=c" True/False	"a+b=c" Solving	"c=a+b" True/False	"c=a+b" Solving	"a=a" True/False	"a=a" Solving	"a+b=c+d" True/False	"a+b=c+d" Solving	average
relational-structural	13.5	44.3	21.9	45.1	18.6	83.1	66.2	22.8	53.8	41.0
relational-computational	60.3	47.3	76.4	49.4	79.3	10.5	30.0	64.4	38.2	50.6
operational	26.2	8.4	1.7	5.5	2.1	6.3	3.8	12.9	8.0	8.3

operational relational-computational relational-structural

Figure 2 indicates that by the start of Grade 5, the majority of tested students possessed relational understanding of the equal sign. Across all number sentence items, at least 87% of responses fall at either relational-computational or relational-structural levels. Furthermore, the results shows that a substantial number of students used a strategy of applying quantity relations to fill the missing number in number sentences, without full calculations. For instance, 53.8% of answers to solving the number sentence type $a + b = c + d$ are classified as relational-structural level. For example, many students explained their approach to answer 53 + 31 = <u>63</u> + 21 as "since 21 is 10 smaller than 31, to keep two sides the same, the missing number needs to be 10 greater than 53, so it must be 63". Students with this structural view towards a number sentence have departed from focusing on calculating the results of operations and have started attending to the general structure and relations among terms in a number sentence, which is a hallmark of the early algebraic thinking (Molina & Ambrose, 2008). In this sense, it could be argued that a significant portion of tested students demonstrated their emergence of early algebraic thinking. On the other hand, it is noted that compared to number sentence items, fewer responses to the definition item are classified as relational understanding levels (74% compared to 87%). This result appears to suggest some students, who can apply their relational understanding of the equal sign to solve the problems, described the meaning of the equal sign as being coded as 'operational'. The possible explanation is that when expressing the meaning of the equal sign, some students stated, "the equal sign is to connect between the answer and the calculation". According to the coding scheme, this kind of responses was categorised as 'operational'. However, the use of words like "connection" and "between" implied that these students may hold the relational understanding, so they can apply it in solving number sentence items. There was an ambiguity in their explanation of their conceptions. This finding tended to suggest that students who have developed certain mathematical thinking possibly have difficulties in articulating this thinking verbally or in written form due to a lack of appropriate mathematical vocabulary. This was echoed by Kranda (2008), who showed many students had difficulties in updating their mathematical language to adapt to the new situations.

While an earlier study (Sun et al., 2023) documented that approximately half of the participating Chinese Grade 3 students still possessed an operational understanding of the equal sign, in this study, the majority of tested students, who had recently completed Grade 4, demonstrated their conception of the equal sign reached the relational level. Given that participating schools in both studies had the similar context (e.g., region, SES, academic ranking), it seems that by the start of Grade 5, Chinese students' relational understanding of the equal sign has been enhanced, compared to those in Grade 3. As will been seen later, teachers' comments in the interview also endorsed this claim. However, this claim may need more careful longitudinal evaluation (see Conclusion and Limitation section).

Teacher Interviews

In interviews, six teachers shared their experiences in facilitating students' development of the relational understanding of the equal sign. These experiences enabled this study to have an explorative understanding of factors promoting students' relational understanding of the equal sign. All teachers commented they would not formally highlight the definition of the equal sign in everyday teaching. However, teachers recognised that some learning activities, introduced not specifically for fostering students' relational understanding of the equal sign, actually promoted it. For instance, teachers revealed that students were exposed to number sentences such as '2 + 4 = 5 + __' and '__ + 3 = 5 − 1' sometimes during classroom practice, homework or tests. For completing these tasks, students needed to notice the equivalence of both sides. Three teachers used the word "embedding" to describe this experience, saying "embedding the conception of the equal sign into daily teaching practice." These words are concurred with by Cai et al. (2013), who showed that in Chinese primary school, the emergence of early algebraic thinking was immersed in everyday teaching and learning on arithmetic.

Furthermore, this study considers that an interesting finding from the interview is that five out of six teachers mentioned the introduction of solving simple formal equations (e.g., $2x + 1 = 5$, $5 = x - 2$) in Grade 4 contributed to reinforcing students' relational conception of the equal sign, and the revealed how they believed learning equations enhanced relational understanding. First, using a balance model to demonstrate the equation solving is frequently mentioned by teachers. For instance, Teacher A commented:

> When introducing equations, we used the balance model. Students visualised the similarity between the abstract equations and the concrete balance. So, they easily comprehend the equal sign as indicating equivalent relationship of both sides, like the balance beam.

Similarly, Teacher C said, "the use of balance model can help them to visualise the meaning of the equation and understand equal sign as … just like a balance, indicating two sides have the same weight".

Teachers' responses tended to suggest that the resemblance between the concrete representation (balance beam) and formal equation visually supports a conception that the equal sign refers to the equivalence of both sides. This supporting process is further elaborated by some teachers. For instance, Teacher B mentioned when learning equations, students engaged in a hands-on play with the balance, they saw the beam tilting if one side was heavier, and they observed it balancing again when they adjusted the weights to make both sides weight equally. This dynamic process led students to attend to the simultaneous changes of both sides of an equation, so as to further reinforce the conception of the equal sign that is to represent the equivalence of two sides. This kind of teachers' comment appeared to reveal that that students' hands-on interactions with the balance served as means to promote students' relational understanding of the equal sign. Teacher B stated, "students had learnt the equal sign in the context of comparing quantities at earlier age, but this comparison was static. In contrast, the dynamic play of balance model could press the meaning of the equal sign more". These words tended to highlight that compared to simply comparing quantities on two sides, the interactive play with the balance model is richer, enabling students to grasp a clearer understanding.

Other than the balance model, teachers commonly commented that, when learning equations, compared to earlier grades, students are exposed to more non-conventional form of arithmetic expressions, helping them depart from the operational mindset. As Teacher D stated:

> Before learning equations, students experienced traditional arithmetic expressions for years, so they tend to have a mindset that the equal sign connects a calculated result. When learning equations,

> they see many operators and the unknowns can appear on either side of the equal sign, this breaks this mindset.

Likewise, Teacher C and E said:

> Students normally do lots of the arithmetic operation questions with traditional form of number sentences, like the number to be answered on the right side. When they are learning to solve equations, they start experiencing many operations with unknown number on either side. They need to understand equal sign as 'two sides are equivalent' to make sense to these equations. (Teacher C)

> Before learning equations, students predominately see and think about calculating results, but when they were learning equations, they were pushed to consider the equivalent relations of quantities more than calculating results. (Teacher E)

The teachers' responses demonstrated that they recognised that overcoming a focus on calculating results (left to right) is an important premise for developing the relational understanding of the equal sign. They consider when learning formal equations in Grade 4, students were exposed to many non-conventional arithmetic expressions with operators and unknowns in different positions, providing students with ample opportunities to be aware of the equivalence of quantities on both sides. This finding concurs with other researchers (e.g., McNeil et al., 2015) who proposed non-conventional form of arithmetic expressions are effective to support students' relational understanding of the equal sign. Furthermore, teachers' comments suggested that during equation learning, students were facilitated to depart from focusing on calculating results, and they started noticing the relationships between the quantities, this focus on relationship might explain why a substantial number of students had started applying a structural strategy to evaluate or solve number sentence problems.

Taken together, teacher interview tended to show that the introduction of the formal equation could: (1) allow students to visualise the connection between the formal equation and the balance model through the hands-on experiences; (2) expose students to many non-conventional arithmetic expressions with operators and unknowns in different positions, whereby it promoted students' relational conception of the equal sign. This finding echoes an early study (Cai et al., 2013), which put forth that learning formal equations supported students in attending to quantity relationships in number sentences in Chinese primary school. Notably, recently researchers in western countries also start paying attention to use equations with the balance model to support students' relational understanding of the equal sign. For instance, Stephens et al. (2022) showed that equation solving activity with the assistance of a pan balance can promote US Grade 2 students' relational understanding of the equal sign. Therefore, it could be understood that the literature on utilising equation learning activity to promote students' relational understanding of the equal sign is emerging and is worth further exploration.

Finally, Sun et al. (2023) showed that many Chinese students' conceptions of the equal sign possibly revert to an operational level during the first three grades, and they speculated that this reversion might be to the excessive arithmetic operation drill practice. Teachers' comments in this research confirmed this speculation. Many teachers reported that students need to drill an extensive amount arithmetic expressions that are in the traditional 'left to right' form in early grades, this can press a 'show result' conception of the equal sign to students. However, all teachers agreed that most students are able to view equal sign as indicating the equivalence of two sides after equation learning in Grade 4. This aligns with the claim made earlier: at the start of Grade 5, students' relational understanding of the equal sign has been strengthened.

Conclusion and Limitations

In this study, the MEA test result suggested that by the start of Grade 5, the majority of tested Chinese students demonstrated a relational understanding of the equal sign, based on understanding levels categorised by Stephens et al. (2013), with some of the students exhibiting structural thinking that is a hallmark of early algebraic thinking. Teacher interview revealed that students' exposure to simple formal equations with the balance model in Grade 4 can strengthen their relational understanding of the equal sign. When learning formal equations, the concrete representation (balance model) visually aided students to perceive equal sign as indicating the equivalence of both sides. Also, students encountered many non-conventional arithmetic expressions during equation learning, which pushed them to notice the equivalence relationship instead of focusing on calculating results. With these findings, this study contributes to the research gap in literature on Chinese students' understanding of the equal sign, exploratively explaining how students were supported to step towards a robust relational understanding of the equal sign since Grade 3.

The claim that the relational understanding of Grade 5 is enhanced compared to Grade 3 students is obtained by comparing the test results of Grade 5 students in this report with the results of Grade 3 students in a similar research context (Sun et al., 2023), and this claim was echoed by teacher interview. However, in the future, a longitudinal study that tracks the same cohort of students from Grade 3 to Grade 5 will be highly desirable to make a more rigorous conclusion. Also, as stressed early, the findings of this study are explorative, therefore, a further investigation is worth conducting to understand more about how exposure to simple formal equations can enhance students' relational conception of equal sign in details, possibly with some in-depth classroom research.

Acknowledgments

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References

Cai, J., Jiang, C., & Nie, B. (2013). Penetration, introduction and development of algebraic concepts in primary curriculum in China: Comparison of textbooks between China and the United States. *Curriculum, Teaching Material and Method*, *33*(6), 57–61.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Heinemann.

Jones, I., Inglis, M., Gilmore, C., & Dowens, M. (2012). Substitution and sameness: Two components of a relational conception of the equals sign. *Journal of Experimental Child Psychology*, *113*(1), 166–176.

Kranda, J. (2008). *Precise mathematical language: Exploring the relationship between student vocabulary understanding and student achievement* [Master's Thesis]. University of Nebraska-Lincoln.

Li, X., Ding, M., Capraro, M. M., & Capraro, R. M. (2008). Sources of differences in children's understandings of mathematical equality: Comparative analysis of teacher guides and student texts in China and the United States. *Cognition and Instruction*, *26*(2), 195–217.

McNeil, N. M., Fyfe, E. R., & Dunwiddie, A. E. (2015). Arithmetic practice can be modified to promote understanding of mathematical equivalence. *Journal of Educational Psychology*, *107*(2), 423.

Molina, M., & Ambrose, R. (2008). From an operational to a relational conception of the equal sign. Thirds graders' developing algebraic thinking. *Focus on Learning Problems in Mathematics*, *30*(1), 61–80.

Simsek, E., Jones, I., Hunter, J., & Xenidou-Dervou, I. (2021). Mathematical equivalence assessment: Measurement invariance across six countries. *Studies in Educational Evaluation*, *70*, 101046.

Stephens, A., Sung, Y., Strachota, S., Torres, R. V., Morton, K., Gardiner, A. M., Blanton, M., Knuth, E., & Stroud, R. (2022). The role of balance scales in supporting productive thinking about equations among diverse learners. *Mathematical Thinking and Learning*, *24*(1), 1–18.

Stephens, A. C., Knuth, E. J., Blanton, M. L., Isler, I., Gardiner, A. M., & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *The Journal of Mathematical Behavior*, *32*(2), 173–182.

Sun, J., & Gu, Y. (2023). A further investigation to introducing the equal sign in China. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 483–491). Newcastle, Australia.

Sun, X., Ma, Y., & Chen, N. (2023). Understanding and teaching of the concept of equal sign for low-grade students of primary school. *Journal of Mathematics Education*, *32*(2), 37–43.

Yang, X., Huo, Y., & Yan, Y. (2014). Primary school students' understanding of equation structure and the meaning of equal sign: A Chinese sample study. *Research in Mathematical Education*, *18*(4), 237–256.

Out-of-School PSLE Mathematics Practice Books in Singapore

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At the end of Primary 6, students in Singapore schools take a national examination, the Primary School Leaving Examination (PSLE). Society at large view the PSLE as a high-stakes examination. In addition to out-of-class work assigned by mathematics teachers for students to prepare for the PSLE mathematics, parents may also draw on out-of-school practice books to further support their children's preparation for the examination. A study of two main types of such books show that these books do generally support students in reviewing content knowledge for the examination and test preparation.

The Primary School Leaving Examination (PSLE) is a national examination in Singapore. It is conducted by the Singapore Examinations and Assessment Board (SEAB). It is taken by 12-year-olds at the end of their primary schooling. The examination tests student's proficiency in the English language, their respective mother tongue languages (typically Chinese, Malay or Tamil), mathematics and science. The purpose of PSLE is to emplace students in a course of study that is suited to their learning ability in a secondary school. Singaporeans place great emphasis on education and examinations are gatekeepers to educational opportunities throughout the Singaporean educational education system (Gregory & Clarke, 2003). As such the PSLE is viewed as a high-stakes examination by society at large (Loke, 2016).

Though out-of-school mathematics practice books, particularly those for the PSLE, have been available for the last three decades or more, there appears to be no apparent study on these books in Singapore. This study that is part of a larger study (Teo, 2023) attempts to explore the research question: How do the PSLE Mathematics practice books support test-taking preparation and test-taking skills for the PSLE Mathematics?

Examination Preparation

In addition to out-of-class work assigned by mathematics teachers for students to prepare for examinations, like the PSLE, parents may also draw on out-of-school practice books to further support their children's preparation for examinations. The PSLE examination for mathematics (PSLE Mathematics), is a paper-based examination, i.e., students respond individually to a set of questions in writing under supervision of examiners. To be successful, students require the skill or ability to do the mathematics within a given time frame. The development of these skills or abilities—influenced by the nature of the examination appears to be pertinent in examination preparation (Minott, 2020).

Examination preparation with a goal to enhance learning may take several forms, such as review sessions and or practice examinations (Bord, 2008). Review sessions allow students to clarify their questions about content, procedures, etc., that they will be tested on (Gurung, 2005). Practice examinations would involve students doing mock examination or past examination papers in a similar manner to that of the examination. Balch (1998) noted that practice examinations in contrast to review sessions led to positive gains in student performance in final examinations. In the study by Muchiri and Mawira (2020), it is evident that mathematics practice has positive impact with or without teacher's intervention as it gives "students an opportunity to attempt and familiarise themselves with questions and the way they are set in an exam" (p. 539).

Naujoks et al. (2022) noted that practice tests function as a (repeated) retrieval opportunity, which has the potential to empower future learning, recall and recognition. In addition, it is also

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 519–526). Gold Coast: MERGA.

suggested that practice test participation establishes potentials for learning processes like metacognitive judgement accuracy (Karpicke & Roediger, 2008). Students who do practice tests have an idea of how well they would do on an exam without any further practice or studying and this allows them to adjust their studying accordingly or focus on areas that may need more of their attention (Balch, 1998). A practice exam engages the student in self-assessment and metacognition. They can see their scores on the practice exam, evaluate their confidence with the tasks and assess what they know and what they need to study further or seek help with (Bord, 2008).

Students must be proficient in test-taking for the test to adequately serve as a measure of their mastery of a learning content (Dixon & Erinosho, 2020). It is possible to have knowledge of the content of a subject but perform poorly on a test due to poor test-taking strategies. The ability to utilise appropriate strategies to demonstrate competence and perform at optimal level on a test is referred to as test-taking skills. Such skills enable students to recognise what to do before-test, during-test, and after-test to achieve success (Bicak, 2013; Dodeen et al., 2014). As noted by Dixon and Erinosho (2020), "before-test skills include the study strategies prior to a test such as practicing past test / examination questions or self-quizzing; during-test skills are the strategies employed while taking a test which include time management, structure and organization, correctness of information, control of test anxiety and test-wiseness; whereas after-test skills include use of hints and feedback on test answers or scoring rubrics for managing future testing situation" (p. 4287). In their study that examined the interplay of test taking skills and performance in an open-ended mathematics test among secondary school students in Nigeria, Dixon and Erinosho (2020) found that test preparation comprising before, during and after test skills contributed significant positive variance to the prediction of students' performance on the tests.

PSLE Mathematics

The SEAB guide (SEAB, 2022) for the PSLE Mathematics outlines the format, content and assessment objectives of the examination.

The Format

PSLE Mathematics comprises two written papers, Paper 1 and Paper 2, that are taken at one sitting with an hour's break between them. Paper 1 is divided into two parts, Booklet A and Booklet B. Booklet A has fifteen multiple-choice questions while Booklet B has 15 short-answer questions. The use of calculators is not allowed for Paper 1 and students are expected to complete the paper within the one-hour allocated. Paper 2 consist of 5 short-answer questions and 12 structured questions. Calculators are allowed in Paper 2 and the duration for this paper is 1hour and 30mins. The allotment of marks for each paper is shown in Table 1.

The Content

The PSLE Mathematics covers three mathematics content strands, namely, i) Number and Algebra, ii) Measurement and Geometry, and iii) Statistics. The weightage of the strands and topics in the strands are shown in Table 2.

Assessment Objectives

The assessment objectives (AOs) for the PSLE Mathematics reflect the emphasis of the syllabuses and describe what students should know and be able to do with the concepts and skills learned in the syllabus. The AOs framed in three levels of increasing cognitive demand are: (i) AO1—recall mathematical facts, concepts, rules and formulae and perform straightforward computations, (ii) AO2—interpret information, understand and apply mathematical concepts and skills in a variety of contexts, and (iii) AO3—reason mathematically; analyse information and make inferences; select appropriate strategies to solve problems. The weightage of the AOs in the PSLE mathematics papers is shown in Table 3.

The Study

The study adopted a qualitative research design involving a single case study approach (Creswell, 2007). It attempted to examine a specific aspect of out-of-school support for students preparing for the PSLE Mathematics. It used a purposeful survey to collect information about out-of-school PSLE Mathematics practice books and again purposeful sampling to carry out content analysis (Patton, 1990) of a sample of the practice books.

Object of Study

A survey of out-of-school PSLE mathematics practice books, available at the largest school bookstore in Singapore, was undertaken in December 2022. December is the month of the year when parents of students going on to Primary six in the following year are most active sourcing for resources to support their children's preparation for the PSLE Mathematics. During this period most practice books are available. We limited the scope of our survey. It excluded books that were wrapped and could not be examined by us, books with solution parts sold separately and any other book not specifically focused on PSLE Mathematics practice.

Data and Analysis

A total of 76 practice books were surveyed. Ten of the books had PSLE mathematics mock exams for timed practice only. Forty-nine of them comprised topical revision exercises and PSLE mathematics mock exams. Seventeen focused on problem solving strategies and solving of challenging problems. All the books had stepwise solutions provided. Thirty-nine had worked examples as part of the topical exercises and three had detailed marked scheme provided including examiners comments.

To answer the research question that guided the study reported in this paper, we limited our purposeful selection of the PSLE Mathematics practice books to those with mock exams only and those with topical revision exercises and mock exams. The rationale for this was that books in these two categories were most suited for the general preparation for the PSLE Mathematics. The books selected for study were:

- Lau & Yang (2018). Countdown to PSLE Maths Primary 6. Marshall Cavendish;
- Lee (2002). Mathematics weekly revision for Primary 6. Educational Publishing House Pte Ltd.

The first book: Countdown to PSLE Maths Primary 6, based on the 2018 PSLE examination guidelines and 2013 revised mathematics syllabus, comprises mock PSLE Mathematics exams. It has detailed marking schemes for self-assessment by learners. The second book, Mathematics weekly revision for Primary 6, was first published in 2011.The third edition (present one), published in 2018, is based on the 2018 PSLE examination guidelines and the 2013 revised mathematics syllabus. It comprises both topical revision exercises and mock PSLE exams.

To examine how the mathematics practice tasks in the mock PSLE Mathematics exams compared with the SEAB guide for PSLE Mathematics we analysed the first and last mock PSLE exams in the book Countdown to PSLE Maths Primary 6. There were 8 mock exams in the book and we purposively selected the first and last of these. As in all practices, easy questions precede difficult ones so as to allow the confidence of test takers to develop. In a similar manner we anticipated that the first and last mock exams would be the easiest and toughest respectively and allow us to see it's comparability with the SEAB guide for PSLE Mathematics.

The items in the mock exams were coded for type, content, and assessment objective. The papers were coded independently by the first author and a Primary 6 mathematics teacher for type, content, and assessment objectives. There was 100% agreement for the type and content of all the test items. However, this was not the case for the assessment objectives, where the agreement was only 80%. For items where there were disagreements both the coders reviewed the codes during a meeting and mutually agreed on a code.

To study how the books facilitated test-taking preparation and test-taking skills we examined the organization and contents of the book: Mathematics weekly revision for Primary 6 (Lee, 2002).

Findings

How did the Mock PSLE Exams Compare With the SEAB Guide?

Table 1 shows that for Paper 1 the spread of the item types for the mock exams1 and 8 was similar. The number of items per item type and mark allocated was exactly as in the SEAB guide. However, for Paper 2 though the overall numbers per item type were as the SEAB guide, there was variation for the structured answer and long answer items. In mock exam 1, there were more long answer items compared to the structured answer items, while in mock exam 8, there were more structured answer items than the long answer items.

Table 1

Type and Mark of Mathematics Test Items in the Mock PSLE Exams 1 and 8

Paper	Type of test item	SEAB guide		Mock PSLE exams			
				Exam 1		Exam 8	
		No of items	Mark per question	No of items	Mark per question	No of items	Mark per question
1 (Booklet A)	MCQ	10	1	10	1	10	1
		5	2	5	2	5	2
1 (Booklet B)	Short answer	5	1	5	1	5	1
		10	2	10	2	10	2
2	Short answer	5	2	5	2	5	2
	Structured answer	12	3, 4 or 5	5	4, 5	8	3, 4, 5
	Long answer			7	3, 5	4	3
	Total	47	100	47	100	47	100

Table 2 shows that the spread of the strand and topics in the mock exams 1 and 8 were within a margin of ± 5% and consistent with the SEAB guidelines.

Table 3 shows that the spread of the assessment objectives for the mock exams 1 and 8 was skewed towards assessment objectives 2 and 3. The spread of assessment objectives in both mock exams was not consistent with the SEAB guidelines. Compared to mock exam 1, mock exam 8 was more challenging with 39% of the marks being allocated for test items with assessment objective 3, instead of 26%.

Table 2

Weightage of Topics in the SEAB Guide and Mock PSLE Exams 1 and 8

		SEAB guide		Mock PSLE exams			
				Exam 1		Exam 8	
Strand	**Topics**	**%**	**Total %**	**%**	**Total %**	**%**	**Total %**
Number and algebra	Whole numbers, fractions, decimals	25	45	27	46	32	49
	Ratio, percentage	10		9		10	
	Rate, speed	5		5		5	
	Algebra	5		5		2	
Measurement and geometry	Measurement	20	40	20	40	21	41
	Geometry	20		20		20	
Statistics	Statistics	15	15	14	14	10	10
Total		100	100	100	100	100	100

Table 3

Weightage of Assessment Objectives in the SEAB Guide and Mock PSLE Exams 1 and 8

Assessment objective	**SEAB guide**	**Mock PSLE exams**	
		Exam	**Exam 8**
	%	%	%
1	25	13	11
2	40	61	50
3	35	26	39
Total	100	100	100

A total of six items in the mock exams were inappropriate for PSLE mathematics. The first item required the students to draw on the concept of congruent triangles and find the area of a figure. The concept of congruent triangles is part of the secondary school mathematics curriculum and hence this item was beyond the mathematical knowledge of the students. The assessment objectives of the second, third and fourth items did not match the marks allocated for them. The fifth item tested a skill, drawing bar graphs, that was no longer in the PSLE mathematics syllabuses. The last and sixth item was a poorly constructed test item.

How did the Mock PSLE Exams Provide Support for Self-Assessment?

The mock PSLE exams provided students with support for self-assessment. For every practice the answer key for the multiple-choice questions, stepwise solutions for short answer questions and stepwise solutions together with mark schemes for questions 6–17 of paper 2 were provided. At times additional support in terms of explanations specific to a suggested strategy was also provided.

How was the Topical Revision PSLE Mathematics Practice Book Structured?

In tandem with the school calendar year, the topical revision book was organised into four terms. Each term was again organised into weeks. Terms 1, 2, and 3 had ten weeks each while term 4 had only 6 weeks.

How did the Content of the Topical Revision PSLE Mathematics Practice Book Support Test Preparation and Test-Taking Skills?

Table 4 shows the content of weekly practices for term 1. The weekly practices review essential concepts and skills tested during the PSLE Mathematics. In weeks 1, 2, and 3 the revision was focused on two or three topics each week. In week 4, the revision included all the topics worked through in weeks 1, 2, and 3. This provides students with an opportunity to revisit topics they did in the past weeks. This pattern of revising topics carries on for the first nine weeks. For all the nine weeks, the practices comprise of ten short answer questions and 4 structured / long answer questions. There are no multiple-choice questions. Also, no marks are allocated for any of the questions.

In week 10, students do a mock exam (Practice 10) that is modelled like the PSLE Mathematics comprising two papers. Paper 1 has two booklets, with booklet A comprising 15 multiple choice questions and booklet B comprising 15 short answer questions. Paper 2 has 17 questions, the first 5 are short answer questions and the next 12 are structured or long answer questions. The marks allocated for all the questions mirror PSLE Mathematics questions.

For term 2 (weeks 1–10) and term 3 (weeks 1–10) the practices are similar in structure with that of term 1. In term 4, there are only 6 weeks of practice. In week 1, there is revision comprising 10 short answer questions and 4 structured / long answer questions on topics that preceded term 4. The following five weeks provide students with opportunities to hone test-taking skills through timed mock PSLE mathematics exams.

Table 4

Content of Topical Revision PSLE Mathematics Practice Book for Term 1

Week	Content	Focus	Item types	Time
1 & 2	Whole numbers, Algebraic expressions	Revision of content	Section A–10 SAQ Section B–4 Str A / LA Q	No time limit
3	Solid figures, Nets, Volume of cuboids	Revision of content	Section A–10 SAQ Section B–4 Str A / LA Q	No time limit
4	*Revision* Whole numbers, Algebraic expressions, Solid figures, Nets, Volume of cuboids	Revision of content	Section A–10 SAQ Section B–4 Str A / LA Q	No time limit
5, 6 & 7	Fractions, Ratio	Revision of content	Section A–10 SAQ Section B–4 Str A / LA Q	No time limit
8	Whole numbers, Fractions, Ratio	Revision of content	Section A–10 SAQ Section B–4 Str A / LA Q	No time limit
9	*Revision* Whole numbers, Algebraic expressions, Solid figures, Nets, Volume of cuboids, Fractions, Ratio	Revision of content	Section A–10 SAQ Section B–4 Str A / LA Q	No time limit
10	*Term 1 Revision* Whole numbers, Algebraic expressions, Solid figures, Nets, Volume of cuboids, Fractions, Ratio	Test taking preparation and test-taking skills	Paper 1–Booklet A (15 MCQ) Booklet B (15 SAQ) Paper 2–5 SAQ, 12 Str / LA Q	Paper 1–1 hour Paper 2–1 hour 30 minutes

How did the Topical Revision PSLE Mathematics Practice Book Provide Support for Self-Assessment?

The support for self-assessment provided in the book was in the form of stepwise detailed solutions for both the topical practices and the mock PSLE mathematics practices. For the mock PSLE mathematics exams marks were also indicated at milestone steps.

Discussion and Conclusion

Past research suggests that learning may be enhanced through practice for examination preparation (Bord, 2008). Therefore, the two types of PSLE mathematics practice books examined as part of the study reported in this paper are critical in providing practice for the PSLE Mathematics. However, of concern is how closely do the practices in such books match the PSLE mathematics as outlined in the SEAB guide.

We found the mock exams 1 and 8 papers less adherent to the SEAB guide and past PSLE Mathematics papers. The test item types were according to the SEAB guide and of interest were the number of long answer items compared with the structured answer items. The SEAB guide just states that the total number of both type of items should be 12. In the mock exam 1, there were 5 structured answer and 7 long answer items and in the mock exam 8, there were 8 structured answer and 4 long answer items. The distribution of structured answer and long answer items in the mock PSLE Mathematics practice papers were not representative of the trend in the 2020 and 2021 PSLE Mathematics (see Teo, 2023), which is at most 3 long questions in paper 2.

The spread of the strand and topics in the mock exams 1 and 8 were generally consistent with the SEAB guide. The spread of the assessment objectives for both exams was not consistent with the SEAB guide. It is apparent from the spread that the items in the mock exams were more challenging than stipulated in the SEAB guide. In addition, there were several items in the mock exams that were inappropriate as they: were testing concepts in the secondary school mathematics curriculum; had a mismatch between assessment objective and mark allocated; were poorly constructed; had test objectives that were no longer in the PSLE Mathematics syllabuses. Otherwise, the mock exams were framed as per the PSLE Mathematics papers with time limits of papers just like the PSLE Mathematics.

Bearing in mind that the intent of such practices is to provide students with practise for a high-stakes examination like the PSLE, these books need to be representative of the examination in all aspects. Such practices do provide students with test-taking experiences that facilitate proficiency (Dixon & Erinosho, 2020).

Other than doing mock PSLE Mathematics exams, students preparing for the PSLE Mathematics may also need to engage with “before-test skills” which would include review of past knowledge (Bicak, 2013; Dodeen et al., 2014). In the book examined as part of the study reported here, i.e., Mathematics weekly revision for Primary 6, preparation for the PSLE mathematics was spread over the academic year and detailed by terms. There was emphasis on both mastery of content and test-taking skills. Mastery of content was through bite-sized exercises and development of test-taking skills was through periodic mock PSLE Mathematics exams that gradually extended the content coverage sequentially.

In both types of practice book the support provided by the books through worked solutions for the test items allowed the students to develop ‘after-test’ skills (Dixon & Erinosho, 2020). This involved using the stepwise solutions and mark schemes to engage in self-assessment and metacognitive activity to identify gaps in knowledge and remediate before doing more practice for the PSLE Mathematics. In conclusion, it is recommended that more studies be carried out on how students use such books for examination preparation. Teachers’ perspectives of such books would also be helpful for parents who may need resources to support their children's PSLE Mathematics preparation.

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References

Balch, W.R. (1998). Practice versus review exams and final exam performance. *Teaching of Psychology*, *25*(3), 181–185.

Bicak, B. (2013). Scale for test preparation and test taking strategies. *Educational Sciences: Theory and Practice*, *13*(1), 279–289.

Bord, D. (2008). Enhancing learning and exam preparation. *Association for Psychological Science: Observer*, *21*(1), 1–5. https://www.psychologicalscience.org/observer/enhancing-learning-and-exam-preparation

Creswell, J. (2007). *Qualitative inquiry and research design: Choosing among five approaches* (2nd ed.). Thousand Oaks, CA: Sage.

Dixon, O. O. & Erinosho, S.Y. (2020). How do test type and test-taking skills affect mathematics performance among high school students: A Nigerian example. *International Journal for Cross-Disciplinary Subjects in Education (IJCDSE)*, *11*(2), 4286–4293.

Dodeen, H. M., Abdelfattah, F., & Alshumrani, S. (2014). Test-taking skills of secondary students: The relationship with motivation, attitudes, anxiety and attitudes towards tests. *South African Journal of Education*, *34*(2), 1–18.

Gregory, K. & Clarke, M. (2003). High stakes assessments in England and Singapore. *Theory Into Practice*, *42*(1), 66–74. https: //doi.org/10.1207/s15430421tip4201_9

Gurung, R. A. R. (2005). How do students really study (does it matter)? *Teaching of Psychology*, *32*, 39–41.

Karpicke, J. D., & Roediger, H. L. (2008). The critical importance of retrieval for learning. *Science*, *319*(5865), 966–968. https://doi.org/10.1126/science.1152408

Lau, B. P. W. & Yang, M. (2018). *Countdown to PSLE maths primary 6*. Marshall Cavendish.

Lee, H. (2002). *Mathematics weekly revision for Primary 6*. Educational Publishing House Pte Ltd.

Loke, H. Y. (2016). *High stakes exams and tuition in Singapore, Asia: Still stressful*? ASEAN Today. https://Aseantoday.com/2016/06/exams-tuition-singapore-asia

Minott, M. (2020). Exam preparation: Concerns of London upper secondary school students with implications for teachers and teaching. *Journal of Educational Research and Practice*, *10*(1), 338–350. https://doi.org/ 10. 10.5590/JERAP.2020.10.1.21.

Muchiri, D. K. & Mawira, P.Z. (2020). Investigating the effectiveness of a math practice paper in teaching and learning of mathematics. *International Journal of Scientific and Research Publications* (IJSRP), *10*(7), 537–542. https://dx.doi.org/10.29322/IJSRP.10.07.2020.p10357.

Naujoks, N., Harder, B. & Handel, M. (2022). Testing pays off twice: Potentials of practice tests and feedback regarding exam performance and judgement accuracy. *Metacognition and Learning*, *17*, 479–498. https"//doi.org/10.1007/s11409-022-09295-x

Patton, M. (1990). *Qualitative evaluation and research methods*. Beverly Hills, CA: Sage.

Roediger, H.L. & Butler, A.C. (2011). The critical role of retrieval practice in long-term retention. *Trends in Cognitive Sciences*, *15*(1), 20–27. https://doi.org/10.1016.j.tics.2010.09.003

Singapore Examinations and Assessment Board (SEAB). (2022). *PSLE mathematics*. https://www.seab.gov.sg/docs/default-source/national-examinations/syllabus/psle/2022_psle_subject_info/0008_y22_sy.pdf

Teo, P. P. (2023). *A study of out-of-school mathematics practice books for the PSLE in Singapore*. [Unpublished Master's thesis]. Nanyang Technological University, Singapore.

How Children who Speak Marathi Respond to the Introduction of Uncertain Language in a Statistical Investigation

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This study was conducted in Maharashtra, India with children studying in a regional medium (Marathi) government school. In Marathi, the translation of the word ‘about’ is not very commonly used. The aim of the study was to see how the children used uncertain language about prediction while engaged in a statistical investigation and how children would respond to the uncertain language introduced by the researcher. The findings suggest that children did not use the equivalent word for ‘about’ without prompting from the researcher. The study has the potential of exploring and impacting the influences of language on the learning of statistics in a non-Western culture.

Being able to use uncertain language has always been an important aspect of informal statistical inference (Makar & Rubin, 2009). This includes both acknowledgement of variation in the data and to express uncertainty regarding future events. However, language is a social phenomenon and social interactions play an important role in the development of language, particularly social interactions with a significant adult (Vygotsky, 2012). It is reasonable to assume that children negotiate the usage of syntax and semantics of uncertain language through various interactions in contexts where they have to reason about uncertainty.

This paper explores the interactions between a researcher and children a school in Maharashtra, India while the children were engaged in a statistical investigation. The medium of instruction was Marathi, which is the same language that the children spoke at home, although children’s home dialect was slightly different than the standard dialect used in school textbooks. The children belonged to a socio-economically disadvantaged section of the society. In this study, we explore the challenges faced by the researcher and the children in arriving at a shared understanding of the uncertain language used by the researcher.

Literature Review

In this section, we look at language difficulties in statistics that may influence how non-English learners articulate their understanding of likelihood. We also consider how language encodes culture, power and equitable access to learning.

Articulating Uncertainty

Expressions of uncertainty are often vague and subjective in nature. They can carry challenges conceptually in aligning the intended meaning with the context in which probabilistic phrases are used, as well as navigating the variety of ways to express likelihood and uncertainty due to variability and estimations in future predictions (Karelitz & Budescu, 2004). “The situation in which random variation is met influences how people think about probability” (Pratt & Kazak, 2018, p. 214). Researchers noticed that students often articulated predictions under uncertainty by either being overly deterministic or overly relativistic (Ben-Zvi et al., 2012; Rubin et al., 1990). In making predictions from data, the primary students in Ben-Zvi et al.’s (2012) study oscillated between certainty-only (deterministic) and uncertainty-only (relativistic) statements before further exploration assisted them to express uncertainty with emerging probabilistic language (about, maybe). Therefore, attending to expressions of uncertainty requires attention to understanding of uncertainty, how it is expressed and importantly, the relationship with the context.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 527–534). Gold Coast: MERGA.

When considering how non-English language learners articulate uncertainty, challenges may be related to statistical concepts, distinctive statistical language, both or neither. It can be difficult to tease out where the challenges lie. For example, one challenge of exploring statistical ideas with uncertain language in Marathi (the language in this study) is that the use of some words in English do not translate well. For example, there is no commonly used translation of the word ‘about’ (such as in ‘I will sleep for about 7 hours every day’). It is on this issue of linguistic diversity that the researchers conducted a statistical investigation with children.

In research with non-English speaking students, additional constraints arise as how one expresses statistical ideas is also related to the specifics of the language or dialect. Lesser and Winsor (2009) worked with Spanish-speaking pre-service teachers in an English-speaking environment. They argued that with second language learners, not enough emphasis is placed on statistical *register*, a subset of language relevant to a particular context and purpose. Learners’ proficiency in everyday language may be quite different from de-contextualised academic situations, which can lag behind by three or four years. Lesser and Winsor’s research found that even if learners are exposed to an academic term, they may be more likely to draw on their everyday register than to adopt the academic term. Their research recommended attending to how a student’s language articulates an idea and to incorporate their language, community, and culture into learning activities rather than see these as limitations. Other recommendations included using wait time, embedding instruction across several contexts, seeking multiple ways to express and represent a new idea, and encouraging collaboration. In related work, Lesser and his colleagues (2013) focused on different aspects of statistical register (field, mode, tenor) to identify how students in different language groups understood statistical concepts that were articulated in each language. Unfortunately, all of these studies took place in English-language contexts, even when the students were non-English speakers. This creates a major obstacle when seeking to locate and build on research in non-English speaking contexts.

Beyond Language as Articulation: Culture, Power, and Equitable Access

Cultural dimensions of language can contribute significantly to learning statistics. For example, Chauhan’s (2013) research showed that children with fatalist beliefs may not see the purpose of estimating likelihood as “ultimately everything depends on luck” (p. 153). Language evokes tacit assumptions about the speaker, particularly their command of the dominant language (Benson, 2013). In education, if a learner is not fluent in the dominant language of instruction in school, the “monolingual habitus causes us to view a learner in deficit” (p. 284). The difference in status between the language of society and the language of school can exacerbate this issue, particularly if content is taught and assessed in the dominant language.

Kulkarni (1981, p. 55), while talking about the barriers that children face learning scientific concepts, especially children from lower socio-economic communities, stated that “More relevant is the role of language in classroom instruction at the school level and the comparison of language in and out of school”. He makes this point not when the medium of instruction in school was an entirely different language, but when the kind of language (formality, syntax etc) used in school was different than what the child used at home. They noted that in changing the language of textbooks to better suit the home language, children’s scholastic performance improved significantly; a significant finding was the removal of disparity in the performance of students coming from different social economic backgrounds.

Encouraging children to use uncertain language in statistics classrooms, especially in socioeconomically and linguistically diverse communities is a challenging process. Teachers are faced with the dual challenge of encouraging the natural expression of children along with introducing the children to the language used by dominant groups, who hold more power. For example, Delpit (1988) argued that we should accept children’s home expression and also teach them standard (academic) language in order not to be disadvantaged. This issue is especially relevant as learning is thought to be a largely social process where interactions with a significant adult has a large impact on learning (Vygotsky, 2012). It stands to reason that children's ideas about statistics concepts are influenced by the language used by their teachers in the classroom and might be highly relevant when the children speak a different dialect than the standard one spoken by the teachers or expected in textbooks. This was a key issue that the researchers sought to address when it comes to natural expressions of uncertainty.

Methodology

The activity was conducted with four students in Grades 5 and 6 (ages 9–11). It took approximately 2.5 hours to complete. The researcher told the children that there was to be a competition in the district where multiple schools would participate. The competition would involve throwing a ball into a bucket and whoever got the ball in the greatest number of times would win the competition. The researcher told the students that they had to recommend one student among them who would represent their school in the competition. The children spent the first part of the activity discussing what they would have to do in order to recommend someone to the competition. With a nudge from the researcher, they decided that they would collect data several times and then make a decision. The children played 10 rounds each with 10 chances per child and collected data for the number of times out of 10, the ball actually landed in the bucket. Based on these data the children had a discussion facilitated by the researcher in order to decide who to send to the competition.

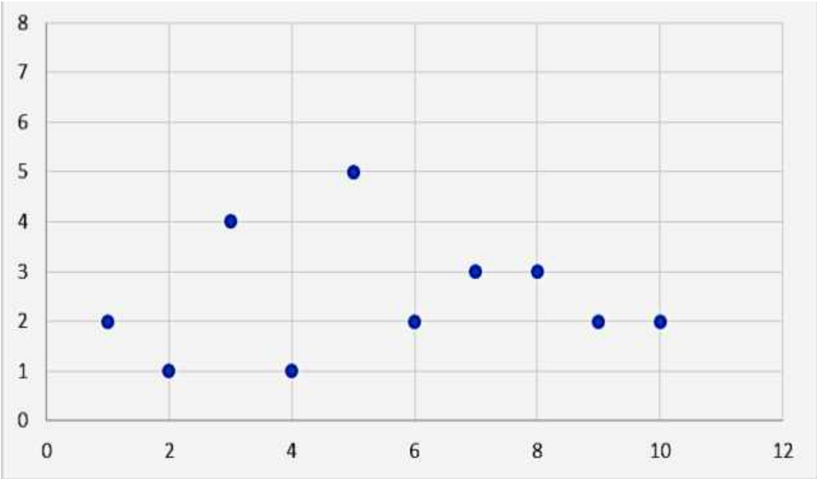
The data collection consisted of audio and video of the activities, artefacts that were created (e.g., tallies or graphs) and field notes. The analysis was based on an adaption of Powell et al.'s (2003) analysis of video data. The focus of the analysis was to identify excerpts in the data in which the language of uncertainty arose or did not arise (but was expected). Initially, the audio and video data were reviewed and briefly summarised with timestamps for content. Potential critical events were highlighted during this process and then transcribed. A critical event was identified as an interaction in which a language feature raised a challenge or insight into the children's articulation of uncertainty. The pieces of transcript were coded to identify concepts that arose. These excerpts were discussed to locate a storyline—an articulation of key ideas told through a coherent linking of the transcripts. The transcripts were reviewed again to edit them for brevity of expression, adding further insights of analysis to create a narrative, before final editing of the results section presented below.

Results

The following is an interaction between the researchers and students after they had drawn a time series graph for Nikhil (Figure 1). The researcher has asked, "About how many balls does Nikhil get in?" While asking the question, the researcher has used the Marathi rough equivalent for the word 'about' which is '*sadharan*' (साधारण). In the researcher's experience, the word sadharan is commonly understood by adults who speak the standard dialect/variant of Marathi.

Figure 1

Plot for Nikhil: The x Axis Shows the Number of the Trial While the y Axis Shows the Number of Balls That Went in the Bucket in That Particular Trial



Researcher:	About (sadharan) how many balls does Nikhil throw? He has ten chances right?	साधारण किती टाकतो. त्याला १० chances दिलेत ना?
Vandana:	25	25
Researcher:	He has 10 chances right? About (sadharan) how many balls will he get in?	त्याला १० chances दिले ना? तो दहापैकी किती टाकेल?
Vandana:	He got 2 in the first round, 3 in the second round, 4 in the third round, 1 in the fourth round, 5 in the fifth round…	पहिल्या फेरी २ मिळाले, दुसऱ्या फेरीत ३. तिसऱ्यात ४, चौथ्या फेरीत १, पाचव्या फेरीत ५...
Priya:	So he got 25 overall	सगळे मिळून २५ झाले

Because questions in textbooks generally ask students to count frequency, or add up values, it seemed likely that children were trying to apply the same technique in this context. It was apparent that the children were not used to hearing the word sadharan, likely because it was not a part of daily use in their dialect. Considering that children did not adopt the word sadharan, the researcher tried to use another word as a substitute for sadharan which is the word '*andaje*' 'अंदाजे'. The word sadharan is almost an exact translation of the word 'about' for describing the data, as well as for expressing uncertainty about events (including future events). Andaje is a broader word better suited to expressing uncertainty about events rather than data, sometimes used as a combination of guess and estimate. For example, the predictive statement, "It will take me about (andaje) 2 hours to reach my destination", but not the descriptive statement "An average child scored about (andaje) 30 marks on the test".

Researcher:	My question is a little different. Out of ten, about (andaje) how many balls will Nikhil get in?	माझा प्रश्न थोडासा वेगळा तो दहापैकी अंदाजे किती बॉल टाकेल.
Nikhil:	4	4
Priya:	5	5
Researcher:	Why do you think 5?	तुला का असं वाटतंय ५ टाकेल?
Priya:	His aim is very good	त्याचा नेम चांगला आहे

The children used their experience seeing Nikhil throw the ball and were optimistic in predicting how many Nikhil would throw next time. Since the children used neither word as a translation of the word 'about' as the researcher expected, the researcher decided to explicitly introduce the word sadharan and model how it is used in statistics. The following excerpt shows how the researcher led the discussion to create good conditions for introducing the word.

Researcher:	Let's have a bet on how many balls Nikhil will get in. What number will you bet on?	आपण पैज लावूया हं. निखिल किती बॉल टाकेल याची. तर कुठल्या नंबर वर पैज लावायची?
Vandana:	5	5
Priya:	3 (Immediately changed her answer from 5 to 3)	3
Researcher:	Why do you think so?	तुला असं का वाटत आहे.
Priya:	His aim is not that good	कारण की जास्त नेम नाही लागलेला

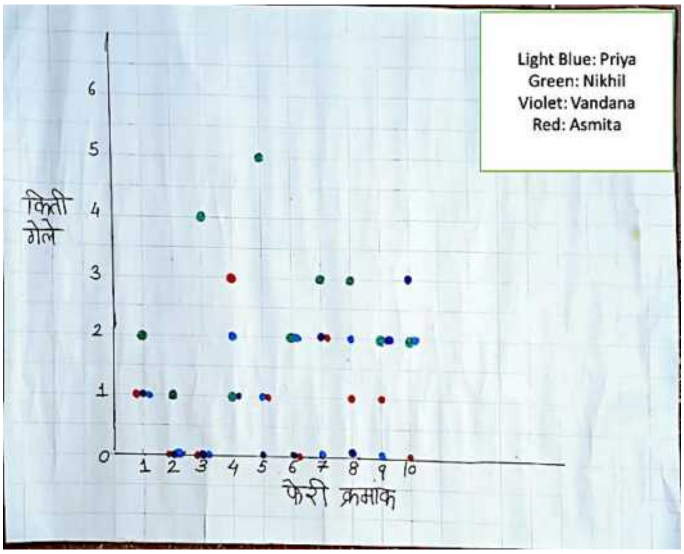
This time, rather than use their personal beliefs about how many Nikhil would throw, the children gravitated towards using the central tendency of the data when the researcher phrased the question about making a bet (preduction). The researcher and the children had a discussion on which number to bet on, if Nikhil gets 10 chances to throw the ball in the bucket. Then the researcher introduced the word sadharan as they were curious to see how the children would react to the introduction of the word (which is reasonably common in Marathi but apparently not used by the children). The following conversation is from the later half of the activity.

Researcher:	Suppose I bet that Nikhil throws 7. Will I win or lose?	समजा अशी मी पैज लावली की निखिल सात टाकेल. तर मी जिंकी का हरीन?
Chorus:	Lose	हरेन
Researcher:	Suppose I bet that Nikhil scores a two. Then will I win or lose?	समजा मी अशी पैज लावली की निखिल दोन टाकेल. तर मी जिंकीन का हरिन
Chorus:	Win	जिंकेन
Researcher:	Can I say Nikhil throws about (sadharan) 6 balls in?	तर निखिल साधारण दहा बॉल टाकतो असा पण म्हणू शकतो का?
Chorus:	No	नाही
Researcher:	Why	का
Asmita:	Because he has not thrown 6 even once.	कारण त्यांनी एक पण सहा टाकलेला नाही आहे
Researcher:	About (sadharan) how many balls does Nikhil throw?	हा तर निखिल साधारण किती टाकतो?
Priya:	2 or 3	दोन नाहीतर तीन
Researcher:	Yes, we can say that Nikhil throws about (sadharan) 2 or 3 balls. Meaning, he throws 2 or 3 balls, give or take. (Around 2 or 3 balls)	निखिल साधारण दोन किंवा तीन टाकतो असा आपण म्हणू शकतो. म्हणजे काय की दोन तीन च्या आसपास टाकतो

The researcher explicitly introduced uncertain language *(*sadharan*)* to the children in order to see how they would respond to hearing such language and whether they would be able to use the language once the researcher modelled its use in the given context. The researcher then drew the time series plot for the data of the second child, Priya (Figure 2).

Figure 2

Graph of the Number of Successes (y Axis) in Each Round (x Axis) for All the Children (Colour)



The following is a conversation that takes place after the researcher has drawn the time series graph for Priya.

Researcher	About (sadharan) how many balls does Priya throw	प्रिया साधारण किती टाकते बॉल
Asmita	2	2
Vandana	2	2
Nikhil	1 to 2	एक ते दोन
Sadhana	Even I think 1 to 2	मला एक ते दोनच वाटतय
Vandana	I think 2	मला दोन वाटतं
Researcher	Now don't say about 1 to 2. Which is better. About (sadharan) 0, about (sadharan) 1, about (sadharan) 2, about (sadharan) 3	आता एक ते दोन असं नाही द्यायचा उत्तर. साधारण शून्य साधारण एक साधारण दोन किंवा साधारण तीन
Vandana	(Practising the phrasing) About 2	साधारण 2
Asmita	About 0	साधारण शून्य

The researcher discouraged the use of "1 to 2" and asked that children use "about *(*sadharan*)* 0" or "about *(*sadharan*)* 1" in order to see if the children would adopt the implied uncertainty in the word "about *(*sadharan*)*". But it would appear that the children likely used "about" as a label; that is, they just attached the word "about" *(*sadharan*)* before any number or they used it to indicate some value in the range. From this it would seem that the children have not yet articulated a sense of the uncertainty implied by the word "about" *(*sadharan*)*.

Researcher:	Let's see how to use the word sadharan (about). Its about give or take or around (using scale, plot and gestures). If I say about (sadharan) 2, are her dots around 2. They are down (gesture) but are they up (gesture) With scale	बघा हा आपण साधारण शब्द कसा वापरणार होतो. की आसपास gestures. मी साधारण दोन असं म्हणलं तर दोनच्या आसपास आहेत का तिचे. इकडे आहेत (gesture down) पण इकडे आहेत का.

Chorus:	No	नाही
Researcher:	Then I can’t say about 2. Can I say about 0?	मग मी साधारण दोन नाही म्हणू शकणार. साधारण शून्य असं म्हणू शकीन का?
Nikhil:	No	नाही
Asmita:	We can say about 1	साधारण एक असं म्हणू शकतो
Researcher:	(Looking at Nikhil) Why?	(निखिल कडे बघून) का बरं
Nikhil:	Not many are near 0	कारण शून्याच्या जवळ नाही आहेत
Researcher:	Most of them are above 0, right. Then what can we say?	हा बाकीचे सगळे शून्याच्या वरचे आहेत. मग मी काय म्हणू शकते?
Asmita:	About 1	साधारण एक

This exchange suggests that the children were developing the concept of uncertain language, but not yet adopting the terminology, even when the researcher introduced the word in multiple ways. The following conversation followed after the researcher introduced the word using the graph. This is where the children seemed to be playing with the word to understand how it was being used in the context introduced by the researcher. They are using other language also which is going more towards uncertainty. They are looking more specifically at the data.

Researcher:	Then how much does Priya throw?	मग प्रिया किती टाकते?
Chorus:	About 1	साधारण एक
Researcher:	And Vandana?	आणि वंदना किती टाकत होती?
Nikhil:	1-2	एक ते दोन
Asmita :	(Correcting Nikhil) About 2. No, about 0	साधारण दोन. साधारण शून्य

The children went back and forth between using the language, not using it, and using the word as a label. Rather than being an authentic use, they were likely responding to the researcher’s expectation. This experimentation may have allowed them to test using “about” (sadharan) in context. The following conversation took place after displaying Asmita’s graph.

Researcher:	About how many does Asmita throw	अस्मितासाधारण किती टाकते
Asmita:	1	1
Vandana:	1	1
Nikhil:	1	1
Chorus:	About 1	साधारण एक

Even after modelling the use of the word “about”, we did not see any spontaneous use of the word in the expression of the children. They later remembered they were supposed to use the word but likely used it as a label, again in response to the researcher’s expectation.

Discussion

In Marathi the exact translation of the word ‘about’ is sadharan and we could see that none of the children in the study were particularly comfortable in using the word or any other alternative suggested by the researcher. This is an unexpected complication while navigating the expression of uncertainty that may not be easily foreseeable in research conducted in classrooms where English is the dominant language. The trajectory that the children followed while reasoning about data seemed to be the same one described in research. However, it may have been unreasonable to assume the children would pick up the word introduced by the researcher. This is a challenge while informally dealing with statistical ideas. It was clear that even though the researcher and children spoke the same language, a language barrier existed between the researcher and the children. In the state of Maharashtra there are multiple dialects of the state language of Maharashtra. There is a standard variant used in textbooks which is generally understood by a large percentage of the population. However, adults and children speak variants of this language while conversing with each other. Although the researcher did not use any technical terms, the language of the researcher was closer to the dominant language written in textbooks and the children's dialect did not appear to utilise the same idea. While we completely acknowledge that there will exist other ways of expressing uncertainty that the children may be familiar with, the lack of similarity to dominant language of school may pose pedagogical challenges in introducing the informal ideas such as informal inference.

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References

Ben-Zvi, D., Aridor, K., Makar, K., & Bakker, A. (2012). Students' emergent articulations of uncertainty while making informal statistical inferences. *ZDM Mathematics Education*, *47,* 1107–1120. https://doi.org/10.1007/s11858-012-0420-3

Benson, C. (2013). Towards adopting a multilingual habitus in educational development. In C. Benson, & K. Kosonen (Eds.), *Language issues in comparative education* (pp. 283–299). Sense Publishers.

Chauhan, B. (2013). *Development of a mathematics curriculum: A culturally responsive approach*. [Unpublished doctoral dissertation]. Department of Education, University of Delhi.

Delpit, L. D. (1988). The silenced dialogue: Power and pedagogy in educating other people's children. *Harvard Educational Review*, *58*(3), 280–298.

Karelitz, T. M., & Budescu, D. V. (2004). You say "probable" and I say "likely": Improving interpersonal communication with verbal probability phrases. *Journal of Experimental Psychology: Applied*, *10*(1), 25–41. DOI: 10.1037/1076-898X.10.1.25

Kulkarni, V. G., & Gambhir, V. G. (1981). The effect of language barrier on science education. *Indian Educational Review, 16*, 48–58.

Lesser, L., Wagler, A., Esquinca, A., & Valenzuela, M. (2013). Survey of native English speakers and Spanish-speaking English language learners in tertiary introductory statistics. *Statistics Education Research Journal*, *12*(2), 6–31.

Lesser, L., & Windsor, M. (2009). English language learners in introductory statistics: Lessons learned from an exploratory case study of two pre-service teachers. *Statistics Education Research Journal*, *8*(2), 5–32.

Makar, K., & Rubin, A. (2009). A framework for thinking about informal statistical inference. *Statistics Education Research Journal*, *8*(1), 82–105.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, *22*(4), 405–435.

Pratt, D., & Kazak, S. (2018). Research on uncertainty. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 193–227). Springer.

Rubin, A., Bruce, B., & Tenney, Y. (1990). Learning about sampling: Trouble at the core of statistics. In D. Vere-Jones, S. Carlyle, & B. P. Dawkins (Eds.), *Proceedings of the third international conference on teaching statistics* (ICOTS-3) (pp. 314–319). Dunedin, New Zealand: International Statistical Institute.

Vygotsky, L. (2012). *Thought and language.* MIT Press.

Social Mathematical Practices in Multi-Digit Multiplication

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Despite substantial research exploring multiplicative thinking and students' difficulty in the domain, the topic of multi-digit multiplication is under-researched. In this paper, I share a learning trajectory for multi-digit multiplication that combined social and cognitive perspectives of learning. Using Design Research methods and involving 45 Year 5 (9–11-year-olds) students from two different schools, an instructional sequence based on the trajectory was implemented. Findings led to the refinement of a trajectory that has implications for teaching practice.

Multiplicative thinking is widely recognised as an important understanding in students' mathematical development (Clark & Kamii, 1996; Park & Nunes, 2001). Multiplicative reasoning is the ability to think and reason using a deep conceptual understanding of the multiplicative structure, which underlies important concepts including fractions, decimals, and proportional reasoning (Siemon, 2013). Concerningly, evidence suggests that students' 10-15 years of age struggle to think multiplicatively (Siemon, 2013). Students' fluency with multi-digit multiplication develops from their ability to think multiplicatively. Although multiplicative thinking is well-researched, there is limited work in the field of multi-digit multiplication (Hickendorff et al., 2019).

In the last two decades, there has been an increased research focus on using learning trajectories to study the development of learning in mathematical domains. In this paper, a learning trajectory is presented for multi-digit multiplication that fosters the development of multiplicative thinking. Unlike other trajectories in the domain of multiplication, this one combines cognitive and social perspectives of learning. The taken-as-shared learning route that a class community may follow is described and a means by which an individual student's learning might be supported as s/he participates in and contributes to the collective learning of the class.

Literature

Developing understanding in multiplication is complex and acquired over multiple years (Clark & Kamii, 1996). Multiplication is a binary operation (Barmby et al., 2009) that requires the coordination of composite units. Multiple theories have been presented through relevant literature into early multiplicative understanding. In his study of the counting scheme, Steffe (1994) explained that understanding of multiplication is based on the construction of a composite, iterable unit. Other researchers have argued that this repeated addition model of multiplication is incomplete (Clark & Kamii, 1996; Park & Nunes, 2001) and that repeated addition is a procedure for solving multiplicative problems, not a conceptual basis. Park and Nunes (2001) present an alternate theory, stating that multiplicative understanding is defined by an invariant relationship between two quantities.

Much less is known about the development of understanding in multi-digit multiplication compared to single-digit multiplication (Hickendorff et al., 2019). In one study, Larsson (2016) described understanding in multi-digit multiplication as the result of connections between three elements: the arithmetic properties of commutativity, distributivity and associativity; models of multiplication; and strategies for solving multiplicative calculation. Although there is not a developmental hierarchy evident in multiplicative properties or models, Larsson (2016) reports an observable progression in students' solution strategies for multi-digit multiplication,

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 535–542). Gold Coast: MERGA.

developing from addition-based strategies such as repeated doubling through to strategies that draw on multiplicative thinking, including distribution and decomposition. Similar strategy progressions relating to multi-digit multiplication are reported in other studies (Ambrose et al., 2003; Barmby et al., 2009; Izsak, 2004).

Student difficulties in multi-digit multiplication result in many relying on additive strategies for prolonged periods of time (Ambrose et al., 2003; Barmby et al., 2009; Izsak, 2004). Evidence suggests that there is scope to extend additive strategies to more efficient ways of calculating. For example, students' intuitive use of repeated doubling to solve multi-digit multiplication problems has supported students to develop more complex multiplication strategies that draw on the associative property (Ambrose et al., 2003; Tripet, 2019). On the flip side, there is a danger that overgeneralising addition strategies can impede students' conceptualisation of the binary nature of multiplication (Larsson, 2016).

The use of effective multiplicative strategies demonstrates a shift in students' understanding of multiplication (Larsson, 2016) and their efficiency in performing calculations (Hickendorf et al., 2019). These strategies for multi-digit multiplication draw on the associative and distributive properties. Students recognise the practicality of partitioning and grouping numbers to solve more complex multiplication problems (Barmby et al., 2009). Ambrose et al. (2003) found that students in Years 3 to 5 instinctively used partitioning strategies to solve multiplication and division problems, concluding that many students hold an intuitive understanding of the distributive property. Izsak (2004) reported similar findings, noting that the coordinated rows and columns of the array supported students' calculations.

Although the associative property is an important multiplicative understanding, there is very limited research based on this property (Ding et al., 2013). In one notable study, Ding et al. (2013) evaluated primary pre-service teachers' understanding of the associative property of multiplication. Their work showed general misunderstandings around the associative property, with many pre-service teachers confusing the associative and commutative properties. They concluded that students' conceptual understanding of the associative property would be impeded by teachers' misconceptions.

Learning Trajectories

Learning trajectories were introduced by Simon (1995) as a description of "what teaching might be like if it were built on a constructivist view of knowledge development" (p. 115). Simon's (1995) original trajectory was based on a single instructional episode. Others have since applied the construct to sequences over longer periods of time.

As "a vehicle for planning learning of a particular concept" (Simon & Tzur, 2004, p. 93), trajectories refocus teaching from content transmission to students' cognitive constructions (Gravemeijer, 2004) and help provide focus and direction for instruction (Wright et al., 2006). Although most trajectories are concerned with an individual's cognitive development, few acknowledge the complementary nature of social and individual aspects of learning (Stephan & Rasmussen, 2002). Recognising that student learning is rarely uniform, Cobb et al. (2011a) adapted the construct of a learning trajectory as a sequence of "mathematical practices" (p. 80). Mathematical practices are descriptions of the collective learning pathway of the class community, and individual learning is understood in terms of individual students' reasoning as they participate in and contribute to the collective mathematical practices of the class (Cobb et al., 2011b). Describing social and individual learning provides a more comprehensive description of the learning that takes place in the classroom (Stephan & Rasmussen, 2002).

The intent of this study was to better understand how students construct understanding in multi-digit multiplication in the social setting of the classroom and how this process of understanding might be reflected through a learning trajectory. My work was guided by the following question: How can the social and cognitive aspects of learning be accounted for in a learning trajectory for multi-digit multiplication?

Methodology

Theoretical Perspective

The theoretical perspective for this study draws on Cobb and Yackel's (1996) emergent perspective, in which learning is recognised as both a social and individual endeavour. From this perspective, a reflexive relationship exists between the constructions of the individual and the social culture of learning in the classroom. Individuals construct new knowledge and understandings through mathematical activity while participating in the social role of learning in the classroom, and, in turn, students' interactions and contributions influence the evolving learning culture in the classroom. As such, the social and cognitive aspects of learning cannot exist in isolation. Concern is not about which perspective is more dominant, rather how the two aspects work together to support the development of students' mathematical learning.

Methods

Design Research methods were employed that allowed observation of students' thinking and reasoning of multi-digit multiplication first-hand (Cobb & Gravemeijer, 2008). In the preparatory phase, a domain-specific instructional theory was developed based on a detailed analysis of literature (Bobis & Tripet, 2023; Tripet, 2019). The teaching experiment phase involved a cyclic process of designing, testing, and refining the implementation of the instructional theory in the classroom setting. The experiment was conducted in two different Year 5 (9–11-year-olds) classes in Sydney, Australia: 23 students in Class 1 and 22 in Class 2, creating a sample size of 45 students. The same instructional theory was used as the basis for teaching in both classes and was implemented over a two-week period. The author was the primary researcher and adopted the role of teacher in each experiment, with the regular class teacher present to help facilitate student activity.

Data Collection and Analysis

The experimental phase involved the collection and ongoing analysis of data. Data collected included student work samples, classroom video recordings and field notes taken by the primary researcher. Following each lesson, the new data were compiled with existing data and reviewed. To coordinate the differing cognitive and social aspects, the dataset was iteratively analysed from three perspectives: the social learning of the class, the cognitive constructions of individual students, and then the relationship between the two.

Analysis of data from the social perspective identified the emerging mathematical practices (Cobb & Yackel, 1995). Cobb et al. (2011b) explain that mathematical practices comprise three interrelated and interdependent mathematical norms: mathematical activity, argumentation, and ways of reasoning with tools and symbols. Based on this, analysis of data focused on mathematical activity, reasoning and argumentation that became taken-as-shared in the classroom. A practice was considered taken-as-shared when most students were observed acknowledging acceptance and/or employing the practice. Along with analysis of student work samples after each lesson, video footage of class discussions was viewed to identify regularities and patterns in the way students acted, reasoned, or spoke mathematically.

Each mathematical practice identified was attributed to conceptual events. Events were considered conceptual when a shift in the collective reasoning of the class was observed. Conceptual events were considered significant when they were observed over multiple instances and influenced the collective knowledge of the class.

The cognitive perspective was informed by individual students' reasoning and their participation in, and contribution to, the collective ways of acting, reasoning and arguing mathematically that emerged in the class. Students' participation and contributions were documented on three levels: students' emerging use of representation, the pathways of strategy reinvention and the associated key understandings for these strategies, and through their discourse as they explained and justified their thinking.

The final stage of analysis in the teaching experiment was to consider the relationship between the social and cognitive aspects of learning, that is, how students' cognitive constructions contributed to the emerging social mathematical practices in the class, and then how students' participation in these practices led them to more sophisticated mathematical thinking. Analysis of video footage of each instructional episode examined how the actions of the teacher, the classroom culture and the role of the context supported student cognition and the development of social mathematical practices.

Results

Four socially constructed mathematical practices were identified in the study (Table 1). Each mathematical practice was linked to two conceptual events. The following section presents a detailed description of one of the conceptual events that led to the negotiation of the first mathematical practice (MP1). The conceptual event describes individual students' mathematical reasoning and argumentation, which serve as illustrations from both classes.

Table 1

The Four Mathematical Practices and Associated Conceptual Events

Mathematical practices (MP)	Conceptual events (CE)
MP1—The array as a sense-making tool: Partitioning the array	CE1—Using complete rows and columns to partition the array CE2—The use of place value across strategies
MP2—The array as a sense-making tool: Rearranging the array	CE1—Noticing a relationship between the numbers CE2—Using factors to manipulate the array
MP3—Working mathematically: Thinking multiplicatively	CE1—Recognising and adding all partial products CE2—Differing between additive and multiplicative compensation
MP4—Working mathematically: Using friendly numbers	CE1—Looking for efficiency CE2—Use of multiple strategies

MP1—The Array as a Tool for Sense-Making: Partitioning the Array

The instructional sequence followed the narrative of a cupcake bakery. The first teaching episode presented the narrative: *A baker makes and sells eight different flavours of cupcakes. The cakes are baked in a tray that has four rows with six cakes in each row. He bakes one tray of each flavour. How many cupcakes does he bake each day?* Students were asked to answer the question and explain how they obtained their answer. To assist in this process, students had access to different manipulatives, including a graphic of the cakes in an array.

A whole class discussion formed the inciting incident for the first conceptual events central to the negotiation and acceptance of the first mathematical practice. The intent of the discussion was to compare different student strategies, and for individuals to use these observations as a stimulus for modifying and refining their own strategies.

MP1 Conceptual Event 1—Using Complete Rows and Columns to Partition the Array

Three different strategies that focused on partitioning the array were presented during the class discussion. Zoe and Lucille (Figure 1) demonstrated that the structure of the array allowed for a straight partition to create a group of 20 and a group of 4.

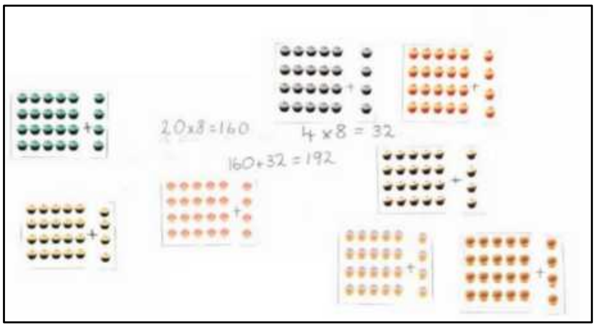
Zoe: We cut it down here (pointing to an array on their poster) to make a group of 20 and a group of 4.

Teacher: Why did you use 20 and 4?

Zoe:	This one is 5 times 4 and this one is 1 times 4. You can do it just by cutting down here (showing the cut made on one of the trays of cakes) … also we thought that 20 and 4 would be easy.

Figure 1

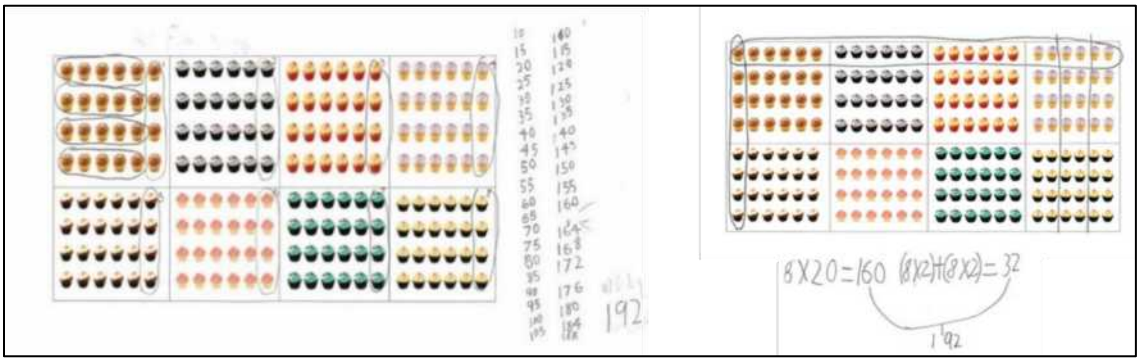
Strategy Used by Zoe and Lucille



Jake (Figure 2) explained his strategy of skip counting fives and then adding on the remaining fours. Jasper (Figure 3) explained how he used the larger array to partition 20 × 8 and two groups of 2 × 8, as these were multiplication calculations that he could perform mentally.

Figure 2 and 3

Strategies Used by Jake (1) and Jasper (2)



Students examined the similarities and differences between the three strategies. The partitioning methods were identified as a key difference between strategies: Jake circled groups of five, Zoe and Lucille had physically cut the array, and Jasper had drawn lines. One student identified the similar way in which Jasper, Zoe and Lucille used columns to partition multiple rows. Jake commented that he could have used columns to partition rows as well:

Teacher:	Could you have split your array in a similar way to Zoe, Lucille and Jasper?
Jake:	Yes … I think … I could have just cut down there and … just cut it like Zoe and Lucille's. I think our ways are sort of the same … more than Jasper's anyway.
Teacher:	Why is your way more like Zoe and Lucille's?
Jake:	Well, where I circled is sort of like… it's just like where they cut. They are just the same really.
Teacher:	Do people agree with Jake? Do you think his strategy is like Zoe and Lucille's? … Frederique?
Frederique:	Well, I think it is sort of. You can see that they both used the 4s at the end …
Lucy:	Jake could use 20 too because he has 20 with his 5s.

Jake's contribution was an important element in the first mathematical practice becoming taken-as-shared in the class. There was verbal consensus in the class that Jake could partition multiple rows using columns, rather than focusing on just one row at a time. Similarly, there was verbal consensus that Jake's multiple rows of fives were like Zoe and Lucille's partitioning of the smaller arrays into 4 × 5 and 4 × 1. The similarity between Jake's strategy and Zoe and

Lucille's strategy was accepted and reinforced by other class members, who indicated how the partitioning used by Zoe and Lucille was evident in Jake's skip counting.

Students were asked to compare Zoe and Lucille's work with Jasper's work. One student observed that both used 8 × 20 and 8 × 4, although Jasper had partitioned his 8 × 4 into two groups of 8 × 2, which became a focus for discussion.

Teacher: How could they both use the same calculations?

Luke: If you look at Jasper's there is 24 across the top and there is 8 [motioning down the column]. You can just make 20 by putting in the line and then there is 4 on this side. And then in that one [pointing at Zoe and Lucille's] they have 20 and then they have 4 and … if you count … there are 8 of them.

The class indicated their agreement with Luke's explanation and that Jasper's method of partitioning was more efficient. Significantly, the students' reasoning centred on the array which indicated that the representation was a useful sense-making tool. The array provided a means for students to reason conceptually, not just procedurally, about multiplication. Students' argumentation, even their gesturing, centred on the array. In subsequent activity, a shift was observed in students' strategy use. For example, when asked to calculate 8 trays of 16 cakes, most students partitioned 16 into 10 and 6. This shift in students' strategy use demonstrated an acceptance of the power of the distributive property to aid efficient computation.

Discussion

The purpose of this research was to articulate a learning trajectory for multi-digit multiplication that accounted for the social and cognitive aspects of learning. The final trajectory is presented in Table 2. The trajectory is structured on Cobb's et al. (2011b) description of mathematical practices of the interrelated and interdependent elements of mathematical activity, argumentation, and ways of reasoning with tools and symbols. The first section in the trajectory identifies the four social mathematical practices that emerged through the experiment. Initially, students used the array as a tool for sense-making as they explored partitioning the array and using factors to manipulate the array. The two subsequent practices related to ways that students worked mathematically, as they transitioned to thinking multiplicatively and identifying 'friendly' numbers within more complex problems.

Students' cognitive advances were primarily seen through use of more sophisticated strategies, changing interactions with the array, and their mathematical argumentation. These aspects form the other three categories listed in the trajectory. The computational strategies observed in this study were comparable to those reported in previous studies (Ambrose et al., 2003; Barmby et al., 2009; Izsak, 2004; Larsson, 2016). The array proved integral to students' reasoning. As Gravemeijer explains (2004), mathematical models bridge informal and formal mathematics. In this instance, the array bridged students' experimental strategies to the properties of distributivity and associativity. It also formed a bridge across students' strategies and their argumentation. Students were able to make sense of others' thinking and argumentation using the array. This was particularly evident in the class discussion that formed the inciting incident for the first conceptual events central to the negotiation and acceptance of MP1. As students participated in this discourse, they were reorganising their own thinking, resulting in cognitive shifts.

This trajectory offers significant implications to teaching practice. Typically, learning trajectories focus on the mathematical learning of individual students, suggesting (whether intentional or not) that students will engage in tasks in similar ways and develop mathematical insights at similar times. The uniform descriptions of learning can mean that the diversity of students' reasoning fade into the background (Cobb et al., 2011a). This presents a challenge to teachers: how can one use a trajectory to inform instruction while still accounting for and responding to the diversity of student thinking?

The power of this trajectory and its potential impact on classroom practice is realised in the reflexive relationship between the social mathematical practices and the development of individual cognitive learning; social and individual aspects of learning are interrelated and interdependent, one cannot exist without the other (Cobb et al., 2011b). In this study, students' participation in and contribution to classroom activity shaped the mathematical practices, and through their participation and contributions, students reorganised their own thinking resulting in cognitive advances. The mathematical practices in the trajectory provide teachers with instructional directionality (Wright et al., 2006). By guiding the establishment of mathematical practices, teachers are simultaneously supporting individual students' cognitive advances.

Table 2

A Learning Trajectory for Multi-Digit Multiplication

Mathematical practices (MP)	Shared purpose	Reasoning with tools and symbols	Mathematical discourse
MP1 The array as a sense-making tool: Partitioning the array	Calculate the total number of cakes on 8 trays of 24	Partitioning the array by physically cutting or drawing lines to make smaller, more friendly parts to aid computation Informal recording with symbols	The distributive property—Noticing the similarity between the different partitioning strategies used by students
MP2 The array as a sense-making tool: Rearranging the array	Calculate the total number of cakes—16 boxes with 12 in each box	Physically cutting the array and then rearranging all parts, ensuring the rectangular structure is maintained Informal recording with symbols	The associative property —The array can be rearranged using factors and multiples
MP3 Working mathematically: Thinking multiplicatively	Calculate the area of two trays	Using the array to see: • all partial products formed need to be added together. • only factors and multiples can be used to rearrange the array More formalised symbolic recording	Additive v multiplicative thinking—Noticing the 2D structure of the array and the way it impacts the working of strategies
MP4 Working mathematically: Looking for friendly numbers	Calculate the money raised through cake orders	Reduced use of the array Formalising methods of symbolic recording	Identifying friendly numbers in calculations and determining which strategy to use based on the numbers to be multiplied

Conclusion

Students' fluency with multi-digit multiplication develops from their ability to think multiplicatively. In this paper, a learning trajectory for multi-digit multiplication is presented. Uniquely, the trajectory coordinates social and cognitive aspects of learning and, as such, provides a rich description of learning over the course of the instructional sequence. Practically, this is significant. The trajectory provides a viable theory that teachers can use to provide directionality for their teaching while also acknowledging student diversity.

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References

Ambrose, R., Baek, J. M., & Carpenter, T. (2003). Children's invention of multiplication and division algorithms. In A. J. Baroody, & A. Dowker (Eds.), *Studies in mathematical thinking and learning. The development of arithmetic concepts and skills: Constructive adaptive expertise* (pp. 305–336). Erlbaum.

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2009). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics*, *70*(3), 217–241.

Bobis, J., Tripet, K. (2023). Situating teacher learning in the mathematics classroom and everyday practice. In R. Tierney, F. Rizvi, & K. Ercikan (Eds.), *International encyclopedia of education* (4th ed., pp. 518–527). London: Elsevier.

Clark, F., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1–5. *Journal for Research in Mathematics Education*, *27*(1), 41–51.

Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes. In A. Kelly, R. Lesh, & J. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 68–95). Routledge.

Cobb, P., Gravemeijer, K. & Yackel, E. (2011a) Chapter 6: Introduction. In E. Yackel, K. Gravemeijer & A. Sfard (Eds.), *A journey in mathematics education research: Insights from the work of Paul Cobb* (pp. 75–84). Springer.

Cobb, P., Stephan, M. & Bowers, J. (2011b) Chapter 8: Introduction. In E. Yackel, K. Gravemeijer, & A. Sfard (Eds.), *A journey in mathematics education research: Insights from the work of Paul Cobb* (pp. 109–115). Springer.

Cobb, P., & Yackel, E. (1995). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3–4), 175–190.

Ding, M., Li, X., & Capraro, M. M. (2013). Preservice elementary teachers' knowledge for teaching the associative property of multiplication: A preliminary analysis. *Journal of Mathematical Behavior*, *32*(1), 36–52.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, *62*(2), 105–128.

Hickendorff, M., Torbeyns J. & Verschaffel L. (2019), Multi-digit addition, subtraction, multiplication, and division strategies. In A. Fritz, V. G. Haase, & P. Räsänen (Eds.), *International handbook of mathematical learning difficulties*. Springer.

Izsak, A. (2004). Teaching and learning two-digit multiplication: Coordinating analyses of classroom practices and individual student learning. *Mathematical Thinking and Learning*, *6*(1), 37–79.

Larsson, K. (2016). *Students' Understandings of Multiplication*. Stockholm University.

Park, J. H., & Nunes, T. (2001). The development of the concept of multiplication. *Cognitive Development*, 16(3), 763–773.

Siemon, D. (2013). Launching mathematical futures: The key role of multiplicative thinking. In S. Herbert, J. Tilley & T. Spencer (Eds.), *Mathematics: Launching futures. Proceedings of the 24th biennial conference of the Australian Association of Mathematics Teachers Inc.* (pp. 36–52). AAMT.

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, 6(2), 91–104.

Steffe, L. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3–39). SUNY Press.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behaviour*, *21*(4), 459–490.

Tripet, K. (2019). *A journey to understanding: Developing computational fluency in multi-digit multiplication* [Doctoral dissertation, The University of Sydney]. SES. http://hdl.handle.net/2123/20763

Wright, R., Stanger, G., Stafford, A. & Martland, J. (2006). *Teaching number: Advancing children's skills and strategies*. Sage Publications.

Preservice Primary Teacher Pedagogical Content Knowledge of Fractions Using the Refined Consensus Model

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Fractions remains a difficult topic for primary preservice teachers (PST). This paper analyses a PST teaching episode using the *Refined Consensus Model* (RCM) of Pedagogical Content Knowledge (PCK) to explore challenges faced by PST in acquiring specialised knowledge for effective fractions teaching. Data include semi-structured interviews highlighting relationships between collective and personal PCK with a videotaped teaching episode allowing an interrogation of her enacted PCK. Key findings illustrate the relationships among a PST collective, personal, and enacted PCK, and the importance of the reflection phase of enacted PCK.

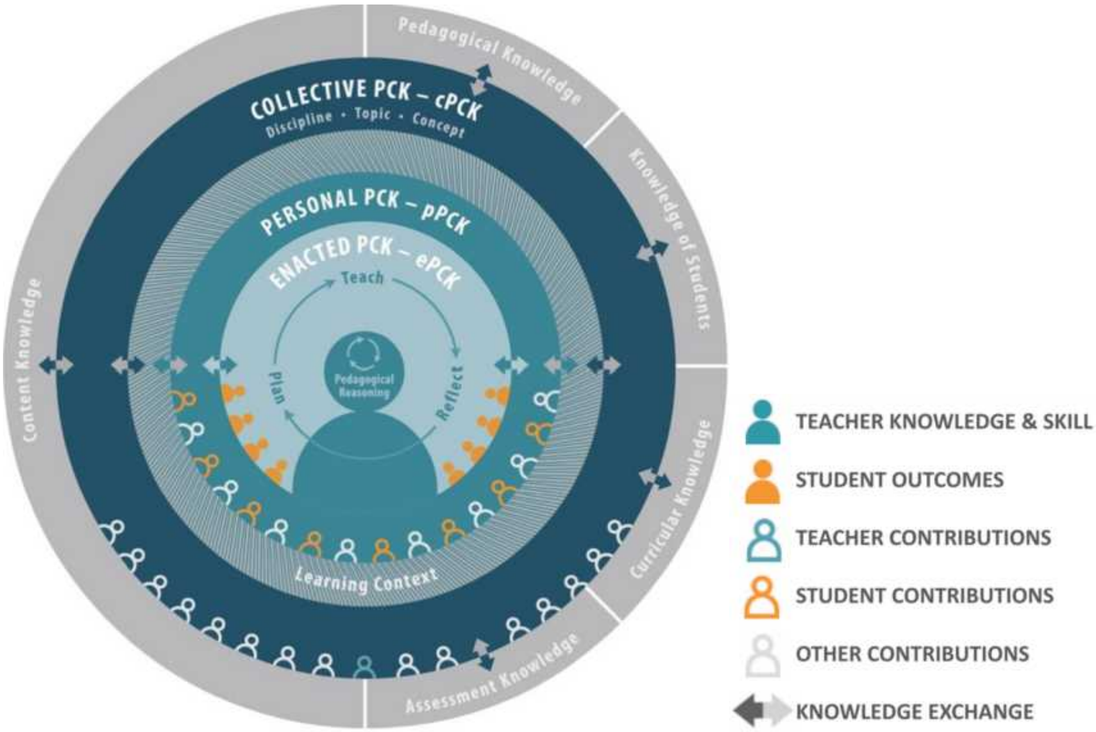
Empirical work has shown the significant relationship between primary teachers' mathematical content knowledge and their students' mathematics achievement (Campbell et al., 2014). Teaching demands deep knowledge of "subject matter for teaching which consists of an understanding of how to represent specific subject matter topics and issues appropriate to the diverse abilities and interest of learners" (Shulman & Grossman, 1988, p. 9). This deep knowledge is what Shulman termed 'Pedagogical Content Knowledge' (PCK), a framework that has been taken up widely in order to better understand knowledge development for teaching with both inservice and preservice teachers (PST). However, there are differing conceptual viewpoints of PCK in the extant literature. Some scholars position PCK as the knowledge of expert teachers that preservice teachers begin to accumulate during initial teacher education (ITE), that is, knowledge *for* teachers. For example, Ball et al. (2008) included PCK as a standardised and measurable part of mathematical knowledge *for* teaching. Alternatively, knowledge *of* teachers describes a more responsive and adaptive kind of knowledge that is particular to each teacher. Whilst acknowledging the central role that the specific learning context plays alongside the unique composition of each individual teacher's knowledge, the collective canonical knowledge in the field as developed through empirical and theoretical work is also key to understanding a teacher's PCK. Thus, we consider both knowledge *for* and *of* teachers to be important to understand how PST knowledge develops.

Research calls for greater clarity and rigour in PCK models (e.g., Abell, 2007), recommending further interrogation and refinement of models of teachers' PCK to better illustrate the relationships between teachers' knowledge and experience and how these impact practice and student outcomes. Building on the 2012 Consensus Model of PCK (Berry et al., 2015), the *Refined Consensus Model* (RCM) of PCK now encompasses three distinct realms of a teacher's PCK (Carlson & Daehler, 2019). The three realms and their proposed interrelationships are summarised in Figure 1. Collective PCK (cPCK) is the professional knowledge held in the field by the full range of educators and educationists. Personal PCK (pPCK), as the name suggests, is more personalised, residing in the individual teacher. Enacted PCK (ePCK) describes the knowledge subset upon which the teacher draws to guide pedagogical reasoning in planning, teaching, and reflecting on lessons taught.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 543–550). Gold Coast: MERGA.

Figure 1

Refined Consensus Model of PCK (Carlson & Daehler, 2019, p. 82)



Also depicted in the RCM of PCK is the two-way knowledge exchange ( ) that occurs between the knowledge realms. In this way, the model shows how teacher knowledge and skills are filtered or amplified by teacher attitudes and beliefs, thus shaping their pPCK over time. For example, a teacher's attitudes and beliefs about "students, the nature of content knowledge, or the role of the teacher" (Carlson & Daehler, 2019, p. 82) can amplify and/or filter the teacher's developing pPCK. The RCM of PCK has been taken up widely in science education research (Mientus et al., 2022), yet relatively few studies of mathematics teachers' PCK have adopted the RCM of PCK (e.g., Botha et al., 2023), and even fewer have used this model to interrogate *primary* school teachers' PCK (e.g., Amador et al., 2022). In the current paper, we use the RCM of PCK to explore its potential to conceptualise the relationships between PCK as both integrative (ePCK as knowledge *of* teachers) *and* transformative (cPCK as knowledge *for* teaching) as PST learn mathematics for teaching.

Knowledge for, and of, Pre-Service Teachers: Fractions

Often cited, Shulman's (1986) original description of PCK as "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9) makes clear that representations are key to effective teaching. In mathematics, this includes a collective knowledge base about effective ways of representing fractions and teaching about fraction representations. PST experience fraction instruction in school as children yet the content knowledge teachers need goes beyond that required by students. Fractions are one of the most complex areas of the primary school curriculum, both to learn and to teach. Whilst children's knowledge of fractions has been studied extensively (e.g Roesslein & Codding, 2019), primary PST have similar difficulties themselves (Vula & Kingji-Kastrati, 2018). This is concerning as limited teacher knowledge is related to children's difficulties in learning fractions (Van Steenbrugge et al., 2014) and indeed, may go some way in accounting for children's problems. The development of PST knowledge during ITE for teaching mathematics is complex. PST are learning the mathematical content and how to teach it simultaneously. In framing PST knowledge, it is important to conceptualise both the knowledge needed for teaching as well as PST emerging understandings, hence the value of a theoretical perspective, such as RCM, on teacher knowledge and its development. Additionally, PST bring with them prior experiences from their own education, including beliefs about the nature of mathematics and how to teach mathematics (Johnson & Olanoff, 2020).

Despite the wealth of research into primary PST knowledge of fractions, few studies illustrate what this knowledge looks like as teachers draw on it while teaching (Thanheiser et al., 2014). Additionally, most research has focused on what these teachers do or do not know (Olanoff et al., 2014) rather than how they apply the knowledge to teaching (Mewborn, 2001). In the current paper, we use the constructs of the RCM to explore the relationships among cPCK, pPCK, and ePCK for teaching fractions. Thus, the following research question was posed:

- How can the *Refined Consensus Model of Pedagogical Content Knowledge* be used to examine *PST Pedagogical Content Knowledge* of fractions?

Methods

The research took place in a post-graduate primary ITE program at a regional university in Australia. As part of a larger study, this instrumental case study (Stake, 1995) enables particular attention to key elements of the RCM. We chose Fiona, a female in her late 20s, as a single case in this study because (1) she taught fractions on her Professional Experience (PEx) and (2) her prior mathematics knowledge was typical of many PST in this program. Analysis through the RCM thus offers opportunity to unpack the relationships between layers of PST PCK during an ITE program. The instrumental case also has the potential to refine theory, which we address later in the paper. Fiona had completed a Bachelor of Arts in the year prior to enrolling in the ITE program. In high school, she had not studied any mathematics subjects in her final two years and thus was required to complete a 14-week pre-university access program that included fundamental mathematics skills. Also, prior to enrolling in the ITE program, Fiona had worked at an early childhood centre, which inspired her to pursue a career in primary school teaching.

In the ITE program, PST took two sequential mathematics subjects both of which included numeracy content and pedagogy for teaching Kindergarten to Year 6 children (ages 5–12). Fraction content was addressed in three 2-hour lectures and four 1-hour tutorials. The instruction followed a *Representational Reasoning Teaching and Learning* approach (see Thurtell et al., 2019 for an elaboration of this approach). Key representations of fractions included area models, linear models (such as number lines), and discrete models that were used to develop PST understanding of different fraction ideas. To assist PST in constructing knowledge of fraction concepts and operations, discussions of PST representations and explanations, including the misconceptions evident, were scaffolded. These discussions also addressed the difficulties children experience with an explicit focus on fraction representations. The ITE program also included three PEx opportunities of varied duration for PST to teach in primary school classrooms: PEx 1 (3 weeks); PEx 2 (3 weeks); and PEx 3 (5 weeks).

Data Collection and Analysis

Data for the study include 30–60 minute semi-structured interviews held at four points of the teacher education program: before and after PEx 1 (Int. 1 and 2) and after PEx 2 and PEx 3 (Int. 3 and 4). Additionally, a segment of Fiona's teaching on PEx 2 was videotaped and transcribed alongside photographs of the fraction representations constructed by Fiona and her students. Interviews focused on general perceptions of mathematics, fractions, and experiences in the subjects and/or on PEx. As the study progressed, interviews explored how Fiona's fraction knowledge developed, and Int. 3 and 4 included explicit questions interrogating the teaching episodes. The interviews were analysed using the RCM of PCK framework to identify the ‘realms’ of knowledge (ePCK, pPCK, and cPCK) as *a priori* categories. Working definitions of each realm were developed from Carlson and Daehler (2019) and then used to identify examples of these layers of Fiona’s PCK.

Since PST fraction representations serve as indicators of ePCK (Thurtell et al., 2019), we observed how Fiona tailored her teaching to specific children and their mathematical work and selected the teaching action of *responding to children about their representations* as the unit of analysis for the current case study. There were 12 such instances in the videotaped teaching episode; eight of these involved co-constructed representations. Only four responses addressed a child’s own fraction representation and all of these came in the final lesson segment. Further analysis of Int. 2, 3, and 4 sought additional indicators of Fiona’s ePCK.

In the current paper, pPCK is represented by Fiona’s perceptions of her knowledge of mathematics, fractions, and fractions representations. Thus, Int. 1–4 were analysed to identify Fiona’s PCK related more broadly to teaching mathematics and fractions. Further, in alignment with the RCM of PCK (Carlson & Daehler, 2019), potential filters/amplifiers were identified as Fiona’s beliefs about and attitudes towards teaching fractions and herself as a learner and teacher of mathematics. Finally, cPCK is the specialised professional knowledge independent of a specific learning context generated and held by multiple professionals. The present paper focused on the *concept-specific* idea (Carlson & Daehler, 2019) of *equality of parts* of fractions. In identifying this concept in the data, we also drew on research about learning and teaching fractions, fractions as represented in curriculum documents and teaching handbooks, and the fraction content of the ITE program to identify illustrative examples of Fiona’s cPCK.

Examining Fiona’s Pedagogical Content Knowledge through the RCM

Reflecting on the *content knowledge* aspect of her pPCK prior to the ITE program, Fiona was not confident in the quality or depth of mathematics in general. Fiona stated her knowledge “wasn’t that great” (Int. 3) and she made the “mistakes that sometimes kids make” (Int. 4). The knowledge of fraction content and instruction Fiona had carried from her own schooling was limited. When faced with fractions in school, she stated she “just tried to learn the rules, like when to flip, when not to flip. We just learned that [and] all that stuff just makes me go *err*” (Int. 4). Although, she did remember “doing more of the basic stuff, halves, quarters” she felt that if the fraction content “got any more difficult than that I would be struggling” (Int. 1). In fact, Fiona continued to find the *content knowledge* aspect of pPCK required for teaching fractions difficult, “worrying about fractions, well worrying about maths in general” (Int. 1). She felt her procedural skill was the most important aspect of pPCK needing improvement, stating she “still just need[s] a lot of practice, just going over things” needing to learn “the actual maths of it … like doing fractions—actually adding and subtracting. Sometimes I just need to remember, like go back over it and do more of them” (Int. 2). This was also what she felt was needed to teach (i.e. for ePCK), “if I was to teach it to, say, a Year 6 or something, I wouldn’t be confident. I’d need to just prepare, go over it again” (Int. 2). Her lack of confidence was consistent with her beliefs about the nature of fractions as synonymous with fraction notation, typically believing that “you can still use other stuff as well to help it, but I guess you’ve got to be able to do the symbolic to do the maths” (Int. 1).

The Teaching Episode and Fiona’s ePCK

Because ePCK is the knowledge an individual teacher draws on in a particular setting with a particular child or children to support a specific concept, we now contextualise the teaching episode as an example of Fiona’s ePCK. This episode occurred during PEx 2 in the final week of a 3-week block of teaching and was the second of two lessons about fractions. The class was a Year 1/2 composite class comprised of sixteen children (aged 6–8). The focus of the lesson was fractions needing equal parts using scenarios of sharing food. The lesson began with a 10-

minute, whole class introduction where individual children were invited to partition images on the interactive whiteboard, sharing a rectangular chocolate bar among two, four, and eight people, then sharing three circular lollipops between two people. The main segment of the lesson was completed at desks with children working individually. Children were given playdough with which to roll out and cut circles, partitioning to "share a cake" first between two then four people. Fiona showed two quarters "equals" one half, asking children to place the half next to two-quarters of the circle. We focus on the final moment in the lesson where children were invited to share a new 'cake' among eight people.

The Teaching Moment

There were four strategies used by children to partition their circles (see Table 1, one child did not produce a model) and Fiona responded directly to three children (Strategies 1–3).

Table 1

Children's Strategies for Partitioning Circles into Eighths and Fiona's Responses

Strategy	No. of children[α]	Work sample	Fiona's response
1. Three partitions along the diameter to cut the circle into halves, then quarters, then eighths	5 e.g., from Kai		"Great work Kai, good"
2. Individual cuts along the circle's radii with unequal parts	5 e.g., from Li		"I like it, Li. Well done"
3. One partition along the diameter and three perpendicular partitions, creating a grid-like pattern	3 e.g., from Jo		"That's a good try, Jo. Did you see Li's? Try and make it look like that. Okay? Maybe roll it back together and start again. But good try"
4. A combination of strategies that did not result in eighths	2 e.g., from Sam		Fiona did not respond

[α]Pseudonyms used for all children.

A teacher's ePCK might prompt a response to the unequally sized pieces, however Fiona made no mention of this important issue of partitioning with Li. The one instance of Fiona prompting a child to correct their representation directly was in response to Jo who used Strategy 3. Here, Fiona directs them to copy Li's model, despite its unequal partitions. Interpreting Fiona's ePCK in this episode, her own understanding of partitioning is poor (e.g., endorsing Li's model as appropriate). Fiona later addressed the idea of equality of parts with the whole class but made no mention of the children's models. Fiona partitioned a circle on the interactive whiteboard into eight roughly equal parts (using Strategy 1) and asked children to confirm that their models resembled the displayed image, noting she "saw some people did it another way, but we need to make sure that they're getting the same amount of cake, don't we?" She demonstrated the equality of the slices of cake but did not capitalise on specific examples of unequal partitioning that more than half the children produced (Strategies 2–4) and thus showed limited ePCK of the common misconception(s) around equivalency. Strategy 2 highlights the difficulty of using the central angle to preserve the equality of its parts, where Strategy 3, the grid pattern, could be demonstrated to show that these horizontal and vertical partitions did not represent eighths as the pieces produced were not equal. Modelling an example of Strategy 2 and/or 3 demands pPCK of the problems associated with these methods of partitioning a circular area model.

Fiona's ePCK in Lesson Reflections During Interview 3

Fiona described the teaching as a "real[ly] basic lesson" whose motivation was to "find out how much they knew". Although Fiona reasoned that the children were already "down with sort of halves, and some of them could kind of understand the concept of quarters", the concept of eighths was new to them and she had not "necessarily planned to do the eighth part" but spontaneously included it "because some of them were really getting it". Citing the tutorials and assignments as inspiring the implementation of the area models chosen for her lesson, Fiona's pPCK had clearly been informed by the ITE program. Those teaching and learning activities "helped because I learned sort of all this stuff and why [the fractions] make sense, and why … that helps before you're doing the symbolic stuff". When asked to provide a rationale for her choice of fraction representations, Fiona drew on more *general pedagogical* knowledge rather than PCK, stating the playdough was "fun", "concrete", and "hands-on", wanting children to "see sort of the number of people so that they had a reference … so there was a visual to look at". Further, Fiona did not believe teaching this lesson had developed her own PCK as it was "so simple anyway, I kind of got it anyway". However, she added "maybe if I had had to do something more difficult I wouldn't have been like 'oh my god'. And I would have had to really make sure I was clear on what I was doing before I went into those lessons". Fiona is hinting at her awareness of the link between pPCK as developed in the ITE program and enacting the knowledge during teaching episodes.

Knowledge for Teaching Fractions: Children's Misconceptions as cPCK

Fiona noted some specific aspects of cPCK that enabled her to feel more confident for teaching fractions such as being "prepared before the lesson and know[ing] all the types of areas that kids might go wrong" (Int. 4), citing the most helpful source of this knowledge as "the assignments about different misconceptions, like conceptual things" (Int. 4). The ITE program explored cPCK of fractions specifically including research about the potential difficulties that circular fraction models present for children (Gould, 2005). Drawing on this *concept-specific* cPCK, Fiona claimed awareness of the problems in partitioning circular area models and asserted that course content outlined children might partition circles in a grid-like pattern:

> It was funny to see because we did use circles, it sounds weird, but it's just like what they said in my maths textbook about how they, rather than cutting like a pizza, they would go to do it like that [a grid] into those weird square shapes out of the circle. … In the lecture or something, they said if you used a circle that would happen. (Int. 3)

However, being aware of difficulties that children were likely to experience with the circular area model (cPCK) did not prevent Fiona from employing it in her lesson (ePCK). When asked later in the same interview whether the children had difficulties with any of the representations, Fiona responded "Not really. … I think because I used circles and those basic kind of shapes" (Int. 3). Fiona believed she had addressed the children who had used the grid strategy to partition circles, "because it was playdough we could pick up and move the pieces, so it was kind of like we need to look at this piece and that piece and when you cut it that way, they're not the same size, are they? So yeah, from there they just rolled it and cut it out again" (Int. 3). In the *reflect* phase of ePCK, Fiona identifies the problem with the grid strategy as not representing eighths as equivalent pieces. Yet in the teaching episode, Fiona's ePCK was visible in directing a child holding the misconception to copy another inappropriate representation. Here, Fiona's reasoning in the teaching moment, *reflection in action*, is not as robust as her reasoning about the episode afterwards, *reflection on action*, despite being part of her ePCK. Fiona's case shows the complexity of ePCK and the relationships between the reasoning that occurs during the lesson, reflection *in* action, and the reasoning before (*plan* phase) and after (*reflection* phase) the lesson, reflection *on* action (Carlson & Daehler, 2019). Fiona had only planned to address the concept of quarters with her class (*plan* phase) yet adapted the lesson while teaching for children who she considered to be ready for eighths (*teach* phase). Here, there is potential for the *plan* and *teach* phases of Fiona's ePCK to be more strongly integrated.

Discussion and Implications

Fiona's case illustrates the utility of the RCM of PCK to show the complexities in PST knowledge *of* and *for* teaching fractions. The case showed that even when there is some evidence of cPCK as part of a teacher's pPCK (in this case, being aware of the difficulties a particular fraction representation might cause children), this may not be held in ways that adequately inform the *teach* phase of ePCK to enable a robust response to children's misconceptions. Knowing *about* children's potential difficulties can be insufficient as knowledge to *address* the misconceptions. This highlights the importance of PST having pPCK not just of likely difficulties of children but also the strategies to address these, probably best drawn from cPCK. Although anticipating children's responses should mean teachers notice and attend to children's responses when teaching (Vale et al., 2019), teachers also need to develop pPCK of ways *to address* children's misconceptions. The case of Fiona suggests there could be filters and/or amplifiers at work between a teacher's pPCK and their ePCK in a specific teaching moment. Behling et al. (2022) found that "the worse preservice teachers' knowledge-based reasoning, the smaller the transformation from pPCK to ePCK" (p. 592). Many PST do not emphasise the equality of the parts when explaining the part-whole concept in fractions (Magdaş et al., 2023) and their representations are often limited to rectangular and circular area models when demonstrating this relationship (Castro-Rodríguez et al., 2016). Responding to children's mathematics demands teacher pPCK comprising robust *conceptual knowledge* (Kahan et al., 2003) but disconnections in the *plan-teach-reflect* phases of ePCK were highlighted when Fiona spontaneously introduced eighths with circle models. It is possible that further developing Fiona's pPCK with more specific cPCK about children's misconceptions about fraction representations could inform the *plan* phase of ePCK, allowing her to prepare teaching moves to address children's understanding of the concept of equal parts more directly in the moment. In the *reflect* phase of ePCK, Fiona drew from the *concept-specific* cPCK to subsequently identify an appropriate response to the child regarding the need for size equivalency. Her post-lesson reflection was thus a critical element of her developing pPCK consistent with Carlson and Daehler (2019), "the insight a teacher takes away from each interaction with students further informs the teacher's pPCK" (p. 85). Analysis in this paper of the interactions among pPCK, cPCK, and ePCK as illustrated in the teaching episode in the case of Fiona demonstrate the value of such analysis and the utility of the RCM of PCK as a framework for unpacking the complexity of teaching and learning. We argue particularly that this small case study illustrates the importance of the *reflect* phase of ePCK during ITE and points to the need in ITE programs to enable this reflective work at key points of the program.

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References

Abell, S. K. (2007). Research on science teacher knowledge. In S. K. Abell & N. G. Lederman (Eds.), *Handbook of research on science education* (pp. 1105–1149). Lawrence Erlbaum.

Amador, J. M., Rogers, M. A. P., Hudson, R., Phillips, A., Carter, I., Galindo, E., & Akerson, V. L. (2022). Novice teachers' pedagogical content knowledge for planning and implementing mathematics and science lessons. *Teaching and Teacher Education*, *115*, 103736.

Ball, D. L., Thames, M. H., & Phelps, G. C. (2008). Content knowledge for teaching: What makes it so special? *Journal of Teacher Education*, *59*(5), 389–407.

Behling, F., Förtsch, C., & Neuhaus, B. J. (2022). The refined consensus model of pedagogical content knowledge (PCK): Detecting filters between the realms of PCK. *Education Sciences*, *12*(9), 1–21.

Berry, A., Friedrichsen, P. J., & Loughran, J. (2015). *Re-examining pedagogical content knowledge in science education*. Routledge.

Botha, H., Coetzee, C., & Zweers, L. (2023). Teaching measurement: The role of mathematics teachers' enacted PCK on gain in learner outcomes. *Journal of Mathematics Education*, *12*(2), 307–322.

Campbell, P. F., Nishio, M., Smith, T. M., Clark, L. M., Conant, D. L., Rust, A. H., Neumayer DePiper, J., Frank, T. J., Griffin, M. J., & Choi, Y. (2014). The relationship between teachers' mathematical content and pedagogical knowledge, teachers' perceptions, and student achievement. *Journal for Research in Mathematics Education*, *45*(4), 419–459. https://doi.org/10.5951/jresematheduc.45.4.0419

Carlson, J., & Daehler, K. (2019). The refined consensus model of pedagogical content knowledge in science education. In A. Hume, R. Cooper, & A. Borowski (Eds.), *Repositioning pedagogical content knowledge in teachers' professional knowledge for teaching science* (pp. 77–94). Springer.

Castro-Rodríguez, E., Pitta-Pantazi, D., Rico, L., & Gómez, P. (2016). Prospective teachers' understanding of the multiplicative part-whole relationship of fraction. *Educational Studies in Mathematics*, *92*(1), 129–146. https://doi.org/10.1007/s10649-015-9673-4

Gould, P. (2005). Really broken numbers. *Australian Primary Mathematics Classroom*, *10*(3), 4–10.

Johnson, K., & Olanoff, D. (2020). Using transformative learning theory to help prospective teachers learn mathematics that they already "know". *Mathematics Enthusiast*, *17*(2–3), 725–769.

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education*, *6*(3), 223–252.

Magdaş, I., Henry, J., & Magdaş, A. (2023). A concept inventory to identify fractions misconceptions among prospective primary and preschool teachers in Romania. *Acta Didactica Napocensia*, *16*(1), 1–11.

Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in the US. *Mathematics Teacher Education and Development*, *3*(1), 28–36.

Mientus, L., Hume, A., Wulff, P., Meiners, A., & Borowski, A. (2022). Modelling STEM teachers' pedagogical content knowledge in the framework of the refined consensus model: A systematic literature review. *Education Sciences*, *12*(6), 1–25.

Olanoff, D., Lo, J.-J., & Tobias, J. M. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on fractions. *The Mathematics Enthusiast*, *11*(2), 267–310.

Roesslein, R. I., & Codding, R. S. (2019). Fraction interventions for struggling elementary math learners: A review of the literature. *Psychology in the Schools*, *56*(3), 413–432. https://doi.org/10.1002/pits.22196

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14. https://doi.org/10.2307/1175860

Shulman, L. S., & Grossman, P. (1988). *Knowledge growth in teaching: A final report to the Spencer Foundation*. Stanford University.

Stake, R. E. (1995). *The art of case study research*. Sage.

Thanheiser, E., Browning, C., Edson, A. J., Lo, J. J., Whitacre, I., Olanoff, D., & Morton, C. (2014). Prospective elementary mathematics teacher content knowledge: What do we know, what do we not know, and where do we go? *Mathematics Enthusiast*, *11*(2), 433–448.

Thurtell, E., Forrester, T., & Chinnappan, M. (2019). Building conceptual knowledge of fraction operations among pre-service teachers: Effect of a representation-based teaching approach within a teacher education program. *Mathematics Teacher Education and Development*, *21*(1), 100–124.

Vale, C., Widjaja, W., Doig, B., & Groves, S. (2019). Anticipating students' reasoning and planning prompts in structured problem-solving lessons. *Mathematics Education Research Journal*, *31*(1), 1–25.

Van Steenbrugge, H., Lesage, E., Valcke, M., & Desoete, A. (2014). Preservice elementary school teachers' knowledge of fractions: A mirror of students' knowledge? *Journal of Curriculum Studies*, *46*(1), 138–161.

Vula, E., & Kingji-Kastrati, J. (2018). Pre-service teacher procedural and conceptual knowledge of fractions. In G. Stylianides & K. Hino (Eds.), *Research advances in the mathematical education of pre-service elementary teachers. ICME-13 Monograph*s (pp. 111–123). Cham: Springer.

Student Problem-Posing During Open Mathematical Inquiry

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Problems presented in mathematics classrooms often focus on routine tasks with students practicing mathematical techniques demonstrated by the teacher. However, this does not reflect the problem-solving process in the real world, and students often find it difficult to connect school mathematics with authentic contexts. This paper discusses findings from a study of Year 5 student perceptions of problem-posing during a two-week open mathematical inquiry. While the semi-structured interviews indicated the students perceived themselves to be skilled at problem-posing, triangulation of the video observations and work samples told a different story.

Inquiry-based learning (IBL) in mathematics has gained prominence in Australian educational policy and curriculum as policy makers and educators seek to increase student engagement and achievement in mathematics. For example, an increase in available educator resources such as The Australian Academy of Science's Mathematics ReSolve project (Australian Academy of Science, 2023), coupled with a growing number of presentations focusing on inquiry approaches at Mathematics Education Research Group of Australasia (MERGA) conferences (e.g., Fielding et al., 2023; Wadham et al., 2023), suggests an increased interest in applying this approach in classrooms. The need to provide students with opportunities to transfer their mathematics learning to authentic situations and to connect the 'real world' to the 'mathematics classroom' is imperative and has been noted in the current version of the Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2023a). One way to achieve this is through open mathematical inquiry, which engages students in problem posing; indeed, an aim of the Australian curriculum is for students to:

> Develop proficiency with mathematical concepts, skills, procedures and processes, and use them to demonstrate mastery in mathematics as they pose and solve problems, and reason with number, algebra, measurement, space, statistics and probability. (ACARA 2023b, para. 2)

Inquiry Approaches and The Australian Curriculum

Bruder and Prescott (2013) discuss different studies that examined IBL in both science and mathematics, highlighting three types of inquiry:

> *Structured Inquiry:* The teacher gives the students a problem or question to be solved as well as the appropriate method and materials to solve it.
>
> *Guided Inquiry:* The teacher provides the students with the problems or questions and the necessary materials. The students' task is to find the appropriate problem-solving strategies and methods. The teacher guides students through the problem-solving process.
>
> *Open Inquiry:* The students' task is to find problems or questions they would like to solve and answer. They also decide upon the methods and materials they would like to use. (p. 812)

Makar's (2012) approach to guided mathematical inquiry is slightly different to what is described by Bruder and Prescott (2013) as, in her approach, students are presented with a well-considered, ill-structured question, and are supported by the teacher to identify and define important parts of the question and to determine what evidence is needed for them to answer the question. Much of the more recent research in Australia regarding inquiry-based pedagogy

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in mathematics has focussed on guided mathematical inquiry (e.g., Fielding et al., 2023; Fry & English, 2023), with little classroom-based research related to open mathematical inquiry, which was the approach that the students in this study undertook.

During open inquiry, students must devise questions based on a stimulus, locate additional information to answer their questions, and make decisions regarding the methods or materials they want to use. While having students formulate and pose their own problems is not a new phenomenon, it is less commonly used by teachers than other, non-inquiry based pedagogical approaches. Problems presented in mathematics classrooms are often presented by the teacher, and commonly focus on students finding a particular answer in a prescribed way, or practicing a mathematical technique demonstrated by the teacher through routine exercises (Schoenfeld, 2017).

In addition to being noted in the aims of curriculum, connections to problem-posing are also evident in the proficiency strands (ACARA, 2023c) and under Mathematical Processes: Mathematical modelling (ACARA, 2023a). The curriculum requires students from Year 3 onwards to use mathematical modelling to solve practical problems, formulate problems, and choose operations and strategies, as well as interpret and communicate solutions (ACARA, 2023d). We argue that open mathematical inquiry meets these curriculum requirements as, during open mathematical inquiry, the students must identify practical problems in situations, mathematise their ideas, solve the problem, and share their solutions.

Singer et al. (2015) argue that support of problem-posing in school mathematics can be considered from at least two vantage points, the first historical, demonstrating that problem-posing is an agent of change, and the second futuristic, with the knowledge economy and increasing societal pressure on educational systems. While both perspectives are beneficial, and the policy and curriculum interest in problem-posing is strong, researchers are still developing an understanding of the effects of problem-posing activities on engagement and achievement, as well as conceptualising the instructional strategies that support their effective implementation (Cai et al., 2015). Although the understanding of these processes may be somewhat limited, researchers have found that "problem-posing can enhance learners' mathematics conceptual understanding, dispositions on mathematics, and mathematical creative thinking" (Xie & Masingila, 2017, p. 102). Furthermore, research has shown that open mathematical tasks promote engagement in mathematical thinking, communicating, problematising, and creativity (Attard, 2013; Xie & Masingila, 2017). These claims suggest that research on open mathematical inquiry is needed to support teachers in providing students with authentic problem posing and problem-solving opportunities.

This paper reports on one aspect of a study (Zorn et al., 2022) that sought to understand the ability of a group of Year 5 students to problem-pose by investigating their perceptions of engagement and ability to create their own mathematics investigations based on a video stimulus. The research question for this paper is How do Year 5 students perceive their ability to problem-pose using a video prompt as stimulus during an IBL mathematical problem-posing investigation?

Method

The qualitative findings reported in this paper are taken from a single instrumental case-study. Case-study methodology provides a framework to investigate various participant perspectives and discover patterns, relationships, and themes (Holosko & Thyer, 2011). A single instrumental case-study was appropriate as this study was conducted with one class (n = 17) and findings were developed based on the student perspectives from that one 'bounded' case. The first author led a two-week open mathematical investigation that required students to develop their own investigation questions based on a video stimulus centred on a tennis theme.

Seventeen students worked in collaborative pairs or groups of three to investigate their own questions and present their findings.

Data Collection and Analysis

Semi-structured interviews that were held with students to identify their perceptions of their own problem-posing ability after the open mathematical inquiry were triangulated with video observations and student work samples. The semi-structured interviews were recorded, transcribed, and analysed by the researcher, and this process allowed for multiple exposures to the data. Thematic analysis was used to identify, analyse, and interpret patterns as it provided an accessible and systematic way for generating codes and themes from the qualitative data (Clarke & Braun, 2017). Following thematic analysis, the video observations, student work samples, and literature related to the phenomenon, were used to triangulate and substantiate the findings.

Participants

The research was conducted at a co-education, independent school in South-East Queensland, Australia. The participating class consisted of 17 students and one teacher. Due to student absence, only 16 out of 17 students participated in a semi-structured interview. The school delivered the Australian curriculum through individualised pathways and students were engaged in a range of pedagogies throughout each day. From Year 5 the students learnt mathematics through 'Maths Pathways' as the school leadership believed it supported teachers to deliver personalised learning. Students engaged in online tutorials and questions based on their readiness and took fortnightly paper tests to identify growth and areas for improvement. As part of the program students also engaged in rich tasks and projects; however, the regularity of this was determined by each classroom teacher. Students had some experience with IBL pedagogy, and regularly posed questions for personal investigations in other curriculum areas; however, this was not done in mathematics.

Findings and Discussion

The data from the semi-structured interviews revealed that the students felt confident to problem-pose and develop their own mathematical investigations. When asked whether they found it challenging to create their own investigation and questions based on the video stimulus, 12 students reported that they did not find it challenging, two reported that it was slightly challenging, one student did not respond to the question, and one student's response was inaudible due to the student mumbling. Many of the student's responses were simple and direct, with minimal or no elaboration to explain whether it was initially challenging for them to pose questions to investigate. For example, responses included, 'no, not really', 'nah, not actually', 'a bit'.

However, triangulation of the interview data, video observations, and student work samples indicated that the students' perceptions of their ability were not aligned with their actual ability to independently problem-pose. The video observations and work samples revealed that the students had difficulty problem-posing and that they required teacher support to make connections between the video and mathematics and to mathematise their ideas (Zorn, 2022).

In the beginning of the unit, after watching the video stimulus, the students engaged in the Visible Thinking Routine—'see, think, wonder' (Lowe et al., 2013). The students were specifically encouraged to pose mathematical questions during the 'wonder' phase and were required to post their questions to a shared Padlet. While the students recorded 54 questions, only four were linked to mathematical concepts (e.g., area, perimeter, and money) for investigation in the classroom. This revealed how challenging it was for the students to connect mathematics to the stimulus. Most of the questions focussed on the emotions of the people in the video or were closed, subjective questions (see Table 1).

Table 1

Student Question Examples

Mathematical questions	Examples of non-mathematical questions
What is the total perimeter of an average tennis court?	I wonder if they play it professionally?
How big is a tennis court?	How are they feeling?
How much does it cost to buy a tennis court?	Do they get worried before a tournament?
I wonder how much money they earn?	Is there a certain technique to playing tennis?

In addition to the 54 questions, two students explicitly questioned the connection to mathematics. For example, Nick wrote, “How does tennis relate to math?” A later comment from Nick in the semi-structured interview suggested a shift in his thinking during the two-week investigation:

> Well, it was kind of out of the blue when you showed us the tennis video. Also, I thought how can this relate to maths. And when we actually go into it, I realise how like, how much maths is involved in everything.

Other comments described the impact of scaffolding during the investigation. The scaffolding helped the students to connect mathematics to the ‘real-life’ situation presented in the video, and opened their eyes to a world that is full of mathematics:

> They [teachers] should have taught us that pretty much everything can be math. (Matt)

> Yeah, how you wouldn’t think it would be that involved in math, but it is and you have to use your brain in a way that you wouldn’t think to use it. And yeah, that was just really cool to experience. (Milly)

These findings are consistent with the findings of Singer et al. (2015) who found that it was challenging for students to problem-pose, and that students need scaffolding to mathematise questions and ideas. In response to their problem-posing efforts, and to help the students in this study to see the mathematics in their daily life and in the mathematics in the video stimulus, Math Curse, a picture book by Jon Scieszka and Lane Smith (1995), was read to the class. The book follows a boy through his week and, as the story unfolds, he realises that he is surrounded by mathematics problems everywhere he goes. Table 2 illustrates a sample of the ideas as they developed through the two weeks.

Table 2

Developing Ideas

Initial question/idea	Developing questions
Are they doing [sic] a tournament?	Are there winners, how much prize money? How many courts and how much space is needed? Hitting the ball—what angles are involved? How many seats on each court and price per seat?
Is Roger Federer the best tennis player of all time?	How long has he played for? What are his scores and points? Who is the 1st, 2nd, 3rd and how do people vote?
I wonder if we could have a tennis court at school?	What area would we need and how long to build? How much money would it cost?
How is a tennis racket made?	What is the cost of the materials? The size—measurements, different sized rackets

Although their ideas developed during the two weeks, and their questions became more mathematically focussed, the work samples and video observations demonstrated that some groups still found it difficult to mathematise their investigation. For example, one group decided to work on ‘ticket prices’ and started to create a list with different types of tickets, randomly selecting a dollar figure that they felt was appropriate. After conferencing with the teacher, these students changed their approach and began researching the costs of hiring a local tennis court, umpires, and commentators for their tournament. They then used the information to calculate total costs for the tournament, and then worked out ticket prices based on covering expenses and making a profit. In doing this they also made decisions about the number of players, the number of games, and the number of courts that would be needed. During their class presentation, at the end of the two weeks, these students noted that they had started to create ticket prices but “it didn’t work out so well, so we had to restart.” This indicated that they were aware of the challenges they faced in formulating questions and solving the problem, and contradicted their perceptions shared during the interview that problem-posing was easy.

It was beyond the scope of this project to fully understand why some students felt confident to problem-pose even though the evidence indicated they needed scaffolding and support. However, the kind of teacher support provided may have contributed to their confidence. It was observed that the students were able to maintain a sense of autonomy and competence, resulting in high self-efficacy. Self-efficacy has been found to increase during IBL (Dunlap, 2005); however, the degree to which it is influenced by teacher involvement is unclear (Tawfik et al., 2020). The student-centred environment created in this study encouraged students to be active participants in their learning, with teachers often adopting more of a facilitative role, using skilful questioning, scaffolding thinking, and avoiding the temptation to quickly impart knowledge. To illustrate this point, the following student comments, shared during semi-structured interviews, articulate the level of support provided by the teacher:

> You and Miss E checked around and looked if we had any troubles or not. And if we did, you guys would sit down and help us, and if we didn’t, you guys would just walk off, and be like, they don’t need any help, they’re fine and go see if anybody else needs help. (Koby)

> Just looking over us just to make sure that we’ve got the right criteria of what we needed to accomplish. But other than that, we were mainly doing it ourselves and that’s what I like. (Freya)

The idea of the teacher as a facilitator is often associated with student-centred learning (Goodyear & Dudley, 2015), and is not a new phenomenon; however, it remains a common misconception of open mathematical inquiry, and other constructivist, student-centred approaches, that the teacher just creates a task and leaves the students to work together to learn (Goodyear & Dudley, 2015). That said, there does need to be a balance between active teacher involvement and student-directed learning, and it is essential that teachers provide support in autonomy-supportive ways.

Cheon et al. (2020) used self-determination theory principles to develop a teacher intervention, which supports teachers in creating an autonomy-supportive classroom environment. They found that when teachers allow students to work in their own way, at their own pace, provide explanatory rationales, and use invitational language, that these actions support students to meet their autonomy need satisfaction. This was further supported by providing structure through clearly communicating expectations, scaffolding progress, offering help, and providing constructive feedback that allowed students to experience self-confidence and competence need satisfaction.

In the semi-structured interviews, when the students were asked if they could pose an additional problem to investigate, all the students responded positively with the belief that they could do so. However, most of the proposed ideas reflected the ideas of their peers that were shared at the end of the two-week period. For example, Matt thought they would “probably [plan] a tournament”, and Nick agreed that “scheduling a tournament and building it up” was a good idea. While their ideas were investigable topics, they were all previously presented to the class by their peers, lacked originality and were not mathematical questions, suggesting that the students may not be able to problem-pose as well as they perceived they could. It would have been beneficial to provide the students with a second video stimulus to develop insights into their evolving ability to problem-pose as it was difficult for them to move beyond their peers' ideas. Other ideas suggested by the students also indicated a contradiction between the students' perceived and actual ability to problem-pose. At the end of the inquiry, the students were asked if they could pose an additional problem related to the original prompt. Gemma and Eric's additional problems demonstrated a limited understanding of mathematical concepts and processes, and/or a limited understanding of the idea of mathematical problem-posing:

> I'd probably come up with something like, what top 10 players would like, put into an even tennis match or something. But it's like, it's not like you're going to have someone like, David Smith or something against like Roger Federer, because that'd be extremely unbalanced. (Eric)

> Writing a letter to the school to try and build one. Once we are done that, if they let us build it, we could host a tournament. (Gemma)

Eric's idea lacked logic or an understanding of the task, and did not relate to mathematics, rather it focussed on something that Eric found interesting. Gemma's response focussed on what she'd like to do next in relation to her initial investigation rather than an additional mathematics question.

Conclusion

This paper focussed on problem-posing during an open mathematical inquiry. The findings indicated that while the students perceived themselves to be able to problem-pose effectively, the student work samples and video observations contradicted this. During the study the students reported experiencing high levels of autonomy and competence need satisfaction (Zorn et al., 2022) and this may have contributed to high levels of self-reported efficacy, confidence, and ability to problem-pose (Zorn, 2022). Confidence and self-efficacy have been linked to student growth and positive academic outcomes. Students experiencing these positive emotions are more likely to take academic risks, increasing the likelihood of improving their problem-posing abilities. This reinforces the need to create an environment that ensures students feel supported and safe to work autonomously and feel competent in doing so.

The challenge with problem-posing in this study possibly reflects the fact that this was the first time the students had been exposed to the idea of problem-posing or creating their own mathematical investigations. The support they were given assisted the students to develop their initial investigation topics and questions, and it is unknown whether they would be able to work more independently to pose problems that could be solved using mathematics with a new video stimulus. It was beyond the scope of this study to provide the students with an additional stimulus to evaluate whether this initial experience of problem-posing would support them to develop problems based on new video stimulus.

Although some suggestions have been made regarding why the students may have self-reported feeling confident to problem-pose, future research could involve a longer-term study investigating what teacher practices support students to problem-pose effectively, and how teacher practices change over time to support student engagement and achievement during problem-posing.

Acknowledgments

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References

Attard, C. (2013). "If I had to pick any subject, it wouldn't be maths": Foundations for engagement with mathematics during the middle years. *Mathematics Education Research Journal*, *25*(4), 569–587. https://doi.org/10.1007/s13394-013-0081-8

Australian Academy of Science. (2023). *Teaching resources. reSolve: Maths by inquiry*. Australian Academy of Science. https://resolve.edu.au/teaching-resources

Australian Curriculum Assessment Reporting Authority (ACARA). (2023a). *Understanding this learning area: Mathematics: Mathematical processes: Mathematical modelling*. ACARA. https://v9.australiancurriculum.edu.au/teacher-resources/understand-this-learning-area/mathematics

Australian Curriculum Assessment Reporting Authority (ACARA). (2023b). *Understanding this learning area: Mathematics: Aims*. ACARA. https://v9.australiancurriculum.edu.au/teacher-resources/understand-this-learning-area/mathematics

Australian Curriculum Assessment Reporting Authority (ACARA). (2023c). *Understanding this learning area: Mathematics: Proficiencies: Problem-solving*. ACARA. https://v9.australiancurriculum.edu.au/teacher-resources/understand-this-learning-area/mathematics

Australian Curriculum Assessment Reporting Authority (ACARA). (2023d). *Foundation to year 10 curriculum*. ACARA. https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics

Bruder, R., & Prescott, A. (2013). Research evidence on the benefits of IBL. *ZDM Mathematics Education, 45*(6), 811–822. https://doi.org/10.1007/s11858-013-0542-2

Cai, J., Hwang, S., Jian, C. & Silber, S. (2015). Problem-posing research in mathematics education: Some answered and unanswered questions. In F. M. Singer, N. F. Ellerton & J. Cai., *Mathematical problem posing: From research to effective practice* (pp. 3–34). New York: Springer. https://doi.org/10.1007/978-1-4614-6258-3

Cheon, S. H., Reeve, J., & Vansteenkiste, M. (2020). When teachers learn how to provide classroom structure in an autonomy-supportive way: Benefits to teachers and their students. *Teaching and Teacher Education*, *90*, 103004. https://doi.org/10.1016/j.tate.2019.103004

Clarke. V., & Braun. V. (2017). Thematic analysis. *The Journal of Positive Psychology*, *12*(3), 297–298.

Dunlap, J. C. (2005). Problem-based learning and self-efficacy: How a capstone course prepares students for a profession. *Educational Technology Research and Development*, *53,* 65–85. https://doi.org/10.1007/BF02504858

Fielding, J., Adeniji, S., & Baker, P. (2023). Throw away the script: Examining the introduction of a guided mathematical inquiry unit. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 27–35). Newcastle: MERGA.

Fry, K. & English, L. D. (2023). How big is a leaf? Mathematical modelling through STEM inquiry. *Mathematics Teacher: Learning and Teaching PK–12*, *116*(2), 99–107.

Goodyear, V., & Dudley, D. (2015). "I'm a Facilitator of Learning!" Understanding what teachers and students do within student-centered physical education models. *Quest*, *67*(3), 274–289. https://doi.org/10.1080/00336297.2015.1051236

Holosko, M. J. & Thyer, B. A. (2011). *Pocket glossary for commonly used research terms*. Thousand Oaks: SAGE Publications.

Lowe, G. M., Prout, P., & Murcia, K. (2013). I see, I think, I wonder: An evaluation of journaling as a critical reflective practice tool for aiding teachers in challenging or confronting contexts. *Australian Journal of Teacher Education*, *38*(6). https://doi.org/10.14221/ajte.2013v38n6.6

Makar, K. (2012). The pedagogy of mathematical inquiry. In R. Gillies (Ed.), *Pedagogy: New developments in the learning sciences* (pp. 371–397). New York: Nova Science.

Schoenfeld, A. H. (2017). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics [Reprint]. *Journal of Education*, *196*(2), 1–38. doi:10.1177/002205741619600202

Scieszka, J. & Smith, L. (1995). *Math curse*. Viking Press.

Singer, F. M., Ellerton, N., & Cai, J. (Eds.). (2015). *Mathematical problem posing: From research to effective practice*. New York, NY: Springer.

Tawfik, A. A., Hung, W., & Giabbanelli, P. J. (2020). Comparing how different inquiry-based approaches impact learning outcomes. *The Interdisciplinary Journal of Problem-Based Learning*, *14*(1). https://doi.org/10.14434/ijpbl.v14i1.28624

Wadham, B., Pearce, E., & Hunter, J. (2023). Learning to notice algebraically: The designed instructional material on student thinking. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 27–35). Newcastle: MERGA.

Xie, J., & Masingila, J. O. (2017). Examining interactions between problem posing and problem solving with prospective primary teachers: A case of using fractions. *Educational Studies in Mathematics, 96*(1), 101–118. https://doi.org/10.1007/s10649-017-9760-9

Zorn, K. (2022). *Student perceptions of engagement and problem-posing during an inquiry-based learning mathematical investigation.* [Master's thesis]. Griffith University.

Zorn, K., Larkin, K., & Grootenboer, P. (2022). Student perspectives of engagement in mathematics. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys. Proceedings of the 44th annual conference of the Mathematics Education Research Group of Australasia*, July 3–7. (pp. 570–577). Launceston: MERGA.

Round Tables

Supporting Out-of-Field Secondary Mathematics Teaching in NSW: A Multifaceted Project Design

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Understanding the extent of out-of-field (OOF) secondary mathematics teaching in department schools in NSW is key to addressing teacher shortages and designing professional learning (PL) support to ensure students have opportunities to develop deep learning in mathematics (Shah et al., 2020). Leveraging existing datasets as well as collecting new survey and interview data from 118 current in-field and 96 OOF mathematics teachers, 100 head teachers and other school leaders in 48 schools has provided initial data to inform PL design. PISA data from 2275 teachers revealed 20% of teachers who were teaching grade 10 mathematics in NSW in 2015 were teaching OOF—they were more likely to be women, younger, on fixed-term contracts, working part-time, and have less than three years' experience (Watt et al., 2023). Preliminary new survey data analyses reveal that OOF teaching appears slightly more pronounced in smaller schools, in lower socioeconomic areas, in schools with fewer resources, and lower parental participation. Interview data from fifteen case-study schools has provided in-depth information about schools' approaches to supporting OOF mathematics teaching including new models of curriculum design, new approaches to recruiting and sustaining their mathematics teacher workforce, particularly in rural and regional areas, where staffing shortages are most acute. Designing PL that meets OOF mathematics teachers needs is challenging given the diversity of the cohort and the variety of contexts within which they work (Vale et al., 2021). While we are aware OOF mathematics teaching can occur in all secondary grades and at all levels of mathematics learning, we are focusing our PL provisions on teaching of grades 7 and 8 classes where most OOF mathematics teaching currently occurs.

This roundtable provides opportunities to explore findings from this Department of Education funded project. Other education jurisdictions and tertiary providers have offered programs to address OOF mathematics teaching (Barker et al., 2022). We anticipate the discussion will enable participants to share their experiences and insights to enrich our project.

References

Barker, M., Goos, M., & Coupland, M. (2024). *Relieving out-of-field teaching in Australian secondary mathematics: Analysis of out-of-field secondary mathematics teacher upskilling initiatives in Australia*. Australian Mathematical Sciences Institute.

Shah, C., Richardson, P., Watt, H. (2020). Teaching 'out of field' in STEM subjects in Australia: Evidence from PISA 2015. *GLO Discussion Paper, No. 511* [rev.], Global Labor Organization, Essen. http://hdl.handle.net/10419/217484

Vale, C., Campbell, C., & White, P. (2021). Beliefs and practices of secondary teachers crossing subject boundaries to teach mathematics out-of-field. *Mathematics Education Research Journal, 33*, 589–612.

Watt, H. M. G. et al. (2023, November 26–30). Teaching mathematics out-of-field: Who, where, why, to whom, and does it matter [Conference paper]? *The annual conference of the Australian Association for Research in Education*, Melbourne.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 561). Gold Coast: MERGA.

If Not With Fennema in the 1970s, Then When?

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It has generally been considered that Elizabeth Fennema was a trail-blazing researcher in the field of gender and mathematics education. Fennema's (1974) literature review encompassed 36 U.S. studies of mathematics achievement from 1960 onwards. She claimed that "such studies were difficult to find because often when sex is used as an independent variable, it is treated and discussed casually, if at all" (p. 127). Fennema's research question was "Are there sex differences in mathematics achievement?" (p. 136), and, by examining this body of work she wanted to determine whether boys' superiority in mathematics learning was a myth or a reality. In the final section of the manuscript, Fennema posed seven questions; these became the springboard for her subsequent research and inspired other researchers in the field.

Not taking anything away from Fennema's (1974) review or her subsequent ground-breaking research in the field, it would appear that there is quite a large body of research prior to 1960 in which gender differences (then termed "sex differences") in mathematics learning were reported. Leder's (1992) review of the literature in the two decades prior makes mention of some of work published before 1960, "Then, as now, gender differences in mathematics learning were attributable glibly by some… to differences in innate abilities whereas others… to differences in … interests and perceptions of future usefulness of the subject" (p. 598).

The narrative in the field of gender and mathematics research typically involves a 'start' date of the 1970s (despite Leder's (1992) references to four studies from before 1960), usually with reference to the Fennema (1974) review or to her subsequent research with Julia Sherman (e.g., Fennema & Sherman, 1977). When searching for articles to add to the IOWME publications page (Hall, 2024), Jennifer stumbled upon an article (Lambert, 1960) about mathematics ability and masculinity. Following back from the references in that article, we continued to find earlier research about gender and mathematics, dating back more than a century ago. As experienced researchers in the field, we were surprised to find so many historical articles of which we had not been aware.

In this session, we will explore the lessons that we learnt from our serendipitous finding of the trove of research on gender and mathematics learning outcomes prior to 1960. In fact, the research stretched back as far as the late 19th century. In light of our experience, we will also highlight the challenges that researchers face in their own fields of expertise in mathematics education, as well as suggesting ways to overcome these challenges.

References

Fennema, E. (1974). Mathematics learning and the sexes: A review. *Journal for Research in Mathematics Education*, *5*(3), 126–139. https://doi.org/10.2307/748949

Fennema, E., & Sherman, J. (1977). Sex-related differences in mathematics achievement: Spatial visualization and affective factors. *American Educational Research Journal*, *14*(1), 51–71. https://doi.org/10.3102/00028312014001051

Hall, J. (2024, April). Gender/sexuality and mathematics publications. International Commission on Mathematical Instruction. https://www.mathunion.org/icmi/organization/affiliate-organizations/iowme/gender/sexuality-and-mathematics-publications

Lambert, P. (1960). Mathematical ability and masculinity. *The Arithmetic Teacher*, *7*(1), 19–21. https://doi.org/10.5951/AT.7.1.0019

Leder, G. C. (1992). Mathematics and gender: Changing perspectives. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 597–622). New York: Macmillan Publishing.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 562). Gold Coast: MERGA.

Arresting the Decline in Secondary School Mathematics Enrolments

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There are substantial and ongoing concerns in the international secondary and tertiary education sectors about the number of students choosing Advanced Mathematics in secondary school. Declining enrolments in the last two years of secondary school Intermediate and Advanced Mathematics is seen as a major concern for the future of STEM education:

> Falling participation in higher maths and physics, especially, is a worry to me…We need to understand why this is happening, and work to turn it around so that Australia remains a science and research powerhouse (Office of the Chief Scientist, 2023).
>
> Student uptake of fundamental subjects in secondary school such as higher mathematics, intermediate mathematics and science subjects has declined nationally over recent decades (Office of the Chief Scientist, 2020).
>
> This (Advanced Maths numbers) is the lowest level recorded in more than twenty years (Australian Mathematical Sciences Institute, 2019).

Nationally, the number of students studying Intermediate and Advanced Mathematics as a percentage of the number of students completing Year 12 of secondary school has declined over the past 25 years. However, the patterns across states and territories vary greatly. Following on from the *Maths? Why Not?* project (McPhan et al., 2008), Jennings (2022, 2023) investigated enrolment trends in Queensland, New South Wales, and Victoria, and also reasons why Queensland students choose to / not to study Advanced Mathematics in the last two years of secondary school.

In this round table, I invite people to discuss strategies for turning around the number of students enrolling in Intermediate and Advanced Mathematics in the last two years of secondary school.

References

Australian Mathematical Sciences Institute. (2019). *Year 12 mathematics participation in Australia 2008–2017*. Australian Mathematical Sciences Institute. https://amsi.org.au/preview-year-12-mathematics-participation-in-australia-2008-2017/

Jennings, M. (2022). Advanced mathematics enrolment numbers: A crisis or fake news? *Australian Mathematics Education Journal, 4*(1), 33–45.

Jennings, M. (2023). Issues in the transition from secondary school to university mathematics: student, teacher, and lecturer perspectives [Doctoral dissertation, The University of Queensland]. https://doi.org/10.14264/a3d9362

McPhan, G., Morony, W., Pegg, J., Cooksey, R., & Lynch, T. (2008). *Maths? Why Not?* https://simerr.une.edu.au/maths-whynot/

Office of the Chief Scientist. (2020). *Mapping university prerequisites in Australia*. Australian Government, Canberra.

Office of the Chief Scientist. (2023, November 28). *Chief Scientist calls on schools to improve science studies*. The Australian. https://www.theaustralian.com.au/nation/chief-scientist-calls-on-schools-to-improve-science-studies/news-story/bc5fa99688ded139426abc8e48ce4571

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 563). Gold Coast: MERGA.

Reflective Encounters Between Primary Pre-Service Teachers and a Mathematics Teacher Educator to Explore Critical Mathematics Teaching Approaches

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Traditional mathematics teaching practices built around standardisation do not result in socially just and equitable outcomes when working with diverse student populations (O'Keeffe & Paige, 2021). Pre-service teachers (PSTs) tend to enter their initial teacher education program having experienced traditional mathematics teaching, and these experiences form the basis for their beliefs and views of mathematics teaching and learning (Ma & Singer-Gabella, 2011). Teacher educators must consider the ways that their practice and positionalities shape the experiences and perspectives of the PSTs with whom they engage. Past experiences and beliefs systems impact the practice of teacher educators (Kalinec-Craig et al., 2019). Leibowitz and Bozalek (2016) indicate that both teacher educators and PSTs need to undergo a process of learning and unlearning and of challenging their practices and assumptions to open up the possibilities for transformation of their practice. Therefore, in the field of primary mathematics education, it is important that both preservice teachers and teacher educators engage in self-study of their beliefs and teaching practices.

This project aims to explore the process of deconstruction and imagining possibilities for social justice in mathematics teaching practice undertaken by a group containing primary PSTs and a mathematics teacher educator. Hitherto, a pilot study has been undertaken through encounters between the researcher and two primary PSTs, with a focus on interaction and dialogue to support reflection, along with the researcher using a reflective journal.

In the session, I aim to share impressions and experiences from the study thus far and open discussion to explore teacher educator experiences of self-study and supporting PSTs to deconstruct and imagine possibilities for social justice in mathematics teaching. In particular, I am seeking guidance from experienced teacher educators to shape the ongoing development of the project.

References

Kalinec-Craig, C., Chao, T., Maldonado, L. A., & Celedón-Pattichis, S. (2019). Reflecting back to move forward: Using a mathematics autobiography to open humanizing learning spaces for pre-service mathematics teachers. In T. G. Bartell, C. Drake, A. R. McDuffie, J. Aguirre, E. E. Turner, & M. Q. Foote (Eds.), *Transforming mathematics teacher education an equity-based approach* (pp. 135–146). Springer, Cham.

Leibowitz, B., & Bozalek, V. (2016). The scholarship of teaching and learning from a social justice perspective. *Teaching in Higher Education*, *21*(2), 109–122. doi:10.1080/13562517.2015.1115971

Ma, J. Y., & Singer-Gabella, M. (2011). Learning to teach in the figured world of reform mathematics: Negotiating new models of identity. *Journal of Teacher Education*, *62*(1), 8–22. doi:10.1177/0022487110378851

O'Keeffe, L., & Paige, K. (2021). Re-highlighting the potential of critical numeracy. *Mathematics Education Research Journal*, *33*(2), 285–299. doi:10.1007/s13394-019-00297-8

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 564). Gold Coast: MERGA.

Threshold Concepts in Primary Mathematics

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Mathematics requires teachers to make decisions daily and often spontaneously about meaningful connections between concepts and instructional activities. Various curricula have attempted to simplify this process, while recognising that teaching is complex and nuanced. Curricula have centred around 'Big Ideas and Understandings' (Randall, 2005), 'Powerful Ideas' (Schwartz, 2008), and 'Key Developmental Understandings' (Simon, 2006), 'Big Ideas in Number' (Siemon, 2012), all endeavour to provide a framework for teachers to facilitate students in accommodating and assimilating new ideas.

This current study uses the Threshold Concept Framework (Meyer, 2003) to identify concepts that are transformative for learners to become mathematical thinkers. These concepts are not necessarily the same as 'core,' 'fundamental', or 'central,' rather they irreversibly alter the way the learner thinks about the subject knowledge. Very often, these concepts are troublesome to learn as they may be counterintuitive, are understood in a particular way, and they connect other concepts across the discipline. This project aims to explore how primary teachers recognise and use these concepts in their everyday classroom teaching.

A literature search revealed minimal evidence of Threshold Concepts having been identified in primary mathematics; therefore, the initial stages of this study used a Delphi survey to identify Candidate Threshold Concepts. A content analysis of the Australian curriculum determined where and how often these concepts occurred. The remaining questions are:

- To what extent do teachers recognise these Threshold Concepts in the curriculum?
- To what extent is the teachers' knowledge of these Threshold Concepts reflected in their pedagogy?

The purpose of the roundtable is to share the Candidate Threshold Concepts and to discuss how these concepts, e.g. partitioning, might be used as threads to pull together a wide and encompassing curriculum in an effective and manageable manner for a generalist primary teacher.

References

Meyer, J. H. F., Land, R. (2003). Threshold concepts and troublesome knowledge 1: Linkages to ways of thinking and practising. In C. Rust (Ed.), *Improving student learning—Ten years on* (pp. 412–424). Oxford: OCSLD, Oxford Brooks University.

Randall, I. C. (2005). Big ideas in mathematics. *Journal of Mathematics Education Leadership,* NCSM Journal, *7*(3), 9–24.

Schwartz, J. E. (2008). *Elementary mathematics pedagogical content knowledge: Powerful ideas for teachers*. Boston: Pearson/Allyn and Bacon Publishers.

Siemon, D., Bleckly, J., & Neal, D. (2012). Working with the big ideas in number and the Australian curriculum: Mathematics. In B. Atweh, M. Goos, R. Jorgensen, & D. Siemon (Eds.), *Engaging the Australian national curriculum: Mathematics—Perspectives from the field* (pp. 19–45). MERGA. https://merga.net.au/common/Uploaded%20files/Publications/Engaging%20the%20Australian%20Curriculum%20Mathematics.pdf

Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8*(4), 359–371.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education*. *Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 565). Gold Coast: MERGA.

Exploring the Culture of Out-of-Field Professional Education for Mathematics Teachers

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The current teacher shortage facing Australia and many other countries is leading to desperate measures to get teachers in front of classes. A common solution has been to assign teachers to teach subjects or levels of schooling for which they are not qualified—that is, to teach out-of-field (OOF) (Hobbs, 2013). Recent data suggests that potentially 40% of Mathematics teachers are teaching OOF. OOF teaching has been linked to low academic performance of students (Van Overschelde, 2022), teacher attrition (Sharplin, 2014), and poor teacher confidence and sense of belonging (Du Plessis, Carroll & Gillies, 2015). Existing upskilling programs, like graduate certificates and micro-credentials, may have limited impact on the phenomenon due to cultural norms that do not recognise or value teacher specialisation (Hobbs et al., 2022).

In this round table, we wish to share insights gained as we work to map the educational ecosystem that creates and supports the OOF phenomena, inviting the audience to contribute to mapping this complex ecosystem. Participants will be asked to share their insights about the out-of-field teaching phenomenon, knowledge of local responses and their opinions as to what might influence teachers to seek professional learning.

References

Du Plessis, A. E., Carroll, A., & Gillies, R. M. (2015). Understanding the lived experiences of novice out-of-field teachers in relation to school leadership practices. *Asia-Pacific Journal of Teacher Education*, *43*(1), 4-21. https://doi.org/10.1080/1359866X.2014.937393

Hobbs, L. (2013). 'Teaching 'out-of-field' as a boundary-crossing event: Factors shaping teacher identity', *International Journal of Science and Mathematics Education*, *11*(2), 271–297.

Hobbs, L., Du Plessis, A. E., Oates, G., Caldis, S., McKnight, L., Vale, C., O'Connor, M., Rochette, E., Watt, H., Weldon, P., Richardson, P., & Bateup, C. (2022). National summit on teaching out-of-field: Synthesis and recommendations for policy, practice and research. https://static1.squarespace.com/static/57f5c6e0414fb53f7ae208fc/t/62b91e6f96a09659f90a3b79/1656299310264/TOOFNational-Summit-Report.doc

Sharplin, E. D. (2014). Reconceptualising out-of-field teaching: experiences of rural teachers in Western Australia. *Educational Research*, *56*(1), 97–110.

Van Overschelde, J. (2022). Value-lost: The hidden cost of teacher misassignment. In L. Hobbs & R. Porsch (Eds.), *Out-of-field teaching across teaching disciplines and contexts* (pp. 71–97). Springer.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 566). Gold Coast: MERGA.

Effective Pedagog(ies) in Mathematics: The Current State of Mathematics Education Practice and Research

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Currently we find ourselves in turbulent waters within the context of mathematics education research and in debates about current policy, curriculum, and practice reform. The debate that pits 'traditional' approaches to teaching mathematics against 'reform-oriented' approaches has recently been making waves, yet these debates are certainly not new. We seek to advance our understanding of *effective pedagogies* to help researchers, policy-makers, and practitioners navigate these waters.

In Australia, the national mathematics curriculum for years F–10 aims to support students to "become confident, proficient and effective users and communicators of mathematics, who can investigate, represent and interpret situations in their personal and work lives, think critically, and make choices as active, engaged, numerate citizens" (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2022). However, defining the kinds of knowledge and skills students need to achieve this goal and the pedagogical approaches that most effectively support their development remain contentious. Important work has begun advancing the debate through clarifying the positions of the two 'sides' of instruction. For example, Munter et al. (2015) adopted *adversarial collaboration* in which experts who held opposing views clarified their arguments about the importance and role of talk and collaboration and the nature of mathematical instructional tasks and representations. Yet we see potential to bridge the divide between these approaches rather than conceptualising them as dichotomous. To this end, there is a need to "contemplate instructional methods within the broader context of instructional goals. It is only in this context that it can be meaningful to do so" (Kuhn, 2007, p. 112). The MERGA Special Working Group for Effective Pedagogies in Mathematics invites you to this round table discussion where we will ask:

- Where are you seeing the tensions about pedagogies of mathematics in your context? For example, do concepts and procedures need to be taught before introducing problem solving tasks? What is the purpose and role of student autonomy?
- What do you see as the purpose(s) of mathematics education and what are the implications for pedagogy (and stakeholders)?
- Where are the research gaps and how do we see our work (currently or potentially) contributing to this field?

References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2022). *Understand this learning area: Mathematics (v9.0)*. ACARA. https://v9.australiancurriculum.edu.au/teacher-resources/understand-this-learning-area/mathematics

Kuhn, D. (2007). Is direct instruction an answer to the right question? *Educational Psychologist*, *42*(2), 109–113. https://doi.org/10.1080/00461520701263376

Munter, C., Stein, M., & Smith, M. (2015). Dialogic and direct instruction: Two distinct models of mathematics instruction and the debate(s) surrounding them. *Teachers College Record: The Voice of Scholarship in Education*, *117*(11). https://doi.org/10.1177/016146811511701102

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 567). Gold Coast: MERGA.

World-Centred Mathematics Education: Theorising with Biesta

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School mathematics continues to suffer from an identity crisis. It is too often portrayed as a collection of facts and calculation skills to be acquired, often by mimicking the procedures demonstrated by the teacher and textbooks, and where correctness and speed matter (Boaler, 2016). This is substantially different form the discipline of mathematics, where specific ways of doing, communicating, and looking at the world brand one as being mathematical, and where criteria of judging success foreground beauty and intricacy of one's ideas. In some ways, one could say, school mathematics remains un-mathematical. The difference remains despite the focused advances in mathematics education research and curriculum shifts which advocate for a better alignment of the school and the disciplinary mathematics (e.g., ACARA, NCTM). Attempts to foreground mathematical competencies and proficiencies at times provide little more than additional processes to be itemised, committed to one's memory, and assessed (e.g., direct teaching of problem solving). Under these circumstances we wonder whether perhaps a different language, or an additional theory grounding, could guide our work of supporting the teaching of mathematics in classrooms, especially if school mathematics that is capable of turning students towards mathematics, rather than away from it, is the goal.

In this round table, we follow work of Gert Biesta and return to some of the core questions, including What is education for (Biesta, 2020, 2022). We contemplate the potential of Biesta's theorising in addressing the problems in mathematics education outlined above. Biesta posits that besides the educational purpose of qualification, which is well attended to in schooling today, there are two additional domains of educational purpose. The purposes of socialisation (e.g., into practices of doing mathematics) align well with Freudenthal's (1971) pointing out the importance of treating mathematics as a human activity when we bring it to classrooms. Finally, the purposes of subjectification are those of awaking/provoking in children the wish/aspiration to be a 'subject' in their own life (and, in a mathematics classroom, to be a subject in their mathematics activities) rather than being an 'object' responding to the requests and desires of others. Biesta associates being a subject (of one's life) with both one's freedom and one's responsibility to make choices. He also points out how being an object brings one a comfort and security of not having to make them. It is in these terms that we would like to look into a mathematics classroom, and into the work of mathematics education researchers who aim to support school mathematics. We invite participants to consider: How do Biesta's ideas fit with your current view of mathematics education? Do Biesta's views of purposes of education provide an alternative path to advocating for why and how school mathematics should become more mathematical? What does this mean for initial teacher education?

References

Boaler, J. (2016). *Mathematical mindsets*. San Francisco, CA: Jossey-Bass.

Biesta, G. (2020). Risking ourselves in education: Qualification, Socialization, and subjectification revisited. *Educational Theory*, *70*(1). 89–104.

Biesta, G. (2022). *World-centred education: A view for the present.* New York: Routledge.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*, 413–435.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education*. *Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 568). Gold Coast: MERGA.

Write Like a Reviewer: MERGA Conferences and Beyond

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There is little doubt that writing with the intention of publishing is one of the core practices in our field. As a result, concerted attention is devoted to supporting HDR students' and early career researchers' writing, as well as to establishing peer-based writing support structures. In this round table, we explore another avenue for improving the expression of mathematics education research ideas through writing: learning by participating in the review process.

Regularly reading others' manuscripts to highlight the strengths and to suggest ways to address weaknesses provides a fresh perspective on one's own writing. But what is required to become a reviewer? How do we learn to review? And what does a good review look like? We will examine the review processes for the annual MERGA conference and the *Mathematics Education Research Journal* (MERJ), a high-ranked journal in our field. Emphasising that the aim of reviewers is to help authors get their work published, the written reviews have two main purposes: (1) Providing the editor with justification for the review decision; and (2) Presenting the author with constructive feedback and support for producing a stronger manuscript (Messa et al., 2021).

Our discussion will focus both on the issues that good reviews should address and how the reviewers may need to go about addressing these issues.

Through examples of review comments, we will consider their accuracy and usefulness, focusing on review principles of constructiveness, sensitivity and humility. We will discuss reviewing texts written in English by authors for whom English is a non-dominant language (Geiger et al., 2022) and reviewing manuscripts written from a theoretical position different from that of the reviewer. We will share different MERGA-related review opportunities and encourage participants to engage in these. Participants will be invited to share their questions, experiences, and suggestions related to reviewing with the aim of building the strength of our community.

While the awareness gained through reviewing is said to benefit one's writing, it is also true that the more one writes, the better one can review. This round table will thus round off with the reassurance that to write better—and to review better—we all need to keep writing.

References

Geiger, V., Delzoppo, C., & Straesser, R. (2022). Supporting English non-dominant language authors' efforts to publish: Perspectives from the editors-in-chief of highly recognised journals in mathematics education. *Educational Studies in Mathematics*, *111*, 543–565.

Mesa, V., Bakker, A., Venkat, H., et al. (2021). Writing reviews: Perspectives from the editors of Educational Studies in Mathematics. *Educational Studies in Mathematics, 108,* 419–428.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 569). Gold Coast: MERGA.

University Lecturers in Early Childhood Mathematics Education: Who are They? What are Their Professional Needs?

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This Round Table is in fact the inaugural meeting of Early Childhood Mathematics Education Special Working Group (ECME-SWG), sanctioned by the MERGA executive. The purpose of the SWG is to provide opportunity and structure for the collaboration of interested MERGA members to identify and articulate issues in early childhood mathematics education and take research-based action towards resolving the issues. All interested conference attendees are welcome.

The Issue: The proposed first issue for investigation is the current lack of understanding of the backgrounds and professional needs of our colleagues lecturing and researching in the field of early childhood mathematics. While there is substantial research activity around the characteristics of Early Childhood (EC) educators working directly with young children, there is a profound lack of research about tertiary level mathematics educators teaching future EC pre-service teachers, as a scan of the previous issue of Research in Mathematics Education in Australasia (Way et. al., 2020) will attest. The positive influence that EC educators with tertiary qualifications have on the mathematical development of toddlers and pre-schoolers has been established (e.g., MacDonald et. al, 2023), but little is known about those who teach pre-service courses leading to such qualifications. Anecdotal evidence suggests that many EC Mathematics lecturers are either primary mathematics specialists without strong EC teaching experience, or EC specialists without strong mathematics backgrounds. The actual 'EC Mathematics specialists' seems to be a rare phenomenon.

The Proposal

The ECME-SWG will conduct a study to build a profile of EC Mathematics university lecturers and produce a report, with recommendations, for the MERGA Executive, and potentially produce a publication. Participants attending the Round Table will launch a data-gathering exercise by:

- Completing an informal mini-survey and considering the data collected;
- Sharing perspectives on the proposed issue for investigation and discussing the design of a more comprehensive survey;
- Discussing how to conduct the study and who will be active investigators.

The outcome of the ECME-SWG (aka Round Table) will be an action plan for the study.

References

MacDonald, A., Deehan, J., & Lee, P. (2023). Relations between early childhood educators' qualifications and experience and their beliefs about mathematics education for babies and toddlers. *Australian Journal of Education*, *67*(3), 253–269. https://doi.org/10.1177/00049441231193776

Way, J., Attard, C., anderson, J., Bobis, J., McMaster, H., & Cartwright, K. (Eds.). (2020). *Research in mathematics education in Australasia 2016–2019*. Springer Nature. https://doi.org/10.1007/978-981-15-4269-5

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 570). Gold Coast: MERGA.

Short Communications

Using a Visual Representation Framework with Pre-Service Teachers to Analyse Place Value Representations

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Numerous studies in teacher education programmes have revealed that pre-service teachers do not have sufficient knowledge to teach primary school mathematics (Taylor, 2021). For example, pre-service teachers enter their mathematics education courses with knowledge of how to implement the standard written algorithm with little conceptual place value knowledge (Fasteen, et al., 2015). Place value helps learners understand the position and value of the number (Kortenkamp & Ladel, 2014). Considering the difficulties of PV for learners, visual representations should assist in developing their understanding of PV (Ainsworth, 2006).

VRs assist learners in understanding and interpreting concepts, and they transmit information in an aesthetically pleasing way (Arcavi, 2003). When selecting textbooks, it is important to pay attention to the quality of the learning opportunities in the text and whether the teaching and learning activities are based on the prescribed curriculum. Much research has been done on the content of texts, however, there is little research that focuses on analysing the appropriateness of the visual representations in texts (Mathews et al., 2014). This research aims to assist pre-service teachers in choosing suitable visual representations when teaching place value using a visual representation framework. The above has led to the following research question: *How might (or not) a Visual Representation Framework assist pre-service teachers in analysing and selecting visual representations from texts for teaching place value?*

This research explores how a visual representation framework can assist pre-service teachers in analysing textbook visuals. The data collection includes document analysis, observations, task analysis, reflections, and interviews. The research consists of three phases: Phase 1 focuses on developing a Visual Representation Framework for analysing PV visual representations. Phases 2 and 3 focus on the use of the VRF to assist pre-service teachers' in analysing and selecting visual representations for teaching PV. The Artifact-Centric Activity Theory (ACAT) will be used to analyse the data (Larkin et al., 2019).

References

Ainsworth, S. (2006). Deft: A conceptual framework for considering learning with multiple representations. *Learning and Instruction*, *16*(3), 183–198. https://doi.org/10.1016/j.learninstruc.2006.03.001

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*, 215–241. https://doi.org/10.1023/A:1024312321077

Fasteen, J., Melhuish, K., & Thanheiser, E. (2015). Multiplication by 10_{five}: Making sense of place value structure through an alternate base. *Mathematics Teacher Educator*, *3*(2), 83–98.

Kortenkamp, U., & Ladel, S. (2014). Flexible use and understanding of place value via traditional and digital tools. In P. Lilhedahl, S. Oesterle, C. Nicol, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME 36* (Vol. 4, pp. 33–40). Canada. International Group for the Psychology of Mathematics Education.

Larkin, K., Ladel, S., Kortenkamp, U., & Etzold, H. (2019). Developing student understanding of place value and supporting teachers' confidence in teaching place value via a digital tool. *Australian Primary Mathematics Classroom*, *24*(3), 21+. link.gale.com/apps/doc/A655088724/AONE?u=anon~1dd3be79&sid=googleScholar&xid=903582f1

Mathews, C., Mdlulu, M., & Ramsingh, V. (2014). The use of workbooks in South African Grade 3 mathematics classrooms. *South African Journal of Childhood Education*, *4*(1), 80–94.

Taylor, Nick. (2021). The dream of Sisyphus: Mathematics education in South Africa. *South African Journal of Childhood Education*, *11*(1), 1–12. https://dx.doi.org/10.4102/sajce.v11i1.911

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 573). Gold Coast: MERGA.

Use of Newman Error Analysis Guidelines to Identify Pupils' Errors in a Word Problem Involving Fractions

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In Singapore, the model-method is adopted as a problem-solving approach by primary mathematics teachers with the intention to help pupils visualise word problems correctly. It is the use of rectangles and numbers to encapsulate both the quantitative and relational information in a mathematical problem (Kho, Yeo & Lim, 2009). Model drawing provides a form of scaffold in the development of proportional reasoning (Ng & Lee, 2005). However, one of the challenges faced by pupils would be the intricacy of obtaining a diagram that reflects the mathematical intention of the question. For a model to satisfy this condition, it must have sufficient relational accuracy for one to determine the operations required. Furthermore, the model must be drawn such that any relationship between the units and any known parts may be deduced. Meeting these conditions indicate that the word problem is being understood. Successful problem-solving activities using models require: (a) the ability to translate the word problem into models; (b) the ability to use the correct operations from the model; and (c) to put these together. Lower progress pupils tended to have difficulty in identifying the correct operations (Poh, 2007).

The Newman Error Analysis Guidelines (NEAG, 1977a) offers a structured approach in examining how a pupil might think when facing a word problem. It comprises five hierarchical stages, namely reading recognition, general and specific comprehension, transformation, process skills and encoding. In order to examine the difficulties encountered when solving a one-step fraction word problem using the model method, grade 5 low attaining pupils in mathematics were interviewed on their thought processes behind their models drawn and the mathematical equations written using the NEAG.

The findings were similar to earlier research conducted with students of different age groups and nationalities (Prakitipong & Nakamura, 2006) with most of the errors being committed at the comprehension and transformation stages. Therefore, it is important for mathematics teachers to be mindful of how they can better scaffold students' learning at these stages so that pupils would be able to solve these mathematical tasks accurately.

References

Kho, T. H., Yeo, S. M. & Lim, J. (2009). *The Singapore model method for learning mathematics*. Singapore: EPB Pan Pacific.

Prakitipong, N. & Nakamura, S. (2006). Analysis of mathematics performance of grade five pupils in Thailand using Newman procedure. *Journal of International Cooperation in Education*, *9*(1), 111–122.

Newman, M. A. (1977a). An analysis of sixth-grade pupils' errors on written mathematical tasks. In M. A. Clements, & J. Foyster (Eds.), *Research in mathematics education in Australia* (pp. 269–287). Melbourne: Swinburne College Press.

Ng, S. F. & Lee, K. (2005). How primary five pupils use the model method to solve word problems. *The Mathematics Educator*, *9*(1), 60–83.

Poh, B. K. (2007). Model method: primary three pupils' ability to use models for representing and solving word problems [Unpublished Master's Thesis]. National Institute of Education, Nanyang Technological University, Singapore.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 574). Gold Coast: MERGA.

Supporting EAL/D Learners in Mathematics

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Mathematics is sometimes referred to as a 'universal language', implying anybody with mathematical understanding can solve mathematical problems regardless of the language they speak (Adoniou, 2014). Teachers are required to be adaptive experts in the teaching of mathematics as they cater for the differing needs of learners in their class. In addition to this, they need to also consider the diversity of learners, in particular students whose first language is a language other than Standard Australian English (EAL/D learners).

Teachers play a critical role in supporting EAL/D Learners and their acquisition and use of mathematical language. Each teacher is responsible to know each learner, including their phase of English language acquisition, to give them the opportunity to learn, understand and develop the disposition to use mathematics. The challenge for EAL/D students is that they need to concurrently learn English, learn through (or in) English and learn about English (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2015). These students require support to engage in the mathematical ideas and to communicate their thinking using mathematical language. Moschkovich (1999) found that rather than focussing on the acquisition of technical language in isolation it was more beneficial for EAL/D students to participate in both verbal and written practices, such as explaining solution processes, describing conjectures, proving conclusions, and presenting arguments (Moschkovich, 1999).

When supporting a school in the Catholic Schools Parramatta Diocese with 76% of students identified as EAL/D, the focus of the work was intentional planning to scaffold language for students in mathematics to improve reasoning and communication processes of Working Mathematically. The teachers engaged in collaborative planning facilitated by the numeracy leaders. The focus was on anticipating possible strategies, students' misconceptions that may arise and carefully consider the mathematics they want students to know, and the language students may need. They planned specific teaching strategies to scaffold mathematical language and engage students in more purposeful classroom talk. Teachers are now more intentional in their planning and have seen an improvement in the level of purposeful talk about mathematics.

There is little research on which specific language features impact greatest on EAL/D students' performance in mathematics. This opens the opportunity for future research concerning this and the support these students require to access challenging tasks and progressing mathematical reasoning and communication in mathematics.

References

Adoniou, M. (2014). Language, mathematics, and English language learners. *Australian Mathematics Teacher*, *70*(3), 3–13.

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2015). *English as an additional language teacher resource: EAL/D Learning Progression Foundation to Year 10*. ACARA. https://docs.acara.edu.au/resources/EALD_Overview_and_Advice_revised_February_2014.pdf

Moschkovich, J. (1999). Supporting the participation of English language learners in mathematics. *For the Learning of Mathematics*, *14*(1), 11–19. https://flm-journal.org/Articles/6A4CFB710B22B177F9E631D46FD89F.pdf

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 575). Gold Coast: MERGA.

Supporting Out-of-Field Mathematics Teaching in Middle Years of Schooling

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Despite an understanding that mathematics is critical for students' future academic and career options, as well as the perception that mathematics can be a 'difficult' subject for some students, out-of-field mathematics teaching remains most prevalent in regional and remote schools in Australia. Out-of-field mathematics teaching has been implicated with poorer student outcomes, including student disengagement and lower levels of participation and achievement (Weldon, 2016; Vale et al., 2021). Research also reveals that transition from year 6 to 7 poses a big challenge for both teachers and students. This transition challenge is further exacerbated in classrooms with out-of-field mathematics teaching (Sniedze-Gregory, 2021).

If out-of-field mathematics teaching is prevalent and inevitable in regional and remote schools, then teachers need effective support to succeed, so that their students can succeed too. There is a need to reimagine the type of support that out-of-field teachers need to succeed. This pilot project investigated the support strategies that are required for out-of-field mathematics teaching in middle years of schooling. The study incorporated an ethnography research approach to capture both in-field and out-of-field mathematics teachers' perceptions of the type of support required for out-of-field mathematics teaching in middle years of schooling (from year 7 to 9) in Far North Queensland during two professional learning workshops.

In this presentation, we share three preliminary findings from the pilot. Firstly, both in-field and out-of-field teachers articulated that with a well-designed and coordinated inhouse program, it can take approximately two years for out-of-field teachers to upskill on-the-job and become in-field mathematics teachers. Secondly, the inhouse program should aim to develop the out-of-field teachers' confidence, knowledge, and skills gap with teaching multiplicative, proportional, algebraic, and functional reasoning strategies from year 5 to 9. Thirdly, that Head of Departments do not have adequate time and resources to mentor and address these issues. There is need for a coordinated approach across schools in the district. These findings will inform the next phase of the project.

References

Sniedze-Gregory, S., Felgate, R., O'Grady, E., Buckley, S., & Lietz, P. (2021). *What Australian students say about transition to secondary school* [Final report]. Australian Council for Educational Research. https://doi.org/10.37517/978-1-74286-644-4

Vale, C., Campbell, C., & White, P. (2021). Beliefs and practices of secondary teachers crossing subject boundaries to teach mathematics out-of-field. *Mathematics Education Research Journal*, *33*, 589–612. https://doi.org/10.1007/s13394-020-00323-0

Weldon, P. R. (2016). Out-of-field teaching in Australian secondary schools. (Policy Insights; n.6). Melbourne: Australian Council for Educational Research (ACER). https://research.acer.edu.au/policyinsights/6/

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 576). Gold Coast: MERGA.

Revisiting Pedagogical Design Capacity: Mathematics Teachers' Agency in Designing Instructional Materials

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Knowing how to appropriate curricular resources (e.g., textbooks, workbooks) for teaching is an important skill for teachers to develop. To describe teachers' ability to do so, Brown (2009) introduced the notion of teachers' *pedagogical design* capacity (PDC), their capacity to "perceive and mobilise existing resources in order to craft instructional contexts" (p. 24). While PDC has been predominantly used to explore teachers' decision-making during instruction, emergent research on teachers' PDC during lesson planning suggests *teacher agency* (Cong-Lem, 2021) is an important factor of if, when, and to what extent teachers choose to appropriate curricular resources (Amador, 2016; Chin et al., 2023). This is demonstrated by secondary mathematics teachers in Singapore, where beyond adapting from curricular resources these teachers commonly design their own instructional materials (IMs) (Leong et al., 2022).

In this context, these teachers explicitly demonstrate a high level of teacher agency. Instead of adopting or starting with the agenda of the curricular resources, these teachers often bring with them their own agendas and actively draw on their PDC across lesson planning and instruction, and across different grain sizes (Chin et al., 2023). In this short communication, we revisit the notion of PDC in the context of lesson planning through the design of IMs and discuss preliminary findings on the different ways teacher agency is demonstrated through non-trivial design decisions. We invite feedback from participants on how teacher agency and PDC are developed through teacher education and professional development programs in their respective institutions.

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References

Amador, J. M. (2016). Mathematics pedagogical design capacity from planning through teaching. *Mathematics Teacher Education and Development*, *18*(1), 70–86.

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). Routledge.

Chin, S. L. (2023). Design principles for raising students' awareness of implicit features of ratio: Creating opportunities to make and catch mistakes. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 163–170). MERGA.

Cong-Lem, N. (2021). Teacher agency: A systematic review of international literature. *Issues in Educational Research*, *31*(3), 718–738. https://www.iier.org.au/iier31/cong-lem.pdf

Leong, Y. H., Yeo, B. W. J., & Choy, B. H. (2022). Instructional materials as a site to study teachers' planning and learning. *Mathematics Education Research Journal*, *34*, 575–598. https://doi.org/10.1007/s13394-022-00430-0

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 577). Gold Coast: MERGA.

Investigating High School Students Understanding of Decomposition Techniques in Mathematics

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In completing arithmetic and algebraic problems during mathematics lessons, students are required to proficiently manipulate numbers and expression. Decomposition of numbers and algebraic terms is an important skill as part of this manipulation. This study shows that while High school students can complete mathematics problems in arithmetic and algebra, there is little conceptual understanding of the mathematical laws and properties that involve decomposition, which hinders deeper thinking about mathematics.

The need for students to decompose numbers and algebra is a skill that is seen in many areas of mathematical computation.

In a literature review for this investigation, three themes became evident; students number sense is impacted by the lack of understanding of the associative, distributive and commutative laws, the teacher has a large influence on student choice of strategies when solving mathematical problems and there are implications for student arithmetic thinking on algebraic thinking. This review revealed that there had been some research in this area in primary schools but there has been minimal research in high school settings.

As part of this study students from two high schools (*n*=37) completed a written computational task named Computational and Algebraic Fluency Task (CAFT). Some students were also selected to participate in follow-up interviews. The written task was based on the Mental Computational Fluency Measure – Addition and Multiplication as developed by Downton, Russo and Hopkins (2019, 2020).

The results from the data indicate that the themes from the literature review were evident in work from high school students. Many students could recognise the mathematics in the CAFT but could not explain their reasoning. This demonstrates that many students even in high school are using rules without reasons in mathematics. The impact of the teacher on the student understanding of these decomposition techniques was evident in some of the responses, but also evident in that no student in the CAFT or the interviews specifically used the terms associative, distributive, or commutative. The impact of arithmetic concepts on algebraic items in the CAFT also demonstrated that students were not generalising arithmetic concepts with algebraic applications.

This study shows that while students struggle with formalising many arithmetic and algebraic concepts, there is the indication the teacher can influence student deep understanding and high school teachers, like primary colleagues, may need more training/inservicing in relation to explicitly teaching strategies related to arithmetic.

References

Downton, A., Russo, J., & Hopkins, S. (2020). Students' understanding of the associative property and its applications: Noticing, doubling and halving, and place value. *Mathematics Education Research Journal*, *34*, 437–456. https://doi.org/10.1007/s13394-020-00351-w

Hopkins, S., Russo, J., & Downton, A. (2019). Mental computation fluency: Assessing flexibility, efficiency and accuracy. In H. V. M. Graven, A. Essien, P. Vale (Eds.), *Proceedings of 43rd Psychology of Mathematics Education conference* (Vol. 2, pp. 376–383). Pretoria, South Africa: PME.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education*. *Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 578). Gold Coast: MERGA.

Students of Different Engagement and Achievement Levels Responses to Mathematics Lessons Involving Challenging Tasks

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Student engagement plays a significant role in their achievement in mathematics (Martin et al., 2012). However, little is known about the learning preferences of students of different engagement and achievement levels. Although Skilling, Bobis, & Martin (2021) explored the engagement of 37 grades 6–7 students with high and low achievement levels using semi-structured interviews and questionnaire data, they did not adopt an intervention design nor obtain qualitative information specific to the students' perceptions of different lesson structures that incorporate challenging tasks. Extending the existing body of knowledge, the focus of this study is on how students of different engagement and achievement levels respond to different lesson structures (Task-first and Teach-first) involving challenging tasks. Adopting a qualitative, exploratory design with multiple data sources, students (n = 18) from two composite Year 3 and 4 classes (aged 8–10 years) completed surveys and participated in 5 individual semi-structured interviews following a series of lessons over a 4-week period as part of this intervention study. Thematic analysis indicated similar enjoyment response patterns to the different lesson structures and the challenging tasks regardless of differences in student enjoyment and achievement levels. However, closer inspection of the reasons as to why students preferred a lesson structure over another provide insights into the learning preferences of students with different engagement and achievement profiles. The results have implications for teachers to help engage students of diverse engagement and achievement backgrounds during mathematics lessons. Theoretically, findings from this study will extend existing theories of learning and of instruction by deepening our understanding of how students effectively learn challenging mathematics.

References

Martin, A. J., Anderson, J., Bobis, J., Way, J., & Vellar, R. (2012). Switching on and switching off in mathematics: An ecological study of future intent and disengagement among middle school students. *Journal of Educational Psychology*, *104*(1), 1–18. https://doi.org/10.1037/a0025988

Skilling, K., Bobis, J., & Martin, A. J. (2021). The "ins and outs" of student engagement in mathematics: Shifts in engagement factors among high and low achievers. *Mathematics Education Research Journal*, *33*(3), 469–493. https://doi.org/10.1007/s13394-020-00313-2

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education*. *Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 579). Gold Coast: MERGA.

Building Parental Capital in Supporting Early Literacy and Numeracy Learning

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This communication explores the parental learning outcomes of an early numeracy focused intervention with parents, caregivers and learners from disadvantaged communities in South Africa. The Family Maths Storytime Project (FMSP) aimed to build the early literacy in numeracy skills of young children up to 8 years of age. The learners have limited resources in the home, so a comprehensive range of materials were developed (i.e., number story books and linked resources). The storylines and pictures in the books were designed to serve as a stimulus for: counting, subitising, counting on, skip counting, number facts, recognising and predicting patterns, number word recognition and numeral recognition.

Three workshops were conducted with parents and caregivers of Grade R (reception year pre-Grade 1) learners from two different schools in Makhanda in the Eastern Cape in South Africa. Almost all learners spoke isiXhosa or Afrikaans in their homes while the language of instruction in school was English. Resources were provided in English and in the language spoken at home. The workshops aimed to scaffold parent and caregiver learning on how to support their children's early numeracy learning. The first author demonstrated dialogic reading (Doyle & Bramwell, 2006) of number stories with learners, engaging them in retelling the stories and playing mathematical games with cards, dice and finger puppets of the story characters.

Data was gathered in the form of end of programme interviews with 20 parents and caregivers. Interviews were transcribed and all utterances were coded using Nvivo. Most of the interview data focused on children's learning in terms of new practices and dispositions in the homes and new learner numeracy and literacy knowledge. However, when answering questions about their children's learning and use of the FMSP resources, parents provided a range of comments relating to reflections on *their own* changing practices and dispositions in the home. In addition, the interview question: "What do you think you have learned by participating in this project, in terms of helping your child with reading or with numbers?" prompted rich data that exceeded our expectations. Thus, within our coding we had a category of responses of parent self-reflections which included utterances (n) about: engaging children in new ways and practices (n = 34); learning about numeracy and literacy learning (n = 55) and stated benefits of participating in the FMSP (n=16) (Graven and Jorgensen, 2023).

In this communication we share the findings from our analysis of the carer self-reflection data with the aim of understanding how the FMSP may have built parental capital in supporting early numeracy and literacy learning. We draw on the work of Bourdieu (1977) to theorise the parental capital building enterprise that was an unexpected outcome of the study.

References

Bourdieu, P. (1977). *Outline of a theory of practice*. Cambridge University Press.

Doyle, B., & Bramwell, W. (2006). Promoting emergent literacy and social–emotional learning through dialogic reading. *The Reading Teacher, 59*(6), 554–564.

Graven, M., & Jorgensen, R. (2023). Early numeracy opportunities through number stories with marginalised families. *ZDM–Mathematics Education*, 1–15.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 580). Gold Coast: MERGA.

What Would Make Mathematics More Interesting? Junior Secondary Student Perspectives

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Student engagement is a multi-faceted construct and can be characterised as active involvement in learning, which includes the mental (cognitive), physical (operative) and emotional (affective) aspects of learning (Munns & Martin, 2005). There is evidence to suggest that students begin to disengage with mathematics as they move through secondary school (Collie et al., 2019). A negative educational experience in secondary school mathematics can be a burden throughout life and has the potential to negatively influence the uptake of the subject in the later years of schooling as well as impact on the post-school career choices students make (Gemici et al., 2014; Bourgeois & Boberg, 2016).

This presentation will focus on a section of qualitative data drawn from a larger study that investigated the key factors influencing students' mathematics engagement and participation in NSW secondary government schools. This session will draw on the opinions of 183 students across 41 focus groups in 21 secondary government schools in NSW who were asked 'What would make mathematics more interesting?

Findings suggest that students are looking for more opportunities to engage with mathematics in interactive ways including learning mathematics outside of the classroom. They would like more time spent working in groups and for their teachers to make mathematics relevant to their lives. Students also reported that they would like more time to learn mathematics as well as teachers who cater for their preferred ways of learning, and who would help them to connect more with the subject.

These findings have implications for teachers and schools in relation to how they go about planning for the teaching and learning of mathematics with the aim of increasing student interest, engagement and participation in the subject.

References

Bourgeois, S., & Boberg, J. (2016). High-achieving, cognitively disengaged middle level mathematics students: A self-determination theory perspective. RMLE Online: *Research in Middle Level Education*, *39*(9), 1–18. https://doi.org/10.1080/19404476.2016

Collie, R. J., Martin, A. J., Bobis, J., Way, J., & Anderson, J. (2019). How students switch on and switch off in mathematics: Exploring patterns and predictors of (dis)engagement across middle school and high school. *Educational Psychology*, *39*(4), 489–509.

Gemici, S., Bednarz, A., Karmel, T., & Lim, P. (2014). *The factors affecting the educational and occupational aspirations of young Australians*. Adelaide, Australia: NCVER. https://www.ncver.edu.au/research-and-statistics/publications/all-publications/the-factors-affecting-the-educational-and-occupational-aspirations-of-young-australians

Munns, G., & Martin, A. J. (2005). It's all about MeE: A motivation and engagement framework. *Paper presented at the Australian Association for Academic Research focus conference*. Cairns: AARE.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 581). Gold Coast: MERGA.

Teacher Professional Learning: The Interplay of External Stimuli, Social Dynamics, and Institutional Dimensions

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The complex nature of onsite teacher professional learning has been well documented (e.g., Goldsmith et al., 2014). Fundamental to recent perspectives is the idea of schools as learning communities (Clarke & Hollingsworth, 2002). Studies draw attention to the critical role external experts or researchers play within professional learning communities, in particular how they interact with teachers to afford opportunities for learning (Arzarello et al., 2014; Timperley et al., 2007).

Developing a model to theorise the complexity of learning processes presents challenges. Various studies have adopted the Interconnected Model of Professional Growth (IMPG) developed by Clarke and Hollingsworth (2002) to guide research in mathematics. While analysis of change sequences is recognised as a strength of the model, some studies have suggested modifications to support analysis of their data (e.g., Lomas, 2018). Others suggest drawing upon more than one model to analyse onsite learning processes (e.g., Wilkie, 2017).

In this short communication, I draw upon one aspect of the third cycle of a designed-based research study with Year 3 teachers in which the IMPG was used to analyse change pathways for individual teachers in a collaborative project. Retrospective analysis of the data indicated influences on learning within two of the domains of the IMPG, which could not be explained by mediating processes of reflection and enactment within the model. Building on the work of Wilkie (2017), I will illustrate how a second complementary model, the Meta-Didactical Transposition model (MDT) developed by Arzarello et al. (2014), provided a lens to analyse changes in teacher knowledge, practice and dispositions during professional learning. The findings highlight the interplay between key variables, namely the researcher, teachers, and the institutional dimensions as critical to the development of teacher practice. The study contributes to empirical research on professional learning and suggests a theoretical framing that combines two models may be necessary to capture the complexity of teacher learning.

References

Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N., et al. (2014). Meta-didactical transposition: A theoretical model for teacher education programmes. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 347–372). Springer.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, *18*(8), 947–967. https://doi.org/10.1016/S0742-051X(02)00053-7

Goldsmith, L. T., Doerr, H. M., & Lewis, C. C. (2014). Mathematics teachers' learning: A conceptual framework and synthesis of research. *Journal of Mathematics Teacher Education*, *17*, 5–36. https://doi.org/10.1007/s10857-013-9245-4

Lomas, L. (2018). Proposed structural refinements to the interconnected model of teacher professional growth. In J. Hunter, P. Perger, & L. Darragh (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 495–502). MERGA.

Timperley, H., Wilson, A., Barrar, H., & Fung, I. (2007). *Teacher professional learning and development: Best evidence synthesis iteration*. Ministry of Education, New Zealand. https://www.educationcounts.govt.nz/publications/series/2515/15341

Wilkie, K. (2019). The challenge of changing teaching: Investigating the interplay of external and internal influences during professional learning with secondary mathematics teachers. *Journal of Mathematics Teacher Education*, *22*(5), 95–124. https://doi.org/10.1007/s10857-017-9376-0

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 582). Gold Coast: MERGA.

Uncovering the Complexities of Mathematical Problem-Solving Instruction: An In-Depth Analysis of a Mathematical Problem-Solving Lesson for Low-Progress Primary School Students

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An important aspect of mathematics instruction in schools is its focus on presenting mathematics as sense-making and problem-solving rather than as a collection of facts, rules, and procedures (Schoenfeld, 1992). Despite the importance of developing mathematical problem-solving pedagogies and infusing them into lessons, many teachers struggle with the notion of teaching mathematics through problem-solving (Kaur et al., 2019). Given its nature, teachers often have to find a balance between scaffolding and challenging low-progress students such that their learning is maximised. Recognising the benefits of problem-solving, the key question that I will be focusing on in this study is: How might we then infuse mathematical problem-solving in classes so that low-progress school students can think creatively and independently while shifting away from a teacher-directed approach? A baseline study, which is part of a larger project, was conducted at a local primary school to examine the complexities of teaching problem-solving to low-progress students. A three-camera approach (Kaur et al., 2019) was used to capture the teacher's view, the students' view, and the whole class' view of the lesson. Data collected hence included video recordings, as well as a post-lesson interview with the teacher. I will examine the lesson in-depth and analyse the critical incidents observed during the lesson. An analysis of the critical incidents provides a detailed account of the significant incidents that take place in the lesson (Randall, 2002). Reflecting on these events is beneficial in enhancing teaching practices that strengthen students' mathematical thinking (Choy, 2014). In this presentation, I will share about these critical incidents observed while relating them to the theoretical underpinnings of mathematical problem-solving in extant literature. I will then invite feedback from participants and suggest how the lesson may be modified differently such that it can maximise the affordances of a mathematical problem-solving lesson for low-progress primary students. This study will support the development of a novel pedagogical model for teaching through problem-solving for low-progress students in future research.

References

Choy, B. H. (2014). Noticing critical incidents in a mathematics classroom. In J. Anderson, M. Cavanagh, & A. Prescott (Eds.) *Curriculum in focus: Research guided practice. Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia* (pp. 143–150). MERGA. https://files.eric.ed.gov/fulltext/ED572572.pdf

Kaur, B., Toh, T. L., Lee, N. H., Leong, Y. H., Cheng, L. P., Ng, K. E. D., Yeo, B. W. J., Yeo, K. K., Wong, L. F., Tong, C. L., Toh, W. Y. K., Safii, L. (2019). *Twelve questions on mathematics teaching*. Singapore: National Institute of Education, Nanyang Technological University.

Randall, M. (2002). Using critical incidents in journal writing. *REACT, 21*(2), 111–119 https://hdl.handle.net/10497/3858

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). MacMillan.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 583). Gold Coast: MERGA.

Understanding Australian Teachers' Conceptualisation of Angle

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In geometry, the concept of angle is complex and multi-faceted. One way of understanding this complexity was provided by Mitchelmore & White (1998) when they argued that three different classes of angle definition are present in schools at years three to eight. These are: the dynamic concept of turning about a point between two lines; and the static concepts of a pair of rays with a common endpoint and a region formed by two intersecting half-planes. These definitions are referred to as *turn, geometric* and *region* concepts for angle.

Vinner's (1991) theory of concept image and *concept definition* provides a framework for understanding angle complexity. He argues that students learn geometry primarily through the formation of concept image that comprises mental pictures and their associated properties and processes, as well as word and symbol strings that students associate with a concept. Formally structured mathematics is deductive, being based on theorems and axioms. Although concept image is central to geometry learning, it is not sufficient. The mathematics teachers' role is to ensure that student concept image interacts with concept definitions so that difficulties and misconceptions can be overcome as students learn.

Mitchelmore & White (1998) argue that no single definition can completely define angle. To understand angle students need to operate the different concepts for it simultaneously. This understanding needs to be acquired through experiencing many different angle contexts and abstracting what is common between them. Eventually students develop a concept image for angle which Mitchelmore and White (1998) refer to as the *standard angle concept* of "two lines meeting at a point with some relation between them (p. 5)." This concept image can be used in all angle situations as a mental model to identify and measure angle magnitude.

Research has shown that teachers and students have a shallow and fragmented understanding of angle (Smith & Barrett, 2017). The author's PhD research studies how Australian teachers conceptualise angle and how enhancements to that understanding might impact teaching practice. The research involved a survey of 43 Australian teachers (primary and secondary), which establishes a baseline of understanding for the study of how teachers conceptualise angle, how they view problems with angle teaching referred to in extant research and their interpretation of aspects of the curriculum related to angle.

This presentation reports on findings from the survey of teachers related to their concept image and concept definition for angle. Although the majority (72%) of teachers in the survey gave a written definition for angle used in their teaching, only 33% defined angle in both static and dynamic terms, as conceptualised by Mitchelmore and White (1998). Teachers' concept image for angle was classified in the authors' survey. The presentation discusses some apparent disconnects that were found between the concept image and concept definitions for angle reported by survey participants. Discussion will include the potential for teaching a broader and deeper understanding of geometry through enhanced concept image of angle.

References

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. *Advanced Mathematical Thinking.* D. Tall. Dordrecht; Boston, Kluwer Academic Publishers**:** 65-81.

Smith, J. P., & Barrett, J. E. (2017). The learning and teaching of measurement: Coordinating quantity and number. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 355–385). NCTM.

Mitchelmore, M. and P. White (1998). Development of angle concepts: A framework for research. *Mathematics Education Research Journal*, *10*(3): 4-27.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 584). Gold Coast: MERGA.

Supporting the Development of Language and Collaborative Competencies for International Students in 1st Year Mathematics

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Research shows that both language and the unfamiliar expectations of a new educational context can be challenging for international students in first-year mathematics. Barton et al. (2005) showed how some multilingual learners in undergraduate mathematics courses at the University of Auckland were disadvantaged by their levels of English proficiency. They report that struggles with language in first-year mathematics are often masked by high levels of prior knowledge and the low language requirements of first-year mathematics courses. However, difficulties might surface at a later stage when mathematical knowledge needs to be extended to cope with the requirements of second- and third-year courses (Barton et al., 2005). My own doctoral research shows that some first-year international students might be constrained from accessing course resources such as spoken lecture content, one-to-one help from lecturers or tutors, and collaborative peer discussions in English. I show how both language proficiency and cultural interpretations of the student role can create barriers for international students when studying mathematics at a foreign host university (Locke et al., 2023a, 2023c, 2023b).

Literature urges universities to meet the learning needs of their international students through a range of different strategies. Barton et al. (2005) suggest that mathematics departments implement strategies such as staff professional development, first language tutorials and resources, and specific mathematical English courses. Andrade (2010) supports the idea of linking courses with language support to enhance subject specific vocabulary. In this short communication I will discuss some ideas for a series of *English for Mathematics* workshops aimed to support the development of a more comprehensive mathematical register and facilitate collaborative practices for first-year international students.

References

Andrade, M. S. (2010). Increasing accountability. *Journal of Studies in International Education*, *14*(3), 221–239. https://doi.org/10.1177/1028315308331295

Barton, B., Chan, R., King, C., Neville-Barton, P., & Sneddon, J. (2005). EAL undergraduates learning mathematics. *International Journal of Mathematical Education in Science and Technology*, *36*(7), 721–729. https://doi.org/10.1080/00207390500270950

Locke, K., Kontorovich, I., & Darragh, L. (2023a). Transforming mathematical identity: Changes in one international student's positioning during first-year mathematics tutorials. *International Journal of Mathematical Education in Science and Technology*, *54*(9), 1785–1803. https://doi.org/10.1080/0020739X.2023.2259917

Locke, K., Kontorovich, I., & Darragh, L. (2023b). How mathematical identity informed collaborative learning opportunities for a first-year international student from China. In P. Drijvers, C. Csapodi, H. Palmer, K. Gosztonyi, & E. Konya (Eds.), *Proceedings of the thirteenth congress of the European Society for Research in Mathematics Education* (CERME13) (pp. 2443–2450). Alfréd Rényi Institute of Mathematics and ERME. https://shs.hal.science/CERME13/hal-04410111v1

Locke, K., Kontorovich, I., & Darragh, L. (2023c). Secondary-tertiary transition of international students: One student's efforts to overcome the challenge of learning mathematics in English. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel, & M. Tabach (Eds.), *Proceedings of the 46th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 291–298). https://www.igpme.org/publications/current-proceedings/

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 585). Gold Coast: MERGA.

On the Necessity of Multimodal Semiotic Approaches for the Analysis of Young Children's Mathematical Drawings

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Mathematics education research has classically considered several possible representations produced by students while engaging in mathematical activities: semiotic approaches have been developed to analyse students' products and interactions. The analysis of students' products becomes particularly challenging when they are so young and written language is not available as means of expression for them. Researchers in the field of early years mathematics are discussing theoretical frameworks and methods for addressing the complexity of communication of mathematical ideas (e.g., Maj-Tatsis et al., 2023).

Drawings have often been used as means for understanding children's representation of mathematical ideas. They can depict the context of mathematical activity (characters, objects, etc.) together with dynamic, pictographic, iconic, written, and symbolic marks (e.g., arrows, letters, numerals, etc.) (Carruthers & Worthington, 2006). Some researchers use drawings solely as a data source, but we believe that this might result as misleading and we argue that methodologically, multimodal approaches are preferrable. With the term 'multimodality' we are referring to the full range of means of representation which are observable in the mathematics class (or in kindergarten) including gesture, speech, bodily movements, writing, gazes, and tone of voice (Arzarello et al., 2009).

Graphical representations of mathematical ideas can serve to communicate the process of problem solving to others, but also to oneself. The different components of a drawing not only represent the process of reasoning, but they are part of it together with other means of representation. All of these must be considered to fully understand the child's mathematical ideas, and this could only be realised by taking into consideration both how the different means of representation evolve along time (diachronic analysis) and how they are related (synchronic analysis). We suggest Arzarello et al.'s (2009) semiotic bundle is suitable construct. A semiotic bundle is a system of signs produced by one or more interacting subjects while solving a problem and/or discussing a mathematical question. It is a bundle in the sense that different signs (speech, gestures, drawings) are produced and modified during time, but they are not separated: they are bundled and evolve together over time. Data taken from the study presented in (Downton & Maffia, 2023) supports us in showing that, in the case of children representing division word-problems, a multimodal semiotic approach allows to discern different strategies of problem solving, while the analysis of solely the final product would point to more similarities than differences.

References

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*(2), 97–109.

Carruthers, E., & Worthington, M. (2006). *Children's mathematics: Making marks, making meaning*. Sage.

Downton, A., & Maffia, A. (2023). Preschool children's representation of division word problems through drawings. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel, & M. Tabach (Eds.) *Proceedings of the 46th PME conference* (Vol. 2, pp. 235–242). University of Haifa & IGPME.

Maj-Tatsis, B., Bjorklund, C., Lembrer, D., Maffia, A., Levenson, E., & Tzekaki, M. (2023). An introduction to TWG 13: Early years mathematics. In P. Drijvers, C. Csapodi, H Palmér, K. Gosztonyi, & E. Kónya (Eds.), *Proceedings of the 13th congress of the European Society for Research in Mathematics Education* (CERME13) (pp. 2018–2021). Alfréd Rényi Institute of Mathematics and ERME.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 586). Gold Coast: MERGA.

Analysing the Use of Hands-on Tasks to Develop Student Talk and Collaboration

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In this short communication we will share our analysis of the use of tasks with young (6-to 7-year-old children) as part of our larger Australian Research Council funded project on developing talk in mathematics classrooms. The project involved four teachers across one year through a participatory design (Hennessy, 2014) to engage low attaining students in key mathematical ideas through collaboration and talk. A key focus of discussion in the research workshops was how to develop effective tasks that would support collaboration with lower attaining students who often had limited language and confidence to share their thinking.

Based on previous research (Murphy, 2011), we encouraged teachers to introduce tasks that would provide hands on experiences and limited recording. In the example presented here, two of the teachers explored multiplicative thinking based on the text The Doorbell Rang (Hutchins, 1989). The teachers used models of cookies and plates and phased the students' exploration in relation to the storyline.

Our analysis focuses on both the mathematical thinking that arose and also on how the task enabled students to collaborate and share thinking. The concrete models allowed students to share their thinking without needing to rely on language. As such the concrete models became tools for dialogue and thinking was often shared through gestures and actions.

References

Hennessy, S. (2014). *Bridging between research and practice: Supporting professional development through collaborative studies of classroom teaching with technology*. Brill.

Hutchins, P. (1989). *The doorbell rang*. A Mulberry Paperback Book.

Murphy, C. (2011). Reviewing the effectiveness of mathematical tasks in encouraging collaborative talk with young children. In J. Clarke, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices. Proceedings of the 34th annual conference of the Mathematics Education Research Group of Australasia* (pp. 557–564). MERGA.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 587). Gold Coast: MERGA.

Mathematics Homework: The Importance of Pitch

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Defined as any "school-prescribed tasks undertaken by children and usually under the supervision of an adult", homework, once the labour of older children, is now emerging as a regular and somewhat expected activity for younger children (Farrell and Danby, 2015, p. 250). Homework is a longstanding and widely used instructional practice (Murillo and Martinez-Garrido, 2017) and is also one of the few visible and explicit overlaps between home and school practices. Yet, for some children and their families, homework experiences are, as discussed by O'Keeffe et al. (2023), sites of intergenerational negativity.

In their study, O'Keeffe et al. (2023) shared examples of families struggling to help their children with mathematics homework, some of whom were already falling behind their peers. Issues for the families emerged when the adult helping with homework found they did not fully understand the mathematical concept or the particular strategies and approaches their child was working with. The parents/guardians in the study felt they were in the position of having to try and teach their children mathematical concepts that their child didn't fully grasp from their classwork where they had the advantage of being guided by a teacher. This suggests that the homework may not have been well aligned with the children's current capabilities.

In this presentation, we focus on the importance of the purpose and pitch of mathematics homework. Drawing on a set of problem-solving activities, we raise questions about the potential for some homework to unfairly predispose children to negative experiences with mathematics. Our initial analysis of the problem-solving activities highlights a fundamental issue of inequity. Children with access to mathematical support and resources are more likely to benefit from engaging in mathematics homework that extends and enhances their mathematical learning experiences, thereby bolstering their confidence in their mathematical abilities. In contrast, children who lack appropriate support at home may struggle with mathematical tasks that are confusing and thus convey a sense that mathematics is too hard. When this occurs these children are prevented from accessing the 'extra' learning opportunities available to their peers. Such disparity underscores the crucial role of accessible and appropriately pitched mathematics homework in fostering positive mathematical learning experiences for all children.

References

Farrell, A., & Danby, S. (2015). "How does homework 'work' for young children? children's accounts of homework in their everyday lives." *British Journal of Sociology of Education*, *36*(2), 250–269. https://doi.org/10.1080/01425692.2013.814532

Murillo, F., & Martinez-Garrido, C. (2014). Homework and primary-school students' academic achievement in Latin America. *International Review of Education*, *60*(5), 661–681.

O'Keeffe, L., Clarke, C., McDonald, S., & Comber, B. (2023) Mathematics homework and the potential compounding of educational disadvantage. *British Journal of Sociology of Education*, *44*(7), 1144–1160. https://doi.org/10.1080/01425692.2023.2240530

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 588). Gold Coast: MERGA.

Enrolment Trends in Queensland Senior Advanced Mathematics: Comparisons for Socio-Economic Advantage and Regions

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Enrolment into advanced level mathematics in Australian high schools has been in steady decline (Wienk, 2022). Mathematics enrolment data from 2020 show a significant decline that takes mathematics participation in Year 12 in Australia to an unprecedented low. Jennings (2022,2023) suggests that contrary to the rest of the country, Queensland had an increase from 2006-2018 due to the implementation of a two-point bonus scheme for students who finished Year 12 having studied higher level mathematics (Maths C) for two years. Since the departure of the bonus point scheme, and the implementation of a new senior syllabus in line with the Australian Senior Secondary Curriculum, Queensland's data has seen significant decline on par with the remaining states and territories (Jennings, 2022). For this study, advanced level mathematics in a Queensland context groups together Queensland Curriculum and Assessment Authority (QCAA), (2018) subjects *Mathematical Methods* and *Specialist Mathematics*. Queensland is a large state with many varied educational regions. These regions vary from inner city/metropolitan to regional/remote areas. In Australia, the Index of Community Socio-Educational Advantage (ICSEA) is used as a scale to allow fair and reasonable comparisons amongst schools (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2024). In this study, school's ICSEA scores were correlated against their *Mathematical Methods* and *Specialist Mathematics* enrolment numbers in Unit 1 (Year 11) from 2019–2024. A bivariate analysis was completed with a sample size of n=156 using variables: ICSEA score and subject enrolment (as a percentage of the school's cohort size). Analysis of retention rates from Unit 1 (start of Year 11) to Unit 4 (end of Year 12) was also calculated using percentage enrolment in Unit 1 and Unit 4. This was also compared with ICSEA score and across different Queensland Department of Education regions. The study aimed to consider the research question: 'What impact of socio-educational advantage and education region can be seen in enrolment numbers of advanced mathematics courses in Queensland Schools?'

Variance in socio-educational schools, based on their ICSEA scores has an impact on student enrolment into advanced level Mathematics in Queensland. For *Mathematical Methods* and *Specialist Mathematics*, a weak positive correlation exists. Enrolment and retention also vary between educational regions. Findings of this study will be shared and discussed.

References

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2024). *MySchool.* ACARA. https://www.myschool.edu.au/

Jennings, M. (2022). Advanced mathematics enrolment numbers: A crisis or fake news? *Australian Mathematics Education Journal*, *4*(1), 33–45.

Jennings, M. (2023). *Issues in the transition from secondary school to university mathematics: Student, teacher, and lecturer perspectives* [Unpublished doctoral dissertation]. The University of Queensland.

Queensland Curriculum and Assessment Authority (QCAA). (2019). *Mathematics senior subjects*. https://www.qcaa.qld.edu.au/senior/senior-subjects/mathematics

Queensland Department of Education. (2024). *Region maps* [Online]. https://education.qld.gov.au/contact-us/region-maps

Wienk, M. (2022). *Year 12 mathematics participation report card. Mathematics enrolments reach all-time low* [Online]. Australian Mathematical Sciences Institute. https://amsi.org.au/?publications=year-12-participation-in-calculus-based-mathematics-subjects-takes-a-dive-2

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 589). Gold Coast: MERGA.

Froebel Meets OpenSCAD: Pre-Service Teachers Form Units in Notes for Children and in Instructions for 3D Cube Constructions

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Recent studies in Australia and Germany have shown that block-play activities and especially play with the so-called “Froebel’s Gifts” offer profound opportunities to develop spatial sense in preschool and in primary school (Livy et al., 2018; Reinhold et al., 2017). These Froebel’s Gifts, specific education toys introduced by the German pedagogue Friedrich Froebel in the late 18th century, were also used as a tool to investigate Australian children’s geometrical concept knowledge (Downton et al., 2019). Based on these and related research experiences, the study we introduce in our presentation questions how pre-service teachers identify and address part-whole relationships within symmetrical arrangements made of cubes from Froebel’s Gift 3 (called “forms of beauty” by Froebel), when they are asked to give instructions for primary students. We were also interested to find out if these notes differ from pre-service teachers’ handling of instructions in tasks using a digital tool for the design of the same “forms of beauty”. Therefore, data collection included the use of *Open Spatial Computer Aided Design* (OpenSCAD, https://www.openscad.org/), an open program for 3 D constructions.

First analyses show that pre-service teachers’ handwritten notes not only differ in terms of the length of the descriptions, but also vary since some pre-service teachers stay very close to the (mental) construction process, guiding a fictive grade 2 recipient of the description “step by step” through the anticipated process of reconstruction. Others invite the fictive recipient to think backwards, or at least to use an underlying structure for reconstructing details, later on. In a wide range of structuring strategies, which we identified in the handwritten free texts, some of the instructions reveal a clearly holistic view on the form of beauty—starting with a general statement concerning the gestalt, which is then described step by step with a focus on different subsets of the whole, always referring back to the entire array. We suggest to call this a *global approach with analysis of units*, and contrast it to a *local approach*, where the cube arrangement appears to “break apart”, isolating merely single cubes during the description of a reconstruction process in a “piece by piece” manner. In comparison, the digital tool OpenSCAD challenges and somehow seems to urge the pre-service teachers to always keep in mind the final product. We suggest to discuss these differences and potential obstacles for children which should be addressed when introducing these tasks in pre- and in-service teacher education.

References

Downton, A., Livy, S., & Reinhold, S. (2019). Grade 3/4 students’ understanding of geometrical objects: Australian case studies on (mis)conceptions of cubes. In U. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European Society for Research in Mathematics Education* (pp. 857–864). The Netherlands: Utrecht University.

Livy, S., Downton, A., Reinhold, S., & Wöller, S. (2018). Insights into a year 4 student’s spatial reasoning and conceptual knowledge of rectangular prisms. In J. Hunter, L. Darragh, & P. Perger (Eds.), *Making waves, opening spaces. Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 487–494). Auckland: MERGA.

Reinhold, S., Downton, A., & Livy, S. (2017). Revisiting Friedrich Froebel and his gifts for kindergarten: What are the benefits for primary mathematics education? In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia* (pp. 434–441). Melbourne: MERGA.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 590). Gold Coast: MERGA.

Scaffolding Structured Inquiry Learning Through 'Spotlighting'

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When implementing challenging tasks through structured inquiry pedagogies, concerns are sometimes raised by teachers about the level of support provided to students during task exploration (Sullivan et al., 2010). Enabling prompts can be employed to augment tasks and scaffold the learning experience (Sullivan et al., 2009), however such prompts may not always be sufficiently utilised and appropriately interpreted by students to be effective (Russo & Hopkins, 2019). In fact, the issue of sufficiently scaffolding mathematical thinking when students are exploring a challenging task is one of the reasons why some teachers are hesitant to utilise task-first lesson structures when teaching with such tasks, preferring instead to use more teacher-directed approaches (Calleja et al., 2023). Although structured-inquiry pedagogies do emphasise experience before instruction, this should not be taken to imply that the teacher's role is passive. However, a teacher's lack of clarity about their role during key stages of the explore phase, particularly just after a problem-solving task has been launched, could be a further reason why a teacher may resort to premature 'telling' beyond concerns about struggling students. One potentially useful practice for providing more structure, support, and clarity to this phase of the lesson, for both teachers and students, is that of spotlighting.

A spotlight can be described as the practice of "calling students to a momentary pause in order to share and briefly discuss an example of student thinking that will be of benefit to the rest of the class" (Hubbard et al., 2023, p. 32). Spotlighting generally involves the teacher facilitating the sharing of the spotlighted student(s)' mathematical work with a particular purpose in mind. These purposes include but are not limited to: how the student has chosen to represent the problem mathematically, a partial solution or first step in the problem; highlighting a misinterpretation or misconception; or contrasting two different student approaches. In this short communication, we share some of the preliminary findings of our recent research into spotlighting such as how frequently teachers use the practice in a given lesson, how many students are the focus of a typical spotlight, and the reported affordances and constraints of this pedagogical move for supporting the learning of mathematics through structured inquiry. We conclude by discussing potential future research directions, as well as considering implications for teacher professional learning and classroom practice.

References

Calleja, J., Foster, C., & Hodgen, J. (2023). Teachers' structuring of mathematical inquiry lessons: shifting from "task-first" to "scaffolded inquiry". *Research in Mathematics Education*, 1–34. https://www.um.edu.mt/library/oar/handle/123456789/112464

Hubbard, J., Russo, J., & Sullivan, P. (2023). Spotlighting: Helping to scaffold the 'explore phase' when teaching with challenging tasks. *Australian Primary Mathematics Classroom, 28*(1), 31–37.

Russo, J., & Hopkins, S. (2019). Teaching primary mathematics with challenging tasks: How should lessons be structured?. *The Journal of Educational Research*, *112*(1), 98–109.

Sullivan, P., Clarke, D. M., Clarke, B., & O'Shea, H. (2010). Exploring the relationship between task, teacher actions, and student learning. *PNA*, *4*(4), 133–142.

Sullivan, P., Mousley, J., & Jorgensen, R. (2009). Tasks and pedagogies that facilitate mathematical problem solving. In B. Kaur (Ed.), *Mathematical problem solving* (pp. 17–42). Singapore/USA/UK, World Scientific Publishing: Association of Mathematics Educators.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 591). Gold Coast: MERGA.

Mathematics Education Researchers' Perspectives on Implications of Their Research for Primary Teachers

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The Mathematics Education Research Group of Australasia (MERGA) has contributed a wealth of information about the teaching and learning of mathematics over the years. However, little research has included data collection from this particular community. This study provided an opportunity for the MERGA community to share their perspectives on what they believed were the most important implications for mathematics education in primary schools that have arisen from both their own research and mathematics education research more generally. Specifically, 86 members of the MERGA community responded to an online questionnaire probing their views about how mathematics education research translates into practice.

Although the principle motivation for undertaking this research was to share our findings with those directly involved in providing primary mathematics education to students (teachers and mathematics leaders), we believe that there is value in presenting preliminary results back to the MERGA community who provided the data in the first instance. This will help offer clarity to the community as to what its members view as the most seminal contributions their research has made (and could potentially make) to mathematics teaching practice at the primary school level, as well as provide insights around how the community believes research should be disseminated to, and created/ consumed by, school-based teachers and leaders.

Related to this latter point, one notable finding was that approximately 90% of respondents thought that it was important or very important that primary school teachers and leaders engage directly with research through undertaking their own action research projects in their schools and/ or through being collaborators on university-based academic research projects. By contrast, only around half of respondents thought that it was important or very important that these school-based professionals engage directly with research through other means, such as attending research conferences or reading research journals. Together these results imply that the MERGA community believes that teachers should strive to be active participants in the research process, rather than merely passive consumers of research findings. Moreover, whereas approximately 90% of respondents thought that it was important or very important that teachers and leaders engaged indirectly with research through reading practitioner journals (e.g., Australian Primary Mathematics Classroom) and/ or attending practitioner conferences where academics are sharing research findings, indirect engagement with research through books that synthesised research (59%) or following social media accounts of prominent academics (39%) were both less likely to be viewed as important. This implies that the MERGA community may be inclined to value 'gatekeepers' (e.g., journal editors, mathematics associations responsible for organising and coordinating conferences) when it came to considering how teachers accessed research indirectly.

The intention of presenting preliminary findings arising from this study as a short communication at the current MERGA conference is to provoke critical discussion and reflection, as well as to seek the community's input into how this data might be most meaningfully disseminated within the practitioner and researcher communities.

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Enhancing Problem-Solving Skills Among Low-Progress Students in Singapore: Leveraging Variation Theory in Mathematics Education

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Problem-solving serves as a central focus of Singapore's Mathematics curriculum, and one common mathematics instructional practice is the use of worked examples. This practice not only promotes mastery of learning, but also reduces unproductivity and cognitive overload in learning (Kaur et.al, 2021). To further enhance problem-solving competency, an alternative strategy is to teach mathematics through problem-solving, which involves presenting problems for students to solve before introducing related mathematical concepts (Takahashi, 2021). However, tension arises between students' inclination toward instrumental understanding and teachers' aim to cultivate relational understanding of mathematics. This tension complicates the cultivation of problem-solving skills, particularly for low-progress students who may face obstacles due to mathematics self-efficacy issues (Leong, 2023). Another challenge lies in the designing of problem-solving tasks such that it effectively emphasises the critical aspects students need to grasp while solving problems. Variation theory provides a way for teachers to think about how these critical aspects can be incorporated into problem-solving tasks. By discussing these crucial aspects during problem-solving sessions, teachers can help students build a bridge between their instrumental understanding and relational understanding of mathematics.

A baseline study, conducted at a local secondary school, analysed the dynamics of teaching mathematics through problem-solving for low-progress students. Data collected included video and voice recordings of a lesson, teacher post-lesson interviews, and lesson materials. I will explore the challenges and the complexities involved when teachers try to teach mathematics through problem-solving for low-progress students. Greater attention will be given to dissecting the task design and classroom dynamics between students and teachers in upcoming discussions. These insights will inform recommendations to improve future mathematics lessons, particularly when teaching through problem-solving for low-progress students. Following this, participants will be encouraged to offer their perspectives to further refine such lessons to maximise learning outcomes for low-progress students. These discoveries will aid in crafting an inventive pedagogical framework designed to enhance problem-solving skills among low-progress students among low-progress students in forthcoming research endeavours.

References

Kaur, B., & Leong, Y. H. (2021). *Mathematics instructional practices in Singapore secondary schools*. Springer Singapore Pte. Limited. https://doi.org/10.1007/978-981-15-8956-0

Leong, Y. H. (2023). Contours of self-efficacy across nested mathematical domains: A case of a Singapore student with a history of low performance in mathematics. *Mathematics Education Research Journal, 35*(Suppl 1), 171–192. https://doi.org/10.1007/s13394-021-00394-7

Takahashi, A. (2021). *Teaching mathematics through problem-solving: a pedagogical approach from Japan*. Routledge.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 593). Gold Coast: MERGA.

We all Know Fun Maths—But Where is the Theoretical Foundation?

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Tuohilampi and Attard (2024) identified an interesting void in the realm of teaching mathematics: while there are several millions of teaching resources available on the Internet presented as ‘fun maths’, peer-reviewed research literature rarely addresses this key term. Fun maths is mentioned in peer-reviewed studies that address games or gamified learning (Darragh, 2021). Fun as easing the learning of mathematical content (Fouryza et al., 2019), versatile learning activities, such as playful, physical learning (Bustamante et al., 2022) and mathematics being embedded in real life phenomena (Eubanks-Turner & Haji, 2015). Fun maths seems to have a place in the lexicon of teachers of mathematics; in this short communication, we will discuss how to take a systematic look at fun maths in research literature.

Fun Maths is a Practical Term

Amazon gives over 50,000 results for ‘fun maths’ (tested 24th of August 2023); using web-based search engines, such as Google, one can find several million hits (952,000,000 results, tested 31st of January 2023 using Google). Interestingly, using the same search term in research databases does not reflect this abundance of teaching resources. ERIC lists just 97 peer reviewed journal articles with the same keyword from the last 20 years.

To start unravelling this misalignment, Tuohilampi and Attard (2024) conducted an initial review of four peer reviewed research articles published after 2019, all using the key term fun maths. This initial review identified that: (1) fun maths can be used to describe the nature of an activity, learning experience or teaching approach; (2) it is not always clear how fun maths can contribute to learning ; and (3) the term can be used non-purposefully, e.g., having it in the title but never using the it in the body text. The first overview suggests a practical, all-encompassing definition: fun maths could feature almost any positive aspect of mathematics learning.

Multiple questions remain unanswered: Who is the intended audience for fun maths activities or approaches? Is there any theoretical framework or pedagogical principles behind fun maths? When fun maths is connected to any outcomes, is there any evidence or data regarding the effectiveness? What specific activities or approaches are proposed to connect with fun maths? Without a clearly defined concept it is difficult to review the needs for ‘fun maths’. In this short communication, I will discuss the challenges related to the ubiquitous, yet vaguely defined concept of fun maths. What are we talking about, when we talk about fun maths?

References

Bustamante, A. S., Begolli, K. N., Alvarez-Vargas, D., Bailey, D. H., & Richland, L. E. (2022). Fraction ball: Playful and physically active fraction and decimal learning. *Journal of Educational Psychology, 114*(6), 1307. https://doi.org/10.1037/edu0000714

Eubanks-Turner, C., & Hajj, N. (2015). Mardi gras math. *Mathematics Teaching in the Middle School, 20*(8), 492–498.

Fouryza, D., Amin, S. M., & Ekawati, R. (2019). Designing lesson plan of integer number operation based on fun and easy math (FEM) approach. *International Journal of Evaluation and Research in Education, 8*(1), 103–109. https://doi.org/10.11591/ijere.v8i1.15514

Tuohilampi L, & Attard C. (in press). What are we talking about, when we talk about ‘fun maths’ [Conference paper]? *International Congress on Mathematical Education* (ICME–24), Sydney, 07 Jul 2024–14 Jul 2024. ICME.

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 594). Gold Coast: MERGA.