



Mathematics Education: Expanding Horizons

Proceedings of the 35th Annual Conference of the
Mathematics Education Research Group of Australasia

Edited by
Jaguthsing Dindyal
Lu Pien Cheng
Swee Fong Ng



Mathematics Education: Expanding Horizons
Electronic Book Edition

Proceedings of the 35th Annual Conference of the Mathematics Education Research Group of Australasia

Edited by Jaguthsing Dindyal, Lu Pien Cheng & Swee Fong Ng

© Mathematics Education Research Group of Australasia Inc. 2012

ISBN 978-981-07-2527-3



Published by

MERGA Inc.
GPO Box 2747
Adelaide, SA 5001
www.merga.net.au

Printed by Welon Printing Co. Pte Ltd for: Local Organising Committee, MERGA 2012, MME, National Institute of Education, 1 Nanyang Walk, NTU, Singapore.

The 35th Annual Conference of the Mathematics Education Research Group of Australasia was held at the National Institute of Education, Nanyang Technological University, Singapore from 2 – 6 July 2012.

Local Organising Committee

Co-chairs Jaguthsing Dindyal & Lu Pien Cheng

Advisor Keng Cheng Ang

Committee Members Chun Ming Eric Chan, Weng Kin Ho, Berinderjeet Kaur, Phong Lee Koay, Swee Fong Ng, Kok Ming Teo, Sook Fwe Yap, Cynthia Seto, Huei Wuan Low-Ee

Conference Programme Jaguthsing Dindyal, Lu Pien Cheng, Berinderjeet Kaur

Cite as:

John, M. (2012). XXXXXX. In J. Dindyal, L. P. Cheng & S. F. Ng (Eds.), *Mathematics education: Expanding horizons*, (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia, eBook, pp. xx). Singapore: MERGA, Inc.

PREFACE

We have compiled the proceedings of the 35th Annual Conference of the Mathematics Education Group of Australasia (MERGA 2012) in two volumes. The conference is held for the first time outside Australia and New Zealand at the National Institute of Education in Singapore and it is good to note that the conference is attended by participants from 14 different countries. The theme of this year's conference is *Mathematics Education: Expanding Horizons*, and is consistent with the expanding influence of MERGA in the region. A unique feature of this year's annual MERGA conference is the Special Teachers Day (STD), one day for primary school teachers and the other for secondary school teachers, running in parallel with the regular research sessions. The presentations cover a wide range of topics in mathematics education, at various levels of schooling, from both within and outside the Australasia region. The papers highlight the increasing interest of researchers and other mathematics educators to improve the teaching and learning of mathematics at all levels.

Each research paper and symposium paper was blind peer-reviewed as were the abstracts for short communications, round tables, and posters. As in previous MERGA conferences, research papers were accepted for presentation only or for both presentation and publication in the proceedings. Volume 1 of the proceedings contains the keynote papers, the practical implications award paper and about half of the research papers that were accepted for presentation and publication. Volume 2 contains the remaining research papers and the abstracts of the short communications, round tables and posters, as well as the symposium papers. Most of the authors used the standard template for MERGA conferences and adhered to the 8-page limit for the Research Papers. We have tried our best to format those papers that did not meet some of the formatting requirements. In doing so, we did not bring any modifications to the text of these papers, except for some obvious editing of figures, tables and correction of some grammatical and typographical errors. The authors remain solely responsible for the contents of their papers. We are hopeful that the proceedings will be good resources for all participants as well as for all those having an interest in mathematics education.

We would like to take this opportunity to thank everybody who helped us in one way or the other with the organisation of the conference, the preparation of the conference handbook, and the publication of these proceedings. In particular, we are extremely thankful: to the Director of the National Institute of Education, Professor Lee Sing Kong for strongly supporting this event; to the Head of the Mathematics and Mathematics Education Academic Group (MME), Dr Ang Keng Cheng (Advisor); to all members of our Local Organising Committee and our fellow colleagues from MME; to Shirley Seet from our secretariat, and above all to our support staff: Low Chwee Tee Nee Sze, Quek Siew Eng Michelle, Rodziah Binte Abdul Rahman, Mohamed Fadzli Bin Mohamed Ibrahim, Ang Kheng Yeong and Ngo Hong Tat for their more than generous help.

Thank you Merrilyn, Bobbie and the MERGA family!

Jaguthsing Dindyal

Lu Pien Cheng

Swee Fong Ng

MERGA 35: Reviewers and Judges

Anderson Dayle	Lomas Gregor
Anderson Judy	Lynch Peggy
Anthony Glenda	MacDonald Amy
Attard Catherine	McChesney Jane
Averill Robin (Panel Chair)	McMaster Heather
Barton Bill	Mousley Judy
Begg Andy	Muir Tracey
Beswick Kim	Mulligan Joanne (Panel Chair)
Bobis Janette	Oates Greg
Brady Kathy	Page Shaileigh
Burgess Tim	Paterson Judy
Callingham Rosemary	Perry Bob (Panel Chair)
Carmichael Colin	Prescott Anne
Cavanagh Michael	Reaburn Robyn
Chick Helen (Panel Chair)	Stephenson Brett
Chinnappan Mohan	Stewart Sepideh
Clark Julie (Panel Chair)	Thomas Mike (Panel Chair)
Dockett Sue	Thornton Steve
Drake Michael	Walshaw Margaret
Ell Fiona	Way Jenni
Goff Wendy	White Bruce
Greenlees Jane	White Paul
Grootenboer Peter	Wilson Sue
Highfield Kate	Winn Tiffany
Kinnear Virginia	Wong Monica
Klymchuk Sergiy	Wood Leigh
Linsell Chris	Yoon Caroline
Logan Tracy	

CONTENTS

Preface	
MERGA 35: Judges and Reviewers	

KEYNOTES

The Beginnings of MERGA	3
MERGA 2012: Where We've Been, Where We Are, and Where We're Going	4
<i>M. A. ("Ken") Clements</i>	
Mathematics Education as a Multicultural Field of Research and Practice: Outcomes and Challenges	20
<i>Mich �e Artigue</i>	
What Can and Should We Learn from International Studies of Mathematics Achievement?	34
<i>Frederick K.S. Leung</i>	

PRACTICAL IMPLICATIONS AWARD

A Learning Community for Pre-service Secondary Mathematics: Learning With and From Each Other	63
<i>Michael Cavanagh</i>	

RESEARCH PAPERS

Exploring the Use of iPads to Engage Young Students with Mathematics	75
<i>Catherine Attard & Christina Curry</i>	
Teaching Algebra Conceptually: The Process of Bringing Research to Practitioners	83
<i>Megan Anakin & Ayelet Lazarovitch</i>	
Influences of Self-Perceived Competence in Mathematics and Positive Affect toward Mathematics on Mathematics Achievement of Adolescents in Singapore	91
<i>Shaljan Areepattamannil & Berinderjeet Kaur</i>	
Socially Response-able Mathematics Education: Lessons from Three Teachers	98
<i>Bill Atweh & Kate Ala'i</i>	
Improving First Year Mathematics Teaching Through Making Connections: An Action Research Approach	106
<i>Jo Balatti & Shaun Belward</i>	
Teaching Mathematics in a Project-Based Learning Context: Initial Teacher Knowledge and Perceived Needs	114
<i>Kim Beswick, Rosemary Callingham & Tracey Muir</i>	

Stories From the Classroom: The Developing Beliefs and Practices of Beginning Primary Mathematics Teachers	122
<i>Kathy Brady</i>	
Teacher Subject Matter Knowledge of Number Sense	130
<i>Hannah Briand-Newman, Monica Wong & David Evans</i>	
Mathematics as it Happens: Student-Centred Inquiry Learning	138
<i>Chris Brough & Nigel Calder</i>	
How Does Teacher Knowledge in Statistics Impact on Teacher Listening?	146
<i>Tim Burgess</i>	
An Experienced Teacher's Conceptual Trajectory for Problem Solving	154
<i>Barbara Butterfield</i>	
Mathematical Knowledge for Teaching of MERGA Members	162
<i>Rosemary Callingham, Kim Beswick, Julie Clark, Barry Kissane, Pep Serow & Steve Thornton</i>	
Emotions and the Development of Statistical Literacy	170
<i>Colin Carmichael</i>	
The Effects of Creating Rich Learning Environments for Children to Measure Mass	178
<i>Jill Cheeseman, Andrea McDonough & Sarah Ferguson</i>	
Developments in Pre-service Teachers' Mathematics for Teaching of Fractions	186
<i>Mohan Chinnappan, Tricia Forrester & Elise Thurtell-Hoare</i>	
Mathematics Knowledge for Teaching: Evidence from Lesson Study	194
<i>Mohan Chinnappan & Ui Hock Cheah</i>	
Characteristics of Problem Posing of Grade 9 Students on Geometric Tasks	202
<i>Puay Huat Chua & Khoon Yoong Wong</i>	
Alternative Starting Point for Teaching Fractions	210
<i>Jose Luis Cortina, Jana Visnovska & Claudia Zuniga</i>	
Concept Cartoons as a Way to Elicit Understandings and Encourage Reasoning about Decimals in Year 7	218
<i>Samone Davidson & Mike Askew</i>	
Block Model Approach in Problem Solving: Effects on Problem Solving Performance of the Grade V Pupils in Mathematics	226
<i>Niño Jose P. de Guzman & Rene R. Belecina</i>	
Constructing and Consolidating Mathematical Entities in the Context of Whole-Class Discussion	234
<i>Thérèse Dooley</i>	
Male Students' Perspectives Concerning the Relevance of Mathematics – Pilot Study Findings	242
<i>Michael Easey, Elizabeth Warren & Vince Geiger</i>	

Spatial Metaphors of the Number Line	250
<i>Cris Edmonds-Wathen</i>	
Reinventing the Wheel: Historical Perspectives on Theories for Interpreting Discourse Patterns in Mathematics Classrooms	258
<i>Nerida F. Ellerton, Pongchawee Vaiyavutjamai & M. A. (Ken) Clements</i>	
Young Children's Metarepresentational Competence in Data Modelling	266
<i>Lyn English</i>	
Challenges in Responding to Scaffolding Opportunities in the Mathematics Classroom	274
<i>Sarah Ferguson</i>	
Using Classroom Episodes to Foster Prospective Teachers' Didactical Knowledge: Issues for Teacher Education	282
<i>Rosa Tomás Ferreira, Luís Menezes & Maria Helena Martinho</i>	
Interpreting Graphs: Students Developing an Understanding of Covariation	290
<i>Noleine Fitzallen</i>	
Young Pedestrians' Gendering of Mathematics: Australia and Spain	298
<i>Helen Forgasz, Gilah Leder & Inés M^aGómez-Chacón</i>	
The Progress of Grade 1 Students Who Participated in an Extending Mathematical Understanding Intervention Program	306
<i>Ann Gervasoni, Linda Parish, Teresa Hadden, Carole Livesey, Kate Bevan, Melissa Croswell & Kathie Turkenburg</i>	
Auditing the Numeracy Demands of the Australian Curriculum	314
<i>Merrilyn Goos, Shelley Dole & Vince Geiger</i>	
Gesture Types for Functions	322
<i>Sandra Herbert</i>	
Evaluating Middle Years Students' Proportional Reasoning	330
<i>Annette Hilton, Geoff Hilton, Shelley Dole, Merrilyn Goos & Mia O'Brien</i>	
Singapore Students' Performance on Australian and Singapore Assessment Items	338
<i>Siew Yin Ho & Tom Lowrie</i>	
Developing Teacher Understanding of Early Algebraic Concepts Using Lesson Study	346
<i>Jodie Hunter</i>	
Designing Opportunities for Prospective Teachers to Facilitate Mathematics Discussions in Classrooms	354
<i>Roberta Hunter & Glenda Anthony</i>	
Professional Learning for Teaching Assistants and its Effect on Classroom Roles	362
<i>Chris Hurst & Len Sparrow</i>	
Curriculum Leadership: Reforming and Reshaping Successful Practice in Remote and Regional Indigenous Education	370
<i>Robyn Jorgensen</i>	

Digital Games for Learning Mathematics: Possibilities and Limitations	378
<i>Robyn Jorgensen (Zevenbergen) & Tom Lowrie</i>	
Influences of Metacognitive and Self-Regulated Learning Strategies for Reading on Mathematical Literacy of Adolescents in Australia and Singapore	385
<i>Berinderjeet Kaur & Shaljan Areepattamannil</i>	
Identifying Stages in a Learning Hierarchy for Use in Formative Assessment – the Example of Line Graphs	393
<i>Kaye Stacey, Beth Price & Vicki Steinle</i>	
Get Into Vocational Education (GIVE): Motivating Underperforming Students	401
<i>Gillian Kidman, Tom Cooper & David Nutchey</i>	
Mathematical Proficiency and the Sustainability of Participation: A New Ball Game through a Poststructuralist Lens	409
<i>Mary Klein</i>	
Developing a Culture of Collaboration	417
<i>Janeen Lamb & Jana Visnovska</i>	
K-2 Make it Count Students' Views of Mathematics	425
<i>Gilah Leder & Helen Forgasz</i>	
Variation and Mathematics Pedagogy	433
<i>Allen Leung</i>	
Supporting Secondary Novices' Efforts to Implement Student- and Discourse-Centred Pedagogical Practices	441
<i>Gary Lewis</i>	
The Hammer-and-Nail Phenomenon in Mathematics Education	449
<i>Kien H. Lim</i>	
Impulsive-Analytic Disposition in Mathematical Problem Solving: A Survey and a Mathematics Test	457
<i>Kien H. Lim & Amy Wagler</i>	
Teaching Algebra Conceptually: Student Achievement	465
<i>Chris Linsell, Lynn Tozer, Megan Anakin, Anna Cox, Rachel Jones, Eric McAuslan, Donna Smith & Garry Turner</i>	
Does Knowing More Advanced Mathematics Ensure Effectiveness of Working Towards Demonstrating Specialised Mathematical Content Knowledge of Second-Year Pre- Service Teachers?	473
<i>Sharyn Livy</i>	
The Development of an Assessment Tool: Student knowledge of the Concept of Place Value	481
<i>Karen Major</i>	
Projects, Puzzles and other Pedagogies: Working with Kids to Solve Local Problems	489
<i>Margaret Marshman</i>	

The Impact of a Professional Learning Intervention Designed to Enhance Year Six Students' Computational Estimation Performance	497
<i>Paula Mildenhall & Mark Hackling</i>	
An Exploration into Growing Patterns with Young Australian Indigenous Students	505
<i>Jodie Miller & Elizabeth Warren</i>	
The Four-Three-Four Model: Drawing on Partitioning, Equivalence, and Unit-Forming in a Quotient Sub-Construct Fraction Task	513
<i>Annie Mitchell</i>	
Virtual Mathematics Education: Using Second Life to Model and Reflect upon the Teaching of Mathematics	521
<i>Tracey Muir</i>	
Developing Pedagogical Strategies to Promote Structural Thinking in Early Mathematics	529
<i>Joanne T. Mulligan & Michael C. Mitchelmore</i>	
The Influence of Gender, Parents, and Background Variables on Perceived Usefulness of Mathematics among Grade 7 Students in Mozambique	537
<i>Adelino Murimo</i>	
Problem Categorisation in Ratio – A Closer Look	545
<i>Norhuda Musa & John Malone</i>	
Student Understanding of Large Numbers and Powers: The Effect of Incorporating Historical Ideas	554
<i>Mala S. Nataraj & Michael O. J. Thomas</i>	
The Concept of Generalised Number: Valuable Lessons from the History of Algebra	562
<i>Mala S. Nataraj & Michael O. J. Thomas</i>	
Mathematics Anxiety in Secondary School Students	570
<i>Lay Keow Ng</i>	
Students' Summaries of Mathematical Lectures: Comparing the Discourse of Students with the Discourse of Lectures	578
<i>Magnus Österholm</i>	
Identity and Ethnomathematics Projects in Papua New Guinea	586
<i>Kay Owens</i>	
Let's Count: Evaluation of a Pilot Early Mathematics Program in Low Socioeconomic Locations in Australia	594
<i>Bob Perry, Ann Gervasoni & Sue Dockett</i>	
Trialling a Professional Statistical Literacy Hierarchy for Teachers	602
<i>Robyn Pierce, Helen Chick, Jane Watson, Michael Dalton & Magdalena Les</i>	
The Classicist and the Frequentist Approach to Probability within a TinkerPlots2 Combinatorial Problem	610
<i>Theodosia Prodromou</i>	

Diffusion of the Mathematics Practical Paradigm in the Teaching of Problem Solving: Theory and Praxis	618
<i>Khiok Seng Quek, Yew Hoong Leong, Eng Guan Tay, Tin Lam Toh & Jaguthsing Dindyal</i>	
Coordinating Known and Unknown Quantities in a Multiplicative Context: Problem Conceptualization Affordances and Constraints	625
<i>Ajay Ramful</i>	
Strategies Used by Students to Compare Two Data Sets	633
<i>Robyn Reaburn</i>	
Exploring Student Reflective Practice during a Mathematical Modelling Challenge	640
<i>Trevor Redmond, Raymond Brown, Joanne Sheehy & Harry Kanasa</i>	
Steps in Developing a Quality Whole Number Place Value Assessment for Years 3-6: Unmasking the “Experts”	648
<i>Angela Rogers</i>	
On Diagnosis and Intervention: Helping Students with Special Needs Learn Fraction Ideas Involving Decimal Numbers	656
<i>Rebecca Seah</i>	
Chinese Primary Students’ Mathematical Task Types Preferences	664
<i>Wee Tiong Seah & Anastasios (Tasos) Barkatsas</i>	
Mathematical Modelling for Singapore Primary Classrooms: From a Teacher’s Lens	672
<i>Cynthia Seto, Thomas Mary Magdalene, Kit Ee Dawn Ng, Chun Ming Eric Chan & Wanty Widjaja</i>	
Empirical Evidence for Niss’ Implemented Anticipation in Mathematising Realistic Situations	680
<i>Gloria Stillman & Jill P. Brown</i>	
Supporting Teachers in Choosing and Using Challenging Mathematics Tasks	688
<i>Peter Sullivan, Doug Clarke, Debbie Michaels, Angela Mornane & Anne Roche</i>	
Insights into Ways that teachers plan their Mathematics Teaching	696
<i>Peter Sullivan, David Clarke, Doug Clarke, Peter Gould, David Leigh-Lancaster & Gerard Lewis</i>	
Students’ Ways of Knowing and Learning Mathematics and Their Ways of Interacting with Advanced Calculators	704
<i>Hazel Tan</i>	
Pedagogical Content Knowledge in Mathematical Modelling Instruction	712
<i>Liang Soon Tan & Keng Cheng Ang</i>	
Developing Mathematical Knowledge through Social Justice Pedagogy with Young Adult Arab Women	720
<i>Mohammed Goma Tanko & Bill Atweh</i>	
Developing Mathematical Resilience among Aboriginal Students	728
<i>Steve Thornton, Joanne Statton & Sophie Mountzouris</i>	

Use of Practical Worksheet in Teacher Education at the Undergraduate and Postgraduate Levels	736
<i>Pee Choon Toh, Tin Lam Toh, Foo Him Ho & Khiok Seng Quek</i>	
The Effect of Small-Group Game Play Activities on Number Sense Performance	744
<i>Rashidah Vapumarican & Manu Kapur</i>	
Average Revisited in Context	753
<i>Jane Watson & Helen Chick</i>	
Teaching for Abstraction: Collaborative Teacher Learning	761
<i>Paul White, Sue Wilson & Michael Mitchelmore</i>	
Australian Pre-Service Teachers Overseas Tour: Implications for Mathematics Teaching and Learning	769
<i>Allan Leslie White</i>	
Investigating Pre-service Teachers' Mathematics Anxiety Using the Revised Mathematics Anxiety Scale (RMARS)	777
<i>Sue Wilson</i>	
A Revolving Model of Pre-service Teacher Development in Mathematics	785
<i>Susanna Wilson</i>	
Nature of an Attitudes toward Learning Mathematics Questionnaire	793
<i>Khoon Yoong Wong & Qian Chen</i>	
Profiling Students' Capacities to Link Number and Algebra in Years 5, 6 and 7 in Nanjing, China	801
<i>Wenbin Xu, Max Stephens & Qinqiong Zhang</i>	
Problem Posing in Mathematical Investigation	809
<i>Joseph B. W. Yeo</i>	
Predictive Validity of Numeracy Entry Requirements for University: Pre-service Teachers' Mathematics Knowledge and Attitudes	817
<i>Jenny Young-Loveridge, Brenda Bicknell & Judith Mills</i>	
SYMPOSIA	
Working Through the Practice Architectures of First Year University Mathematics Teaching	827
<i>Jo Balatti & Shaun Belward</i>	
Monitoring and Analysing Attendance in First Year University Mathematics Tutorials	831
<i>Patrick Higgins & Wayne Read</i>	
Are Online Quizzes an Effective Tool for Mastering Basic Algebra?	835
<i>Wayne Read & Patrick Higgins</i>	
Doing it Differently: The Ups and Downs of Peer Group Learning	839
<i>Shaun Belward & Jo Balatti</i>	

ROUND TABLE DISCUSSIONS

Students' Conceptions of Equality	845
<i>Megan Anakin, Chris Linsell & Jeffrey Smith</i>	
Teaching Mathematics Respectfully: Preparing Culturally Responsive Mathematics Teachers	845
<i>Robin Averill & Megan Clark</i>	
Developing Self-Regulated Teacher Learners to Improve Student Outcomes in Mathematics	846
<i>Janette Bobis, Jenni Way & Judy Anderson</i>	
Professional Conversations among Mathematics Educators	847
<i>Rosemary Callingham, Kim Beswick, Helen Chick, Julie Clark, Merrilyn Goos, Barry Kissane, Pep Serow & Steve Thornton</i>	
Teachers' Mathematical Knowledge and Practice	847
<i>Mohan Chinnappan</i>	
Models and Modelling for the Future	848
<i>Lyn D. English, Richard Lesh, Kit Ee Dawn Ng & Gloria Stillman</i>	

SHORT COMMUNICATIONS

Early Childhood Mathematics: The Case of More or Less	853
<i>Mohan Chinnappan & Amy Chan</i>	
Critical Reflection as an Important Pathway to Pre-service Teachers' Development (A Snapshot)	853
<i>Kwee Gek Chua</i>	
The Use of Open-Ended Tasks as an Instructional and Assessment Tool	854
<i>Kum Fong Foo, Leng Low, Yen Ping Pang, Wai Leng Lye & Cherng Ginn Kenneth Lui</i>	
Investigating the Interrelationships Between Teachers' Pedagogical Content Knowledge and Student Achievement Within Vocational High School Context	854
<i>Vesife Hatisaru</i>	
Seeing is Believing: Building Mathematical Knowledge for Teaching Through Demonstration Lessons	855
<i>Louise Hodgson</i>	
Harnessing the Power of Cloud Computing for Mathematical Modelling in Two Different Schools	856
<i>Bock Boon Lim, Mei Chuen Chen, Trevor Redmond & Joanne Sheehy</i>	
iMPaCT-Math: Programming as a Means to Motivate Exploration of Foundational Algebraic Concepts	856
<i>Kien H. Lim, Eric Freudenthal, Art Duval & Amy Wagler</i>	

Translation of Word Problems by Year 6 Lower Ability Students: An Action Research Project	857
<i>Min Chern Lim & Kai Kow Joseph Yeo</i>	
Developing an Intervention Program for Students at Risk Drawing on the Strengths of Successful Existing Programs	857
<i>Bernadette Long</i>	
Classroom Goal Structure, Achievement Goals, and Achievement: A Multilevel Mediation Analysis of Longitudinal Data	858
<i>Wenshu Luo & David Hogan</i>	
Transforming Children's Mathematical and Scientific Development: A Longitudinal Study	858
<i>Joanne Mulligan, Lyn English, Michael Mitchelmore & Nathan Crevensten</i>	
Examining Opportunities and Constraints in the Use of Context Based Experiences for Engaging Indigenous Australian Students in the Learning of Numeracy	859
<i>Kate Naughtin</i>	
Developing Statistical Literacy: Student Learning and Teacher Education	859
<i>Hélia Oliveira, Rosa Tomás Ferreira, Ana Henriques, João Pedro da Ponte, Carolina Carvalho, Ana Paula Canavarro & Susana Colaço</i>	
Alleviating Maths Anxiety through Mentoring in an Emotional Intelligence Framework	860
<i>Timothy Perkins</i>	
Teachers' Perceptions of Student Engagement and Disengagement in Mathematics	860
<i>Karen Skilling, Janette Bobis, Andrew Martin, Judy Anderson & Jenni Way</i>	
Using Students' Algebraic Thinking to Support Teacher Learning	861
<i>Shikha Takker</i>	
Developing Computational Fluency in Multi-Digit Multiplication: A Learning Trajectory Approach	861
<i>Kristen Tripet, Janette Bobis & Jenni Way</i>	

POSTERS

Do not Call on Me: Mathematics Anxiety among Students with Learning Differences	865
<i>Michelle L.W. Bower & Suriza van der Sandt</i>	
Problem@Web: A Project around an On-line Problem Solving Mathematical Competition in Portugal	865
<i>Susana Carreira, Hélia Jacinto, Jaime Silva, Juan Rodriguez, Nélia Amado, Nuno Amaral, Rosa Tomás Ferreira, Sandra Nobre, Isa Martins & Sílvia Reis</i>	
A Longitudinal Study Preparing Preservice Teachers to Learn and Teach Mathematics with Technology	866
<i>Leah A. Nillas</i>	
An Analysis of Students' Strategies for Area Measurement and its Curricular Implications	866
<i>Jeenath Rahaman</i>	

KEYNOTES

The Beginnings of MERGA

Preamble to the Annual Clements/Foyster Lecture

In the middle of 1976 John Foyster, who was then based at the Australian Council for Educational Research (ACER), came to see me at Monash University, where I was in charge of the Mathematics Education program. John talked about how the Australian Science Education Research Association (ASERA) had recently been established, with Professor Richard Tisher (then of Monash University) as the prime mover. John wondered whether the time was ripe for a similar national group interested in *mathematics* education research to be established, and asked whether he and I might take steps to establish such a group.

My immediate reaction was yes, we should do it. Then came the doubts and reservations. How would the Australian Association of Mathematics Teachers (AAMT) react to such an initiative? After all, AAMT already had a “Research Committee.” In any case, would there be enough mathematics educators in Australia, interested in such a group to make it a viable proposition? Who would provide the funds likely to be needed for the establishment of such a group?

It was John’s and my opinion that the AAMT Research Committee had not reached out to embrace most of the people lecturing in mathematics education in Australia at teachers colleges or in universities at that time. Intuitively, I thought Australia needed a group like the one John was proposing. My intuition told me that AAMT was not the organisation to move towards the establishment of such a group.

John assured me that he would put up any funds needed to get the group going (and, hopefully, any group that was established would be able to pay him back within a few years). Hence we decided to proceed with the idea of establishing the group and to strike while the iron was hot, so to speak, by conducting a national conference at Monash University in the middle of 1977. I came up with the name “Mathematics Education Research Group of Australia” which John liked because of the acronym MERGA, which suggested a “merging together.” We sent out notices of our intention to form MERGA late in 1976. Neither of us knew many of the people who might be interested in joining such a group, so the notices were addressed to the “Mathematics Lecturers at ...”

Soon after we had decided to go ahead, I heard of the existence of a group, based in New South Wales, called the Mathematics Education Lecturers’ Association (MELA). John and I talked about whether MERGA and MELA might become one from the outset, but we decided that the aims of MELA seemed to be sufficiently different from those that we envisaged for MERGA, focused far more on research than lecturing, that we should proceed with the MERGA idea.

And so it came to be that in May 1977, the first of what was to become the annual conference of MERGA took place. About 100 people attended, with papers frenetically being read from 9 am to about 10 pm, for three days, in a Rotunda Theatre at Monash University. Professor Richard Tisher was present at the start of the Conference, and talked of his experiences in establishing ASERA. Frank Lester, of Indiana University, was among those present. In the event, two volumes of papers read at the Conference were produced (the first volume being available on the first day of the Conference, and the second several months later).

At a post-Conference meeting it was decided that, yes, MERGA should be formed, that the second meeting would be at Macquarie University in May 1978, and that an annual conferences should be held each year at a different academic institution. At that second conference it was decided by those present that MERGA should continue and a constitution and election of offices would be decided on at the third conference to be held at the then Brisbane College of Education. And so MERGA was born.

Ken Clements

MERGA 2012: Where We've Been, Where We Are, and Where We're Going

M. A. ("Ken") Clements
Illinois State University
<clements@ilstu.edu>

Why have an organisation like MERGA? This question will be addressed from past, present and future perspectives (1976, 2012, and 2025). One focus of the paper will be the need to improve mathematics curricula, and to improve the teaching and learning of mathematics, at all levels. I shall argue that we have not done enough to make sure that MERGA has delivered, is still delivering, and will continue to deliver the goods on such basic curriculum/teaching/learning issues. Part of the difficulty is that we researchers have not reached agreement on what we mean by "improvement". That is as much a political issue as anything else, of course, but the MERGA community needs to do more to make sure that the responsibility for defining what improvement means, and how it is assessed, is not in the wrong hands. A second focus of the paper will be some reflections on what the "A" in MERGA might represent. This Conference is being held in Singapore, and the challenge is for a wider vision of MERGA's role in Asia to be formulated and implemented.

This paper will present an analysis of the history of influence in international mathematics education from a non-European, non-North American perspective, and especially from Asian and Australasian perspectives. The focus will be on events and developments from the beginning of the twentieth century to the present time (2012). In the short space available I will attempt to re-interpret events, in a manner not unlike what Nerida Ellerton and I recently did in our book *Rewriting the History of School Mathematics in the United States 1607–1861* (Ellerton & Clements, 2012).

Although this paper is likely to generate heat, especially from some European and North American scholars, it is meant to be constructive rather than destructive. The paper draws special attention to developments which have taken place around the world over the past 50 years which have placed the international mathematics education community in its best position ever to establish much more equitable ways of operating.

Where We've Been (I): International Mathematics Education in a Colonialist Environment

The International Mathematics Education World in 1914

In 1914, when the world was on the brink of a Great War, colonialism in Africa had reached its zenith. There had been an unseemly "scramble for Africa", and virtually the entire continent was "held" by one European nation or another (see Table 1, which is based on information in Weir, 2009, p. 99). The map of the continent of Africa in 1914 was remarkable for the fact that only two nations—Liberia and Abyssinia—could really claim independence, the latter having gained its independence from Italy only in 1913.

It would be inappropriate in this paper to interpret the situation so far as colonial activity was concerned in Eastern Asia in 1914. It suffices to say that the period was

also characterised by a scramble for new markets and territories and has been dubbed the “New Imperialism” (see, e.g., Cribb, 1993; Porter, 1996).

Table 1
European Colonisation of Africa Around 1914

“Controlling” European Power	African Colonies
Great Britain	Basutoland, Bechuanaland, British Somaliland, British East Africa, Egypt, Gold Coast, Gambia, Nigeria, Northern Port Guinea, Rhodesia, Nyasaland, Sierra Leone, Southern Rhodesia, Sudan, Swaziland, Uganda, Union of South Africa
France	Algeria, French Equatorial Africa, French Somaliland, French West Africa, Guinea, Ivory Coast, Morocco, Senegal, Togo, Tunisia, Upper Volta
Belgium	Belgian Congo
Portugal	Angola, Mozambique
Germany	Cameroon, German East Africa, German South-West Africa
Spain	Rio de Oro, Spanish Morocco, Spanish Sahara
Italy	Eritrea, Italian Somaliland, Libya

Australia’s achievement of federation notwithstanding, around 1900 there was a conscious move from European nations to assert formal colonial control of large overseas territories. Established European powers in Asia—like the United Kingdom, France, and the Netherlands—succeeded in placing under their influence huge numbers of people in South East Asia, the Middle East, and the Indian Subcontinent. Around the same time the Japanese and German empires, Tsarist Russia; and the United States also began to emerge as new imperial powers in East Asia and in the Pacific Rim.

Relevance to mathematics education? But what might the above have to do with mathematics education and, in particular, what is the relevance for mathematics educators in Asia, in Australia, and in New Zealand in 2012? The answer is that the colonial masters had the responsibility of defining and controlling school education, including school mathematics, in the colonies that they administered. This control generated a colonialist mentality which dominated world thinking about mathematics education for much of the twentieth century.

The analysis presented here will indicate that in 1914 those seeking to internationalise mathematics education operated from a mindset which saw male mathematicians in Western Europe and the United States as the experts in both mathematics and the teaching and learning of mathematics. A small group of European and North American mathematicians took it upon themselves to coordinate a move to develop improved approaches to the teaching and learning of mathematics. These mathematicians believed that their strong mathematical backgrounds and their own educational experiences in top-class schools and universities qualified them to take on this responsibility (Curbera, 2009; Singh & Ellerton, 2012).

That kind of thinking in education was a direct extension of European and North American attitudes towards international relations. The Canadian philosopher, John Watson (1919), in *The State in Peace and War*, supported the need for “enlightened”

imperialism. According to Watson, well-qualified outsiders believed that they had a legitimate authority to take over and run the affairs of under-developed, troubled, countries provided that they exercised their authority in a way that would benefit the developing nations. Thus, for Watson, the concept of empire was not inherently morally offensive, but rather something that could uplift peoples and nations in need (Sullivan, 1983).

Moves towards the internationalisation of mathematics education. Early in the twentieth century there was not much agreement within and between nations, on the purposes of school mathematics. Some of the more polarised points of view were reflected in debates about desirable curricula and teaching methods that occurred in Europe and the United States, and also in far-flung parts of the world—like Victoria, Australia (Clements, 1975). In England, John Perry, a London professor of applied mathematics with experience working in Japan, was advocating a form of school mathematics, which came to be known as “Perryism”. That approach emphasised the use of laboratory methods in mathematics classes, the widespread adoption of the concept of function, and the coordination of school mathematics with the physical sciences, especially physics. In 1902, Perryism was strongly supported by the Chicago-based Eliakim Hastings Moore, then President of the American Mathematical Society, who took steps to introduce the approach into Chicago schools (see Clements, Keitel, Bishop, Kilpatrick & Leung, 2012).

But Perry’s and Moore’s radical, mathematical-modelling approach to school mathematics was condemned by Professor David Eugene Smith, of Teachers College, Columbia University in New York. Smith, a mathematics educator with close links to the mathematics community, was an author of North America’s best-selling school mathematics textbook series, and therefore had much to lose if Perryism was taken up on a large scale (Clements et al., 2012). His statements in opposition to Perry’s and Moore’s ideas pitted the mid-Western form of Perryism against what he claimed were the views of mathematics teachers on the East Coast of the United States. Smith’s language was direct and conservative to the point of being reactionary.

When summarizing the attitudes to Perryism of teachers in Eastern United States, Smith (1905) could not have adopted a more conservative stance when he wrote:

1. Any effort to introduce physical experiments into the classes in mathematics has no support whatever from either the teachers of mathematics or those of physics. ...
2. Any effort to seek the applications of mathematics chiefly in physics or in science generally, has not met with favor, and is not likely to find advocates. The consensus of opinion is that the number of applications of algebra to physics, for example, is exceedingly small, those to business being considerably larger, even though these are not numerous. ...
3. The attempt to have algebra and geometry appear to the pupil as having any considerable application to science or to business aside from a few special propositions will not be made. (p. 207)

Smith (1905) went on to say that teachers in the eastern states (of the United States) not only wanted their students “to love mathematics for its own sake” (p. 208), but also to be well prepared for rigid system of public examinations. According to Smith, teachers in the Eastern states would disagree strongly with the proposition that “no equation should be given without a genuine application, that no problem be assigned

without a genuine application, that no problem should be assigned without a physical context, that no topic shall be considered save as it bears upon life” (p. 208). He added that teachers in the eastern states wanted to develop “pure mathematics” laboratories in which pupil activity took place in the mind rather than with physical apparatus (Clements et al., 2012).

The above ideas expressed by Smith were important in the early history of international cooperation in mathematics education because, with Felix Klein, a leading German mathematician, Smith would become the major force behind early moves to establish an international community of mathematics educators. It was Smith and Klein who had the greatest influence on the formation and early work of the International Commission on Mathematical Instruction (ICMI)—a structure created in 1908 to internationalise mathematics education (Donoghue, 2008; Furinghetti, 2008).

Singh and Ellerton (2012) argue, in their chapter in the *Third International Handbook of Mathematics Education*, that the ICMI structure created in 1908 for the purpose of internationalising mathematics education was, from the outset, decidedly Eurocentric, with a strong bias towards European and North American mathematicians, and with little input from women or school teachers. ICMI was created at a meeting of the International Congress of Mathematicians (ICM) held in Zurich in 1908, and although the conference organisers had hoped to gather in Zurich “mathematicians from all countries on earth” (Curbera, 2009, p. 9), the 208 who actually attended (204 males, 4 females) came from just 16 nations, 15 of which were European. The non-European nation was the United States of America, which had seven conference attendees. Much power was given to a handful of people who were on a central committee which decided that ICMI's decisions would need to be ratified by 36 voting delegates: three each from Austria, France, Germany, Great Britain, Hungary, Italy, Russia, Switzerland, and the United States of America, and one each from Belgium, Denmark, Greece, The Netherlands, Norway, Portugal, Romania, Spain, and Sweden. Thus, 33 of 36 delegates (i.e., persons with voting rights at ICMI meetings) were to be from European nations, and the other three were to be from the United States (Schubring, 2008).

What is striking is how the early governance structure of ICMI was dominated by European mathematicians and educators, and subject to the control of ICM. It is easy to understand how this happened, of course, but that does not hide the fact that although ICMI identified eight “associated countries” (Australia, Brazil, Bulgaria, Canada, Cape Colony, Japan, Mexico, and Serbia), the voting nations for ICMI had a total population of about 480 million, or only about 30 percent of the world's population. Voting considerations aside, the following nations were among those not represented at ICMI meetings: China, India, Indonesia, Iran, Korea, Nigeria, The Philippines, Thailand, and Vietnam—nations that accounted for more than half the world's population.

Singh and Ellerton (2012) did not mince words when discussing the matter:

From the outset, ICMI was not representative of the worldwide body of mathematics teachers. Despite the best intentions, the central committee and its voting members were largely white, European, and male, and mathematics teachers in schools were not represented. Although ICMI intended that its vision would be *international*, its structure belied that vision. The principal form of collaboration in ICMI operations was little more than a sharing of information between its members. It was not truly global in its outreach, and its restricted mission arose out of an unfortunate internal belief that only certain mathematically advanced people in certain

mathematically advanced nations possessed sufficient knowledge and wisdom about mathematics and mathematics education to be worthy of being admitted to ICMI's "inner circle". This belief would permeate ICMI for more than 50 years. (p. 810)

When Singh and Ellerton (2012) studied the mathematics curricula, examination and teaching methods produced for ICMI-member nations and associates, they were struck by a report on the teaching of mathematics in Japan (Fujisawa, 1912).

Whereas the other national reports (mostly from European nations) were, from, from Singh and Ellerton's perspective, mostly tedious summaries of recent curricular and examination changes (see, for example, Young, Osgood, Smith, & Taylor, 1915), the Japanese report concentrated on pedagogical matters. It offered rich commentary on methods for teaching difficult topics in mathematics (e.g., fractions, ratio, elementary algebra)—comments which continue to have freshness and relevance even a century after they were written. These reports came from school teachers, mathematicians, school and college administrators, and government officials from all over Japan. The English version of the final Japanese report comprised 15 chapters, prepared by 17 educators and mathematicians, and occupied 238 pages. Within Japan, this was truly a collaborative effort, and was an achievement made all the more remarkable by the fact that the Japanese respondents were given very little time to prepare the report and have it translated into French and English (Fukisawa, 1912).

According to Singh and Ellerton (2012), there was much to be learned from the Japanese report, but the general method of collaboration adopted by ICMI meant that the important ideas in the Japanese report were not discussed outside of Japan. The issue of whether the West had much to learn from the East was not to be taken very seriously in a largely Eurocentric organization like ICMI, which seemed to be convinced that the best mathematicians, and therefore, probably, the best teaching of mathematics, were to be found mainly within Europe or North America. ICMI was officially part of ICM, and during its early existence, in the first half of the twentieth century, it remained firmly under the wing of mathematicians (Furinghetti, 2008).

Fast Forward More than 50 Years: New Mathematics, Royaumont, and a Revival of an Exclusive Club

The Soviet Union's successful launching in 1957 of the satellite *Sputnik* brought a strong reaction from Western European and American governments. During the period 1957 through 1959 the U.S. government worked with the Organisation for European Economic Cooperation (OEEC) to organize an international conference, which would be held in Royaumont, France, in 1959, for the purpose of developing new thinking in mathematics and mathematical education (Furinghetti, Matos & Menghini, 2012; Moon, 1986). Attendance at this conference was by invitation, with each participating country being asked to send three delegates: a mathematician, a mathematics educator, and a secondary school teacher. In fact, most of those who attended the Royaumont Conference were university-based mathematicians or mathematics educators, with the leadership coming from mathematicians.

The Royaumont conference was attended by representatives from 18 countries, 16 of which were European, the other 2 being Canada and the United States (Singh & Ellerton, 2012). From an international mathematics education perspective, the structure and attendance closely resembled that of the first ICMI conferences held before World War 1. African, Asian and Australasian nations were not invited to send

delegates who would make presentations to the conference. Singh and Ellerton (2012) saw this episode as evidence of a revival of the old-boy, mathematics-dominated, European/North American club which had controlled the only moderately-successful earlier attempts at international collaboration in mathematics education.

Most of the Stimulus for Change Came from “Outside”

But a change in thinking would gradually emerge during the second half of the twentieth century. And, significantly, the first stimulus for change would come from outside the worlds of mathematics and mathematics education. That source was a group of psychometricians—not including any well-known mathematicians or mathematics educators—who in 1958 met at a UNESCO office in Germany to discuss problems of school and student evaluation. These scholars decided to gather international data on school students’ knowledge of key mathematical concepts, their attitudes towards mathematics, and the extent of their participation in mathematics. They decided to risk asking participating nations to pay for the costs of the studies. In 1967, this group became formally known as the International Association for the Evaluation of Educational Achievement (hereafter IEA) (Postlethwaite, 1967).

From the outset the IEA studies relied on careful design and rigorous statistical analysis. Pencil-and-paper tests were developed that corresponded to curricula adopted in participating nations. The delicate issue of translating tests so that, as far as possible, equivalent tests in different languages would be produced, was also faced, although on that matter Ellerton and Clements (2002) have argued that solutions reached were inadequate. Participating nations had to agree to meet strict stratified random sampling requirements. Not surprisingly, often mathematicians and mathematics educators complained that they had not been adequately consulted in what were, clearly, international mathematics education research studies (Keitel & Kilpatrick, 1999).

Data for IEA’s First International Mathematics Study (FIMS) were collected in the early 1960s. FIMS targeted 13-year-old students and pre-university students, and participating nations were Australia, Belgium, England, Finland, France, Germany (FRG), Israel, Japan, the Netherlands, Scotland, Sweden, and the United States. When the results were released, in the mid 1960s, it was found that, relatively speaking, Japanese students performed extremely well, and those from Australia and Israel performed at least as well as, if not better than, European students (Husén, 1967). Immediately the world was faced with the compelling question: was the teaching and learning of school mathematics better in Japan than in Europe and North America?

Governments of participating nations were satisfied with the IEA’s work on FIMS, and were content to leave future IEA studies into international achievement in school mathematics—and specifically the Second International Mathematics Study (SIMS), and the Third International Mathematics and Science Study (TIMSS)—largely in the hands of IEA psychometricians and psychologists. Admittedly, these “outsiders” were assisted by carefully-chosen curriculum and content authorities, including some well-regarded mathematics education researchers (Postlethwaite, 1993).

SIMS data were gathered between 1977 and 1981. Participating nations were Belgium, Canada, England and Wales, Finland, France, Hong Kong, Hungary, Israel, Japan, Luxembourg, Netherlands, New Zealand, Nigeria, Scotland, Swaziland, Sweden, Thailand, and the United States. Analyses of SIMS data showed that at the

13-year-old level, students from Japan and Hong Kong achieved the highest means (Robitaille & Garden, 1989; Travers & Westbury, 1989). The concern raised by FIMS was now magnified almost to the extent of paranoia, and Western mathematicians, educators, parents and governments began to demand that something be done to improve the situation in their countries (Clements, 2003).

Data for TIMSS were gathered in the early 1990s from half a million students in more than 15,000 schools in 45 participating nations. Students were mainly at three levels, Grade 4, Grade 8, and end-of-secondary school. Supposedly equivalent tests were developed in more than 30 different languages, and strict sampling procedures were followed within most of the participating nations. In addition, a TIMSS video study, focusing on eighth-grade mathematics classes in Germany, Japan and the United States, was conducted (Hiebert, Stigler, & Manaster, 1999). TIMSS analyses indicated that at both the fourth- and eighth-grade levels, the four best-performing nations were the four participating Confucian-heritage nations: Singapore, Korea, Japan, and Hong Kong. The mean for U.S. fourth-grade students was slightly above the international average, and the mean for students from England and Wales was slightly below the mean. At the eighth-grade level, the means for U.S. students and for students from England and Wales were below the international mean. The lowest national mean scores were from South Africa.

TIMSS results, together with results from other international comparative studies, especially the PISA (Programme for International Student Assessment) studies conducted, since 2000, by the Organisation for Economic Co-operation and Development (OECD) (see Neubrand, 2005), confirmed that students in Confucian-heritage nations such as Singapore, Hong Kong, Japan and Korea, tended to perform better than students from North America and most European nations (an exception, perhaps, being Finland) (Stacey, 2010). Western European politicians and educators struggled to cope with the situation. For example, in 2002, Edelgard Bulmahn, then German Federal Minister for Education and Research, in reacting to Germany's below-average placement on the PISA international league table, exploded:

The findings of the OECD Programme for International Student Assessment are alarming. A country with the economic and political significance of Germany belongs at the top of the league and cannot be satisfied with an education system performing at the OECD average level—never mind below it.

But PISA should not mislead us into starting discussions about reforming our education system from scratch again. In fact, many of the shortcomings PISA brought to light were not new. Earlier studies had already drawn attention to the weaknesses of our educational system and prompted me to set a new educational reform process in motion. (Bulmahn, 2004, p. 1)

By the time the Mathematics Education Research Group of Australia (MERGA) held its first annual conference, in 1977, there had already been three International Congresses of Mathematics Education (ICMEs), conducted through ICMI. But these first three ICMEs had all been held in Europe (Lyon, France, in 1969, Exeter, UK, in 1972, and Karlsruhe, Germany, in 1976). The fourth ICME would be held in San Francisco, California, in 1980. The first annual conference of the International Group for the Psychology of Mathematics Education (PME1) was held in the Netherlands in 1977, and the next four annual PME conferences were held in Germany (1978), the United Kingdom (1979), the United States (1980), and France (1981). The fact that scholars from all nations were invited and welcomed at these conferences could not hide the lingering Eurocentric and North American biases.

All this would change—the last four annual PME conferences have been held in Mexico (2008), Greece (2009), Brazil (2010), and Turkey (2011), and the next one is about to be held in Taiwan (2012). A life-member, and former president of MERGA, Gilah Leder of Australia, was awarded the Felix Klein Award for 2009 by ICMI; a former editor of the *Mathematics Education Research Journal*, Bill Barton, of New Zealand, served as president of ICMI between 2008 and 2012; Lyn English, an Australian member of MERGA, was the founding editor (and continues as editor) of the American-based journal *Mathematical Thinking and Learning* (Singh & Ellerton, 2012).

Much more could be said, but one thing is clear—the old assumptions that characterised thinking about relative excellence in school mathematics during the first 60 years of the twentieth century are, in the thinking of most of today’s mathematics educators, dead and buried. Today, only the brave would contradict an assertion that the world has much to learn from Eastern Asia about mathematics curriculum management, and about the teaching and learning of mathematics (Leung, Graf, & Lopez-Real, 2006).

Where We’ve Been (II): 1976, and Reflections on Developments after the Founding of MERGA in 1977

As Clements (2007) and Mousley (2009) have pointed out that, the seeds of an idea for the establishment of an Australian national group in mathematics education research were sown in Melbourne, by John Foyster and me, in 1976. The Mathematics Education Research Group of Australia (MERGA) was formally established after the first MERGA annual conference held at Monash University in 1977. Judy Mousley (2009) and I have claimed that MERGA was the world’s first national mathematics education research group to be formally created, and that claim has been repeated by Hodgson, Rogers, Lerman and Lim-Teo (2012). In the early 1980s the group changed its name to the Mathematics Education Research Group of Australasia, and from the early 1990s New Zealand mathematics educators became active participants in MERGA. At its 2011 annual meeting, the ICMI Executive Committee officially approved the affiliation of MERGA to ICMI as a multinational organisation involved in mathematics education (Hodgson et al., 2012).

This is not the place to summarize the unusual set of events marking the creation of MERGA in 1977. It is important, though, to see the establishment of MERGA in the context of the history of the internationalisation of mathematics education. In 1974 when I accepted an appointment as Lecturer in Mathematics Education at Monash University—I had been a secondary mathematics teachers for 10 years before that—those responsible for mathematics teacher education programs in Australia were isolated. I knew personally, only one tertiary-level mathematics educator—George Booker—outside of the state of Victoria (although I had met a few others at a biennial meeting of the Australian Association of Mathematics Teachers). I was only barely aware that several Australian scholars (e.g., Roly Mortlock of Western Australia, Graham Jones of Queensland, and Merv Dunkley and John Conroy of New South Wales) had gained doctorates in mathematics education (I think all of them from the United States). At Monash University, I used printed reference lists passed on to me by Roly Mortlock, and wondered about the possibility of establishing a national mathematics education research association, with annual conferences and a journal.

In 1975, the international world of mathematics education was hardly known to me. I had never attended an overseas conference, and I looked to the United Kingdom and to North America for my inspiration. Although I had heard of scholars like Zoltan Dienes, Geoffrey Matthews, George Polya, Alan Bishop and Richard Skemp, Dienes was the only one whose ideas affected my teaching at Monash in a big way. Yet, interestingly, walking around the Monash corridors with me at that time was a group of young scholars interested in becoming fully-qualified mathematics education researchers. Among this group were the following—I list the names in alphabetical order—Barbara Clarke, David Clarke, Doug Clarke, Phil Clarkson, Gilah Leder, Charles Lovitt, Anne Newman, and Dianne Siemon. These were all young and extremely impressive. At the time, I sensed that an organisation like MERGA was sorely needed if we were to have any chance of unlocking the massive potential which existed at Monash. We needed an organisation that brought together mathematics education scholars—admittedly a poorly defined group at that time—from all across the nation, on a regular basis. And so, in 1977, at the first annual MERGA conference, a national association for mathematics education researchers was born.

One problem was that nobody outside of Australia knew or cared much about mathematics curricula and the teaching and learning of mathematics in Australia. When, in 1977, Hans Freudenthal looked around for someone to write a statement on school mathematics in Australia for publication in *Educational Studies in Mathematics*, the person he chose was Professor Larry Blakers, a Western Australian mathematician (Blakers, 1978). As far as I know, the fledgling mathematics education research community in Australia was not consulted on who should represent Australia. Blakers undoubtedly had an interest in school mathematics, but the fact that he was chosen, by a European scholar, to represent Australian mathematics education, despite the fact that his *main* scholarly interests and experience were in mathematics and not mathematics education, suggested that the type of thinking that had characterized international mathematics education between 1900 and 1970 was still alive and well. Blakers' (1978) article was competent, but had a top-down tone.

As a result of the creation of MERGA, members of the previously isolated and scattered group of would-be Australian mathematics education researchers was able to come together and, for the first time, feel as one. Despite some opposition to the idea, a majority decided to have an annual conference, and spoke of creating a natural and empathetic outlet for publishing summaries of their work. By the early 1980s it had become standard practice for promising young Australian mathematics education researchers to complete their graduate degrees in Australia, and not, as previously, in the United Kingdom or the United States. Every four years MERGA published summaries of mathematics education research completed by Australian mathematics education researchers. When New Zealand became an integral part of MERGA in the early 1990s, the work of the Group was enriched, and soon the transformation to a vibrant internationally-oriented research community was complete. Every year, at annual meetings of the International Group for the Psychology of Mathematics Education, MERGA was extremely well represented in the list of those presenting papers. The fifth ICME conference was held in Australia in 1984 and, in that same year, PME also had its annual conference in Australia.

The horizons of many MERGA members were beyond Australia and New Zealand. Over the past 30 years there has always been a leaven of mathematics education researchers in Australia and New Zealand who have made conscious efforts

to become aware of, identify with, and contribute to literature associated with mathematics curricula, mathematics teaching and learning, and mathematics education research in Southeast Asian nations. The cover of Nerida Ellerton's and my *Mathematics Education Research: Past, Present and Future*, published by UNESCO in 1996, showed an image of the earth in the form of a globe, with a spotlight shining on the Southeast Asian region (Ellerton & Clements, 1996). In our Preface to that book we indicated that there were more references in the book to reports on research carried out in the Asia-Pacific region than to research carried out in the United States of America. It was argued that the writings of Piaget and Vygotsky, and the Continental *didactique* research may not have much relevance "to the nations in the Asia-Pacific region where linguistic and cultural patterns are typically very different from those in Europe" (p. vii). The book included passages like:

After all, it was on the basis of Piagetian stage theory that researchers in the period 1950–1980 carried out and reported investigations which purported to show that many children in Asia-Pacific nations—such as students in Papua New Guinea Community Schools, and Aboriginal Australian children—thought in "prelogical" ways and were three or four years "behind" Western children from the perspective of cognitive development. Such conclusions can now be seen as degrading and highly value-laden. (p. vii)

The assumption that it is reasonable to accept a form of mathematics education that results in a large proportion of school children learning to feel incompetent and helpless so far as "Western" Mathematics is concerned should be rejected. Alternative forms of mathematics education by which greater value should be accorded to the cultural and linguistic backgrounds of learners should be explored. (p. 178)

The present international mathematics education research community needs to move proactively so that full and equal participation is possible for mathematics educators in countries which are currently under-represented in the community. (p. 188)

It is hardly surprising that although Ellerton and Clements's (1996) book has been, and still is, widely used in universities in the Middle East (e.g., in Iran), it has not won much recognition in Continental Europe or North America.

That said, it should be acknowledged that over the past 40 years, ICMI has made many moves towards achieving equal and equitable participation in the international mathematics education research community. For example, six years ago a 596-page, 40-chapter book entitled *Mathematics Education in Different Cultural Traditions—A Comparative Study of East Asia and the West* was published as an outcome of the 13th ICMI Study Conference, which had been held in Hong Kong in 2002 (Leung, Graf, & Lopez-Real, 2006). This volume had authors from every continent, and offered important commentary on historical, cultural and contextual background factors. The authors' well-argued contributions provided important input on issues like whether the West had much to learn from the high performance of Confucian-background students in international comparative studies.

It would be naïve, however, not to recognise that during the period 1977 to 2012 Western European nations (especially France, Germany, Italy, Spain and the Netherlands), and the United States have maintained very influential power blocs within the international mathematics education research community. *JRME* and *ESM* have remained the dominant journals, and from my perspective it is desirable that that situation changes. Although we would all want *JRME* and *ESM* to continue to be important journals, we need high-class, international, refereed mathematics education research journals coming out of Asia, Africa, the Middle East, and South America.

Slowly but surely *MERJ* has achieved a solid international recognition, and we are now ready for the next step. Analyses of attendance figures at the annual PME conferences over a period of 35 years would reveal that, overall, the numbers of scholars attending from Australia or New Zealand, or from Israel, are higher than from any other single nation except the United States (a fact which has not been reflected in the choice of keynote speakers). If one looks carefully at the national affiliations of authors in the three Kluwer/Springer *International Handbooks of Mathematics Education*, then one finds that Australian and New Zealand authors have been more numerous than authors from any other nation except the United States of America, and the number of authors from Eastern Asia has steadily grown (Singh & Ellerton, 2012).

Where Are We Now? New Scenarios and Opportunities, 2012

The paranoia that swept across Europe and North America resulting from the strong performance of Confucian-heritage nations on FIMS, SIMS and TIMSS, and on PISA, has created an opportunity for mathematics educators in Asia, Australia, New Zealand (and indeed many other countries) to work to achieve greater equity and participation in the increasingly important international world of mathematics education research. From that perspective, it is timely that MERGA is now seen as a well-respected international organization (Hodgson et al., 2012), with a high-quality refereed research journal, the *Mathematics Education Research Journal (MERJ)*, published by Springer,

However, there is still a way to go. The two top international mathematics education research journals are the North American-based *Journal for Research in Mathematics Education (JRME)* and the European-based *Educational Studies in Mathematics Education (ESM)*. Other highly regarded international mathematics education research journals include MERGA's own *Mathematics Education Research Journal (MERJ)*, the Canadian-based *For the Learning of Mathematics (FLM)*, the US-based *Mathematical Thinking and Learning (MTL)*, *School Science and Mathematics (SSM)*, the *Journal of Mathematical Behavior (JMB)*, the German-based *ZDM*, and the French-based *Recherches en Didactique des Mathématiques*. There are others—I hope that I will be forgiven for my sins of omission.

The trouble is, as Hodgson et al.'s (2012) review makes clear, *MERJ* is still being viewed in Europe and North America as being a regional journal rather than an international journal. One only has to look at the workplaces of the authors of *MERJ* articles over the past 10 years to recognise that such a view is unfair, but nevertheless the perception remains. Hodgson et al. (2012) have written:

MERJ is clearly among highly regarded research journals in mathematics education. However, its ambition to address the total Australasian context is still partly problematic, in particular with respect to the origins of its editorial board members, reviewers, and authors. (p. 866)

Notice the emphasis in this statement on *MERJ* addressing “the total Australasian context.” Robyn Jorgenson, the current *MERJ* editor, has taken steps to ensure that *MERJ* is seen as being genuinely international. From my perspective, it is important that MERGA seeks to do all that is necessary to make *MERJ* as influential, from an international research perspective, as *JRME* and *ESM*. That leads me to my next set of points, which will be the most controversial, and most difficult to achieve, of the points I make in this paper.

Where We're Going? The Mathematics Education Research Group of AustralASIA

It seems to me that the mathematics education research community in Southern and Eastern Asia, and in Australia and New Zealand would benefit greatly from a new merger, by which MERGA (a name which I consciously created to indicate a merging of previously separated bodies that had like interests) would formally come to represent Southern and Eastern Asia, Australia, New Zealand and the surrounding Pacific islands. It occurred to me that that a solid preparation for such a move would be, formally, to decide that MERGA is to be an acronym for “AustralASIA” and not “Australasia”.

This Conference is the first MERGA annual Conference to be held outside of Australia and New Zealand. I commend the present MERGA committee for its wisdom and courage to make such a move. I also commend the Singapore mathematics education research community for contributing to what could become a watershed event in the history of the international mathematics education community. If my proposal were to be listened to, I would imagine the MERGA annual conferences being located, each year, at a different Southern or Eastern Asian venue, or in Australia, New Zealand, or a surrounding Pacific island. The MERGA committee would reflect the new, extended emphasis, with elected members being based in any of India, Pakistan, Sri Lanka, Myanmar, Indonesia, Thailand, Malaysia, Singapore, Vietnam, the Philippines, China, Japan, Korea, ..., Australia, New Zealand, Fiji, etc.

When I was President of MERGA (1987–1993), we developed a policy whereby MERGA's annual conferences would take place in the full range of states and regions in Australia and New Zealand. We reasoned that, in time, such a policy would expand the Group's membership by making mathematics education researchers *across* Australia and New Zealand feel that they “owned” MERGA. The acronym *MERJ* for the journal (which, as President, I recommended to the 1988 annual meeting of MERGA) was also meant to suggest a process of “merging”, of offering “collaborative ownership”. I did not completely win a battle within the MERGA committee that raged for a while on desirable emphases for *MERJ* at that time, because, from the outset, I was opposed to the idea that *MERJ* should have a special focus on Australian and New Zealand interests and issues. I wanted *MERJ* not only to be fully international, *but also to be seen as* fully international. The decision to compromise my vision at that time may have been wise—I'm not sure about that—but what we have now are comments by Hodgson et al. (2012), quoted above, that, in my view, unfairly, but understandably, detract from the hard-won internationalism of our lead journal.

If my idea were to be accepted by a majority of current MERGA members then there would need to be some constitutional changes, for MERGA is an incorporated entity (thanks to the excellent work to achieve that status by Gilah Leder and Mike Mitchelmore in the mid-1990s). However, I think that if my proposal for the “A” in MERGA to stand for AustralASIA were to be accepted, then such constitutional changes could be kept to a minimum.

2025: A Vision

If the proposal that I have outlined above were to be accepted so that from the 2013 annual meeting of MERGA onwards, say, MERGA would “belong” to a much

wider base of international mathematics education scholars, then I foresee the following being in place by 2025, at the latest:

1. The nations that would be part of MERGA would form one of three major power blocs in international mathematics education research—the other two being North America and Western Europe.
2. *MERJ* would become the natural journal for submission of research carried out by scholars from the expanded set of MERGA nations. It would quickly (within a few years) achieve the same status as *JRME* and *ESM*, and in time would achieve a higher status than either of those two journals. After all, the nations associated with the new Group would have about one-half of the world's total population.
3. Just as, during the period 1976–2012, when MERGA became a family uniting scholars from all Australian states and from New Zealand, the new MERGA would become a family of scholars—only this time the family would unite scholars from a much wider-ranging group of nations. These scholars would get to know each other well, and respect each other—just as current MERGA members based in Australia and New Zealand currently know and respect each other.
4. Collaborative international research projects involving mathematics education researchers from the various MERGA member nations would be facilitated. Quite simply, there would be much more opportunity for MERGA researchers to include in their investigations a wider-than-ever-before range of cultural, religious, socio-economic, political and linguistic variables.

The above four benefits would quickly arise from the establishment of the new MERGA. The new MERGA committee members could meet (via Skype) on a regular basis, and debate proposals such as:

1. Should we establish annual (or biennial) MERGA research awards for excellence in research in MERGA nations? Such an award might be called the “Ben Nebres Award”.
2. Should we work with the Southeast-Asian Ministers of Education Organization to develop its mathematics education centres (e.g., RECSAM, in Penang, Malaysia), so that they become centres of research and professional development excellence, with MERGA-member nations providing high-level research appointments to the centres. Note that Singh and Ellerton (2012) recently argued that RECSAM, in Malaysia (which is almost 50 years old) is unique, in the sense that that there is nothing like it in other parts of the world.
3. What should MERGA do to ensure that high quality, culturally appropriate, mathematics education opportunities are available to all persons in all its member nations?
4. What should MERGA do to ensure that the potential benefits of the findings of high-quality mathematics education research are being realised in educational institutions at all levels, and in technologically-appropriate distance-education settings, throughout MERGA member nations.

Closing Comments

We, who form the MERGA community, should be proud of our achievements over the past 35 years. There are, and have been, some great people, and great mathematics education researchers who have helped give MERGA the influence and reputation it

has today. I know that although I have been away from my homeland for 14 years, I will always be proud of MERGA.

Most professional adults at the age of 35 have reached a level of maturity that allows them to ask themselves what they should now do to maximise the benefits of all that they've achieved thus far. I believe that MERGA members should be asking the same kind of question now with respect to the future of MERGA. The door of opportunity is open, but we need to be brave enough to walk through that door into a wider world.

Of course, the radical agenda that I have placed before us in this paper will also require the agreement of other scholars, in other nations before it can be operationalised. It would be entirely proper for Asian nations to reject the vision that I've put forward. Let us hope that we can all look to the future, and seek to establish a way of working together that will draw mathematics education researchers in this part of the world together in realistic, fulfilling ways.

Thank you, Singapore, for providing this venue for this highly significant, perhaps groundbreaking, event.

I wish to close this paper with the following lengthy quotation from my editor's introduction to the *Third International Handbook of Mathematics Education* (Clements et al., 2012):

Mathematics is one of the few areas in an individual's life in which she or he is required to spend between three and five hours per week (and, in addition, more hours on homework or with a tutor), for between 10 and 12 years (at least) studying a curriculum defined by others. What a waste of everyone's time, energy, and money, if students do not learn school mathematics as well as they possibly can, so that they develop an interest in the subject and an appreciation of its power to help them deal efficiently with important everyday problems. Furthermore, I believe that success with the subject is likely to be associated with greater satisfaction in later life (because successful students are more likely to take up vocations of their choice, or gain entry to a wider range of courses in higher education institutions). From a national perspective, the benefit of having a mathematically-competent citizenry is, it is often asserted, likely to result in strong economic performance (or, at least, stronger than would be the case if most citizens were not mathematically competent). Thus, it is important that research be conducted which will take into account students' attitudes towards mathematics, as well as their mathematical problem-solving and problem-posing performances.

But if mathematics education research is important, then how well are we doing in fostering the highest possible quality of mathematics learning as a result of our mathematics education research? Let's not put our heads in the sand on this matter. There is certainly a lot of room for improvement! The nation which has the most qualified mathematics education researchers is probably the United States of America—yet, many indicators (including results on international comparative studies) suggest that many U.S. students fail to learn mathematics well. How could that be the case, considering the amount of research that has been conducted, and published, within the United States, over so many years?

It is well known that many students, in most nations (perhaps all nations), experience difficulty in understanding fractions, the four operations with integers, and elementary algebra. We need to face the reality that many learners experience much difficulty in mathematising situations for which mathematical approaches to problem solving would be informative and efficient. Why has there not been a marked improvement, given the large amount of mathematics education research conducted around the world, and over a very long period of time, with respect to such fundamentally important curriculum matters? Should our standard curricula, and teaching approaches, be problematised and reconceptualised? (p. 6)

I'll stop there, for I've probably said too much already. It's over to you. Thank you for the honor you have all paid me for asking me to give this address.

References

- Blakers, A. L. (1978). Changes in mathematics education since the 1950's—Ideas and realisation (Australia). *Educational Studies in Mathematics*, 9(2), 147–158.
- Bulhamn, E. (2002). PISA: The consequences for Germany. *OECD Observer*, 231–232, 1.
- Clements, M. A. (1975). Are there some lessons to be learnt from the introduction of the New Maths into Victorian secondary schools in 1905? In D. Williams (Ed.), *Perspective in teaching mathematics* (pp. 20–56). Melbourne, Australia: Mathematical Association of Victoria.
- Clements, M. A. (2003). An outsider's view of North American school mathematics curriculum trends. In G. M. A. Stanic & J. Kilpatrick (Eds.), *A history of school mathematics* (pp. 1509–1580). Reston, VA: National Council of Teachers of Mathematics.
- Clements, M. A. (2007). The beginnings of MERGA. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice*. Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Hobart (Vol. 1, p. 2). Adelaide, Australia: Mathematics Education Research Group of Australasia.
- Clements, M. A., Keitel, C. Bishop, A. J., Kilpatrick, J., & Leung, F. (2012). From the few to the many: Historical perspectives on who should learn mathematics. In M. A. Clements, A. Bishop, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Third international handbook of mathematics education* (pp. 10–44). Dordrecht, The Netherlands: Springer.
- Clements, M. A., & Ellerton, N. F. (1996). *Mathematics education research: Past, present and future*. Bangkok, Thailand: UNESCO.
- Cribb, R. (1993). Development policy in the early 20th century. In J. P. Dirkse, F. Hüskens, & M. Rutten (Eds.), *Development and social welfare: Indonesia's experiences under the New Order* (pp. 225–245). Leiden, The Netherlands: Koninklijk Instituut voor Taal-, Land- en Volkenkunde.
- Curbera, G. P. (2009). *Mathematicians of the world, unite! The International Congress of Mathematicians: A human endeavor*. Wellesley, MA: A. K. Peters.
- Donoghue, E. F. (2008). David Eugene Smith and the founding of the International Commission on the Teaching of Mathematics. *International Journal for the History of Mathematics Education*, 3(2), 35–46.
- Ellerton, N. F., & Clements, M. A. (2002). Translating pencil-and-paper mathematics tests from one language to another: The myth of equivalence. In H. S. Dhindsa, I. P.-A. Cheong, C. P. Tendencia, & M. A. Clements (Eds.), *Realities in science, mathematics and technical education* (pp. 245–254). Gadong, Brunei Darussalam: Universiti Brunei Darussalam.
- Ellerton, N. F., & Clements, M. A. (2012). *Rewriting the history of school mathematics in the United States 1607–1861*. Dordrecht, The Netherlands: Springer.
- Fukisawa, R. (Ed.). (1912). *Summary report of the teaching of mathematics in Japan*. Tokyo, Japan: Department of Education.
- Furinghetti, F. (2008). Mathematics education in the ICMI perspective. *International Journal for the History of Mathematics Education*, 3(2), 47–56.
- Furinghetti, F., Matos, J. M., & Menghini, M. (2012). From mathematics and education, to mathematics education. In M. A. Clements, A. Bishop, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Third international handbook of mathematics education* (pp. 211–234). Dordrecht, The Netherlands: Springer.
- Hiebert, J., Stigler, J. W., & Manaster, A. B. (1999). Mathematical features of lessons in the TIMSS video study. *Zentralblatt für Didaktik der Mathematik*, 31(6), 196–201.
- Hodgson, B. R., Rogers, L. F., Lerman, S., & Lim-Teo, S. K. (2012). International organizations in mathematics education. In M. A. Clements, A. Bishop, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Third international handbook of mathematics education* (pp. 848–877). Dordrecht, The Netherlands: Springer.
- Husén, T. (Ed.). (1967). *A comparison of twelve countries: International Study of Achievement in Mathematics* (Vols. 1–2). Stockholm, Sweden: Almqvist & Wiksell.
- Keitel, C., & Kilpatrick, J. (1999). The rationality and irrationality of international comparative studies. In G. Kaiser, G. Luna, & I. Huntley (Eds.), *International comparisons in mathematics education* (pp. 241–256). London, UK: Routledge.
- Leung, F. K. S., Graf, K.-D., & Lopez-Real, F. J. (Eds.). (2006). *Mathematics education is different cultural traditions—A comparative study of East Asia and the West: The 13th ICMI Study*. New York, NY: Springer.
- Moon, B. (1986). *The “new maths” curriculum controversy: An international story*. London, UK: Falmer Press.

- Mousley, J. (2009). *A history of MERGA*. Retrieved from <http://www.merga.net.au/node/75>
- Neubrand, M. (2005). The PISA study: Challenge and impetus to research in mathematics education. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 79–82). Melbourne, Australia: International Group for the Psychology of Mathematics Education.
- Porter, B. (1996). *The lion's share: A short history of British imperialism 1850–1995*. London, UK: Longman.
- Postlethwaite, T. N. (Ed.). (1967). *School organisation and student achievement: A study based on achievement in mathematics in twelve countries*. Stockholm, Sweden: Almqvist & Wiksell.
- Postlethwaite, T. N. (1993). Torsten Husén (1916–). *Prospects: The Quarterly Review of Comparative Education*, 23(3/4), 677–686.
- Robitaille, D. F., & Garden, R. A. (Eds.). (1989). *The IEA study of mathematics II: Context and outcomes of school mathematics*. Oxford, UK: Pergamon Press.
- Schubring, G. (2008). The origins and the early history of ICMI. *International Journal for the History of Mathematics*, 3(2), 3–33.
- Singh, P., & Ellerton, N. F. (2012). International collaborative studies in mathematics education. In M. A. Clements, A. Bishop, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Third international handbook of mathematics education* (pp. 806–839). Dordrecht, The Netherlands: Springer.
- Smith, D. E. (1905). Movements in mathematical teaching. *School Science and Mathematics*, 5(3), 205–209.
- Stacey, K. (2010). Mathematics and scientific literacy around the world. *Journal of Science and Mathematics Education in Southeast Asia*, 33(1), 1–16.
- Sullivan, E. P. (1983). Liberalism and imperialism: J. S. Mill's Defense of the British Empire," *Journal of the History of Ideas*, 44(4), 599–617.
- Travers, K. J., & Westbury, I. (Eds.). (1989). *The IEA study of mathematics I: Analysis of mathematics curricula*. Oxford, UK: Pergamon Press.
- Watson, J. (1919). *The state in peace and war*. Glasgow, Scotland: James Maclehose.
- Weir, S. (2009). *History's worst decisions and the people who made them*. New York, NY: Metro Books.
- Young, J. W. A., Osgood, W. F., Smith, D. E., & Taylor, E. H. (Eds.). (1915). *Mathematics in the middle commercial and industrial schools of various countries represented in the International Commission on the Teaching of Mathematics*. Washington, DC: Government Printing Office.

Mathematics Education as a Multicultural Field of Research and Practice: Outcomes and Challenges

Michèle Artigue,
LDAR, Université Paris Diderot – Paris 7
<artigue@math.jussieu.fr>

Mathematics education seen both as a field of research and as a field of practice is a multicultural field. For a while, this essential characteristic of the field has been underestimated but this is no longer the case. Both theoretical frameworks and empirical research help us today better approach this phenomenon, understand its impact on the field, reflect on research outcomes and the value of knowledge progressively built as well as on the challenges that we globally face as a community. In this lecture I would like to share with the audience my vision of this evolution and of the potential it offers for the field of mathematics education, relying on my personal experience as a researcher raised in a specific culture but also involved in many international collaborations, and on my engagement in ICMI, the International Commission on Mathematical Instruction, an institution which, for more than one century, has tried to contribute to the development of mathematics education through international exchanges and collaboration.

Introduction

Mathematics education seen both as a field of research and as a field of practice is a multicultural field. For a while, this essential characteristic of the field has been underestimated but this is no longer the case. From the pioneering work of Ubiratan d'Ambrosio, ethnomathematics has developed as a specific domain in mathematics education, demonstrating the diversity and richness of the mathematical cultures which have accompanied and still accompany the development of human communities all over the world, and investigating how to make mathematics education benefit from this richness and diversity (D'Ambrosio, 2008). But the sensitivity to the cultural dimension of the field has extended far beyond the ethnomathematics community, thanks to the increasing influence of socio-cultural (Lerman, 2000), anthropological (Chevallard, 1992) and more recently socio-political approaches (Gutiérrez, 2010), thanks also to the multiplication of research and innovative projects crossing cultural and educational contexts. As a researcher engaged in mathematics education for more than 30 years, I have experienced this evolution and observed how, progressively, it has impacted the field, research interests and practices. I have also experienced that there is a huge distance between the intellectual acknowledgement that mathematics education is a multicultural field and a practice which pays the necessary attention to contexts and cultures, which emancipates from predominant cultural views. In the first part of this text, I will review from this perspective ICMI history, showing how the process has been slow, even for such an institution created for fostering cultural exchanges and communications. I will then trace this progressive evolution in my own trajectory before evoking some recent projects which directly address this issue. I will end with some more global comments, balancing the problematic and positive dimensions of this cultural dependence.

ICMI: A Slow Progression towards True Internationalization

ICMI, the International Commission for Mathematical Instruction¹, was founded in 1908 at the International Congress of Mathematicians (ICM) in Rome, following an initial proposal made by the US mathematics educator David Eugene Smith in the journal *l'Enseignement Mathématique*, in 1905 (Menghini, Furinghetti, Giacardi & Arzarello, 2008). At that time of intense curricular reforms and debates regarding mathematics education, the ambition of the promoters of ICMI was clear: fostering international communication and exchange for stimulating reflection on sensitive issues such as:

- school organization, teaching methods and assessment modes;
- content issues, such as the relationships between geometry and algebra courses, the introduction of Calculus in secondary schools, the place to be given to applications with respect to pure mathematics, the differentiation of syllabi according to students' orientations;
- the preparation of teachers...

During the first years of its existence, under the Presidency of Felix Klein, the commission developed an intense activity. National subcommittees were created in 18 fully participating countries and 15 associated countries. These committees collected a huge amount of data, and produced more than 300 reports. This was certainly an impressive success but more in terms of data collection and descriptive texts than in terms of analysis and interpretation. Moreover, this was essentially a Western enterprise. Among the 18 fully participating countries, 16 were European countries, the two others being Russia and the USA. Asia was poorly represented with three countries only, China, India and Japan, all among the 15 associate members with no voting right. Moreover, until the Second World War, all the members of the ICMI Central Committees were from Europe and the USA.

After a period of reduced activity between the two World Wars, the Commission was re-established after the Second World War, becoming in 1952 a permanent sub-commission of the International Mathematical Union (IMU) charged with the conduct of the activities of IMU bearing on mathematical or scientific education. At that time, ICMI became more international with 27 member countries including India, Japan, Malaysia-Singapore, Pakistan, the election of the first non-Western member to its Executive Committee in 1954: Ram Behari from India, followed by Yasuo Akizuki from Japan and Aleksandr Danilovic Aleksandrov from USSR in 1958, and the development of actions in Latin-America, Africa and Asia. The first ICMI regional network: the *Comité Interamericano de Educación Matemática*, CIAEM, was established in 1962, at the initiative of ICMI President Marshall H. Stone after a conference held in Bogota in 1961. The links established at that time with UNESCO contributed to the strengthening of international exchanges and perspectives. Beyond the collaboration in the publication of the well-known series *New Trends in Mathematics Education* whose first volume was published in 1967, the support provided to the establishment of the first ICME (International Congress in Mathematics Education) in 1969 and to the CIAEM Conferences, ICMI and UNESCO co-operated for instance in the organization of two meetings held in Africa, one in Nairobi in 1974 on Language and Mathematics Teaching and one in Khartoum in 1978 on Developing Mathematics in Third World Countries.

With Hodgson's (2009) analysis, a new step was achieved when Shokichi Iyanaga became vice-President in 1971, and then President of ICMI in 1975, the General-Secretary being

¹ At that time, the commission was designated by CIEM, ICTM or IMUK according to the language used, French, English or German.

Yukiyosi Kawada². That period coincided with the founding of the Southeast Asian Mathematical Society in 1972 and of national mathematical societies in Southeast Asia, these societies showing from the beginning evident concern for school mathematics and teacher education (Nebres 2009). This situation led to the organization of the first Southeast Asian Conference on Mathematics Education by the young Southeast Asian Mathematical Society with the support of ICMI in Manila in 1978, following a recommendation from Kawada. SEACME conferences were then held regularly every three years until 2002. In 1980s, when China rejoined IMU and ICMI, formal collaboration started to develop in East Asia beyond that existing in South East Asia with two ICMI – China Regional Conferences on Mathematics Education in Beijing in 1991 and then Shanghai in 1994, leading to the first EARCOME Conference in Korea in 1998, these conferences being held now every two years. Nebres pointed out that this situation

introduced Southeast Asian mathematicians and mathematics educators to cultures with very successful approaches to mathematics education, which had significant differences from the approaches of the West. This was important as it invited our mathematics educators to ask whether we should not develop approaches more in conformity with our situations, rather than simply imitating trends from the developed countries. (2008, p.152)

East Asia was indeed emerging as a new center for mathematics education, due to the outstanding performance of its countries in international comparative studies such as SIMS and TIMSS (Robitaille & Garden, 1989), (Mullis & al., 1997). East Asian educational practices were valued and quickly became object of systematic study, collaborative projects began to develop with East Asian countries. The participation of East Asia in ICMI activities progressively increased leading in particular to:

- the organization of annual conferences of the ICMI Affiliated Study Group Psychology of Mathematics Education in East Asia: Tsukuba (Japan) in 1993, Hiroshima (Japan) in 2000, and Seoul (Korea) in 2007);
- the organization of an ICME congress for the first time in Asia: ICME-9 in Makuhari (Japan) in 2000;
- the launching of the East/West ICMI Study in 2000.

Extending ICMI outreach beyond the Western world and affluent countries was also a major ambition of Miguel de Guzmán, when he became President of ICMI in 1991. He was at the origin of the Solidarity Fund that permitted ICMI to take effective actions in Central America, Palestine, and some African countries, in the nineties. He was also at the helm of the Solidarity Task force held at ICME-8, Sevilla in 1996, which proposed a general policy of reserving a portion of the registrations fees for ICME congresses (typically 10%) to provide grants assisting the participation of delegates from non-affluent countries.

Under the presidency of Hyman Bass, Michèle Artigue and that of Bill Barton, extending ICMI outreach is still a major focus in the ICMI agenda (about 90 countries are today ICMI members), associated with the conviction that, beyond extending its outreach, ICMI had to find ways to give more exposure to the voices of those from outside the Western world and from the periphery on the international scene. Eventually, in 2000, ideas already expressed by Akizuki

Oriental philosophies and religions are of a very different kind from those of the West. I can therefore imagine that there might also exist different modes of thinking even in mathematics. Thus I think that we should not limit ourselves to applying directly the methods which are currently considered in Europe and America to be the best, but should study mathematical instruction in Asia properly. Such a study might prove to be of interest and value for the West as well as for the East (1959).

² Since that time, East Asian countries have always been represented in the ICMI EC.

were implemented in the ICMI Study 13 mentioned above comparing mathematics education cultures from East Asian countries of Confucian tradition to those of Western countries (Leung, Graf & Lopez-Real, 2006). For the first time with ICMI Study 20 on multilingual issues in mathematics education, the responsibility of chairing an ICMI Study was given to two researchers from the periphery: Maria do Carmo from Brazil and Mamokgethi Setati from South Africa. For the first time also, in 2008, an ICME Congress took place in an emergent country: Mexico. Moreover, two more regional networks were created: EMF (*Espace Mathématique Francophone*) and AFRICME (*Africa Regional Conference of ICMI on Mathematics Education*). EMF is built on a notion of ‘region’ defined in linguistic rather than geographical terms, French being a common language among participants. It aims at supporting the participation to international exchanges of Francophone countries which suffer from the increasing domination of English as the international language of communication, and specific attention is paid to Africa. The sites selected for the conferences of this network which alternate developing and developed countries evidence this point: Grenoble, France (2000), Tozeur, Tunisia (2003), Sherebrook, Canada (2006), Dakar, Senegal (2009), Geneva, Switzerland (2012), Algeria (2015). The second regional structure is the African network, AFRICME, launched in 2005 for Anglophone countries of Africa. Using these new networks to facilitate contacts inside Africa, independently of the language, remains however a challenge to be overcome. The new program established in 2011 under Bill Barton’s presidency, in collaboration with UNESCO: CANP (Capacity and Networking Program in the Mathematical Sciences) is the last outcome of this policy, involving the different communities in charge of teacher education in a given regional area, paying specific attention to regional context and needs, and fostering regional networking. This philosophy was perfectly reflected by the first realization in Bamako, Mali for Francophone sub-Saharan countries, in September 2011, paving the way for the next two planned in Costa Rica in 2012 and Cambodia in 2013.

There is no doubt that thanks to this evolution, ICMI has become progressively more and more sensitive to the cultural and social dimensions of mathematics education, and to the real challenges faced by mathematics education in a number of countries. At the ICMI Centennial Symposium in Rome, Nebres pointed out that

The opening of ICMI to countries from Asia, Africa and Latin America also began to bring up concerns different from the central concerns coming from Europe and North America. Among the major concerns and issues that have entered the mathematics education discourse:

- Ethnomathematics, especially from Latin America and, in particular, the work of Ubiratan d’Ambrosio
- Mathematics for All, coming in great part from UNESCO
- The impact of society, economics and culture on mathematics education.

Voices from Latin America and from Africa have emerged in the ICMEs and other international and regional conferences. (2008, p. 152)

Despite all these efforts, the representation of the different cultures in ICMI activities remains today seriously unbalanced. This is even true, as pointed out in (Leung, 2008) for the Asiatic cultures mentioned above, despite the international interest they raise, and confirms that breaking with the prevalence of dominant voices and dominant cultures, which takes both visible and invisible forms, is not something easy to achieve. This is necessarily a long term enterprise, and a battle that we can never be sure to have definitively won. International institutions such as ICMI certainly can play an important role but, as individuals and members of more local communities, we have also to incorporate this sensitivity, and make it more than an intellectual idea. This leads me to the second part of this text.

Reflecting on a Personal Trajectory

My career began at the time of the New Math period in the early seventies, and while preparing my doctorate in logic, I was asked by André Revuz, one of the professors at the Ecole Normale Supérieure in Paris to participate in the activities of the Institute of Research in Mathematics Education (IREM) recently created at University Paris 7 and headed by him. The IREMs³ were and still are a very specific structure, where mathematicians, secondary teachers, teacher educators, researchers in mathematics education work together part time, developing experimentation and innovation, producing educational resources and organizing professional development activities for teachers. Some years later, André Revuz obtained the creation of an elementary experimental school attached to the IREM in the south of Paris, and with two other colleagues, I accepted to pilot the teaching of mathematics in this school. This creation was influenced by the on-going experience of Guy Brousseau at the COREM in Bordeaux (Brousseau, 2008), and we used and adapted many situations he had built there before developing our own constructions. This first experience was for me very rewarding. Everything seemed to work perfectly. Pupils were learning, teachers played their role, parents were happy... The emergent theory of didactical situations (Brousseau, 1997) provided an effective framework for the construction of situations, their management in the classroom, and the analysis of their outcomes, and our research, for instance on pupils' conceptions of the idea of circle contributed to its consolidation and development (Artigue & Robinet, 1982).

I could say nearly the same of my second important experience some years later when together with a group of physicists, we decided to create an experimental course for first year university students tightly connecting the teaching of mathematics and physics, introducing multi-disciplinary projects and computer laboratory sessions, which was very innovative at that time (1980). Everything worked fine except the planned common lecture on differentials, for which the mathematician and the physicist in charge were not able to conciliate their positions. With the physics educators of the team, we thus decided to take this theme of differential processes in mathematics and in physics as an object of research, combining historical and epistemological approaches, the study of didactical transposition in the two disciplines, the investigation of students' conceptions and difficulties, and the design and experimentation of teaching resources in the two disciplines. The project was also extended to integral processes with colleagues from Grenoble University (Artigue & al., 1988), (Artigue, Menigaux & Viennot, 1990). Once again, the theoretical frameworks we used combining the affordances of didactics of mathematics and physics efficiently supported our work, and conversely beyond its local results our research led to consolidate and refine them.

At that time, the early eighties, the French community of didacticians had decided to create specific institutions for supporting its collective work and development: a journal *Recherches en Didactique des Mathématiques*, a national seminar taking place two consecutive days three times per year, and a summer school of two weeks each alternative year. Thanks to these institutions, communication and exchanges were easy; capitalization of knowledge was systematically organized in a friendly atmosphere, which did not prevent fierce scientific debates. I had the feeling of being part of a community which was addressing important social issues, was creating efficient conceptual and methodological tools for addressing these issues in close connection with the terrain, and was accumulating results which constituted a coherent landscape. Interesting and challenging questions regularly emerged from research work, from the terrain, from curricular changes or technological advances, giving to the field an evident dynamics.

³ See the portal www.univ-irem.fr/

At that time, I already attended some international events but these had a marginal impact on my vision of the field, all the more as Spanish being my first foreign language, my understanding of English was very limited. The situation progressively changed for me in the late eighties, with the contribution to master and doctoral courses in Spain, the regular participation to PME and to its organization in 1989, my election at its international committee the next year, and a first collaborative project with Brazilian colleagues. But, looking retrospectively at this period, I would say that this internationalization was rather superficial. The courses I gave in foreign countries presented the French didactic culture. In PME I was exposed to other educational cultures but, at that time, essentially Western cultures, and regarding my first collaboration with Brazilian colleagues, I would confess that it did not escape the trap of cultural neo-colonialism.

Substantial changes occurred later. My increased engagement in ICMI activities (congresses and studies), and more especially my participation to the ICMI executive from 1998, played a crucial role. As mentioned above, we shared the ambition of extending ICMI outreach and of giving more voice to countries on the periphery, and in particular to African countries which, with the exception of South Africa, were poorly represented in ICMI. I learnt a lot from the privileged contacts I had with Jill Adler within the ICMI EC, and from the activities of the AFRICME and EMF regional networks. I also learnt a lot from the discussions around ICMI Study 13 launched in 2000 and from the resulting volume when it came out, and of course from my increasing participation in international research projects where the multiculturalism of the field was an essential issue, and where specific theoretical constructs and methodologies were used and developed for addressing it. All these experiences made an evolution possible, and the cultural nature of mathematics education something more for me than an intellectual claim.

Looking retrospectively at this journey, there is no doubt that this is a long and slow journey still unfinished. The fact that I began my career in a Western and developed country at a time when cultural issues and approaches were not on the top of the educational agenda, at a time when international exchanges and collaborations were not developed as they are today, certainly contributed to it. Moreover I conjecture that the strength of the didactic culture I was acculturated to and the home-grown nature of the theoretical frameworks progressively built in it were not facilitating conditions. Today, nevertheless, I find a powerful support for addressing the challenge of multiculturalism in this didactic culture and especially in the anthropological approach due to Chevallard (Chevallard, 1992), (Bosch & Gascón, 2006). In the next part, I will try to share this experience with the MERGA community, hoping that, despite the cultural distance, my discourse will make sense for it and my concerns will resonate at least partly with its preoccupations.

A Situated Anthropological Perspective

As any sensitivity, sensitivity to the cultural dependence of mathematics education, must be supported by appropriate constructs and methodological tools for being productive. With the development of socio-cultural approaches, the field of mathematics education today offers a diversity of theoretical frameworks and constructs for such a purpose. As many French colleagues, due to my cultural environment, I have found a support in the Anthropological Theory of Didactics (ATD). In this theory initiated by Chevallard, indeed, an initial postulate is that human knowledge emerges from practices which are institutionally situated thus a fortiori culturally situated. What it means to know fractions or functions for instance is not a universal but depends on the norms and values of the institution at stake. Quite early, I have experienced the power of this approach for addressing transition issues, supervising Brigitte Grugeon's thesis (Grugeon, 1995) on the transition between vocational and general education. She tried to understand why motivated and talented students from vocational education generally failed when

they were offered the possibility of integrating general education. In reaction to the common educational discourse attributing this failure to students' cognitive characteristics, she conjectured that the main obstacle faced by these students and their teachers was the gap between the algebraic cultures of vocational and general high schools, a gap all the more difficult to detect as the syllabuses at stake seemed rather close. The success of the didactic strategy she developed from this institutional analysis proved the power of this approach. She decided indeed to value the vocational students' algebraic culture based on formulas, to consolidate it and organize its transition towards the functional world, instead of offering them the usual and unsuccessful transition courses focusing on the manipulation of algebraic expressions and solving of equations, two genres of task emblematic of the entrance in the algebraic culture in general education.

Nevertheless, it was only recently that I used a similar approach for addressing cultural differences between countries. I did it first in collaboration with Carl Winslow in a meta-analysis of comparative studies (Artigue & Winslow, 2010) and then in a Brazilian-French comparative study (Alves Dias, Artigue, Campos, & Jahn, 2010). I will present and discuss the latter example first.

A Brazilian-French Comparative Study

This comparative study was part of a bigger project involving two institutions in each country, addressing both the issue of teachers' representations and practices, and the design and appropriation by teachers of educational resources. One of its foci aimed at the collaborative development of digital resources for the secondary-tertiary transition and, due to its mathematical and educational importance, it was decided to focus on the concept of function. Our sensitivity to cultural issues led us to consider that a preliminary step should be the elucidation of the functional culture in the two countries and of the characteristics of the secondary-tertiary transition in that area. For this purpose, we decided to rely especially on the ATD notions of praxeology and of hierarchy of levels of co-determination which had proved to be productive in the meta-study mentioned above. I introduce these notions which are not necessarily familiar to the MERGA audience below.

Praxeologies and hierarchy of levels of co-determination: In ATD all human practices, thus mathematical and didactical practices, are modelled in terms of praxeologies. In its simplest form, a praxeology is pointwise and structured around a *type of task* (for instance, finding an algebraic expression for a quadratic function given by its graphical representation), a *technique* (a way, not necessarily algorithmic, for solving this task), a *technology* which is a discourse used to describe, explain and justify the technique, and a *theory* within which the technological discourse itself can be justified. A praxeology thus combines a practical part formed by the first two components and a theoretical and discursive part. A fundamental assumption is that any praxeology necessarily has such a discursive component even if this is difficult to access or embryonic, and that it is especially the case for school praxeologies. The progression of knowledge goes along with the progressive development and structuration of praxeologies. Pointwise praxeologies group into local praxeologies sharing a common technology and local praxeologies into regional praxeologies sharing the same piece of theory. Mathematical praxeologies differ from one institution to another, defining what it means to know such or such piece of mathematics in a given institution.

For understanding the *raison d'être* of a particular praxeology in a given institution, its current functioning and potential evolution, one must investigate the conditions and constraints governing its ecology. These may be situated at very different levels, which is the source of the

hierarchy of levels of co-determination (Chevallard, 2002), structured into nine different levels which, ordered from the lowest to the highest level, are the following: subject, theme, sector, domain, discipline, pedagogy, school, society, civilization (cf. Figure 1). This last level must be understood as the level corresponding to conditions and constraints which transcend one given society. For instance, ICME Study 13 mentioned above has made clear the influence on East-Asia mathematics education of the Confucian culture, and no one would negate that today international evaluations such as TIMSS and PISA exert an increased influence on educational systems worldwide.

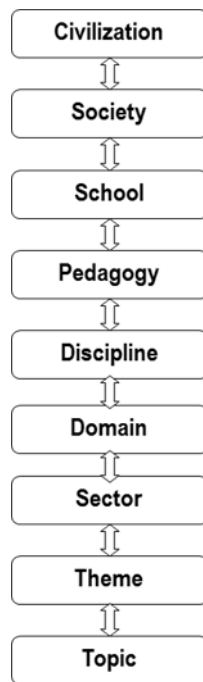


Figure 1. Hierarchy of levels of co-determination

For instance functions, a subject in grade 12 in France can be the variation of exponential functions, this subject being part of the theme of exponential functions, itself included in the sector of functions of one real variable, itself included in the domain of Analysis, a sub-domain of the mathematics discipline. But current teaching practices regarding this topic cannot be understood without considering influences situated at higher levels of codetermination, for instance, the insistence put by the CNP (Conseil National des Programmes) on the necessity of better connecting the teaching of scientific disciplines in France, in the nineties. This led to the introduction of exponential functions as solutions of differential equations modelling phenomena such as radioactive decay in the 2000 high school reform and influenced the teaching of the theme. Organizations, such as the one just described regarding exponential functions, are meaningful or not according to the educational context. Understanding the influence on the mathematical organizations observed in a given context of higher levels of co-determination is thus of crucial importance for making visible the constraints shaping mathematical and didactical praxeologies, for making these intelligible, and also for understanding what can be changed and how.

Coming back to the Brazilian-French study, for identifying the functional culture in the two countries, we tried to identify functional praxeologies and the conditions and constraints shaping these at different levels of co-determination. For that purpose, we collected a lot of data: curricular documents, textbooks, state or national examinations at the end of secondary school and also at university entrance in Brazil. I cannot here enter into the details of the analysis, but the resulting picture was that of two very different functional cultures, more than could be expected considering the common influences we are submitted to in this globalized world.

For instance, the traditional habitat of functions in senior secondary education in Brazil is the domain of algebra while in France it is calculus and analysis. This affects the curricular organization of subjects, themes and sectors related to functions, and the associated mathematical praxeologies in the two countries. In Brazilian secondary education, this algebraic tradition leads to give a particular importance to polynomial functions, and especially quadratic functions (theme level), and to the algebraic resolution of equations involving such functions (subject level). Coherently with this perspective, the introduction of parameters plays an essential role in the complexification of students' activity for a given praxeology and in access to generalization. In the French context, the study of functions in the two last years of high school is part of analysis (domain level). This contributes to the emphasis put on the study of the variation of functions including asymptotic behavior and the search for extrema (subject level), and also to exponential and trigonometric functions beyond polynomial functions (theme level). Effects of these differences in domains of inscription are even more visible when passing to the sector of sequences. For instance, in the Brazilian context, for the central theme of recurrent sequences a typical task is the determination of terms and reason for arithmetic and geometric sequences, on the basis of some partial information. The associated praxeologies are based on algebraic techniques and technology. Such a task is nearly nonexistent in France. A typical task is the study of the convergence of such sequences both qualitatively and quantitatively. The associated praxeologies use analytic techniques and technology, for instance theorems of convergence for monotonic sequences, the theorem ensuring that, if f is a continuous function, the limit of a recurrent sequence $u_{n+1}=f(u_n)$ is necessarily a fixed point of f , or analytic techniques for overestimating the distance between two successive terms of the sequence or between the general term and the limit.

Such differences cannot not be understood without considering the highest levels of the hierarchy of co-determination. For instance, in France, differential and integral calculus has been part of the secondary curriculum since the 1902 reform whose ambition was to give scientific humanities a similar position to classical humanities (school and society levels). Achieving a better balance between quantitative and qualitative approaches towards convergence was an explicit aim of the 1982 reform rejecting the values of the New Math period, and its emergence was non independent on the facilities offered by technological advances (Artigue, 2011). In Brazil, the observed characteristics also reflect a tradition of algebraic teaching in secondary schools whose origin can be traced up to the turn of the nineteenth century. It emphasizes transformation rules for algebraic expression and the solving of equations (Miorim et al., 1993). During the New Maths reform, the algebraic domain even gained in importance and set theoretical visions whose influence is still visible impacted the teaching of functions, but analysis remained a domain for tertiary education. The fact that mathematics education is not differentiated in Brazil at senior secondary level, contrary to France, that in France the scientific section is the "elite" section and that its syllabus influences those of other sections, certainly also contributes to this characteristic. The early influence of technological advances mentioned above for France did not have its counterpart in Brazil until recently, and one cannot doubt that higher levels of co-determination (society level) contribute to this difference. Even today, calculators are

forbidden at most entrance tests to university in Brazil. Such a situation impacts lower levels, from the school one, and certainly contributes to the importance attached to the mastery of techniques for computations involving numbers or algebraic expressions. Another characteristic contributing to the stability of the algebraic Brazilian tradition is the importance traditionally attached to analytical geometry which also uses algebraic techniques (discipline level). This contrasts with the French situation where, due to a long term tradition, the teaching of geometry is more oriented towards synthetic geometry.

Our research also showed that, beyond these differences, the two traditions now have to cope with international perspectives promoting the values of an education for democracy and citizenship. Functional praxeologies are affected by this context in the two countries. In Brazil, this influence is especially visible in the tasks proposed at the national test ENEM taken at the end of high school, and in recent textbooks. Most of the ENEM tasks that we analyzed for instance, and all those involving functions, were linked to a real world context, and often societal or environmental issues. In France, this influence is also visible but it impacts less the teaching of functions at the end of high school, especially in scientific sections.

I will not enter into more details, but supported by these theoretical constructs, the efforts put at understanding these two functional cultures and their *raisons d'être*, deeply impacted our vision of the whole research project, and of the way we could envisage a productive collaboration in the production of educational resources for supporting the secondary / tertiary transition in the two countries. They revealed strengths and weaknesses in the two cultures, logics which were certainly different but equally coherent and could not be boiled over. Comparing with the way we had approached collaboration in the first project CAPES-COFECUB two decades before, this made a tremendous difference.

The ReMath Project

The second example I would like to present is the European project Representing Mathematics with Technology, also known as ReMath. With this project, a new dimension enters the scene: the dependence of educational research itself on educational contexts and cultures. This project, led by Chronis Kynigos from Greece and mobilizing six teams from four European countries, has now officially ended but we are still working on the data collected and deepening our analyses⁴. Its ambition was to develop an integrated vision of the potential offered by digital representations for the learning of mathematics, and we considered that this goal could not be achieved without building appropriate connections between the different theoretical frameworks we were relying on. As made visible on figure 2 were the main theoretical frameworks used by ReMath teams are presented, there was an important theoretical diversity, which reflected the diversity of our respective research cultures. This diversity affected our vision of the representative potential of digital technologies for mathematical learning, and of the design and use of digital artifacts.

⁴ Main deliverables, digital and educational resources produced in the frame of ReMath are accessible on the website: <http://remath.cti.gr>

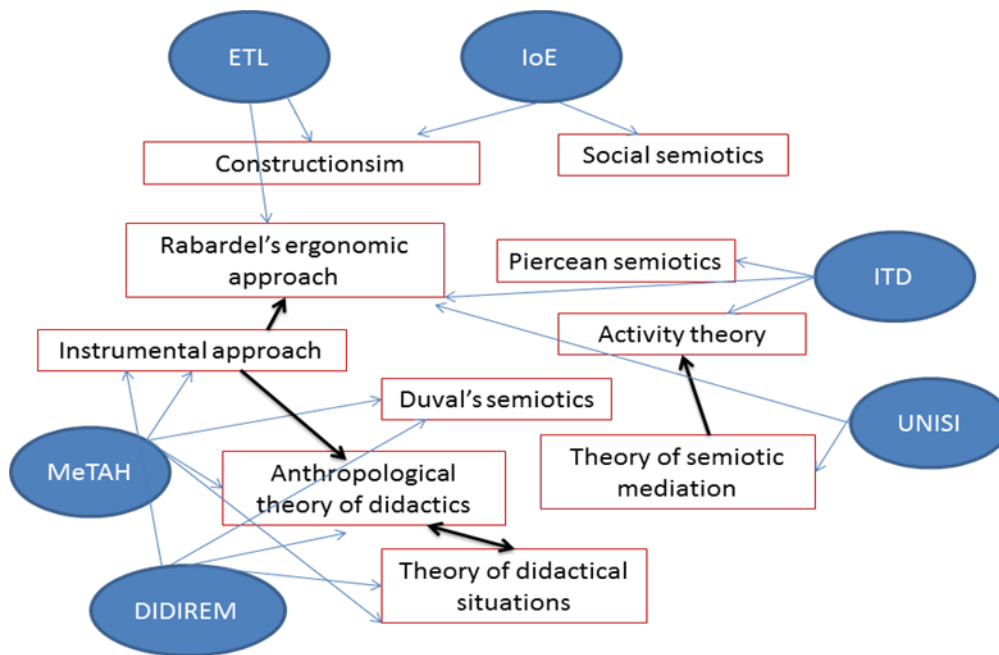


Figure 2. The initial theoretical landscape

In figure 2 above, the shaded ovals correspond to teams: ETL (Greece), IoE (UK), ITD and UNISI (Italy), MeTAH and DIDIREM (France). Note that this landscape already mentions connections, but these were internal to a given team or a given culture. The ambition of ReMath was to make networking progress beyond these cultural frontiers.

We had experienced in a previous collaborative work within the European network of excellence Kaleidoscope the limitations of strategies of theoretical networking based on mutual readings, exchanges and discussions, and learnt the importance of creating an appropriate *milieu*⁵ for interaction. This is what we did, taking our research and development practices - within the language of ATD research and development praxeologies (Artigue, Bosch & Gascón, 2010) - as objects of study, investigating how our respective theoretical frameworks impacted them. This was not an easy task for several reasons:

- the fact that theoretical frameworks and constructs, when familiar, tend to be used in a naturalized way, the technological discourse accompanying their use progressively fading;
- the fact that theoretical frameworks and constructs are often used more as metaphors than as operational tools, which makes the exact role they play in research and development decisions more difficult to identify;
- the fact that many decisions escape theoretical control.

These characteristics make the relationship between the practical and theoretical block of research praxeologies problematic. For taking up this difficulty, we developed a specific language, that of didactical functionalities and concerns, and a specific methodology, that of cross-experimentation and cross-case studies that I detail below.

In ReMath, each team (with one exception) was in charge of developing a Dynamic Digital Artefact (DDA) which was innovative in terms of representations, and in charge of two

⁵ The term *milieu* is used here with the technical meaning it has in the theory of didactical situations (Brousseau, 1997), that is to say the meaning of antagonist milieu offering researchers retroactions but also some resistance for preventing them to adapt and even distort theoretical frameworks and constructs in their desire of establishing connections.

experimentations, one with its own DDA and one with a DDA developed by another team from another country. The experimentation process was piloted by a very precise guide, from the design and a priori analysis of the experimentation, making explicit didactical functionalities and concerns, to its a posteriori analysis. Interactions between teams, and especially those experimenting the same DDA were also closely monitored as well as the cross-case studies.

Didactical functionality and concerns had been already introduced in TELMA with the ambition of approaching theoretical frameworks through their functionalities. They contributed to the creation of what Radford referring to what Lotman calls a *semiosphere*, that is

an uneven multi-cultural space of meaning-making processes and understandings generated by individuals as they come to know and interact with each other (2008, p.318),

and what he claims to be a condition for developing productive theoretical networking.

More precisely, the notion of didactical functionality had been introduced for supporting the instrumental vision of the theoretical reflection concerning both the design of digital artifacts and the design and analysis of their use. For a given artifact, it aims at capturing its properties (or characteristics) and their modalities of employment, which may favour or enhance teaching/learning processes according to a specific educational aim. For that purpose, it is structured around three key elements:

1. a set of features / characteristics of the digital artifact
2. an educational aim
3. modalities of using the tool in a teaching/learning process referred to the chosen educational aim.

Artigue and Mariotti (2012) pointed out that

Designers of educational artefacts have didactical functionalities in mind and these orientate design, but as evidenced by research users have also at this level a creative role and contribute to the didactical functionalities of artefacts through the modalities of use they envisage and implement. As, for both designers and users, the didactical functionalities identified are dependent on their theoretical culture, didactical functionalities can be used as windows on the instrumental dimension of theories. Hence, the importance attached to didactical functionalities in the ReMath methodology.

Beyond that, it was supposed that different teams involved in a common project as was the case for ReMath, even when living in different contexts and cultures and relying on different theoretical frameworks, more or less face similar problems and ambitions, and have shared sensitivities. The notion of concern was introduced for capturing these commonalities taken as a possible basis for communication and integration of perspectives. A list of key concerns was attached to each component of didactical functionalities, and the different teams were asked to rate these on a scale from 0 to 5, to make explicit the theoretical frameworks or constructs associated with if any, and to make clear if their use was metaphorical or operational. It was of course expected that the distribution of key concerns would be different from one team to another. Moreover it was expected that even when a particular concern would be given a similar importance by two teams, it would not necessarily be associated with the same theoretical constructs. Understanding these differences and their possible impact on design decisions was considered crucial for the success of the networking enterprise.

Another important ingredient of this methodology was that a common question for investigation through the cross-experimentation process had been fixed. It was the following:

How can the representations identifiable in the DDAs be put in relationship with the achievement of specific educational goals?

The researchers of each team were asked to rephrase this question in their own language, and to complement it by questions of more specific interest for them. Thanks to this system of common and specific questions, we created in fact a new research praxeology on top of the research praxeologies proper to the different teams, and coordinated with these. This coordination was made in such a way that the praxeologies which became object of study were not distorted by the research praxeology created on top of them. Thanks to this construction, each cross-experimentation became both a research praxeology for the two teams involved, rooted in their specific cultures, and a research praxeology which took these as object of study for answering questions related to networking.

This original construction was essential to the success of ReMath whose outcomes cannot be summarized in a few sentences. It helped us understand better our respective research cultures and the ways these impacted our vision of technological affordances in terms of representation, to build a consensus around these affordances structured around 10 assertions, each of these supported by experimental evidence and expressed in a language accessible to a wider audience, to enrich with trans-cultural connections the initial landscape, to identify productive complementarities but also to point out some limitations to networking efforts and understand the reasons for these.

Final Comments

In this text, starting from the fact that the field of mathematics education is a multicultural field, and relying on my personal experience of this multiculturalism, I have tried to show that accepting the full consequences of such an intellectual position is not so easy as it could appear. I did it first at an institutional level, taking as an illustrative example the history of ICMI, the International Commission for Mathematical Instruction, and showing its slow and still unachieved progression towards true internationalization, despite its foundational aim of international communication and exchange. From this institutional and international level, I skipped then to an individual level, reflecting on my personal trajectory and on the difficulties I had to face. I also tried to show how the global evolution of the field of mathematics education makes that fortunately the situation is quite different for the new generations. This is certainly due to the intensification of international exchanges and to the fact that the dominant educational cultures of the past are today challenged by other educational cultures, for instance those of East Asia. But this is also more fundamentally due to the fact that the field of mathematics education as a field of research has drastically evolved, moving from the cognitive study of the single learner to the analysis of the complex joint action of teachers and students (Sensevy, 2011), seen as a socially and culturally situated process, and to the analysis of the condition and constraints which up to the most global levels shape the functioning of didactical systems and of their actors. This is also because we have accepted to acknowledge that the field of mathematics education is a field where knowledge cannot be separated from values. Acknowledging the multiculturalism of mathematics education is in fact a political position.

References

- Alves Dias, M., Artigue, M., Jahn, A.P., & Campos, T. (2010). A comparative study of the secondary-tertiary transition. In Pinto M.F. & Kawasaki T.F. (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*. Vol. 2, pp. 129-136. Belo Horizonte, Brazil: PME.
- Artigue M. & Robinet J. (1982). Conceptions du cercle chez des enfants de l'école élémentaire. *Recherches en Didactique des Mathématiques*, vol. 3.1., 5-64.
- Artigue M. et al. (1988). The thème « différentielles » : un exemple de coopération math-physique dans la recherche. In G. Vergnaud, G. Brousseau (eds), *Actes du Colloque du GRECO Didactique et Acquisition des Connaissances Scientifiques du CNRS*. Sèvres, mai 1987. 5-35. Grenoble : La Pensée Sauvage.

- Artigue M., Menigaux J., & Viennot L. (1990). Some aspects of students' conceptions and difficulties about differentials. *European Journal of Physics*, Vol 11, 262-267.
- Artigue, M. (Ed.) (2009). Connecting Approaches to Technology Enhanced Learning in Mathematics: The TELMA Experience. *International Journal of Computers for Mathematical Learning*, 14(3).
- Artigue M., Bosch, M., & Gascón J. (2011). Research praxeologies and networking theories. In, M. Pytlak, T. Rowlad, & E. Swoboda (Eds.). *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education*, pp. 281-2390. University of Rzeszów, Poland.
- Artigue, M., & Winslow, C. (2010) International comparative studies on mathematics education: a viewpoint from the anthropological theory of didactics. *Recherche en Didactique des Mathématiques*, 30/1, 47-82.
- Artigue, M., (2011). Les questions de développement curriculaire à travers un exemple : l'enseignement de l'analyse en France au lycée depuis le début du XXème siècle. *Quadrante*, Vol. XX, n°1, 7-29.
- Artigue M., & Mariotti M.A. (2012). *Networking theoretical frames: the ReMath enterprise*. (Preprint).
- Bosch M., & Gascón J. (2006). 25 years of the Didactic Transposition, *Bulletin of the International Commission on Mathematical Instruction* 58, 51-65.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer Academic Publishers.
- Brousseau, G. (2008). Research in mathematics education. In, M. Niss & E. Emborg (Eds.), *ICME-10 Proceedings*, pp. 244-254. Roskilde University, Denmark.
- Chevallard, Y. (1992). Concepts fondamentaux de la didactique : perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques*, 12(1), 77-111.
- Chevallard, Y. (2002). Organiser l'étude 3. Écologie & régulation. In: Dorier, J. L. et al. (Eds.), *Actes de la 11^e école de didactique des mathématiques*. pp. 41-56. Grenoble: La Pensée Sauvage.
- Clarke, D. J., Keitel, C., & Shimizu, Y. (Eds.) (2006). *Mathematics classrooms in twelve countries: The insider's perspective*. Rotterdam: Sense Publishers.
- D'Ambrosio, U. (2008). *Etnomatemática. Eslabón entre las tradiciones y la modernidad*. Mexico: Limusa.
- Grugeon, B. (1995). *Etude des rapports institutionnels et des rapports personnels des élèves dans la transition entre deux cycles d'enseignement*. Thèse de Doctorat. Université Paris 7.
- Hodgson B. (2009). ICMI in the post-Freudenthal era: moments in the history of mathematics education from an international perspective. In Bjarnadóttir, K., Furinghetti, F., & Schubring, G. (eds.), "Dig where you stand". *Proceedings of the conference "On-going research in the History of Mathematics Education"*. Reykjavik: University of Iceland
- Gutiérrez, R. (2010). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, 41, 1-32.
- Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning*. Westport: Ablex.
- Leung, F. K. S., Graf, K.-D., & Lopez-Real, F. J. (Eds.) (2006). *Mathematics education in different cultural traditions: A comparative study of East Asia and the West* (New ICMI Study Series 13). New York: Springer.
- Leung F. (2009). The significance of the ICMI Study on mathematics education in East Asia and the West. In, M. Menghini, F. Furinghetti, L. Giacardi, & F. Arzarello (Eds.), *The first century of the International Commission on Mathematical Instruction (1908-2008). Reflecting and shaping the world of mathematics education*, pp. 223-229. Istituto della enciclopedia Italiana. Roma.
- Menghini, M., Furinghetti, F., Giacardi, L., & Arzarello, F. (Eds.) (2008). *The first century of the International Commission on Mathematical Instruction (1908-2008). Reflecting and shaping the world of mathematics education*. Istituto della enciclopedia Italiana. Roma.
- Miorim, M. A., Miguel, A., & Fiorentini, D. (1993). Ressonâncias e dissonâncias do movimento pendular entre álgebra e geometria no currículo escolar brasileiro. *Zetetiké* 1, 19-39.
- Mullis I.V.S., Martin M.O., Beaton A.E., Gonzales E.J., Kelly D.L., & Smith T.A. (1997) *Mathematics Achievement in the Primary School Years*. Centre for the Study of Testing, Evaluation and Educational Policy, Boston College, Boston.
- Nebres B. F. (2008). Centres and peripheries in mathematics education. In, M. Menghini, F. Furinghetti, L. Giacardi, & F. Arzarello (Eds.), *The first century of the International Commission on Mathematical Instruction (1908-2008). Reflecting and shaping the world of mathematics education* (pp. 149-162). Istituto della enciclopedia Italiana. Roma.
- Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM – The International Journal on Mathematics Education*, 40(2), 317-327.
- Robitaille D.F., Garden R.A. (Ed.) (1989). *The IEA Study of Mathematics II: Contexts and Outcomes of School Mathematics*. Pergamon Press, Oxford.
- Sensevy G. (2011). *Le sens du savoir. Éléments pour une théorie de l'action conjointe en didactique*. De Boeck Editeur.

What Can and Should We Learn from International Studies of Mathematics Achievement?

Frederick K.S. Leung
The University of Hong Kong
<hraslks@hku.hk>

International studies of mathematics achievement have profound influence on mathematics education worldwide in the past 15 years. Results of studies such as TIMSS and PISA have dominated the agenda of discussion in the mathematics education community as well as among policy makers. Much attention however has been paid on the ranking of countries in the league tables generated from such studies, often without due consideration of the nature of these studies, as well as the contextual factors that affect the performance of students from different countries. In this paper, the nature of these international studies of mathematics achievement will be examined. Based on an understanding of the nature of these studies, how the results should be interpreted will be discussed. This includes a proper understanding of the meaning and significance of ranking of countries; the suitability of drawing causal relationships between various variables and student achievement; and the appropriateness or otherwise of learning from the educational practices of high ranking countries, etc. It will be argued that cultural value may be an important factor in explaining differences in educational practices and student achievement. Without due consideration of the cultural and other contextual differences, passing judgement on the performance of students in different countries based on results of international studies is very misleading, and may even be damaging. In learning from other countries, one must first evaluate the cultural values and educational context in one's own country before deciding on how much can be learned from other countries. Examples of lessons that can be and should be learned from international studies of mathematics achievement will be discussed, and implications for policy makers in education as well as for school mathematics teachers in their classroom practices will be explored.

Introduction

Ever since the results of the 1995 Third International Mathematics and Science Study (TIMSS, which was later renamed as Trends in International Mathematics and Science Study) were released in 1996 (Beaton, et al., 1996; Mullis, et al., 1997; Mullis, et al., 2000), international studies of mathematics achievement such as TIMSS and PISA (Programme for International Student Assessment; OECD, 2001; 2003) have dominated much discussion on mathematics education policies and effectiveness of schools systems within the mathematics education community, among policy makers and the general public. But do these discussions do justice to the nature of these international studies? For example, have such discussions taken into account the purposes that these studies set out to achieve? Are the conclusions and suggestions made based on results of these studies warranted by the methodologies adopted in these studies?

Take the results of the 2007 round of TIMSS and the 2003 round of PISA as examples. Much discussion has been based on results of the studies from the following tables in the international reports (Mullis, et al., 2008; OECD, 2004):

Table 1
Trends in International Mathematics and Science Study (TIMSS) 2007

Exhibit 1.1 TIMSS 2007 Distribution of Mathematics Achievement

TIMSS2007 Mathematics 4th Grade

Country	Mathematics Achievement Distribution	Average Scale Score	Years of Formal Schooling ^a	Average Age at Time of Testing	Human Development Index ^{b,c}
Hong Kong SAR		507 (3.6)	4	10.2	0.917
Singapore		509 (3.7)	4	10.4	0.912
Chinese Taipei		516 (3.7)	4	10.2	0.912
Japan		508 (2.1)	4	10.5	0.951
¹ Kazakhstan		540 (7.1)	4	10.6	0.794
Russian Federation		544 (4.9)	4	10.8	0.811
England		541 (2.9)	5	10.2	0.946
¹ Latvia		537 (2.3)	4	11.8	0.851
¹ Netherlands		535 (2.1)	4	10.2	0.955
¹ Lithuania		536 (2.4)	4	10.8	0.862
^{2,1} United States		529 (2.4)	4	10.3	0.951
Germany		525 (2.3)	4	10.4	0.915
¹ Denmark		523 (2.4)	4	11.8	0.949
Australia		516 (3.5)	4	9.9	0.902
Hungary		510 (3.5)	4	10.7	0.874
Italy		507 (3.1)	4	9.8	0.901
Austria		505 (2.8)	4	10.3	0.940
Sweden		503 (2.5)	4	10.8	0.956
Slovenia		503 (3.8)	4	9.8	0.917
TIMSS Scale Avg.		500			
Armenia		500 (4.3)	4	10.6	0.775
Slovak Republic		496 (4.5)	4	10.4	0.861
¹ Scotland		494 (2.2)	5	9.8	0.946
New Zealand		491 (2.3)	4.5–5.5	10.8	0.901

SOURCE: IEA's Trends in International Mathematics and Science Study (TIMSS) 2007

Table 2
Trends in International Mathematics and Science Study (TIMSS) 2007

Table 2 Trends in International Mathematics and Science Study (TIMSS) 2007

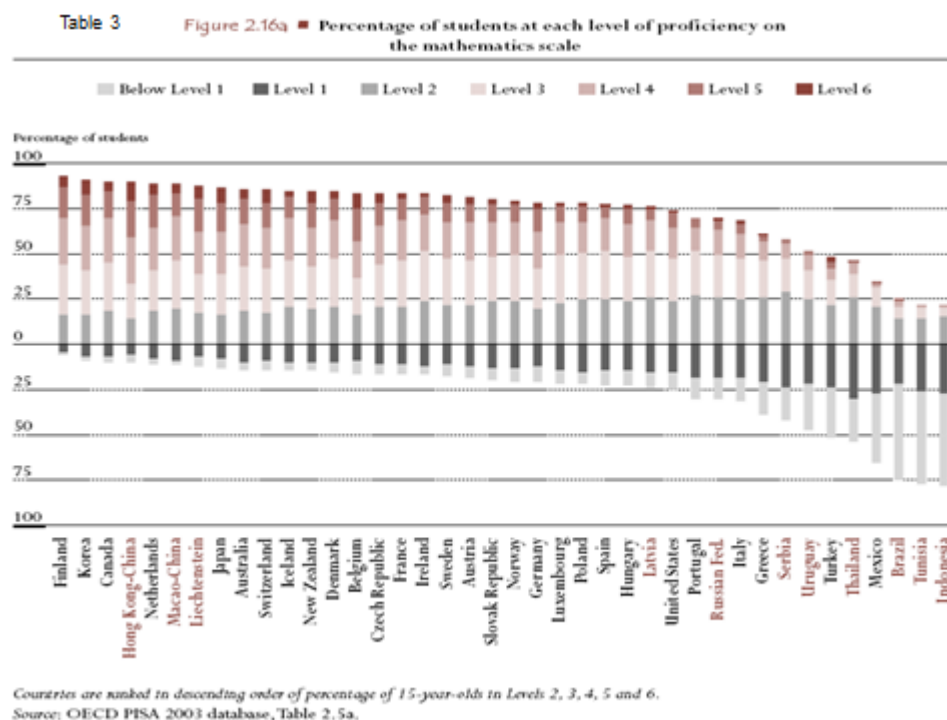
Exhibit 1.1 TIMSS 2007 Distribution of Mathematics Achievement (Continued)

TIMSS2007 Mathematics 8th Grade

Country	Mathematics Achievement Distribution	Average Scale Score	Years of Formal Schooling ^a	Average Age at Time of Testing	Human Development Index ^{b,c}
Chinese Taipei		596 (4.5)	8	14.2	0.932
Korea, Rep. of		597 (2.7)	8	14.3	0.921
Singapore		593 (3.8)	8	14.4	0.932
¹ Hong Kong SAR		572 (5.8)	8	14.4	0.937
Japan		570 (2.4)	8	14.5	0.953
Hungary		537 (3.5)	8	14.6	0.874
¹ England		513 (4.8)	9	14.2	0.946
Russian Federation		512 (4.1)	7 or 8	14.6	0.802
^{2,1} United States		508 (2.8)	8	14.3	0.951
¹ Lithuania		506 (2.3)	8	14.9	0.862
Czech Republic		504 (2.4)	8	14.4	0.891
Slovenia		503 (2.3)	7 or 8	13.8	0.917
TIMSS Scale Avg.		500			
Armenia		489 (3.5)	8	14.9	0.775
Australia		486 (3.9)	8	13.9	0.902
Sweden		481 (2.3)	8	14.8	0.956
Malta		480 (1.2)	9	14.0	0.878
¹ Scotland		467 (3.7)	9	13.7	0.946
^{1,2} Serbia		466 (3.3)	8	14.9	0.830
Italy		460 (3.8)	8	13.9	0.941
Malaysia		474 (5.8)	8	14.3	0.831
Norway		469 (2.8)	8	13.8	0.908
Cyprus		465 (3.4)	8	13.8	0.903

SOURCE: IEA's Trends in International Mathematics and Science Study (TIMSS) 2007

Table 3
Percentage of Students at Each Level of Proficiency on the Proficiency Scale



What can we learn from the data in these tables? Given the way the data are presented, for many people, the natural thing to look for is the positions, and perhaps the scores as well, of their own country in the table. Another obvious thing to notice is the names of the countries on the top, and perhaps also those at the bottom, of the tables.

This is indeed how the results of these studies were reported in the media, and often by the politicians as well, in many countries. For example, shortly after the international results of TIMSS 2007 were released in December 2008, the 11 December 2008 issue of *The Guardian* reported on UK's performance in the study, naming TIMSS "the Olympics of education" with the headline "*The Olympics of education: Fresh efforts to boost the UK in maths and science league tables will also help our economic health*". The article went on to report as follows:

"This week the results were announced for the Olympics of education. As they have for many years now, Singapore, South Korea and Taiwan took home the medals. The contest was the quadrennial match of brainpower called the Trends in International Mathematics and Science Study, known fondly as the TIMSS"

The *New Zealand Press Association* had the following report on 10 December 2008:

"Land Lampooned By Borat Outsmarts Nz Pupils In Maths, Science

An international report on maths and science shows New Zealand primary school pupils lagging behind their counterparts in Kazakhstan, portrayed as a nation of simpletons by comic character Borat. New Zealand students scored below the international average for maths in the TIMSS Report, and just ahead of the Czech Republic and the Ukraine, and another 10 countries, but lagged behind 22 other nations -- including Kazakhstan, Slovakia, Slovenia, Latvia and Lithuania. The league table was led by the usual suspects in Asia: Hong Kong, Singapore, Taiwan and Japan."

In Japan, *Kyodo News* on 10 December 2008 reported the following:

“Japanese students rank high in global math, science surveys

Japanese students maintained their relatively high ranks broadly in global achievement tests in mathematics and science, the latest results of a 2007 survey showed, prompting the Japanese education ministry to say the declining trend in their scholastic abilities has been arrested. Eighth-grader students in Japan ... ranked fifth in math, as they did in the last test, in the fourth Trends in International Mathematics and Science Study, or TIMSS. In the 2003 test, Japanese fourth-graders dropped to fourth from third both in science and arithmetic among 36 countries and regions”

As can be seen from the news reported above, what concerned the media is mainly the ranking of the countries that took part in the test. But is ranking of countries the primary purpose of such studies? Are TIMSS and PISA competitions, the “Olympics of education”? What do these international studies purport to do? What data are collected by these studies? What methodologies are employed in these studies, and what are the limitations on the findings as a consequence of the methodologies? What can and should we learn from such studies?

The Purposes of International Studies of Mathematics Achievement

Before we are to understand and discuss the findings of international studies of mathematics achievement, it is just reasonable that we first look at what these studies purport to do. Take TIMSS 2007 as an example. The values and goal of the study are clearly laid down in the Foreword and Introduction of the TIMSS 2007 international report (Mullis, et al, 2008) as follows:

“For policymakers, the TIMSS 2007 report contains a wealth of information about key instructional, curricular, and resource related variables that are fundamental in understanding the teaching and learning process. This extensive information about trends in students’ achievement and the contexts for teaching and learning mathematics and science should help ensure that TIMSS continues to be widely recognized as the most influential study of its type. The information should be of great value in guiding educational decision making and practice in the areas of mathematics and science around the world.” (p.2) “The goal (of the study) is to provide comparative information about educational achievement across countries to improve teaching and learning in mathematics (and science)” (p.14)

But it seems that this “wealth of information about key instructional, curricular, and resource related variables” is rarely reported or discussed, not in the media at least.

And what about PISA? What are the goals of PISA, and how are they different from those of TIMSS? These two most influential international studies of educational achievement, TIMSS and PISA, have been causing confusion to many in the public, and even to some educators in the field as well. The two studies have been described as “two watches” (Hutchiso & Schagen, 2007), referring to the saying that “The man with one watch always knows what time it is – the man with two watches is never quite sure”. Indeed, when a country’s performances differed in the two studies, it is difficult to explain the discrepancy, and so one is never quite sure how well the country performed. But in the documents published by OECD (2003), the goals of PISA are clearly different from those of TIMSS.

PISA claims to measure mathematics literacy. “Mathematical literacy is an individual’s capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual’s life as a constructive, concerned and reflective citizen Rather than being limited to the

curriculum content students have learned, the assessments focus on determining if students can use what they have learned in the situations they are likely to encounter in their daily lives” (p.24).

So rather than assessing what students have learned from the school curriculum, as TIMSS does, “PISA seeks to assess how well 15-year-olds are prepared for life’s challenges. ... focusing on young people’s ability to use their knowledge and skills to meet real-life challenges, rather than merely on the extent to which they have mastered a specific school curriculum” (OECD, 2004, p.20).

Of course, even given these proclaimed differences in the official documents of the two studies, the confusion perceived by the public is understandable. One of the most important purposes of the school curriculum is to prepare students for adult life, and no serious educator would argue for schooling for schooling’s sake. So TIMSS is not just measuring what is confined in the school curriculum, but is predicting how well students are prepared for adult life as well. Similarly, the knowledge and skills for meeting real-life challenges, as PISA seeks to assess, obviously come mainly from the school curriculum. So in reality, the purposes of TIMSS and PISA are hard to be clearly differentiated.

What is clear from the official statements of the two studies is that there is a clear absence of any mention of ranking or competition among countries. The major purpose of these international studies, as purported by the organisers of the studies, seems to be on identifying variables which explain achievement, which in turn will be useful for informing policy decisions. But can the “instructional, curricular, and resource related variables” really explain student achievement, and hence the results able to guide “educational decision making and practice in the areas of mathematics (and science)” (Mullis, 2008)?

The Relation between Student Achievement and Other Variables

In the TIMSS 2007 international report, some of the “instructional, curricular, and resource related variables” are presented in such a way that that a relation between these variables and achievement is hinted. Table 4 below (taken from Mullis, 2008, p.274) shows that the grade 4 class size in different countries together with the achievements of students in classes of different sizes:

Table 4
Achievement and Class Size for Mathematics Instruction

Exhibit 7.2 Achievement and Class Size for Mathematics Instruction

TIMSS2007 4th Grade Mathematics

Table 4

Country		1–19 Students		20–32 Students		33 or More Students	
		Percent of Students	Average Achievement	Percent of Students	Average Achievement	Percent of Students	Average Achievement
Algeria	r	11 (2.8)	588 (14.2)	60 (4.3)	378 (7.0)	29 (4.0)	383 (9.4)
Armenia	s	24 (3.3)	526 (14.1)	50 (3.8)	499 (7.3)	26 (3.6)	484 (6.0)
Australia		19 (1.0)	510 (9.0)	80 (3.0)	521 (4.3)	2 (1.2)	—
Austria		17 (2.9)	506 (3.1)	61 (2.9)	505 (2.7)	0 (0.0)	—
Chinese Taipei		3 (1.2)	548 (12.8)	45 (3.7)	570 (3.2)	51 (3.4)	583 (2.4)
Colombia		19 (3.3)	342 (13.7)	24 (4.7)	347 (14.0)	57 (4.4)	365 (8.1)
Czech Republic		11 (3.5)	482 (5.9)	69 (3.5)	489 (2.9)	0 (0.0)	—
Denmark		14 (3.9)	529 (4.4)	66 (3.9)	521 (2.9)	0 (0.0)	—
El Salvador		20 (2.7)	307 (10.7)	17 (4.1)	318 (9.1)	41 (3.8)	352 (4.2)
England		8 (1.9)	556 (9.6)	80 (3.0)	519 (3.2)	12 (2.4)	546 (9.0)
Georgia		17 (1.8)	454 (7.3)	50 (4.5)	428 (6.6)	13 (2.2)	454 (6.3)
Germany		21 (2.4)	512 (5.6)	79 (2.4)	528 (2.2)	0 (0.0)	—
Hong Kong SAR		1 (0.7)	—	25 (3.3)	588 (5.5)	74 (3.4)	616 (3.8)
Hungary		11 (1.7)	482 (6.5)	67 (3.7)	525 (4.7)	0 (0.0)	—
Iran, Islamic Rep. of		25 (2.7)	381 (6.5)	59 (3.8)	406 (5.3)	16 (2.9)	421 (11.6)
Italy		44 (2.6)	506 (4.3)	56 (2.6)	507 (4.5)	0 (0.0)	—
Japan		7 (1.5)	538 (8.5)	47 (2.9)	569 (3.4)	45 (3.2)	569 (2.9)
Kazakhstan		10 (4.5)	550 (20.2)	68 (4.6)	548 (5.5)	3 (1.2)	577 (29.4)
Kuwait	s	7 (2.8)	330 (18.1)	88 (3.4)	314 (5.0)	5 (1.9)	302 (11.9)
Latvia		44 (2.4)	525 (3.9)	49 (3.8)	550 (2.6)	6 (2.0)	551 (9.3)
Lithuania		17 (3.0)	511 (4.7)	61 (3.0)	541 (3.1)	0 (0.0)	—
Morocco	r	17 (3.3)	352 (17.7)	42 (4.3)	343 (11.4)	41 (3.9)	338 (7.7)
Netherlands		27 (3.3)	531 (4.3)	71 (3.5)	535 (2.9)	2 (1.3)	—
New Zealand	s	13 (2.1)	489 (8.7)	81 (2.4)	497 (3.0)	6 (1.7)	524 (11.7)
Norway		42 (3.3)	473 (4.4)	53 (3.6)	474 (3.5)	5 (1.9)	467 (10.6)
Qatar	r	8 (0.1)	301 (4.3)	75 (0.2)	296 (1.4)	17 (0.2)	316 (3.4)
Russian Federation		11 (2.7)	511 (10.5)	67 (2.7)	551 (3.8)	0 (0.0)	—
Scotland	r	16 (2.8)	492 (9.4)	79 (3.0)	491 (3.1)	5 (1.6)	506 (14.0)
Singapore		0 (0.0)	—	6 (1.1)	514 (13.5)	94 (1.1)	605 (3.5)
Slovak Republic		14 (2.5)	497 (6.6)	65 (2.6)	496 (5.7)	1 (0.6)	—
Slovenia		46 (2.9)	497 (2.7)	53 (3.0)	506 (2.6)	1 (0.6)	—
Sweden		16 (1.4)	505 (4.5)	60 (3.6)	504 (3.2)	4 (1.6)	512 (12.4)
Tunisia		20 (2.8)	303 (12.2)	69 (3.8)	314 (5.0)	11 (2.7)	354 (21.3)
Ukraine		10 (3.3)	445 (4.9)	65 (3.5)	480 (3.8)	5 (1.4)	472 (13.4)
United States		26 (2.4)	521 (4.1)	69 (2.8)	533 (3.3)	5 (1.3)	522 (8.0)
Yemen	r	9 (2.1)	262 (18.5)	17 (4.0)	227 (16.4)	74 (4.1)	219 (7.7)
International Avg		24 (0.5)	462 (1.8)	58 (0.4)	471 (1.1)	18 (0.4)	460 (2.3)

SOURCE: IEA Trends in International Mathematics and Science Study (TIMSS) 2007

Has a relation been established between class size and student achievement according to the data? From Table 4, it can be seen that for many countries (e.g., Austria, Italy), class size does not make any difference to student achievement. For some countries (e.g., Armenia, Kuwait), the smaller the class size, the higher the student achievement. But for the majority of the countries (e.g., Chinese Taipei, Colombia, New Zealand), the bigger the class size, the higher the student achievement. Moreover, if we compare the pattern across countries, it can be seen that all the high achieving countries (e.g., Singapore, Korea, Hong Kong) have large class sizes. How do these results guide “educational decision making and practice”? Are we going to suggest increasing class size in order to raise the achievement of students? I think the best conclusion that we can draw from these results is that class size may not be the most important factor as far as student achievement is concerned. But this conclusion actually applies to most, if not all, the variables under study. So are we going to conclude that none of these variables can account for student achievement?

Limitations of Large Scale International Studies

This difficulty of drawing conclusion from large scale international studies is perhaps not unexpected. The nature of studies such as TIMSS and PISA itself poses serious limitations on what can be learned from these studies. There are technical problems such as translation of instruments and issues related to the vastly different contexts among the participating countries.

The technical issue of language equivalence in the translation of instruments is something that all comparative studies need to confront. TIMSS and PISA involve more than 60 countries operating in more than 30 languages, so the complexity of the task is understandable. Some argued that not all ideas or concepts can be fully translated between languages (Walker, 2002). In particular, for mathematics, languages differ in their

explicitness in expressing mathematics concepts and ideas, and so a literal translation of a test item means that the item may differ in terms of its difficulty or even in terms of its nature in the two languages. Some items become meaningless after translation. For example the item “How many sides are there in a heptagon?” when translated into Chinese will mean something like “How many sides are there in a seven-sided polygon?”! Although TIMSS and PISA have already tried very hard to overcome this problem of translation, the nature of the translation task itself poses great challenges on the comparability of the results in different countries.

Another technical difficulty concerns item difficulty, a limitation that is common in setting test items in any study. But in a large scale international study, as countries may differ tremendously, an item which is deemed difficult in one country may not be deemed so in another. TIMSS and PISA demand that items need to satisfy stringent statistical requirements, and for items to “behave well” statistically, they should not be too difficult in general. As a result, some items may prove to be too easy for some countries, and hence may not be a good measure of the achievements of their students – the results may be very reliable but may not be too meaningful for those countries.

Another limitation is that TIMSS and PISA rely on paper-and-pencil tests in measuring achievement. As a result, competencies that cannot be easily demonstrated through a paper-and-pencil test tend to be neglected. In the 1995 round of TIMSS, a Performance Assessment component of the study was included to tackle this problem (the Appendix shows an example a Performance Assessment task; Harmon, et al., 1997), with limited success. The exercise was very expensive and labour intensive, as a result not too many countries participated in this component of the TIMSS 1995 study. Some also doubt the comparability of the results because of the difficulty of ensuring standardized testing conditions. The fact that this component of Performance Assessment stopped after the 1995 round of the study shows that truly genuine testing of non-paper-and-pencil competencies is difficult to achieve.

The more serious problem, in contrast to the technical problems discussed above, is the contextual differences among the countries that participate in these large scale international studies. Take the differences in the school context as an example. The starting ages of schooling in different countries of the world differ by as much as three years, and so if we are testing students of a certain grade, their ages may differ by as much as three years as well. In TIMSS 2007, students’ ages of starting schooling ranged from 4.5 to 8.0, and by 2007 when they took the TIMSS test, the ages of grade 4 students ranged between 9.7 and 11.2 years. For grade 8 students, their ages ranged between 13.7 and 15.8. For PISA 2003, we had the reciprocal problem. Since students of age 15 were targeted for study, so a narrow age range of 15.7 to 15.9 was achieved in the sample. But because students started schools at different ages, their length of schooling at the time of testing differed by up to three years. This poses a serious question in terms of the comparability of the findings.

Some items cannot be meaningfully set because of contextual differences. For example, application items on the use of decimals involving currencies are very common in many countries. But for some countries (e.g., Japan), decimals are not used at all in their currency, and so it is not possible to set items of this kind which are fair and meaningful to all participating countries. Some tried to overcome this problem by creating an imaginary currency, but of course it won’t overcome the fact that students in some countries are more familiar with using decimals in their currencies than students in other countries, and the creation of an imaginary currency defeats the whole purpose of testing students’ achievement in tackling real-life problems.

The root of the problem is that in international studies, we often compare student achievement and other educational practices across cultures, using the world as “a natural educational laboratory” (Drent, Meelissen & van der Kleij, 2011). The rationale is that many variables within a country or culture are uniform and cannot be manipulated, and so if we want to study the impact of those variables on say student achievement, we have to collect data in different cultures, where the variables differ. But the problem is, not only are those variables of interest differ, a host of other variables are vastly different as well, and usually these variables exist as a bundle. So it is often difficult, if not impossible, to control for all the other variables in studying the variables of interest. And we are never sure whether we have taken all relevant variables into account. As Husen (1983) remarked, in international studies, “we are comparing the incomparables”!

What Can We Learn from International Studies of Mathematics Achievement?

If there are so many limitations to these international studies, what, if any, can we learn from these studies? To be fair to these studies, it must be mentioned that very rigorous methodologies are adopted in studies such as TIMSS and PISA, and hence within the limitations posed by the nature of these studies, they provide rather reliable results concerning student achievement in the participating countries. However, the nature of these large scale studies means that only broad-brush pictures about achievements in different countries are painted by the results, and one has to be very careful in drawing causal relations among specific variables.

One broad-brush picture that evolved from the results of these studies over the years (which perhaps came as a surprise for many) is that there is a cluster of countries which consistently performed very well in these studies. These countries, named East Asian countries thereafter in this paper, include Chinese Taipei, Finland, Hong Kong, Japan, Korea, and Singapore. Is there anything that we can learn from this phenomenon?

Following the purposes of these studies mentioned above, let us look at what these high achieving countries have in common to see whether we can identify shared “instructional, curricular, and resource related variables” that can be used to explain student achievement, and hence inform policy and practice. Tables 5 to 19 below show a host of variables collected for a subset of the TIMSS 2007 countries, including the five East Asian countries.

Table 5
GNI Per Capita

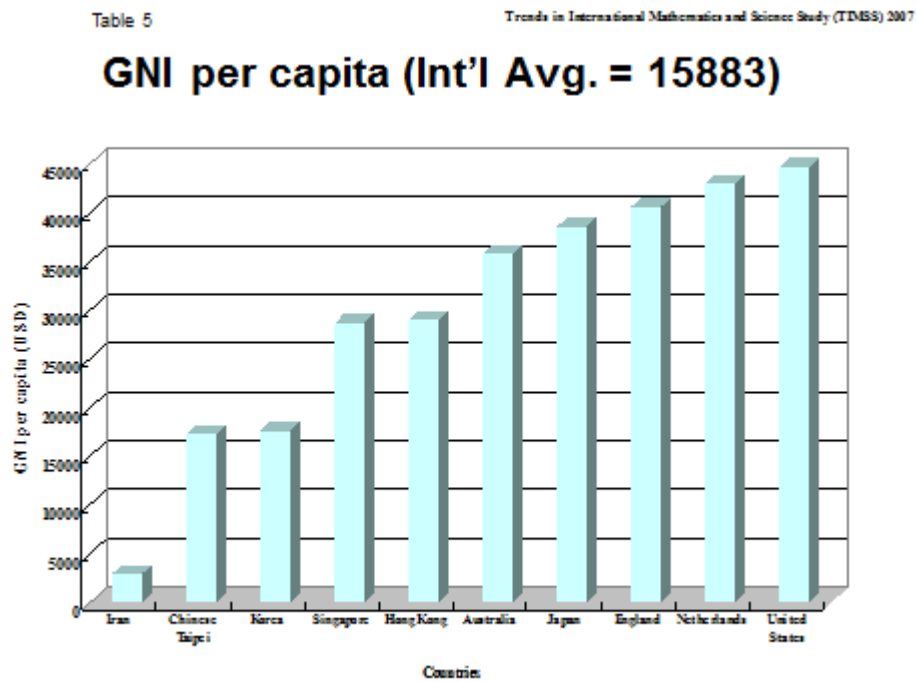


Table 6
Parent with University Degree or Above

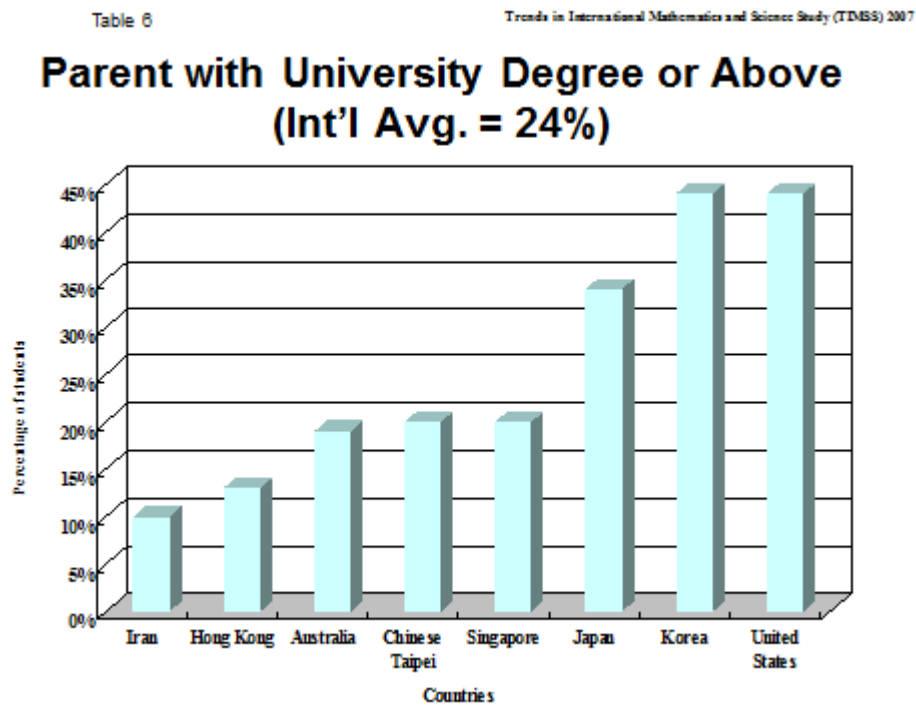


Table 7
Mathematics Teacher with University Degree or Above (Grade 4)

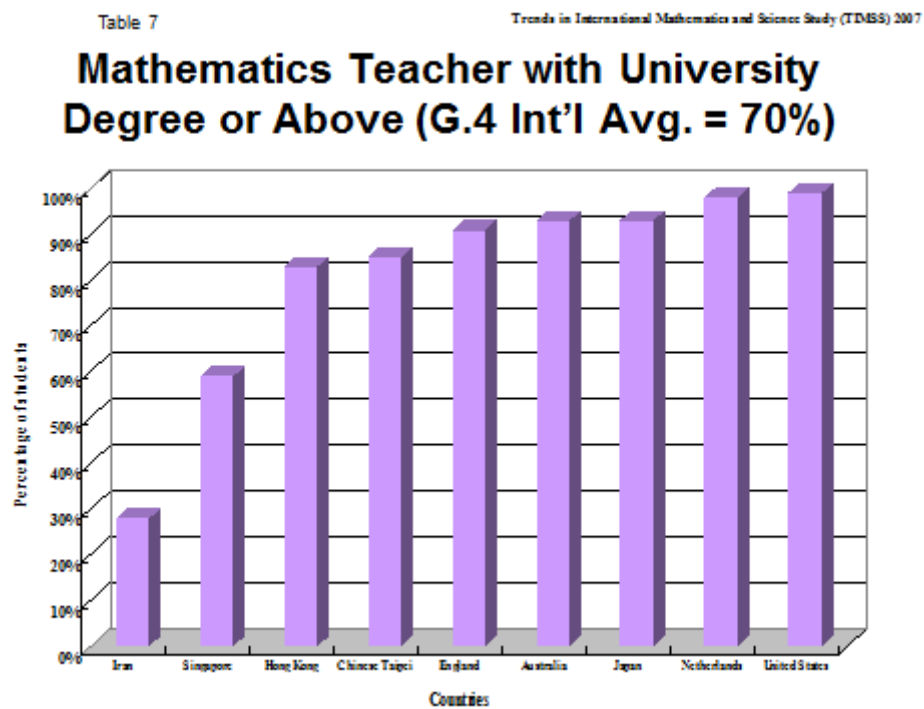


Table 8
Mathematics Teacher with University Degree or Above (Grade 8)

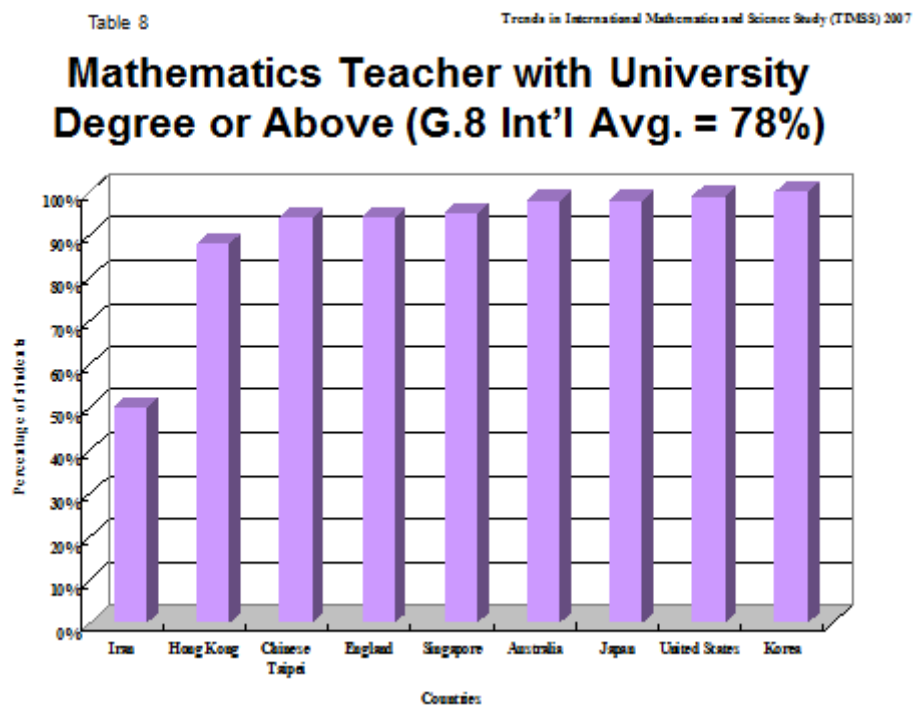


Table 9
Primary Student Teacher Ratio

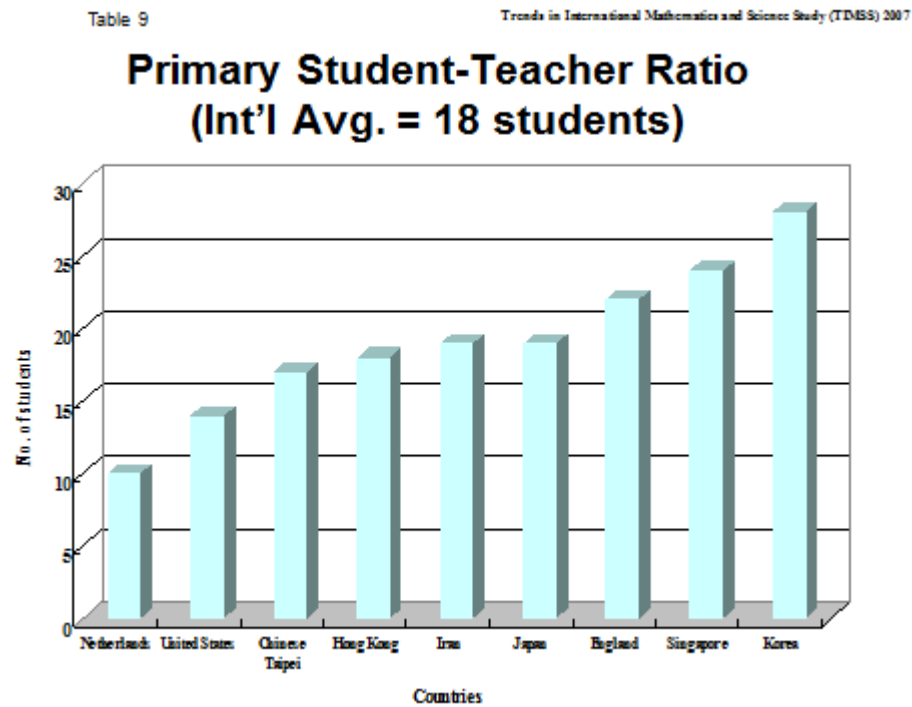


Table 10
Class Size for Mathematics Instruction (Grade 4)

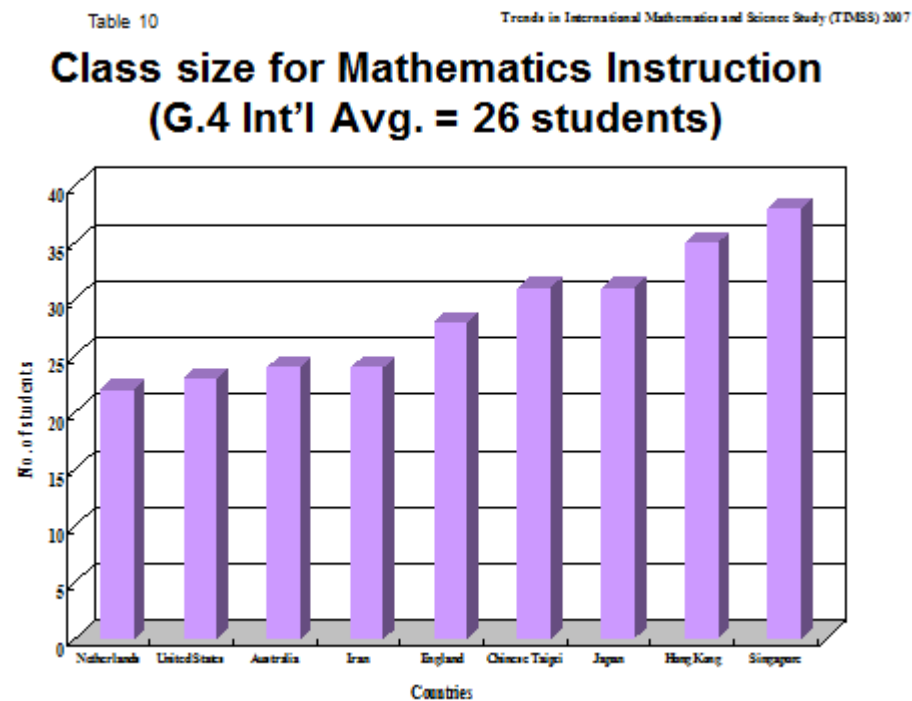


Table 11
Class Size for Mathematics Instruction (Grade 8)

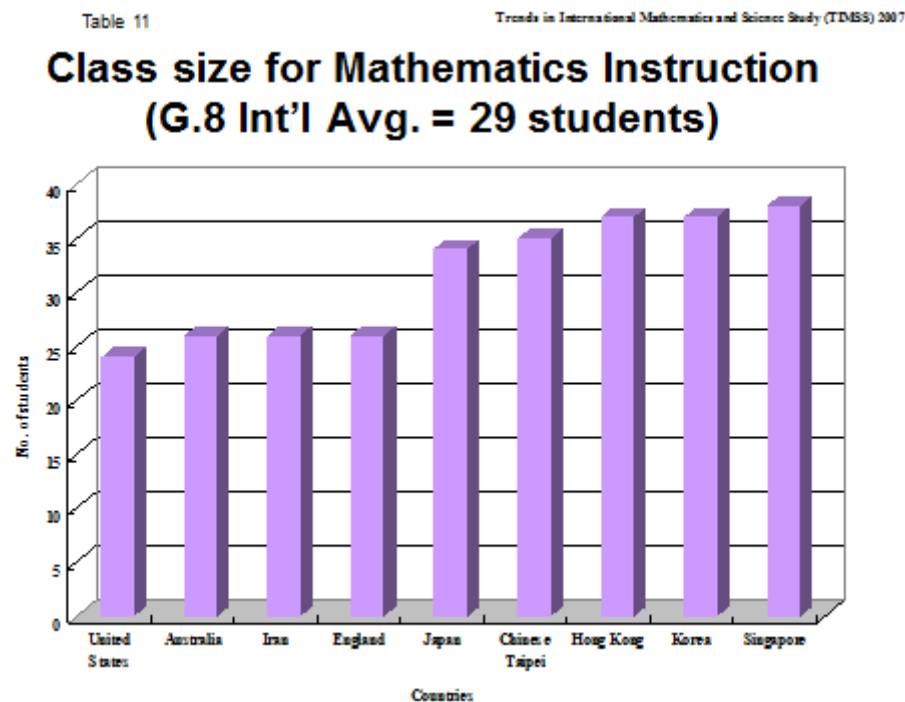


Table 12
Mathematics Instructional Time (Grade 4)

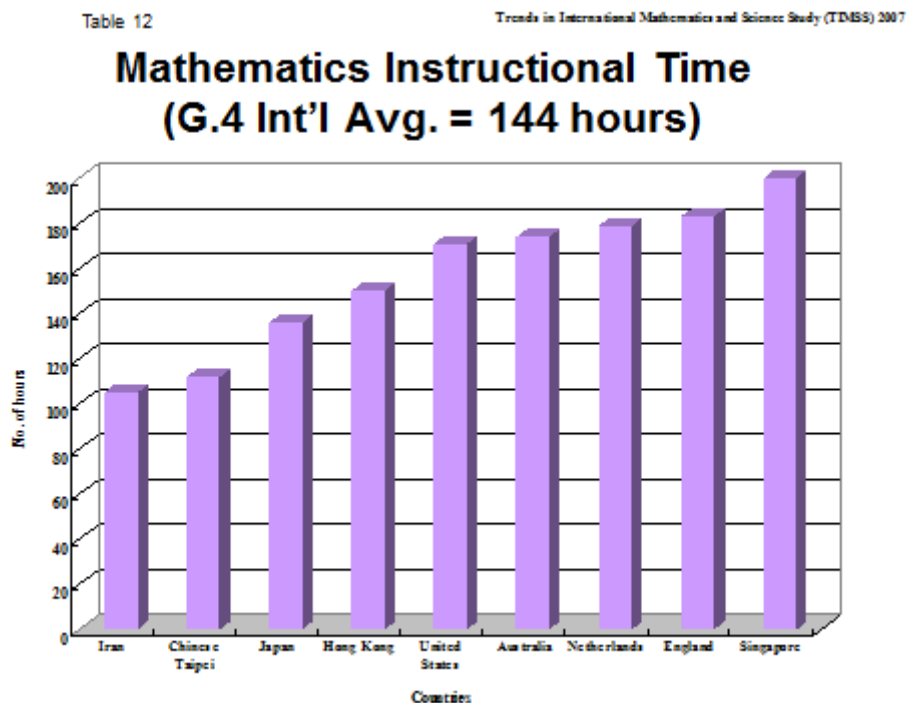


Table 13
Mathematics Instructional Time (Grade 8)

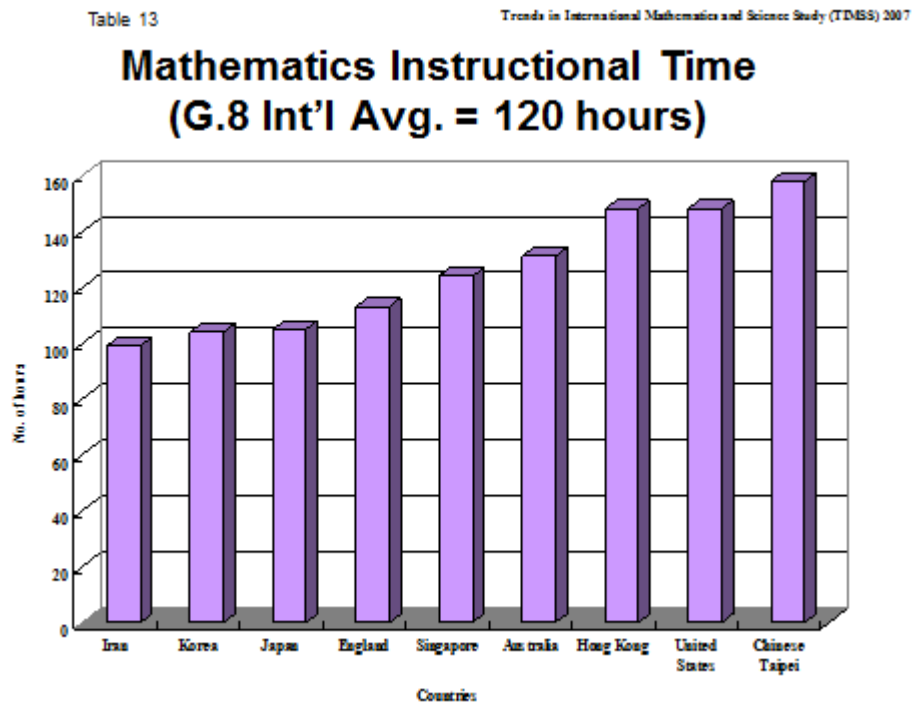


Table 14
Percentage of Implemented Instructional Time for Mathematics (Grade 4)

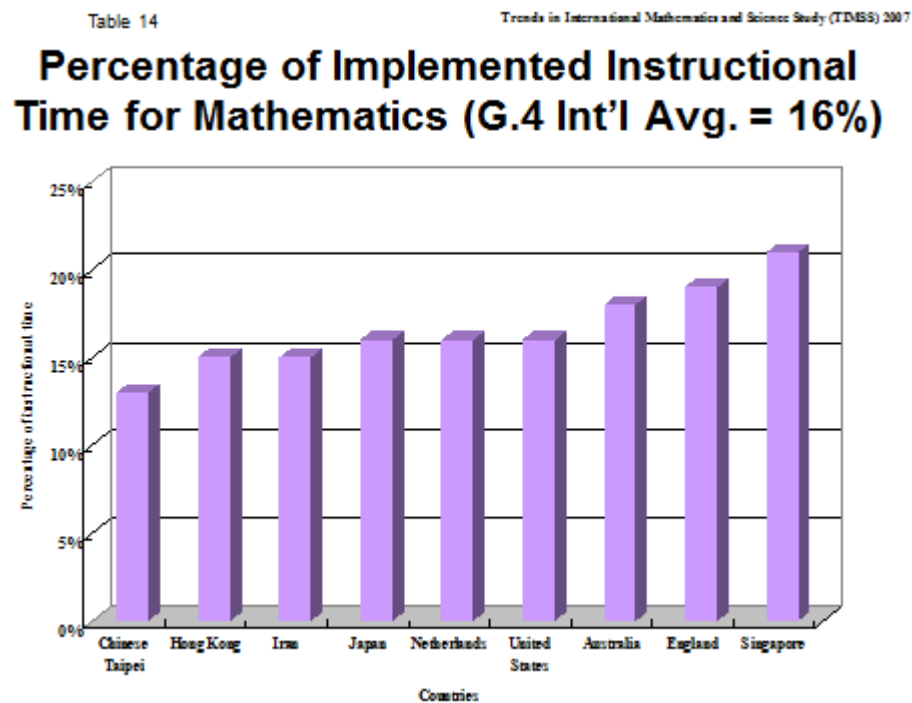


Table 15

Percentage of Implemented Instructional Time for Mathematics (Grade 8)

Table 15

Trends in International Mathematics and Science Study (TIMSS) 2007

Percentage of Implemented Instructional Time for Mathematics (G.8 Int'l Avg. = 12%)

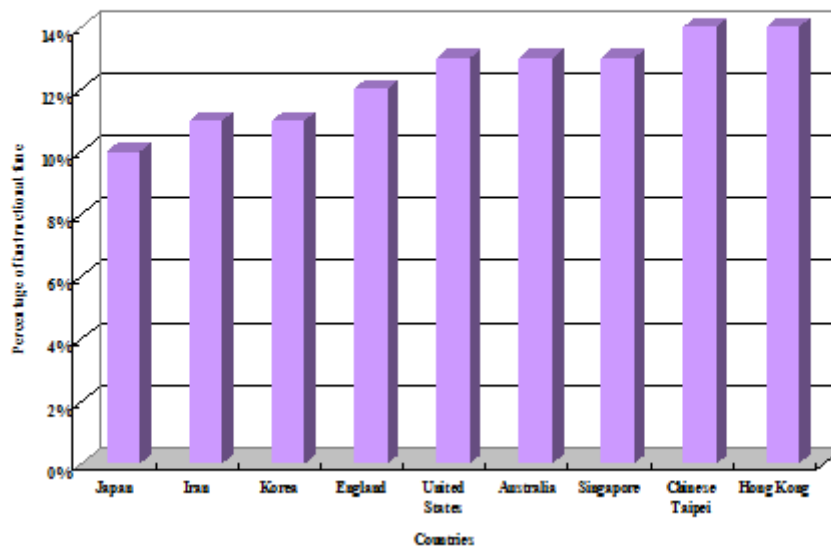


Table 16

High Emphasis on Mathematics Homework (Grade 4)

Table 16

Trends in International Mathematics and Science Study (TIMSS) 2007

High Emphasis on Mathematics Homework (G.4 Int'l Avg. = 13%)

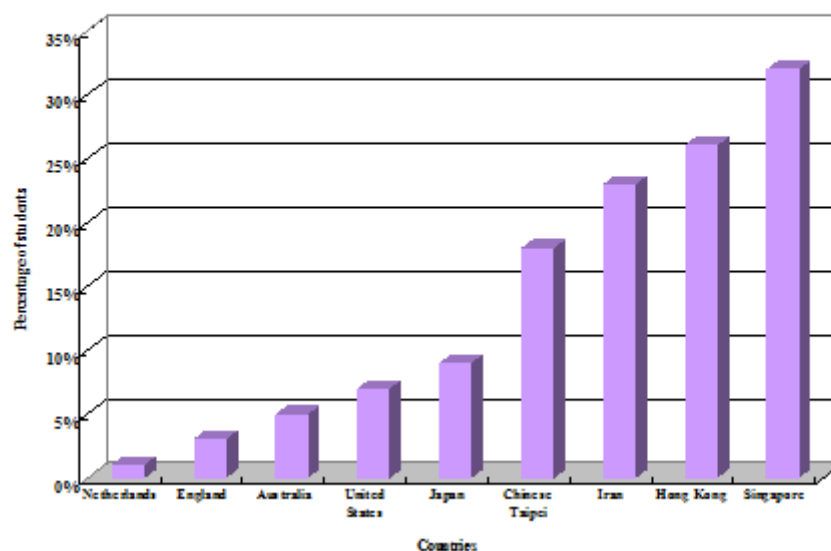


Table 17
High Emphasis on Mathematics Homework (Grade 8)

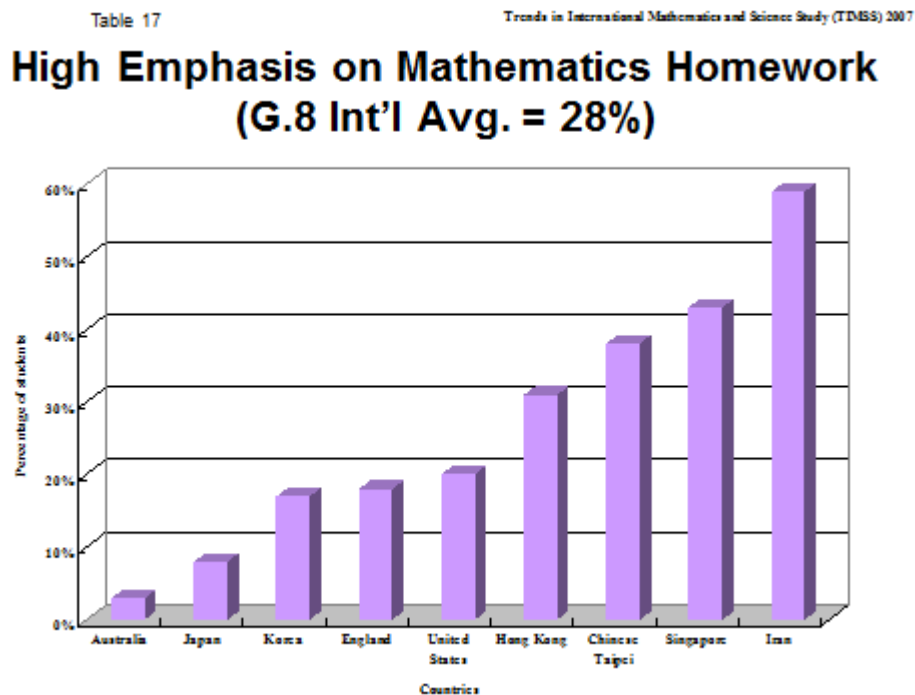


Table 18
School Resources for Instruction (Grade 4)

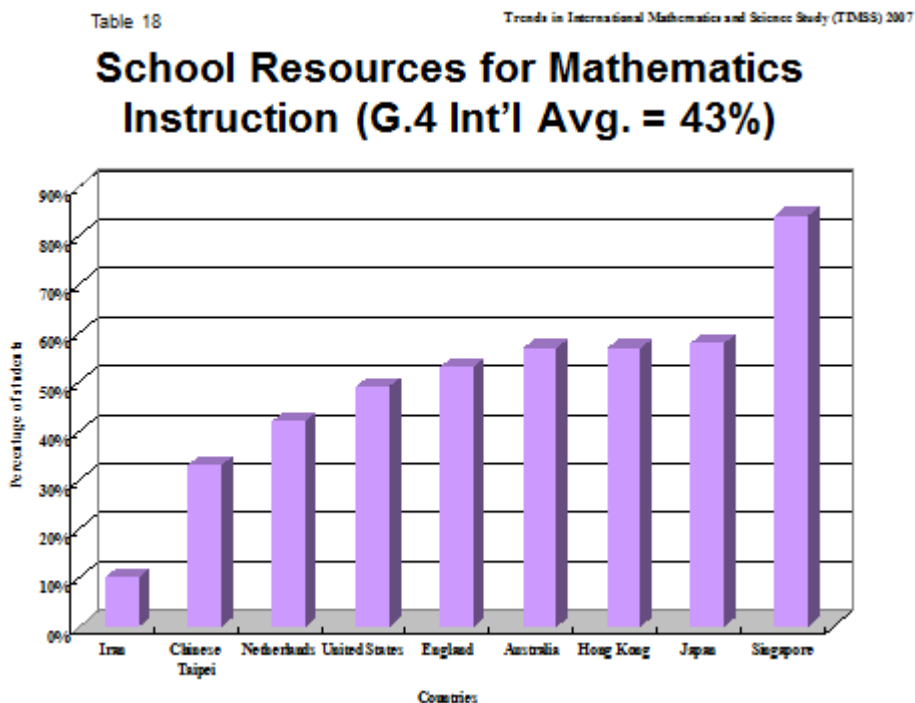


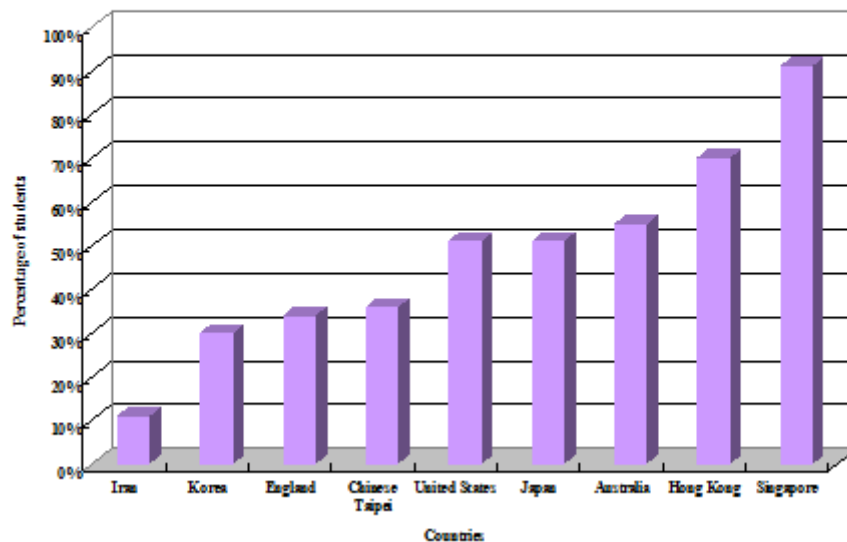
Table 19

School Resources for Mathematics Instruction (Grade 8)

Table 19

Trends in International Mathematics and Science Study (TIMSS) 2007

School Resources for Mathematics Instruction (G.8 Int'l Avg. = 32%)



What can we learn from these results? If we intend to find patterns among these variables which can be used to explain the high achievements of the countries concerned, then it is rather clear that no obvious relationship exists. For example, there is no obvious relationship between the wealth of a country and the achievement scores of its students. For the five East Asian countries, although Japan, Hong Kong and Singapore are relatively affluent as measured by GNI per capita, Taiwan and Korea are below average in wealth among the TIMSS countries. Parents of students from Japan and Korea are highly educated, but those from Hong Kong, Singapore and Taiwan are not. Most mathematics teachers in the five countries (except grade 4 Singapore teachers) have a degree, but so do many other countries. Student/teacher ratio is not favourable in Korea and Singapore, and class size in all five countries is large compared to Western countries. Time for mathematics and instruction varies in the five countries, but it is not particularly high compared to other countries. Singapore stresses homework a lot, but this is not the case for Japan and Korea. Schools in Singapore, Hong Kong and Japan are well resourced, but not those in Taiwan and Korea.

From the data above, we can conclude that student achievement cannot be accounted for totally, or even to a significant extent, by these background factors. Some factors (e.g., SES) may explain variations in achievement within a country, but they may not account for across country differences. So although background characteristics may be important factors for explaining student achievement, but they are not the only factors – and may not even be the most important factors.

Attitudes of East Asian Students towards Mathematics

Paradoxically, it is also found that the students in this cluster of countries hold rather negative attitudes towards mathematics and mathematics learning. Tables 20 to 24 below display some of the results of the attitudes of students in TIMSS 2007.

Table 20
High Positive Affect towards Mathematics (Grade 4)

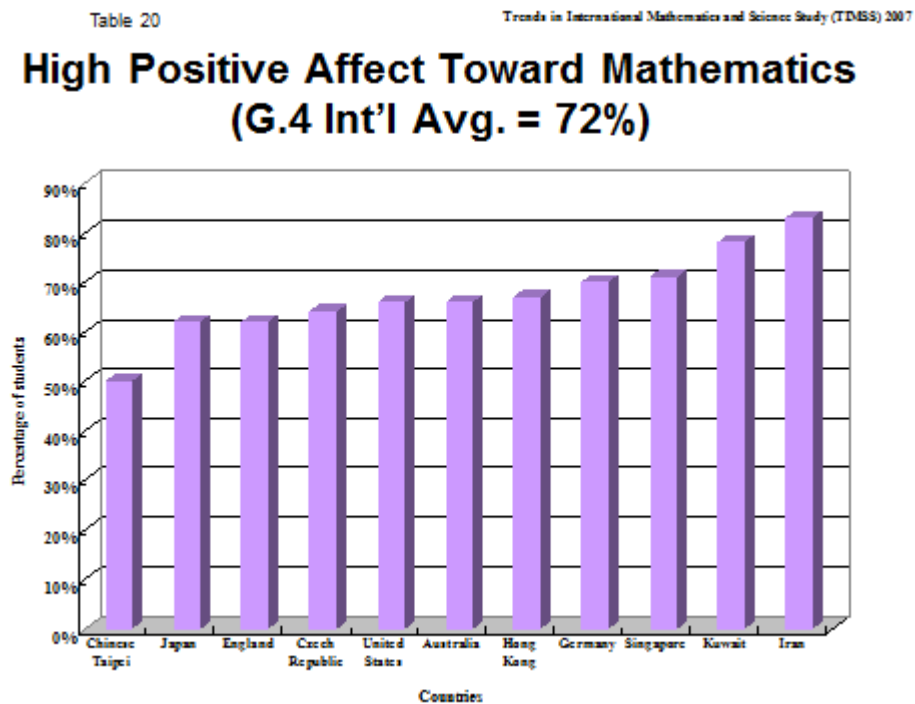


Table 21
High Positive Affect towards Mathematics (Grade 8)

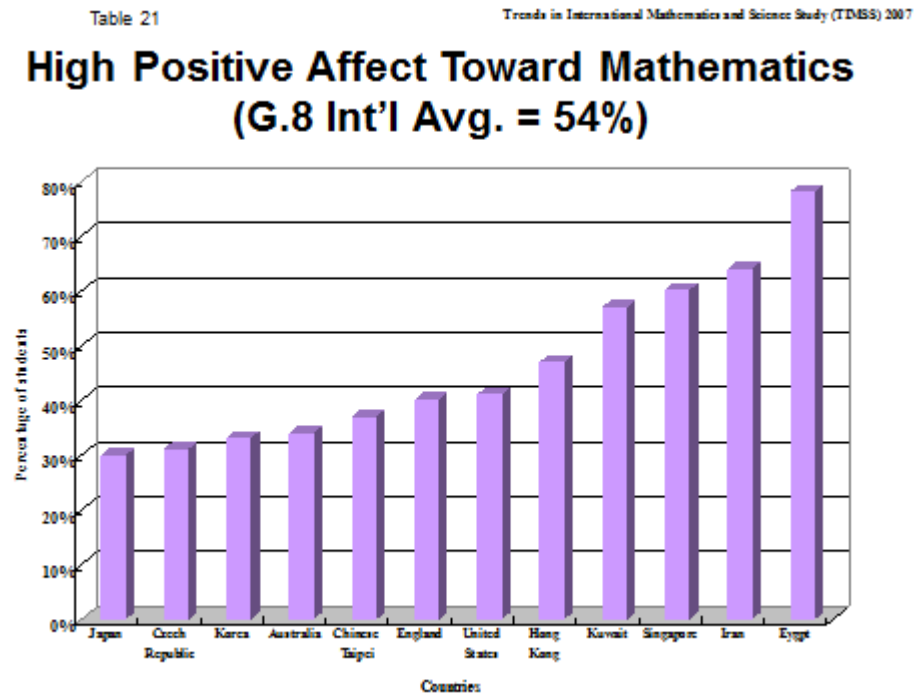


Table 22
Students Valuing Mathematics Highly

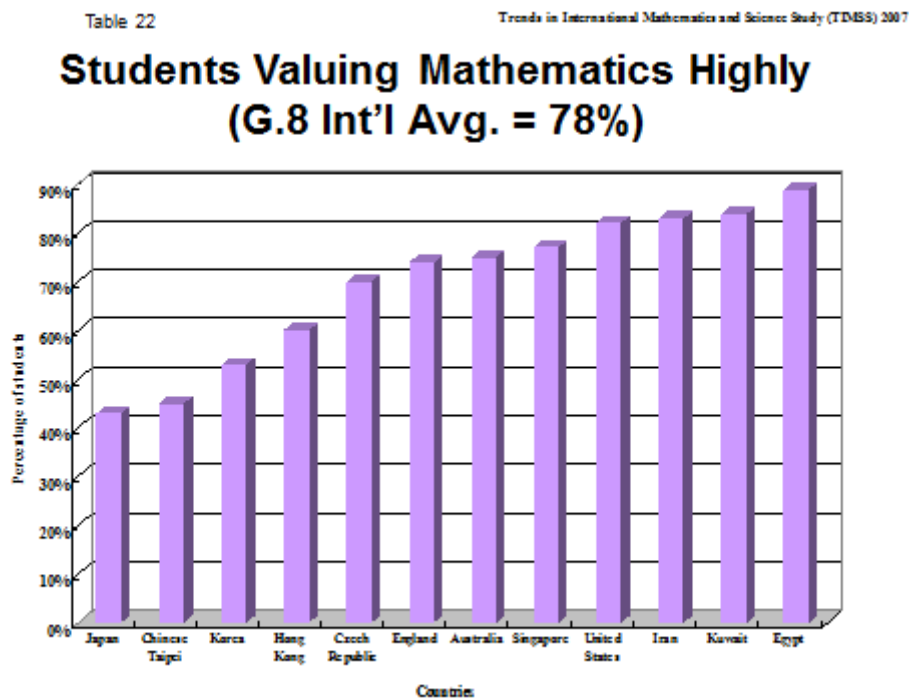


Table 23
High Self-Confidence in Learning Mathematics (Grade 4)

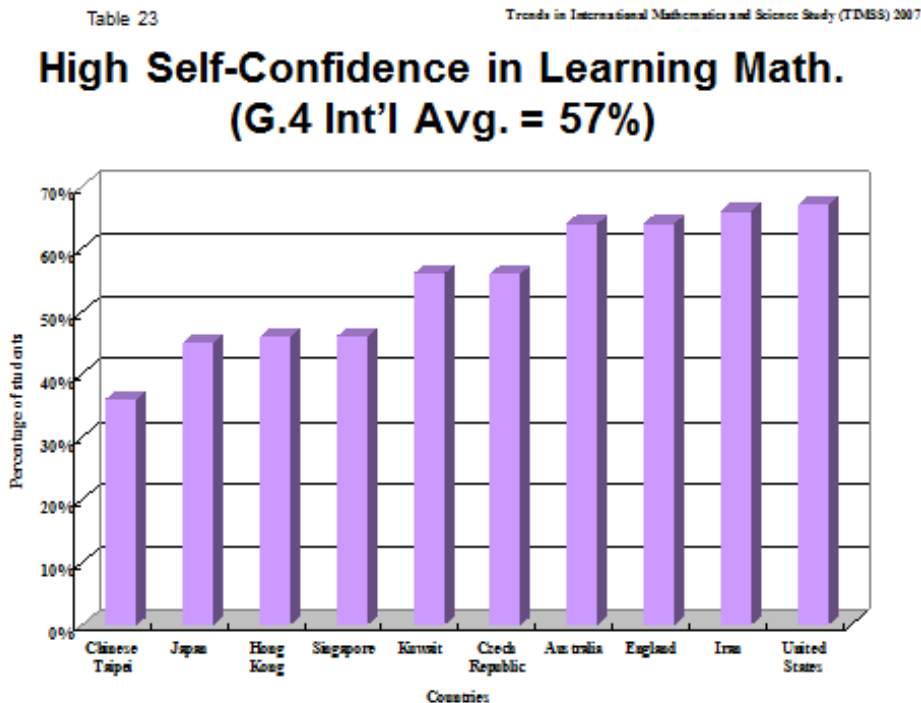
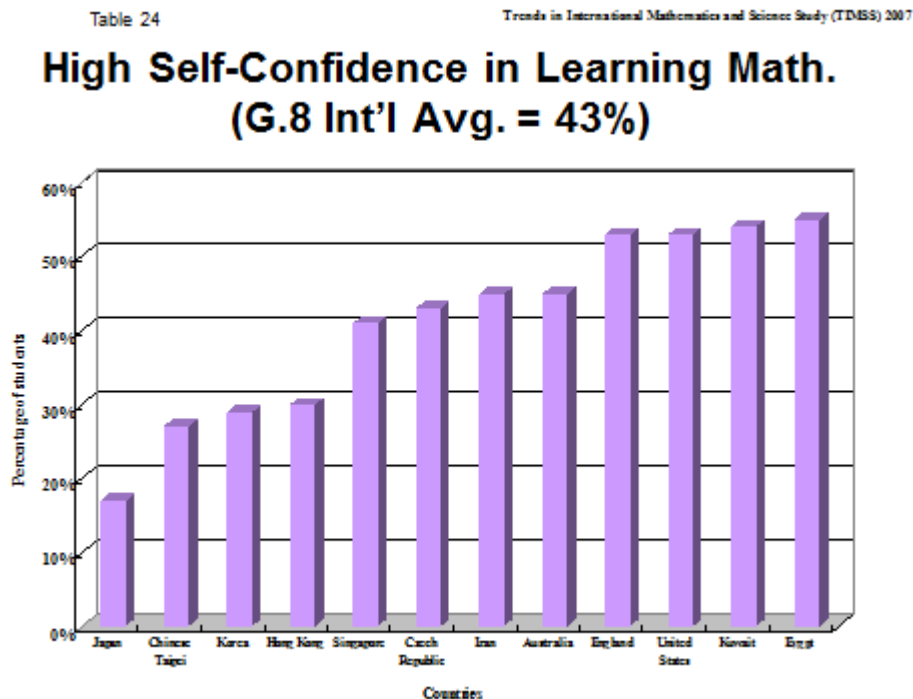


Table 24
High Self-Confidence in Learning Mathematics (Grade 8)



Culture as a Possible Explanation for Student Achievement

Why is the superior achievement of East Asian students not accompanied by a correspondingly positive attitude towards study? What is common among the five countries? The five countries are all in East Asia, but there are other East Asian countries which do not perform well. On closer examination, it is found that the five countries share a common culture, namely the Confucian heritage culture or CHC (Biggs, 1986). So in this paper, “East Asian countries” is a cultural rather than geographic demarcation. I have argued elsewhere (Leung, 2006) that cultural values held by students and teachers are an important factor which is often ignored in the discussion of student learning and achievement. Some of the relevant East Asian cultural values include:

- a strong emphasis on the importance of education
- a high expectation for students to achieve
- an examination culture
- a stress on diligence, practice and memorization
- a pragmatic philosophy
- the virtue of modesty, and
- the Chinese language

Implications

The discussion above hints that the underlying cultural values may be an important factor in explaining student achievement, but we should not take it as establishing a simple causal relation between culture and student achievement. Low achievement in a certain

country does not necessarily imply the need of total revamp of the curriculum or instructional practices. Complicated cultural factors might have affected classroom practices and student achievement, and so drastic changes should not be undertaken until such factors are thoroughly examined. Any changes in educational policy must ensure that the strengths in a country are not lost in the process. Simple transplant of policies and practices from high achieving countries to low achieving ones would not work, because one cannot transplant the practices without regard to the cultural differences. Culture by definition evolves slowly and stably with the passage of long periods of time, and there is simply no quick transformation of culture. In learning from these international studies and from other countries, we need to identify not only the superficial differences in educational practices, but also the intricate relationship between practice and the underlying culture. It is through studying these relationships that we understand the interaction between educational practices and culture. And through identifying commonalities and differences of both educational practices and the underlying cultures, we may then determine how much can or cannot be borrowed from another culture.

If culture is indeed a cause for the differences in achievements in international studies of academic achievement, and since culture is something that is by definition stable and cannot be easily changed over a short period of time, does it mean that nothing can be learned from these studies?

What Should Not Be Learned from these International Studies?

Ranking of Countries

Given the various limitations of international studies of educational achievement discussed above, what can we learn from these studies? As pointed out at the beginning of this paper, the media and the politicians (and perhaps some educators as well) have paid too much attention to the ranking of the countries that took part in these international studies, quite contrary to what these studies purport to achieve. But methodologically speaking, the data of these studies do allow us to rank countries according to the performance of their students. However, we should not take ranking too seriously. For one thing, the participating countries in these studies change from one cycle to another. So a rank of say 20th in a certain cycle may not mean the same thing as a rank of 20th in another cycle.

Also, when we compare the relatively rankings between two particular countries, we should be careful in taking the standard error of measurement into consideration. For example, compared to TIMSS 2003 (Mullis et al., 2004), grade 8 students in Singapore may be seen as “dropping” from the first place to the third place in TIMSS 2007. But if we take the standard errors of measurement into consideration, the differences between the score for Singapore and those of Korea (rank 2) and Chinese Taipei (rank 1) in 2007 are not statistically significant. That is, from a statistical point of view, we cannot say that the scores of Chinese Taipei and Korea are higher than that of Singapore. Because of these two reasons (change of the participating countries from cycle to cycle and the standard error of measurement), we should not be too sensitive about fine changes in ranking from cycle to cycle - it is usually not too meaningful to say that a country’s ranking has dropped from say 15th to 18th without further qualification.

Causal Relations

Another point to be remembered is that these international studies are surveys, and not experiments. So we have to be extra cautious in drawing conclusion about casual relations. In most instances, the best that we can conclude is that a certain variable A *may* have caused or impacted student achievement, based on the correlations between the measure of variable A and the achievement scores, since it is unlikely or illogical that achievement leads to changes in variable A. But of course, we cannot rule out the possibility that there is a third variable which influences both variable A and achievement, causing variable A and achievement to be correlated with each other. The problem of course is that there are so many possible “source” variables that may have influenced both variable A and achievement, so we can never be sure whether it is variable A which really impacts achievement or that both achievement and variable A are caused by some unidentified factors. One example is the relationship between amount of homework and achievement. Students may have better achievement because they do more homework, but students may need to do more homework because they have low achievement. It is therefore not surprising that there is no clear relationship between student achievement and the amount of homework students do.

Teacher Attributes

If we need to be careful in drawing any conclusions about student attributes (achievement, attitudes, etc.), we need to be even more careful in drawing conclusions about teacher attributes in different countries. This is because in TIMSS, we do not have a representative sample of teachers, so all references to teachers are from the student sample. For example, we cannot say that 83% of the primary school teachers in Hong Kong have a university degree, all we can only say is that 83% of the primary (four) school students have teachers with a university degree.

Implications for Education Policies

Despite the aims of these international studies in providing information for “guiding educational decision making and practice in the areas of mathematics (and science)”, we should be extremely cautious in suggesting changes in policy and practice based results of these studies, given the dubious nature of drawing causal relations from the findings. Actually in many instances, curriculum changes claimed to be based on results of these international studies were more excuses rather than rational decisions based on a rigorous examination of the results.

What Can and Should We Learn from these International Studies?

Despite all the limitations of these international studies discussed earlier in this paper, the rigorous methodologies adopted in these studies do provide us with a reliable, “‘objective’, accurate and meaningful measure of student achievement to answer the question of effectiveness” of an education system (Leung, 2011, 391). And since these studies are “international (studies) with endorsement from a large number of countries” (Leung, 2011, 391), they provide benchmarks against which different countries may measure the achievement of their students.

In this regard, some people pointed out the “unfair” nature of these international studies as another limitation of these studies. This limitation is particularly acute in TIMSS as it

aims at testing students' competence according to the school curriculum, and given the scale of the study the test items inevitably match the curricula in some countries better than others. For example, while there is 100% coverage of the TIMSS 2007 grade 8 test items in the US and Hungarian curriculum, the coverage for the Dutch curriculum is only 71%. Is that a big problem? If TIMSS is rightly not regarded as a competition, then "fairness" may not be the most important criterion in the design of test items for these international studies. (TIMSS actually has measures to tackle this problem – the test-curriculum matching analysis, see Mullis et al., 2008, p.439.) One may even argue that it will be more interesting to find out how well our students perform in areas that they have not learned, rather than in those that they have. Whether it is fair or unfair boils down to what the purposes of these international studies are.

For those countries which have participated in more than one cycle of the studies, a very instructive piece of information is the change of scores (rather than change of ranking) across different cycles, because scores in these studies are standardized across years and are thus theoretically comparable. But of course we should realize that different students participated in different cycles, and so the change of scores just gives a rough indication of "trends" – these are not truly longitudinal studies. Even for TIMSS when the scores of say TIMSS 2003 grade 4 students in a certain country are compared to the TIMSS 2007 grade 8 students, the students come from the same cohort but not the same students were taking the tests in 2003 and 2007, so any "gain" in scores should be interpreted in that light. Notwithstanding this limitation, this rough information on trends of performance should be informative to educators in the country, especially when there are major curriculum changes taking place in between the cycles of study.

There should also be more attention given to the relative strengths in different strands of mathematics (e.g., geometry versus statistics), in addition to the overall mathematics achievement. This information will inform us of the relative strengths and weaknesses of our students, especially in light of the performance of students in another country or internationally, and this in turn will be informative for both curriculum developers and practising teachers. More importantly, in the TIMSS test, there are measures of students' performance in the cognitive domain, in addition to the content domain. This information will be very useful to curriculum developers and teachers alike.

In the scoring of open-ended items of the TIMSS test, a two-digit scoring code is used, the first digit records the marks given to that item (partial correct answers are reflected by the marks awarded), while the second digit categories how the student arrives at the right or wrong answer. The former will inform us of the typical way the item is solved in a country or a school, while the latter will inform us of typical misconceptions concerning that item. These are extremely useful information for teachers, but unfortunately the second digit code is hardly used at all in the analysis of the results.

Another area that is worth paying more attention to is the attitudes of students towards mathematics and mathematics learning. When we talk about the attained curriculum, we often think of it in terms of the test scores only. But actually students' attitudes are an important component of the attained curriculum, since in all school systems, students' positive attitudes are one of the goals of mathematics education. In this era when life-long learning is so much stressed, some people think that a positive attitude is even more important than attaining high test scores. A positive attitude will motivate students to continue to learn even after they have left school.

Conclusion

As mentioned at the beginning of this paper, international studies of mathematics achievement have profound influence on mathematics education worldwide in the past 15 years or so. Much attention however has been paid on the ranking of countries in the league tables generated from such studies, often without due consideration of the nature of these studies, as well as the contextual factors that affect the performance of students from different countries. In this paper, the nature as well as the limitations of these large scale international studies have been discussed. It has also been argued that cultural values may be an important factor in explaining differences in student achievement. Without due consideration of the cultural and other contextual differences, passing judgement on the performance of students in different countries based on results of international studies is very misleading, and may even be damaging. In learning from other countries, one must first evaluate the cultural values and educational context in one's own country before deciding on how much can be learned from other countries.

It is also important that when results of these international studies are discussed, we should identify the strengths of our students, not just their weaknesses. It is through identifying strengths that we can make sure that these strengths are not compromised in any measures suggested to overcome the weaknesses identified. We don't want to "throw the baby out together with the bath water!"

Results of international studies should serve as mirrors for us to better understand our system. In the process, we must bear in mind that education is a complex endeavour – we cannot expect international studies to produce answers for all our national problems in education. International studies provide rich dataset for individual countries to seek answers for their own issues. In so doing, we need wisdom, and not just data!

References

- Beaton, A.E., Mullis, I.V.S., Martin, M.O., Gonzalez, E.J., Kelly, D.L. & Smith, T.A. (1996). *Mathematics Achievement in the Middle School Years*. Chestnut Hill, MA: International Study Center, Boston College.
- Biggs, J.B. (1996). Western Misconceptions of the Confucian-Heritage Learning Culture. In D.A. Watkins & J.B. Biggs (Eds.), *The Chinese Learner*. Hong Kong: Comparative Education Research Centre.
- Drent, M., Meelissen, M. & van der Kleij, F. (2011). *The Contribution of TIMSS to Theories of Educational Effectiveness: A Systematic Review of Literature*. Symposium paper presented at the European Educational Research Association Conference, Berlin, 2011.
- Harmon, M., Smith, T.A., Martin, M.O., Kelly, D.L., Beaton, A.E., Mullis, I.V.S., Gonzalez, E.J. & Orpwood, G. (1997). *Performance Assessment in IEA's Third International Mathematics and Science Study*. Chestnut Hill, MA: International Study Center, Boston College.
- Husen, T. (1983). Are Standards in U.S. Schools Really Lagging Behind Those in Other Countries?, *Phi Delta Kappan*, 64, 455-461.
- Hutchison, G., & Schagen, I. (2007). Comparisons between PISA and TIMSS—Are we the man with two watches? In T. Loveless (Ed.), *Lessons learned—What international assessments tell us about math achievement*. Washington, DC: The Brookings Institution.
- Leung, F.K.S. "Mathematics Education in East Asia and the West: Does Culture Matter?", book chapter in Leung, Graf and Lopez-Real (Eds.) *Mathematics Education in Different Cultural Traditions: A Comparative Study of East Asia and the West, The 13th ICMI Study*, New York: Springer, 2006, 21-46
- Mullis, I.V.S., Martin, M.O., Beaton, A.E., Gonzalez, E.J., Kelly, D.L. & Smith, T.A. (1997). *Mathematics Achievement in the Primary School Years*. Chestnut Hill, MA: International Study Center, Boston College.
- Mullis, I.V.S., Martin, M.O., Gonzalez, E.J., Gregory, K.D., Garden, R.A., O'Connor, K.M., Chrostowski, S.J. & Smith T.A. (2000). *TIMSS 1999 International Mathematics Report*. Chestnut Hill, MA: International Study Center, Boston College.

- Mullis, I.V.S., Martin, M.O., Gonzalez, E.J., & Chrostowski, S.J. (2004). *TIMSS 2003 International Mathematics Report*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.
- Mullis, I.V.S., Martin, M.O. and Foy, P. (2008). *TIMSS 2007 International Mathematics Report*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.
- Organisation for Economic Co-operation and Development (2001): *Knowledge and Skills for Life: First Results from PISA 2000*. Paris: OECD Publications.
- Organisation for Economic Co-operation and Development (2003): *Literacy Skills for the World of Tomorrow — Further Results from PISA 2000*. Paris: OECD Publications.
- Organisation for Economic Co-operation and Development (2004). *Learning for Tomorrow's World – First Results from PISA 2003*, OECD.
- Walker, M.A. (2002). Prolegomena to Any Future Philosophy, *Journal of Evolution and Technology*, Vol 10

Appendix

Appendix

Eighth Grade

AROUND THE BEND

At this station you should have:

- Two rectangles of white card, A and B, which are models of pieces of furniture
- 1 cm squared graph paper to make different rectangles to be models of other pieces of furniture
- Scissors
- A 30 cm ruler
- Plastic bag and labels
- Paper clips
- A model representing a corridor in an apartment

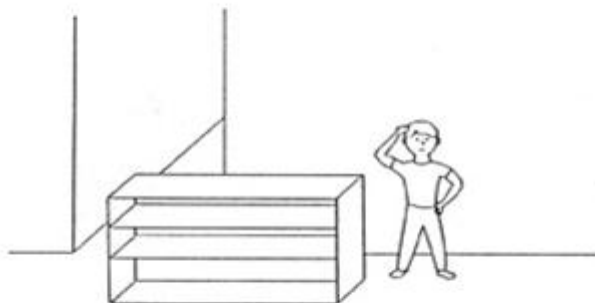
Your task:

Find out what sizes of furniture can be moved around the bend in the corridor.

Read this before answering the questions:

Ray is to move into an apartment which has the main rooms around a bend in the corridor leading from the front door.

What sizes of furniture will go around the bend in the corridor?

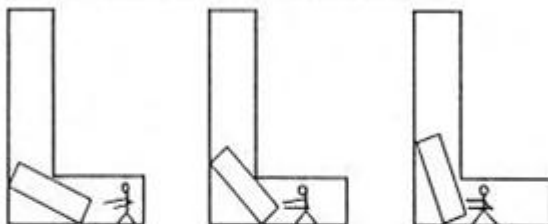


Ray wants to get some large pieces of furniture around the bend the right way up. He does not want to turn the pieces of furniture on their sides. He uses the models of the corridor and furniture to find out which pieces of furniture will go around the bend.

page 1

TASK M4-P2

Here are some pictures (not to scale) showing what could happen.



The rectangles representing furniture and the model of the corridor in Ray's apartment are drawn to scale. Scale: 4 cm represents 1 m.

1. Measure the lengths and widths of the two models of pieces of furniture in cm.

A is 8cm cm long and 4cm cm wide.

B is 4cm cm long and 2cm cm wide.

2. What are the lengths and widths of the two pieces of furniture in meters?

A is 2m m long and 1m m wide.

B is 1m m long and 1/2m m wide.

3. Here is a list of furniture:

single bed
cot

coffee table

3-seater couch

armchair

double bed

dining table

2-seater couch

sideboard

Judging from their sizes:

What piece of furniture is A most likely to be? Single bed

What piece of furniture is B most likely to be? coffee table

4. Which piece(s) of furniture (A or B or both) will go around the bend in Ray's apartment and which will not?

piece A will not go around the bend.

piece B will go around the bend.

Please turn the page.

TASK M4-P2

page 2

5. Use the graph paper to make other models of pieces of furniture to the sizes listed in the table below. The sizes are all given in meters.

In the second column of the table suggest what the furniture could be.

In the third column find out if the piece of furniture will go around the bend, and check the correct answer.

	Furniture Size		What Furniture could be:	Goes around the bend?		
	Length (m)	Width (m)		Yes, easily	Yes, just	No
C	0.5	0.5	bedside table	✓		
D	1.5	0.5	2 seater couch	✓		
E	2	0.5	3 seater couch		✓	
F	1	1	dining table		✓	
G	1.5	1	double bed			✓
H	2	1	king size bed			✓

6. Whether or not a piece of furniture goes around the bend of Ray's corridor depends on its length and width. Look at the results you have for all the pieces of furniture A, B, C, D, E, F, G and H.

- Try to find a rule for working out from their lengths and widths whether or not a piece of furniture goes around the bend.

If the width is half as long
as the length it will not go
around the bend. But it will if
the width & length are the same or
if the width is less. It will

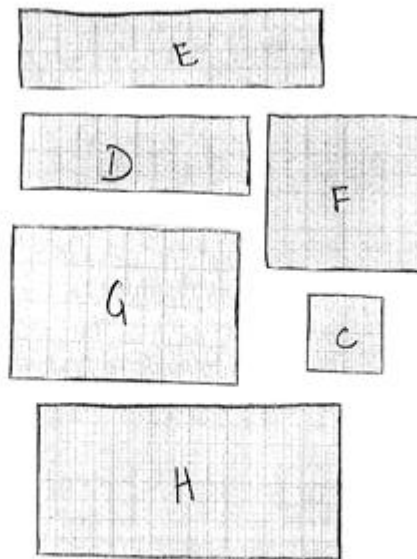
PUT THE PIECES OF FURNITURE YOU MADE IN THE PLASTIC BAG AND PUT
YOUR NAME ON THE LABEL.

FASTEN THE BAG TO THIS PAGE WITH A PAPER CLIP.

LEAVE THE MODELS A AND B AT THE STATION.

page 3

TASK M4-P2



Shown at actual size

PRACTICAL IMPLICATIONS AWARD

A Learning Community for Pre-service Secondary Mathematics: Learning With and From Each Other

Michael Cavanagh
Macquarie University
<michael.cavanagh@mq.edu.au>

This paper reports the impact of a collaborative professional experience model on pre-service secondary mathematics teachers' perspectives and practices within a learning community. Nine pre-service teachers made 12 school visits over one year to observe and co-teach problem-solving lessons in two Year 8 classes. They discussed the lesson with the teacher and the university supervisor, and posted reflective comments to an online forum. Data from questionnaires, interviews, and reflections indicate participation in the learning community helped pre-service teachers link theory and practice, learn from each other, and become more reflective.

Many pre-service teachers come to university with firm beliefs about learning and teaching (Scherrf & Singer, 2012) and they invariably want to teach as they were taught (Cooney, Shealy, & Arvold, 1998). This is particularly the case for secondary mathematics because pre-service teachers typically learned mathematics in a traditional manner (Cooney & Weigel, 2003). The situation is exacerbated because, rather than challenge prior understandings, some teacher education programs have been found to reinforce them (Zeichner, 2010). These courses “do not provoke students to confront their naïve notions of teaching mathematics” (Lerman, 2001, p.48) by supporting a ‘cycle of tradition’ (Frid & Sparrow, 2009). The quality of professional experience supervision can be inconsistent and has traditionally been *evaluative* rather than *educative* (Blanton, Berenson, & Norwood, 2001) with an over-emphasis on classroom management and organisation. Hence there is limited support for pre-service teachers to “explore, discuss, and reflect on their developing understandings” (Sim, 2006, p. 78).

In contrast, effective professional experience programs enable student teachers to learn *in* and *from* practice (Ball & Cohen, 1999). They expose student teachers to the complex nature of the classroom (Goos, 2008) and provide opportunities to implement alternative approaches (Leiken, 2008), discuss their experiences, and learn from each other (Mewborn, 2000). However, a recent report found there was “little consensus on the best structure for practicum” (Commonwealth of Australia, 2007, p. 67) and called for the development of new professional experience models that are “flexible and encourage innovation” (p. 74). Zeichner (2010) also recently noted that there were few research studies on the impact of new models of professional experience programs. The present paper seeks to address this gap in the literature by reporting the impact of a professional experience learning community for pre-service secondary mathematics teachers.

Theoretical Framework

Ponte et al. (2009) develop the notion of a *learning community* as a group of people “involved in some kind of activity that learn together and, more importantly, learn from each other” (p. 197). The authors describe four key issues for a learning community: (i) the purpose of the group and how closely the members identify with that purpose; (ii) the knowledge that develops from the activity of the group; (iii) how learning happens in the group; and (iv) the roles and relationships of group members, especially their mutual involvement and commitment to the group's progress.

In this learning community, the purpose for the pre-service teachers was to develop their professional practice through observation and co-teaching (Bobis, 2007), and reflection on practice through group discussion (Sim, 2006) and individual reflections (Ebby, 2000). The learning community encouraged the mutual involvement of participants through *reciprocity* or “the development within learning communities of learners’ commitment to and responsibility for their own learning as well as that of other members of the community” (Le Cornu & Ewing, 2008, p. 1808). Reciprocity requires a shift in focus from the individual to a shared experience among pre-service teachers and supervisors. Supervision is seen as collaborative, with university and school staff working closely together. Reciprocity therefore acknowledges the sociocultural aspects of learning to teach (Goos, 2008; Lerman, 2001).

The design of the learning community focused on *teaching* rather than *teachers* (Hiebert, Morris, & Glass, 2003) to develop an educative rather than an evaluative professional experience (Blanton et al., 2001) with on-going engagement of the university supervisor to link methods workshop activities with the pre-service teachers’ classroom experiences (Borko & Mayfield, 1995). The learning community also reflected important aspects of the *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2006), particularly Standard 2.2 “active exploration of new teaching ideas” and Standard 2.3 “actively engaging and collaborating with colleagues ... sharing insights, practices and resources; supporting and mentoring others; and providing feedback” (AAMT, 2006).

The research questions for the study focus on the second and third issues identified by Ponte et al. (2009): What kinds of knowledge about practice did the pre-service teachers develop through their participation in this professional experience learning community? What features of the learning community facilitated the development of this knowledge?

Method

Participants

The study sought to describe the learning community from the participants' perspective so a phenomenological case study design was used. Phenomenological research “seeks the individual’s perception and meaning of a phenomenon or experience” (Mertens, 2005, p. 240). Nine pre-service teachers (five female and four male), the entire cohort for secondary mathematics in a one-year Graduate Diploma of Education, participated in the study. The cooperating teacher was the head teacher of mathematics; he had taught for 27 years and supervised 20 pre-service teachers. His previous supervisory practice accorded with the evaluative supervisory roles described by Blanton et al. (2001). He was purposefully selected for the project because of his desire to showcase problem-solving activities in his lessons. The school was a comprehensive government secondary school with an enrolment of about 1 100 boys. The university supervisor (the author of this paper) had been a secondary mathematics education methods lecturer for seven years. He had previously taught secondary mathematics for 20 years and been a cooperating teacher.

The cooperating teacher and the university supervisor designed the learning community program together as a partnership between the school and the university. The pre-service teachers formed two mixed groups (of four and five) matched to a Year 8 class (the highest ability or second-highest ability class) and made six fortnightly school visits each semester, for a total of 12 visits per group over the year. The university supervisor attended all school visits for both groups. The Year 8 lessons were 80 minutes long and there were approximately 30 students in each class. In Semester One, student teachers observed the

cooperating teacher. In Semester Two, they each co-taught two or three lessons with a different partner each time. Bessette (2008) describes co-teaching as “jointly planning, coordinating, implementing, and evaluating ... as co-teachers form bonds, teach and learn from one another, and provide mutual support” (p. 53).

Following each class, the cooperating teacher led a 15-minute discussion about the lesson by asking the pre-service teachers to comment on student engagement and learning, why the teacher(s) had acted in a particular way at certain key points in the lesson, and anything else they had noticed. After this, the university supervisor shared his thoughts and posed some questions for the pre-service teachers to consider on their own. Within one or two days, the pre-service teachers wrote a personal reflection about the lesson and posted it to the university online discussion forum. There they could read and comment upon the reflective posts of their peers. Neither the cooperating teacher nor the university supervisor contributed to the online forum. Many of the online posts drew on ideas already discussed in the group after the lesson, but they also included reactions to posts from other pre-service teachers and reactions to the questions raised by the university supervisor during his contribution to the post-lesson group discussion.

Pre-service Program

The methods workshops emphasised constructivist learning theories and challenged pre-service teachers’ views about the efficacy of traditional teacher-centred pedagogies by encouraging them to consider reform-oriented approaches such as those recommended in key policy documents (e.g., AAMT, 2006). In alternate weeks, the methods workshops incorporated discussions about the school visits so that the pre-service teachers could further reflect on what they had observed at the school. To provide opportunities for further reflection and analysis of the lessons, the university supervisor identified pertinent comments from the online forum which focussed on features of non-traditional classroom practice evident in the problem-solving lessons. He asked the student teacher to elaborate their ideas and the discussion continued for about 20 minutes as other students shared their reactions and reflections. Even though the student teachers had not all witnessed the same lesson, the problem-solving activity was identical for both classes, so it was possible for everyone to contribute their ideas and be understood by the group. Comparing the variations in the lessons and the student learning outcomes in each case made for some of the most interesting exchanges.

Data Collection and Analysis

A research assistant who was not part of the learning community collected all of the data from pre-service teachers. Pre-service teacher data comprised anonymous questionnaires at the end of each semester and an audio-recording of a 35-minute focus group interview with the nine student teachers held at the end of the year. The author conducted a 20-minute audio-recorded interview with the cooperating teacher at the end of the year. The questionnaires comprised four questions about the pre-service teachers’ views on the learning community and how it compared to their other professional experience activities. The group interview stimulated discussion from multiple perspectives to explore the range of participant views (Bogdan & Biklen, 2006).

Data analysis occurred at the end of the year. The author and a research assistant independently analysed the questionnaires and interview recordings. They read the questionnaires multiple times and made detailed notes as they listened to the interview recordings. They used open coding (Strauss & Corbin, 1990) to examine and categorise the

data to identify emerging themes. Later, they compared their initial codes to contrast and refine them. Axial coding (Strauss & Corbin, 1990) helped to establish connections between the refined categories and develop the major themes reported in this paper.

Results

Learning to Link Theory and Practice

The pre-service teachers were unanimously positive in their evaluation of the learning community approach, with many regarding it as the single most important and worthwhile feature of the entire Teacher Education Program.

New teaching practices. The learning community lessons were “unconventional” and “very different from other teachers observed”. This contrasted their other school experiences, which they described as “chalk and talk” practices which had little relationship to the constructivist theories they were studying in their methods workshops. They noted that the teaching methods of the learning community classroom “closely reflected theories taught at uni” and began to “realise [that the] teaching theories we learn at uni have real applications”. The concurrent classroom and workshop activities of the learning community provided pre-service teachers with a chance to integrate theoretical and practical knowledge and “test a different teaching strategy that otherwise couldn’t have been done during normal prac”.

The problem-solving lessons of the cooperating teacher demonstrated that constructivist theories could be implemented in realistic classroom settings so pre-service teachers could see that “there is another way to teach maths”. They noted how the problem-solving lessons they observed productively engaged students and the evidence of effective student learning encouraged the pre-service teachers to implement some of the learning activities and teaching strategies in their other school placements. “I taught Year 9 one of the [problem-solving] activities. They liked it, they loved it, and the supervising teacher liked it too”.

Theorising about teaching. Participation in the learning community also helped pre-service teachers theorise about classroom practices. By observing the cooperating teacher and their peers they understood “there are several ways of teaching mathematics” and “having many approaches brings out great ideas from the students”. By reflecting on the successful elements of these lessons they recognised problem solving as a productive mathematical learning experience because “a sense of challenge and purpose to solve a problem increases engagement”. And by adopting some of the cooperating teacher’s strategies in their co-taught lessons the pre-service teachers became more closely attuned to the notion of teachers as facilitators of learning since teaching is “not only teaching standing at the board and explain[ing]”.

Collaborative Learning within the Learning Community

Through observing, co-teaching, discussing and reflecting on each other’s lessons, participants had many opportunities to share their experiences and expand their repertoire of pedagogical approaches.

Co-teaching and peer observation. The experience of co-teaching was “empowering” and highlighted how much they could “learn off their peers”. In lesson preparation, co-teaching “brings more ideas and approaches to the table” and “encouraged experimentation” because the process of designing the lesson was “less isolated”. In delivering the lesson, co-teaching was “less scary with someone for support” and provided a unique opportunity for

pre-service teachers to “learn how to collaborate with another colleague to produce a lesson”. Peer observation was valuable because it allowed the student teachers to “see different ways of teaching”. Their peers were “at the same level” so it was possible to imagine how they could “put into practice things you liked or thought were effective in others’ lessons”. Being observed by peers allowed pre-service teachers to receive “quality feedback from different perspectives” and to hear “different aspects of myself from others’ point of view”.

Group discussions. The group discussions following each lesson were characterised by their richness and variety: “One thing that really comes out clear is that some people have different views on the same thing”. The diversity of opinions allowed “alternate perspectives that may not have been previously considered” to emerge and opened up “many different alternative courses of action”. Pre-service teachers learnt “many different ways” and “more options” to “enrich the teaching of mathematics” which promoted experimentation at their other school placements. They reported incorporating problem-solving tasks in their lessons, becoming more aware of the need to make lessons relevant to students, and adopting a more constructivist style. Even if some of their initial attempts were unsuccessful, they were encouraged to persevere by repeated exposure to the positive outcomes they discussed in the learning community classroom.

The cooperating teacher. Reciprocity occurred not only among the pre-service teachers but also for the cooperating teacher. He “learnt a great deal” from observing the lessons co-taught by pre-service teachers in the second semester. He explained that he was inspired by their innovative and creative lesson introductions and the ways they used technology to motivate and engage students. The cooperating teacher also commented that “comparing how they teach the same lesson with the way I taught it gives me a lot to think about” and he added that he intended to adapt his teaching of some of the problem-solving activities in the following year. He also found the post-lesson discussions useful because the pre-service teachers “have a perception of things that’s really worthwhile [and] it’s interesting to hear their ideas about my lessons”. He said the group discussions were “very different” to his previous experiences supervising pre-service teachers because “usually discussing a lesson is just a sideline, an occasional thing and then you move straight on to the next class, the next urgent thing that has to be done”.

Individual Learning within the Learning Community

The learning community provided opportunities for individual reflection as pre-service teachers wrote a personal critique of the lessons via the online forum.

Becoming more reflective. The pre-service teachers recognised the importance of critical reflection in helping to “increase awareness” about learning and teaching by “making me focus on specifics of the lesson” and “making me think through what I have observed”. They found the individual writing task challenging and rewarding because written reflections “provided the opportunity to pause and deeply reflect on the observed lesson”. The deliberative act of writing “forces pre-service teachers to pause and think”, encouraging them to evaluate learning and teaching outcomes and more acutely consider their strengths and weaknesses. Identifying the positive elements of each other’s lessons was affirming, while constructively critiquing the less successful aspects was beneficial as “reflection, particularly writing it down, allows one to identify areas for improvement”.

The written reflections allowed student teachers to begin processing the outcomes of the group discussions in a more considered way than they might otherwise have done. Their

reflective practice developed not only through hearing the views of others but also in thinking more deeply about the ideas: “Digesting others’ point of view opens up my thinking”. These reflections often raised many more questions for the pre-service teachers to consider about the lessons they observed and or taught, such as: “What went wrong? How could things be done better?”

Reflection in action. Reading each other’s reflections increased pre-service teachers’ awareness of alternative viewpoints and helped them make sense of their classroom experiences as they observed their peers’ lessons and critiqued the lessons they co-taught. The group discussions and individual reflections from peers on the co-taught lessons also had an impact on individual pre-service teacher’s practice beyond the learning community classroom: “I can take at least one piece of advice and use it in normal prac”. Another pre-service teacher summarised her developing reflective practice in terms of a three-stage process “Try, Reflect, Re-model” indicating a connection between her ability to experiment with new pedagogies, think deeply about the lesson observations and discussions, and enact further changes in her classroom practice.

Discussion

Although they found the cooperating teacher’s lessons worthwhile, the pre-service teachers reported that co-teaching, observing, discussing, and reflecting on their peers’ lessons were the most valuable aspects of the program. In co-teaching, they took greater risks than they could have made done on their own. They were more forthright in their analysis of each other’s lessons because it was less threatening to critique the teaching of their peers than an experienced practitioner. They also regarded the successes of their peers as achievable goals which pointed the way towards each person’s improved classroom practice. Co-teaching was therefore central in promoting innovative classroom practice as these lessons demonstrated how student-centred teaching approaches could be successfully implemented in a secondary mathematics classroom. Observing and co-teaching these lessons helped to disrupt the ‘cycle of tradition’ (Frid & Sparrow, 2009) and encouraged the pre-service teachers to develop their professional knowledge of reform-oriented pedagogies.

The combination of learning community activities was carefully constructed as educative rather than evaluative (Blanton et al., 2001) in nature. There was no formal evaluation of the pre-service teachers’ co-taught lessons and their reflective writing forum postings were not assessed. Since the problem-solving lessons were distinct from the rest of the Year 8 Mathematics program there was no pressure to cover the syllabus content in preparation for examinations and pre-service teachers had greater freedom in planning and teaching their co-taught lessons. They could experiment more freely with reform-oriented pedagogies than in their other school placements where pressure to complete the syllabus often constrained the implementation of new teaching strategies.

Reflecting on teaching can be difficult for pre-service teachers (Star & Strickland, 2008). As Sim (2006) observed, the key to becoming more reflective lies in exploring and discussing fundamental aspects of practice. The present study confirms this finding and demonstrates two other important features of reflective activities: their quality and synchrony. High quality reflection was achieved through a combination of individual and group tasks, both oral and written, which encouraged pre-service teachers to think more deeply about their experiences. Synchrony of reflection occurred by interweaving the school visits with follow-up activities in methods workshops to mirror the classroom experiences and reinforce the practice of reflection.

Literature on improving the quality of teacher education programs argues for a closer alignment between university studies and field experiences (e.g., Ebby, 2000). Previous research has identified irregular supervisory visits (Blanton et al., 2001) and giving too much attention to classroom management and organisation (Goos, 2008) as problematic. The cooperating teacher and university supervisor were equal partners in the design and development of the learning community; they conceived the program together and developed a supervisory partnership as it progressed. The cooperating teacher shared his experience with problem-solving lessons and the university supervisor provided a reform-oriented theoretical lens through which to interpret and discuss the classroom episodes.

The cooperating teacher and the university supervisor participated in the learning community as co-constructors of knowledge (Lerman, 2001) with the pre-service teachers. The cooperating teacher facilitated the post-lesson discussions so that the student teachers could share their observations and react to each other's ideas and the university supervisor used the ideas proposed by the pre-service teachers to stimulate further discussion and reflection in the workshops. Their aim was to prepare graduates who would engage in the kinds of professional learning envisioned in the *National Professional Standards for Teachers* (Australian Institute for Teaching and School Leadership [AITSL], 2011). In particular, Standard 6.3, which recognises that teachers should "contribute to collegial discussions and apply constructive feedback from colleagues to improve professional knowledge and practice" (AITSL, 2011, p. 18).

Conclusion

All of the learning community participants valued the experience and were able to identify some benefits they gained from taking part in it. Everyone developed a greater appreciation of the importance of mathematical problem solving as a practical way of implementing the reform agenda for secondary mathematics. Intermingling the activities between the classroom and the methods workshop assisted pre-service teachers in establishing strong links between theory and practice. The opportunities for co-teaching and peer observation allowed the participants to collaborate and support each other's learning. Finally, posting individual comments to the online forum helped the pre-service teachers develop a more reflective stance on their personal classroom practice. Above all, the learning community helped to position the pre-service teachers as collaborative partners who recognised the value of learning with and from each other.

References

- Australian Association of Mathematics Teachers. (2006). *Standards for excellence in teaching mathematics in Australian schools*. Adelaide, SA: AAMT.
- Australian Institute for Teaching and School Leadership. (2011). *National professional standards for teachers*. Carlton, Vic.: Education Services Australia.
- Ball, D. L., & Cohen (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3–32). San Francisco, CA: Jossey-Bass.
- Bessette, H. J. (2008). Using students' drawings to elicit general and special educators' perceptions of co-teaching. *Teaching and Teacher Education*, 24, 1376–1396.
- Blanton, M. L., Berenson, S. B., & Norwood, K. S. (2001). Exploring a pedagogy for the supervision of prospective mathematics teachers. *Journal of Mathematics Teacher Education*, 4, 177–204.
- Bobis, J. (2007). Empowered to teach: A practice-based model of teacher education. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential Research, Essential Practice* (Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 61–70). Adelaide, SA: MERGA.

- Bogdan, R. C. & Biklen, S. K. (2006). *Qualitative research for education: An introduction to theories and methods*. New York: Pearson.
- Borko, H., & Mayfield, V. (1995). The roles of the cooperating teacher and university supervisor in learning to teach. *Teaching and Teacher Education*, 11, 501–518.
- Commonwealth of Australia. (2007). *Top of the class: Report on the inquiry into teacher education*. Canberra: House of Representatives Standing Committee on Education and Vocational Training.
- Cooney, T., Shealy, B. E., & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics education*, 29, 306–333.
- Cooney, T. J., & Weigel, H. (2003). Examining the mathematics in mathematics teacher education. In A. J. Bishop, M. A. Clements, D. Brunei, C. Keitel, J. Kilpatrick, F. K. S. Leung (Eds.), *The second international handbook of mathematics education* (pp. 795–828). Dordrecht: Kluwer.
- Ebby, C. B. (2000). Learning to teach mathematics differently: The interaction between coursework and fieldwork for preservice teachers. *Journal of Mathematics Teacher Education*, 3, 69–97.
- Frid, S., & Sparrow, L. (2009). “You just have to take a bit of a risk sometimes”: Breaking the ‘Cycle of Tradition’ in primary mathematics. *Mathematics Teacher Education & Development*, 11, 36–53.
- Goos, M. (2008). Sociocultural perspectives on learning to teach mathematics. In B. Jaworski & T. Wood (Eds.), *The mathematics teacher educator as a developing professional* (pp. 75–92). Rotterdam: Sense Publishers.
- Hiebert, J., Morris, A. K., & Glass, B. (2003). Learning to learn to teach: An “experiment” model for teaching and teacher preparation in mathematics. *Journal of Mathematics Teacher Education*, 6, 201–222.
- Le Cornu, R., & Ewing, R. (2008). Reconceptualising professional experiences in pre-service teacher education...reconstructing the past to embrace the future. *Teaching and Teacher Education*, 24, 1799–1812.
- Leiken, R. (2008). Teams of prospective mathematics teachers: Multiple problems and multiple solutions. In T. Wood, & K. Krainer (Eds.), *International handbook of mathematics teacher education: Vol. 3. Participants in mathematics teacher education: Individuals, teams, communities, and networks* (pp. 63–88). Rotterdam, The Netherlands: Sense Publishers.
- Lerman, S. (2001). A review of research perspectives on mathematics teacher education. In F.-L. Lin & T. Cooney (Eds.), *Making sense of mathematics teacher education: Past, present and future* (pp. 33–52). Dordrecht: Kluwer Academic Publishers.
- Mertens, D. M. (2005). *Research and evaluation in education and psychology: Integrating diversity with quantitative, qualitative, and mixed methods*. Thousand Oaks: Sage.
- Mewborn, D. S. (2000). Learning to teach elementary mathematics: Ecological elements of a field experience. *Journal of Mathematics Teacher Education*, 3, 27–46.
- Ponte, J. P., Zaslavsky, O., Silver, E., Carvalho Borba, M., Heuvel-Panhuizen, M., Gal, H., Fiorentini, D., Miskulin, R., Passos, C., Rocque Palis, G., Huang, R., & Chapman, O. (2009). Tools and settings supporting mathematics teachers’ learning in and from practice. In R. Even & D. Ball (Eds.), *The professional education and development of teachers of mathematics*, (pp. 185–210). New York: Springer.
- Scherrf, L., & Singer, N. R. (2012). The preservice teachers are watching: Framing and reframing the field experience. *Teaching and Teacher Education*, 28, 263–272.
- Sim, C. (2006). Preparing for professional experiences—incorporating pre-service teachers as ‘communities of practice’. *Teaching and Teacher Education*, 22, 77–83.
- Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers’ ability to notice. *Journal of Mathematics Teacher Education*, 11, 107–125.
- Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications.
- Zeichner, K. (2010). Rethinking connections between campus courses and field experiences in college and university based teacher education. *Journal of Teacher Education*, 61(1–2), 89–99.

Practical Implications

The results of the study have some important practical implications for the future design of professional experience programs. Although the pre-service teachers benefitted from observing and discussing the problem-solving lessons of the cooperating teacher, they valued most the opportunities they had for co-teaching and peer observation. The lessons of their peers encouraged them to experiment with reform-oriented pedagogies in their normal school placements and it catalysed their reflective thinking.

Co-teaching allowed pre-service teachers to challenge each other in planning and implementing more creative and engaging lessons. In lesson planning, they could critique each other's ideas and keep focusing on improving students' learning outcomes. Co-teaching provided a safe environment for pre-service teachers to adopt reform-oriented teaching strategies and expand their pedagogical horizons. They could take risks together they were not always willing to take on their own and they could practise new skills with a partner who was there to support them. Their experiences of co-teaching enriched their professional practice and gave them the confidence to try out new teaching approaches that they could take into the classroom when they taught on their own.

Observing their peers teach a lesson provided pre-service teachers with a plentiful source of material for discussion about learning and teaching. They were far more willing to deconstruct and critique each other's lessons than those from the cooperating teacher, perhaps because they did not feel comfortable evaluating the practice of a more experienced teacher who was also acting in a supervisory role. Peer observations not only provided valuable feedback for the pre-service teachers who conducted the lesson, they also allowed the observers to hone their ability to notice key aspects of classroom practice.

The other significant outcome of co-teaching and peer observation during the learning community program relates to the ways that these activities encouraged the pre-service teachers to become more reflective practitioners. It can be easy to neglect the processes of lesson evaluation and reflection on learning and teaching during professional experience where the demands of the school environment are keenly felt. However, collaborative teaching and peer observation create a sense of mutual accountability to others so that each person makes a greater effort to share his or her insights during the processes of lesson planning, teaching, and post-lesson analysis. The results of the present study demonstrate that accountability to the professional growth of others was central to the mutual involvement and commitment of pre-service teachers to the progress and success of the learning community.

The author has already begun to create a sense of mutual accountability and commitment by implementing co-teaching and peer observation in the Teacher Education Program for secondary mathematics at his university. Importantly, this is being done without the need for any extra involvement from cooperating teachers or university staff than has typically been the case in the past. As a pilot study, some of the secondary mathematics pre-service teachers are currently completing their entire professional experience in pairs so that they have ongoing opportunities for co-teaching, peer observation, and collaborative reflection on learning and teaching. The combination of classroom and university experiences also continues through a series of School Experience Discussions in which each pair of pre-service teachers report on critical incidents from their practice during the methods workshops. The university supervisor and the other pre-service teachers in the methods workshop support the analysis of these critical incidents.

RESEARCH PAPERS

Exploring the Use of iPads to Engage Young Students with Mathematics

Catherine Attard

University of Western Sydney

<c.attard@uws.edu.au>

Christina Curry

University of Western Sydney

<c.curry@uws.edu.au>

One of the most significant influences on student engagement is the teacher's pedagogical practices, including the incorporation of technology into the teaching and learning of mathematics. This paper reports on a qualitative study investigating how the incorporation of iPads into a Year 3 primary classroom during a six month trial influenced teaching and learning practices and student engagement with mathematics. All of the students appear to have had a positive experience during the trial and the classroom teacher believed their engagement with mathematics had improved as a result. Although there were challenges involved in integrating the iPads into mathematics lessons, some teaching practices were adapted to accommodate the technology. The integration of the iPads highlighted the need for teacher professional development and the importance of developing strong Technological Pedagogical Content Knowledge.

Introduction

It has been well documented that there is widespread concern over lowered levels of student engagement with mathematics during both primary and secondary schooling in Australia (see for example, Commonwealth of Australia, 2008; Sullivan & McDonough, 2007) and internationally (Boaler, 2009; Douglas Willms, Friesen, & Milton, 2009; McGee, Ward, Gibbons, & Harlow, 2003). One of the most significant influences on student engagement is the teacher's pedagogical practices, including the incorporation of technology into the teaching and learning of mathematics (Hayes, Mills, Christie, & Lingard, 2006; NSW Department of Education and Training, 2003). While there are many primary and secondary schools investing in a range of newer, more mobile technologies such as iPads and iPods, teachers are often expected to integrate the technologies into teaching and learning without the support of professional development, particularly in relation to using the technology to enhance teaching, learning and student engagement. To date there is little evidence of whether the implementation and integration of technologies such as iPads serve to improve student engagement with mathematics in the early years of primary schooling or how their integration influences teacher practices.

This paper reports on a study exploring how the introduction and incorporation of iPads into a Year 3 primary classroom affected one teacher's teaching and learning practices and his students' engagement with mathematics. The study was conducted under the umbrella of a larger study, the *Fair Go from the Get Go* project, investigating exemplary early career teachers and their engaging pedagogies. The theoretical framework underpinning this paper is based upon the technological pedagogical and content knowledge (TPACK) framework (Mishra & Koehler, 2006) and current definitions of student engagement. A brief overview of the literature is now provided.

Technology and the Mathematics Classroom

The use of information and communication technologies (ICT) has the potential to change teaching and learning by acting as a source of knowledge, a medium for transmitting content and a resource that fosters dialogue and exploration (Levin & Wadmany, 2008).

Mishra and Koehler (2006) argue that it is not enough to simply introduce technology to the educational process. In order for teachers to make the transition from a traditional mathematics teaching approach to one in which ICT plays an integral role, a commitment to learning how and when to use the technology is required (Goos & Bennison, 2008; Pierce & Ball, 2009). A framework that addresses what teachers need to know to successfully integrate technology into teaching and learning is TPACK (Mishra & Koehler, 2006), which builds on Shulman's pedagogical content knowledge (PCK) framework (1986).

Koehler and Mishra (2009), argue that there are three essential components at the heart of good teaching with technology: content, pedagogy, and technology. "The interactions between and among the three components, playing out differently across diverse contexts, account for the wide variations seen in the extent and quality of educational technology integration" (p. 62). When applied to the teaching and learning of mathematics, the TPACK framework extends beyond ways in which to use a specific tool and its operation to ways the technology can be used to improve teaching and learning (Guerrero, 2010).

Guerrero (2010), built on the original framework (Figure 1) by identifying four components that characterise TPACK in mathematics:

1. Conception and use of technology: this includes ways in which a teacher can conceptualise the use of specific technologies to support teaching and learning mathematics;
2. Technology-based mathematics instruction: this includes the teacher's ability to make changes to pedagogy and recognise the need for flexibility in instruction that results from the use of technology;
3. Management: this includes a range of issues relating to implementation of technology including maintaining student engagement, dealing with the physical environment and hardware issues, and dealing with behaviour management; and
4. Depth and breadth of mathematics content: this component deals with the teacher's knowledge base in terms of the mathematics content and a willingness to allow students to explore mathematical content that may arise during students' investigations using technology.

A developmental model for mathematics TPACK developed by Niess et al. (2009) recognises that teachers require guidance in the development of their ability to effectively use technology to enhance the teaching and learning of mathematics. The model identifies five steps that teachers progress through when integrating a new technology such as the iPad. The steps illustrate the progression from simply recognising the alignment of the specific technology with mathematics but not integrating it into teaching and learning, through to accepting, adopting, exploring and evaluating the decision to integrate the appropriate technology with the teaching and learning of mathematics. The authors of the model claim the emergence of each new technology requires teachers to move through the steps of the model each time, strengthening the argument that new technologies such as the iPad should be supported by appropriate professional development.

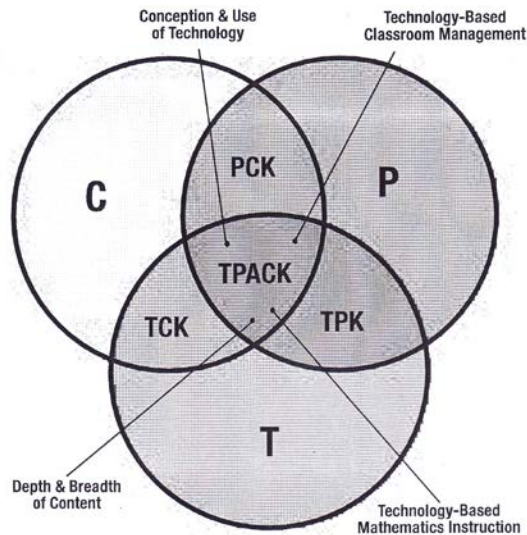


Figure 1. Four central components of mathematics-related TPACK (Guerrero, 2010, p. 134).

One important aspect of the development of TPACK for teaching mathematics is the teacher's ability to maintain positive student engagement once the novelty of the new technology has worn off. Although Guerrero (2010) claims technology has been proven to have positive effects on student engagement, when implemented inappropriately, too often, or too infrequently, it can result in student disengagement. At this point and because a focus of this paper is the effect of the introduction of iPads on the participants' engagement with mathematics, a definition of engagement is introduced.

Engagement

The concept of engagement has been defined in several different ways. Some view engagement at a purely behavioural level (Hickey, 2003), and others define it as "a deeper student relationship with classroom work" (Fair Go Team NSW Department of Education and Training, 2006, p. 9). In this paper, engagement is viewed as a multi-faceted construct which operates at three levels: cognitive, affective and behavioural (Fredricks, Blumenfeld, & Paris, 2004). On a general level, cognitive engagement involves the idea of investment, recognition of the value of learning and a willingness to go beyond the minimum requirements. Affective engagement includes students' reactions to school, teachers, peers and academics, influencing their willingness to become involved in school work. Finally, behavioural engagement encompasses the idea of active participation and involvement in academic and social activities, and is considered crucial for the achievement of positive academic outcomes.

When translated into a mathematics classroom context, engagement occurs when all three facets come together. This occurs when students are procedurally engaged during mathematics lessons and beyond, they enjoy learning and doing mathematics, and they view the learning and doing of mathematics as a valuable, worthwhile task, useful within and beyond the classroom (Attard, 2011). For the purposes of this study and from this point on, the concept of engagement will encompass behavioural, cognitive and affective engagement, leading to students valuing and enjoying school mathematics and seeing connections between school mathematics and their own lives. This view of engagement, combined with the TPACK framework, will be used to address the central questions of this study: How does the introduction of iPads influence teacher practices in the mathematics

classroom? Does the introduction of iPads into mathematics teaching and learning improve student engagement?

Methodology

The participants in this study were of a mixed gender and ability Year 3 class (the fourth year of primary school in NSW) and their teacher (who was given the pseudonym of Mr Milroy) at a government primary school in Sydney. The site was chosen as they were involved in a Department of Education and Communities iPad trial that involved the loan of 30 iPads for a period of six months and because Mr Milroy was already a participant in the larger, Fair Go from the Get Go study. Mr Milroy's willingness to participate was the result of a desire to improve his pedagogical practices and improve his students' low levels of engagement with mathematics. At the time of data collection Mr Milroy was in his second year of teaching and was also the school's technology coordinator.

The school population consisted of 240 (15% indigenous) students of low to mid socio-economic status. Prior to the iPad trial the only ICT accessible to the students was a class set of laptops supported by a specialist ICT teacher. In the two years leading up to the iPad trial the Year 3 group had undertaken a one-hour computer class once a week. The focus of these lessons was to enhance the students' computer skills.

The data collection method consisted of a semi-structured interview with Mr Milroy at the commencement and again at the completion of the trial to gain an understanding of his experiences with the iPads and the impact they had on his classroom practices. Mr Milroy also recorded written reflections detailing his thoughts following the trial. Students were selected by Mr Milroy for a focus group comprising of mixed ability and mixed gender to allow for a representative sample. The focus group met once at the beginning and again at the end of the trial. During the focus groups the students discussed their engagement with mathematics both prior to their use of iPads and following the six month trial. An observation of a mathematics lesson featuring the iPads was carried out in the final weeks of the trial. All interviews and focus groups were transcribed and participants' responses were analysed and coded using open and selective procedures into categories to identify emerging themes which were used for interpreting the data.

While the data collected is not representative of the perceptions and practices of all teachers and students across all schools, the insights gathered from the study are valuable for furthering our understandings of the relevant issues related to the use of iPads for teaching and learning mathematics within a primary school setting.

Results and Discussion

Over the course of the six-month iPad trial, the students and Mr Milroy experienced a range of changes to their mathematics lessons. This resulted in a number of challenges to Mr Milroy's pedagogical practices and a perceived improvement in student engagement. The following is a brief discussion of the results of this study.

Conception

The data and discussion of Mr Milroy's experiences with introducing the iPads will be framed around the TPACK framework and Guerrero's (2010) four components that characterise TPACK in mathematics. The first component, conception, involves how Mr Milroy conceptualised the use of the iPad to support teaching and learning within his classroom. When interviewed at the start of the study, Mr Milroy envisioned using the iPads

as an additional source of information that his students could access when a teacher was not available. He saw the affordance of having access to the Internet and to instructional video tutorials as an opportunity for students to access direct instruction on demand, and not necessarily be limited to the confines of mathematics lesson time. It was Mr Milroy's aim to increase his students' independence through having access to the iPads: "I am hoping that technology will get that idea across to them that hey, you can teach yourselves and guess what, you can use that to teach each other."

Mr Milroy also identified another affordance in the form of mathematics applications (apps) that students could use to practice mathematical concepts and skills. However, early in the study he found that some students did not have the skills or understandings required to successfully engage with the specific apps he had selected. This was evidenced during a classroom observation where one student appeared to be guessing the answers to a mathematics game, claiming "this is the wrong answer game". This scenario is evidence that the teacher's knowledge of the individual needs of students, the appropriate pedagogy and the appropriateness of using technology is an important aspect of the development of TPACK and is an important consideration when planning teaching and learning activities.

As the study progressed, other affordances of the iPad were identified by Mr Milroy as being of benefit to mathematics teaching and learning such as potential to provide interactivity and instant feedback via the use of mathematics game apps. These affordances will be discussed further throughout the paper.

Technology-based Mathematics Instruction

The second component, technology-based mathematics instruction, involved Mr Milroy making significant changes to his pedagogical practices which initially relied heavily on the use of worksheets, mathematics investigations and a commercial, computer-based program that provided drill and practice of computation skills. Mr Milroy admitted it was a challenge to plan mathematics lessons in terms of sequencing content in an engaging manner, indicating his level of pedagogical content knowledge was still developing, possibly as a result of his status as a beginning teacher. This challenge may have had some impact on the integration of the iPads into his mathematics lessons.

During the course of the study Mr Milroy experimented with different ways of using the iPads within his mathematics lessons and in other subject areas. Initially, the iPads were used for group activities based on apps that were fundamentally game based. Mr Milroy then revised his use of the iPads and extended the ways in which he used them, including the incorporation of online tutorials accessed via the Internet. Once students had participated in an online tutorial covering a specific content area or skill, they then moved on to using apps that required them to practice the skills and content, often in the form of a game. Using the iPads in this way also resulted in lessons that were based on a rotation of group activities (not necessarily cooperative groups) rather than whole class lessons.

Management

Although the iPads were used regularly and in a variety of ways, their integration was not without problems. The third component, management, saw restrictions in the ways the iPads could be used and their initial set up and maintenance was found to be a cumbersome burden that was perceived by Mr Milroy to impact on the effectiveness of their implementation. The fact that the iPads had to be shared with students from another class group also meant that they could not become an integral part of teaching and learning because they were not always available for use in Mr Milroy's classroom. Mr Milroy

perceived a different aspect of the management component, student engagement, to have been improved as the result of introducing the iPads, stating:

...in comparing and contrasting how I taught initially without and then with the technology of iPads, wifi, IWB and USB document camera, I can unequivocally claim I successfully engaged the kids in more content creation, criteria based projects in a reflective and learning-cultured class environment. The students mostly used higher-order thinking skills, reflection and analysis throughout the entire day aided heavily by the above technology.

An affordance of the iPads mentioned above was the immediate feedback students were able to receive through the incorporation of mathematics games. Rather than wait for the teacher to correct answers as would normally occur when worksheets are used, the games provided an immediate response. The benefit of instant feedback is reflected in this student's comment:

Well it makes me feel happy because if you touch it and you make a mistake it just like takes it away, straight away, not like if it is on paper and then it is an exam with no rubbers, if it's on the iPad you can just go oh, that's wrong and you can take it away.

The affordance of instant feedback highlights the iPad's potential for building students' confidence in terms of risk taking and feeling safe to make mistakes and try again, building persistence. This in turn promotes positive affective, behavioural and cognitive engagement.

Students made several comments specifically comparing the iPads to the use of worksheets, with this comment typical of the group's sentiments: "I think learning with the iPad is better because some children don't concentrate when it's on paper but when it's on an iPad because it's electronic and modern they can concentrate more." However, students could also see a slight disadvantage with using the iPads in that they were aware that they could be distracting, making it difficult to focus and stay on task. This issue highlights the importance of teacher supervision and management skills as well as the importance of being aware of individual students' capabilities, an important component of effective TPACK.

For most of the students the games appeared to improve affective engagement and for many, cognitive engagement because of the level of challenge within the games. Many students claimed lessons were more 'fun'. This could be attributed to the mathematics games the students were playing and the level of interactivity required, increasing behavioural engagement. It is not possible to conclude if it was purely the game aspect that made the lessons fun (behavioural and affective engagement) or whether the students were cognitively engaged and actually learned mathematics as a result of playing the games. As stated earlier, during a classroom observation, one student was observed to be disengaged from the mathematics whilst still playing a game and interacting with the iPad. Clearly although it appeared all students were behaviourally and affectively engaged, not all were engaged on a cognitive level possibly due to a mismatch between their ability and the given task. Although an affordance of using iPads is the potential to address diversity by differentiating learning through a range of activities, it appears that during the trial their use during mathematics lessons was limited to the use of a small number of mathematics based apps and the on-line mathematics tutorials.

Depth and Breadth of Mathematics Content

The final component regarding Mr Milroy's depth of understanding of mathematics content and the potential for students to explore mathematics appeared to be the most challenging aspect of introducing the iPads into mathematics lessons. A benefit of using the iPads was the increased opportunity to differentiate learning by providing a broader range of activities for students to engage with when compared to the limited range of activities used

prior to the introduction of the iPads. Before the trial, the students perceived their mathematics activities to be based on worksheets. The iPads provided students with opportunities to work in groups on a range of activities within each lesson and to do so in a more flexible learning space, allowing students to move around the room from one activity to another. Arguably the increased variety within their mathematics lessons would have contributed to the increased engagement of the students.

During Mr Milroy's final interview at the completion of the trial, he indicated that in future he would plan mathematics activities using the iPads that were based more on mathematical investigations and problem solving. This was the result of his 'trial and error' approach and his self-evaluation of the effectiveness of the iPad trial. Throughout the six months it is evident Mr Milroy progressed through the developmental model described by Niess et al. (2009), and this process may be repeated until a stronger level of TPACK is achieved.

Implications and Conclusion

In this study, the introduction and integration of iPads into mathematics teaching and learning appears to have had a positive impact on the teaching and learning of mathematics for the participants involved. The iPads allowed the teacher to introduce a wider range of teaching strategies that included group work and a rotation of tasks within each lesson as opposed to whole-class, worksheet based lessons. The use of the iPads appears to have increased student engagement by providing a resource that promoted interactivity, immediate feedback, challenge and fun.

All of the participants in the Year 3 class appear to have had a positive experience during the trial. The classroom teacher, Mr Milroy, believed their engagement with mathematics had improved, with students displaying increased enthusiasm and higher levels of participation during mathematics lessons. The iPad integration provided an opportunity for Mr Milroy to reflect upon and adjust his pedagogies and although he was already a confident user of the technology, he felt compelled to spend a significant amount of time researching apps for use within the classroom, therefore taking some responsibility for his own professional development. The findings in this study highlight potential difficulties when implementing new technologies, particularly for beginning teachers who are still developing their pedagogical content knowledge. Further support in the form of opportunities to develop his TPACK through formal professional development may have been beneficial and could have avoided the 'trial and error' approach.

The fast-paced development of mobile technologies such as the iPad and their potential for use in the classroom requires careful professional development that highlights the importance of developing strong TPACK. Pre-service teacher education and the development of TPACK is also an issue that may benefit from further research.

Although this study has several limitations in that it was conducted with only one group of students in one school, it provided the opportunity to investigate a common occurrence in our schools today. That is, the implementation of new technologies without appropriate support through professional development opportunities that explore and develop teachers' technological pedagogical content knowledge. The findings suggest that further, much deeper investigation into the use of iPads and other mobile technologies to teach and learn mathematics is warranted and timely. Research conducted over a longer period of time with a more representative sample would be of benefit and would highlight further the benefits and disadvantages of using mobile technologies.

References

- Attard, C. (2011). "My favourite subject is maths. For some reason no-one really agrees with me": Student perspectives of mathematics teaching and learning in the upper primary classroom. *Mathematics Education Research Journal*, 23(3), 363-377.
- Boaler, J. (2009). *The elephant in the classroom: Helping children learn and love maths*. London: Souvenir Press Ltd.
- Commonwealth of Australia. (2008). *National numeracy review report*. Canberra, ACT: Human Capital Working Group, Council of Australian Governments.
- Douglas Willms, J., Friesen, S., & Milton, P. (2009). *What did you do in school today?* Toronto, ON: Canadian Education Association.
- Fair Go Team NSW Department of Education and Training. (2006). *School is for me: Pathways to student engagement*. Sydney: NSW Department of Education and Training, Sydney, Australia.
- Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research*, 74(1), 59 -110.
- Goos, M., & Bennison, A. (2008). Surveying the technology landscape: Teachers' use of technology in secondary mathematics classrooms. *Mathematics Education Research Journal*, 20(3), 102-130.
- Guerrero, S. (2010). Technological pedagogical content knowledge in the mathematics classroom. *Journal of Digital Learning in Teacher Education*, 26(4), 132-139.
- Hayes, D., Mills, M., Christie, P., & Lingard, B. (2006). *Teachers and schooling making a difference*. Sydney: Allan & Unwin.
- Hickey, D. T. (2003). Engaged participation versus marginal nonparticipation: A stridently sociocultural approach to achievement motivation. *The Elementary School Journal*, 103(4), 401-429.
- Koehler, M. J., & Mishra, P. (2009). What is technological pedagogical content knowledge? *Contemporary Issues in Technology and Teacher Education*, 9(1), 60-70.
- Levin, T., & Wadmany, R. (2008). Teachers' views on factors affecting effective integration of information technology in the classroom Developmental scenery. *Journal of Technology and Teacher Education*, 16(2), 233-263.
- McGee, C., Ward, R., Gibbons, J., & Harlow, A. (2003). *Transition to secondary school: A literature review*. Ministry of Education, New Zealand.
- Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record*, 108(6), 1017-1054.
- Niess, M. L., Ronau, R. N., Shafer, K. G., Driskell, S. O., Harper, S. R., Johnston, C., . . . Kersaint, G. (2009). Mathematics teacher TPACK standards and development model. *Contemporary Issues in Technology and Teacher Education*, 9(1), 4-24.
- NSW Department of Education and Training. (2003). *Quality Teaching in NSW Public Schools*. Sydney: Professional Support and Curriculum Directorate.
- Pierce, R., & Ball, L. (2009). Perceptions that may affect teachers' intention to use technology in secondary mathematics classes. *Educational Studies in Mathematics*, 71(3), 299-317.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *American Educational Research Journal*, 15(2), 4-14.
- Sullivan, P., & McDonough, A. (2007, July). *Eliciting positive student motivation for learning mathematics*. In Watson, J. & Beswick, K., *Mathematics: Essential Research, Essential Practice*. (Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia, Hobart, Tasmania). Adelaide: MERGA.

Teaching Algebra Conceptually: The Process of Bringing Research to Practitioners

Megan Anakin
University of Otago
<megan.anakin@otago.ac.nz>

Ayelet Lazarovitch
University of Otago
<ayelet.cohen@otago.ac.nz>

Using a case study, the complex challenge of making mathematics education research accessible to secondary mathematics teachers was addressed with two questions. How can we design a method that will meet the challenge of making research usable for mathematics teachers? And what would this method be? To address this challenge we describe a process and product that emerged from using the stages of design-thinking. We invite fellow researchers to join us in the collective mission of bridging the gap between mathematics research and practice, furthermore, we seek to stimulate conversations about what counts as a research output.

Introduction

Linking research and practice is fraught with challenges and has become a high profile topic in the education research community. For example, the theme of the American Educational Research Association (AERA)'s annual meeting is, "*Non Satis Scire To Know is Not Enough*" (2012). In essays commissioned prior to AERA's 2012 annual meeting, researchers highlight the gap between research and practice and suggest that we need to think about this problem in different ways. Carol Lee suggests that "there is significant evidence that the relationship of research to practice is mine-filled, inconsistent, and incomplete" (2012, p. 4) and we must expand our efforts "to understand complex cultural systems embedded in multi-tiered ecologies" (p. 7). In another essay, Cynthia Hudley and Barbara Wells express a need for "better translation of research to practice to research in an ongoing loop structured by the knowledge and experience of researchers and practitioners" (2012, p. 5).

In New Zealand's mathematics education community, there is considerable effort being made to link research and practice for mathematics teachers. Efforts include the publication of documents, internet accessible websites, and collaborative projects to promote research-informed practice (e.g., Education Counts, 2012; NZMaths, 2012; Teaching and Learning Research Initiative [TLRI], 2012a), however, there remains a gap between communicating these findings to teachers and teachers changing their practice to incorporate these effective teaching strategies. (Nuthall, 2004). In this paper we describe a process for bridging this gap and the results of our efforts. First we describe the research that we wanted to make accessible for mathematics teachers. Then we outline how we designed a method to make this research accessible. Lastly, we describe the significance and implications of the method we designed.

The Research: Teaching Algebra Conceptually (TAC)

In 2010 and 2011, a research team consisting of five mathematics teachers from two Dunedin secondary schools and three researchers from the University of Otago studied the impact of teaching approaches used to help students acquire the algebraic knowledge and strategies. Preliminary findings have been reported by Linsell et al. (2011) and a brief interview with a teacher researcher, Anna Cox about her involvement in the project can be

found on the TLRI (2012b) website. The project produced a refined diagnostic assessment tool and a collection of teaching approaches that mathematics teachers in secondary schools can use to support their students' learning of algebra. Another project goal was to bring the research finding to other secondary mathematics teachers across New Zealand and this goal provides the focus of this paper.

The TAC research was an investigation of mathematics teaching practice; it was intended to feed back into practice and to inform the teaching of algebra. Considering previous attempts to bridge the gap between research and practice and the notion that "to know is not enough" (AERA, 2012), the team faced a complex challenge; how to develop an effective way to communicate our research findings to practitioners and improve teachers' practice in light of these findings.

Designing a Solution to the Challenge

Part 1: Considering Effective Communication with Practitioners

To find a solution to the first part of the challenge, "effectively communicate our research findings to practitioners", the teachers involved in the project were consulted. They reported finding the more theoretically-oriented research articles (i.e., Sfard, 1991) challenging to read because of the dense academic language and lack of explicit links to their practice, whereas they expressed a preference for the more practitioner-oriented papers (i.e., Barnes & Hamon, 2010) because of the more accessible language and vivid classroom examples described in those texts. The teachers also revealed that they had very little time to read any research literature at all because their days were filled with teaching duties. The team realised that any resource we created would have to be 'practice-oriented' and integrate seamlessly into teachers daily activities. The team suggested an on-line resource and the teachers on the team emphasised that it should be practical in its nature and linked to websites they visit to inform their planning and assessment practices (i.e. TKI, 2012b; NZMaths, 2012). The decision to create an on-line resource led us to consider design-thinking as process to create a useful product for bridging the gap between research and practice.

Part 2: Applying the Design-Thinking Process

Fostering teachers to change their practice to incorporate research-informed teaching strategies is a complex challenge (Hudley & Wells, 2012; Lee, 2012; Nuthall, 2004), and the research team wanted to use a process that promoted expansive learning (Roth & Lee, 2007). The team had access to staff at the University of Otago with expertise in design-thinking; therefore, a partnership was formed between the two authors of this paper to bring knowledge of the research project together with the design-thinking process. The design-thinking process is a problem-solving protocol used by designers to solve complex challenges (Dorst, 2011). These challenges often do not have one right answer, but rather a collection of better and worse possibilities (Owen, 2007). In order to find better solutions, design-thinking uses a process of divergent and convergent thinking (Ambrose & Harris, 2010) and this structured approach often leads to both creative and functional solutions. The design-thinking process used in the TAC project is summarised in Figure 1.

Identify the challenge. The first stage of the design-thinking process required us to identify the overall vision and ultimate goal for the on-line resource. Through different exercises, we realized that the ultimate goal of the resource stretched beyond changing the

practice of teachers and extended to improving the mathematical understandings and achievement of students. By explicitly stating that our goal was student understanding and achievement, it became possible to see that our framework should not only include the “what” or content, and the “how” to teach algebra conceptually, but also “why” this endeavour is important. We decided to emphasise the value that the research findings can add to a teachers’ practice.

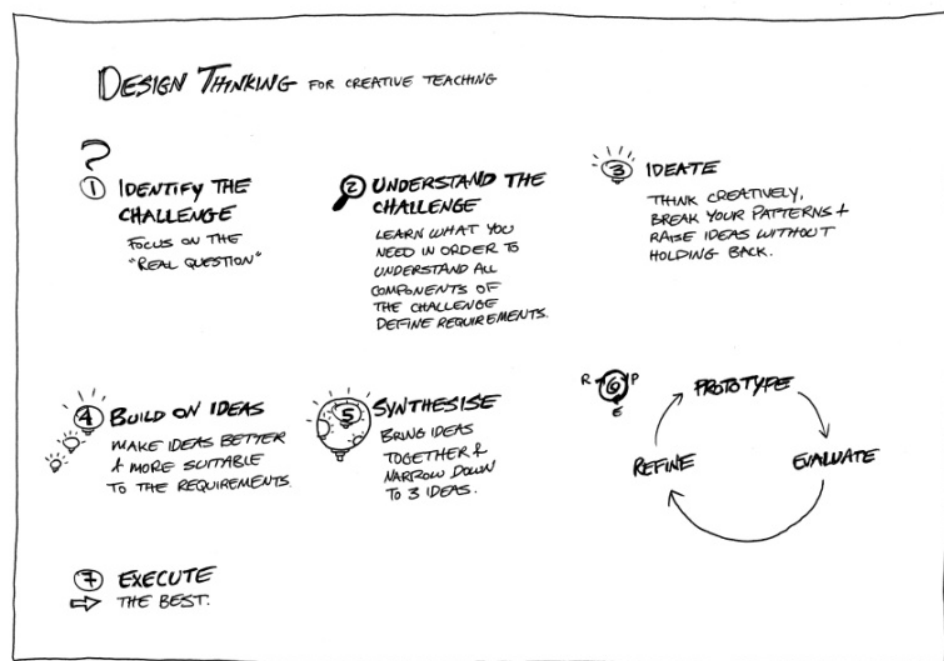


Figure 1. An early draft of the design-thinking process for teachers (Cohen, 2011).

Understand the challenge. The second stage of the process involved understanding the different aspects of the challenge and defining the parameters of the on-line resource. Among the different aspects considered were the audience for the on-line resource, the context in which the resource would be used, and what the related literature suggested as viable approaches. To generate the parameters of the resource, a thinking exercise was used to generate lists of what must/should/could be included in the solution. The “musts” are the absolute requirements, without them the solution would not work. The “shoulds” are the parameters that would make the solution better; the solution would work without them. The “coulds” are possibilities that come to mind as the “musts” and the “coulds” are generated. “Coulds” are considered optional extras; again the solution would work without them. An early version of the parameters produced in the second phase of design thinking is shown in Figure 2.

Looking at the literature, we wondered if theories about research dissemination could be used to inform the development of the on-line resource. *Research dissemination* refers to the process by which findings from research studies are moved from their site of origin, such as a research project, to be put into action by practitioners, in this case, secondary mathematics teachers who then incorporate novel ideas or strategies into their practice with the goal of enhancing the achievement of their students (Hutchinson & Huberman, 1994). A variety of knowledge-for-action models of research dissemination were consulted and we considered

how our choices would foreground different aspects involved in changing a teachers' practice of teaching mathematics (Ottoson, 2009). Research dissemination is a difficult process to theorise when we consider who might use the resource and for what purpose, how might the resource be interpreted by different users, and that information can flow in two directions (Nutley, Jung, & Walter, 2008).

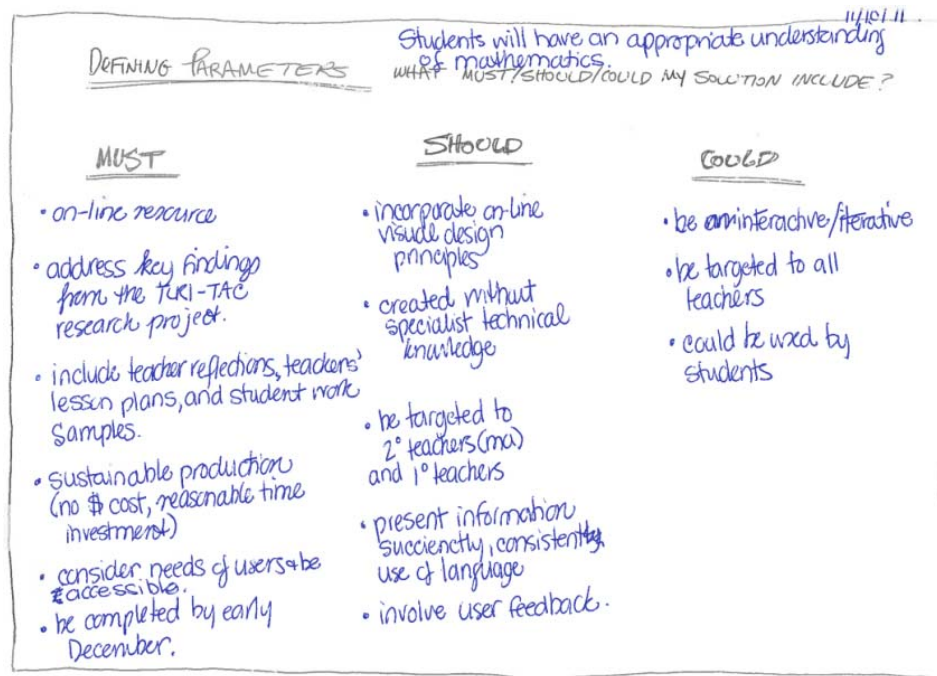


Figure 2. An early version of the parameters of the TAC output.

Ideate and Build on Ideas. After deepening our understanding of the research dissemination literature and developing the parameters for the resource, we moved to the third and fourth steps of the process: creating ideas for solutions and building on them in order to develop better solutions. Rather than choosing a possible solution and building upon it, we used creative thinking techniques and cycles of divergent exercises for idea generation and convergent exercises for idea refinement. An example of several ideas produced by this process is shown in Figure 3.

By this stage of the design thinking process, the initial picture we had in our minds for the resource faded away. Instead of a collection of static pages, the idea of a two-way interaction between all stakeholders began to take a central role in our possible solutions. The notion of a two-way interaction was reinforced by learning theories developed to support research-informed practice (Nutley, et al., 2008; Roth & Lee, 2007). The parameters for a solution were defined in a new light. Firstly, the resource had to be inviting and accessible to teachers who teach algebra in diverse settings to students in New Zealand. Secondly, the resource had to provide teachers with practical examples of how to teach algebra conceptually and engage them with the research findings at practical and intellectual levels. Thirdly, the resource had to allow for interaction and collaboration so that teachers could offer feedback from their experiences and suggest variations or new examples. The platform that seemed to best meet these parameters was a blog because it combines “static” and collaborative content with interactions mediated by commenting and discussion functions (Wordpress.com, 2012).

Synthesise. With the idea of creating a blog, we were able to move to the synthesis stage of the design-thinking process. Engaging with a number of synthesis exercises allowed us to generate a structure for the blog. The result of one such activity is shown in Figure 4.

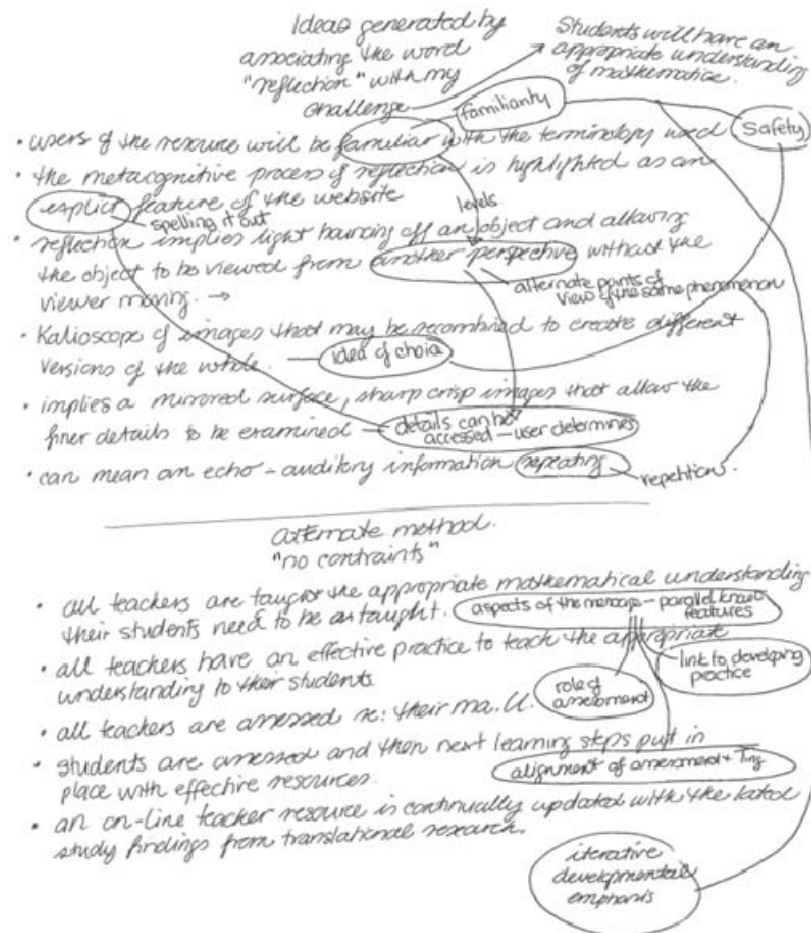


Figure 3. Ideas generated by two divergent thinking tools (shown as bullet points) and a subsequent convergent thinking tool (shown as interlinked circled words and phrases).

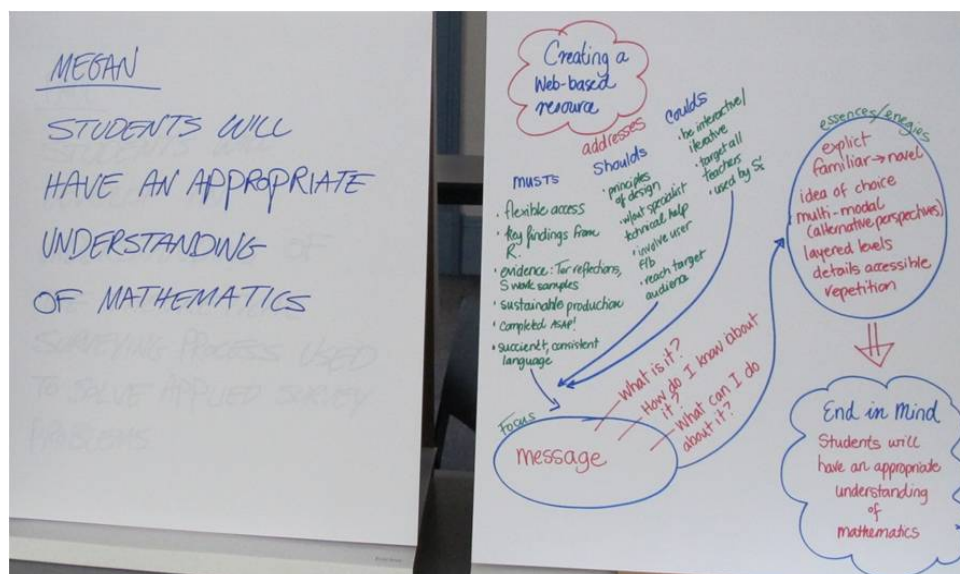


Figure 4. The structure of the blog emerging from the result of a synthesis activity.

Prototype, Evaluate, Refine. The interactive website is now in the prototype and pilot stage of construction. The structure of the interactive website is stable and the final details are being added before “going live” in June 2012. A single question phrased from a teachers’ perspective greets users on the blog’s homepage and asks, “How can we create engaging classroom lessons that help our students learn algebra?” (Linsell et al., 2012, Home: Teaching Algebra Conceptually). The phrasing was deliberate so the blog would be targeted at secondary mathematics teachers rather than researchers. The blog also contains images of research teachers and their students showing them working on classroom activities to emphasise the key findings of the TAC project in practical ways. The site itself is divided into different sections which are framed around four elements: “presenting challenges our students face as they learn algebra, showing how we’ve identified those challenges, offering some practical ways to help students learn algebra, and encouraging teachers to share challenges and suggest practical ways to help students learn algebra” (Linsell et al., 2012, Home: Teaching Algebra Conceptually).

The first section is structured to hook teachers’ interest by identifying the key algebraic concepts students struggle to learn. The challenges students face when they learn algebra were identified by teachers involved in the TAC project and written in the words they said to or heard from their students. The second section is structured to engage teachers intellectually with the findings. It introduces the algebra framework, stresses the importance of diagnostic assessment, and offers classroom-tested assessment tools and techniques as downloadable materials. The third section is structured around practice; teachers participating in the TAC project provided practical activities to engage teachers with the findings. The practical activities include images, video clips, links, and downloadable materials for use with students in secondary mathematics classrooms. The fourth section emphasises the interactive nature of the blog and is designed to invite discussion, suggestions, critique and further contributions by those who use it. Comment functions have been embedded at the bottom of each webpage. The home page suggests a pathway so that first time users can experience the complete structure of the blog (see Figure 5).

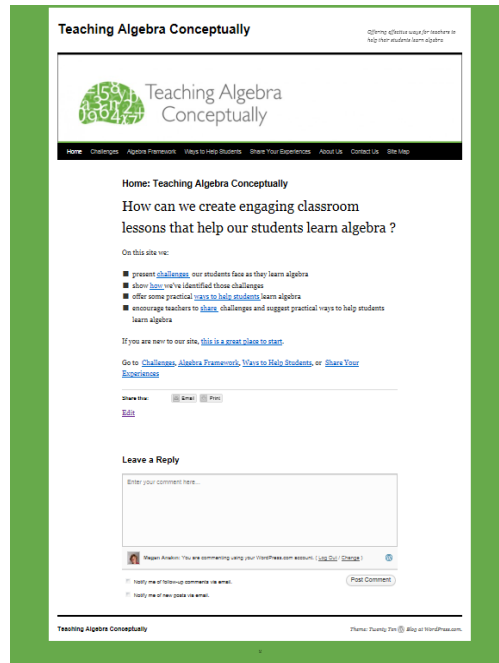


Figure 5. Teaching Algebra Conceptually home page (Linsell et al., 2012)

Significance and Implications

In this paper, we used the TAC project as a case study to demonstrate how the complex challenge of making mathematics education research accessible to secondary mathematics teachers was addressed. We described a process and product that emerged from using the stages of design-thinking specifically targeted at fostering secondary mathematics teachers in New Zealand to use recent research about teaching algebra conceptually. The design thinking process helped us understand the challenge we faced, be clear about the parameters for a solution, and finally, to produce a set of possible solutions, from which we could choose the most suitable one. The method has resulted in an on-line resource in the form of a collaborative and interactive blog. The blog communicates the research findings, contains examples suggesting how to link the findings to practice, and provides opportunities for teachers to add comments and share their own relevant teaching experiences.

We feel that the design of a practitioner-oriented research-informed blog is an important contribution to the mathematics education community. We hope that the interactive options that are built into the blog, such as the commenting system and discussion board, will be used by teachers to continually demonstrate the link between the research findings and current practice. By choosing a blog as a dissemination medium we seek to complement traditional dissemination avenues available to us, which tend to be removed from the lived richness of the research project and experiences of mathematics teachers in secondary schools across New Zealand.

A significant question raised during the production of the TAC blog is the value that the research community may assign to this form of output, since a blog is typically considered a social rather than a scholarly medium. We seek to stimulate conversations about what counts as a research output as we explore alternate modes of interaction and collaboration between researchers, practitioners and learners. We hope that the process we described in this paper will encourage our fellow mathematics education researchers to join us in searching for creative ways to bridge the gap between research and practitioners.

Acknowledgement

This work was supported by a Teaching and Learning Research Initiative TLRI grant from the New Zealand Council for Educational Research.

References

- Ambrose, G., & Harris, P. (2010). *Design Thinking*. Lausanne, Switzerland.
- American Educational Research Association. (2012, March 23). *Annual meeting 2012 details*. Retrieved from <http://www.aera.net>
- Barnes, R., & Hamon, S. (2010). Proof & prealgebra. *Mathematics Teacher*, 103(8), 597-602.
- Cohen, A. (2011). Increasing Creativity in Teaching. In K. Shephard, T. Harland, & A. Cohen (Eds.), *Proceedings of the Spotlight on Teaching and Learning Colloquium*, (p. 16). Dunedin, New Zealand: HEDC, University of Otago.
- Dorst, K. (2011). The core of 'design thinking' and its application. *Design Studies*, doi:10.1016/j.destud.2011.07.006
- Education Counts. (2012). *BES (iterative best evidence synthesis) programme*. Retrieved from <http://www.educationcounts.govt.nz>
- Hudley, C., & Wells, B. (2012). *The case for translational research in education*. Retrieved from http://www.aera.net/Portals/38/docs/Annual_Meeting/CynthiaHudley_AERA%20essay%20June%2011.pdf
- Hutchinson, J. R., & Huberman, M. (1994). Knowledge Dissemination and Use in Science and Mathematics Education: A Literature Review. *Journal of Science Education and Technology*, 3(1), 27-47.
- Lee, C. (2012). *Implications of cultural, complex ecological systems for what we think we know and how such knowledge can enhance teaching and learning*. Retrieved from http://www.aera.net/Portals/38/docs/Annual_Meeting/Lee_Essay_rmc.pdf
- Linsell, C., Tozer, L., Anakin, M., Cox, A., Jones, R., McAuslan, E., Smith, D., Turner, G. (2011). *Teaching algebra conceptually: Preliminary findings*. Paper presented at the New Zealand Association for Research in Education.
- Linsell, C., Tozer, L., Anakin, M., Cox, A., Jones, R., McAuslan, E., Smith, D., Turner, G. (2012, March 23). *Teaching algebra conceptually*. Retrieved from <http://teachingalgebraconceptually.wordpress.com>
- Nutley, S., Jung, T., & Walter, I. (2008). The many forms of research-informed practice: a framework for mapping diversity. *Cambridge Journal of Education*, 38(1), 53-71.
- Nuthall, G. (2004). Relating classroom learning to student learning: A critical analysis of why research has failed to bridge the theory-practice gap. *Harvard Educational Review*, 74(3), 273-306.
- NZMaths (2012). *Annual research and evaluation reports and compendium papers*. Retrieved from <http://www.nzmaths.co.nz>
- Ottoson, J. M. (2009). Knowledge-for-action theories in evaluation: Knowledge utilization, diffusion, implementation, transfer, and translation. *New Directions for Evaluation*, 124, 7-20.
- Owen, C. (2007). Design Thinking: Notes on its Nature and Use. *Design Research Quarterly*, 2(1), 16-27.
- Roth, W.-M., & Lee, Y.-J. (2007). "Vygotsky's neglected legacy": Cultural-historical activity theory. *Review of Educational Research*, 77(2), 186-232.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as two sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.
- Teaching and Learning Research Initiative. (2012a). *Home*. Retrieved from <http://www.tlri.org.nz/>
- Teaching and Learning Research Initiative. (2012b). *TLRI research involvement gets teachers thinking*. Retrieved from <http://www.tlri.org.nz/about-tlri/news-archive/tlri-research-involvement-gets-teachers-thinking>
- Te Kete Ipurangi. (2012b). *The New Zealand Curriculum on-line*. Retrieved from <http://nzcurriculum.tki.org.nz>
- Wordpress.com. (2012). *Get a Free Blog Here*. Retrieved from <http://wordpress.com/>

Influences of Self-Perceived Competence in Mathematics and Positive Affect toward Mathematics on Mathematics Achievement of Adolescents in Singapore

Shaljan Areepattamannil
National Institute of Education, Singapore
<shaljan.a@nie.edu.sg>

Berinderjeet Kaur
National Institute of Education, Singapore
<berinderjeet.kaur@nie.edu.sg>

This study, drawing on data from the Trends in International Mathematics and Science Study (TIMSS) 2007, examined the influences of self-perceived competence in mathematics and positive affect toward mathematics on mathematics achievement of adolescents in Singapore. Ordinary least squares (OLS) regression analyses revealed the positive influences of self-perceived competence in mathematics and positive affect toward mathematics on mathematics achievement of adolescents in Singapore. Implications of the findings for policy and practice are discussed.

Numerous studies have examined the influences of domain specific academic self-concepts (e.g., mathematics self-concept and verbal self-concept) on their corresponding measures of achievement. However, academic self-concept is multidimensional in nature, and domain specificity is one of the facets of multidimensional academic self-concept (see Arens, Yeung, Craven, & Hasselhorn, 2011; Marsh, Craven, & Debus, 1999). Marsh (1992) posited that students' self-concept may comprise of affective as well as competency dimensions. Later, Marsh et al. (1999) found empirical support for the multidimensional facets of academic self-concept at the within-domain level. Specifically, the authors found empirical evidence from two separate studies for the separation of competence and affect components of mathematics, reading, and school self-concepts among school children in Australia.

Recently, Arens et al. (2011) found empirical support for the two-fold multidimensionality of academic self-concept (mathematics, verbal, and general school self-concepts)—domain specificity and separation between competence and affect components—among school children in Germany. Theories pertaining to motivation, self-determination theory (see Deci & Ryan, 2000), and expectancy-value theory (see Eccles & Wigfield, 1995) as well postulate that students' self-perceived academic competence and affect toward learning be treated as two separate constructs. Thus, the multidimensionality of academic self-concept also includes two separate dimensions of academic self-concept: a self-evaluative/cognitive dimension—competence—and an affective/motivational dimension—affect (within-domain level; Arens et al., 2011; Marsh et al., 1999).

Although a substantial body of research has examined the relationships between domain-specific academic self-concepts and their corresponding measures of achievement (see Marsh & Martin, 2011, for a review), only one study to our knowledge has specifically explored the relationships between competence and affect components of academic self-concept and achievement using Marsh and colleagues' theoretical model of academic self-concept. Arens et al. (2011) explored the latent factor correlations between competence and affect components of academic self-concept and achievement among school children in Germany. Mathematics and verbal achievement were more strongly correlated with students' self-perceived competence in mathematics and German than with their affect toward mathematics and German.

Shen and Pedulla (2000), drawing on data from the Third International Mathematics and Science Study and employing Bandura's (1997) self-efficacy theory, explored the relationships between self-perceived competence in mathematics and science and student achievement. Within country data suggested a positive relationship between self-perceived competence in mathematics and science and student achievement. Nonetheless, the authors used a single item ("I usually do well in mathematics and science") to measure students' self-perceived competence in mathematics and science. Neither Arens et al. (2011) nor Shen and Pedulla (2000) have examined the predictive effects of competence and affect components of academic self-concept on student achievement.

However, studies that employed other theoretical frameworks, such as self-determination theory (see Deci & Ryan, 2000) and achievement goal theory (see Maehr & Zusho, 2009), have demonstrated the positive predictive effects of self-perceived academic competence on achievement outcomes (e.g. Cho, Weinstein, & Wicker, 2011; Ferla, Valcke, & Schuyten, 2010). Similarly, studies have also documented the positive effects of students' affective self-perceptions on their learning and achievement (e.g. Lau & Roeser, 2008; Nieswandt, 2007).

Nevertheless, no study to date has examined specifically the predictive effects of competence and affect components of mathematics self-concept on mathematics achievement. A better and deeper understanding of the effects of competence and affect components of mathematics self-concept on mathematics achievement may help educators to develop appropriate educational interventions with a view to enhancing students' achievement in mathematics. As the Organization for Economic Cooperation and Development (OECD, 2007) notes:

The performance of a country's best students in mathematics and related subjects may have implications for the role that the country will play in tomorrow's advanced technology sector, and for its overall international competitiveness. Conversely, deficiencies among lower-performing students in mathematics can have negative consequences for individuals' labour market and earnings prospects and for their capacity to participate fully in society. (p. 323)

Given the importance of mathematical competence and the dearth of research on the impact of competence and affect components of academic self-concept on academic achievement, the purpose of the present study was to explore the influences of self-perceived competence in mathematics and positive affect toward mathematics on mathematics achievement of adolescents in Singapore. Specifically, the study addressed the research question: How well do self-perceived competence in mathematics and positive affect toward mathematics predict mathematics achievement among adolescents in Singapore?

Method

Data

Data for the study were drawn from the Trends in International Mathematics and Science Study (TIMSS) 2007 dataset. TIMSS 2007, sponsored by the International Association for the Evaluation of Educational Achievement (IEA), is the fourth in a series of comparative international studies assessing the mathematics and science performance of both fourth and eighth graders. In Singapore, 4599 eighth graders (male = 2353, female = 2246) from 164 schools took part in TIMSS 2007. The mean age of participants was 14.39 years.

Measures

Outcome variable. Mathematics achievement score (five plausible values) at eighth grade level on the TIMSS 2007 assessment was the dependent variable in the study. Since students were administered too few items in each specific content area, the TIMSS 2007 developed plausible values as a computational approximation to obtain consistent and precise estimates of students' ability (IEA, 2009a). "Plausible values are multiple imputations of the unobservable latent achievement for each student" (Wu, 2005, p. 114). The mathematics achievement score is reported on a scale from 0 to 1,000, with the TIMSS scale average set at 500 and standard deviation set at 100 (Olson, Martin, & Mullis, 2008). The IEA IDB analyzer (Version 2.0; IEA, 2009b)—a plug-in for the Statistical Package for the Social Sciences (SPSS) that allows the user to combine and analyze data from TIMSS 2007—along with IBM SPSS Statistics 19 was used to run models for each of the five plausible values, producing their average values and correct standard errors.

Predictor variables. The TIMSS 2007 had 8 items pertaining to students' affect toward mathematics and their self-perceived competence in mathematics. The current study used these 8 items to measure adolescents' multiple dimensions of mathematics self-concept. All items were rated on a four-point Likert scale ranging from 1 (*disagree a lot*) to 4 (*agree a lot*). Negatively worded items were reverse scored. Maximum-likelihood extraction with oblique rotation was performed on all the 8 items. Two factors were extracted from the 8 items, and these two factors accounted for 73% of the variance. Loadings of all variables on factors were above the cutoff of 0.45 (20% of variance). The factor solutions met multiple additional criteria such that each factor contained a minimum of three items, exhibited sufficient internal consistency, and was interpretable. Cronbach's α was used to measure the internal consistency of the scale items. The two factors were *positive affect toward mathematics* (four items; $\alpha = 0.90$) and *self-perceived competence in mathematics* (four items; $\alpha = 0.82$). The student demographic variables were gender (1 = female; 0 = male), language spoken at home (speaking the language of the assessment at home; 1 = *never* to 4 = *always*), and parental education (1 = *less than lower secondary education* to 5 = *university education*).

Results

The descriptive statistics and the correlation between outcome variable and predictor variables are presented in Table 1. All predictor variables were positively correlated with mathematics achievement of adolescents in Singapore. To address the purpose of the study, ordinary least squares (OLS) regression analyses were conducted (see Table 2). Mathematics achievement was the dependent variable. Gender, language spoken at home, parental education, self-perceived competence in mathematics, and positive affect toward mathematics were the independent variables. Given the two-stage stratified cluster sample design of the TIMSS 2007 assessment, replicate weights—based on the jackknife repeated replication technique—were used to generate unbiased parameter estimates and their correct standard errors.

Table 1
Descriptive Statistics and Correlations

	<i>M</i>	<i>SD</i>	Mathematics Achievement (<i>r</i>)
Gender	0.51	0.50	0.05
Language spoken at home	2.65	0.93	0.21
Parental education	3.51	1.17	0.31
Self-perceived competence in mathematics	2.69	0.78	0.44
Positive affect toward mathematics	2.96	0.83	0.33

Note. All correlation coefficients (*r*) are statistically significant, $p < 0.05$.

The overall regression model predicting mathematics achievement from student demographic variables, self-perceived competence in mathematics, and positive affect toward mathematics was statistically significant, adjusted $R^2 = 0.27$, $F(5, 3606) = 266.30$,

$p < 0.001$, 95% confidence interval (CI) [0.24, 0.29], $f^2 = 0.37$. Student demographic variables were positively associated with mathematics achievement. Female adolescents scored statistically significantly higher than their male counterparts on mathematics achievement. Adolescents who often spoke the language of the assessment at home had statistically significantly higher scores on the TIMSS 2007 mathematics assessment than their peers who infrequently spoke the language of the assessment at home. Similarly, adolescents whose parents had higher levels of education tended to have higher scores on the TIMSS 2007 mathematics assessment than their peers whose parents had lower levels of education.

Self-perceived competence in mathematics and positive affect toward mathematics were also positively associated with mathematics achievement. Adolescents who perceived themselves to have higher mathematical competence performed statistically significantly better in mathematics than did their peers who perceived themselves to have lower mathematical competence. Likewise, adolescents who reported higher levels of positive affect toward mathematics performed statistically significantly better in mathematics than did their peers who reported lower levels of positive affect toward mathematics.

Table 2
Ordinary Least Squares (OLS) Regression Analyses Predicting Mathematics Achievement

	<i>B</i>	<i>SE</i>
Constant	380.39	11.35
Gender	14.99	3.93
Language spoken at home	9.12	2.07
Parental education	16.28	1.47
Self-perceived competence in mathematics	41.17	2.37
Positive affect toward mathematics	7.15	2.71
Adjusted R^2		0.27
f^2		0.37
<i>F</i>		266.30*

Note. All coefficients are statistically significant, $p < 0.05$. * $p < 0.001$.

Discussion

The purpose of the study was to examine the influences of self-perceived competence in mathematics and positive affect toward mathematics on mathematics achievement of adolescents in Singapore. The study revealed the positive influence of self-perceived competence in mathematics on mathematics achievement. Because self-perceived academic competence precedes intrinsic motivation (Losier & Vallenard, 1994), there is growing evidence that greater self-perceived academic competence would enhance students' intrinsic motivation, which, in turn, would promote their academic achievement (Zisimopoulos & Galanaki, 2009).

Moreover, unlike students with low self-perceived academic competence, students who are mastery goal-oriented and who perceive themselves to have high academic competence may employ a more adaptive learning strategy (Cho et al., 2011). The use of an adaptive learning strategy may help students with high self-perceived academic competence to perform significantly better than students with low self-perceived academic competence. Further, students with high self-perceived academic competence tend to adopt high-performance approach goals, which, in turn, enhance their academic achievement (Cho et al., 2011).

The findings of the study also revealed the positive influence of positive affect toward mathematics on mathematics achievement, suggesting that boosting students' positive affect toward mathematics may help to improve their achievement in mathematics. There is mounting evidence that self-concept enhancement intervention programs would help to improve the self-concepts of adolescents in educational settings (e.g. O'Mara, Green, & Marsh, 2006; O'Mara, Marsh, Craven, & Debus, 2006).

However, self-concept enhancement intervention programs are not generally designed to cater to the needs of students in terms of their self-perceived competence and affect. In other words, such educational interventions often fail to identify competence and affect as two separate dimensions of academic self-concept. The findings of the present study point out the need to treat competence and affect as two separate components of academic self-concept. Hence, self-concept enhancement intervention programs may need to redesign such interventions by incorporating competence and affect components of academic self-concept as two separate constructs.

To enhance students' self-perceived competence in mathematics, mathematics teachers may need to build students' confidence in their ability to do well in mathematics. Insofar as students experience feelings of autonomy and competence during classroom instruction, they tend to be intrinsically motivated toward school and academics (Jang, Reeve, & Deci, 2010). When students are intrinsically motivated, they tend to learn better in school and become more creative on tasks requiring conceptual understanding (Niemic & Ryan, 2009).

Because teacher autonomy support during classroom instruction may help enhance students' perceived academic competence and academic engagement (see Reeve, Jang, Carrell, Jeon, & Barch, 2004, for a review), it is important to develop autonomy-supportive mathematics teachers rather than controlling teachers. To create autonomy-supportive classrooms, teachers may need to gain knowledge, skills, and training in how to be more autonomy supportive toward their students during classroom instruction (Reeve & Assor, 2011). Meta-analyses have demonstrated the effectiveness of such intervention programs in developing an autonomy-supportive motivating style among teachers (e.g., Su & Reeve,

2011). Hence pre-service and in-service teacher training programs may need to design and implement appropriate autonomy-supportive intervention programs with a view to developing autonomy-supportive mathematics teachers, who are capable of fostering their students' perceived competence in mathematics. Given the positive influences of positive affect toward mathematics on adolescents' mathematics achievement, the mathematics teachers may also need to develop and implement appropriate educational interventions to boost their students' positive affect toward mathematics.

Self-concept enhancement intervention programs, however, may not alone effectively help students to improve their achievement in mathematics. To be effective, skills training interventions need to be designed and implemented in conjunction with self-concept enhancement interventions (see O'Mara et al., 2006). Indeed, skills training interventions aimed at developing mathematics skills may benefit underachieving students.

In conclusion, the results of the study suggest the key roles that self-perceived competence in mathematics and positive affect toward mathematics may play in influencing the mathematics achievement of adolescents in Singapore. Developing and implementing appropriate educational interventions with a view to enhancing adolescents' self-perceived competence in mathematics and positive affect toward mathematics may not only improve adolescents' achievement in mathematics but also their labour market outcomes.

References

- Arens, A. K., Yeung, A. S., Craven, R. G., & Hasselhorn, M. (2011). The twofold multidimensionality of academic self-concept: Domain specificity and separation between competence and affect components. *Journal of Educational Psychology*. doi:10.1037/a0025047. Advance online publication.
- Bandura, A. (1997). *Self-efficacy: The exercise of control*. New York: Freeman.
- Cho, Y., Weinstein, C. E., & Wicker, F. (2011). Perceived competence and autonomy as moderators of the effects of achievement goal orientations. *Educational Psychology*, 31, 393–411.
- Deci, E. L., & Ryan, R. M. (2000). The “what” and “why” of goal pursuits: Human needs and the self-determination of behaviour. *Psychological Inquiry*, 11, 227–268.
- Eccles, J. S., & Wigfield, A. (1995). In the mind of the actor: The structure of adolescents' achievement task values and expectancy-related beliefs. *Personality and Social Psychology Bulletin*, 21, 215–225.
- Ferla, J., Valcke, M., & Schuyten, G. (2010). Judgments of self-perceived academic competence and their differential impact on students' achievement motivation, learning approach and academic performance. *European Journal of Psychology of Education*, 25, 519–536.
- International Association for the Evaluation of Educational Achievement (IEA) (2009a). *TIMSS 2007 international mathematics and science reports*. Retrieved from <http://timss.bc.edu/TIMSS2007>
- International Association for the Evaluation of Educational Achievement (IEA) (2009b). *IEA IDB Analyzer* (Version 2) [Computer software]. Retrieved from <http://www.iea.nl/>
- Jang, H., Reeve, J., & Deci, E. L. (2010). Engaging students in learning activities: It's not autonomy support or structure, but autonomy support and structure. *Journal of Educational Psychology*, 102, 588–600.
- Lau, S., & Roeser, R. W. (2008). Cognitive abilities and motivational processes in science achievement and engagement: A person-centered analysis. *Learning and Individual Differences*, 18, 497–504.
- Losier, G. A. & Vallenard, R. J. (1994). The temporal relationship between perceived competence and self-determined motivation. *Journal of Social Psychology*, 134, 793–801.
- Maehr, M. L., & Zusho, A. (2009). Achievement goal theory: The past, present, and future. In K. Wentzel & A. Wigfield (Eds.), *Handbook of motivation at school* (pp. 77–104). New York: Routledge.
- Marsh, H. W., Craven, R. G., & Debus, R. (1999). Separation of competency and affect components of multiple dimensions of academic self-concept: A developmental perspective. *Merrill-Palmer Quarterly*, 45, 567–601.
- Marsh, H. W., & Martin, A. J. (2011). Academic self-concept and academic achievement: Relations and causal ordering. *British Journal of Educational Psychology*, 81, 59–77.
- Niemiec, C., & Ryan, R. (2009). Autonomy, competence, and relatedness in the classroom. *Theory and Research in Education*, 7, 133–144.

- Nieswandt, M. (2007). Erratum: Student affect and conceptual understanding in learning chemistry. *Journal of Research in Science Teaching*, 44, 908–937.
- Olson, J. F., Martin, M. O., & Mullis, I. V. S. (Eds.). (2008). *TIMSS 2007 technical report*. Chestnut Hill, MA: TIMSS & PIRLS International Study Centre.
- O'Mara, A. J., Green, J., & Marsh, H. W. (2006). Administering self-concept interventions in schools: No training necessary? A meta-analysis. *International Education Journal*, 7, 524–533.
- O'Mara, A. J., Marsh, H. W., Craven, R. G., & Debus, R. (2006). Do self-concept interventions make a difference? A synergistic blend of construct validation and meta-analysis. *Educational Psychologist*, 41, 181–206.
- Organization for Economic Cooperation and Development (OECD) (2007). *PISA 2006 science competencies for tomorrow's world*. Paris, France: OECD.
- Reeve, J., & Assor, A. (2011). Do social institutions necessarily suppress individuals' need for autonomy? The possibility of schools as autonomy-promoting contexts across the globe. In V. Chirkov, R. M. Ryan & K. Sheldon (Eds.), *Human autonomy in cross-cultural context: Perspectives on the psychology of agency, freedom, and well-being* (pp. 111–132). New York: Springer.
- Reeve, J., Jang, H., Carrell, D., Jeon, S., & Barch, J. (2004). Enhancing students' engagement by increasing teachers' autonomy support. *Motivation and Emotion*, 28, 147–169.
- Shen, C., & Pedulla, J. J. (2000). The relationship between students' achievement and their self-perception of competence and rigour of mathematics and science: A cross-national analysis. *Assessment in Education*, 7, 237–253.
- Su, Y., & Reeve, J. (2011). A meta-analysis of the effectiveness of intervention programs designed to support autonomy. *Educational Psychology Review*, 23, 159–188.
- Wu, M. L. (2005). The role of plausible values in large-scale surveys. *Studies in Educational Evaluation*, 31, 114–128.
- Zisimopoulos, D. & Galanaki, E. P. (2009). Academic intrinsic motivation and perceived academic competence in Greek elementary students with and without learning disabilities. *Learning Disabilities Research and Practice*, 24, 33–43.

Socially Response-able Mathematics Education: Lessons from Three Teachers

Bill Atweh
Curtin University
<b.atweh@curtin.edu.au>

Kate Ala'i
Curtin University
<k.alai@curtin.edu.au>

This paper examines the experiences of three teachers who were engaged in a project designed to enhance student engagement in mathematics through the development of real-world context activities which enabled students to use mathematics as a tool for responsible citizenship and the pursuit of social justice.

In a previous publication, Atweh and Brady¹ (2009), discussed an approach to mathematics education based on ethics informed by critical mathematics education (Frankenstein, 1983; Skovsmose, 1994), ethnomathematics (D'Ambrosio, 1985; Powell & Frankenstein, 1997) and social justice pedagogy (Gutstein, 2006). As Atweh and Brady noted, ethics is generally associated with moral decisions, dogma and codes of behaviour and not seen as relevant to teaching and learning mathematics. Here we understand ethics differently. Using Levinas (1969), for us ethics is not predetermined principles or commitment that is either manifested in a particular behaviour or not; rather, ethics is an adjective that describes an existing relationship with the other that precedes our knowledge of the other. Chritchley (2002, p. 11) puts it this way: "It is the relation which is ethical, not an ethics that is instantiated in relations" (p. 12).

The construct of responsibility, on which we based this approach, is a key component of ethical discourse. Puka (2005) argues that the distinction some feminists make between responsibility and "response-ability" is a significant contribution to ethical thinking. Response-ability highlights the ability to respond to the demands of our own wellbeing and the ability to respond to the demands of the other. This is similar to what Roth (2007) points out, that responsibility "etymologically derives from a conjunction of the particles *re-*, doing again, *spondere*, to pledge, and *-ble*, a suffix meaning "to be able to."

Atweh and Brady assert that mathematics can contribute effectively to student response-ability if it engages with the world of the students. Further, this engagement should be not only with the physical world and the economic world, but also with the social world; it should aim at developing knowledge not only of mathematics but also of the world. Finally, such engagement should aim at not only *reading the world* but also, whenever possible, at *writing the world* (Gutstein, 2006). The implication here is that students can learn about their social world while they are learning mathematics and, at the same time, learn mathematics as they are engaging with the world.

A socially response-able approach to mathematics education necessarily involves concern about social justice. Questions of social justice in mathematics have often been raised in the discipline under different labels including those of equity and diversity. However, arguably, they have mainly been concerned with social justice *in* mathematics education (Burton, 2003; Secada, 1989) rather than attempts of achieving social justice *through* mathematics education (Gutstein, 2003).

¹ Brady is the previous name of the second author in this paper.

Response-able Teachers

In this section we discuss stories of three teachers, their backgrounds and attempts to change their teaching practices in their classroom to incorporate Socially Response-able Mathematics activities in their teaching. The three teachers volunteered for an 18 month collaborative project between university staff and six Western Australian schools. The Project commenced by a professional development workshop, incorporating theory and practical exemplars of approaches to develop students' responsibility through mathematics. Several workshops and school visits by university staff and teacher-consultants were planned to support the teachers in their design and implementation of their school-based projects. Teachers were provided with a few opportunities to be critical friends for each other through comments on each other's projects. The Project culminated in teachers presenting their findings at the AAMT conference.

The Story of Alex².

Alex commenced his 25 years of teaching experience as a primary school teacher, shifting to middle school mathematics teaching when he moved with his young family to a small tourist town, a considerable distance from the capital. The school caters for fewer than 300 students from K-12, and won recognition by a leading national newspaper as one of the top ten schools in Australia in curriculum innovation and community partnerships.

While Alex showed the greatest enthusiasm for the Project, he was also believed that there was not enough flexibility in the curriculum for him to pursue this kind of pedagogy. Through the professional development workshops he encountered the theoretical justification for how he always envisaged mathematics education ought to be, but he had never had the support or opportunity to implement it. The focus on social justice matched his own personal passion in line with his role as union representative dedicated to improving the working conditions of teachers.

So strong was his enthusiasm to the Project that he developed three school-based projects for his grade 8, 9 and 10 students – although, in this discussion here, we will concentrate on the grade 8 project. His motivation for choosing this particular project was based on his perception of grade 8 students, and in particular as youth in a small secluded town, as basically “egocentric” with very limited experience about global issues. He utilised a set of pictures from a book called the Hungry Planet (Menzel & D’Aluisio, 2007) which represented families from many nations around the world posing in front of the food that they would consume in one week. He describes his experience as “jaw dropping” and adds “I want to shake [the students’ narrow views] and make students see themselves as part of a bigger more dynamic world ... but also tie some good mathematical learning to it”. He called the project “Feed Me”.

The project required the students to keep a record of the food they ate in one week. The first class activity, costing the food consumed by each family, was a source of surprise to all students. Students learnt how to use spreadsheets to enter their results. The next phase of the project involved teaching the statistical concepts of different averages that can summarise the data including minimum and maximum values – all part of the content of the curriculum at that level. Alex believed that mathematics was only one way to examine the pictures. He used de Bono’s six thinking hats strategy, which was new to the students, to look at the information depicted from a variety of perspectives. These perspectives were used to introduce additional mathematics in terms of Venn diagrams, also part of the grade 8

² All names used here are pseudonyms.

mathematics curriculum. This activity included a discussion as to whether happiness and wealth are necessarily related. The mathematical data generated in these activities were all incorporated in the grade 8 end of semester exams. Another session required students to collect information about their chosen countries from the internet to decide if the family represented was somewhat representative of the economy of that country. Finally, students tied in the activities to the “food pyramid” to design a healthy menu for their own family for one day and consider reasons why it may or may not be feasible for their family by considering the health, environmental and economic implications of their choices. The project concluded with reflections such as:

Give some definition of what rich and poor mean.

How do you feel about the lives of people from less wealthy countries?

How would youth of other nations comment about your lifestyle?

What does the comment ‘with privilege comes responsibility’ mean?

What impact has this maths project had on your understanding of the world and how you fit in it?

The Story of Mary Lee

Mary Lee was a recent migrant to Australia and the youngest of the teachers in the Project with four years of experience as a science teacher. This was her first year teaching mathematics. Mary Lee worked in a fairly new middle school (grades 8-10) with 158 enrolments situated at the rural outskirts of Perth. In many ways, the school is an experimental school intended to develop alternative ways to assist teaching and learning to young adolescents most of whom are considered to be disengaged in traditional education. The school has a significant number of Aboriginal and non-English background students.

The school structure and management were influenced by the Big Picture Schools approach to teaching and learning which offers “personalised curriculum” and “real world connections”. However, class teaching was not totally abandoned – often they were single gender classes.

Mary Lee described her all-girl grade 9 students: “they don’t place education as a high priority and will mostly enter a trade-based occupation or apprenticeship. Very few go to university”. Mary Lee was very quick to decide on a school based project at the conclusion of the professional development workshop. As with many young people around the world, McDonald’s is a significant part of the daily life of students at this school – both as consumers and employees. Mary Lee envisaged that this project may contribute to the students’ perceptions that mathematics is not disjointed from decisions in daily life. The discussion around the project by the other teachers and facilitators of the Project was very enthusiastic about the possibility of the project to both provide important social issues that may interest the students as well as give rise to significant mathematics teaching and learning. We discussed the potential of considering the economic implications to society, and the world, of the expansion of the McDonald’s chain of restaurants balanced with their contribution to training young people and community charity work. In addition, the health related issues of McDonald’s food would provide significant ties to other aspects of the curriculum and would assist students to become more informed about in their dietary decisions.

Of the two possible foci identified above, Mary Lee’s activities tended to concentrate on the health aspects – arguably since it was easier to obtain data on this aspect of McDonald’s and reflecting her background as science teacher. It also fell within her comfort zone to deal with social issues. Mary Lee’s project consisted of 9 “discrete, staged learning activities” presented as worksheets collated into a booklet. The first activity consisted of watching the movie Super-Size Me. The teacher identified a few mathematical type statements from the

movie (e.g. McDonald's spent \$1.4 billion worldwide on advertising) and asked students to reflect on their implications to society and the environment. The second activity included published tables about the number of McDonald's restaurants in top ten countries as well and a table of the annual income from the 2007 Annual Report of the McDonald's Corporation. Students were given specific questions which required reading graphs and calculating percentages and fractions.

The third activity involved the energy content of McDonald's food. Published tables were provided to students and a set of questions which led to answering the bigger question of whether the higher energy content of the burger implied a heavier weight. The next two activities included studying the recommended daily intake of the energy from food and type of fats and their effect on health. Interesting to note that the data was given without a reference to its source or raising issues about its authenticity or applicability to the different social groups. This information was used to create a menu "If you had to eat breakfast, lunch and dinner at McDonald's, what will you order?" The next activity centred on "Is MacDonald value for money? Is being bigger better?" where prices and weights of hamburgers from 2008 and 2010 were compared. The last activity was graphing how often the students' families eat at McDonald's which lead to representing the data on a bar graph.

The Story of Peter

Peter is a trained geologist from the United Kingdom who migrated to a small regional centre and commenced teaching in the school in 1970s. At the time of his involvement in a project, he was the head of the middle school and the mathematics department. Peter's school was established in 1906 by an order of nuns in a town in a regional centre in Western Australia. It is a Catholic school, with a majority of lay teachers and enrolments of fewer than 300 mixed gender students from kindergarten to grade 10. The school's mission balances academic excellence and care for inclusion of disadvantaged students, in particular Aboriginal students.

Peter is a dedicated teacher who placed high value on higher level mathematics and at the same time was able to gain the respect of his students and colleagues at school. In this project, he involved his small group of "higher ability" grade 10 students in mathematics.

Peter's interest in joining the project was due to his concern about making mathematics relevant to students: "to find something that actually meant something to them". Even though he came from a school system that focused on issues of social justice, he did not feel comfortable when dealing directly with issues of social justice in mathematics. In his own words, "if [the activities] go too far [from] the mathematics curriculum - that would worry me". His concern was that such discussion of social justice and values might lead to the imposition of the teacher's values on the students.

Peter planned a school-based project in which students attempted to calculate the necessary height of a storm surge wall around the town to safeguard it from the effect of cyclones that frequent the area. Concerns about damage of cyclones were a "real community talking point" for the students and their parents. The last major cyclone hit the town about four hours before high tide. Hence, the question attempted by this activity was how high the wall should be to safeguard the town from flooding if the cyclone hit the town right at high tide what would happen to their own classroom in that case.

To shed some light on these questions, the teacher provided the students with data about the height of the normal tide every four hours, together with data from the day that the cyclone hit the town. Plotting the two graphs, they were able to learn about the trigonometric sine function as well as learn about subtraction of functions, which they were able to accomplish using a spreadsheet created by the teacher. By translating the scale of the

two functions, they determined that if the cyclone had hit during high tide, the storm surge would have been 6.5 metres. To answer the second main question about the height of their classroom above sea level, the students had to learn the use of a surveyor's levelling instrument. They found out that the height of their classroom floor was 4.6 metres. The actual height of the existing wall protecting that part of the town ranges from 1.9 to 2.7ms.

At the conclusion of the school-based project, the Shire Chairman and the Chief Engineer were invited to the school to hear the students' presentation on their findings. The Engineer congratulated the students and indicated that he had employed exactly same techniques and obtained similar results within few centimetres of the measures obtained by the students. The look on the students' faces was a testimony of the great sense of pride in what they have done. Not often are school students taken so seriously by people in authority. The Engineer went on to identify the complications associated with the decision about the height of a storm wall needed for the town. For example, he pointed out that the cost of building a 6.5 metre wall would have been astronomical. Secondly, the town prided itself in being a coastal town with easy access to the sea and scenic views, so living behind a wall 6.5 metres high was not a preferred option for this reason. Lastly, if the storm wall is too high, at the time of a cyclone, it would trap the water in town and have to be knocked down to allow water to recede. The council had adopted an alternative partial solution to the problem. Current building regulations demand that new constructions should be built on an elevation of least 6 metres.

Supporting Teacher Response-ability

During the course of this Project, discussions with teachers have highlighted several challenges faced by them in their attempts to adapt mathematics classrooms to incorporate a focus on Social Response-ability. Due to time limitations in the Project, only some of these challenges were directly addressed. Similarly, the experience of the teachers in meeting these challenges varied. Here we discuss two main challenges. The first relates to the challenges to the curriculum and pedagogy practices in the classroom. The second relates to finding the balance between the focus on the mathematics and on the social issues.

Curriculum and Pedagogy

At several times in the early stages of the Project's implementation, many teachers were uneasy that using project work with their students would require a significant amount of time and may lead into difficulties in covering the expected curriculum. These concerns are not unique to these teachers and are well founded in many instances. One of the concerns behind the development of the new national Australian Curriculum: Mathematics (Atweh, et al. 2012) has been the need to unclog the curriculum to allow teachers to deal with mathematics topics in depth rather than breadth. Further, the increasing role that national testing is playing in all the participating schools implies that failing to cover certain topics of the curriculum may have serious implications for the students, their schools and the teachers themselves.

Undoubtedly, direct instruction on mathematics topics and providing opportunities for students to practice the intended skills and procedures gives the teacher more control over time allocations to assure that all the curriculum topics have been covered. This, however, does not guarantee the amount and depth of student learning.

Discussion with the teachers highlighted that a Socially Response-able approach to the teaching of mathematics should not be seen as mere application of content learnt in the traditional ways; rather as an alternative way to develop the expected outcomes with the

students. As the above descriptions of the school-based projects demonstrate, all projects developed by the teachers were directly related to the content that was prescribed in the official curriculum. They were convinced that students not only had a chance to encounter the intended content, but also demonstrate deeper understanding and engagement in the class. In their presentations on their projects at the national conference, teachers showed what can be described as evangelistic commitment to this approach and a belief in its efficacy. In all cases, teachers have taken a risk in providing alternative ways of teaching mathematics based on projects that included discussion and outcomes that are seen as non-mathematical. Arguably, those calculated risks have been supported by university staff and critical friends.

However, there is room to continue to examine the nature of projects that teachers could use to develop students response-ability in terms of the level of mathematics demonstrated. While, there was some evidence that some activities involved higher level mathematics (as in the example of Peter's activities) some activities remained at low level such as simple calculations and reading information from graphs.

Atweh and Brady (2009) discussed the implication of this approach to mathematics teaching on pedagogy. There we argue that developing students' response-ability cannot be achieved merely by changing the content of the activities but by giving students more control over their learning. Similarly, dealing with real world phenomena where there is no unique answer to problems, and where mathematics is only one of the contributing factors to their solution, implies a more open environment where students are involved in decisions on what mathematics is needed and on how to develop arguments using mathematics towards a particular decision. In other words, we had the vision of using Freire's construction, a teacher sharing decision making with the students as a co-learner.

The experiences of the teachers in this Project demonstrated significant steps that these teachers have taken to open up their classrooms for student involvement. In our discussion with the students, they often talked about different relationships that they have established with their teachers. Peter's students pointed out that one of significant aspect of the project for them was that the teacher did not appear to know what the answer was and he was finding it out with them. Similarly, all three groups of students have participated in discussions about the role of mathematics in understanding or shaping the real-world and indicated their growing perception that mathematics does not always give the right answer. Peter's students, in discussion with Council personnel, learnt that measuring the height of the storm wall is only a part of what needed to be considered when making the decision on what is possible/desirable to construct. Alex's students learnt that the concepts of rich and poor are not quantitative categories. Mary Lee's students learnt that being aware of the dietary benefits or detriments of a particular food is no guarantee that people will reach right decision about its consumption.

However, by examining the way the teachers implemented their projects, it is evident that there is further room for experimenting with more open ended pedagogies towards increasing students' response-ability. For example, the teachers differed in the ways they provided their students with the data and information they needed to make their observations and calculations. Both Peter and Mary Lee provided students with the data they needed. Alex, on the other hand, required his students to collect their own data about their eating habits as well as searching for information from the internet about the different countries. Further, Mary Lee's McDonald's booklet consisted of tables to be filled by the students in a familiar methodological way – consistent with many text and workbooks in commonly available in schools.

It is important here to point out that these comments are not meant to evaluate the teachers' work in the project. Only the teacher is able to determine the level of support that students need/expect in order to carry out their activities. Further, a certain level of confidence of the teacher in both content and pedagogy is required before they open up their classroom for student-led explorations. The intention here is to present these as possible areas of support needed to increase the response-ability of the teachers themselves.

Balancing Mathematical and Social Learning

Jacobsen and Mistele (2010), in their investigation of the challenges preservice teachers face with teaching for social justice concluded that the major challenge was what they called the "problem of balancing" (p. 8). In other words, the problem of how to get a balance between the mathematical concepts and the social justice issues addressed in given tasks. They provided four possible manifestations of the "problem of balance" (1) the use of mathematics without mathematics instruction by the teacher (2) use of traditional methods of teaching and/or non-challenging mathematics (3) trivializing of social issues, and (4) disconnect or artificial connections between social issues and the mathematics. In this section, we make some observation on how these teachers have balanced the focus on mathematics and the focus on social justice in their projects.

First we note that this balance is subject to the personal value system of the individual teacher. From the above discussion, Peter, who developed a very successful project to teach rather abstract concepts in the mathematics curriculum was hesitant to introduce social justice discussion in his classes based on his personal fear of indoctrination of the students. His belief of what constitutes worthwhile mathematics outcomes has directly been reflected in his project. His focus was on developing high level mathematical skills in his students – and arguably he has achieved this well. Similarly, however, due to lack of confidence rather than a political stance, there was very little discussion in Mary Lee's workbook about social issues. Alex, however, highlighted social discussions in most of his activities, while it is possible that there may be room for higher order thinking mathematical problem solving activities. Nevertheless the students have had meaningful contexts to deal with several topics in the curriculum.

A related challenge is that while social studies teachers receive professional development to be able to deal with controversial issues without, to use Peter's term, "indoctrinating" students into particular lines of thinking, mathematics teachers receive no such training. As we have seen above, this deficit led Peter to avoid controversial social justice issues in his activities. Alex did not have this problem. He felt comfortable in dealing with issues of poverty and happiness in his class. Mary Lee on the other hand, may have chosen a safer discussion of diet and health than the discussion of McDonald's as a business. However, in negotiation with the English teacher, she passed on the data that she used together with the students' work in order for students to develop an essay about their conclusions. Peter also recruited the English and Religious Education teacher to coach the students in their presentations – an area where he did not feel he had much experience and expertise.

Conclusion

The stories of the three teachers above demonstrate that engaging mathematics with social response-ability depends, to a large extent on of the teachers themselves. Teachers' beliefs about the nature of mathematics, its role in the curriculum and in the real world determine their readiness to take risks in changing classroom practices. Similarly, more

confident and experienced teachers are in better position to take larger steps towards that goal. Teachers need support and scaffolding in their risk taking. In particular, mathematics teachers require further development in dealing with issues that are seen as controversial and depend on personal and social values. Mathematics teachers who are not confident in dealing with these interdisciplinary capacities may benefit from working collaboratively with teachers in other subjects in the school curriculum. Lastly, these teachers have dealt with the balance between quality mathematics and quality social justice issues differed. It is our hope that through further reflection and self-critique of their practices, with further support, all teachers would be able to negotiate this balance to their students, and their own empowerment towards making mathematics learning a socially response-able activity.

References

- Atweh, B. & Brady, K. (2009). Socially Response-able Mathematics Education: Implications of an ethical approach. *Eurasia Journal of Mathematics, Science and Technology Education*, 5(3), 135-143
- Atweh, B.; Goos, M.; Jorgensen, R. & Siemon, D. (Eds.). (2012). *Engaging the Australian Curriculum - Mathematics: Perspectives from the field*. Mathematics Education Research Group of Australia. Online book.
- Burton, L. (Ed.). (2003). *Which way social justice in mathematics education?* London: Praeger.
- Critchley, S. (2002). Introduction. In S. Critchley & R. Bernasconi (Eds.), *The Cambridge companion to Levinas*. Cambridge, UK: Cambridge University Press.
- D'Ambrosio, U. (1985). Sociocultural basis for mathematics education. In M. Carss (Ed.), *Proceedings of the fifth international congress on mathematics education* (pp. 1-6). Boston: Birkhäuser.
- Frankenstein, M. (1983). Critical mathematics education: An application of Paulo Freire's epistemology. *Journal of Education*, 165(4), pp. 315-339.
- Gutstein, E. (2006). *Reading and writing the world with mathematics: Towards pedagogy for social justice*. New York: Routledge.
- Jacobsen, L. J., & Mistele, M. J. (2010). Please Don't Do "Connect the Dots": Mathematics Lessons with Social Issues. *Science Education and Civic Engagement*, 2(2), Summer 2010.
- Levinas, E. (1969). *Totality and infinity: An essay on exteriority* (A. Lingis, Trans.). Pittsburgh, PA: Duquesne University Press.
- Menzel P; D'Aluisio F. (2007) *Hungry planet: What the world eats*. Berkeley: Ten Speed Press
- Powell, P. & Frankenstein, M. (1997). *Ethnomathematics: Challenging eurocentrism in mathematics education*. Albany: State University of New York Press.
- Puka, B. (2005). Teaching ethical excellence: Artful response-ability, creative integrity, character opus. *Liberal Education*, 91(3), 22-25
- Roth, W. R. (2007). Solidarity and the ethics of collaborative research. In S. Ritchie (Ed.), *Research collaboration: Relationships and praxis* (pp. 27-42). Rotterdam: Sense Publishers.
- Secada, W. (1989). *Equity in education*. Philadelphia: Falmer.
- Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Dordrecht: Kluwer Academic Publishers.

Improving First Year Mathematics Teaching Through Making Connections: An Action Research Approach

Jo Balatti

James Cook University
<Josephine.Balatti@jcu.edu.au>

Shaun Belward

James Cook University
<Shaun.Belward@jcu.edu.au>

Four university lecturers at an Australian university have been undertaking an action research project for almost two years to improve first year mathematics teaching. The project is analysed here through the single idea of “connection”. Making new connections or changing the nature of existing connections with colleagues and especially with students is leading to different ways of teaching. Beliefs and practices that had remained, until now, unexamined, are being abandoned, modified, or at the very least, questioned.

In certain traditions, practitioners of meditation believe that focussing exclusively on the breath leads to the centring of the mind and to a better understanding of the interconnectedness of all things. Similarly, in this paper, we focus exclusively on one concept through which to reflect on a complex, collaborative human experience that we have shared for almost two years. The concept is that of “connection” and the experience is that of an action research project in which we have worked at improving the way we teach first year mathematics. At a time in Australia when tertiary mathematics enrolments are not meeting the graduate demand of STEM (Science, Technology, *Engineering*, and *Mathematics*) related professions (Brown, 2009; Dobson, 2007), mathematicians are becoming more aware of how they are teaching (Nardi, 2008; Wood et al, 2011). A review of our experience to date indicates that the quality of “connection” powerfully permeates the ways in which we talk/think, act, and relate (Kemmis, 2009) in our action research group. We argue that “connection” is fundamentally contributing to the changes we are making in the teaching and learning practices of first year mathematics in the lecture theatre and the tutorial room.

The Latin origin of the verb “to connect” means to join or to tie. This meaning has been taken into modern English where “to connect” means to bind or fasten together; or to join or unite (Macquarie Dictionary, 1997). To these meanings, the thesaurus in the WORD software package adds the following synonyms: to relate, to associate, to bring together, to link up, and to network. In our context, connections refer to the links we make in our action research project among people, ideas, and practices.

The human capacity “to connect” is integral to a range of theories that explain how we communicate, build community, and learn. For example, albeit in different ways, the concept of connection is as essential to those learning theories that focus on the cognitive to explain how learners construct and organise knowledge as it is to those theories that view learning as a social activity. Not surprisingly therefore, the element of connectedness appears in pedagogical models (Biggs & Tang, 2007) where connection can refer to how teachers help students develop and make appropriate links among concepts; to how they have students learn from one another; or to how they might encourage students to connect what they are learning in the classroom to the broader social contexts in which they live. While many of these theories would be useful frameworks for analysing the connections discussed in this paper, we use Wenger’s community of practice theory (1998) because its emphasis on the social aligns with the collective nature of action research. Wenger’s (1998) theory is a social theory of learning in which learning is viewed as social participation where identity transformation, practice, and social structure are mutually constitutive.

Lave and Wenger (1991, p. 98) introduced the term “community of practice” in a study of learning through apprenticeships to refer to “a set of relations among persons, activity, and world, over time and in relation with other tangential and overlapping communities of practice”. They then described the connection between the practice of the community and the learning of its members as follows: “The social structure of this practice, its power relations, and its conditions for legitimacy define possibilities for learning”. Learning, they argued, is about participation in social communities and because learning changes who we are and what we can do, it is ultimately to do with transformation of identity. From this perspective, identity, and community membership are interrelated. In a later book, Wenger (1998) details the interconnectedness of identity, community, practice, meaning, and learning, all of which, he explains, are mutually defining.

Wenger (1998) describes practices as the collective learnings of a group or community of people whereby meanings are negotiated and “created over time by the sustained pursuit of a shared enterprise” (p. 45). Communities of practice then are “shared histories of learning” (p. 86) that develop during a time span of indeterminate length, but in which there is sufficient mutual engagement in a common purpose to produce some significant learning. Wenger (1998) argues that practice is a source of community coherence. This comes about in three ways: through the mutual engagement of participants; through the negotiation of a joint enterprise; and through the development of a shared repertoire of resources accumulated over time.

Mutual engagement refers to the interactions and relations built up between community members as they negotiate meanings in pursuit of a common purpose. Mutual engagement requires drawing on one’s own resources as well as drawing on those of others. It involves “defining identities, establishing who is who, who is good at what, who knows what, [and] who is easy or hard to get along with” (p. 95). Negotiating a joint enterprise involves struggling to define the enterprise and reconciling conflicting views and interpretations. Because it is ongoing, the process produces relations of mutual accountability among the community members. These include documented rules, policies, standards, and goals, but they also include norms and values that can go unarticulated. Relations of accountability are central “in defining the circumstances under which, as a community and as individuals, ... they attempt, neglect, or refuse to make sense of events and to seek new meanings” (Wenger, 1998, p. 81). The third element, that of a shared repertoire, refers to the shared resources that the community has developed for negotiating meaning. The repertoire ranges from routines, stories, and symbols to the styles and discourses used in practice. Wenger (1998) uses the term “repertoire” to emphasise that while these resources are made available through a history of mutual engagement, they are also used to create new meanings. Compatible with the idea of communities of practice is the notion of social capital. It, too, has connectedness at its core.

Social capital theory proposes “that networks of relations are a resource that can facilitate access to other resources of value to individuals or groups” (Balatti & Black, 2011, p. 66). According to Coleman (1988), the main aspects of social relations in which the social capital inheres are the obligations, expectations, and trust set up within the relationships; the information channels created in the social structure; and the norms and sanctions operating within the collectivity. In our case, the collectivity is the action research team.

The social relations included in the definitions above have been described in the social capital literature as ties, or more specifically, as bonding, bridging, and linking ties (Gittell & Vidal 1998; Granovetter, 1973). Bonding ties are the ties that build a common purpose and cohesion within groups such as within an action research group. Unlike bonding ties

that are the strong ties within a group, bridging ties are the relatively weaker ties that exist between groups or that an individual has with members of groups that are not his/her main groups. For example, in the university setting, bridging ties may be those ties that exist between schools or between members of different teams within the one school. The third kind of tie is the linking tie. These are the ties that people have with institutions, systems, or “faceless” entities that are important to their capacity to participate more fully in their groups or individually. Linking ties for members in an action research group may include knowing how to communicate with administrators of online databases or software.

To describe the action research team in this study as a community of practice allows us to analyse the connections among its members in terms of Wenger’s (1998) theory of learning. Doing so also provides a framework for discussing the connections among the participants from a social capital perspective thus focussing on the ties and on the resources that are shared and generated by virtue of those ties.

The action research approach adopted in this case follows Kemmis and McTaggart’s (1988, p. 5) model where they describe action research as

a form of collective self-reflective enquiry undertaken by participants in social situations in order to improve the rationality and justice of their own social or educational practices, as well as their understanding of these practices and the situations in which these practices are carried out.

In our project, our purpose is to better understand how first year students experience university mathematics; to reflect on our assumptions and beliefs about what constitutes good teaching; to make considered changes to our practice; and to evaluate the impact of those changes. The repeated action research spiral of plan, act, observe and reflect (Kemmis & McTaggart, 1988, p.11) shapes the interaction among the team members.

Action research as an approach to changing professional practice at the grassroots level has gained currency in a range of professions including mathematics teaching (Atweh, 2004) in the school and university settings. The literature is rich in guidelines on how to undertake action research but much of the research-based literature focuses on the outcomes of the approach rather than on the approach itself. The approach is generally dealt with, sometimes perfunctorily, in the “methods” section of papers which then proceed to report and analyse the results. Hannah, Stewart and Thomas’s study (2011) involving two mathematics educators and one mathematician that reports the development of the mathematician’s lecturing pedagogy is an example of this type. In the smaller body of research that has attended more to the dynamics of the action research approach, Males, Otten and Herbel-Eisenmann (2010) explore the nature of collegiality and “challenging interactions” (p. 461) amongst participants in a long-term action research project of several years involving eight mathematics teachers and three researchers. This paper also contributes to this smaller body of literature by exploring the element of connections in the action research approach adopted by our team.

The Action Research Method

The action research project that this paper draws on is taking place in the Mathematics Discipline in the School of Engineering and Physical Sciences at James Cook University. It began in July 2010 when at a meeting of the six mathematicians in the discipline, staff decided that the fail rate in the two first year mathematics subjects was unacceptable and that changes needed to be made to the teaching and learning in those subjects. The head of discipline called for volunteers who wished to investigate the issues further and improve the learning experience of first year students.

After some early changes, the four member team that continues to the present was established. The three mathematicians in the team are Patrick, the coordinator/lecturer of the first year subject offered in first semester; Shaun, the head of discipline who is also the coordinator/lecturer of the first year subject offered in the second semester; and Wayne, the coordinator/lecturer of the second year subject offered in first semester. Each has extensive experience (over 10, 20 and 30 years respectively) teaching university mathematics including first and second year subjects. Two of the mathematicians completed their undergraduate and postgraduate studies at the university and all three began their professional careers at the university.

Jo, the fourth member of the team is the secondary mathematics educator from the School of Education. She was invited to join the group because of her experience in facilitating action research projects, her knowledge of pedagogy, and her familiarity with conducting education research.

The team continues the practice initiated at the outset of meeting at least weekly during the semester. There is full attendance at almost all meetings. At the beginning of each meeting, an agenda is drawn up from the group. Meetings are audio-recorded and minuted. The minutes document reflections about any actions taken and the results of those actions; issues raised; decisions made and their rationale; and any concerns or insights shared by the members. The recordings and minutes are posted on our community site on the university's Blackboard learning management system.

The online community site also includes other material pertinent to our work. Participants contribute to the research folder that comprises journal articles organised in categories. The material prepared by the high school teacher who delivered professional development in ICTs to the members of the action research team is on the site. Also present are the written reflections that the mathematics lecturers posted in the first twelve months of the project on their experience using unfamiliar teaching strategies in their tutorials.

Most of the "talk" that occurs at our meetings coalesces around two kinds of endeavours. The first is activity at the subject level which impacts the practice of all the teachers in the team. This may include reflecting on student related data such as test results and student feedback or discussing actions taken or actions proposed. The second is activity at the individual level. This may include a tutor's personal attempts at responding more effectively to the different ability levels within his tutorial group or someone else's efforts to collect evidence to support or disconfirm a hunch. In both kinds of endeavour, evidence based decision-making is paramount, a norm that had previously not existed among the mathematicians with regards to their teaching.

It is within the operational framework described above that the action research team members draw on existing connections, modify others, or build new ones. The types of connections and the quality of those connections that the action research team draws on and/or generates are, using Wenger's (1998) terminology, a function of its common purpose, its membership, and its norms of interaction. The rest of this paper discusses three sets of connections in turn, namely those connections forged amongst the members of the action research team; those between the members and people outside the group; and most importantly, those between the members of the group and the students. We place the emphasis on connections among people because as the discussion demonstrates, these connections link to the connections among ideas and practices that the members of the action research team experience.

Making Connections Within the Action Research Team

The common purpose of the team, which connects the participants and shapes the joint enterprise (Wenger, 1998), is shared strongly amongst the four members in different ways. Improving first year mathematics performance is core business for the head of discipline (Shaun) and for the coordinator/lecturer of one of the subjects (Pat). Student performance is also important for Wayne who, as well as tutoring into the first year subjects, teaches mathematics to students who progress to second year and beyond. Although Jo is external to the discipline she, too, is strongly committed to its purpose. Her interest relates to the secondary mathematics preservice teachers. Historically, the fail rate amongst preservice teachers in first year mathematics is much higher than the average fail rate for the whole cohort. The fail rate in these subjects and subsequent ones has consequences for the number of graduates available in a region where many of the teachers teaching mathematics in the junior secondary school are non-specialists. In conclusion, in different but compatible ways the purpose has and continues to be of genuine importance to all members of the group.

The composition of the action research team also influences the links that are made among the members and the nature of their interaction. Arguably the single most important decision made by the three mathematicians was to be involved in first year mathematics as lecturers and/or tutors. This was a shift from the normal practice of having postgraduates as tutors in first year mathematics. Part of the rationale for the change was the need to develop a co-constructed, shared understanding of the first year context (i.e., a new “resource” in the shared repertoire) which, amongst other things, would lead to the development of an induction program for future tutors in the subjects. This shared experience has led to a stronger common purpose.

The membership of the mathematics educator has also brought a particular set of resources to the team. Unlike the longstanding relationships among the mathematicians, Jo had no shared history with any of the team members other than two positive collaborations that had recently commenced with Shaun in a postgraduate supervision and in the development of a mathematics community website. Resources that the team accesses via the mathematics educator include action research know-how, literature, contacts, and research capacity. This is provided by Honours education students who undertake projects to do with the first year mathematics experience.

The third major element of the community’s practice that influences connections is the frequency and nature of the interaction among the members of the team. Although the mathematicians had known one another for a very long time, they had never met deliberately to talk about their pedagogy. In this community of practice, we meet weekly to talk only about pedagogy, student learning, and related topics. This was a new way of interacting which requires deprivatising practice. It requires talking about our practice; inviting others to observe our teaching; learning to talk to others about their practice; and finally, it requires planning and implementing new ways of teaching—together.

To use a social capital term, the connections that have been described above have lead to “bonding ties” developing among the members of the group. Bonding ties are important to developing cohesion of purpose, trust, and a sense of commitment, reciprocity, and accountability within the group. These are the strengths of bonding ties. The concomitant risk of strong bonding ties is that the group becomes closed and ultimately stops learning.

Making Connections Between the Action Research Team and Others

As much as bonding ties are important for the mutual engagement of a community of practice to develop, bridging and linking ties are also critical, especially if the common

purpose of the community of practice is to learn. It is through the bridging ties that new resources enter the community, resources including new knowledge, skills, contacts, and even new members. It is through linking ties that we access resources such as expertise and opportunities not immediately available to our members. For our team, attending to bridging and linking ties is also important because of its small size.

Bridging ties that are particularly important are those that we have with practising school teachers of mathematics. As a result of this connection, the mathematicians have more familiarity with the senior syllabus and the assessment regime and thus possess a more informed understanding of the skill base that students bring to first year mathematics. The bridging tie has also brought to the teaching team a high school teacher to tutor the preservice teachers. This same teacher also delivered professional development in ICTs to the action research team members over a period of months. For the first time in semester one, 2012, the mathematics lecturer is having his students use Geogebra as a learning tool in a graphing topic.

Bridging and linking ties are also important for the flow of resources in the opposite direction, i.e., going from the action research team out to other communities of practice within the university or elsewhere in the wider community of mathematics teachers. As well as drawing on or creating new links to share our learning with others, these links, especially in the university context, are important for the sustainability of the project. The team is buried within a discipline, within a school, within a faculty in the University. To ensure that colleagues know what we are doing and achieving we take opportunities to present our work to colleagues.

Making Connections Between the Action Research Team and the Students

Because the connections between the action research team and the student body are core to the purpose of our project they warrant being dealt with separately. An analysis of these connections shows that some can be described usefully in the language of social capital. Most of the connections are at the tutor/tutorial group level.

The formal connections between the action research group as an entity and the student body are limited. One has been the interactions as dictated by the university ethics requirements for seeking consent to use student data in our research. Students also see us together when supervising invigilated assessments. The only other occasion when we have presented ourselves as a group was when we co-delivered a one hour induction to the preservice teachers, an initiative that we may develop and extend to the whole cohort in the future.

The three mathematicians of the team however, do collectively connect face to face with the student body in another very important way. This is via the tutorial representatives meetings which were introduced early in the action research project. Twice a semester, feedback meetings are held with a student representative from each tutorial group (approximately 10 groups each time). The lecturer then reports on these meetings to all the students in the following lecture, thus completing the communication loop.

It is in the context of the tutorial group that the social relations between the mathematician and his students are more complex. The introduction of peer group learning with its emphasis on learning from one another and the accompanying de-emphasis on the tutor as the fount of all knowledge (Belward & Balatti, 2012; Higgins & Read, 2012) has changed the traditional connections between students and students and students and tutor. Establishing peer learning groups for home study and increasing the amount of student-student interaction within tutorials have generated different sets of ties among tutors and students. Tutorials have now primarily become the place where questions that stump the

peer study groups are discussed. Via blogs (previously emails), the peer groups communicate their problems to the tutors (and other students) in advance of their tutorials. Bridging ties are thus created between the community of practice that is the tutorial and the smaller communities of practice that are the peer study groups. In the tutorial context, bonding ties amongst all the participants are forged as students and tutor work together in a joint enterprise that produces identities that are markedly different from those that students and tutors developed in the past.

Discussion and Conclusion

The longevity of the project, the frequency of interaction, the continuity of the main purpose, and the make-up of the team might make this action research project unusual when compared with others in mathematics teaching. The narrative of this particular action research project, including the place that connections have in it, is necessarily unique by virtue of its context and membership. This, however, can be said of all action research projects. While we note the limitations of the single case study, we would argue that the importance of connection is applicable to most, if not all, action research projects because action research, by definition, is a collective enterprise underpinned by learning theories such as Wenger's community of practice (1998) that, in different ways, account for connection.

In this paper, we have provided insights into the workings of an action research team through focussing our reflection using the concept of "connection". In so doing, we have revealed the importance of the bonding, bridging and linking ties we make to the quality of the mutual engagement, joint enterprise and shared repertoire (Wenger, 1998) of the action research community of practice. We have also noted, without being able to elaborate, that the repeated spiral nature of the action research approach makes the transformation of the connections among people, ideas, and practices mutually constitutive in a particular way. By this we mean that the evidence based decision-making characteristic of action research influences how the connections are formed and inter-relate. Furthermore, the inter-related nature of the connections among people, thought, and action makes separating action research into process and outcomes too simplistic.

Although we cannot predict the longevity of this particular project, we are continuing to find value in working together. Student performance has not yet reached what we would like it to be and we realise that much more work needs to be done. For example, we have not yet adequately understood or developed the various facets of connection in the curriculum of first year mathematics. With increasing attention being given to the quality of teaching in the STEM related subjects at universities, action research is likely to become a more utilised way of fostering improvement in how lecturers teach mathematics. Attending to the nature of connections among people is an important element within the action research enterprise.

Acknowledgement

We thank Patrick Higgins and Wayne Read for their input in this paper.

References

- Atweh, B. (2004). Understanding for changing and changing for understanding: Praxis between practice and theory through action research in mathematics education. In P. Valero & R. Zevenbergen (Eds.), *Researching the socio-political dimensions of mathematics education: Issues of power in theory and methodology* (pp. 187–205). New York: Kluwer Academic Publishers.
- Balatti, J., & Black, S. (2011). Constructing learners as members of networks. In R. Catts, I. Falk, & R. Wallace (Eds.), *Vocational learning: Innovative theory and practice* (pp. 63-76). Dordrecht: Springer.

- Belward, S., & Balatti, J. (2012). *Teachers' response to peer group learning in first year mathematics: An analysis using transformative learning theory*. Paper presented at the 35th annual conference of MERGA, Singapore.
- Biggs, J., & Tang, C. (2007). *Teaching for Quality Learning at University* (3rd ed.). Buckingham: SRHE and Open University Press.
- Brown, G. (2009). *Review of Education in Mathematics, Data Science and Quantitative Disciplines*. Turner ACT: The Group of Eight. Retrieved from http://www.go8.edu.au/_documents/go8-policy-analysis/2010/go8mathsreview.pdf
- Coleman, J. S. (1988). Social capital in the creation of human capital. *American Journal of Sociology*, 94(Suppl.), S95-S120.
- Dobson, I.R. (2007). *Sustaining science: University science in the twenty-first century*. Melbourne: Australian Council of Deans of Science. Retrieved from <http://www.educationalpolicy.org/pdf/acds.pdf>
- Gittell, R., & Vidal, A. (1998). *Community organizing: Building social capital as development strategy*. Thousand Oaks, CA.: Sage.
- Granovetter, M. (1973). 'The strength of weak ties', *American Journal of Sociology*, 78(6), 1360–80.
- Hannah, J., Stewart, S., & Thomas, M. (2011). Analysing lecturer practice: The role of orientations and goals. *International Journal of Mathematical Education in Science and Technology*, 42(7), 975-984.
- Higgins, P., & Read, W. (2012). *Increasing the learning value of first year university mathematics tutorials*. Paper presented at the 35th annual conference of MERGA, Singapore.
- Kemmis, S. (2009). Action research as a practice-based practice. *Educational Action Research* 17(3), 463-474.
- Kemmis, S., & McTaggart, R. (1988). *The action research planner* (3rd ed.). Geelong, Victoria: Deakin University Press.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.
- Macquarie Dictionary (Revised 3rd ed.) (1997). Macquarie University, NSW: The Macquarie Library Pty Ltd.
- Males, L.M., Otten, S., & Herbel-Eisenmann, B.A. (2010) Challenges of critical colleagueship: Examining and reflecting on mathematics teacher study group interactions. *Journal of Mathematics Teacher Education*, 13(6), 459–471. doi: 10.1007/s10857-010-9156-6
- Nardi, E. (2008). *Amongst mathematicians: Teaching and learning mathematics at university level*. New York: Springer.
- Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. New York: Cambridge University Press.
- Wood, L. N., Vu, T., Bower, M., Brown, N., Skalicky, J., Donovan, D., Loch, B., Joshi, N., & Bloom, W. (2011). Professional development for teaching in higher education. *International Journal of Mathematics Education in Science and Technology*, 42(7), 997-1009. doi: 10.1080/0020739X.2011.608864

Teaching Mathematics in a Project-Based Learning Context: Initial Teacher Knowledge and Perceived Needs

Kim Beswick
University of Tasmania
<Kim.Beswick@utas.edu.au>

Rosemary Callingham
University of Tasmania
<Rosemary.Callingham@utas.edu.au>

Tracey Muir
University of Tasmania
<Tracey.Muir@utas.edu.au>

This paper reports on initial data from teachers at an Australian Year 9-12 secondary school which is attempting to implement a project-based learning model across the entire curriculum. The eight teachers had diverse prior teaching experiences, including work in primary schools, special education, Year 7-10 secondary schools, and technical and further education. None had studied mathematics at tertiary level and just one listed maths as a main teaching area. Results of an initial survey indicate that most had reasonable levels of personal mathematics competence, and could identify relevant mathematical concepts among the affordances of a particular scenario, but that they struggled to articulate how they would work with students to pursue any of the relevant mathematics in depth.

Innovative models of schooling such as those described as project-based or design-based learning (Doppelt, 2009) share an emphasis on learner autonomy and choice, intrinsic motivation (Lam, Cheng, & Ma, 2009; Quek, et al., 2007), and authentic assessment. In such environments, teachers have a largely facilitative role. Benefits have been identified in terms of student engagement with their learning (Lam, et al., 2009), as well as increased opportunities to develop students' abilities to think creatively and innovatively (Lee & Breitenberg, 2010) and to work independently (Doppelt, 2009). Nevertheless, many teachers and parents share a concern that students in these schools might be disadvantaged by not having access to the entire curriculum (Toolin, 2004). To be effective in these contexts teachers need to acquire pedagogical knowledge and skills in addition to those that they have honed in traditional contexts, and this presents difficulties for many (Toolin, 2004). The school that was the site of this study had adopted project-based learning across the entire curriculum under the auspices of Big Picture Education Australia (BPEA). The model of project-based learning used in these schools includes particular emphases, described in the sections that follow that impact the nature of knowledge required by teachers.

Quantitative Reasoning

The participants in the study were interested in how students' quantitative reasoning (QR) could effectively be developed in their context. Thornton and Hogan (2003), in their discussion of cross-curriculum numeracy, link the term quantitative reasoning (QR) to quantitative literacy as used by Steen (e.g., 2004). The concept is very like that of numeracy as defined by the Australian Association of Mathematics Teachers (1997, p. 15), that is, "To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life". The term has been adopted by BPEA to describe one of five major learning goals, three of which include the word reasoning. They define reasoning as "the process of forming conclusions, judgments, or inferences from facts or premises" (Down & Hogan, 2010, p. 63). This definition is consistent with the description of the Proficiency Strand of the same name in the *Australian Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority

[ACARA], 2012), although ACARA describe reasoning in the context of mathematics in greater detail. Down and Hogan are careful to point out that the learning goals, including QR, are not content areas and cannot be substituted for traditional curriculum areas. They acknowledge, however, that the learning goals of BPEA cannot be achieved in the absence of content and, indeed, that important content should be learned in the context of each. The relevant content for QR is largely mathematics and hence, in this paper, QR and mathematics can be used essentially interchangeably.

Teacher Knowledge

In any teaching context teachers of mathematics need not only to understand the relevant mathematical content but also how students develop understanding of specific mathematical content and how that content can be represented and made accessible for students (i.e., pedagogical content knowledge (PCK)) (Ball, Thames, & Phelps, 2008; Chick, 2007). Following Beswick, Callingham and Watson (2011), in the construct of teacher knowledge we also include teachers' relevant beliefs, for example, about their role as teacher, the nature of mathematics, and how mathematics is best taught and learned. Confidence, in both the everyday uses of mathematics and to teach various aspects of the curriculum, is also considered part of teacher knowledge. Confidence to teach is related to content knowledge and PCK, with increasing confidence in relation to specific aspects of the mathematics curriculum associated with higher level of overall knowledge for teaching mathematics (Beswick et al., 2011).

In project-based learning contexts teachers also need to be able to identify the mathematics inherent in a range of contexts and applications, draw students' attention to that mathematics, and engage them with it in authentic ways. Furthermore, they must be able to identify and structure projects that present the necessary opportunities for mathematical learning (Lee & Breitenberg, 2010). In addition, in Big Picture schools teachers encourage students to pursue their own interests and, therefore, rather than providing projects, teachers must guide students as they explore projects of their own choosing (Down & Hogan, 2010). Our hypothesis is that the development of knowledge for mathematics teaching as documented by Beswick et al. (2011), and shown in the left column of Figure 1, might be a longer process involving one or more additional stages in the context of project-based learning, as well as knowledge specific to teaching mathematics in project-based contexts.

Specifically, developing confidence to teach mathematics in project-based contexts may be a longer process because most teachers in these contexts are transitioning from more traditional contexts and must come to terms with substantial differences in their role as teacher, which are central to their beliefs about themselves as teachers, as well as learning to teach mathematics (Beswick, 2007). In particular, they need to move from seeing themselves as content and curriculum experts responsible for assigning tasks and managing students' engagement with them, to reconceptualise their role as facilitators of individual students' explorations of ideas and contexts that are of interest to them. The development of relationships is central to the teacher's task, "to know students well and to provide the right measure of challenge and support for each student in each activity to promote academic and social growth" (Down & Hogan, 2010, p. 34). The transition to project-based learning thus requires an increase in teachers' knowledge regardless of the extent of their knowledge at that time and, as shown in right column of Figure 1, in the nature of knowledge required in accordance with the differing demands of teaching in project-based learning contexts compared to traditional contexts. Understanding the knowledge required is vital given the

established importance of the teacher to the effectiveness of project-based learning (Quek, et al., 2007).

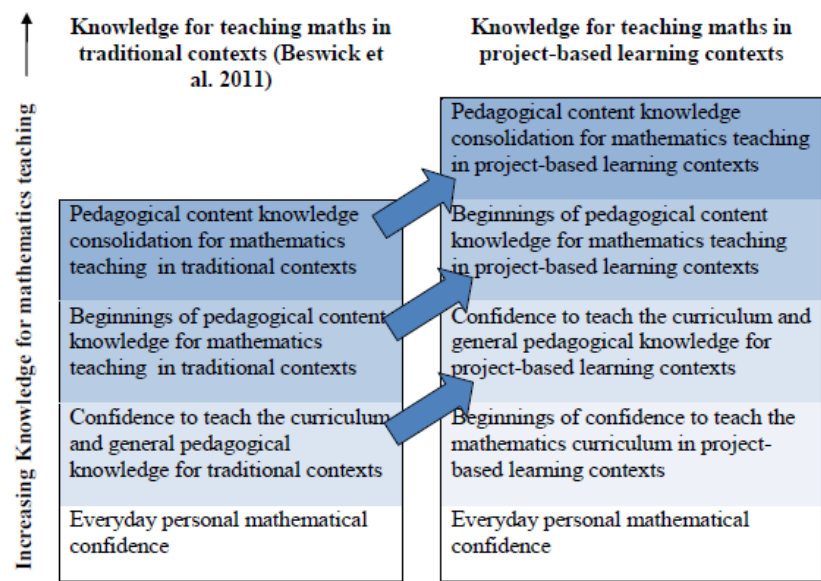


Figure 1. Hypothesised model of knowledge required in traditional and project-based contexts.

The Study

BPEA is the umbrella organisation for a group of 28 Australian schools in which project-based learning has been adopted in whole or part. The Year 9-12 school in which this study was conducted had been in operation for just one year prior to the start of this project. The school caters for the full range of student abilities including those intending to pursue vocational and academic paths after Year 12.

The study was designed to find out about the initial knowledge for mathematics teaching of teachers in the school at the start of a 1-year project investigating the development of their knowledge to teach mathematics in a project-based learning context. The project was initiated by the school principal and constructed as a genuine partnership in which, consistent with BPEA principles, all of the participants, including the researchers, are learners bringing their differing expertise to the problem of how QR can be best taught in a project-based context. The focus of this paper is the mathematics content knowledge, PCK, and confidence that the teachers brought to the project and the outcomes for which they hoped as a result of their participation.

Participants

Six of the school’s eight teachers, the school principal, and another Department of Education leader with responsibility for innovative schools across the state completed the initial survey. For the purposes of this paper all are henceforth referred to as teachers. The teachers’ backgrounds were diverse; six had bachelor degrees in (general) education, one had a Diploma of Teaching and Graduate Certificate in special education, and another was enrolled in a Bachelor of Education (Applied Learning) and had two trade certificates. None had tertiary mathematics qualifications. Four of the teachers reported having taught in primary schools for from 2 to 18 years, seven had secondary school teaching experience of 2 to 28 years. In addition, four reported having taught in other contexts including technical and further education, K-12 contexts, or special schools. One was in her first year of

teaching. In terms of a main teaching area, English, literacy, and history were each named by two teachers, and special education, arts, science/maths, humanities, manual arts, drama, and sport were each named once. One teacher declined to nominate a main teaching area.

Instrument

In addition to asking about teachers' qualifications, teaching experience, and main teaching area, the initial teacher survey comprised 15 Likert-type items, 2 multiple-choice items, and 9 open-response items. Many were taken or adapted from similar items used by Callingham and Beswick (2011). Of the Likert-type items, five asked teachers to indicate the extent of their agreement with statements about their personal use of quantitative reasoning and ten concerned their teaching of QR. The multiple-choice items concerned only mathematical content knowledge, and eight of the open-response items concerned both this and the teachers' PCK. Some open-response items required respondents first to solve a mathematical problem and then to suggest opportunities it afforded for QR or to critique it in terms of being the basis of a project in which QR could be developed. The final item asked teachers about what they hoped to gain from their participation in the project. The results presented here relate to the open-response items that related to the teachers' content knowledge, the item that most authentically assessed their PCK in relation to project-based mathematics teaching, and the items asking about their aspirations for the project's outcomes. The relevant items or parts thereof are shown in Table 1.

Table 1

Relevant Items from the Initial Teacher Survey (pictures and formatting removed)

Item no.	Item
1a	A student is telling you about her weekend. She says, "We went cycling. After 45 minutes we had only completed 15 km and we still had 25 km to go. Guess how long it took us." Work out a mathematical answer to the problem in any way that you like.
4	An upright 1-metre stick casts a shadow that is 60 centimetres long. At the same time, a flagpole casts a shadow that is 5.4 metres long. How high is the flagpole?
6	Below are two currency conversion graphs [graphs omitted]. How many Brunei dollars are equal in value to 50 British pounds?
7	Which one of the following contains a set of three fractions that are evenly spaced on a number line? Tick one box only. [Options:] (a) $\frac{3}{6}$, $\frac{3}{5}$, $\frac{3}{4}$; (b) $\frac{3}{4}$, $\frac{19}{24}$, $\frac{5}{6}$; (c) $\frac{4}{5}$, $\frac{5}{6}$, $\frac{7}{8}$; (d) $\frac{3}{4}$, $\frac{19}{24}$, $\frac{7}{8}$
8	I think of a number, multiply it by 5 and add 7 to get an answer of 52. If my number was x , what equation represents this? Tick one box only. [Options:] (a) $5(x + 7) = 52$; (b) $5x + 7 = 52$; (c) $7x + 5 = 52$; (d) $7(x + 5) = 52$; (e) $x = 52 \times 5 + 7$
9	A [cross-shaped] skydiving target is made from 5 squares the same size, as shown [diagram omitted]. The perimeter of the cross is 48m. What area in square metres (m^2) is covered by the cross?
10a	Write, in as much detail as you can, about the QR opportunities that could arise from this photograph [photograph omitted].
10b	Choose one of the QR possibilities you have identified and describe how you might encourage / assist a student to engage with that idea.
11	What are the main things that you would like to get out of your participation in this project?

For space and copyright reasons diagrams, graphs and photographs have been omitted. The photograph that was part of Item 10 depicted motorcycles at various stages of approaching and leaving a ramp to land on a truck mounted platform. This question was

designed to provide insights into the teachers' abilities to identify QR opportunities in a context that could be of interest to students and to explain how they might use it to develop QR (i.e. aspects of their PCK in a project-based learning context).

Procedure

The survey was administered towards the end of a one-and-one-half day professional learning workshop facilitated by two of the authors. The focus of the workshop was on identifying and raising teachers' awareness of contexts in which QR might be developed, finding resources to support their own and their students' learning of relevant mathematics, and building their confidence to identify QR opportunities and facilitate their students' exploration of them. Teachers completed the survey individually, using as much time as they wished, and were able to use calculators.

Results and Discussion

Mathematics Content Knowledge: Items 1a, 4, 6, 7, 8 and 9

No teacher answered all of these items correctly. One provided only correct answers but omitted Item 8. A further five provided just one incorrect answer, although three of these also declined to answer Item 8. One teacher provided three incorrect answers and another, four.

All but one teacher answered Item 1 correctly. The teacher who answered this item incorrectly provided a correct response only to Item 6; the reasoning employed to obtain the incorrect answers was unclear and involved various choices and/or manipulations of numbers in the items. Two other teachers also provided incorrect answers to Item 4. Both of these showed evidence of attempts to use proportional reasoning but, in one case, exhibited confusion about exactly which lengths were in proportion and, in the other, revealed an apparently unsuccessful attempt to set up the standard proportions. The first of these obtained the intuitively plausible result of 8.55m whereas the second arrived at 3.24m. One of two teachers who answered Item 6 incorrectly did not provide an explanation. The other correctly identified the proportional relationships between British pounds and Australian dollars, and between Brunei and Australian dollars but appeared to have manipulated these rather than using the graphs to solve the problem. Seven teachers made the correct choice for Item 7 and of the four who attempted Item 8, three were correct. Item 9 was answered correctly by two teachers with a further two showing correct working ending with an incorrect calculation of 16×5 . Another provided sound reasoning but included the statement that $4 \times 4 = 12$, while another correctly calculated the area of a single square but did not multiply by the number of squares. Two of the teachers provided no evidence of understanding the relevant concepts (perimeter and area). One of these added the statement, "I need a formula in front of me to work from."

Most of the teachers demonstrated understanding of mathematical ideas related to speed and time (Item 1), proportional reasoning (Items 1, 4, 6 and 7), reading graphs (Item 6), equivalent fractions (Item 7) and area and perimeter (Item 9). However, only one (whose only error was stating $16 \times 5 = 70$) was consistently successful with the mathematical items even though they required less sophisticated mathematics than some of their students would be studying. Some of the approaches used, such as manipulation of numbers in the problem, reflect those identified by Muir, Beswick and Williamson (2008) among Year 6 problem solvers termed "naive." Consistent with literature described by Mewborn (2001), most of the teachers lacked conceptual understanding of one or more topics, including area and

perimeter, and aspects of proportional reasoning. Many of these teachers appeared to have been taught mathematics in a procedural way resulting in a reliance on ill-remembered procedures. The fact that four of the eight did not attempt the algebra question (Item 8) is also instructive. This was the only question that was not attempted by all teachers and therefore appears to have been judged the most difficult by these teachers.

Pedagogical Content Knowledge: Item 10

The teachers' responses to Item 10 are shown in full in Table 2.

Table 2

Responses to Item 10 (motor bike picture)

Teacher	Response to Item 10a	Response to Item 10b
1	Height, distance, length, speed, cost blah, blah, blah [diagram of triangle with length and height labelled]	Too many to discuss!!!
2	Acceleration, speed needed to clear the jump; velocity and measurement, horizontal, vertical [arrows]; angle of ramp; weight of bike and rider; distance of run up needed	Plotting angles on a graph; Get them to walk up road and get weight of same type of motorbike - weigh themselves; get starting info. for equation needed.
3	Plotting trajectory and estimating landing. Estimating speed required; crash helmet strength? Why angle of jump doesn't equal path of bike, what other factor at play? Wind direction of flags and calculated speed of wind assist; Added weight of bike x height - is the mattress thick enough to cushion fall	Plot the trajectory and work with them - to find other examples of little cars and jumps
4	Where is this? Why are they doing it? What is your prediction of what is happening? Trajectory, curves, speed, power of bikes? Distances; 2 stroke motors, 4 strokes	1. What makes motor bikes go faster? 2. How much are motor bikes - new? Second hand? 3. If you were buying a second hand bike what would you need to consider? Could go on and on.
5	Degree; if you alter the degree of the ramp will the bikes land safely on the truck still? Weight of riders/bike, does weight of riders and bike affect the height the bikes need (how much air can they achieve?); how much money does a bike cost? How much is it to maintain? How many litres is the tank? How much does fuel cost per litre?	Fuel tank capacity and cost of fuel e.g., 45 litres x \$1.36 per L =; compare different fuel tank sizes and the types of fuel each motorbike uses - unleaded or diesel? What would be the most efficient and cheapest fuel? Money; duration fuel last for - TIME (graph and table results)
6	Bike speed, weight, height, angles	HARD!!! Speed and angle and what happens. No idea how to do this.
7	Angle, velocity, speed up and down, distance travelled, wind factor; What speed would you have to reach to be successful? What time would each rider need to wait before going? Wheel rotation, petrol consumption	Speed of leaving ramp compared to height, how could this be tested without having to actually ride? Can it be tested?
8	Speed required to get them safely to the landing ramp; speed vs. height to perform tricks; speed vs. height - trajectory; questions about power (cc's) of motorbike to accomplish.	Mass vs. power vs. speed to determine rate of fall; determine amount of force associated with stuffing up trick and landing on ground.

The responses to Item 10a show that the teachers had no difficulty identifying relevant quantitative aspects of the context depicted in the photo. However, responses 4, 5, and 7 also included less directly related aspects, including the location and reasons for the event, and costs involved in running a motor bike.

Although Item 10b asked teachers to choose just one possibility only responses 3, 6, and 7 did so. Responses 4 and 5 focused on tangential aspects involving money, running costs, and fuel tank capacity. Simulations were suggested in responses 3 and 7 but none provide clear evidence that the teachers had any knowledge of the mathematics involved in any of the QR opportunities that they identified.

In light of the difficulty posed by the elementary algebra item, the responses to the motor bike photo are unsurprising. The photo was chosen because of the opportunities we believed it afforded for exploring some of the more complex mathematics that many students in Years 9-12 are required to learn: for example quadratic relationships, rates of change, trigonometry, differential calculus, and vectors. The fact that none of the teachers mentioned any of these ideas is of concern. Nevertheless, the fact that most recognised the opportunity the context afforded to ask questions that could lead to these ideas means that there is a basis from which to work in helping the teachers to build the required mathematical knowledge.

Desired Outcomes of Participation in the Project: Item 11

Item 11 asked teachers about what they hoped to gain from the project. The desire most commonly mentioned (by 5 teachers) was for ideas for making QR engaging and interesting for their students, with four mentioning the importance of applied or real world contexts. Three expressed a desire to deepen their understanding of QR and three also mentioned improved confidence with respect to QR. Other aspects, mentioned once each, were links to the *Australian Curriculum: Mathematics* (ACARA, 2012), resources at an appropriate level to which students could be directed, appropriate language with which to talk to parents about QR, and deep understanding of students' learning of QR and appropriate pedagogy for individual students.

Conclusion

These initial data provide tentative support for the hypothesised model of knowledge development for teaching mathematics shown in Figure 1. Overall, the teachers in this study demonstrated a degree of mathematical knowledge and skill that warrants confidence in their personal use of mathematics. This is not, however, evidence of adequate knowledge to teach the subject (Beswick et al., 2011). Apart from one teacher, all had taught for no more than a single year in project-based contexts and confidence to teach mathematics (QR) in that context was lacking for most of the teachers (as evidenced by responses to Item 10b). Progress in mathematical understanding will need to take account of a parallel learning curve in relation to general pedagogy for project-based learning. It may, therefore, be more difficult and hence slower than in traditional contexts. Indeed, equipping teachers with the resources and dispositions to be able to learn new mathematics as required is likely to be a more realistic but no less challenging aim.

Completing the survey was a confronting but potentially valuable experience for some teachers. In addition to anecdotal evidence to this effect, one teacher, in response to Item 11, articulated a need to,

... really increase my capabilities when I do not have a formula to work from or an example to view first (I have not undertaken any maths or taught it above primary level for over 8 years) Tests make me feel overwhelmed and today I realised I have lost most of my understanding and I have identified the large emphasis I have placed on text books and learning via these.

None of the teachers in this project had any qualifications for teaching secondary mathematics and could not be expected to know or understand all of the relevant

mathematics content. It is also not reasonable to expect that a 1-year project can equip teachers with all of the knowledge that they need. The teachers expressed a need to learn more about QR as well as to develop their pedagogical knowledge in relation to QR in the highly student-focussed project-based learning context. The challenge for this project is to find ways to equip them to build their own mathematics understanding while simultaneously acquiring the knowledge to teach it and attending to the entire education of the students in their care.

Acknowledgements

This project was funded by University of Tasmania Research Enhancement Grant No. B0019931, with additional funding from the Education Faculty and the participating school. A/Prof. Helen Chick is also a member of the project team and contributed to the paper.

References

- Australian Association of Mathematics Teachers. (1997). *Numeracy = everyone's business*. Adelaide: AAMT.
- Australian Curriculum Assessment and Reporting Authority. (2012). *The Australian curriculum: Mathematics*. Retrieved from <http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10>
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it so special? *Journal of Teacher Education*, 59(5), 389-407.
- Beswick, K. (2007). Teachers' beliefs that matter in secondary mathematics classrooms. *Educational Studies in Mathematics*, 65(1), 95-120.
- Beswick, K., Callingham, R., & Watson, J. M. (2011). The nature and development of middle school mathematics teachers' knowledge. *Journal of Mathematics Teacher Education*. Retrieved from <http://www.springerlink.com/content/th22781265818125/fulltext.pdf>. doi:10.1007/s10857-011-9177-9
- Callingham, R. & Beswick, K. (2011). Measuring pre-service teachers' knowledge of mathematics teaching. In J. Wright (Ed.), *Proceedings of the 2011 annual conference of the Australian Association for Educational Research*.
- Chick, H. L. (2007). Teaching and learning by example. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, pp. 3-21). Sydney: MERGA.
- Doppelt, Y. (2009). Assessing creative thinking in design-based learning. *International Journal of Technology and Design Education*, 19(1), 55-65.
- Down, B., & Hogan, J. (2010). *Big Picture Education Australia: The school and network research framework*. Retrieved 12 March, 2012, from http://www.bigpicture.org.au/files/BP_SNRf_10May_new1.pdf
- Lam, S. F., Cheng, R. W. Y., & Ma, W. Y. K. (2009). Teacher and student intrinsic motivation in project-based learning. *Instructional Science*, 37(1), 565-578.
- Lee, H. K., & Breitenberg, M. (2010). Education in the new millennium: The case for design-based learning. *International Journal of Art and Design Education*, 29(1), 54-60.
- Mewborn, D. S. (2001). Teachers' content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Teacher Education and Development*, 3, 28-36.
- Muir, T., Beswick, K., & Williamson, J. (2008). "I'm not very good at solving problems": An exploration of students' problem solving behaviours. *Journal of Mathematical Behavior*, 27(3), 228-241.
- Quek, C.-L., Wong, A. F. L., Divaharan, S., Liu, W.-C., Peer, J., & Williams, M. D. (2007). Secondary school students' perceptions of teacher-student interaction and students' attitudes towards project work. *Learning Environments Research*, 10(3), 177-187.
- Steen, L. A. (2004). *Quantitative literacy: Why numeracy matters for schools and colleges*. Retrieved 24 June, 2004, from www.maa.org/features/ql.html
- Thornton, S., & Hogan, J. (2003). *Numeracy across the curriculum: demands and opportunities*. Paper presented at the Conference of the Australian Curriculum Studies Association. Retrieved from http://www.acsa.edu.au/pages/images/thornton_-_numeracy_across_the_curriculum.pdf
- Toolin, R., E. . (2004). Striking a balance between innovation and standards: A study of teachers implementing project-based approaches to teaching science. *Journal of Science Education and Technology*, 13(2), 179-187.

Stories From the Classroom: The Developing Beliefs and Practices of Beginning Primary Mathematics Teachers

Kathy Brady
Flinders University
<kathy.brady@flinders.edu.au>

This study examines the developing beliefs and practices of six beginning primary teachers. Their accounts reveal practices indicative of contemporary approaches to teaching and learning in mathematics. Additionally, a consistency appears to exist between the beliefs and practices of the beginning teachers, and the ideals for mathematics teaching formed during pre-service education. This is noteworthy as it is contrary to the literature that has described how the challenges of beginning teaching can have a significant negative impact on pre-existing beliefs and classroom practices.

Introduction

Mathematics teacher educators are currently faced with the task of preparing beginning teachers who can “break the cycle of tradition” in mathematics teaching and learning (Frid & Sparrow, 2007, p. 295). This is best achieved by encouraging pre-service teachers to reflect upon their existing beliefs and supporting them in forming newly developed ideals that they will take into their classrooms. However, the socialisation of beginning teachers can have a significant impact on the beliefs that will ultimately inform their practice. The literature suggests that the influences of pre-service teacher education are nullified in beginning teachers, and that they assume the beliefs and values dominant in the existing culture of the school (Brown & Borko, 1992; Zeichner & Tabachnick, 1981). If the cycle of tradition is to be broken, the newly developed beliefs of beginning teachers need to remain resilient even when they are challenged during the socialisation phase.

The focus of this study is to describe the developing beliefs and practices of a group of beginning primary mathematics teachers, and determine the factors that enabled, or challenged, these teachers in maintaining the ideals and visions for primary mathematics teaching that they formed during their pre-service teacher education.

Literature Review

Pre-service Teacher Beliefs about Mathematics Teaching and Learning

Recent research has found experiences as students in mathematics classrooms significantly influences the beliefs of pre-service teachers regarding mathematics teaching and learning (Lloyd, 2006; Sliva & Roddick, 2001). Typically pre-service primary teachers have negative attitudes and a lack of confidence regarding teaching mathematics as a result of disappointing and discouraging experiences learning mathematics (Brown et al., 1999; Drake et al., 2001). Such personal classroom experiences will also inform how pre-service teachers will ultimately teach mathematics. The participants in Brady’s (2007) study were clear that they would either call upon their prior positive classroom experiences to influence how they would approach teaching primary mathematics, or were adamant that previous negative experiences would not be repeated in their future classrooms.

Teachers' Beliefs and Classroom Practices

A number of researchers (Beswick, 2003; Cross, 2009; Nisbet & Warren, 2000) have investigated the relationship between teachers' beliefs and the classroom practices, which is often not clear-cut. Rather, the relationship is dynamic with each influencing the other (Nisbet & Warren, 2000; Thompson, 1992). Research also highlights the impact of the social contexts of teaching on teacher beliefs. Brown and Borko (1992) contend that personal biography is a strong factor in influencing the beliefs of beginning teachers, together with the impact of their teacher education programs, and their early experiences as teachers. However, Zeichner and Tabachnick (1981, p. 7) argue that the beliefs developed during teacher education are soon “washed out” of beginning teachers. The teacher socialisation literature suggests that when beginning teachers are confronted with the uncertainties of the real world of teaching they assume beliefs and attitudes that are dominant in the existing culture of the school and that they adopt the practices of more experienced teachers (Brown & Borko, 1992). However, in adopting the practices of experienced colleagues, beginning teachers may be accommodating their beliefs regarding ‘fitting in’ rather than actually modifying their teaching beliefs (Steele, 2001).

Beginning Primary Mathematics Teaching

Teaching mathematics is especially demanding for beginning primary teachers and the challenges confronted result from two broad factors. Firstly, although the literature emphasises the importance of beginning primary teachers having a sound mathematical content knowledge, most are typically generalist in their academic preparation and thus lacking in this regard (Kauffmann, 2002). The second broad challenge that beginning primary teachers encounter is managing the complex range of classroom contexts and practices that are required for mathematics to be effectively taught and learned (Frid & Sparrow, 2007; Sparrow & Frid, 2001).

Research Method

Narrative inquiry was selected as the method best suited to the purpose of this study: to describe the developing beliefs and practices of beginning primary mathematics teachers. A number of previous studies have successfully applied narrative approaches to exploring the development of beginning mathematics teachers (Brown, 2005; Kaasila, 2007; Rust, 1999). The participants were six beginning primary teachers mid-way through their second year of teaching (background information about the participants is summarised in Appendix 1). In this study, narrative interviewing was adopted as the means of data collection. The accounts were firstly examined to build a perception of the each participant's individual beliefs regarding of the teachers' role in primary mathematics, and learners and learning. A cross-case analysis of the transcripts then illuminated the commonalities and differences in the beginning teachers' beliefs and practices.

Findings and Discussion

Teacher's Role in Primary Mathematics Classroom

The participants all held firm views regarding the teacher's role in the primary mathematics classroom, some very similar and some distinctly different. One key role of the primary mathematics teacher that Jenny, Sarah and Verity identified was planning for learning. As Verity explained this firstly involves considering their students' existing

understandings: “I’m just a believer [in] making sure children start from where they’re at and move them on from there”. For Jenny this also involved planning for the current and future learning needs of her students: “I find it really hard, unless I have an aim, what do I want the kids to get from this lesson? ... I think you need to know where it’s going as well because then it makes it easier to make the connections”. As a special learning needs teacher Sarah similarly believed that a key aspect of her role was to determine prior understandings in order to plan for future learning:

I see it as meeting the student where they’re at, so working out a baseline understanding and the strategies that are already being used and then moving the student on by carefully selecting resources and manipulatives and concepts that’s within their grasp.

One key point of difference was their view of the teacher-as-expert role of the primary mathematics teacher. In Verity’s opinion, teacher expertise is essential:

... the best maths teaching is having a good understanding of what you’re teaching ... when I came here I thought, “Ah, Reception maths; it’s basic”. But I had to revisit things like base ten understanding and [other] things that I hadn’t necessarily thought about ...

Jenny had a similar belief because the occasions when she has felt less than expert have caused her some anxiety: “I think it is hard to teach if you don’t know what you’re teaching ... I think it is really important”. On the other hand, Donna did not appear to place much importance on the need for anything more than a basic level teacher expertise:

I think you’ve got to have a good understanding of the basic concepts, if I don’t know something I’ll just go and grab a book ... if you’ve got the capabilities to even know that basic concept or learn that basic concept yourself and you’re prepared, I don’t think it’s too bad. I don’t think you need a huge amount of expertise; I think you do need to understand the basic concepts of maths.

Hannah was very content to compensate for her lack of expertise by modelling being a learner to her students.

I go through the concepts myself and make sure that I’m 100% happy with it. If I don’t know exactly what I’m doing I get help before I go into the classroom to do it and make sure I am clear on that. If I come up against something in the classroom that I’m not 100% sure about I won’t lie to the kids ... [When they] ask me a question I’ll say “You know I really don’t know this, maybe we need to look into that to find out”, and I think they appreciate that I’m not a know it all.

Adopting a range of approaches to teaching mathematics is a facet of primary mathematics teaching that many of the participants readily embraced. According to Angela the classroom strategies she selected depended upon what she was teaching: “sometimes it’s whole class direct instruction and sometimes it’s just groups and I have different groups depending on what we’re doing”. On the other hand, Donna saw her role primarily as encouraging student-centred classroom interactions:

... sometimes they can explain it way better than I can explain it to them anyway then if we’re doing other group stuff, I usually try and do mixed abilities just so that there is someone there who knows and then they can help explain it to the others.

Whilst Hannah guided and facilitated her students’ learning by breaking tasks down into smaller steps:

I put a few questions up on the board ... only three or four on the board at a time and then do the first one, talk through it with them what to do and put step by step, usually in a different colour for the different steps so they can say “Oh I’m up to the red step, I’m up to the purple step”.

Sarah’s beliefs in this regard appeared to reflect the perspectives of a special learning needs teacher and were somewhat at odds with those expressed by Hannah, Donna and Angela. Accordingly Sarah believed that direct instruction ought to most definitely form part of the

primary mathematics teacher role.

Directed instruction provides [an] awareness and empowerment and the self-determination that maybe facilitation doesn't. If you're doing something for yourself, you may not always realise what you're actually doing, whereas if you have direction and instruction, that's in combination with the facilitation, then you can actually be made aware of what you're actually doing.

A final aspect of the primary mathematics teacher role that Verity and Sarah identified was the selection and use of teaching resources. Verity held some firm views about the types of resources that were most appropriate to her mathematics classroom, specifically, "I cannot think of a maths lesson I would not use manipulatives". Sarah, conversely, considered the use of manipulatives in an activity-based approach to primary mathematics teaching was not necessarily effective:

... you can have a hands-on lesson but it doesn't necessarily mean that it's going to be successful ... having stations can be great, and let's get into groups and it's all very well, but sometimes I have found that what is missing is a structure. You can't really say to a child, "Here you go, here's some manipulatives, here's an object, here's something to explore, now go away and explore it" ... to make it mathematical and to make it effective, I just think the structure and the idea behind what you're actually wanting them to connect with is a bit lost.

Learners and Learning

Unlike their views regarding the primary mathematics teachers' role, the participants' beliefs about learners and learning were in almost total accord. The key tenet evident was their appreciation that learners have differing mathematical abilities. Accordingly, to accommodate students with a broad range of mathematical abilities Angela attempted to differentiate the learning activities for her students: "I don't think it's fair to hand out a whole class the same work. Even if we do sheet work, the sheets I use are out of three or four different levelled books".

When managing the learning needs of less mathematically able students, both Donna and Hannah found ways of providing extra individual attention. Donna made sure "that they're the first people that I scoot around to spend that little bit of extra time explaining explicitly what they need to do". Hannah used a similar approach: "[When] the lesson is going through some group activities together, I'll go work with those that are having some difficulty with it". Additionally, Verity and Donna also acknowledged that the mathematically more able students have specific learning needs that need to be met. Donna's approach was to "make sure that I've always got something extra for them to do, not necessarily just more work but broader than more". Then again, it had not been easy for Verity to balance the learning needs of all of the students in her class:

I've had some children in here that are very, very bright, particularly in maths, and it's been a challenge to be able to make sure they're being extended and to be able to obviously cater for the children that do struggle.

Most of the participants, however, appeared to give far greater consideration to variances in students' mathematical ability, than the differences in their learning styles. Hannah, though, embraced the notion that learners have differing learning styles that needed to be accommodated by "just knowing how they prefer to learn, like some prefer to learn through the board, not on the floor doing games, whereas others really prefer just to be on the computer doing it". Likewise, Verity was cognisant of accommodating a range of learning styles through the use of various strategies:

... it's so fantastic to have an interactive whiteboard because I can cater to those visual learners. There are some children who learn best through auditory means so we talk about things ... it's just making

sure that you do incorporate a range of things, a range of activities ... some children learn best through doing, through the kinesthetic means [and] you give the opportunity for that.

A direct implication of their appreciation of learner diversity was that all participants identified that primary mathematics learning best occurs when learners are provided with a variety of learning activities. The point of difference between the participants was their expressed preferences for particular approaches to learning. For example, Verity stood by her convictions that learners learn best through inquiry, stating that she is “a big believer in inquiry, inquiry, inquiry”. She also held to the view that learning naturally occurs when learners are engaged in practical activities: “we use lots of manipulatives and things to be able to show understanding ... often I just start by letting them play and do stuff with junk material”. Whilst Angela valued the importance of the practical activities, she also appreciated the place of social engagement in mathematical learning:

... we’ve been doing a lot of hands-on stuff, actually doing that physically, I’m finding that they’ve really enjoyed it, they’ve all been motivated, they’re all joining in ... I do believe that kids work better when they’re doing social stuff together when they’re learning.

A final view regarding learners and learning in primary mathematics expressed by both Angela and Sarah was that learners needed to make their own connections and build their own mathematical understandings. For Sarah this involved “not providing them with information but providing them with the opportunity to give themselves knowledge because that’s how it’s going to be effective and meaningful”. Therefore, in Sarah’s view, if learners discovered and developed their own successful strategies “this needs to be celebrated and nurtured and acknowledged”. Angela was of a similar mind and firm in her view that learners “build that knowledge much more when they have the opportunity to do it at their own pace, and to actually explore what it means, and to go off on tangents. In a rare point of dissention regarding learners and learning, Hannah held a contrary position that teachers ought to “not assume kids can do it on their own and make them work independently”.

Sustaining the Vision

The literature suggests that the socialisation of beginning teachers in school contexts can have a significant impact on the beliefs they developed during their pre-service teacher education. There is no evidence in this study, however, that this has occurred. Many of the participants were able to recollect the visions that they had formed during their pre-service teacher education. For example, Donna recalled that her ideals for mathematics teaching included the use of group work, together with teaching mathematics explicitly, and that in her classroom mathematics would be fun. Hannah remembered that as a pre-service teacher: “I said I was going to [have] visual, hands-on learning space, and I think that’s because that’s what I wanted when I was at school”. Jenny recalled that her vision for primary mathematics teaching was “always being constructivist, always letting them make their own meaning”. Thus, the participants’ accounts of how they approached mathematics teaching and learning seemed to be in accord in the visions that they recalled from their pre-service education, and the findings of Brady (2007).

Donna recognised that her students’ enthusiasm for mathematics had been the most influential factor in implementing the ideals she developed as a pre-service teacher. She observed: “there’s a couple of kids in here that love maths ... so they’re excited about it which makes me more enthusiastic to teach it”. Hannah, on the other hand, considered that she has been able to realise her ideals because she perceived many of her students had a similar learning style to her own.

I was always a visual learner. I still am a visual and hands-on learner as opposed to I could read it in a book ... so I think that's what's made it easier for me to stick to it ... just noticing that my kids are a lot like what I was and still am.

Jenny remained firmly convinced of the efficacy of the ideals for mathematics teaching and learning that she developed as a pre-service teacher and recognised that this had been an important factor in sustaining her vision: "I know this is how I want my kids to learn, I know that this is my idea of teaching maths". Additionally, she acknowledged that positive encouragement from some colleagues has also contributed to her capacity to hold onto her vision for teaching primary mathematics.

Not all of them but quite a few share the same vision. If I didn't have a supportive team, and if I [had] someone else's vision pushed on me, that would make it harder ... but when they're supportive and encouraging it makes it really easy ...

Whilst the participants identified a range of factors that have enabled their pre-service visions to be sustained, they all also described the challenges that they have faced in doing so. For example, Donna believed that the most significant issue that has challenged her capacity to implement her vision for teaching primary mathematics was the degree of student diversity that she needed to manage. Angela identified other constraints: "you get to a classroom that's got a really crowded curriculum or a very tight budget and ... I'd love to be able to do more of that stuff". It had not been easy for Jenny, either, to hold onto her beliefs in the face of colleagues who used more traditional approaches.

... [it's] hard to keep it fun, hands-on, learning ... because it looks like play and it feels like play. You've got kids [and] things everywhere ... sometimes the Principal comes in and I can just see him thinking, "What's going on here?" In the class next door they're always sitting in their chairs, and they're all sitting up and they're writing beautifully and there's not even a sound and then there's my classroom and it's like a wild zoo and you think, "Am I doing it the right way?"

Verity felt parental expectations had impacted on her capacity to realise her ideals.

... parent expectations here are very high ... they expect their child to be fantastic, they expect their child, because they're coming from homes of doctors and pilots, to be successful ... you've got parent expectations and school expectations, it does get pretty full on sometimes, [and] you just go, "How am I going to do this?"

Finally, the most significant impediment that Sarah identified was one associated with the peculiarities of her role as a special learning needs teacher, namely a mismatch between her preferred approaches and those of colleagues.

... the barrier I find is when [what] I'm doing is different to what the classroom teacher is doing in terms of how to present a concept ... if I know that the teacher has presented a concept in one way and then my job is to help the ones that don't really understand it very well to understand it, I have to find the balance between presenting it in the way I want to present it to them ... that can be a real challenge because often, without sounding really negative and pessimistic, in the classroom it's not presented the way I would see it as ideally being presented.

Conclusion

The purpose of this study was to explore the developing beliefs and practices of six beginning primary mathematics teachers, and to determine the factors that enabled, or challenged, these teachers in maintaining the visions for primary mathematics teaching that they formed during their pre-service teacher education. Their accounts revealed classroom practices that indicated they are moving toward breaking the cycle of tradition in mathematics teaching and learning. Additionally, a high degree of consistency between the beliefs for teaching and learning that these beginning teachers had adopted, and the visions for primary mathematics teaching that they recalled having formed during their pre-service

teacher education was evident. This is a noteworthy finding as it is contrary to a body of literature describes how the challenges of school contexts, and the socialisation of beginning teachers, can have a significant impact on their pre-existing beliefs and, ultimately, their classroom practices. The participants' accounts reveal a range of factors that had supported their capacity to sustain their pre-service visions and ideals, and the challenges that they have faced in doing so. In this regard, further research is required to more closely examine the means by which beginning teachers sustain their ideals, and the manner in which they manage the challenges to their visions as they arise. In summary, the outcomes of this study provide a degree of optimism that beginning teachers are able to develop resilient views about mathematics teaching and learning, and that these views can be sustained in order that their visions can become a reality.

References

- Aldridge, S., & Bobis, J. (2003). Implementing beliefs, knowledge and practices: A beginning teachers' story. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, pp. 65-71). Geelong, Vic: MERGA.
- Beswick, K. (2003). Accounting for the contextual nature of teachers' beliefs in considering their relationship to practice. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, pp. 152-159). Geelong, Vic: MERGA.
- Brady, K. (2007). Imagined classrooms: Prospective teachers visualise their ideal primary mathematics classroom. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Hobart, pp. 143-153). Adelaide: MERGA.
- Brown, S. (2005). Emily and Rebecca: A tale of two teachers. *Teaching and Teacher Education*, 21, 637-648.
- Brown, C., & Borko, H. (1992). Becoming a mathematics teacher. In D. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* New York: Macmillan.
- Brown, T., McNamara, O., Hanley, U., & Jones, L. (1999). Primary student teachers' understanding of mathematics and its teaching. *British Educational Research Journal*, 25(3), 299-322.
- Cross, D. (2009). Alignment, cohesion, and change: Examining mathematics teachers' belief structures and their influence of instructional practice. *Journal of Mathematics Teacher Education*, 12, 325-346.
- Drake, C., Spillane, J., & Hufferd-Ackles, K. (2001). Storied identities: Teacher learning and subject matter context. *Journal of Curriculum Studies*, 33(1), 1-23.
- Frid, S., & Sparrow, L. (2007). Towards "breaking the cycle of tradition" in primary mathematics. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Hobart, pp. 295-304). Adelaide: MERGA.
- Kaasila, R. (2007). Using narrative inquiry for investigating the becoming of a mathematics teacher. *ZDM Mathematics Education*, 30, 205-213.
- Kauffmann, D. (2002). *A search for support: Beginning elementary teachers' use of mathematics curriculum materials*. Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA, 1-5 April.
- Lloyd, G. (2006). Beliefs about the teachers' role in the mathematics classroom: One student teacher's explorations in fiction and in practice. *Journal of Mathematics Teacher Education*, 8, 441-447.
- Nisbet, S., & Warren, E. (2000). Primary school teachers' beliefs relating to mathematics, teaching and assessing mathematics and factors that influence these beliefs. *Mathematics Teacher Education and Development*, 2, 34-47.
- Raymond, A. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education*, 28(5), 550-576.
- Rust, F. (1999). Professional conversations: New teachers explore teaching conversation, story, and narrative. *Teaching and Teacher Education*, 15, 367-380.
- Sliva, J., & Roddick, C. (2001). Mathematics autobiographies: A window into beliefs, values, and past mathematics experiences of preservice teachers. *Academic Exchange, Summer*.

- Sparrow, L., & Frid, S. (2001). Dilemmas of beginning teachers of primary mathematics. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, pp. 451-458). Sydney: MERGA.
- Steele, D. (2001). The interfacing of preservice and inservice experiences of reform-based teaching: A longitudinal study. *Journal of Mathematics Teacher Education*, 4, 139-172.
- Thompson, A. (1992). Teachers' Beliefs and Conceptions: A Synthesis of Research. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning: A project of the National Council of Teachers*. New York: Macmillan Publishing Company.
- Zeichner, K., & Tabachnick, B. (1981). Are the effects of university teacher education 'washed out' by school experience? *Journal of Teacher Education*, 32(3), 7-11.

Appendix 1

Background Information of Participants

<i>Name</i>	<i>Age</i>	<i>Profile</i>
Angela	40s	Commenced pre-service teacher education as mature age student. At time of the study on a 12-month contract teaching a Year 2/3 class at a large metropolitan government primary school.
Donna	20s	After a period of relief teaching, at the time of the study on a 12-month contract at government primary school teaching a Year 4/5 class.
Hannah	20s	At time of the study had a permanent position teaching a Year 4/5/6 class in a very small, remote R –12 school. During final undergraduate practicum placement at the school was offered the permanent position.
Jenny	30s	Following graduation accepted a one-term contract at a small, regional R - 12 government school. Had remained at school on contract and at the time of study, was teaching a Year 6/7 class.
Verity	20s	At time of the study held permanent position teaching a Reception class at metropolitan Lutheran primary school in a high SES suburb.
Sarah	30s	At time of study had a permanent position as the Inclusive Education teacher for the primary years at a large, Catholic R - 12 to girls' school located in a high SES suburb.

Teacher Subject Matter Knowledge of Number Sense

Hannah Briand-Newman

University of Sydney

<hnew7092@uni.sydney.edu.au>

Monica Wong

Australian Catholic University

<monica.wong@acu.edu.au>

David Evans

University of Sydney

<david.evans@sydney.edu.au>

Pedagogical content knowledge has been widely acknowledged by researchers and practitioners as a significant factor for improving student knowledge, understanding and achievement. Recently, the knowledge teachers need for teaching has expanded to include teacher horizon content knowledge, “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball, Thames, & Phelps, 2008, p. 403). This study uses a collective case study design, in which three Kindergarten teachers from Greig Heights Primary School participated in a professional learning and development program designed to enhance aspects of their teacher knowledge. This paper will provide an emerging description of the nature of teacher knowledge, and discuss the potential implications this has for catering for the needs of students at-risk of experiencing difficulties in acquiring early numeracy skills (i.e., number sense knowledge).

The acquisition of number sense has been recognised as a fundamental component of learning mathematics. Number sense, or the capacity to make sense of numbers and magnitude, is seen as foundational knowledge needed by children to understand and link quantities to our number system, numerical constructs and mathematical strategies. Children with number sense have developed “meaning” for numbers and their relationships. They are able to recognise the magnitude of those numbers and the “effects of operating” on numbers, and developing “referents” for quantities and measures (Sowder & Klein, 1993, p. 41).

Recent research has confirmed the importance of number sense of young children as they enter school and its association with students’ mathematics achievement in later years (Doig, McCrae, & Rowe, 2003; Gersten & Chard, 1999; Griffin, 2004; Griffin, Case, & Siegler, 1994; Jordan, Kaplan, Ramineni, & Locuniak, 2009). Students commence their first year of school with varying levels of number sense. Students with limited number sense are characterised as developing early counting skills or strategies, relying on procedural knowledge to meet the demands of numeracy in their environment, and having difficulty managing the demands of mathematical language (Evans, Strnadová & Wong, 2007). Number sense therefore needs to be recognised, understood, accommodated and taught effectively. Teacher knowledge of number sense is key to delivering quality early numeracy programs that prevent lasting difficulties with essential skills in our society.

Knowledge for Teaching

The foundational work of Shulman (1986, 1987) provides a conceptual framework and set of analytic distinctions to the knowledge needed for effective teaching. This framework incorporates seven categories of teacher knowledge across the curriculum: general pedagogical knowledge, knowledge of learners, knowledge of educational contexts, knowledge of the purpose of education, content knowledge, curriculum knowledge and pedagogical content knowledge.

Over the past two decades, research into teacher knowledge has focussed on two overlapping and interdependent domains: pedagogical content knowledge and content

knowledge. Good teachers know both content and “how to get it across” to the students they are teaching (Ball & Hill, 2009, p. 68). However, what this knowledge consists of, and whether other types of knowledge are needed is still being debated.

The knowledge teachers require to teach mathematics is “partly domain specific rather than a single construct of general factors such as a teacher’s overall intelligence, mathematical or teaching ability” (Hill, Schilling & Ball, 2004, p. 27). Hill et al. concluded that teacher knowledge of mathematics for teaching is multidimensional and consists of both general knowledge of content and more specific domain knowledge (i.e., specialised content knowledge). The latter is considered a deeper knowledge of specific content such as number and operations, including knowledge of student misconceptions, analysing unusual procedures or algorithms and providing explanations for rules.

The research of Ball et al. (2008) has culminated in the conceptualisation of the “Mathematical Knowledge for Teaching” model shown in Figure 1. This model refines the initial categories of pedagogical content knowledge and content knowledge proposed by Shulman (1986). Subject matter knowledge depicted in Figure 1 includes common content knowledge, specialised content knowledge, with the addition of horizon content knowledge. Ball et al. (2008) speculate that teachers need horizon content knowledge to be able to see the connections within the mathematics they are teaching. Further, it relates to the mathematics being taught and how it connects to the learning horizon, “...the sort of understanding that gives teachers peripheral vision for where they are and where their pupils are heading, to be conscious of the consequences of how ideas are represented, or the later development that is or possibly impeded – by decisions with the current work” (Ball et al., 2009, p. 98).

Common content knowledge on the other hand represents teacher’s knowledge of the mathematics they teach, including the use of correct mathematical terms and notation, and recognise when students give an incorrect answer. This type of knowledge implies that teachers “must be able to do the work that they assign to their students” (Ball et al., 2008, p. 399), including an insight into the mathematical understanding involved, and how this knowledge links to higher order material. Teachers need to recognise an incorrect answer, whereas knowing the nature of the error is specialised content knowledge.

Beswick, Callingham and Watson (2011) extended the work of Shulman (1987) to include measuring the impact of certain aspects of teacher thinking and behaviour. Teachers’ confidence to use and teach various mathematics topics together with their beliefs about mathematics and learning had a marked effect on student learning and mathematical achievement. Beswick et al. suggest that the existence of horizontal content knowledge involves knowing how “current” teacher choices of their knowledge and practice may “facilitate or obstruct” the future learning of their students (p. 2). Vale, McAndrew and Krishnan (2011) highlight the positive impact of knowledge of mathematics on the horizon, knowing where students’ learning of mathematics is heading in the high school context. One teacher stated, “I feel confident in what I am doing... I can identify where they are going with their learning and having that knowledge informs what I am teaching them...” (p. 206). A growing body of evidence supports the need for knowledge at the mathematical horizon, or horizon content knowledge. How teachers develop this knowledge is the subject of ongoing research, and professional learning.

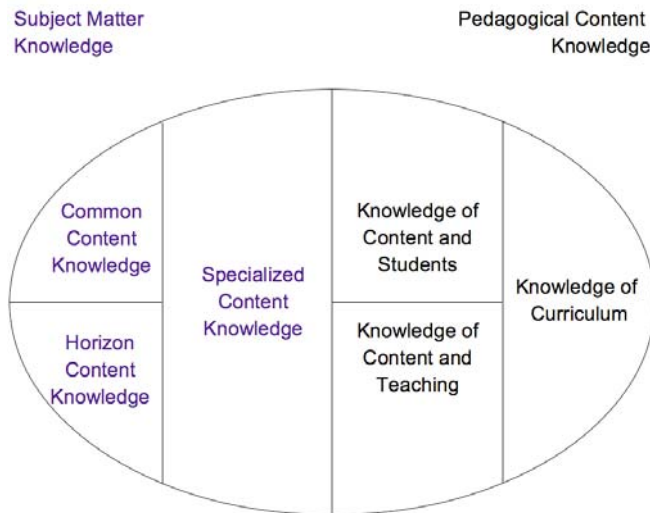


Figure 1. Mathematical knowledge for teaching (Ball et al., 2008, p. 403).

Professional Learning

There is consensus from teachers, researchers, educators and educational bodies that teacher knowledge needs ongoing measurement, refinement and revision so as to understand the many demands of teaching. However, the ongoing process of the professional development for teachers needs to provide, not only theoretical knowledge about teaching and what to teach, but a strong connection to practice (Neubrand & Seago, 2009 p. 211). Clarke and Hollingsworth (2002) argue that professional development programs in the past were based on a ‘deficit - training - mastery’ model. Teachers are “relatively passive participants”, and programs frequently fail to consider the actual process of teacher learning and what constitutes *teacher development and change* (p. 948). Guskey (2002), for example, suggested that opportunities for learning and growth be embedded within a teacher’s everyday professional life and practice, with the most effective learning experiences and processes of change found within their own classroom.

Studies of professional learning have focused on teacher content knowledge (e.g., Vale et al., 2011) and specialised content knowledge (Bray, 2011), drawing their data from a range of sources (e.g., teacher logs, observations, interviews). Hill, Rowan and Ball (2005), for example, investigated the links between teachers professional development and their students’ achievement using of teacher logs and end-of-year questionnaires and student test results. However, few studies have been conducted in the naturalistic environment of the classroom. Recent classroom based studies where teachers critically examining video footage of themselves teaching with a peer have been found to be a powerful method for creating deliberate reflection leading to changes in teaching approaches (Muir, 2010).

This research study focuses on the teacher within their classroom environment. The research explores teachers’ subject matter knowledge (i.e., common content, specialised content and horizon content knowledge) in the area of number sense. These findings will then be examined in regards to how this impacts on the acquisition of number sense by students identified to be ‘at risk’ of failure, low progress or mathematical difficulty during the first year of their school learning.

The Study

Study Design

Aware of the “idiosyncratic and individual nature of teacher professional growth” (Clarke & Hollingsworth, 2002, p. 947) a collective case study design, as shown in Figure 2, was employed in this study. It enabled each of the participating teachers to be considered separately, while exploring the impact of their knowledge and practice on the mathematics learning and achievements of children within their classroom. Although, the design enables the synthesis of the findings from the individual cases to gain an insight into the learning community as a whole, a comprehensive discussion of the learning community is not within the scope of this paper.

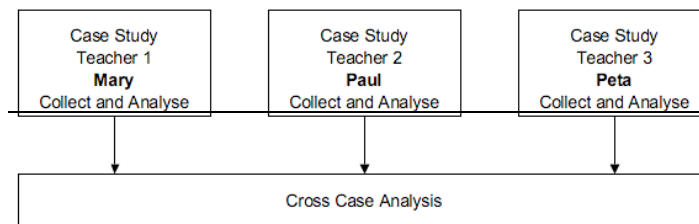


Figure 2. Collective case study design.

Greig Heights Primary School

Three Kindergarten teachers from Greig Heights Primary School, situated in Western Sydney, who were working collaboratively in the development of their mathematics programs and content of their lessons, were recruited to participate in the study. Teacher 1 has over 30 years teaching experience with the last 10 teaching at Greig Heights. Teacher 2 is a beginning teacher teaching Kindergarten for the first time. Teacher 3 has been teaching for over 20 years and the last 7 years at Greig Heights.

Two of the classes were dedicated Kindergarten classes; the third class was a composite Kindergarten/ Year 1 class. Prior to commencing this study, students were assessed on the *Number Knowledge Test* or NKT (Griffin, 2005). Students identified at-risk were unable to complete correctly at least five of nine questions administered in Level 1. Eleven out of 18 (61%) students in Teacher 1’s class, six out of nine (67%) students in Teacher 2’s class, and eight out of sixteen (50%) students in Teacher 3’s class, were identified. These students were able to identify the smaller of two given numbers (e.g., “Which is smaller: 8 or 6?”), but had difficulties with understanding the magnitude of numbers and single digit subtraction (e.g., “How much is 8 take away 6?”). Students operating at this level are considered to still require concrete materials to understand quantity and yet to transition to from concrete to abstract. They are unable to use a “mental counting line” inside their heads and/or their fingers to keep track...” (Griffin, 2005, p. 273).

Case Study Implementation

Each teacher was considered an individual case study; each case study was implemented in accordance with Figure 3. Prior to commencing the study, the first author met with the teachers to orientate them to the study. During the orientation, an overview and aims of the study were outlined. In addition, teachers were provided with an overview of number sense, and its importance with reference to recent research. Finally, a detailed explanation of the professional development cycle was presented to the teachers.

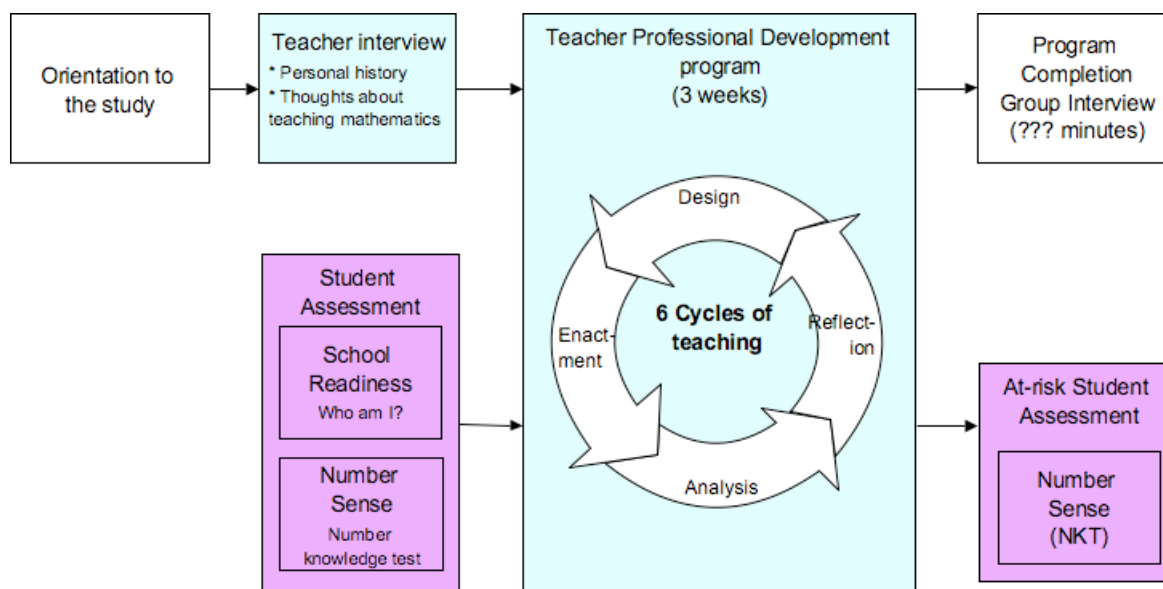


Figure 3. Overview of the study.

Teacher Professional Development

The case study commenced with the first author interviewing each teacher to gain an understanding of their personal and teaching history, along with the participant's thoughts about teaching mathematics in their classroom. Following these interviews, each teacher taught six lessons over three weeks with a focus on developing number sense. The teaching of each lesson occurred within a cycle that enabled the collection of a range of data to capture the complexity of classroom teaching and learning, while allowing a process by which a teacher's professional growth may be enhanced in the context of students' acquisition of number sense. The cycle comprised four elements:

1. Design – teachers collaboratively planned number sense lessons that reflected the Early Stage 1 learning outcomes of the *Mathematics K-6 Syllabus* (Board of Studies NSW, 2002). During each lesson counting to 20 (NES1.1) and the process strand of working mathematically were incorporated. Other lesson content included:
 - Lesson 1, 2- Groups and shares collections of objects (NES1.3);
 - Lesson 3, 4 - Combines, separates and compares collections of objects (NES1.2), and
 - Lesson 5, 6 - Describing and recording halves, encountered in everyday contexts (NES1.4).
2. Enactment: teachers were observed and video-taped teaching each lesson within their classroom, with a focus of teacher practice and interactions with students identified *at-risk*. At the conclusion of each lesson the first author completed a set of field notes to record anecdotal evidence noted during the lesson.
3. Analysis: the first author, who acted as mentor to all three teachers, viewed each video observation and reviewed field notes.
4. Reflection: the teacher and first author met to discuss and reflect on the outcome of each lesson in terms of the teacher's mathematics knowledge, and pedagogical knowledge and practice. These reflections were used as a lead in further reflection and refinement of teacher knowledge and practice, with a focus on students identified at-risk.

Results and Discussion

While the three teacher participants within this study presented as a strong collaborative team each brought with them variations in their awareness and insights into their role of teaching of mathematics, their professional knowledge and practice. An initial cross-case examination of the three teachers highlighted areas of convergence within the three areas of content knowledge.

Common Content Knowledge

All teachers noted the starting point for identifying common content knowledge was the *K-6 Mathematics Syllabus* (Board of Studies NSW, 2002). Teachers referred to specific content from the syllabus (e.g., “counting to 30 by the end of the year”), and their need to continually refer to gain knowledge they needed to teach. Teachers reported, however, that their content knowledge of teachers had been heavily influenced by participation in the professional learning program, *Count Me in Too*.

Teachers noted that they relied on engagement with colleagues to enhance content knowledge. Teacher 3, for example, stated she had developed her content knowledge from observing other teachers, claiming that her teacher training did not include mathematics content knowledge. This same teacher alluded to breadth and depth of knowledge needed to work students. She stated that students come to school able to count, however “they [students] don’t know what they are doing or why they are doing it”. She talked further about the need for students to develop relational understanding, “why 2 comes after 1”.

Horizon Content Knowledge

Each of the teachers revealed evidence of being aware of horizon content knowledge; initial data analysis indicates that this knowledge was embedded within their reflections, with teachers not being overtly aware of its function. Teacher 1, for example, recognised the need for numeracy in everyday life, particularly the mathematical domain of number (i.e., counting and estimation). Furthermore, she acknowledged that knowing what kids can do and “what they need to do next” was essential for teaching mathematics. Teacher 2, an early career teacher, when discussing his classroom practice reflected on his experience as a high school student. He noted he often thinks about what happens to maths in high school and the relevance of students’ early learning: “what did we need that for, but at the same time, we couldn’t have gone there if we didn’t have that, ... they are going to need it later”. This role of horizon knowledge was seen in his lesson videos. He was observed using specific language (e.g., “can you think of another way of doing this?”), prompting students to develop a deeper understanding on mathematics concepts, and hopefully be able to use this concept in future learning.

Teachers discussed how they struggled to grasp an understanding of where students were in their early mathematics development. This discussion viewed horizon knowledge in a different light to that noted by Ball et al. (2009), through examining where students had come from. Teacher 3, for example stated:

For the [kindergarten] kids that hadn’t picked it up you didn’t really know what to do with them. All teachers need that idea of a continuum and how they [students] develop so when they get a child that doesn’t know how to do something, they can look back and see the steps that they need to take to bring that child up...

She was concerned that teachers who don’t have an understanding of where kids have come from, in terms of knowledge development, are unable to address the specific learning needs

of that child. This view of content knowledge had specific implications for students identified as experiencing difficulty in developing an understanding of number sense.

Specialised Content Knowledge

A specific focus of this study was the teaching of number sense. It was found through initial interviews, and observations, that the three teachers were not aware of the role of number sense on longer term mathematical learning. Teacher's skills in focusing on specialised content were varied. Teacher 2 had just commenced targeting students who are exhibiting difficulties in early number. He worked with students individually using knowledge gained from professional learning sessions from *Count Me in Too*. Teacher 1 talked about how teaching the whole class was easier for her compared to small group work, which was usually hands-on. She found that when tasks were hands-on, it was harder to identify students' learning needs and recognise their learning achievements. Teacher 3, demonstrated during class evidence of specialised content knowledge. During an interview, she reflected, "A child's number sense is related to their understanding of the numbers themselves, they can recognise numerals, whether they know of quantities of numbers... counting ... and representing the numbers using various hand-on materials."

Conclusion

This cross-case examination of three Kindergarten teachers highlights the strength of obtaining direct classroom observations, alongside teacher reflections of their teaching, to gain an understanding of teacher knowledge. The breadth and depth of knowledge differed across teachers, and across the areas of content knowledge, specialised content knowledge and horizon content knowledge.

These levels of content knowledge have implications for all students, and the quality of learning they achieve. Teachers in this study used the required syllabus document to underpin common content knowledge. In these case studies, it was apparent that teachers were not aware of the role number sense (specialised content knowledge) plays in the development of early mathematics. Teachers provided evidence of the importance of horizon content knowledge, however, an understanding of where students have come from prior to arriving at school was seen as important. This understanding requires teachers to be able to identified misconceptions, and provide quality instruction that assists students acquire an understanding of key mathematical concepts. This latter understanding is key for students getting off to a good start in mathematics at school. Teachers who build key, conceptual links from existing knowledge to new, provide students with a base on which to become mathematically proficient (Kilpatrick, Swafford & Findell, 2001).

References

- Ball, D. L., & Hill, H. (2009). The curious - and crucial - case of mathematical knowledge for teaching. *Phi Delta Kappan*, 91(2), 68-71.
- Ball, D., Thames, M., Bass, H., Sleep, L., Lewis, J., & Phelps, G. (2009). A practice-based theory of mathematical knowledge for teaching. In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 95-98). Thessaloniki, Greece: PME.
- Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching. *Journal of Teacher Education*, 59(5), 389-407.
- Beswick, K., Callingham, R., & Watson, J. (2011). The nature and development of middle school mathematics teachers' knowledge. *Journal of Mathematics Teacher Education*, 15, 131-157.
- Board of Studies NSW. (2002). *Mathematics K-6 syllabus 2002*. Sydney: Author.

- Bray, W. (2011). A collective case study of the influence of teachers' beliefs and knowledge on error-handling practices during class discussion of mathematics. *Journal for the Research in Mathematics Education*, 42, 2-38.
- Clarke, D., & Hollinsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, 18, 947-967.
- Doig, B., McCrae, B., & Rowe, K. (2003). *A good start to numeracy*: Commonwealth Department of Education, Science and Training.
- Evans, D., Strnadova, I., & Wong, M. (2007). *Examining number sense: An exploratory study of students with additional learning needs*. Paper presented at the International Symposium Elementary Mathematics Teaching (SEMT '07).
- Gersten, R., & Chard, D. (1999). Number sense: Rethinking arithmetic instruction for students with mathematical disabilities. *The Journal of Special Education*, 33(1), 18-28.
- Griffin, S. (2004). Building number sense with number worlds: A mathematics program for young children. *Early Childhood Research Quarterly*, 19(1), 173-180.
- Griffin, S. (2005). Fostering the development of whole-number sense: Teaching mathematics in primary grades. In M. S. Donovan & J. D. Bransford (Eds.), *How students learn: Mathematics in the classroom. Committee on how people learn, a targeted report for teachers* (pp. 257-308). Washington: National Academies Press.
- Griffin, S., Case, R., & Siegler, R. (1994). Rightstart: Providing the central conceptual prerequisites for first formal learning of arithmetic to students at risk for school failure. In K. McGilly (Ed.), *Classroom lessons: Integrating cognitive theory and classroom practice* (pp. 24-49). Cambridge, MA: MIT.
- Guskey, T. (2002). Does it make a difference? Evaluating professional development. *Educational Leadership*, 59, 45-51.
- Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers, mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(2), 371-406.
- Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. [Article]. *Elementary School Journal*, 105(1), 11-30.
- Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: Kindergarten number competence and later mathematics outcomes. *Developmental Psychology*, 45(3), 850-867.
- Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learning mathematics*. Washington DC: National Research Council.
- Muir, T. (2010). Using video-stimulated recall as a tool for reflecting on the teaching of mathematics. In L. Sparrow, B. Kissane & C. Hurst (Eds.), *Shaping the future of mathematics education* (Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia, pp. 438-445). Fremantle: MERGA.
- Neubrand, M., & Seago, N. (2009). The balance of teacher knowledge: Mathematics and pedagogy. In M. Neubrand, N. Seago, C. Agudelo, L. De Blois, R. Leikin, & T. Wood (Eds.), *The Professional education and development of teachers of mathematics* (Vol. 11). New York: Springer.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1-22.
- Sowder, J. T., & Klein, J. (1993). Number sense and related topics. In Owens, D. T. (Ed.), *Research Ideas for the Classroom: Middle Grades Mathematics* (pp. 41-57). National Council of Teachers of Mathematics Research Interpretation Project. New York: Simon & Schuster Macmillan.
- Vale, C., McAndrew, A., & Krishnan, S. (2011). Connecting with the horizon: Developing teachers' appreciation of mathematical structure. *Journal of Mathematics Teacher Education*, 14(3), 193-212.

Mathematics as it Happens: Student-Centred Inquiry Learning

Chris Brough
University of Waikato
<cbrough@waikato.ac.nz>

Nigel Calder
University of Waikato
<n.calder@waikato.ac.nz>

This paper examines how mathematical understandings might emerge through student-centred inquiry. Data is drawn from a research project on student-centred curriculum integration (CI) that situated mathematics within authentic problem-solving contexts and involved students in collaboratively constructed curriculum. Participatory action research (PAR) was the methodology employed and mixed methods were used to collect data. The project took place in three primary school classrooms in New Zealand. The findings indicated that mathematics centred on real-life learning was highly engaging and that the measurement and geometric thinking explored went beyond New Zealand curriculum requirements.

Mathematics and statistics permeate everyday life - they are central to existing and contributing effectively in society. There is a diverse range of everyday practical situations where mathematical thinking is applied, while mathematical and statistical thinking are also applied in other learning areas (Ministry of Education, 2007). Often the application of mathematical knowledge is considered as something that is situated at the end of a learning process, as an application of learnt skills, but it should also be an aspect of the initial engagement (Freudenthal, 1968). Rather than commencing with certain abstractions or definitions to be applied later, some mathematics educators contend that the learning “must start with rich contexts that require mathematical organisation or, in other words, contexts that can be mathematised” (van den Heuvel-Panhuizen, 2010, p. 5). When learners engage in the investigation of a problem, interpretation influences the nature of that engagement. The perspectives that underpin those interpretations are conditioned by socio-cultural experiences (Gallagher, 1992) while the interpretations influence the ongoing understandings that emerge (Calder, 2011). Others contend that understanding emerges when students learn in ways that “engages one’s identity on a meaningful trajectory and affords some ownership of meaning” (Wenger 1998, p. 270). Problem-solving contexts that relate to real-life situations enable this sense of ownership to evolve (Lowrie, 2004). It can also be argued that bringing student’s naturalistic out-of-school experiences into more traditional settings has positive effects on problem solving. Meanwhile, Lowrie and Clancy (2003) maintained that student problem-solving was enhanced when traditional settings were complemented by student out-of-school experiences.

Studies have identified that students’ motivation and persistence with tasks have increased when they engaged in authentic problem-solving situations (Nolan & McKinnon, 2003; Vars, 1997). Other researchers maintain that replicating situations in the classroom that resonate with out-of-school mathematical processes will promote mathematical thinking in real-life situations (e.g., Bonotto, 2002). Meanwhile, the learning environment assumes some primacy and will need to differ substantially from traditional settings for students to connect real-life and problem-solving contexts (De Corte, Verschaffel & Greer, 2000). They advocated that tasks need to be authentic, and as with real problems, that they be diverse and open, enabling multiple approaches and solutions. Others contend that problem solving should involve simulations of real-life community experiences where the mathematical thinking influences the solution of everyday problems (Lesh & Harel, 2003).

Educators are also suggesting that student-centred inquiry, based on problems the students pose, utilise authentic learning contexts and lead to a strong sense of student ownership, enhanced student engagement and understanding, and motivation to learn (Beane, 1997, Brough, 2008, Dowden, 2010). The New Zealand curriculum advocates an inquiry approach and recently classroom environments have emerged that foster authentic inquiry to stimulate high-level thinking. Several new schools have been built and structured specifically to enhance student-centred inquiry learning. Student-centred curriculum integration (CI) encompasses similar pedagogical principles including student-negotiated curriculum and investigation of authentic problems.

Student-centred CI is a democratic teaching approach where relevant and meaningful contexts are central to curriculum design. Students pursue questions, issues or inquiries that are of genuine interest and curriculum is collaboratively co-constructed (Beane, 1997). The power-sharing pedagogy heightens student ownership and raises relevance for students, as they are fully involved throughout the learning process from the initial planning stage through to assessment. Subject area knowledge is employed in order to pursue pertinent issues of inquiry; for example, in mathematics children may explore strategies for calculating money or receive explicit teaching on place value and decimals to determine the costs of going on their class camp. Subject material is repositioned contextually and learning is strengthened, as children are motivated to acquire the skills and knowledge necessary to solve relevant problems. A substantive body of research supports the efficacy of student-centred CI with heightened achievement reported, student positivity and enhanced engagement (Beane, 1997, Brough, in press, Nolan & McKinnon, 2003, Vars, 1997). In addition, the nature of inquiries involves many skills that are not measurable through standardised tests such as the ability to negotiate, create knowledge, think creatively and critically, and work together for the common good. These kinds of competencies align with the New Zealand Curriculum (Ministry of Education, 2007). Contemporary integrative theorist James Beane (1997) advanced these notions further, theorising a curriculum design that used student-generated learning contexts, and involved students and teachers collaboratively constructing curriculum. Meanwhile, others have found that using digital pedagogical media for authentic mathematical inquiry enhances students' engagement (Salsovic, 2009; Calder, 2011).

This paper draws from a participatory action research (PAR) project that explored the implementation of the principles and practices of student-centred curriculum integration in the primary-school setting (Brough, in press). It examines the mathematics learning that was engaged with through the student-centred inquiry approach taken, and contends that the authentic engagement led to powerful learning opportunities.

Methodology

PAR was the methodology of choice, as its emancipatory research process was in keeping with the democratic pedagogy of student-centred CI. Hence, it enabled participants to explore a range of self-determined democratic inquiries. Mixed methods were used to collect data. These included: semi-structured interviews, focus group meetings, informal discussions (electronic and face to face), naturalistic observations, work samples, and photographs. The three teacher participants were early in their teaching careers, while the age range of the children spanned from 5 years of age to eleven. A case study approach was used to document this nine-month inquiry. The principles and practices that the participants explored included how to create more democratic learning environments, the use of relevant learning contexts and collaboratively constructed curriculum. While a broad range of disciplines were required to pursue the meaningful questions arising during the project, the

focus of this paper is the mathematics that occurred. While two scenarios will be the discussed, the first is examined in more detail.

In all three schools, teachers were required to implement mathematics using the numeracy project. Children were grouped based on data attained during a diagnostic interview that assessed children's strategic and mathematical knowledge. The mathematics that evolved naturally as part of student-centred CI took place beyond these times through inquiries triggered by students' questions, classroom conversations, or incidents from within the classroom or the community.

Results and Discussion

The first scenario to be discussed arose as part of an open discussion that sought students' opinions on how they might contribute to decision-making and how their learning might be enhanced in class. After several iterations of discussion and review, a suggestion emerged that learning would be far more effective in a bigger classroom and that the children's small, prefabricated classroom should be extended. Predictably the children wanted to begin designing classroom extensions. Subsequently, conversations led to wanting to calculate the volume of their own classroom suggesting they could then compare it to the larger classroom next door. Over the subsequent week the propositions were explored using both measurement and geometrical thinking.

Teacher (Toni): We started by talking about the space and how to use space in a classroom. We talked about how different shapes might work. They came up with exploring circles, rectangles, squares and other shapes, including irregular ones. I asked them to work in groups to consider different shaped classrooms. How would you make sure you could fit everything in? What gives you the most room for the size? What shapes worked best for doors and the furniture we have? They explored different shapes and drew sketch plans. One group even had a donut shape with the doors opening to the outside. When they wanted to open it into the middle they realised it wasn't going to work very well after all. They saw that there was no flow and the space would be too confined without a hallway off it.

From their initial conceptual drawings, the students soon determined the need to envisage their plans in 3-dimensional space.

Toni: They drew floor plans and explored as 2D. They then set the plans out with multi-link cubes before exploding them up into their height. They built them up with the multilink cubes. They scaled them up with a centimetre cube block representing a cubic metre.

This led them back to the actual structure of their room and the problem of it being too small. They decided to measure the room. One group's discussion initiated an approach to measuring the room.

Sam We need to know how big the room really is now.
Fran How are we going to measure it?
Sam Let's get a ruler and measure the length of the walls.
Tineke And to the ceiling

One student was concerned with making sense of the information.

Sam But what does that mean? How does that show us the space?
Tineke Well we could find the area of the floor and see how much space we have to put stuff.

The nine and ten-year-old students were unfamiliar with the concept of measuring in square metres. The teacher challenged the class to work in small groups and make one

square metre out of paper. This was used to estimate and measure different areas of the school. It helped the children gain an understanding of size and discover how length can be multiplied to determine area.

Toni: The students started with some measuring in groups. They found some lengths for the room, 10m long and 8m wide.

Toni talked about 2D and they identified things that were 2D. Someone suggested the carpet. What about the height of the carpet? The students considered making the 2D things into 3D.

Finn We could turn the flat 2D stuff into things that you could sit on or put stuff on.

Sophie Yeah sort of measure it up to 3D

The teacher prompted them.

Toni How can we use this information? How do we make it into 3D?

The students continued:

Sam What about making a cube like the multilink but bigger.

Ani We could make paper cubes, like the multi cubes, scaling up.

Hine We could roll up the paper to make the sides.

They make the cubes.

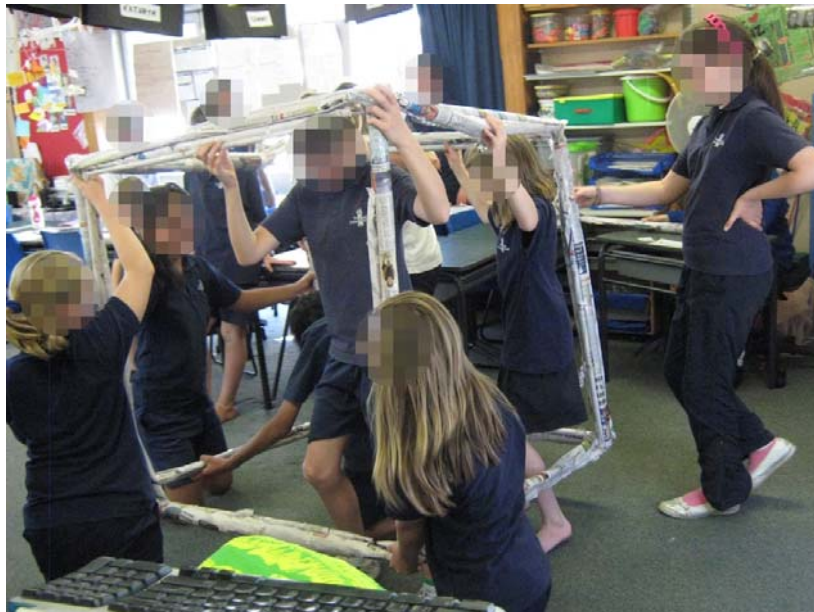


Figure 1. Students measuring with model of cubic metre.

They measure up the size of the room with the paper cubic metres. They rolled the cubes across the floor and marked off how long the classroom is. They started to measure up the wall as well to get the volume. Their classroom had a pitched roof, which presented an additional measuring challenge. They ran into a problem when they hit the sloped angled roof. The children were stuck as to how to solve this problem.

Eli How are we going to measure that?

They tried things and made lots of suggestions:

Ben We could bend down the sides.

Matt Yeah, flatten the sides a bit.

Ash Maybe we could make some other shapes.

Ben People are measuring angles all the time. How are we going to get to it?

Chris We could measure all around the outside.

They tried modelling it with bits of paper to show the profiles. Eventually the teacher gave them some scaffolding to change their approach a bit.

Toni What about cutting the squares down the middle.

The teacher asked them to consider halving the apex triangle shape and seeing what they noticed to determine if that would help them to solve the problem. They went away in groups to discuss the challenge. Several groups cut up the triangle and worked out that if the two halves of the triangle were shifted this would make a rectangular shape which helped them to calculate the total linear area. They drew more squares and tried cutting them down the middle. They physically did it and moved the bits around. The children who generated this solution taught the other members of the class who were still grappling with the challenge.

The next-door class became interested in this activity and decided to measure their classroom using the same approach. The children made an initial estimation then working with the other class they determined the volume of both rooms in cubic metres. The children multiplied the length, width and height from their linear measurements and found the room next door was 11-12 cubic metres bigger. The teacher supplied additional explicit teaching to the children who required additional support to calculate the challenge. Another exploration that evolved from this was: How much water is there in the swimming pool? They decided to measure the sides to get the surface area. They then dropped a stick down into the bottom and marked off the wet mark to get the depth.

In the interim, students had been creating flow charts of the building process they anticipated would be required to bring their proposal to fruition. They also continued to work on the classroom extension plans that included adding on to either side of the building, or both, and second storey options. Recognising they would need help and advice the class arranged to consult with a builder and sought the expertise of a member of the city council who immediately offered to visit.

The councillor presented further mathematical challenges as building costs per metre were shared for single and two storey options. Additional costs were a building permit, architectural plans and any appliances required. The children inquired about wheel chair ramp regulations and sizes of windows to maximise sun and minimise heating costs. The councillor explained about heating cubic metres. The children raised the problem of a protected historic tree situated on one side of the classroom that led to a discussion on the length of roots and drip lines for trees. The children moved outside to estimate and step out the trees distance from the class. Subsequent to this visit the children calculated the costs of different options. They adapted their designs on the tree side of the classroom and reconsidered some of the more ambitious and costly designs.

While budgetary constraints prevented the extension becoming a reality, a large covered veranda has since been added to the classroom. The children reported that they enjoyed the project and the learning that took place. This context extended beyond the mathematics discussed including cross curricula links to literacy, technology and science. Student-

centred CI provided an environment where the children were encouraged to make suggestions and consequently these children identified a learning context that they perceived to be highly relevant to their needs and learning. A high level of motivation and engagement was witnessed throughout the inquiry. In this instance, introducing children into considering volume extended children's mathematical knowledge further and helped children to understand how higher studs and roof apex can give the illusion of space and that additional cubic space does not always result in any additional floor area.

It is likely that the high level of student engagement in the classroom extension inquiry was a result of student involvement in determining the learning context. Student-centred CI was designed with middle-school students in mind (Beane, 1997). The questions or concerns the students have about themselves or their world are organised into similar areas of inquiry and pursued as learning themes. In the primary school setting, younger children are often more concerned with the here and now and consequently issues of immediate interest can be used as potential learning themes (Brough, in press). Themes may be initiated from small incidents that arise as part of daily programme or a community event that has attracted student interest. The second scenario to be discussed arose from such a situation when a drain digger arrived to create a large sump hole to rectify a drainage problem on the field. Spotting the arrival of a digger immediately captured the five-year-old children's attention. Students began asking questions and wanted to observe what was about to happen in the playground. Sensing an excellent learning opportunity the teacher escorted the class out to the field. The children began raising questions and making comments "How will they get the digger off the truck? How deep do you think the hole is? How will they make the hole bigger? How will they get the heavy pipe into the hole? If he digs anymore he will end up in space. Look the dirt is changing colour as he digs." In the initial phases of the project the teacher had focussed on establishing a classroom climate that was democratic, and empowering. She had actively sought student contribution and encouraged curiosity. The teacher achieved this by asking open-ended questions, not solving problems for children, and having high expectations of their independent research. Consequently, the children felt comfortable asking questions, making learning suggestions and solving problems for themselves.

The children took photographs of the digger making the hole and asked if they could talk to the driver so he could provide answers to their questions. The driver told the students the hole was to be four metres deep. He discussed what was happening and why, introduced the children to what porous meant and discussed the layers of the earth. While safety issues prevented students peering fully into the hole as they had requested, the children suggested a viable alternative was to ask the driver to photograph the hole. The children moved to the courts to work out how deep four metres is. The students made an initial estimation then decided they could use metre rulers to work out the problem. The students began measuring the four-metre depth but there were only three rulers, which presented a conundrum for the five-year-olds. Eventually, they determined they could reposition the first ruler to the end and consequently they successfully measured the holes depth. During the investigation one student had a moment of enlightenment when they announced: "Hey did you know these rulers are all the same length." The students lay down end on end and used a variety of other non-conventional measures such as their hands and body length to determine different ways to estimate then measure. They also poured water over various objects to understand what things were porous and what were not. The mathematics learning was invaluable in terms of applying problem-solving skills to a meaningful context and developing estimation and measurement skills, including using conventional and non-conventional measures.



Figure 2. 5-year-olds measuring 4m.

Conclusions

This paper examined mathematics as it emerged through student-initiated inquiries. During this inquiry the students had explored and effectively used two-dimensional representations of 3D objects, enhanced their understanding of area and volume, and made connections between the various measurement representations. The mathematical thinking and knowledge gained through the class extension investigation involved students learning how to apply multiplicative strategies for area and volume, and how to measure using metric units for length, area and volume. Further, the children needed to calculate various building costs that involved students having to apply additive and multiplicative strategies.

It was found that children were highly motivated and engaged with conceptual and mathematical understandings that extended well beyond curriculum year level guidelines, particularly in measurement and geometry. Links were made to other curriculum areas while the key competencies were also achieved including: thinking and problem solving, managing self, and participating and contributing (Ministry of Education, 2007).

Teachers in the project pursued issues of substance that arose on a daily basis and were negotiated alongside students. Teacher flexibility was essential in order for learning to be genuine, on the spot, and authentic. It required teachers to break away from more traditional structured mathematics programmes that commence with abstractions and instead use meaningful contexts that can be mathematised (van den Heuvel-Panhuizen, 2010). Teachers required democratic and empowering pedagogical understandings combined with excellent curriculum content knowledge and a high expectation of students.

In student-centred CI teaching takes place “just in time” as students require skills to solve particular problems. This enables learners to identify the purpose for the acquisition of skills. Teachers require excellent content knowledge to recognise the potential of children’s inquiries and to extend children into new subject material positioned within a meaningful learning context. While teachers of student-centred CI are required to assume a more facilitative and empowering approach to teaching, extending children’s mathematical thinking into areas they may not have considered is frequently required and explicit teaching is still vitally important.

This study was small in scale, and therefore it is difficult to draw generalised conclusions. Nevertheless, it serves to build upon emerging research on the benefits of contextualised learning in mathematics. It also demonstrates that content knowledge is not

lost when using student-initiated learning contexts. Instead, it is repositioned within highly engaging learning contexts, while understanding is facilitated and enhanced.

References

- Beane, J. (1997). *Curriculum integration: Designing the core of democratic education*. New York: Teachers' College Press, Columbia University.
- Bonotto, C. (2002). *Suspension of sense-making in mathematical word problem solving: A possible remedy*. Proceedings of the 2nd International Conference on the Teaching of Mathematics. Crete, Greece: Wiley & Sons Publishers.
- Brough, C. (2008). Student-centred curriculum and The New Zealand Curriculum. Set. *Research Information for Teachers*, 2, 16-21.
- Brough, C. (in press). Implementing the democratic principles and practices of student-centred curriculum integration in primary schools. *Curriculum Journal*.
- Calder, N.S. (2011). *Processing mathematics through digital technologies: The primary years*. Rotterdam: Sense Publishers.
- De Corte, E., Verschaffel, L., & Greer, B. (2000). Connecting mathematics problem solving to the real world. *Proceedings of the International Conference on Mathematics Education into the 21st Century: Mathematics for living* (pp 66-73). Amman, Jordan: The National Center for Human Resource Development.
- Dowden, T. (2010). Curriculum integration in Aotearoa New Zealand: Rediscovering the potential of student-centred curriculum design in the middle years. *Australian Journal of Middle Schooling*, 10(2), 4–10.
- Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics* 1. 3-8.
- Gallagher, S. (1992). *Hermeneutics and education*. New York: State University of New York Press.
- Lesh, R., & Harel, G. (2003). Problem solving, modelling and local conceptual development. *Mathematical Thinking and Learning*, 3(2 & 3), 157-189.
- Lowrie, T. (2004). In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010*. (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia). Townsville Sydney: MERGA.
- Lowrie, T., & Clancy, S. (2003). *Multimodal texts: Numeracy development in naturalistic learning contexts*. Refereed Proceedings of the Annual Australian Association for Research in Education Conference, Auckland, New Zealand.
- Ministry of Education (2007). *New Zealand Curriculum*. Wellington: Learning Media.
- Nolan, P., & McKinnon, D. (2003). Enhancing the middle in a New Zealand Secondary school: integration, experiential learning and computer use. *International Journal of Educational Reform*, 12(3), 230-243.
- Salsovic, A.R. (2009). Designing a WebQuest. *Mathematics Teacher*, 102(9), 666-671.
- Van den Heuvel-Panhuizen, M. (2010). Reform under attack – Forty years of working on better mathematics education thrown on the scrapheap? No way! In L.Sparrow, B.Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia*. Freemantle: MERGA.
- Vars, G. (1997). Effects of integrative curriculum and instruction. In J. Irvin (Ed.), *What current research says to the middle level practitioner* (pp. 179-186). Columbus, Ohio: National Middle School Association.
- Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge, UK: Cambridge University Press.

How Does Teacher Knowledge in Statistics Impact on Teacher Listening?

Tim Burgess
Massey University
<t.a.burgess@massey.ac.nz>

For teaching statistics investigations at primary school level, teacher knowledge has been identified using a framework developed from a classroom based study. Through development of the framework, three types of teacher listening problems were identified, each of which had potential impact on the students' learning. The three types of problems are described, with examples from the classroom along with links to the teacher knowledge framework. It is concluded that teacher knowledge is a necessary condition for avoiding such listening problems.

Teacher listening is known to be a critical factor for successful classroom discourse. The relationship between teacher knowledge and teacher listening, and how the former impacts on the latter, is an important aspect for investigation, and has been the focus of research in mathematics education. Less is known however about the relationship between these two aspects in statistics education. In this paper, teacher listening in relation to teaching statistics through investigations at the primary school level is examined, using data from a classroom based study into teacher knowledge. A number of 'themes' in relation to teacher listening are discussed. Implications of these listening practices are considered in relation to students' learning of statistics.

Literature Review

The importance of establishing effective classroom discourse for enhancing mathematics learning is well recognised. Walshaw and Anthony (2008) indicate that such research in the mathematics education domain is more recent than research about appropriate pedagogical approaches for establishing effective classroom discourse in a generic sense. Teacher practices in relation to listening to students have been found to have an noticeable impact on classroom discourse.

Various aspects of teacher listening have been explored by researchers. For example, Anghileri (2006) identifies types of listening, while Davis (1997) elaborates on levels of listening. However, for teachers to be able to advance students' understanding appropriately, they need to be able deal with students' incorrect responses. This has, according to O'Connor (2001), rarely been the focus of research. Davis's (1997) evaluative listening refers to the teacher being able to compare a student's response with a preconceived answer or standard. With this type of listening, Davis suggests that the teacher is not really interested in what the student is actually saying and is therefore at the lowest level of listening with regard to developing effective classroom discourse. In comparison with evaluative listening, Davis's second level is interpretive listening, in which the teacher makes a more active attempt to connect with and make sense of the student's response, while the highest level which is hermeneutic listening, in which there is a negotiated and participatory interaction between the teacher and student. In the situation of evaluative listening, Davis does not however give consideration to why the teacher does not actively evaluate the student's response. Mercer (1995) describes teacher talk in terms of eliciting comments from students, responding to students, and recapping on students' comments. As part of responding to students, Mercer refers to a range of possible ways of responding such

as confirming, repeating, paraphrasing or reformulating, elaborating, rejecting, and ignoring. Just as Davis (1997) gives no explanation for why a teacher might not actively evaluate a student's response, Mercer does not suggest any reason why a teacher might ignore a student's response.

Types of 'hearing' have been identified and described by Even and Wallach (Even & Wallach, 2003; Wallach & Even, 2005). The types of hearing are: over-hearing - hearing more than the students are actually saying; under-hearing - missing some of what students say; and compatible hearing - making sense of and connecting with what students say; non-hearing - missing the whole message of the student; and biased hearing - the amount heard depends on who is saying it. Additionally, they consider the reasons behind the various types of hearing. One of numerous factors they give for these different types of hearing is teacher knowledge. In a similar vein, Boaler (2000) argues that the practices within a classroom cannot be separated from the teacher's mathematical content knowledge. O'Connor (2001) suggests that the complexity for the teacher in trying to understand what a student is saying, in keeping track of the sequence of contributions from students, and while thinking about how to respond all within a 'conversationally appropriate' two or three second interval, impacts significantly on the teacher's decision making with regard to a response to the student. So, the type of response that a teacher may give to a student's comment may not be the best or most appropriate because of this decision-making under pressure.

To develop students' mathematical understanding requires teachers to scaffold the students' learning through making connections with students' ideas. This clearly is dependent on teacher listening and teacher knowledge. Such scaffolding includes restructuring (Anghileri, 2006), which involves the teacher making contact with the student's understanding and being able to move it forward. If this does not happen, such as when a teacher does not seek further clarification from a student following his or her contribution to a discussion, it can be referred to as 'pseudo-interaction' or 'by-passing' (Bliss, Askew, & Macrae, 1996). In this situation, the conditions for scaffolding are present but not noticed by the teacher, and consequently, no real interaction occurs between students and teacher. Unfortunately, Bliss and colleagues do not suggest why the teacher does not help the students clarify their thinking. O'Connor (2001), however, posits that a lack of comfort when attempting to make sense might be experienced by the teacher, the student who responded, and the other students, which could contribute to the teacher's failure to seek clarification from the student.

The teaching of statistics is known to present some unique challenges attributable to the nature of statistics and statistical thinking in comparison with mathematics and mathematical thinking. One of the main reasons relates to the deterministic nature of mathematics in comparison with the uncertainty and context-laden nature of statistics (e.g., Cobb & Moore, 1997; delMas, 2004; Moore, 1990; Pereira-Mendoza, 2002). To consider teacher knowledge for statistics requires a different approach from that for other areas of mathematics. One such approach resulted in a framework for describing teacher knowledge through combining statistical thinking (Wild & Pfannkuch, 1999) and components of teacher knowledge, such as described by Ball and colleagues (Ball, Thames, & Phelps, 2005; Hill, Schilling, & Ball, 2004). This framework (Burgess, 2006) provides a way of examining teacher knowledge in statistics, and helps make sense of classroom situations involving teacher listening.

This paper therefore explores the following question: How does the knowledge profile of a teacher relate to the teacher listening to and interpreting students' statements in the classroom?

Method

The research for this paper was conducted in four upper-level primary school classrooms (Years 5-8; either single year level or combined year levels). The data consisted of: videos from a sequence of four lessons in each classroom; and recordings of stimulated recall interviews with each teacher following each lesson, using an edited video of ‘episodes’ from that lesson (between 7 and 29 minutes duration for the edited ‘movie’). Ethics approval and consent from all participants (students, parents/caregivers, and teachers) was gained. The lessons involved statistics investigations based on one unit of work from a website, and used multivariate data (four variables of category data or a mix of category and numeric data).

The analysis of the data involved using the teacher knowledge framework for statistics to identify four components of teacher knowledge from the Ball and colleagues (Ball et al., 2005; Hill et al., 2004) - common knowledge of content, specialised knowledge of content, knowledge of content and teaching, and knowledge of content and students, and the statistical thinking components from Wild and Pfannkuch (1999). A teacher knowledge profile was obtained for each of the four teachers. One such example of a profile is shown in Figure 1:

Dimensions of statistical thinking		Statistical knowledge for teaching			
		Content knowledge		Pedagogical content knowledge	
		Common knowledge of content (CKC)	Specialised knowledge of content (SKC)	Knowledge of content & students (KCS)	Knowledge of content & teaching (KCT)
Types of thinking	Need for data				
	Transnumeration				
	Variation				M
	Reasoning with models			M	
	Integration of statistical and contextual				M
	Investigative cycle				M
	Interrogative cycle				
	Dispositions				

Key:		= direct evidence of that knowledge
		= indirect evidence of that knowledge
	M	= missed opportunity related to that knowledge

Figure 1. Example of knowledge profile for one teacher

The shaded cells show aspects of knowledge that were evidenced, either directly through teacher comments/explanations/discussion, or indirectly through other related components of knowledge. Additionally, the profile indicates some ‘missed opportunities’, instances in the classroom that revealed the teacher’s lack of relevant knowledge, and which potentially impacted on the students’ opportunity to learn.

Results and Discussion

Through the analysis and profiling of each teacher and in particular the identification of missed opportunities, a theme emerged in relation to teacher listening. Three different types of listening problems were identified. First, after the teacher had listened to a student's statement or question, the response indicated that the teacher had not interpreted the student correctly, and thereby 'answered a different question' from what the student was asking or saying. Second, the teacher did not evaluate an incorrect or 'off-track' response, and did not follow up with any further question nor seek further elaboration from the student that might have helped the student's understanding to develop. Third, if a student gave an incomplete response, the teacher did not ask for more explanation or clarification from the student in order to make sense of what the student was saying. Each of these three types of listening 'problems' is described, with supporting examples from classroom interactions, and links to relevant literature.

Teacher Responding to Different Question or Statement from the One Asked

In some situations, the teacher did not address the intent of the student's statement or question. One example related to the students considering what data collection question might have resulted in particular data for two different variables. A data card, which showed four data for one student, included the words 'youngest' and 'whistle'. The class were suggesting some possible data collection questions that would have generated these data. The teacher indicated that the correct data question for 'whistle' was, "Can you whistle?" A student questioned this, and the teacher's responded as follows:

Student 1: How do you know that that question is the right one? Like, all these questions could be different, because, like, ... his favourite toy could be whistle ...

Teacher: So you think we might need more for "Can you whistle?" So how could we extend that question to make it more specific, then, if you see a problem with that question? What could we add to it? Because we are trying to find out if the person can whistle. Not get a whistle off the teacher or off a toy and whistle. Can you whistle ... what do we need to add?

Student 2: Can you whistle using only your mouth?

Teacher: So that means you can't use your fingers as well then? Okay then, "Can you whistle using only your mouth?"

The student's question appeared to be suggesting that there were alternative data questions that could have resulted in the data 'whistle', such as, "What is your favourite toy?" The teacher however missed this point, and seemed to think that the student was suggesting something about where the whistle came from. Clearly, neither the teacher nor the student sought further clarification on each other's comments. The process of considering what data question might have generated particular data is not a typical part of the investigative cycle. Instead, part of the planning phase involves developing data collection questions in order to generate particular data to use within an investigation. However, it became clear that the teacher, by responding differently from what the student had asked, missed an opportunity to engage in a discussion that could have helped students develop their understanding of this necessary part of the investigative cycle. The aspect of teacher knowledge relevant to this is specialised knowledge of content related to the investigative cycle. Specialised knowledge of content relates to the teacher being able to draw on their knowledge of statistics to determine whether a student's answer is reasonable from a statistics point of view, and in this case relates particularly to one aspect of the investigative cycle. Later in the series of lessons, the students were to engage in their own investigation, including planning for data

collection, so the teacher missed a learning opportunity in relation to what the students would need at a later stage. Wallach and Even (2005) would classify this teacher listening as ‘under-hearing’. The teacher had heard something about the whistling question, but not heard correctly. It did not result in the teacher misleading students’ understanding of statistics. Even if the teacher had heard correctly and responded appropriately, there is no guarantee that the teacher would have taken the opportunity to develop a discussion/teaching point around the data collection question. As O’Connor (2001) indicated, a teacher’s decision making about responding happens within the pressure of the classroom moment.

The Teacher Does Not Evaluate a Student’s Answer

When a teacher does not evaluate a student’s answer, it could indicate a problem with the teacher’s specialised knowledge of content in relation to various aspects of statistical thinking. One set of classroom exchanges focused on the question, “Are there more whistling right handers or whistling left handers proportionally?”, which was based on a multivariate dataset represented in the following table:

Table 1

Two-way table showing the data related to the question about proportions of whistling right handers and whistling left handers

	Left handed	Right handed	Total
Whistlers	2	15	17
Non-whistlers	1	6	7
Total	3	21	24

After some discussion between some students and the teacher, a student commented:

Student 3: We put 8 out of 21 can whistle for right handed. And 2 out of 3 can [whistle] because there were 3 left handed.

Teacher: There were 3 left handed people? Where did you get your 21 from?

Although the teacher questioned the student about the 21, nothing was raised about the student referring to 8 (which should have been 6) or about the ‘can whistle’ part of the same statement. The student should have said that this number cannot whistle. The conversation continued:

Student 3: um ... out of those ...

Student 4: It is supposed to be 24.

Teacher: Okay.

Student 4: Was it 6 or 8 out of right handed who could whistle?

Teacher: Let's see, we'll check with another few groups.

Student 4: There's 8 so it's $3 \times 8 = 24$, which is $1/3$ of them. And 1 out of 3 left handed ... no 2 out of 3 left handed can whistle.

Teacher: Say that again please.

Student 4: There's 8 right handed people who cannot whistle out of 24 which is $1/3$. And there's 1 out of left handed people that cannot whistle which is $1/3$.

Teacher: So you've looked at the ones that can't whistle in the results as well, out of the group of right handed people. Is that what you've done?

Student 4 continued with the ‘error’ initiated by Student 3, namely the 6 (or 8) right handed people who could whistle, instead of that number being those who could not whistle. However, later in the exchange, Student 4 had self-corrected, without any intervention from the teacher. Another aspect missed by the teacher was that Student 4 compared the **1/3 of the class** who were right-handers and could whistle with **1/3 of the left handers** who could whistle. The comparison was invalid as one was a fraction of the whole class whereas the other was a fraction of a subgroup in the class.

It is clear from the exchanges that the teacher did not refer back to or engage with the data to verify what the students were saying, nor did he ask the students to do the same. Statements involving different parts of two-way classification of data were difficult for the students. As an example, it was not uncommon for students to talk about the proportion of non-whistlers who are boys when they were really referring to the proportion of boys who are non-whistlers (or vice versa). In these situations, the teacher did not necessarily notice and question the students further about it. Creating and referring back to a two-way table representation of the data (such as in Table 1 above) would have assisted the teacher in discussing whether these statements were equivalent. However, such a representation was not used, indicating that specialised knowledge of content in relation to reasoning with models as well as in relation to transnumeration of the data (two cells of the teacher knowledge framework in Figure 1) were missing. It is also an example of what Wallach and Even (2005) refer to as either under-hearing or non-hearing. Wallach and Even suggest that teacher knowledge is a significant factor in the breakdown of teachers really listening to students. To effectively evaluate a student’s response requires specialised knowledge of content.

The Teacher Seeks No Further Clarification from Students

The third type of listening problem identified was when the teacher made no attempt to seek further clarification from a student when it seemed that the answer or explanation was incomplete or quite unclear. One such example was following a question from the teacher as to how it might be possible to investigate whether there was a relationship between the hand-spans and the foot sizes of students. One student suggested that you could “add them together and divide by two”. The teacher did not follow up this suggestion with any questions such as what would be added together or why the average obtained (a meaningless measure in this case) would have helped with investigating the existence of a relationship. Another comment from the student about this was that you could find out “who has the biggest feet ... on average”. Again, this statement did not make sense in connection with the investigative question, but was not challenged by the teacher.

In terms of the teacher knowledge profile, these missed opportunities for elaboration indicated a problem with specialised knowledge of content connected with the interrogative cycle and the investigative cycle. The interrogative cycle relates to students making connections between the problem and the data, through seeking possibilities, generating ideas, judging what might work, etc. When thinking about the problem (namely, is there a relationship between the two variables), students need to consider what might be useful for the analysis phase of the investigative cycle and the data might be transnumerated into a useful representation. So when the students made their vague suggestions, the teacher needed to ‘think ahead’ and be able to link their suggestions to the data and the problem, and whether the suggestions would usefully lead somewhere. By not doing so, specialised knowledge of content was not activated in relation to the interrogative and investigative cycles, as well as to transnumeration of data. Therefore three different ‘cells’ of the teacher knowledge framework were implicated as not activated. As with the previous listening

problem of not evaluating a student's response, this listening problem could be classified as non- or under-hearing (Wallach & Even, 2005).

Conclusions

The classroom examples described in this paper illustrate three different types of listening problems on the part of the teacher, during a series of statistics investigation lessons. These three problems (responding to something different from what the student has said or asked; not evaluating a student's response; and not seeking clarification about an incomplete explanation or answer) are indicative of different aspects of teacher knowledge for statistics that is missing or not activated, and therefore created a missed opportunity for developing students' understanding of statistics. Although Wallach and Even (2005) suggest that there could be a number of factors that contribute to such listening problems, this study illustrates that teacher knowledge has a significant role in avoiding listening problems, and that various types of knowledge are necessary (but not necessarily sufficient on their own) to avoid such problems.

One teacher in the study did not present any listening problems. For this teacher, when for example, a student's explanation was incomplete, the teacher sought further clarification from the student. This teacher therefore showed sound teacher knowledge in most areas of statistical thinking.

Although the listening problems described and exemplified in this paper did not always impact negatively on the students' learning, it is suggested that the learning was not advantaged in ways that might have been possible. It is important that such listening problems are reduced through teachers developing the appropriate teacher knowledge. As a consequence, the classroom discourse practices and opportunities will be enhanced.

References

- Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education*, 9(1), 33-52.
- Ball, D. L., Thames, M. H., & Phelps, G. (2005). Articulating domains of mathematical knowledge for teaching Retrieved May 13, 2005, from http://www-personal.umich.edu/~dball/Presentations/RecentPresentations/041405_MKT_AERA.pdf
- Bliss, J., Askew, M., & Macrae, S. (1996). Effective teacher and learning: Scaffolding revisited. *Oxford Review of Education*, 22(1), 37-61.
- Boaler, J. (2000). Exploring situated insights into research and learning. *Journal for Research in Mathematics Education*, 31(1), 113-117.
- Burgess, T. A. (2006). A framework for examining teacher knowledge as used in action while teaching statistics. In A. Rossman & B. Chance (Eds.), *Working cooperatively in statistics education: Proceedings of the Seventh International Conference on Teaching Statistics (ICOTS 7)*, Salvador, Brazil. Voorburg, The Netherlands: International Association for Statistical Education and International Statistical Institute. [Available online from <http://www.stat.auckland.ac.nz/~iase/publications%5D>.
- Cobb, G. W., & Moore, D. S. (1997). Mathematics, statistics, and teaching. *American Mathematical Monthly*, 104(9), 801-823.
- Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education*, 28(3), 355-376.
- delMas, R. C. (2004). A comparison of mathematical and statistical reasoning. In D. Ben-Zvi & J. B. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning, and thinking* (pp. 79-96). Dordrecht, The Netherlands: Kluwer.
- Even, R., & Wallach, T. (2003). On student observation and student assessment. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia - Geelong, VIC, Vol. 1, pp. 316-323). Sydney: MERGA.
- Hill, H. C., Schilling, S., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, 105(1), 11-30.

- Mercer, N. (1995). *The guided construction of knowledge: Talk amongst teachers and learners*. Clevedon: Multilingual Matters.
- Moore, D. S. (1990). Uncertainty. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 95-137). Washington, DC: National Academy Press.
- O'Connor, M. C. (2001). "Can any fraction be turned into a decimal?" A case study of a mathematical group discussion. *Educational Studies in Mathematics*, 46, 143-185.
- Pereira-Mendoza, L. (2002). Would you allow your accountant to perform surgery? Implications for education of primary teachers. In B. Phillips (Ed.), *Proceedings of the Sixth International Conference on Teaching Statistics (ICOTS 6), Cape Town, South Africa*. Voorburg, The Netherlands: International Association for Statistical Education. [Available online from <http://www.stat.auckland.ac.nz/~iase/publications%5D>].
- Wallach, T., & Even, R. (2005). Hearing students: The complexity of understanding what they are saying, showing, and doing. *Journal of Mathematics Teacher Education*, 8(5), 393-417.
- Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: A review of recent research into mathematics classrooms. *Review of Educational Research*, 78(3), 516-551. doi: 10.3102/0034654308320292
- Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, 67(3), 223-265.

An Experienced Teacher's Conceptual Trajectory for Problem Solving

Barbara Butterfield
University of Wollongong
<butterfi@uow.edu.au>

This study explores how teachers understand and translate curriculum statements into concepts that are better suited to supporting student progress with rich mathematical problem-solving tasks. We report the actions of one experienced teacher of primary mathematics in charting the sequence of concepts and processes relevant to the above issue. We contend that the study embellishes Pedagogical Content Knowledge dimensions of the framework advanced by Ball, Hill and Bass (2005). Implications for field experiences of prospective teachers of mathematics are discussed.

Background

The need to be innovative about programs that are used to support prospective teachers of mathematics continues to be a challenge for tertiary educators. The task is complicated by the ever changing demands of school environments, precipitated by a mobile student and teacher population that is expected to function in a competitive globalised world. In such fluid classroom environments, teachers have to contend with meeting professional standards of performance (NSW Institute of Teachers, 2005), as well as curriculum reforms such as the adoption by the states of the National Curriculum (ACARA, 2010) Mathematics curriculum documents, among other things, identify core content areas and learning outcomes for teachers. It is a teacher's responsibility to translate the statements of content and learning outcomes into appropriate teaching actions and learning activities in the school environment. Against this background, what is the nature of a teacher's knowledge base that is required in making the transition from curriculum statements to action in the classroom?

We explored the above question initially by examining the quality of knowledge that pre-service teachers (PSTs) could bring to the design, solution and teaching of mathematics problems for young children (Authors, 2010, 2011). The results of these studies indicated that PST's tend to have built a reasonably robust body of mathematical content knowledge. However, as expected, they experienced difficulties in comprehending and translating curriculum statements into designing of learning activities that are rich and engaging and showed a degree of alignment with the spirit of the curriculum statements. In particular, we found that PSTs could not unpack the learning outcomes as presented in the K-6 mathematics curriculum (Board of Studies, 2006) in ways that would foreground the underlying mathematical concepts in the design of learning tasks.

In order to better understand the above issue, we sought to draw on studies of teacher expertise with the view to generating information about what this group of teachers faced regarding content and learning outcome statements that appear in K-6 mathematics curriculum. Expert-novice studies of mathematics teachers and teaching have made important inroads into our understanding of teacher cognition and the psychology of mathematics teachers (Chinnappan & Lawson, 2005; Leinhardt, 1987, 1988, 1989; Schoenfeld, 2000). A significant finding from this stream of research is that teachers draw

on robust content knowledge but this knowledge has to be utilised flexibly in supporting learning. This point has been foregrounded in recent works by Ball and colleagues at the University of Michigan.

Objective

The aim of this study was to understand an aspect of mathematics teachers' Pedagogical Content Knowledge (PCK) in the domain of primary mathematics, by describing the knowledge and actions of one experienced primary mathematics teacher as she attempted to develop a problem-solving task on the basis of a K-6 Mathematics curriculum that is adopted in the state of New South Wales (NSW), Australia.

Theoretical Framework

Data analysis and interpretations of our participant were guided by a model of Mathematical Knowledge for Teaching proposed by Ball, Bass, and colleagues (e.g., Ball et al., 2008). Under this model, Mathematical Knowledge for Teaching (MKT) is comprised of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). Pedagogical Content Knowledge is knowledge of suitable teaching strategies for the content, including knowledge of how to arrange components of the content for effective teaching. Subject Matter Knowledge (SMK), on the other hand, is knowledge of content that is not necessarily related to the way that mathematics could be taught. Both types of knowledge are considered relevant to the study of mathematics in the education context, and the division of SMK and PCK allows research to focus in each domain. A diagrammatic representation of the Mathematical Knowledge for Teaching (MKT) model is depicted in Figure 1.

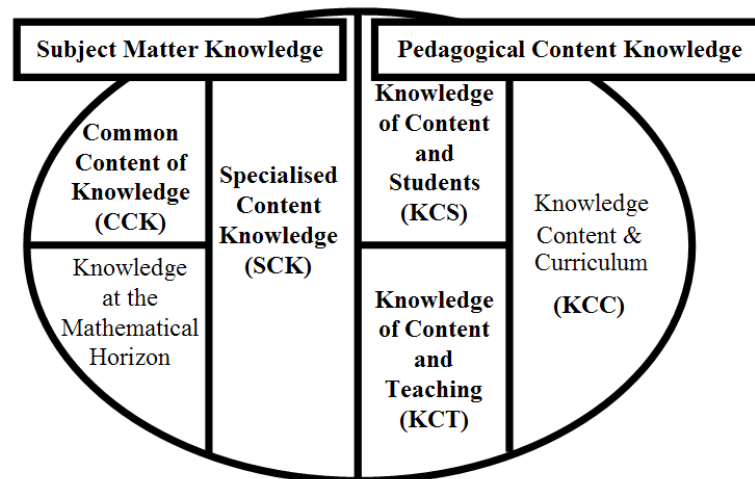


Figure 1. Schematic representation of teacher knowledge for teaching mathematics (Hill, Ball & Schilling, 2008: 174).

In the present study, we focus on the domain of Pedagogical Content Knowledge (PCK). Three dimensions are identified within PCK (see Figure 1).

Knowledge of Content and Students (KCS)

Knowledge that combines knowing about mathematics and knowing about students. Knowledge of how to: anticipate what students are likely to think; relate mathematical ideas to developmentally appropriate language used by children.

Knowledge of Content and Teaching (KCT)

Knowledge that combines knowing about mathematics and knowing about teaching. Knowledge of how to: sequence content for instruction; determine instructional advantages of different representations; pause for clarification and when to ask questions; analyse errors; observe and listen to a child's responses; prompt, pose questions and probe with questions; select appropriate tasks.

Knowledge of Content and Curriculum (KCC)

In a more recent model (Ball, 2010), the dimension of Knowledge of Curriculum was developed into *Knowledge of Content and Curriculum (KCC)* thus emphasising the link between mathematical content knowledge and understanding of mathematics curriculum including statements of learning outcomes, assessment strategies and lesson planning sequences as well as the development of learning sequences that are appropriate for different grade levels.

In this paper we provide data to indicate links between knowledge of mathematics curriculum KCC, KCS and KCT as demonstrated by an experienced teacher.

Method

Design and Participant:

The research design was a case study involving an experienced teacher of primary mathematics. The teacher was a volunteer with 25 years of classroom experience and had been judged by her peers to be an outstanding practitioner in the state of NSW, Australia. This teacher will be referred to as Julia.

Procedure

Julia was invited to participate in two, two-hour interviews. Semi-structured interview questions were used to generate data in this present study. During the first interview, Julia was informed about the purpose and objectives of the study. At this interview Julia was asked to think about how she uses the NSW K-6 Mathematics (2002) curriculum for lesson planning and teaching. Specifically, Julia was encouraged to design a rich learning task that would be appropriate for children in Year 6 (11 year olds) in Australia. During the second interview, Julia was asked to talk about the major strands of the curriculum and links to the learning task that she had designed.

Data Sources

The primary data source for the present study was responses from Julia to the semi-structured interview questions. Responses that are relevant to this study of Pedagogical Content Knowledge are reported in the Results and Conclusions section - Rich Learning

Task (Truss Bridge problem Figure 2), Conceptual Trajectory (Table 1) and Learner Scaffoldings (Table 2).

Results and Conclusions

Rich Learning Task

The Rich Learning Task that was designed consisted of the photograph shown in Figure 2, followed by a set of questions relating to numbers of paddle pop sticks required for the side of this bridge, and the pattern that describes how many paddle pop sticks would be required for any length bridge. The questions also encourage thinking about other situations that may use this type of pattern.



Figure 2. Truss Bridge Problem (TBP)

The solution of the problem involves integration of content knowledge from a number of strands such as number, geometry and algebra, and use of a range of problem-solving strategies, such as reasoning and pattern identification. We see the richness of the Truss Bridge Problem as providing evidence of Julia accessing and integrating content knowledge, in a manner that will encourage students to think deeply about the application of mathematics in a real-life context. This demonstrates the teacher's activation of KCT and KCS.

During the second interview, Julia responded to our questions about the major strands of the curriculum and links to the learning task that she had designed. In order to examine linkage between Knowledge of Content and Curriculum (KCC) and Knowledge of Content and Teaching (KCT) we cross-correlated Julia's responses regarding Content and Process to the curriculum Content and Process Statements. The set of Content and Process data obtained is included as a Conceptual Trajectory in Table 1.

In Table 1, curriculum statements are labelled as 1aES1, 2aES1, 2aS3 and so on. For example, 1aES1 refers to a curriculum statement involving a geometry concept. The corresponding label for algebra is 2aES1. The translations by Julia are labelled as 1bES1, 2bES1 and so on. We note a number of connections in the participant's translations of the curriculum statements to the planned task. Firstly, Julia is very focussed on and explicit about the geometric and algebraic concepts that are played out at each of the Learning Stages (BOS, 2002) for children. For example, for children in Early Stage One (ES1), the curriculum states that 'Students learn about comparing and describing closed shapes and open lines' in the area of geometry (1aES1). This is interpreted by Julia as 'Identifies triangles in pictures and in real-life contexts' (1bES1) in the context of the Truss Bridge Problem. Secondly, she demonstrates the sequencing of concepts as she moves from Early

Stage One (ES1) to Stage Three (S3). Thus, within each Stage of learning Julia attempts to translate the corresponding curriculum statement into a form that is required and used in understanding and solving the Truss Bridge Problem. We contend that the identified activation of Knowledge of Content and Curriculum (KCC) and Knowledge of Content and Teaching (KCT) in this context demonstrates this mathematics teacher's Pedagogical Content Knowledge (PCK) in the domain of primary mathematics.

Table 1
Conceptual Trajectory

	Content and processes statements as per the K-6 syllabus (Board of Studies, 2006)		Julia's translations of the K-6 syllabus content and processes for the designed task	
	Geometry	Algebra	Geometry	Algebra
<i>Early Stage (ES1)</i>	1aES1 - Students learn about comparing and describing closed shapes and open lines	2aES1 – Students learn to ask questions about how repeating patterns are made and how they can be copied on continued	1bES1 – Students identify triangles in pictures and in real-life contexts.	2bES1 – Students recognise, copy and continue repeating patterns using shapes e.g. $\nabla \Delta \nabla \Delta$
<i>Stage 1 (S1)</i>	1aS1 – Students learn about sorting 2-Dimensional shapes by a given attribute.	2aS1 – Students learn to represent number patterns using diagrams, words, symbols	1bS1- Student identify sides and corners of triangles	2bS1 – Students determine a missing element in a number pattern e.g. 3, 6, 9, $\square$, 15
<i>Stage 2 (S2)</i>	1aS2 – Students learn to recognise that a particular shape can be represented in different sizes and orientations	2aS2 – Students learn to pose problems based on number patterns	1bS2 – Student make representations of triangles in different orientations	2bS2- Students make a number pattern using trusses and pose a problem based on that pattern
<i>Stage 3 (S3)</i>	1aS3 – Students learn to explain classification of 2-D shapes	2aS3 – Students learn to use a number of strategies to solve unfamiliar problems	1bS3 – Students compare and describes side properties of equilateral, isosceles and scalene triangles	2bS3 – Students work through a process of building a truss with a series of equilateral triangles, to determine a rule that links the number of trusses and length of any bridge

We also investigated the activation of Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) during the second interview conducted with Julia. Our interview generated response data, consisting of information about potential areas of difficulty that could be encountered by the children while solving the Truss Bridge Problem questions. These data are presented as Learner Scaffolding in Table 2.

Table 2
Learner Scaffolding

Likely Difficulties for Students	Useful Representations to Aid Understanding								
Interpreting a 3D object in a photograph to visualise a 2D side view (elevation).	Photo with triangles highlighted								
Knowledge of equilateral triangles (beams are equal length).	Construct equilateral triangles with paddlepop sticks								
Recognition of an inverted triangle	Manipulation of paper triangles to form alternate pattern								
Recognition of series of beams	Highlight the beams in different colours Build the beams in different colours								
Lack of knowledge of a truss	View real world examples such as crane, bridge, power line towers								
Cannot conceptualise shared side in truss	Highlight shared side between two triangles								
Cannot visualise how the pattern could continue	Draw on the photograph Build the side using paddlepop sticks Touch each stick and count.								
Does not know how to count the beams	Draw the side view. Mark each side while counting (one to one correspondence). Draw table and demonstrate use of the table (students can use supporting representations e.g. drawings, paddlepop stick model to complete the table).								
Does not know how to record patterns clearly and/or efficiently.	<table><tr><td>N° Δ</td><td>1</td><td>2</td><td>3</td></tr><tr><td>N° sides</td><td>3</td><td>5</td><td>7</td></tr></table>	N° Δ	1	2	3	N° sides	3	5	7
N° Δ	1	2	3						
N° sides	3	5	7						
Cannot determine a rule for the pattern from the table	Start with one beam and show that every new triangle requires two more beams one + twice the number of triangles								

The left-hand column of Table 2 shows Julie's anticipated possible student difficulties, while in the right-hand column are Julie's suggestions to assist and support students. For example, if children experience difficulty with recognition of an inverted triangle, Julia suggested that teachers could provide an activity where children could experiment with the different orientations of a triangle (Manipulation of paper triangles to form alternate pattern). We contend that these responses demonstrate activation of KCS and KCT that was triggered by Julia's analysis of difficulties children may experience when solving the Truss Bridge Problem (TBP). There is evidence of our participant decomposing and integrating elements of the TBP. Julie clearly provided an in-depth analysis of potential challenges in solving the Truss Bridge Problem, and appropriate assistance in the form of representations to aid understanding. In addition, Julie alluded to the necessity of timely support for

individual children at their point of need. A result in the classroom of this activation of KCT could be effective time management, as the children could be given suitable time to individually experiment with important concepts and their representations, both in concrete and abstract forms.

Discussion

In advancing frameworks of Mathematical Knowledge for Teaching, Ball (2010) has shown the importance of teachers connecting their Subject Matter Knowledge and Pedagogical Content Knowledge. This study is an attempt to provide insight into links between the components of Pedagogical Content Knowledge (PCK). In designing a rich learning task an experienced teacher draws on Knowledge of Content and Curriculum (KCC) and Knowledge of Content and Teaching (KCT). These are components of Pedagogical Content Knowledge. This study also identifies that there are links between Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT), demonstrated through the development of appropriate learner scaffolding in the context of designing a rich learning task.

The translation of mathematical content knowledge by the teacher is evidenced by the development of a rich learning task and in articulating the pedagogical significance of the task in light of its conceptual underpinnings. This translation lies at the heart of teachers having to restructure their knowledge, which featured strongly in arguments developed by Putnam (1987) and Tamir (1988) about knowing mathematical content in different forms that are germane to teaching.

Although our study was limited to one teacher's knowledge and actions, there are lessons here for tertiary programs that aim to prepare future teachers of primary mathematics (Mewborn, 2001). Specifically, we contend that in supporting pre-service teachers we need to go beyond assessing their content knowledge of mathematics and provide learning and field experiences where they are immersed in activities that enable them to a) discuss curriculum statements, b) draw out the relevant mathematical concepts and c) identify areas of difficulty. Such experiences could then be utilised in the design of rich learning tasks for children. A closer examination into relations between the components of Mathematical Knowledge for Teaching is required in order to support Pre-Service mathematics teachers with innovative programmes.

References

- ACARA (2010). *Report on Trial School Consultation*. Australian Curriculum, Assessment and Reporting Authority. http://www.acara.edu.au/verve/resources/Report_on_Trial_School_Consultation.pdf
- Ball, D. (2010). *Knowing mathematics well enough to teach it: From teachers' knowledge to knowledge for teaching*. http://www-personal.umich.edu/~dball/presentations/042710_ISR.pdf.
- Ball, D. L., Hill, H. C., & Bass, H. (2005). Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 5(3), 43-46.
- Butterfield, B. & Chinnappan, M. (2011). Teacher Knowledge Activated in the Context of Designing Problems. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics Traditions and [New] Practices AAMT/MERGA Conference 2011* (pp. 142-150). Alice Springs N.T.: AAMT & MERGA.
- Butterfield, B. & Chinnappan, M (2010). Walking the talk: Translation of mathematical content knowledge to practice. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education. Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 109-116). Fremantle, WA: MERGA.
- Board of Studies (2006). *Mathematics K-6 Syllabus*. Sydney: Board of Studies New South Wales.

- Chinnappan, M. & Lawson, M. (2005). A framework for analysis of teachers' geometric content knowledge and geometric knowledge for teaching. *Journal of Mathematics Teacher Education*, 8(3), 197–221.
- Department of Education, Employment and Workplace Relations (2008). *Review of Australian Higher Education*. Author,
<http://www.deewr.gov.au/highereducation/review/pages/reviewofaustralianhighereducationreport.aspx>
- Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching*, 15(1), 13-33.
- Hill, H. C., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372–400.
- Leinhardt, G. (1988). Expertise in instructional lessons: An example from fractions. In D. A. Grouws & R. J. Cooney (Eds.) *Perspectives on research on effective mathematics teaching* (Vol. 1 in a series of monographs from conferences of the National Council for Teachers of Mathematics, pp. 47-76). Hilldale, NJ: Erlbaum.
- Leinhardt, G. (1989). Math lessons: A contrast of novice and expert competence. *Journal for Research in Mathematics Education*, 20(1), 2-75.
- Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers. *United States Mathematics Education Research Journal*, 3, 28–36.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA.: Author
- NSW Institute of Teachers (2005), *Professional Teaching Standards*.
<http://www.nswteachers.nsw.edu.au/Main-Professional-Teaching-Standards/>
- Putnam, R.T. (1987). Structuring and adjusting content for students: A study of live and simulated tutoring of addition. *American Educational Research Journal*, 24 (1), 13-28.
- Schoenfeld, A. H. (2000). Models of the Teaching Process. *Journal of Mathematical Behavior*, 18(3), 243-261.
- Tamir, P. (1988). Subject matter and related pedagogical knowledge in teacher education. *Teaching and Teacher Education*, 2, 99-110.

Mathematical Knowledge for Teaching of MERGA Members

Rosemary Callingham

University of Tasmania

<Rosemary.Callingham@utas.edu.au>

Kim Beswick

University of Tasmania

<Kim.Beswick@utas.edu.au>

Julie Clark

Flinders University

<julie.clark@flinders.edu.au>

Barry Kissane

Murdoch University

<kissane@murdoch.edu.au>

Pep Serow

University of New England

<pserow2@une.edu.au>

Steve Thornton

Charles Darwin University

<Stephen.Thornton@cdu.edu.au>

Although there has been considerable interest in teachers' knowledge for teaching mathematics for nearly a quarter of a century, little attention has been paid to the knowledge of mathematics teacher educators. The responses of 57 MERGA members to an online survey addressing beliefs about mathematics and its teaching, mathematics content knowledge and mathematics pedagogical content knowledge are reported. Teacher educators found the items addressing pedagogical content knowledge more difficult than mathematics content questions or endorsing beliefs, and the type of employment appeared to be a more important influence on outcomes than the level of mathematics studied.

Introduction

From the time that Shulman (1987) identified different aspects of teachers' specialised knowledge used in teaching, there has been considerable interest in describing and measuring specific aspects of this knowledge. The knowledge of those who teach pre-service teachers however, has been relatively unexplored (Jasman, Payne, Grundy & Del Borrello, 1998). In this paper the initial findings from a survey undertaken by 57 members of the Mathematics Education Group of Australasia (MERGA) members addressing aspects of their knowledge for teaching mathematics are reported.

Prior Research Base

As suggested, the focus of most studies about knowledge for teaching has been on mathematics teachers. Mewborn (2001) in her useful overview of studies of teachers' knowledge of mathematics indicated that measures such as the number and level of mathematics courses taken during training were not related to students' learning outcomes. Further, studies that focussed on how teachers' knowledge of mathematics influenced their teaching showed that the outcomes were far from straightforward. Effective teachers of primary mathematics are those having a deep connected understanding of the mathematical principles and concepts involved in the curriculum (Askew, Rhodes, Brown, Wiliam & Johnson, 1997; Ma, 1999). Such understanding however goes beyond mathematical content alone.

Shulman's (1987) notion of pedagogical content knowledge (PCK) stimulated considerable research within the mathematics education community into the nature of teachers' knowledge for teaching mathematics. Shulman described PCK as

the blending of content and pedagogy into an understanding of how topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue. (1987, p. 8)

Although content knowledge in mathematics has been measured for many years attempts to develop measures of PCK are relatively new. Hill and her associates (Hill, Schilling & Ball, 2004) developed items to measure teachers' mathematical knowledge for teaching, which included aspects of pedagogical and content knowledge. Chick (2007) provided a useful

framework of categories of PCK in mathematics teaching derived from observation studies. Callingham et al (2011) developed an online instrument that addressed three aspects of teachers' knowledge: beliefs about mathematics and its teaching, mathematics content knowledge (MCK) and PCK. Beswick, Callingham & Watson (2011) showed that a similar instrument comprising beliefs, content knowledge and pedagogical content knowledge in mathematics created a single "thick" construct of teachers' knowledge for teaching mathematics. Internationally, the Teacher Education and Development Study in Mathematics (TEDS-M) (Tatto, Schwille, Senk, Ingvarson, Peck & Rowley, 2008) obtained outcome measures of mathematics content knowledge and mathematics pedagogical knowledge from pre-service primary teachers across 16 countries using a paper-based test, and also collected information about related teacher education courses. One result from this study was that opportunity to learn both mathematics and mathematics pedagogy was critical to program outcomes (Blömeke, Suhl, Kaiser, & Döhrmann, 2012). Baumert et al (2010) found that in German Year 10 classrooms both teachers' content knowledge and mathematics pedagogical knowledge were important, but that mathematics pedagogical knowledge had a greater influence on students' learning. Given this background, and the attempts to link teachers' mathematical knowledge to students' outcomes, it is surprising that so little work has considered the knowledge of mathematics teacher educators.

Background to the Current Study

The study grew out of a 2-year project, funded by the Australian Learning and Teaching Council (ALTC), that aimed to develop and use tools to provide an evidence base for improving mathematics education in universities (Callingham et al, 2011). Surveys had been developed for pre-service teachers and lecturers in order to investigate levels of both content and pedagogical content knowledge. A number of MERGA members expressed interest in examining the survey used with pre-service teachers, and indicated that they would also be willing to undertake a version of the lecturer survey. In addition to questions addressing beliefs about mathematics and its teaching, MCK, and mathematics PCK, a number of questions addressed background variables such as level and type of appointment, experience at teaching in both schools and at tertiary level and mathematics and educational backgrounds. The current study addresses the following research questions:

1. What is the mathematical knowledge for teaching of MERGA members who responded to the survey; and
2. What factors appear to influence this knowledge?

Method

The survey was sent out via the MERGA member list in the second half of 2011. Respondents were required to accept the ethics declaration about voluntary consent at the front of the survey. In order to meet both ethical standards and allow for professional interest, respondents who chose not to accept the declaration were not redirected out of the survey, as is common in online surveys. Instead, any respondents who did not accept the voluntary consent statement were removed from the data set prior to any analysis. A survey identical in content but slightly different in demographic information was provided to the seven universities participating in the project. The two data sets were combined for the purpose of this analysis, and it is assumed that lecturers in participating institutions were MERGA members or eligible for MERGA membership.

The survey comprised four main sections: background information, 10 items addressing beliefs about mathematics, 10 items focussed on MCK and 10 items that dealt with PCK. One

open-ended question was included so that respondents could comment on aspects of the survey if they wished. Both MCK and PCK items included questions appropriate to primary and secondary mathematics. Respondents were asked whether they taught in pre-service primary or secondary courses, and were directed to the appropriate question sets as a result. Examples of primary and secondary MCK and PCK items are shown in Table 1.

Table 1.
Examples of MCK and PCK Items

Item category	Example ¹
MCK Primary	Steve buys a shirt that is discounted by 10% on the ticket. A sign on the rack stated, ‘Discount by a further 15%’. This is the same as a discount of what percentage of the original price of the shirt? A) 12.5% B) 15% C) 17.5% D) 23.5% E) 25%
MCK Secondary	A student picks a value for x , and uses it in the function $f(x) = 2x^2$ to get an answer. He then picks a new value for x , which is 3 times his original choice. His new answer is ... A) 3 times his old one B) 6 times his old one C) 9 times his old one D) 18 times his old one E) 36 times his old one F) Can’t tell how the new answer relates to the old one
PCK Primary	Your class is exploring measurement concepts. Students make the following statements. Which one of these most urgently requires teacher intervention? A) Area is the space inside a shape. B) As the perimeter increases, the area always increases. C) Volume is the amount of space a shape takes up. D) Area is a measurement of the surface.
PCK Secondary	Below is a student’s incorrect attempt to solve a pair of simultaneous equations: $\begin{array}{ll} 3x - y = 7 & (1) \\ x + y = 9 & (2) \end{array}$ $2x = -2 \qquad x = -1 \quad \text{Substitute into (2): } -1 + y = 9 \qquad y = 10$ The student checks her solution via substitution into equation (1) and is surprised to see that it does not make the equation true. Which of the following statements gives the most likely explanation for the student’s error and the most appropriate next step for the teacher to take? A) The student tried to subtract equation (2) from equation (1) but “cancelled” the $-y$ and $+y$ to eliminate this variable. The teacher should tell the student to add the equations instead. B) The student tried to subtract equation (2) from equation (1) but “cancelled” the $-y$ and $+y$ to eliminate this variable. The teacher should ask the student to explain why she chose to subtract rather than add these equations. C) The student tried to subtract equation (2) from equation (1) but “cancelled” the $-y$ and $+y$ to eliminate this variable. The teacher should suggest that the student use the substitution method instead, by finding an expression for x or y from equation (2) and substituting this into equation (1). D) The student has incorrectly added equations (1) and (2). The teacher should ask the student to repeat this addition, ensuring she does it correctly.

¹Note. Formatting removed for space reasons.

All 10 beliefs (BLF) items were presented to every person. MCK and PCK questions were presented randomly from the pool of items developed. Altogether 42 MCK questions

and 31 PCK questions were presented. Some questions had multiple parts based on the same stimulus that were scored as separate items so the final number of items scored was greater than the number of questions presented. In both primary and secondary versions of the survey common questions were included so that the two surveys could be linked using Rasch measurement approaches (Bond & Fox, 2007). Among both MCK and PCK questions there were some that all respondents answered correctly. These questions were removed from the data set. Not every person responded to every item. The final data set for analysis is shown in Table 2.

Table 2.
Analysis Data Set

Scale	Number of respondents	Number of items	Number of link items
BLF	57	10	10
MCK	40	43	21
PCK	44	60	9

The data were analysed using Rasch measurement (Bond & Fox, 2007) and measures of performance in logits—person ability measures—were obtained for each of the three scales of BLF, MCK and PCK. These measures were imported into SPSS for further analysis.

Results

Characteristics of Respondents

A summary of the level of appointment and type of employment of the people who responded to the background information questions is shown in Table 3. The Continuing category tended to have the higher level (Level C and above) staff whereas most fixed-term contracts were at Level B. Those people at higher levels in the Casual category are likely to be retirees who are still active in the mathematics education community.

Table 3.
Summary of Respondents by Nature and Level of Appointment

	Level A	Level B	Level C	Level D	Level E	Total
Continuing	0	8	6	9	0	23
Fixed term	1	11	1	0	0	13
Casual	8	2	1	1	2	14

Over three-quarters of respondents (44/57, 77.2%) held postgraduate qualifications but a surprising proportion (17/57, 29.8%) had not formally studied mathematics beyond school level. Only one person indicated indigenous status and six people spoke a language other than English as their mother tongue. Nearly half of the respondents (26/57, 45.6%) indicated that they had been lecturing in the tertiary sector for 5 years or less, and 13 of these were on continuing appointments. Respondents were also experienced teachers with nearly 40 percent (22/56, 38.4%) having taught in schools for more than 15 years, and across every state and territory in Australia.

Beliefs, MCK and PCK Scales

All three scales showed good fit to the Rasch model with fit indices lying within commonly accepted values of 0.77 to 1.30, indicating that the items worked together to measure the target construct in a valid manner (Bond & Fox, 2007). There was a highly statistically significant correlation between PCK and MCK (Pearson correlation = .49, $p = .001$) and between MCK and BLF (Pearson correlation = .33, $p = .024$) but not between PCK and BLF (Pearson correlation = .17, $p = .28$) suggesting that beliefs about mathematics and its teaching have less association with approaches to pedagogy than with ability to do mathematics.

Performance of Respondents on the Three Scales

The PCK items were, in general more difficult for respondents than either the BLF or MCK items, with a lower mean logit score. In contrast to pre-service teachers, however, there was not a distinct decrease in performance from BLF to MCK to PCK (compare with Callingham et al, 2011). The most likely explanation is that mathematics educators in general found the MCK items very easy and also strongly endorsed the belief statements, whereas the pre-service teachers may not have reflected deeply about their beliefs but accepted the rhetoric, and found the MCK items harder, and the PCK items hardest of all. Figure 1 shows the distribution of mathematics educators' logit scores on the three scales. There was a highly statistically significant difference between the overall mean logit scores on PCK and MCK ($n = 44$, $t = 20.14$) indicating that despite the association between these two variables, the respondents did not perform in the same way on the two sets of items.

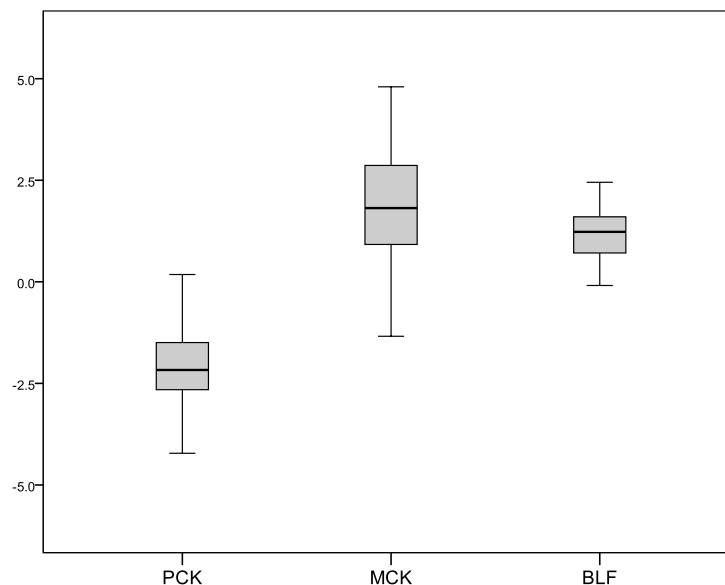


Figure 1. Distributions of mathematics educators' performance measures on scales of PCK, MCK and BELF.

Some interesting differences emerged when logit scores on each scale were analysed by type of appointment. PCK mean logit scores were lower than those on the BLF or MCK scales for all types of appointment but somewhat lower for Casual than other categories of appointment. In contrast, MCK mean logit scores were highest on the three scales for all types of appointment but somewhat higher for Casual than Continuing respondents. The findings are shown in Figure 2, including error bars. One-way ANOVA analysis indicated that there were no statistically significant differences between scores on any one scale

between the three groups but with the small sample sizes this lack of significance is not surprising.

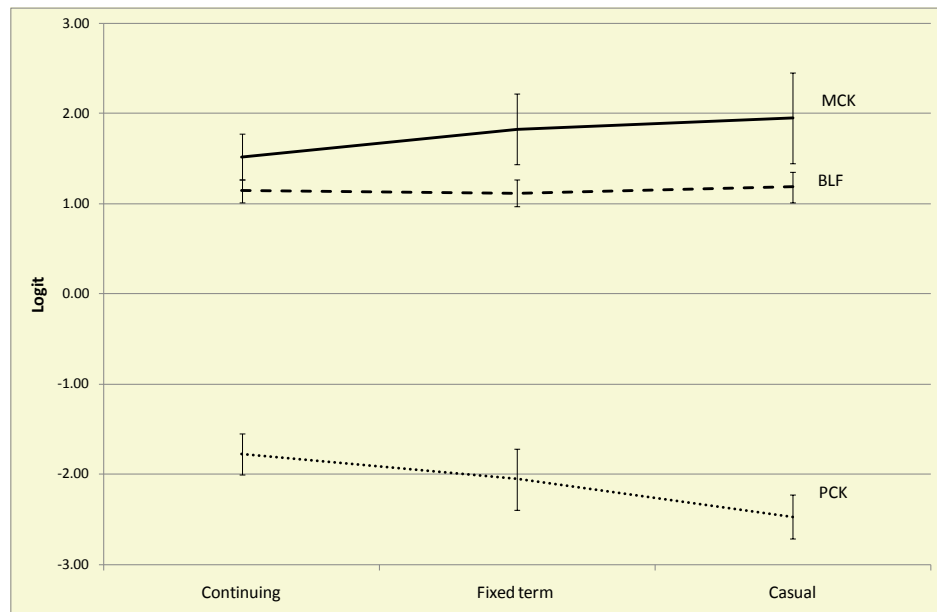


Figure 2. Mean logit scores on PCK, MCK and BLF by type of appointment.

A similar analysis was undertaken for the variable “years of tertiary teaching experience”. Results are shown in Figure 3, including error bars. Of particular interest is the drop in the mean score on the BLF scale, suggesting that highly experienced mathematics educators were less likely to endorse strongly statements about the nature of mathematics and its teaching. The error bars on the BLF scale are also smaller than on the other two scales, suggesting less variation within respondents. PCK scores tended to be higher among experienced lecturers. Again none of the differences between the groups on any one scale was statistically significant using a one-way ANOVA.

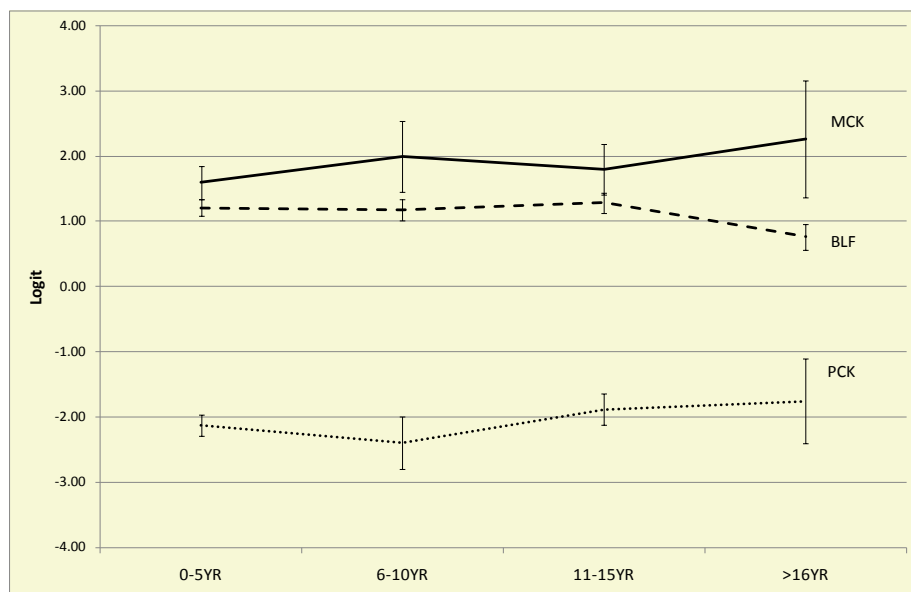


Figure 3. Mean scores on PCK, MCK and BELF by years of tertiary teaching experience.

Discussion

The initial findings from responses by mathematics educators to a survey instrument designed for pre-service teachers suggest some intriguing possibilities for further research. The instrument provided valid measures of mathematics educator knowledge but the three scales did not behave identically to those reported previously for pre-service teachers (Callingham et al, 2011). The results indicate that mathematics teacher educators had high levels of MCK, as would be expected. They did not endorse the beliefs items as easily as the pre-service teachers, possibly indicating a more nuanced understanding of mathematics and its teaching or the result of opportunities for reflection and discussion.

Mathematics educators found the PCK items the most challenging of all scales, as did the pre-service teachers. Although it is acknowledged that the PCK items do not cover the complexity of PCK (see Chick, 2011), it is argued that they do provide a crude measure of something more than MCK alone, especially given the highly significant difference between the performances on the two scales. In designing the PCK questions the ALTC project team engaged in deep discussion and heated argument about what aspects of PCK to incorporate and what kinds of choices to include in the selection of answers. At times it was impossible to choose a “correct” response, and for this reason some questions were scored with partial credit, allocating a 1 for a good response and a 2 for one that was deemed slightly better. As one respondent wrote in response to the open-ended question, “There were other options that I would have liked to be able to include in response to student’s [sic] thinking.” Mathematics educators are likely to have a range of views about what constitutes good mathematics pedagogy, but the finding that experienced mathematics educators had slightly higher performance on the PCK items than did less experienced colleagues may suggest that a level of agreement about PCK develops over time.

The finding that casual teaching staff had the highest level of MCK and the lowest level of PCK may indicate something about the types of people recruited to casual positions in universities. For a mathematics teacher educator position, the first requirement is to know some mathematics but this expectation alone may lead to many students being taught by teaching staff with limited understanding of the PCK needed to teach mathematics. For example, a good high school teacher recruited to take some primary mathematics courses in a teacher education program may have limited understanding about PCK in the primary classroom. The Casual group also included some Level D and E respondents who were highly experienced. It is tempting to surmise that without these people, the PCK knowledge of casual teaching staff might have been lower. The question of what tertiary mathematics teacher educators need to know in order to develop quality mathematics teachers—if you like PCK for teaching PCK—is emerging as an important issue.

The interactions of the BLF scale with the other two scales are worthy of more attention. There appears to be a strong association between beliefs and content knowledge but this is not so between beliefs and PCK. The idea that beliefs strongly influence how people teach may not hold for mathematics educators, but this can only be a conjecture at this stage. In addition, beliefs appear to be similar across all mathematics educators with only highly experienced lecturers showing any difference in the level of response.

This initial study of the knowledge of mathematics educators appears to break new ground. Larger studies and longitudinal measures are needed to identify the development of mathematics educators’ knowledge about mathematics for teaching, and their own PCK for developing PCK in their students.

To conclude, one respondent wrote

It is very difficult to teach something which you are neither competent in nor confident about - and I know that ends in a preposition but I am sure you know what I mean. Subject content knowledge has to be a co-requisite to pedagogical content knowledge for effective teaching.

The challenge for mathematics educators now is to identify what constitutes subject and pedagogical content knowledge in their specialised domain, and then to determine how this affects the outcomes for their students in terms of quality mathematics teaching.

Acknowledgments

This research was supported by the Australian Learning and Teaching Council Priority Projects grant number PP10-1638. In addition to the authors listed, Professor Merrilyn Goos and Assoc. Prof. Helen Chick are members of the project team and contributed significantly to the development and implementation of this study.

References

- Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., Krauss, S., Neubrand, M., & Tsai Yi-Miau (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, 47(1), 133-180.
- Beswick, K., Callingham, R., & Watson, J. (2011). The nature and development of middle school mathematics teachers' knowledge. *Journal of Mathematics Teacher Education*. Available Online First at <http://www.springerlink.com/content/th22781265818125/>
- Blömeke, S., Suhl, U., Kaiser, G., & Döhrmann, M. (2012). Family background, entry selectivity and opportunities to learn: What matters in primary teacher education? An international comparison of fifteen countries. *Teacher and Teacher Education*, 28, 44-55.
- Bond, T. G. & Fox, C. M. (2007). *Applying the Rasch model. Fundamental measurement in the human sciences* (2nd ed.). Mahwah, NJ: Lawrence Erlbaum.
- Callingham, R., Beswick, K., Chick, H., Clark, J., Goos, M., Kissane, B., Serow, P., Thornton, S., & Tobias, S. (2011). Beginning teachers' mathematical knowledge: What is needed? In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.) *Mathematics: Traditions and [New] Practices* (Proceedings of the AAMT-MERGA Conference, July, pp. 900-907), Alice Springs, NT: AAMT-MERGA.
- Chick, H. L. (2007). Teaching and learning by example. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Hobart, Vol. 1, pp. 3-21). Sydney: MERGA.
- Chick, H. L. (2011). God-like educators in a fallen world. In J. Wright (Ed.) *Proceedings of the annual conference of the Australian Association for Research in Education, Hobart*. Available http://www.aare.edu.au/11pap/papers_pdf/aarefinal00667.pdf
- Hill, H., Schilling, S., & Ball, D. (2004). Developing measures of teachers' mathematical knowledge for teaching. *Elementary School Journal*, 105, 11-30.
- Jasman, A., Payne, L., Grundy, S., & Del Borrello, A. (1998). Uncovering the knowledge construction of university based teacher educators: what we learn from/with school based practitioners. In P. Jeffery (Ed.) *Proceedings of the annual conference of the Australian Association for Research in Education, Adelaide*. Available <http://www.aare.edu.au/98pap/jas98305.htm>
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah: Lawrence Erlbaum Associates.
- Mewborn, D. S. (2001). Teachers' content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Teacher Education and Development*, 3, 28-36.
- Shulman, L. (1987) Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1-22.
- Tatto, M. T., Schille, J., Senk, S. L., Ingvarson, L., Peck R., & Rowley, G. (2008). *Teacher education and development study in mathematics (TEDS-M): Conceptual framework. Policy, practice, and readiness to teach primary and secondary mathematics*. East Lansing, MI: Teacher Education and Development International Study Center, College of Education, Michigan State University.

Emotions and the Development of Statistical Literacy

Colin Carmichael
Charles Sturt University
<ccarmichael@csu.edu.au>

Emotions play an important part in learning and are known to influence human development. The positive emotions of enjoyment and interest are thought to contribute to learning in distinctive ways, yet this distinction tends to be blurred in some learning research. This paper explores the role these two emotions play in the development of children's statistical literacy. It focuses on the responses of 220 middle-school children to just six self-descriptions, three assessing interest and three assessing enjoyment. Analysis of these responses suggests that the two emotions are difficult to differentiate empirically, but that differences in reported levels of enjoyment and interest may depend on students' perceptions of competence in the specific tasks with which they are engaged. Implications of this for teachers and researchers are discussed.

"I like using a calculator to work out problems, even though it's not that interesting". Children, such as the Year 7 boy who spoke these words, make emotion laden appraisals of the tasks that they encounter during mathematics lessons. Emotions are an important part of learning, in that children will engage in tasks where they are likely to experience positive emotions (Deci, 1992). Consequently, emotions play a key role in the development of mathematical knowledge. Of interest in this study, is the role that they play in the development of children's statistical literacy, regarded as an ability to critically engage with messages containing statistical elements. Researchers have proposed theoretical hierarchies associated with the development of this literacy that include necessary dispositions, such as a critical stance (Gal, 2002) and skepticism, curiosity and imagination (Watson, 2006). It is argued that these dispositions are more likely to develop when children experience positive emotions during their learning of the concepts and skills associated with statistical literacy. Research has suggested that the emotion of interest, for example, has a positive impact on the development of children's statistical literacy (Carmichael, Callingham, Hay, & Watson, 2010). Enjoyment, however, is also a positive emotion associated with learning. The boy quoted earlier mentioned that he liked using calculators, but didn't find them interesting. The study seeks to explore the role of both emotions in the development of children's statistical literacy.

Background

In her seminal work on human emotions, (Izard, 1977) identified interest and enjoyment as two distinct and fundamental emotions, yet this distinction becomes blurred in the motivational research literature, especially that related to intrinsic motivation. Interest is regarded as an affect with both state and trait like properties. As an emotion, it is defined as "the feeling of wanting to know about something or someone" (Soanes & Stevenson, 2008). Enjoyment, on the other hand is defined as "the state or process of taking pleasure in something" (Soanes & Stevenson, 2008). Both emotions are closely related in that Izard (1977) reported that people asked to describe past experiences of interest were likely to report pleasant experiences, similar to enjoyment. This closeness may have influenced the treatment of these emotions in the motivational literature. In relation to intrinsic motivation, for example, children's motivation to engage in tasks for their intrinsic value is thought to rely primarily on their anticipation of experiencing interest (Deci, 1992). Yet in their study of intrinsic motivation, Marcoulides, Gottfried, Gottfried, and Oliver (2008) conceptualized

intrinsic motivation as being derived from enjoyment only, with interest providing a separate collative motivation (Berlyne, 1960), one that is based on a desire to resolve task-features that are novel, complex and/or uncertain. Other researchers, for example Eccles and Wigfield (2002), argue that both emotions influence intrinsic motivation. These different conceptualizations have influenced the design of instruments that measure this construct. Some instruments contain interest items, for example "I'm interested in", and enjoyment items, for example "I enjoy" (Watt, 2004). Others contain only enjoyment items (Hulleman, Durik, Schweigert, & Harackiewicz, 2008) and then compare these with separate measures of interest. Even in interest based research, the distinction between enjoyment and interest is blurred, with some researchers assessing the emotional dimensions of interest using enjoyment items (Schiefele, 1991). The ensuing discussion attempts to disentangle these emotions in the learning context.

The control-value theory of achievement emotions (Pekrun, 2006) maintains that in learning contexts, interest and enjoyment are distinctive emotions, influenced by the value and difficulty of learning tasks. Enjoyment is thought to occur when students both perceive a level of control in achievement based tasks, and value the task. Perceptions of control are strongest when task competence is regarded as likely. Boredom, or lack of interest, is thought to occur when students do not value the task, irrespective of their level of control. Enjoyment, therefore, is closely related to perceptions of self-competency and value, whereas interest related to value alone. This suggests that irrespective of the value that is placed on a learning task, levels of enjoyment and interest are likely to differ when perceptions of competency are either strongly positive or strongly negative. It is hypothesised that a student who values a task and perceives it as easy will report higher levels of enjoyment than interest, whereas one who values a task but perceives it as difficult will report lower levels of enjoyment than interest.

Studies on the roles of emotions in learning, including flow, suggest that interest and enjoyment play important complementary roles. Interest is thought to play a key role in the learning process, in that it governs initial task engagement and subsequent exploration (Reeve, 1989). If this initial interaction with the task results in positive appraisals of competence then feelings of enjoyment will emerge (Reeve). These feelings will motivate re-engagement with similar tasks. If new tasks do not generate optimum levels of collative motivation, students will soon tire of them (Berlyne, 1960). This interaction between the two emotions may explain individual's attempts to enter and remain in the 'flow-channel' (Csikszentmihalyi, 2002), a location in time where they experience a total absorption of self in the learning task. Csikszentmihalyi maintained that flow occurs when task complexity is close to the skill level of the learner. Students who are confronted with a task that is too complex are motivated to either increase their skill level or engage in a task of less complexity. It is argued that enjoyment and interest work in different ways during this process, in that students' motivation to increase their level of skill emanates from a desire to experience the enjoyment associated with competence, whereas their motivation to engage in more complex tasks lies in a desire to experience interest.

The above theoretical discussion highlights the close relationship that is thought to exist between the emotions of interest and enjoyment in a learning context. Testing this and/or seeking to apply it in a middle school statistics context, is premised on the assumption that it is possible to empirically differentiate between the two emotions. Whereas researchers have achieved this with adult subjects (Reeve, 1989; Turner & Silvia, 2006) none noted have done so with adolescents, who are characterized by emotional instability (Larson, Moneta, Richards, & Wilson, 2002). The study, therefore, aims to determine the extent to which interest and enjoyment can be differentiated in a middle school statistics context.

Research Questions:

1. To what extent do students' levels of enjoyment and interest differ for tasks of varying complexity?
2. Do students' responses to interest and enjoyment items separate onto distinctive interest and enjoyment factors?

Method

Sample

Students from six Queensland schools were invited to participate in the study, which formed a part of a larger interest-based study described in Carmichael et al. (2010) and conducted during 2008 and 2009. The schools were selected for their proximity to the researcher and also to be broadly representative of the types of schools operating in Queensland. Three of the schools were state high schools; one a large metropolitan high school and the other two smaller rural high schools. The other three schools were independent K-12 schools, one of which was a single-sex school. A total of 424 students attending these schools were invited to participate in the study and 221 agreed to participate, a response rate of 52%. The majority of these students were female (62%) and the age of students ranged from 11 to 15 years, with a mean of 12.9 years. Most students were in the first two years of high school (79%), whereas the remainder was in the last year of primary school for that state.

Instrument

Students responded to an instrument containing 40 self-descriptions that assessed a number of constructs related to the acquisition of statistical literacy. Details regarding the instrument are given in Carmichael et al. (2010). Of interest to this study are three pairs of identical self-descriptions, with one in the pair assessing interest through the stem 'I'm interested in' and the other assessing enjoyment through the stem 'I enjoy (or would enjoy)'. These items are listed in Table 1 together with identifying codes that indicate where in the overall questionnaire they were situated. For example item I2, the second item in the questionnaire, assessed interest in using surveys to find out about others, whereas item E38, the 38th item in the questionnaire, assessed enjoyment in using surveys to find out about others. Students responded to each item using a five-point Likert Scale that ranged from 1 (*Not me at all*) to 5 (*Describes me well*).

Table 1

Self-descriptions Used in the Study and Associated Question Codes

Self-description	Interest code	Enjoyment code
Using surveys to find out about others	I2	E38
Working on problems involving data and statistics	I3	E35
Working out the probabilities (or chances) for dice, coins and spinners	I11	E34

The items reflect common tasks that these students should have encountered in their study of mathematics in Queensland (Queensland Studies Authority[QSA], 2004). By the end of Year 5, these students should have designed and conducted a variety of data

collection methods including surveys (QSA, 2004). Consequently many of the students should have competence in using surveys to find out about others. Similarly, students at the end of Year 5 in Queensland should be familiar with the term sample space and have used probability experiments to compare likelihoods. As a result, many of the students should have competence in the context presented in the third item, namely being able to work out probabilities. Although all students should have had experience solving problems, through the "thinking, working mathematically and reasoning" dimension of the syllabus, the very term "problem" suggests a challenge and with this some threat to students' perceived competence.

Given the theoretical role of each emotion in the motivational process and the expectation that enjoyment would exceed interest when perceived competence was high, the nature of these items suggest that reported enjoyment levels should exceed interest levels for the item pairs I2/E38 and I11/E34. However, given the threat to competence posed by the term "problem", reported levels of enjoyment should be lower than reported levels of interest for item pair IE/E35.

Analysis

In order to assess differences between levels of interest and enjoyment, the analysis commenced with the calculation of mean Likert responses for each of the items. In order to assess whether student responses to the six items loaded onto two factors an exploratory factor analysis was undertaken. In addition to this, a confirmatory factor analysis was also applied to the six items using AMOS (Arbuckle, 2008). More specifically, a structural model was established with items I2, I3, and I11 loading onto a latent interest variable, and items E34, E35 and E38 loading onto a latent enjoyment variable. Both latent variables were assumed to be correlated.

Results

Response distributions for each of the six items are shown in Figure 1, which lists plots of the interest items in the left-hand column and enjoyment items in the right-hand column. As is seen from the plots, the mode level of enjoyment was higher than the mode level of interest for item pair I2/E38, but it was lower than the mode level of interest for item pair I3/E35. Modes for the third item pair appear to be similar. In addition to this, means were also calculated for each item. These means are reported in Table 2, which reports the self-description, the mean level of interest in the self-description, and the mean level of enjoyment. As is seen from the table, all means were considerably less than the median Likert value of 3. The closest that these means came to reaching this value was for students' enjoyment in 'using surveys to find out about others', which was 2.92.

Table 2

Mean of Students' Likert Response to Interest and Enjoyment Items

Self-description	Mean interest	Mean enjoyment
Using surveys to find out about others (I2/E38)	2.24	2.92
Working on problems involving data & statistics (I3/E35)	2.40	2.26
Working out the probabilities for dice, coins & spinners (I11/E34)	2.32	2.48

Given the obvious non-normality of response distributions, differences between means were assessed with the Mann-Whitney test using SPSS (SPSS Inc, 2009). Only the enjoyment/interest means for the first item pair (I2/E38) were significantly different at the 5% level of significance ($Z = -5.4, p = 0.00$).

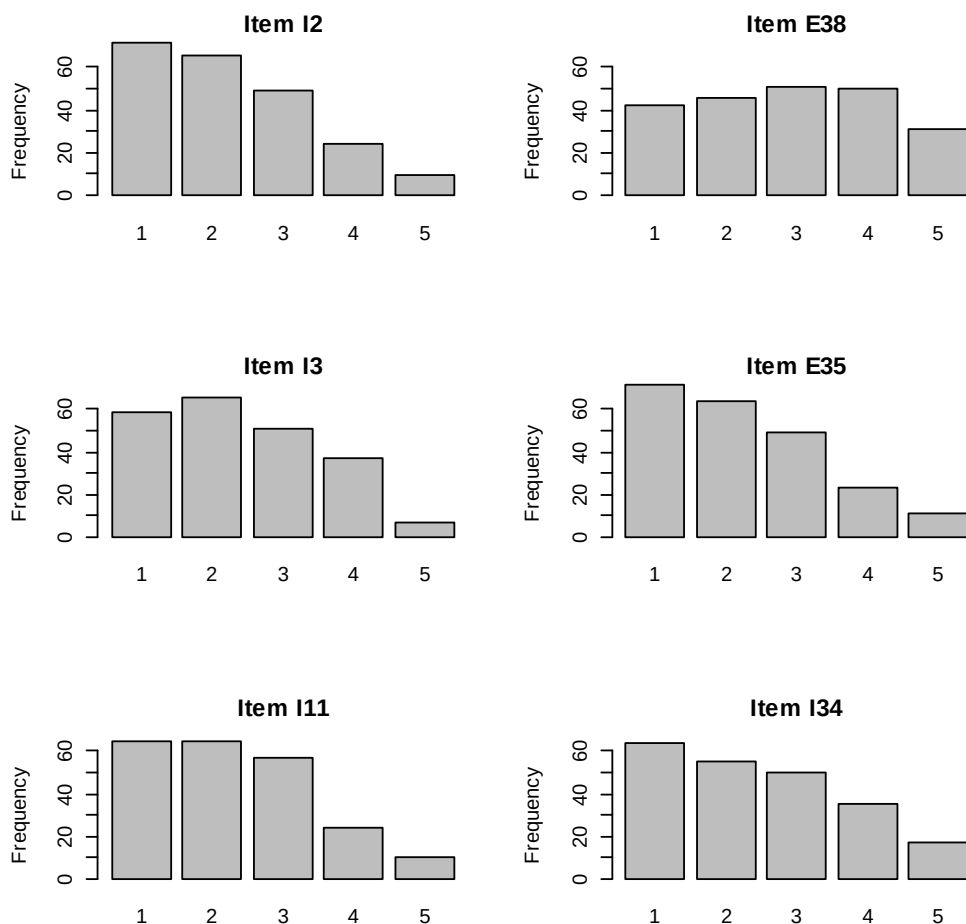


Figure 1. Response distributions for each of the three item pairs

An exploratory factor analysis was then applied to the six items. The scree plot and subsequent parallel analysis implied a two factor solution. Principal axes factoring followed by an Oblimin rotation produced the factor loadings shown in Table 3.

Table 3

Factor Loadings for Enjoyment and Interest Items (pattern matrix)

Item	Factor 1	Factor 2
I2		.840
E38		.787
I3	.749	
E35	.771	
I11	.815	
E34	.704	

Values less than 0.34 are suppressed because there is insufficient statistical power to differentiate them from zero. As shown in the table, the enjoyment items load onto the same factor as their corresponding interest items, indicating that students' ratings of enjoyment and interest for these tasks appear to be indistinguishable. A confirmatory factor analysis was also undertaken as described earlier. Model fit statistics ($RMSEA = 0.173$, $CFI = 0.843$) were outside of acceptable limits (Byrne, 2001), providing insufficient evidence to support the separation of the items into two factors.

Discussion

The reported differences between mean levels of enjoyment and mean levels of interest support the earlier hypothesis that levels of both emotions will differ when perceived task competence is high or low. Of the six items discussed above, using surveys to find out about others, reported the highest mean enjoyment rating, but reported the lowest mean interest rating. This significant discrepancy suggests that irrespective of the value that children placed on this task they viewed it as easy. Moreover, that perceived competence in using surveys positively influenced their levels of anticipated enjoyment.

Differences between the remaining two item pairs, although not statistically significant, were in the anticipated direction. Reported levels of interest in working on problems involving data and statistics (I3/E35) were higher than levels of enjoyment. This is likely a result of the inherent complexity associated with the term "problem". Such complexity induces interest (Berlyne, 1960) but at the same time challenges competency and thus task enjoyment. Similarly, reported levels of enjoyment in working out the probabilities for dice, coins and spinners (I11/E34) were higher than levels of interest. This result is likely a consequence of these children's competence in the task described in the item.

The small magnitude for the reported differences in item pairs I3/E35 and I11/E34 may reflect a lack of detail in the associated task descriptions. Working out probabilities associated with dice, for example, can include tasks with a considerable range of difficulties. Such lack of detail and the small number of items using in the study is a limitation and points to the need for further study using more and clearer items.

The results of the factor analyses did not lend support for the view that the emotions of interest and enjoyment are distinct, but rather that items load onto factors according to the contexts associated with the tasks. As shown in Table 3, the item pair I2/E38 loaded onto the same factor. It is argued that children may use surveys, for example online polls, outside of the classroom and that this factor may be influenced by this non-school context rather than the activity of using surveys. Similarly, the other item pairs appear to load onto a factor with a context that is more associated with the mathematics classroom, in that working on problems and working out probabilities are activities that are likely to occur in the classroom. The inability of the factor analysis to detect two distinct emotions may therefore be a result of the influence of these contexts. In a study of physics students, Haussler (1987) reported that the context associated with given learning tasks explained up to 60% of the variance in students' interest. Such an influence in the statistical literacy domain may invariably muddy the waters, making it difficult to differentiate empirically between two closely related emotions.

The low responses for all six items might also have influenced the results of the factor analyses in that they created a "floor" effect. Only the mean level of enjoyment in using surveys came close to the median Likert response of 3, the remaining means were considerably less. Had the items been able to elicit stronger levels of endorsement, response distributions may have been more symmetrical than those displayed in Figure 1, and factor

results may have differed. The inability of the items to elicit stronger levels of endorsement is another limitation of the study.

The large difference between the mean interest and enjoyment ratings for using surveys to find out about others is noteworthy. Children enjoyed this task but did not find it nearly as interesting. This is surprising, given the relatively large proportion of girls in the sample and evidence that girls tend to find interest in social applications (Jenkins & Pell, 2006). The result suggests that surveys have been used as a means of motivating these students in statistics lessons, but the topics covered in these surveys have lacked value. Statistics lessons can be fun if surveys are used, but they can be fun and interesting if the survey is used to answer a question that is of some value. A nice example of an activity involving data collection, though not specifically one using a survey, has been developed by Watson (2008). It stimulates children's interest through a media claim that is both uncertain and challenging.

Conclusion

It is important that children experience positive emotions when they engage in tasks associated with the development of statistical literacy, because this experience provides the motivation for further engagement. In this paper, theoretical perspectives have suggested that teachers and researchers need to consider specific positive emotions, such as interest and enjoyment. Despite the small sample of items used in the study, results support the view that feelings of enjoyment emerge when students gain competence in tasks. Yet attempts to differentiate between the two emotions were inconclusive, possibly because the contexts associated with statistical literacy tasks induce more emotion than the task itself, minimising the influence of students' perceptions of task competency and task value.

Whereas children need to experience the enjoyment that comes from successfully completing tasks, they also need to value the task. Teachers need to plan for the interest that children might find in tasks. A good starting point is a consideration of the novelty, complexity and/or uncertainty that students can anticipate from the task. As suggested by the first quote in the paper, children need to experience both enjoyment and interest.

References

- Arbuckle, J. (2008). *Amos (17.0.0)*. Crawfordville, FL: Amos Development Corporation.
- Berlyne, D. E. (1960). *Conflict, arousal, and curiosity*. New York: McGraw-Hill.
- Byrne, B. M. (2001). *Structural Equation Modelling with AMOS*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Carmichael, C. S., Callingham, R., Hay, I., & Watson, J. M. (2010). Measuring middle school students' interest in statistical literacy. *Mathematics Education Research Journal*, 22(3), 9-39.
- Csikszentmihalyi, M. (2002). *Flow: The classic work on how to achieve happiness*. London: Rider.
- Deci, E. L. (1992). The relation of interest to the motivation of behavior: a self-determination theory perspective. In K. A. Renninger, S. Hidi & A. Krapp (Eds.), *The role of interest in learning and development* (pp. 43-70). New Jersey: Lawrence Erlbaum Associates.
- Eccles, J. S., & Wigfield, A. (2002). Motivational beliefs, values, and goals. *Annual Review of Psychology*, 53, 109-132.
- Gal, I. (2002). Adult's statistical literacy: Meanings, components, responsibilities. *International Statistical Review*, 70(1), 1-51.
- Hausler, P. (1987). Measuring students' interest in physics-design and results of a cross-sectional study in the Federal Republic of Germany. *International Journal of Science Education*, 9(1), 79-92.
- Hulleman, C. S., Durik, A. M., Schweigert, S. A., & Harackiewicz, J. M. (2008). Task values, achievement goals, and interest: An integrative analysis. *Journal of Educational Psychology*, 100(2), 398-416.
- Izard, C. E. (1977). *Human Emotions*. New York: Plenum Press.
- Jenkins, E. W., & Pell, R. G. (2006). *The Relevance of Science Education Project (ROSE) in England: A summary of findings*: Centre of Studies in Science and Mathematics Education, University of Leeds

- Larson, R. W., Moneta, G., Richards, M. H., & Wilson, S. (2002). Continuity, stability, and change in daily emotional experience across adolescence. *Child Development*, 73(4), 1151-1165.
- Marcoulides, G. A., Gottfried, A. E., Gottfried, A. W., & Oliver, P. H. (2008). A latent transition analysis of academic intrinsic motivation from childhood through adolescence. *Educational Research and Evaluation*, 14(5), 411-427.
- Pekrun, R. (2006). The Control-Value Theory of Achievement Emotions: Assumptions, Corollaries, and Implications for Educational Research and Practice. *Educational Psychology Review*, 18(4), 315-341.
- Queensland Studies Authority. (2004). *Mathematics Years 1 to 10 Syllabus*. Retrieved from <http://www.qsa.qld.edu.au/syllabus/770.html>.
- Reeve, J. (1989). The interest-enjoyment distinction in intrinsic motivation. *Motivation and Emotion*, 13(2), 83-103.
- Schiefele, U. (1991). Interest, learning, and motivation. *Educational Psychologist*, 26(3 & 4), 299-323.
- Soanes, C., & Stevenson, A. (Eds.). (2008) *The concise Oxford English dictionary*. Oxford: Oxford University Press.
- SPSS Inc. (2009). *Statistical Package for the Social Sciences (SPSS): Windows (Version 18.0.0)* [Computer Software]. Chicago: SPSS Inc.
- Turner, S. A., & Silvia, P. J. (2006). Must interesting things be pleasant? A test of competing appraisal structures. *Emotion*, 6(4), 670-674.
- Watson, J. M. (2006). *Statistical literacy at school: Growth and goals*. Mahwah, New Jersey: Lawrence Erlbaum Associates.
- Watson, J. M. (2008). Eye colour and reaction time: An opportunity for critical statistical reasoning. *Australian Mathematics Teacher*, 64(3), 30-40.
- Watt, H. M. G. (2004). Development of adolescents' self-perceptions, values, and task perceptions according to gender and domain in 7th through 11th grade Australian students. *Child Development*, 75(5), 1556-1574.

The Effects of Creating Rich Learning Environments for Children to Measure Mass

Jill Cheeseman
Monash University
<Jill.Cheeseman@monash.edu>

Andrea McDonough
Australian Catholic University
<A.McDonough@acu.edu.au>

Sarah Ferguson
Australian Catholic University
<S.Ferguson@acu.edu.au>

This paper reports on a design experiment regarding young children's concepts of mass measurement. 119 year one and two children were interviewed using a clinical interview both before and after the teaching period comprising five lessons that offered rich learning experiences regarding concepts of mass. The results of the interviews were that the majority of these Year 1 and 2 children moved from using non-standard units to using standard units and instruments for measuring mass.

The teaching of measurement concepts is often sequenced from non-standard to standard units (see for example, Outhred & McPhail, 2000). Traditionally children explored each attribute informally long before encountering standard units or standard tools for measuring (Clements & Battista, 1986). However, more recent studies have questioned whether this sequence is the best approach (Clements, 1999; Clements & Bright, 2003). Studies of children's development in concepts of length for example, found that using standard units better supported children's development of measurement and that non-standard units could interfere with this development (Boulton-Lewis, Wilss & Mutch, 1996). Similarly, withholding standard instruments such as rulers until the end of a teaching sequence has also been questioned (Nunes, Light & Mason, 1993). Nunes et al. found that using a ruler supported children's reasoning about length and improved their performance "clearly profiting from the numerical representation available through the ruler" (p. 46). McDonough and Sullivan (2011) found that although children moved through a sequence of comparing lengths, unit iteration and use of standard units over the first three years of school, there was a great diversity at each year level and suggested that children could be introduced to standard units of length in the second year of school.

Although the design experiment reported in this paper was concerned with children's concepts of mass, the results of studies in other areas of measurement such as length are pertinent, particularly as studies regarding mass are scarce. Furthermore, it is important that research advice that children can and should move more quickly to the use of standard units and instruments in measurement of length be tested in context of mass measurement.

This study builds on earlier research (Clarke, Cheeseman, McDonough, & Clarke, 2003). In an earlier paper (Cheeseman, McDonough & Clarke, 2011) we argued that rich experiences involving measuring mass are needed, particularly at the Year 1 level where the data we reported from interviews with 479 students indicated that little progress appeared to have been made over a year. By rich mathematical learning experiences we mean those in which children are offered opportunities to engage in activities which have the potential to lead to conceptual understanding in mathematics, that challenge children to think, and foster the communication of mathematical reasoning.

The earlier paper also emphasised the importance of teachers assessing children's understandings of mass measurement and structuring learning opportunities to build on and extend those understandings. Having made these recommendations we designed the present

study to further investigate the learning impacts on young children of structured opportunities to learn about mass.

Methodology

Design and Teaching Experiment

The study can be considered design research (Cobb, Confrey, DiSessa, Lehrer & Schauble, 2003). A design experiment was described by Cobb et al. as “engineering particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them” (2003, p. 9). This research involved a classroom experiment in which the research team collaborated with a teacher who was a member of the research team to take responsibility for instruction. The research intent was to assess children’s concepts of mass, “engineer” some rich experiences through one week of lessons and then re-assess the children’s understanding of mass.

We chose this methodology to examine the complexity of the classroom; including the tasks and problems the children were asked to solve, the discourse, the participation in the classroom and the materials with which the children were engaged. We acknowledge that design research is a highly interventionist methodology. In this case it drew on our prior research and gave the research team control over specified learning intentions. However, we acknowledge while the central matters are able to be controlled, many of the ancillary elements of classroom conditions cannot.

Content of the Lessons

A unit of work on the topic of mass measurement was developed for Year 1 and 2 children. Five one-hour lessons were planned to replace the existing school planning in the area of mass. The lessons were designed to follow typical growth of children’s concepts of measurement identified in the literature (Outhred & McPhail, 2000). Accordingly the lessons began with comparing mass and moved toward quantifying mass using standard units of measure. The essential difference here was the more rapid move toward standard units. The lessons were designed to engage the children in rich experiences including real world applications, problem solving and play-based approaches (Baroody, 2009; Copley, 2006). The following discussion gives some insight into the nature of the tasks.

The first lesson, *Party Bag Surprises*, focused on comparing and ordering. The children worked in pairs, choosing and placing objects in opaque party bags to create three bags of different masses. The masses were created and ordered by hefting, that is, by holding a bag in each hand and comparing masses. For each set of three bags, another pair of children was challenged to work out the order of the masses by hefting (without being able to see what was inside each bag). Upon the children’s request, balance scales were made available for further comparison of mass. Much discussion occurred during this task. After using informal units of mass and balance scales in the second lesson, the third lesson of the sequence involved children measuring the mass of various envelopes and packages in a “Post Office”. The children used interlocking plastic cubic centimetres, each identified as having a mass of one gram. The cubes were available in sticks of ten that is, as ten gram weights, and as loose single cubes. The children chose a variety of parcels and using a balance scale and the cubes, measured the weight of each parcel and labelled it with the number and “g” for grams (see Figure 1). A discussion followed about surprises the children had regarding the weight of the parcels and issues such as height, shape or size relating to mass were explored. In the following lesson children were offered a range of experiences

with objects including fruit and vegetables and packaged foods, and using a selection of scales including balance scales and digital and analogue kitchen scales. During the final lesson children made their own set of weights using playdoh, with the purpose of assisting them to develop mass benchmarks.



Figure 1. Children weighing envelopes and packages using Centicubes.

Pre and Post Interview Data

Task-based interviews were used to assess children’s learning before and after the teaching phase of the study. Each child was interviewed twice (see Cheeseman, McDonough & Clarke, 2011) by teachers trained in the use of the protocol. Children proceeded as long as they continued to have success with tasks. Written records were made of each child’s responses for later examination and analysis. Subsequently codes were assigned to the responses to specify the growth point demonstrated by each child.

Coding Analysis

Codes were assigned independently by at least two of the authors according to the framework growth point demonstrated by success on the interview tasks. The tasks were designed to assess the milestones in young children’s thinking about the measurement of mass as defined in the framework of growth points developed in the Early Numeracy Research Project (ENRP, Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough et al., 2002), shown Table 1. The pre-and post-interview data were analysed using SPSS.

Table 1
Mass Measurement Framework (Clarke et al., 2002)

0. Not apparent
<i>No apparent awareness of the attribute of mass and its descriptive language.</i>
1. Awareness of the attribute of mass and use of descriptive language
<i>Awareness of the attribute of mass and its descriptive language.</i>
2. Comparing, ordering, & matching with the attribute of mass
<i>Compares, orders, & matches objects by mass.</i>
3. Quantifying mass accurately, using units and attending to measurement principles.
<i>Uses uniform units appropriately, assigning number and unit to the measure.</i>
4. Choosing and using standard units for estimating and measuring mass, with accuracy
<i>Uses standard units for estimating and measuring mass, with accuracy.</i>
5. Applying knowledge, skills and concepts of mass
<i>Can solve a range of problems involving key concepts of mass.</i>

Teacher-researcher Role of the Authors

The role of the authors was that of active participant-observers in each classroom. The third author, who is a practising primary school teacher, taught lessons during the teaching experiment.

Observational Data

In addition to the quantitative data, observational data were gathered by all authors to provide further insights into young children's thinking about mass measurement. Notes, photographs and audiotapes were used to document actions, comments and thoughts of children during the lessons.

Results

The data in Table 2 show that the children made substantial gains in their learning as measured by the interview results. For the Year 1 students the development in their thinking involved moving from an awareness of the attribute of mass (30%) and some comparison of masses (12%) in the initial interview, to quantifying masses (40%) and using standard units to weigh objects (53%) in the second interview.

The Year 2 students' results show similar development in mass measurement thinking. By their second interview, only 1% of the students were unable to quantify mass, whereas 33% had been unable to quantify mass initially. Further, 89% were using standard units to weigh objects in the second interview. In fact an examination of the data indicated that there was a ceiling effect in the assessment interview as the achievement of Growth Point 5: *Applying knowledge, skills and concepts of mass* was very difficult. This will be discussed later in the paper.

Table 2

Students (%) Achieving Growth Points Pre- and Post-teaching

Interview growth point	Year 1	Year 1	Year 2	Year 2
	Interview 1	Interview 2	Interview 1	Interview 2
	Oct 2011	Nov 2011	Oct 2011	Nov 2011
	(n=57)	(n=57)	(n=62)	(n=62)
Not apparent	2	0	2	0
Awareness of attribute	30	2	15	0
Comparing masses	12	5	16	1
Quantifying masses	49	40	37	10
Using standard units	7	53	27	84
Applying knowledge	0	0	3	5

Examining the data from the original study alongside the present data reveals an interesting distribution of patterns. Table 3 shows the distribution of interview results expressed as percentages. The two sets of results with the most similar patterns are those of the Year 1 students who had been part of the teaching experiment and the Year 2 original students. So it can be seen that after an intensive week of teaching and learning about mass, the Year 1 students had a similar profile of thinking as did Year 2 students from the original (ENRP) study.

Table 3
Students (%) Achieving Mass Growth Points

Interview growth point	Year 1 Nov 2001 (n = 479)	Year 1 Interview 2 Nov 2011 (n=57)	Year 2 Nov 2001 (n = 256)	Year 2 Interview 2 Nov 2011 (n=62)
Not apparent	1	0	0	0
Awareness of attribute	2	2	0	0
Comparing masses	17	5	6	1
Quantifying masses	69	40	50	10
Using standard units	10	53	38	84
Applying knowledge	1	0	6	5

A summary of the change in growth point code from the first to the second interview is shown in Table 4. The first and most startling result to examine is the negative growth by one student. This was a student who achieved growth point 5 at the first interview and when he measured 135grams of rice and was asked, “How many more grams of rice would you need to have one kilogram?” he initially answered 965g but quickly self-corrected to 865g. At the second interview 8 weeks later he was asked the same question as the interview protocol called for students to begin the interview one step before their first error. He calculated the answer as 965g but on the second occasion he did not self-correct.

On reflection the authors believe that having the correct answer to this question as a “gatekeeper” for a growth point code of 5 *Applying knowledge, skills and concepts of mass* was too stringent a requirement. The reason being the skills required to answer the question were more about the mental computation (1000 – 135) than about measurement knowledge.

One third of the cohort of students remained at the same growth point of the mass framework after one week of teaching and learning. This is hardly surprising as a growth point in the major research project from which the framework originated; the Early Numeracy Research Project (Clarke et al., 2002) found that a growth point typically took a student about a year to achieve. The growth points were designed to be major milestones in children’s mathematical thinking.

Approximately one third (38%) of the students had developed their thinking about mass by one growth point after a week of enriching experiences. And 28% of the students made impressive learning gains of 2, 3 or 4 growth points. The mean was 1.1 growth point gain.

Table 4

Difference between 2nd and 1st Interview Results

Change in growth point from Interview 1 to Interview 2 (in growth points)	Frequency (n=119)	Percent
-1	1	1
0	39	33
1	45	38
2	16	13
3	16	13
4	2	2

To illustrate the growth possible from the first to the second interview, we can examine the case of Andrew, a Year 1 boy. In the first interview, Andrew could heft to compare the mass of items and was able to use the language of heavier and lighter. However, Andrew seemed uncertain about using balance scales and was unable to explain how the scales showed items were heavier or lighter. During the first lesson of the teaching week, *Party Bag Surprises* (described previously) it was noted that Andrew worked enthusiastically with his partner to discover the difference in mass of two very similar bags. This pair asked to use digital scales so they could “get a number” that would show which was heavier. During the fourth lesson, Andrew was observed persisting for some time to find the mass of one potato using balance scales and a set of mixed weights. After many minutes absorbed in this task, Andrew was able to proudly say that the potato had a mass of 275 grams. In the second interview, it was clear that Andrew had made remarkable growth over the course of the teaching week. He was able to use the balance scales to measure with informal units and could also accurately weigh objects on the balance using standard weights giving the answer in numbers and grams. Andrew demonstrated that he was beginning to understand how to use kitchen scales and could explain how the scale worked. Perhaps of most note was that Andrew, a quiet unassuming student, became one of the most enthusiastic and eager participants during the teaching week.

Discussion

The Importance of Rich Learning Opportunities

It could be argued that the reason the children in this study made such marked learning gains was that they were taught by an outstanding teacher, the third author. However, this is true for only 3/5 of the children each day. Sarah taught the lesson on three occasions with other teachers present as observers and co-teachers and those teachers then taught the lessons with the children in their classrooms. The demonstration lessons were scheduled on a rotational basis around 5 classrooms over the course of a week. An examination of the data shows that there were no differences in the outcomes for children between teaching groups.

Curriculum Expectations

The curriculum is written in broad terms with explicit curriculum statements about mass in the Measurement and Geometry strand (see Table 5) of the *Australian Curriculum* (Australian Curriculum Assessment and Reporting Authority, 2012).

The first thing to note is that there are no explicit expectations at Year 1 level in the new curriculum documents. Based on the results of this study this would seem to be an oversight by the curriculum developers. Further, the expectations of Year 2 students seem to underestimate the potential of young students to learn about standard units that measure mass.

The speed with which the children in this study adopted the language of mass measurement and facility with use of standard units for measuring mass appeared to indicate their preparedness to use such units. The traditional approach of using informal units for an extended period before the introduction of standard units is challenged by this study as the majority of the children interviewed were able to measure accurately with standard units in the final interview. During the lessons focused on informal units, children often used the language of standard units, albeit incorrectly. For example, one boy described the difference between the masses of two party bags as “just an inch”, gesturing that the difference was

small. McDonald (2011) in examining the measurement concepts of young children also noted “the children show a remarkable awareness of a range of formal units” (p. 490). We agree with Copley who advises that, “teachers should not limit their expectations of young children. Providing a variety of experiences along with reflection and communication about those experiences will result in some surprising results!” (Copley, 2004, p. 2)

Table 5

Australian Curriculum: Mass Measurement

Level	Measurement and Geometry: Using units of measurement
Foundation	Use direct and indirect comparisons to decide which is longer, heavier or holds more, and explain reasoning in everyday language (ACMMG006) p. 16.
Year 1	Measure and compare the lengths and capacities of pairs of objects using uniform informal units (ACMMG019) p. 18.
Year 2	Compare masses of objects using balance scales (ACMMG038) p. 20.
Year 3	Measure, order and compare objects using familiar metric units of length, mass and capacity (ACMMG061) p. 23.
Year 4	Use scaled instruments to measure and compare lengths, masses, capacities and temperatures (ACMMG084) p. 26.
Year 5	Choose appropriate units of measurement for length, area, volume, capacity and mass (ACMMG108) p. 29.
Year 6	Connect decimal representations to the metric system (ACMMG135) Convert between common metric units of length, mass and capacity (ACMMG136) p. 32.

Conclusion

In conclusion we argue that what is important is to offer young children the opportunity to engage with interesting and stimulating contexts for learning to measure mass in which the tasks connect to their lives and the mathematical potential is rich. Of course that is not to underestimate the value of challenging mathematical conversations during the conduct of the lesson (Cheeseman, 2008, 2010; Ferguson, 2012). Nor is it to underestimate the experiences with measurement that some young children bring to school (McDonald, 2011). There is an old saying that experience is the best teacher and the children in this study have shown that they certainly have learned from their rich and thought-provoking experiences with measuring mass.

References

- Australian Curriculum Assessment and Reporting Authority (2012). Australian Curriculum: Mathematics Retrieved 23rd January 2012, from <http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10>
- Baroody, A. J. (2009). Fostering early numeracy in preschool and kindergarten. *Encyclopedia of Language and Literacy Development*, 1-9.
Retrieved from <http://literacyencyclopedia.ca/pdfs/topic.php?topId=271>
- Boulton-Lewis, G. M., Wilss, L. A., & Mutch, S. L. (1996). An analysis of young children’s strategies and use of devices of length measurement. *Journal of Mathematical Behavior*, 15, 329-347.

- Cheeseman, J. (2008). Young children's accounts of their mathematical thinking. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepulveda (Eds.), *Proceedings of the joint meeting of PME 32 and PME-NA* (Vol. 2, pp. 289-296). Morelia, Mexico: International Group for the Psychology of Mathematics Education.
- Cheeseman, J. (2010). *Challenging children to think: An investigation of the behaviours of highly effective teachers that stimulate children to probe their mathematical understandings*. Unpublished doctoral thesis, Monash University, Melbourne.
- Cheeseman, J., McDonough, A., & Clarke, D. (2011). Investigating children's understanding of the measurement of mass. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices* (Vol. 1, pp. 174-182). Adelaide, SA: Australian Association of Mathematics Teachers and the Mathematics Education Research Group of Australasia.
- Clements, D. (1999). Teaching length measurement: Research challenges. *School Science and Mathematics*, 99, 5-11.
- Clements, D. H., & Battista, M. T., (1986). Geometry and geometric measurement. *Arithmetic Teacher*, 33(6), 29-32.
- Clements, D., & Bright, G. (2003). *Learning and teaching measurement 2003 Yearbook*. Reston, VA: National Council of Teachers of Mathematics.
- Clarke, D. M., Cheeseman, J., McDonough, A., & Clarke, B. A. (2003). Assessing and developing measurement with young children. In D. H. Clements & G. W. Bright (Eds.), *Learning and teaching measurement* (Yearbook of the National Council of Teachers of Mathematics, pp. 68-80). Reston, VA: NCTM.
- Clarke, D. M., Cheeseman, J., Gervasoni, A., Gronn, D., Horne, M., McDonough, A., et al. (2002). *Early Numeracy Research Project: Final report, February 2002*. Fitzroy, Victoria: Australian Catholic University, Mathematics Teaching and Learning Centre.
- Cobb, P., Confrey, J., DiSessa, Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9-13.
- Copley, J. V. (2006). "Are you bigger than me?" A young children's mathematical thinking about measurement. Lecture presented at the International Conference on Logical Mathematical Thinking. Retrieved from <http://www.waece.org/cdlogicomatematicas/ponencias/juanitaycopley_pon_ing.htm>
- Ferguson, S. (2012). *Like a bridge: Scaffolding as a means of assisting low-attaining students in mathematics during cognitively challenging tasks*. Unpublished doctoral dissertation, Australian Catholic University, Melbourne.
- McDonald, A. (2011) Young children's representations of their developing measurement understandings. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices* (Vol. 1, pp. 420-490). Adelaide, SA: Australian Association of Mathematics Teachers and the Mathematics Education Research Group of Australasia.
- McDonough, A., & Sullivan, P. (2011). Learning length in the first three years of school. *Australasian Journal of Early Childhood*, 36(3), 27-35.
- Nunes, T., Light, P., & Mason, J. H. (1993). Tools for thought: The measurement of length and area. *Learning and Instruction*, 3, 39-54.
- Outhred, L., & McPhail, D. (2000). A framework for teaching early measurement. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000. Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 487-494). Fremantle, WA: MERGA.

Developments in Pre-service Teachers' Mathematics for Teaching of Fractions

Mohan Chinnappan
University of South Australia
<mohan.chinnappan@unisa.edu.au>

Tricia Forrester
University of Wollongong
<tricia@uow.edu.au>

Elise Thurtell-Hoare
University of Wollongong
<ejth170@uowmail.edu.au>

The mathematics required for teaching is increasingly becoming an important issue for research (Ball, Thames & Phelps, 2008). This study examines the development and quality of the mathematical content knowledge of a cohort of pre-service primary teachers. We commenced the study of the impact of a Model-Based Teaching and Learning (MBTL) approach on the development of pre-service teachers' conceptual knowledge in the domain of fractions (Forrester & Chinnappan, 2011). In this present study we found further evidence for the robustness of MBTL as an effective instructional strategy in promoting conceptual knowledge.

Issue and Context

The quality of teacher knowledge in relation to children's learning outcomes in mathematics (Ball, Hill, & Bass, 2005) has highlighted the need to research and monitor the developing knowledge of mathematics teachers who are in practice and those who are in training. Towards this end the broad area of inquiry focusing on *Mathematics for Teaching* (Ball, Thames, & Phelps, 2008) is significant. Interest in the types of knowledge necessary for teaching has its roots in the work of Shulman (1987) who conceptualised that teaching (in any domain) needs to be supported by subject-matter knowledge that in turn has to be reshaped into pedagogical content knowledge (PCK).

Ball and her associates (Ball, 2003; Ball, et al., 2008), building on Shulman's work, have identified several types of content knowledge necessary for teaching, two of which are particularly relevant to this research, namely, Common Content Knowledge (CCK) and Specialised Content Knowledge (SCK). CCK is the content knowledge used in a range of settings whereas SCK is the content knowledge "unique to teaching" (Ball, et al., 2008, p. 400).

Our research in this area was motivated by the need to better understand the quality SCK of primary teachers of mathematics who are enrolled in a pre-service program. Our previous research was focused on comparing the procedural and conceptual knowledge of fractions of a cohort of Pre-Service Teachers (PSTs) at the end of their first year of undergraduate studies (Forrester & Chinnappan, 2010). This cohort will be referred to as Cohort 1. The results of this analysis demonstrated the dominance of procedural over conceptual knowledge in this group, with almost four times the number of pre-service teachers activating procedural knowledge in comparison to those that demonstrated conceptual knowledge in their solution attempts. About one fifth of responses evidenced neither procedural nor conceptual knowledge.

While both knowledge categories are important, the dominance of procedural over conceptual knowledge would seem to be unhealthy for classroom practice, as teachers will have to support the development of both across all strands of primary mathematics, including fractions. This line of reasoning motivated us to modify our teaching strategies

with the view to enhancing the conceptual component of our pre-service teachers' knowledge of fractions. The following year, we sought to address this lack of conceptual understanding by adopting a Model-Based Teaching and Learning (MBTL) strategy with a second cohort of first year PSTs (Cohort 2). The results of this teaching intervention produced a significant improvement in the balance of the two knowledge categories with about 65% of the students who received MBTL demonstrating conceptual knowledge (Forrester & Chinnappan, 2011).

In the present study we report the results of our continuing work in the effectiveness of MBTL in fostering the construction and development of conceptual knowledge in the domain of fractions by examining changes in the knowledge of Cohort 1 whose knowledge was shown to be predominantly procedural (Forrester & Chinnappan, 2010), as they completed their third year mathematics content and pedagogy subject.

Objective

The aim of this study was to ascertain the impact of a MBTL approach on the development of procedural and conceptual knowledge in the domain of fractions among a cohort of pre-service teachers (PSTs) as they progressed from the first to third year of their undergraduate program. The following research question guided the study:

1. Does a MBTL have an impact on the development of pre-service primary teachers' conceptual and procedural knowledge of fractions?
2. Does a MBTL approach have an impact on the development of pre-service primary teachers' procedural knowledge of fractions?

Relevant Literature

Model Based Teaching and Learning (MBTL)

The focus of our teaching approach was on models and modelling of fractions. By focusing on models, we adopted the view that PSTs construct and reason with entities called mental models (Johnson-Laird, 1983). These are internal representations of the targeted phenomena (fractions) that are being modelled. Such models can be given external representations by the learner in the form of actions, speech and visuals. Our teaching approach was driven by the need to assist learners to engage in developing and expressing mental models. This view is aligned with that of Gobert and Buckley (2000) who argued that model-based teaching 'is any implementation that brings together information resources, learning activities, and instructional strategies intended to facilitate mental model-building both in individuals and among groups of learners' (p. 892). The strategies for MBTL align with the principles suggested by Gobert and Buckley (2000) which describes model-based teaching as "any implementation that brings together information resources, learning activities, and instructional strategies intended to facilitate mental model-building" (p. 892).

Procedural versus Conceptual Knowledge

Procedural knowledge involves understanding the rules and routines of mathematics while conceptual knowledge involves an understanding of mathematical relationships. The relationship between procedural and conceptual knowledge, and the dependency of one on the other, continues to be a legitimate concern for mathematics teachers and researchers alike. Schneider and Stern (2010), in examining potential interconnections between the two, suggested that teaching and learning research needs to examine their parallel developments.

Within the context of primary mathematics, in particular fractions, Mack (2001) suggests that children's use of strategies for representing and solving fractions problems are based on both these knowledge strands. The relationships amongst and the relative roles of these two main dimensions of knowledge, relevant to decoding and solving fractions problems, needs further clarification if we are to better inform teachers about knowledge underlying teaching.

Method

Participants

One hundred and twenty PSTs (14 males and 106 females) participated in the present study. They were enrolled in a third year compulsory subject, which is generally completed in the third year of a four year Bachelor of Primary Education degree. They were completing the second core mathematics content and pedagogy subject in their degree, having completed the first core subject in the first year of their degree. Their responses to fraction exam questions in the first core subject were reported previously (Forrester & Chinnappan, 2010). Prior to entry into the degree course, the participants had a range of mathematical backgrounds.

Model Based Teaching and Learning

The MBTL approach used in this project evolved from the use of animated PowerPoint models provided in lectures and as support materials to develop PSTs' conceptual understanding of fraction concepts and operations. While these models were used to explain fraction concepts and operations, students were not actively engaged in the modelling process and only procedures were examined to assess understanding.

In the 2009 rendition of the first year core subject, efforts were made to focus more specifically on developing PSTs' conceptual understanding of mathematical concepts. The assessment of PSTs required them to demonstrate their knowledge of fraction concepts and operations by providing both a calculation and a model. Exam responses were analysed in terms of the evidence of procedural and/or conceptual understanding and many PSTs evidenced poor conceptual understanding of fraction concepts (See Table 2) with nearly 20% of students evidencing neither forms of understanding, (Forrester & Chinnappan, 2010).

These results led to the introduction of MBTL being introduced into the core mathematics subjects, where the models previously made by the lecturer in demonstrating and explaining mathematical concepts, were also constructed and explained by PSTs to develop their own conceptual understandings. Utilising the representational-reasoning model of understanding theorised in the work of Barmby, Harries, Higgins and Suggate (2009), which maintains that robust mathematical understanding is demonstrated when learners can construct and utilise multiple representations of mathematical ideas and can justify the relationships among representations, we focused on enabling our students to develop and explain models of fraction concepts and operations as well as utilising appropriate algorithms to solve fraction problems.

Tasks

The following fraction problems were provided as questions in the final examination of the core mathematics content and pedagogy subjects in the degree. Tasks 1a and 1b involved subtraction while Tasks 2a and 2b involved multiplication. Structurally, 1b is

similar to 1a; likewise with Tasks 2a and 2b. Task 1a and 2a were completed in 2009 whereas their variants (1b and 2b) were attempted by the same PSTs in 2011. These particular tasks were chosen to examine students' mathematics content knowledge in terms of their conceptual and procedural knowledge of fractions and fraction algorithms.

Task 1a: $1\frac{2}{5} - \frac{5}{6}$

Task 1b: $1\frac{3}{8} - \frac{3}{4}$

Task 2a: $\frac{1}{4} \times \frac{2}{3}$

Task 2b: $\frac{1}{3} \times \frac{3}{5}$

In attempting to solve the problems, PSTs were instructed to i) complete these calculations and ii) model the operations.

Coding Scheme

Students' responses to each of the two problems were analysed in terms of their demonstration of conceptual and procedural knowledge, and coded using a five code scale (see Table 1). The development of the coding scheme was guided by the theoretical framework of Barmby, Harries, Higgins and Suggate (2009) and Goldin's (2008) analysis of problem representations.

Table 1
Coding Scheme

0	No evidence of procedural or conceptual understanding
1	Procedural. Correct algorithm and/or model only of fractions
2	Procedural/conceptual. Correct algorithm and model demonstrates more than a procedural understanding of a concept involved in fraction operations
3	Conceptual. Correct algorithm, model of concept evident
4	Conceptual with explicit reasoning. Correct algorithm, model supported with language

Results

The MBTL approach that underpinned our teaching was aimed at examining changes procedural and conceptual knowledge of PSTs. Quantitative data analyses were conducted with the aid of SPSS version 18. Table 2 shows the percentage figures for the respective codes for the subtraction and multiplication problems respectively for 2009 and 2011, for the entire cohort of participants.

We note a number of patterns that are relevant to our two key research questions about the impact of MBTL on the development of procedural and conceptual knowledge. Firstly, in 2009, for both the problem types, there are higher percentages with codes 0, 1, and 2 (predominantly procedural knowledge) in comparison to codes 3 and 4 (supporting conceptual knowledge). We see the reverse in 2011, for both subtraction and multiplication problems. We thus have preliminary support for our expectation that a MBTL approach has induced among our PSTs the development of conceptual knowledge of fractions.

Table 2
Percentage Figures for the Codes

Code	Subtraction Problem		Multiplication Problem	
	2009	2011	2009	2011
	%	%	%	%
0	17.8	7.0	18.7	7.0
1	53.3	5.2	58.9	4.3
2	17.8	20.0	9.3	17.4
3	11.2	49.6	10.3	43.5
4	0	18.3	2.8	27.8

The above pattern is also reflected in particular cases of modelling as evidenced by solutions provided PST1 and PST 2.

Case of one PST1

In both the responses PST1 produced correct solutions by activating an appropriate algorithm (Figures 1 and 2). However, the model that was constructed in 2009 indicates misconceptions regarding the relationship between parts and wholes, and the notion of equivalence (Figure 1). These misconceptions seem to have been corrected in the model that was constructed in her 2011 solution attempt (Figure 2).

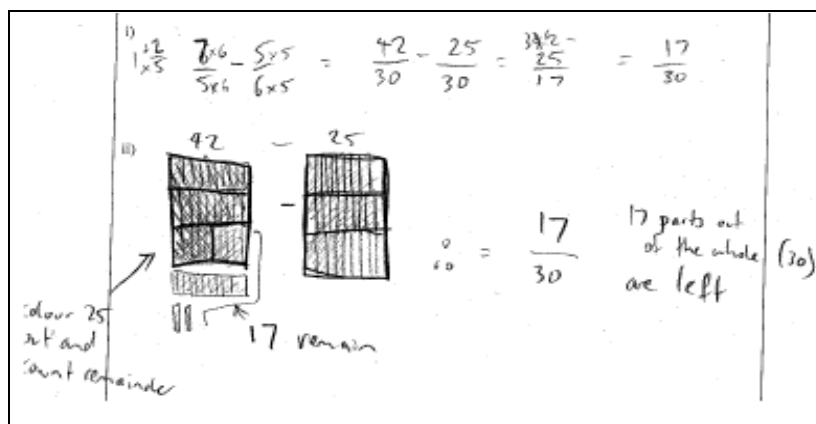


Figure 1. PST1's responses to subtraction task in 2009

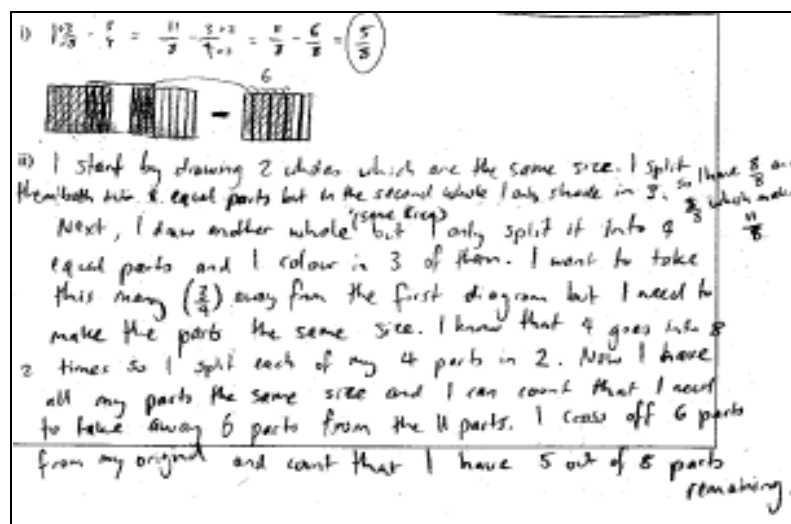


Figure 2. PST1's responses to subtraction task in 2011

PST2 also produced correct solutions to the multiplication problems in both the 2009 and 2011 solution attempts by drawing on and executing a correct algorithm (Figures 3 and 4). We note the model produced in 2011 (Figure 4) engages with the concept of multiplying by a fraction rather than simply representing the procedure diagrammatically. The 2009 model did not engage substantively with the notions of equivalence or the relationship of parts to the whole, whereas the 2011 model demonstrates an understanding of these concepts (Figures 3 and 4). These responses illustrate the types of improvement evident among many PSTs in this cohort following our intervention with a MBTL approach.

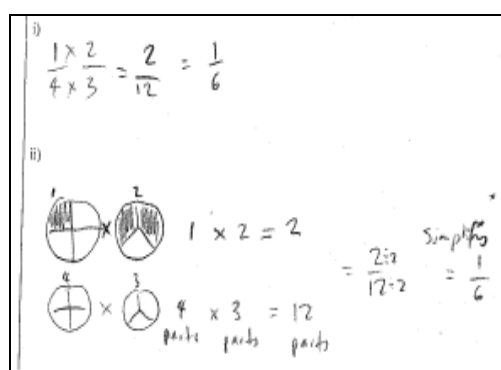


Figure 3. PST2's responses to multiplication task in 2009

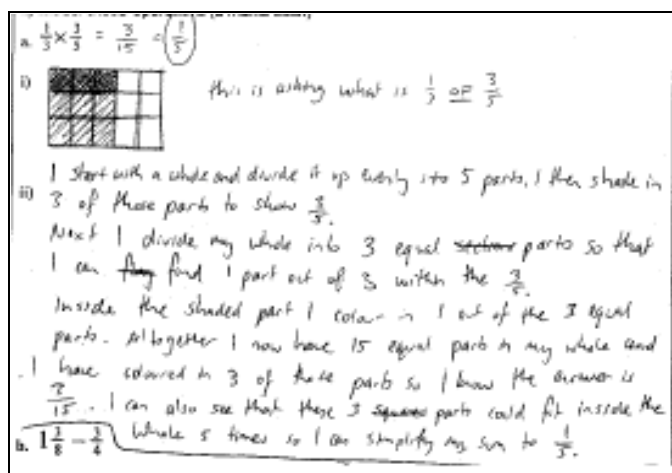


Figure 4. PST2's responses to multiplication task in 2011

Discussion

Ball, Hill and Bass (2005), in their articulation of *Mathematical Knowledge for Teaching*, suggested that teachers need to construct and deconstruct content knowledge in particular forms such that that knowledge is germane to learning. This cluster of knowledge was referred to as Specialised Content Knowledge (SCK). In our study we have attempted to unpack this knowledge further along conceptual and procedural lines. In so doing we argue that conceptual knowledge may subsume procedural knowledge and indeed contribute to a better understanding of related procedural knowledge, and it is important to capture and support both strands of knowledge if we are to develop metrics of SCK both quantitatively and qualitatively. Is conceptual knowledge better than procedural knowledge for practice? We suggest that there has to be a balance and that teachers' SCK ought to exhibit both these characteristics.

The research questions were concerned with the impact a teaching approach based on modelling would have on the development of PSTs' procedural and conceptual knowledge of fractions. The results here suggest that, while there was no tangible effect on procedural knowledge, our teaching had a positive affect on pre-service teachers' conceptual knowledge. The question is how did MBTL influence the conceptual knowledge of the PSTs? Two key facets of MBTL were a) the representation of the fraction symbols into equivalent visual forms and b) the interpretation of the operations in terms of the visuals. We suggest that these separation and subsequent integration was instrumental in developing conceptual knowledge. For example, Task 2a was interpreted as 'one quarter of two thirds'. Such an interpretation was also better aligned with the visual representations of both the fractions.

The positive effect of MBTL on conceptual knowledge may also have a similar impact on the confidence of our PSTs in teaching fractions. This is a hypothesis on our part and is based on the premise that less confident teachers tend to draw on mathematical knowledge that is more procedural in nature. Murphy (2011), for instance, found support for the above contention in her study of prospective teachers' knowledge underlying the concept of area.

There is an on-going debate within the research community about the relations between procedural and conceptual knowledge, regardless of the domain in which these are studies (Rittle-Johnson, Siegler, & Alibali, 2001; Schneider & Stern, 2010). In the present study, we have added to this debate by not isolating the two strands of knowledge in the domain of

fractions but also showing the need for the two to be tightly connected so that we can support better learning among children.

One limitation of the present study is that it was not conducted within an experimental design that involved treatment and control conditions. Thus we are guarded in making claims about the efficacy of MBTL. We suggest that future studies use an experimental design with pre- and post-tests to better measure the effect of MBTL on the development of procedural and conceptual knowledge.

References

- Ball, D. L. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton, Alberta, Canada: Canadian Mathematics Education Study Group (Groupe Canadien d'étude en didactique des mathématiques).
- Ball, D. L., Hill, H. C., & Bass, H. (2005). Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 5(3), 43-46.
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.
- Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2009). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics*, 70, 217-241.
- Forrester, P. A., & Chinnappan, M. (2010). The predominance of procedural knowledge in fractions. In L. Sparrow, B. Kissane & C. Hurst (Eds.), *Shaping the Future of Mathematics Education MERGA33* (pp. 185-192). Freemantle, W.A.: MERGA Inc.
- Forrester, P. A., & Chinnappan, M. (2011). Two avatars of teachers' content knowledge of mathematics. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics Traditions and [New] Practices AAMT/MERGA Conference 2011* (pp. 261-269). Alice Springs N.T.: AAMT & MERGA.
- Gobert, J. D., & Buckley, B. C. (2000). Introduction to model based teaching and learning in science education. *International Journal of Science Education*, 22(9), 891-894.
- Goldin, G. A. (2008). Perspectives on representation in mathematical learning and problem solving. In L. English (Ed.), *Handbook of international research in mathematics education*. New York: Routledge.
- Johnson-Laird, P. N. (1983). *Mental models*. Cambridge, MA.: Harvard University Press.
- Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education*, 32(3), 267-295.
- Murphy, C. (2011). The role of subject knowledge in primary prospective teachers' approaches to teaching the topic of area. *Journal of Mathematics Teacher Education*, DOI 10.1007/s10857-10011-19194-10858.
- Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93(2), 346-362.
- Schneider, M., & Stern, E. (2010). The developmental relations between conceptual and procedural knowledge: A multimethod approach. *Developmental Psychology*, 46(1), 178-192.
- Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1-22.

Mathematics Knowledge for Teaching: Evidence from Lesson Study

Mohan Chinnappan

University of South Australia

<mohan.chinnappan@unisa.edu.au>

Ui Hock Cheah

SEAMEO-RECSAM

<uhcheah@recsam.edu.my>

In this study we take up the issue of teacher knowledge by characterising the quality of one teacher's Mathematics Knowledge for Teaching (Ball, Thames and Phelps, 2008) by examining her Pedagogical Content Knowledge (PCK) that was accessed during a Lesson Study observation in a Chinese primary school. Results show that the participant's PCK was robust in the way she weaved knowledge of content and her students. We explore implications for the development and sharing of this knowledge in the context of her further involvement in our Lesson Study.

Introduction

Mathematics learning should be supported by experiences where students play an active role in the construction of meaning and concept development. While there is little disagreement with the above view, we cannot ignore the role of teachers in developing and using such learning experiences in the classroom. But what are some of the key factors that could have potentially significant impact on the belief and ability of teacher to enact such classroom experiences? This question has been of concern for both researchers and stakeholders in the provision of quality mathematics teaching.

In this context, teacher knowledge continues to be the subject of inquiry (Hill, Rowan, & Ball, 2005; Forrester & Chinnappan 2011; Murphy, 2011). Ball et al. (2005) have been investigating the dimensions and quality of knowledge that teachers need in order to function effectively in the mathematics classroom. This stream of research has yielded evidence for the broad dimension of knowledge called *Mathematics Knowledge for Teaching* (MKT). However, as acknowledge by Ball et al. (2008), such evidences are limited due the tasks that were used in making judgements about MKT. It was argued that research need to widen the contexts against which MKT could be examined more reliably. More specifically, Ball et al. (2008), suggested that research need to examine the quality of MKT in classroom practice.

In the present study, we take up the above issue by investigating the MKT of a primary school teacher as she participated in the Lesson Study (LS) in a Chinese primary school. MKT is multidimensional but our focus was on Pedagogical Content Knowledge (PCK) that was accessed by the participating teacher during the course of the LS. Thus, PCK is seen as a subset of MKT as argued by Ball et al. (2008). A LS is generally driven by a group of teachers with minimal intervention from the researcher. It has been suggested that LSs should be used to examine teacher professional development. For example, Olson, White and Sparrow (2011:56) commented that,

'more research is needed to describe how lesson study and other types of professional development supports sustained teacher change and the impact of new pedagogies supported by professional development on student learning'.

Our study has been aimed at answering the research question: What is the nature of PCK as activated during the course of a Lesson Study in primary mathematics?

Relevant Literature

Lesson Study

In the present study we set out to examine the above issue in the context of Lesson Study. Recent research in mathematics teaching has identified LS as a rich context in which to examine teacher's CK and PCK (Lim 2007; Tepylo & Moss, 2011; Meyer & Wilkerson, 2011). Lesson Studies have generated considerable interest among teachers and mathematics educators. One can advance at least two reasons as to why Lesson Studies have carved out a niche area for mathematics teachers and researchers. Firstly, LS are situated in day-to-day classroom teaching-learning contexts and is largely initiated and driven by teachers as a professional development activity. Thus LS provide a level of authenticity about teacher knowledge that is not available for access in other contexts such as clinical interviews. Secondly, while LS is conducted as a professional development exercise, it does involve the use of research methodology including design and data analysis.

A key feature of LS is teacher collaboration where individual work as individuals and in groups. Knowledge grows out of this interaction among colleagues and LS provides multiple opportunities for teachers to access and modify their knowledge of mathematics and that of their children (Lewis, Perry and Hurd, 2009). Thus, the discourse between the teachers and students before, during and after lessons generate rich source of data that can be used to gain insight into not only the classroom culture of learning but also changes in their knowledge and understandings.

Theoretical Framework

In examining teacher knowledge that was activated during the course of our Lesson Study, we aimed to characterise the knowledge in terms of a number of dimensions relevant to teachers' classroom practice and the profession of teaching. In this regard, we were guided by the following model of teacher knowledge for teaching mathematics that was advanced by Hill, Ball and Schilling (2008: 174). Four dimensions of their framework are shown in Figure 1. Common Content Knowledge (CCK): Mathematical knowledge and skill possessed by a well-educated adult; Specialised Content Knowledge (SCK): Knowledge of how to use alternatives to solve a problem; articulate mathematical explanations; demonstrate representations; Knowledge of Content and Students (KCS)- Knowledge that combines knowing about mathematics and knowing about students, Knowledge of how to anticipate what students are likely to think; relate mathematical ideas to developmentally appropriate language used by children; Knowledge of Content and Teaching (KCT): Knowledge that combines knowing about mathematics and knowing about teaching. In the present study we focussed on KCS which is a component of teachers' PCK (a subset of MKT).

Design and Methodology

Research Design

A mixed method involving qualitative (case study) and quantitative research design was used to identify one teacher's (Kim) emerging PCK as she participated in LS. Johnson and Christensen (2012) argued that such an approach was appropriate for classroom-based research. Data were collected through a video of Lesson Study teaching. Data was also acquired through a simulated recall interview with Kim, the teacher who is the focus of the study. During this interview, the video of the lesson was used to assist Kim to reflect on the

events that happened during the teaching episode. Our analyses for this report are part of a larger study which involved the study of changes in mathematics and pedagogy of four teachers who chose different areas of primary school mathematics during the course of the LS. In the present study we focus on Kim's MKT as she engaged her children in the learning of fractions.

Participants. A group of six primary school teachers in the state of Penang (Malaysia) took part in a LS professional development program. The group was one of three groups of teachers in the school that took part in the program. The first group consists of teachers who were teaching Year 1 mathematics, the second group consists of teachers teaching Year 2 mathematics and the third group were teaching Year 3 mathematics. This paper reports on a teacher who was a member of the third group of teachers who were teaching Year 3 mathematics. Kim, was in her third year of teaching in the school. At the time of this study she was also undergoing a *Vacation Diploma in Teaching Course* (VDTC) to acquire teaching qualifications. Participants of the VDTC were regular teachers in the school but attended classes at the Teacher Education Institutes during the school vacation.

Lesson study as a professional development activity. The LS was conducted over a 6-month period during which all six teachers worked as a team in lesson design, implementation and reflection phase. There was an initial social visit to the school, the purpose of which was to introduce the idea of lesson study to the school teachers. Subsequently there were two more meetings with the teachers before the actual research lesson. During these meetings the teachers were grouped into three groups according to the level they were teaching (year 1, 2, and 3), for lesson planning.

The Year 3 teachers chose the topic on fractions for their lesson plan. The aim of the lesson was to introduce the concept of fractions. The participating teachers were invited to develop a series of lesson plans that were aimed at helping children adopt a problem-solving approach in mathematics. Kim and her team members wanted to develop a sense of problem-seeking and solving approach among her students.

The students were informed about the presence of teacher observers in the lesson. The class was the normal class where Kim was teaching the students for the past six months. The lesson was organised to engage the students using whole-class interaction for most parts of the lesson and small groups (five to six students in a group) during group activities.

The study examines MKT that was accessed by Kim during the course of her 40-minute lesson on fractions. The lesson was conducted in a Chinese primary school in the metropolitan area of Penang, Malaysia. Prior to the lesson observation, Kim had presented her lesson plans to her other five colleagues and invited their feedback. Prior to the lesson observation, the teachers met together with the external "knowledgeable others" twice. During the first meeting the Kim and her five other colleagues were introduced to the idea of lesson study. During the second meeting, the teachers together with the knowledgeable others began discussing the lesson to be observed. Kim volunteered to teach the lesson at the suggestion of the group. Subsequently, Kim and her colleagues in the lesson study group continued planning the lesson together.

Results

Structure of the Lesson

Kim's lesson can be described as consisting of three identifiable phases. In the *Introductory Phase* of the lesson, the teacher called two students to the front of the class and asked them to divide four sweets equally between them. This was a preamble to the lesson

and it was mainly aimed at children's attention and involvement. The idea of dividing equally was an important concept and effective advanced organiser. This activity can be seen as providing an important clue for making connections between students' prior knowledge and a new concept that is to be processed and understood.

In the *Activity Phase* of the lesson, Kim used a combination of whole class teaching and group activity to introduce and develop the core concept of parts-whole and their relations. She showed that a fraction is one or more equally divided parts of a whole. In the first activity, the teacher displayed circles, cut into halves and quarters, on a flannel board. In the ensuing discourse, the teacher guided the class to say that the 'whole is cut into a number of equal parts' and 'the parts joined together makes one whole'. The activity was repeated using rectangles cut into four equal parts. While the first activity was driven by the teacher and therefore somewhat closed, in a subsequent activity, Kim recognised the need to provide space for the children to explore their understanding via an open-ended activity where the students work in groups to colour strips of papers into parts to show different fractions. The *Review Phase* of lesson involved Kim requesting students' views about the lesson and learning that was supported by the lesson. This information was obtained by students responding to a questionnaire.

Introductory Phase Episode

In the interaction below Kim was attempting to get the students to extend their understanding that the numbers of parts a whole can be divided into equal parts and that these parts put together make a whole.

- Teacher: One. Four over four is one, or what we discussed just now, two over two is one. This denotes one. Ok, now you look at the other set of building blocks. What can you notice? How many parts are there? What can you see?
- Student: Four parts.
- Teacher: Four parts. Ok, four parts. Other than seeing it as consisted of four parts, what else can you observe? Teacher: What is that?
- Student: Circle.
- Teacher: Circle. What else? What else can you see? Your..your building blocks. You look at your building blocks. What else can you see?
- Student: Four over four.
- Teacher: Four over four. Oh...any answer else? Any other answer? What else can you see? Yes, any other answer? This group? What else can you see? There are many...there are many...what is that?

We see evidence of the blending of content and emergence of PCK. The content is focussed on part-whole relations. The use of concrete aid (building blocks) to model the relations is indicative of Kim's knowledge of the use of multiple representations to support deeper understandings of the focus concept. Kim originally intended to use the idea of cutting into equal parts but the group suggested using cut pieces of equal size and having the students put the parts together. Kim found that group's idea was more practical and easy because the students would probably be not able to cut shapes into parts of equal size.

During this phase Kim also attempted to draw students' attention to the symbolic representation of fractions ('Four over four'). Comments such as "Any other answer? What else can you see? Yes, any other answer", suggest she was probing students and inviting responses that might provide the teacher with an insight into students' own meanings that were different from that of the teacher. We regard this action, again, as another case of PCK where a teacher shows empathy with her students and anticipates the construction of idiosyncratic meanings about fractions.

Kim explained that she did not want to tell students the answer, instead she would give some clues and let student try to answer. This belief was further reinforced by the lecturers in the VDTC, “My lecturer says that if you want students to learn, we must not give student answer straight away [so that] student will think.” For Kim, asking questions will encourage two-way communication between the teacher and the students thus avoiding one way communication which may lead the students to become bored.

Activity 1 Phase Episode

In this open-ended activity students were given paper strips and asked to fold and shade the fractions (See Figure 2). The students’ responses showed that they were able to fold the paper in many parts. Some students folded the paper strips into two parts while others were able to fold up to sixteen equal parts. Some parts were rectangular in shape while others were triangular.



Figure 2. Folding and colouring paper strips into parts.

In the following interaction, Kim revisits her point about the relations between the number of parts and the whole in a fraction but this time with a model that involved strips of papers rather than circles. Her PCK in this instance indicates Kim’s understanding that modelling of part-whole relations should be not be fixed to any particular object.

- Teacher: No, you don’t have to. Ok, a half. Ok, what else? ... What else? (choosing paper strip) Xiang Yi divided her paper strip into four quarters. Ok, four parts: One, two, three, four. Ok, this is from Xiang Yi. Ok, Ethan’s... his is colourful. Ok. He divided..he divided it into how many parts? One, two, three, four, five, six, seven, eight. He divided it into eight parts. Ok, these are from the first group.
- Teacher: How about the second group? The second group. Ok. Ok, let’s see Jing Rou’s. Jing Rou, see...let’s see Jing Rou’s. Jing Rou, raise your paper strip, turn around and let everyone see. Ok, how many parts did Jing Rou divide her paper strips into? How many? One, two, three, four, five, six, seven, eight. She made it into eight parts and coloured them. Any other else?
- Student: (One of the group student) Mine is eight parts too.
- Teacher: Oi, yours is eight parts too.
- Student: (Another group student pointing to another group student) Teacher, he has extremely many parts.
- Teacher: Hah, Puteri’s one is divided into four..four parts only. One, two, three, four. She divided it into four parts only.
- Teacher: Who else? Do you have others?

Kim explained that she knew that students liked doing art, so if she could she could show the students how to separate the parts, they will be able to learn the idea of fractions.

Review Phase Episode

Table 1 provides responses from the students that was generated at the closure of Kim's lesson.

Table 1
Student Response about Classroom Environment

Statement	Yes (%)	No (%)
I shared with my classmates what I knew in the lesson	100	0
I got help from my classmates	88.5	11.5
I helped students who have trouble understanding the lesson	73.1	26.9
The teacher asked questions	88.5	11.5
I asked the teacher some questions	57.7	42.3
I asked my classmates some questions	65.4	34.6
My classmates talked with me about how to do the activities and problems	76.9	23.1
I showed and explained how I solved a problem to my classmates	76.9	23.1
I learned from my classmates in the lesson	88.5	11.5
The teacher was fair to me and my classmates	100	0
The Math I learned in the lesson can be used at home/the supermarket/store/everywhere.	88.5	11.5
I learned new and interesting things about Math in the lesson	84.6	15.4
What I learned is not useful at home/the supermarket/store/outside school.	76.9	23.1
I understood the lesson.	100	0
The lesson was fun.	96.2	3.8

From the student responses it is seen that most of the items received more than 70% of positive response. Two items, "I asked the teacher some questions" and "I asked my classmates some questions" scored 57.7% and 65.4% respectively.

As indicated in Table 1, the classroom discourse was guided mostly through teacher-initiated questions. It was however observed that Kim made an effort to restrain from giving the answers directly to her questions. Instead, often she probed the class through further questions. It was also observed that the teacher did not capitalise on the students various responses to further deepen the students' understanding of fractions.

Kim expressed her appreciation for being part of her LS group, "Normally you plan alone and you teach alone...[here we] Plan in group. When I am teaching, [I] need to improve. [Other] teachers can let me know [so] I will improve."

Discussion

We commenced our study on the premise that teachers' knowledge play a key role in the decisions they make about the content and delivery of their lessons. In so doing our main interest was the quality of one teacher's PCK that was accessed during her participation in a Lesson Study that was undertaken in a Malaysian Chinese primary school.

One aspect of this teacher's knowledge was her willingness to use a variety of activities in the classroom to engage students to experiment with different representations of the relations between parts and wholes. This engagement of students that was supported by the

introduction of multiple representation gives one insight to her KCS (Knowledge of Content and Students), one of the three components of PCK are suggested by Ball et al (2008). One facet of Kim's KCS indicates that she had become aware of the different ways her students come to understand part-whole relations embedded in both symbolic and conceptual representations.

A second aspect of Kim's KCS can be related more to KCT, rather than KCS. We note the use of a blend of direct instructions and open-ended activities during the lesson. This is indicative of her view that students need clear direction about what to do and needed time to explore the connection between the notion of parts and wholes. On the basis of his work on cognitive processing of information, Mayer (2011:86) argued that teachers needed to provide learning experiences that not only promotes 'prime generative processing but also provide enough guidance to make sure that learners engage in appropriate amounts of essential processing and do not engage in excessive amount of extraneous processing'. We contend that the direct teaching phase of the lesson where Kim models part-whole relations as providing the required *essential processing* while her subsequent use of open-ended activities supporting *generative processing*. The latter activity was conducted in group situations thus allowing for reduction in cognitive load.

Kim's comments on the lesson strongly indicated that she placed high value on the practical aspects of PCK. For Kim, it was important to translate her knowledge into ways of organising student activities that were practical. This was also seen in her appreciation for her group members when they suggested practical and useful ways of managing the student activities.

Lewis, Perry and Murata (2006:13) argued that educational research needs to 'vigorously explore lesson study's potential as a tool of instructional improvement'. In the present study, we have undertaken such a study where a thicker description of a teacher's PCK during in-class action provides important directions about her own strengths and areas of knowledge that needs to be developed. In the long run, we argue that, this information can be expected have significant impact on the quality of mathematics instruction.

The description of Mathematics Knowledge for Teaching that is based on Lesson Study is an emerging field of research. In this study we aimed to capture one teachers' knowledge. It would be interesting to examine what effect this knowledge could have on that of the other teachers who participated in the LS. Equally important is to capture the knowledge of Chinese mathematics teachers that helps them orchestrate lessons that engage students more deeply than traditional classroom learning contexts.

References

- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59, 389–407.
- Doig, B., Groves, S., & Fujii, T. (2011). The critical role of task development in Lesson Study. In Hart, L. C., Alston, A. S., & Murata, A. (Eds.), *Lesson study research and practice in mathematics education* (pp. 181-200). Dordrecht: Springer.
- Forrester, P. A., & Chinnappan, M. (2011). Two avatars of teachers' content knowledge of mathematics. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics Traditions and [New] Practices AAMT/MERGA Conference 2011* (pp. 261-269). Alice Springs N.T.: AAMT & MERGA.
- Hill, H. C., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372–400.
- Hill, H.C. Rowan, B., & Ball, D (2005). Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement *American Educational Research Journal*;42(2), 371-406.

- Hill, H. C., Blunk, M., Charalambous, C., Lewis, J., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An exploratory study. *Cognition and Instruction*, 26, 430-511.
- Johnson, B., & Christensen, L. (2012). *Educational Research: Quantitative, Qualitative and Mixed Approaches* (4th Ed). Thousand Oaks , CA: Sage.
- Lewis, C., Perry, R. R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education*, 12, 285-304.
- Lewis, C., Perry, R., & Murata, A. (2006). How Should Research Contribute to Instructional Improvement? The Case of Lesson Study. *Educational Researcher*, 35(3), 3–14.
- Lim, C. S (2007). Developing Mathematical Thinking in a Primary Mathematics Classroom through Lesson Study: An Exploratory Study. *Proceedings of the APEC-Khon Kaen International symposium on Innovative Teaching Mathematics through Lesson Study (II): Focusing on mathematical thinking* (pp, 141-153), Khon Kaen: Thailand.
- Mayer, R. E. (2011). *Applying the science of learning*. Boston: Pearson.
- Meyer, R. D., & Wilkerson, T. L. (2011). Lesson Study: The impact on teachers' knowledge for teaching mathematics. In Hart, L.C., Alston, A. S., & Murata, A. (Eds.), *Lesson study research and practice in mathematics education* (pp. 15-28). Dordrecht: Springer.
- Murphy, C. (2011). The role of subject knowledge in primary prospective teachers' approaches to teaching the topic of area. *Journal of Mathematics Teacher Education*, DOI 10.1007/s10857-011-9194-8.
- Olson, J.C., White, P., & Sparrow, L. (2011). Influence of Lesson Study on Teachers' mathematics pedagogy. In Hart, L.C., Alston, A. S., & Murata, A. (Eds.), *Lesson study research and practice in mathematics education* (pp. 39-57). Dordrecht: Springer.
- Tepilo, D. H., & Moss, J. (2011). Examining change in teacher mathematical knowledge through Lesson Study. In Hart, L.C., Alston, A. S., & Murata, A. (Eds.), *Lesson study research and practice in mathematics education* (pp. 59-78). Dordrecht: Springer.
- Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Characteristics of Problem Posing of Grade 9 Students on Geometric Tasks

Puay Huat Chua

Ministry of Education, Singapore

<chua_puay_huat@moe.gov.sg>

Khoon Yoong Wong

National Institute of Education, Singapore

<khoonyoong.wong@nie.edu.sg>

This is an exploratory study into the individual problem-posing characteristics of 480 Grade 9 Singapore students who were novice problem posers working on two geometric tasks. The students were asked to pose a problem for their friends to solve. Analyses of solvable posed problems were based on the problem type, problem information, solution type and domain knowledge. With the open-ended task, the students tend to over-condition their problems and to produce more problems with implicit assumption. How the findings can contribute to research in problem posing in schools is discussed.

Introduction and Brief Review of Literature

Complementing the notion of mathematical problem solving is problem posing which often requires imaginative skills (Gonzales, 1998). For the present study, Silver and Cai's (1996) notion of problem posing as the "the generation of a new problem from a mathematical situation or experience" (p. 294) will be used. Various studies have pointed to the importance of students' mathematical problem posing. This is related to students' exploration in mathematics (Cai, 2003) and the teaching and learning of mathematics (Crespo, 2003). Bransford, Zech, Schwartz, and Vye (1996) noted that developing students' ability to formulate their own problems is important for developing the mathematical thinking needed to solve complex problems. Brown and Walter (1993) noted that problem-posing activities in classroom helped in lessening mathematics anxiety, in explicating misconceptions and in fostering group learning and that "we learn mathematics when we were actively engaged in creating not only the solution strategies but the problems that demand them" (p. 187). They also strongly endorsed it as one important element of mathematical proficiency. English (1995) noted that while there was considerable amount of studies on children's ability to solve problems, less work was done on children's ability to pose them. Pirie (2002) noted that in the last two decades, problem posing had only received "sporadic interest within the field of mathematics education" (p. 4). Problem-posing studies tended to revolve around students writing of story problems and with the students coming from the elementary grades. In Singapore, problem posing has also not been emphasized in the mathematics curriculum. One goal of the present exploratory study is to further contribute to the knowledge base in problem-posing research in geometric tasks.

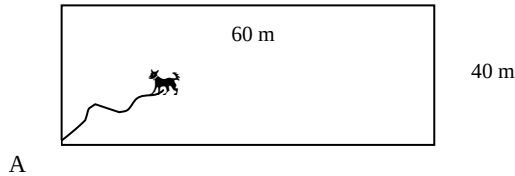
Methodology

Participants and Problem-Posing Tasks

The subjects comprised 480 grade 9 students (222 males, 258 females) from three secondary schools in Singapore. The two tasks were chosen because they were within the curriculum. Students solved their posed problems. The first task involved the writing of a problem for their friends from a known answer: "Write a problem so that the final answer is 60° ." This answer-specific task required the poser to form the initial state, the goal state and

the building of a context for the emerging problem. The task, piloted earlier by Chua and Yeap (2008), contained little context and so afforded itself for students to freely explore options of posing their problems. The second task focused on building and developing a problem from a given context and checked if there were variations in posing. It required the students to pose a problem for their friends based on the situation:

“A goat is inside a 60 m by 40 m rectangular fence in a farm. It is tied to a pole at A by a 30 m rope which could not be stretched.”



Analysis

The problem-posing characteristics were inferred from the posed problems and the solutions. Students' solutions could show how they have conceptualized their problems since the solving and posing processes are complementary (Contreras, 2007). The solution strategies and the use of the mathematical domain knowledge could give insights into the posed problems (Cai, 2003). Drawing from work by English and Halford (1995), problems were first classified as either solvable or non-solvable. A solvable problem has a well-defined initial state, a goal state and an inherent solution path. Solvable problems were coded based on the problem types (*relational, direct recall*), problem information (*edit information, add object, over-conditioning, implicit assumption*), solution type (*multi step, use of algebra*) and domain knowledge (topics). The descriptions of these categories are given below. Binary descriptors were used to indicate the presence or absence of these attributes in each solvable problem. Three raters independently coded the responses with high degree of agreement with the researcher's codings. Crosstabs were then used to investigate the strength of association (*phi*) between problem types, problem information and solution type. A non-solvable problem (92 in Task 1, 98 in Task 2) was either *ambiguous* (no explicit goal state or were non-mathematical in nature) or it involved a *contradiction*. Non-solvable problems were not further analyzed. Missing entries occurred when there were no attempts made (18 in Task 1, 19 in Task 2). Table 1 shows the distribution of the problem types and characteristics.

Table 1

Frequency Table of Problem-Posing Characteristics of Solvable Problems

Problem Type	Problem Characteristics		Task 1(% [^])	Task 2(% [^])
<i>Relational</i> Task 1, n = 198 Task 2, n = 176	i) Problem information	<i>edit information</i>	-	19.3
		<i>add object</i>	-	25.6
		<i>over-conditioning</i>	21.7	0
		<i>implicit assumption</i>	40.4	19.9
	ii) Solution type	<i>multi step</i>	97.0	94.3
		<i>use of algebra</i>	57.1	0
<i>Direct recall</i> Task 1, n = 172 Task 2, n = 187	i) Problem information	<i>edit information</i>	-	9.1
		<i>add object</i>	-	8.6
		<i>over-conditioning</i>	29.7	0
		<i>implicit assumption</i>	22.1	10.2
	ii) Solution type	<i>multi step</i>	15.1	2.7
		<i>use of algebra</i>	53.5	0

[^] as % within problem type

Classification by Problem Type

a) Relational Problem

In this study, a *relational problem* is a problem that involves a relation of at least two concepts in the solution path. Student F341's *relational* problem in Figure 1 involves *angles in a triangle* and *angle circle properties*.

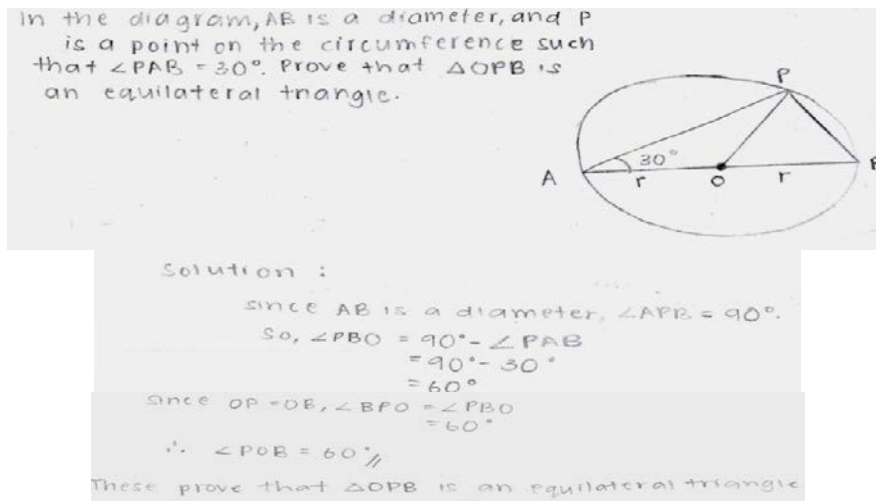


Figure 1. Student F341 posed problem and solution

b) Direct Recall Problem

A *direct recall* problem only has an assignment proposition in its solution path. Work by student T338 in Figure 2 is an example of a *direct recall* problem. It involves the *sum of angles at a point*. Such problems usually produce a direct link between the initial state and the goal state.



Figure 2. Student T338 posed problem and solution

Classification by Problem Information

a) Edit Information

Brown and Walter (1993, p. 23) proposed two problem-posing strategies. In “accepting the given”, the poser poses a problem without changing what is given. The other strategy is “challenging” the given stimulus by making changes. Christou, Mousoulides, Pittalis and Pantazi (2005) referred to this “challenging” strategy as *editing* the information. Student W302 in Figure 3 *edited* the original condition that rope was not stretchable. *Edit information* was not used as a descriptor for Task 1 since there was insufficient context for editing.

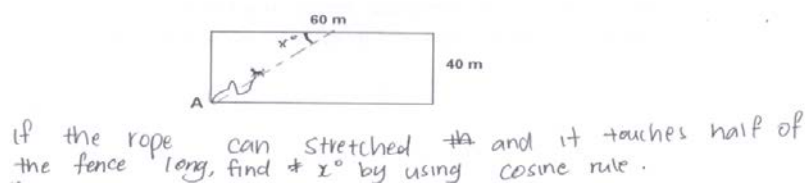
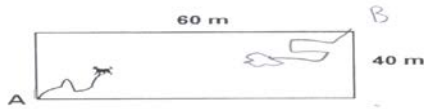


Figure 3. Student W302 posed problem and solution

b) *Add Object*

In Task 2, some students added objects or information into the initial state. Student N341 introduced another goat B into the initial state. This was analogous to the *selecting* process described in Christou *et al.*'s (2005) work when students added objects selectively to the initial state. The descriptor *add object* was only used in Task 2 since the nature of Task 1 had no contextual element to add to.



"Another goat B was tied at B. If goat B could only be able to reach 25% of the grass in the farm and ate the grass, find the length of the rope. Leave your answer to 3 sig fig."

Figure 4. Student N341 posed problem and solution

c) *Over-Conditioning*

Over-conditioning occurs when there is extraneous information which does not contribute to the solution path and to the reaching of the goal state. In student T234's work in Figure 5, the angle 75° at point A is not needed for solving for x . *Over-conditioning* may suggest that the poser is not proficient in linking elements in the initial state to the goal state during posing. *Over-conditioning* was not found in posed problems in Task 2.

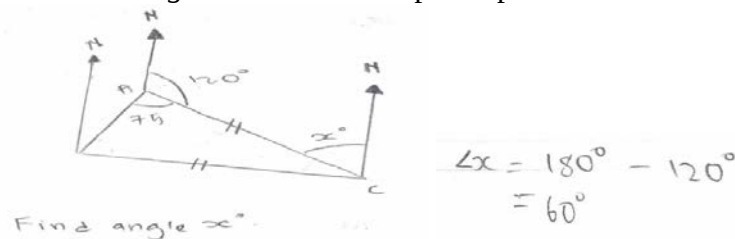


Figure 5. Student T234 posed problem and solution

d) *Implicit Assumption*

Implicit assumption occurs when students make unwritten assumptions about the given initial state. In student T309's work, an assumption was made about the triangle formed as being right angled and without which the problem could not be solved. *Implicit assumption* reflects flaws in the linkages between the initial and the goal states and the students' inability to fully articulate the initial conditions.

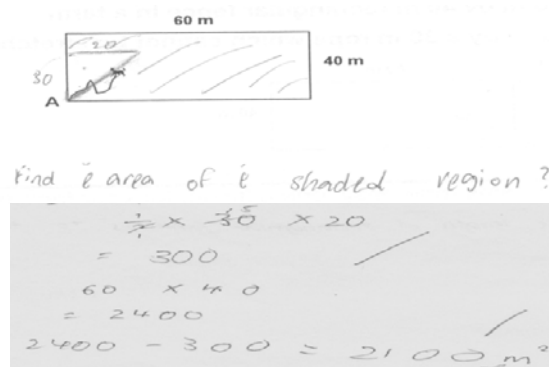


Figure 6. Student T309 posed problem and solution

Classification by Solution Type

a) Multi Step

A *multi step* problem involves more than one operation given by the student either as a repeated application of one operation or a combination of different operations in its solution path. Student N405's *multi step* solution in Figure 7 involves the *area of the sector* and the *area of the rectangle*. Since bridging the initial and the goal states requires the solver to build *multi step* in the solution path, *multi step* problems could be linked to problem complexity. But in the *single step* problem (non- *multi step*) by student T312 in Figure 7 there is only a need to state the answer by using the *interior angle* property of parallel lines.

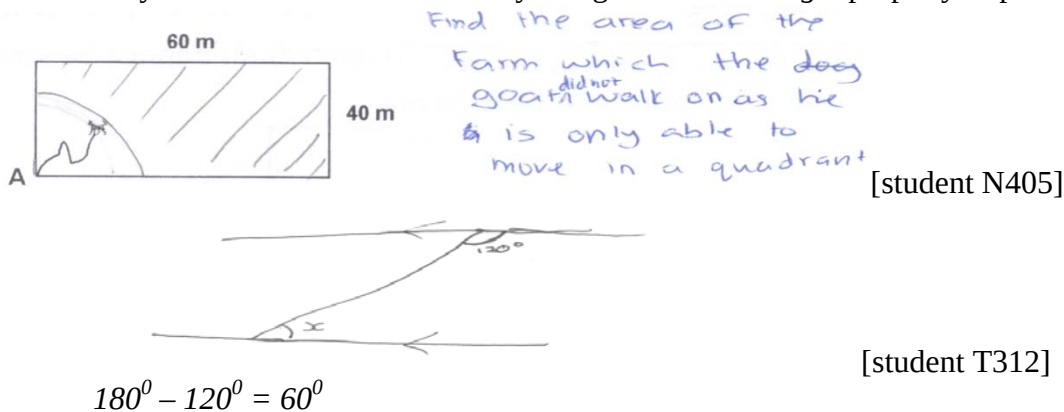


Figure 7. Students N405 and T312 posed problems

b) Use of Algebra

Algebraic variables as symbols in labelling points or vertices, sides and angles of figures were found in students' posed problems and their solutions.

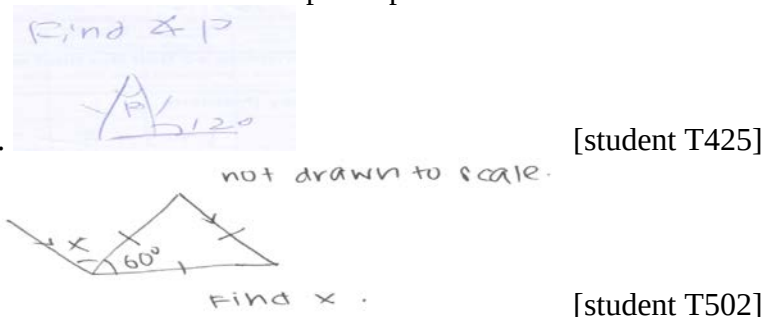


Figure 8. Students T425 and T502 posed problems

In Figure 8, students T425 and T502 had used p and x respectively to refer to “specific unknowns” (Kuchemann, 1981) in their posed problems.

Classification by Domain Knowledge

Posed problems reflected the extent of students' enactment of their learning of the related domain knowledge (Mamona-Downs and Downs, 2002). Table 2 shows students' use of the topics was dependent on the task type. Posed problems in Task 1 had the most number of geometric topics. More students used *Pythagoras Theorem* in Task 2. The topic on *angles in a triangle* was also the most common in Task 1. Students might have associated 60° with a triangle and were unwilling to go beyond this topic. Students also used the topics that they had learned in their earlier grades instead of the topics at their grade

level to pose their problems across the tasks. One possible reason was their familiarity with these earlier topics.

Table 2

Table of Specifications of Topics Used

Main Geometric Domain Knowledge Used and Grade Level	Task 1 (%)	Task 2 (%)
1) Grade 7 Geometry topics		
i) Angles in a triangle	41.6	1.1
ii) Angles in straight line	20.3	0.6
iii) Alternate angles	15.7	0.6
iv) Corresponding angles	5.4	0
v) Vertically opposite angles	6.0	0
2) Grade 8 Geometry topics		
i) Pythagoras' theorem	3.0	18.5
3) Grade 9 Trigonometry topics		
i) Trigonometric ratios	14.6	9.6
ii) Angle circle properties	7.0	0
iii) Cosine / Sine Rule	4.3	2.8
iv) Angle of elevation / depression	3.0	4.1
v) Bearings	3.8	1.4

Association of Problem Type, Problem Information and Solution Type

For both tasks, problem type was found to be strongly associated with solution type. *Multi step* was strongly associated to *relational* problems across Task 1 and Task 2 with $\phi = .830$ and $\phi = .918$ respectively with $p < .001$. For both tasks, *multi step* was more common in *relational* problems than in *direct recall* problems. The involvement of multiple concepts within *relational* problems probably gave rise to more steps in the solution paths. The problem type and problem information (*implicit assumption*) were associated with $\phi = .196$ and $\phi = .137$ with $p < .001$ in Task 1 and Task 2 respectively. In each of the tasks, about 65% of problems with *implicit assumption* were *relational* problems. This perhaps showed students were not proficient in posing *relational* problems.

Implicit assumption was more common in Task 1. Students might have used *implicit assumption* in Task 1 (open-ended) because there were fewer structures to anchor the initial state and this might lead to mistakes in linking the initial and goal states. *Multi step* problems were more common in Task 1 (open).

In Task 2, the use of *edit* characterized the *direct recall* problems more than the *relational* problems. *Add object* was strongly associated with *relational* problems. In Task 2, some students formed *relational* problems by adding information or object to the initial state to create more complex problems. For *direct recall*, students just used *edit* to vary the given initial state. There were fewer *multi step relational* problems in Task 2 because of the lesser need to add context and hence the fewer occurrences of *multi step*. The absence of *over-conditioning* in Task 2 could be attributed to the more contextualized Task 2 and hence, the less need to add more 'information'. Figure 9 and Figure 10 show a summary of the key associations in Task 1 and Task 2 respectively. Table 3 shows a summary of the main characteristics of the posed problems.

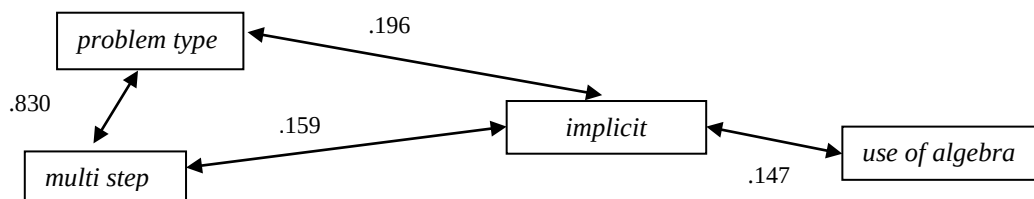


Figure 9. Summary of Associations in Task 1 ↔ significant association (ϕ value)

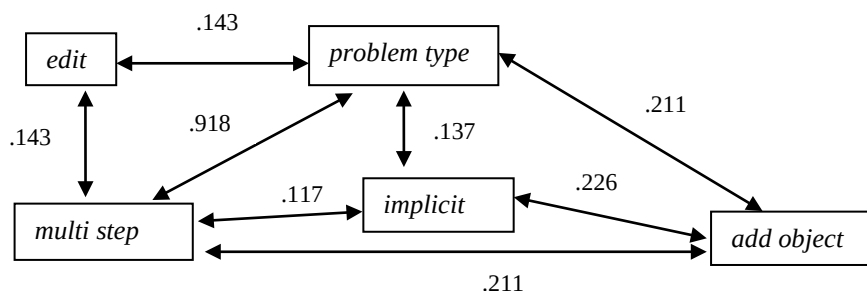


Figure 10. Summary of Associations in Task 2 ↔ significant association (ϕ value)

Table 3

Main Characterization of Problem Types

Problem Type	Description	Task 1 (open)	Task 2 (contextualized)
Relational Problem	Problem Information	i) <i>over-conditioning</i> not associated with problem type and solution type ii) more likely to contain <i>implicit assumption</i>	i) <i>add object</i> more likely in <i>relational</i> ii) more likely to contain <i>implicit assumption</i> iii) <i>over conditioning</i> not present
	Solution Type	i) <i>multi step</i> likely to be present ii) <i>use of algebra</i> not associated to problem type	i) <i>multi step</i> likely to be present
Direct Recall Problem	Problem Information	i) <i>over conditioning</i> not associated with problem type and solution type	i) <i>over conditioning</i> not present
	Solution Type	i) fewer cases of <i>multi step</i> ii) <i>use of algebra</i> not associated to problem type	i) fewer cases of <i>multi step</i> ii) fewer cases of <i>use of algebra</i>

Conclusion

Most of the earlier work on problem posing largely involved arithmetic word problems, so the present study on geometric posing tasks contributes to the knowledge in problem-posing performance. The presence of unsolvable posed problems and the presence of *implicit assumption* in *relational* problems suggest that problem posing may not be a set of natural skills that students possess. This could be due to the less emphasis of such skills in the school mathematics curriculum. That students used more geometric topics in Task 1 suggests that open-ended tasks could elicit from students a wider spread of topics. Students' ability to link various topics in *relational* problems suggests that problem posing can also foster students' ability to see connection across topics. This ability to see connection is

important for interpreting and solving mathematical problems. Like problem-solving heuristics, teachers may have to teach problem posing explicitly as part of the classroom instructional programme.

A suggestion for a research agenda would be to further examine problem-posing responses across task types and involving other domains in school mathematics like statistics or probability. Such work could be useful in the design of intervention studies to promote problem-posing skills. More work may be needed to examine gender and problem-posing performance. There is still the need for more follow-up studies on a bigger sample. The present study nevertheless contributed to knowledge in this area.

References

- Bransford, J. D., Zech, L., Schwartz, D., Barron, B., & Vye, N. (1996). Fostering mathematical thinking in middle school students: Lessons from research. In R. J. Sternberg, & T. Ben-Zeev, (Eds.), *The nature of mathematical thinking* (pp. 285-302). Mahwah, NJ: Lawrence Erlbaum.
- Brown, S. I., & Walter, M. I. (1993). (Eds.), *Problem posing: Reflections and applications*. Mahwah, NJ: Lawrence Erlbaum.
- Cai, J. F. (2003). Singaporean students' mathematical thinking in problem solving and problem posing: An exploratory study. *International Journal of Mathematical Education in Science and Technology*, 34(5), 719-737.
- Christou, C., Mousoulides, N., Pittalis, M., & Pantazi, D. P. (2005a). Problem solving and problem posing in a dynamic geometry environment. *The Montana Mathematics Enthusiast*, 2(2), 125-143.
- Chua, P.H., & Yeap, B.H. (2008). *Problem posing performance of grade 9 students in Singapore on an open-ended stimulus*. Retrieved September 21 2008 from the World Wide Web: <http://tsg.icme11.org/document/get/457>
- Contreras, J. (2007). Unraveling the mystery of the origin of mathematical problems: Using a problem-posing framework with prospective mathematics teachers. *The Mathematics Educator*, 7(2), 15-23.
- Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics*, 52, 243-270.
- English, L. D. (1995). *Children's problem posing in computational contexts*. Paper presented at the annual meeting of the American Educational Research Association (ERIC Document Reproduction Service ED 391 639).
- English, L.D., & Halford, G.S. (1995). Problem solving, problem posing, and mathematical thinking. In L. D. English, & G. S. Halford (Eds.), *Mathematics education – models and processes* (pp. 257-296). Mahwah, NJ: Lawrence Erlbaum.
- Gonzales, N. A. (1998). A blueprint for problem posing. *School Science and Mathematics*, 98(8), 448-456.
- Kuchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics* (pp. 102-119). London: John Murray.
- Mamona-Downs, J., & Downs, M. (2002). Advanced mathematical thinking with a special reference to reflection on mathematical structure. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 174-195). Mahwah, NJ: Lawrence Erlbaum.
- Pirie, S. E. B. (2002). Problem posing: What can it tell us about students' mathematical understanding? *Proceedings of the Annual Meeting of the North American Chapter for the International Group for the Psychology of Mathematics Education*. (ERIC Document Reproduction Service No. ED 471 760).
- Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal of Research for Mathematics Education*, 27(5), 521-539.

Afternote:

This work was done during my attachment at the National Institute of Education, Singapore and that the views expressed here are entirely mine and not that of the Ministry of Education, Singapore.

Alternative Starting Point for Teaching Fractions

Jose Luis Cortina

Universidad Pedagógica Nacional, Mexico

<jose.luis.cortina@mac.com>

Jana Visnovska

The University of Queensland

<j.visnovska@uq.edu.au>

Claudia Zuniga

Universidad Iberoamericana, Mexico

<clamaka@prodigy.net.mx>

We investigate the viability of a new approach to initial fraction instruction. We establish the need to empirically investigate whether the proposed approach shares the strengths of currently used approaches, specifically, whether students will (a) construe problems based on the proposed approach as experientially real, and (b) bring up ideas that could be build upon in subsequent fraction instruction. We then present an analysis of sixteen student interviews from a school in southern Mexico (ages 8 and 9). The analysis supports the conjecture that the proposed approach to initial fraction instruction can be viable, and thus warrants further research attention.

Introduction

In this paper, we build on Freudenthal's (1983) analysis of the fraction concept to propose what we came to view as a viable alternative starting point for fraction instruction. The alternative consists of regarding *comparing*, instead of *fracturing*, as the primary activity from which students are expected to make sense of fractions. It involves using, from the start, problems aimed at orienting students to reason about unit fractions as quantities that account for the relative size of magnitude values,¹ rather than for the size of pieces generated by equally partitioning a food item such as a cake, a candy-bar, a pizza, or a loaf of bread. An example of the problems that, we argue, represent the alternative starting point, orients students to reason about how much milk a cup holds, if so many cups of the same capacity would hold as much milk as a milk carton.

Theoretical Framing

Starting Point for Instruction

Our perspective on instructional design builds on the RME theory (e.g., Gravemeijer, 1994). Central to our perspective is the construct of *hypothetical learning trajectory* (HLT). An important step in formulating a HLT involves choosing a *viable starting point* for instruction. Such a starting point consists of problems that have the potential of fulfilling three characteristics: (a) become experientially real to students, (b) trigger informal ways of reasoning that can be a basis for developing increasingly sophisticated mathematical ways of knowing, and (c) serve as paradigmatic cases in which to "anchor students' increasingly abstract mathematical activity" (Cobb, et al., 1997, p. 159). We use these characteristics as a lens to examine two approaches for introducing students to fraction learning.

¹In this paper we follow Freudenthal in using the terms *magnitude*, *magnitude value* and *quantity*. *Magnitude* is a measurable property of an object; for instance, *length*. In the literature, what we call *magnitude* has also been referred to with the terms *quantitative property* and *dimension*. *Magnitude value* refers to the specific size of a magnitude in an object; for instance, the length of a stick. It has been referred to as *quantitative magnitude* (Thompson & Saldanha, 2003) and *quantity not quantified* (Lamon, 2007). Finally, we use the term *quantity* to refer to the measure of the value of a magnitude that is expressed with a number; for instance, the length of a stick is 12 (centimeters). It has been referred to as *measure*.

Revisiting Freudenthal's Insights on Fraction

In his didactical phenomenology, Freudenthal (1983) distinguished between two different ways of conceiving fractions: *as fracturers* and *as comparers*. In the first of these two ways fractions are conceived as numbers that account for actions that physically modify objects. A notorious example of such actions is partitioning a pizza into equal sized pieces. In contrast, in the second approach, fractions are numbers that account for the relative size of magnitude values. An example is the capacity of a cup to hold milk, relative to the capacity of a milk carton. The actions that take place in order to conduct such comparisons—pouring milk and marking milk levels—serve to gauge a specific property of the objects involved, and need not affect the cohesiveness of the objects.

Fraction as Fracturer

For Freudenthal, the fraction as fracturer approach involves construing the meaning of a unitary fraction as a piece of a whole, which is obtained by equal partition. In this approach, the unit-whole is typically understood as being an object; that is, as a thing (e.g., an actual candy bar) instead of as a property of thing (e.g., the mass or the length that is specific to a certain candy bar). A fraction is understood as something that is produced by fracturing the object, literally. In this interpretation, learners would understand $\frac{1}{5}$ of a candy bar as one of five equal pieces of what used to be a single candy bar (see Figure 1).

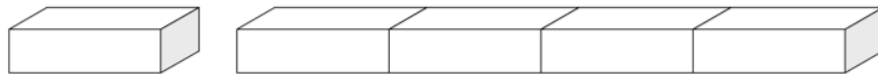


Figure 1. A fifth as a fraction of a unit-whole (i.e., a candy bar), where setting it apart implies disassembling the unit-whole.

The typical instructional situations that are used to introduce fractions to students involve the equal partition of a food item—such as a pizza—so that it can be shared among a certain number of people. From the point of view of a designer with already developed fraction reasoning, the food item is meant to serve as a representation of a unit-whole, of a continuous magnitude (i.e., a mass of food embodying the value of one), and the size of the pieces produced by equally partitioning it, as the entities that unit-fractions quantify. However, the fraction $\frac{1}{b}$ is also meant to be interpreted by learners, in a realistic way, as the *part* of a food item (e.g., a pizza) that someone will get when receiving 1 out of b equal-sized pieces (Clarke & Roche, 2009). In the case of the fraction $\frac{a}{b}$ ($a > 1$), the expectation is that students will come to make sense of the numerator as a number-symbol that accounts for a “number of parts of that name or size” (Clarke & Roche, 2009, p. 136).

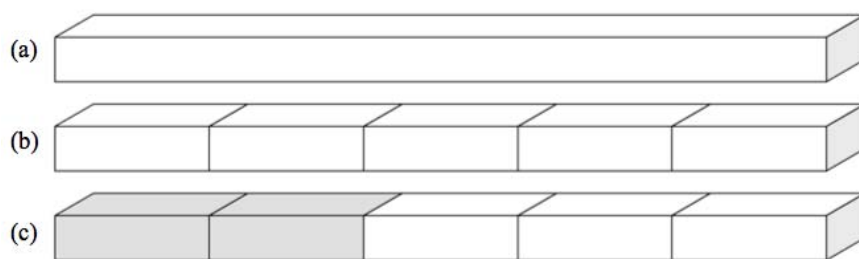


Figure 2. Fractions as numbers that account for equal partitions. From top to bottom: (a) a candy bar as unit-whole, (b) the candy bar partitioned into fifths (i.e., five equal pieces), and (c) $2/5$ of the candy bar shaded.

Situations involving the partitioning and sharing of food-items have been widely used as a context for supporting students in making sense of fractions. These situations appear in textbooks, in specialised teacher-literature, and in research (e.g., Steffe & Olive, 2010; Streefland, 1991; Tzur, 1999). The wide use of these situations can be (at least partially) justified by revisiting the three characteristics of a viable starting point. Much evidence exists that students can readily and meaningfully engage with the food partitioning situations, even from an early age (cf. Pitkethly & Hunting, 1996). In addition, these situations have been shown to trigger informal ways of reasoning, on which students can be supported to make sense of some important fraction relations, such as the relative size of unitary fractions (e.g., $1/3 > 1/4$), and basic fraction equivalencies (e.g., $1/2 = 2/4$; $1/4 = 2/8$; cf. Lamon, 2007).

However, it is with regard to the third criterion—serving as an anchor for students’ reasoning as learning evolves—that Freudenthal’s analysis brings into question the viability of situations involving the equal partition and sharing of food items as starting points for instruction. Freudenthal (1983) judged situations, in which learners operate on objects, that is *fraction as fracturer* situations, to be “much too restricted not only phenomenologically but also mathematically” (p. 144). For him, they are phenomenologically restricted because fractions become consistently “seen or imagined within the whole” (p. 147). As a consequence, in the fracturer approach, fractions become restricted to being “proper fractions only (<1)” (p. 147). In addition, these situations are mathematically restricted because the operation of partition as the basis of equivalence relation always only generates a restricted number of elements in each equivalence class.² As Freudenthal explains, in order to use fractions as a phenomenological source of the rational number, “an equivalence of broader scope is needed, as well as the unrestricted availability of objects in every equivalence class” (p. 147).

Freudenthal’s assessment of the shortcomings of fraction as fracturer is consistent with the concerns expressed by Thompson and Saldanha (2003) about the limitations of traditional fraction instruction:

Students are often instructed, and therefore learn, that the fractional part is contained within the whole, so “A is some fraction of B” connotes a sense of inclusion to them, that A is a subset of B. As a result, statements like “A is $6/5$ of B” make no sense to them (p. 107).

²In instructional terms, a pizza can always only produce four fourths, which makes imagery of $7/4$ problematic. This problem is inherent to how a magnitude is constituted in a system of quantities via operation of partition: if equivalence is defined only by the operation of partition, we have no means to compare quantities of elements produced from different *unit-wholes* (different pizzas).

These authors also contend that students' oftentimes develop an image of fractions, in which they are seen as numbers that account for *so many out of so many*. Thompson and Saldanha (2003) contend that this image too

...possesses a sense of inclusion—that the first “so many” must be included in the other “so many”. As a result, they [the students] will not accept the idea that we can speak of one quantity's size as being a fraction of another's size when they have nothing physically in common. They will accept “The number of boys is what fraction of the number of children?”, but they will be puzzled by “The number of boys is what fraction of the number of girls?” (p. 104).

The concrete imagery that learners develop while working on problems that involve food partitioning and sharing is not consistent with, and thus does not easily lend itself to building more sophisticated fraction and rational number imagery. These situations do not fulfil the third criterion of a viable starting point, serving as an anchor for students' reasoning as learning evolves. Instead, they orient students to develop an initial understanding of fractions as fracturer. As studies on learning difficulties in fractions document, this initial understanding makes it particularly difficult for children to later make sense of fractions as numbers that soundly quantify values bigger than one, as well as the size of something in relation to something else that does not contain it (Hackenberg, 2007). We thus consider it reasonable to regard these situations as inadequate in initial fraction instruction and to seek a viable alternative.

An Alternative to Equal Partitioning

Freudenthal (1983) and Thompson and Saldanha (2003) envisioned a similar alternative to the use of equal partitioning—an alternative in which *ratio comparisons* (i.e., fraction as comparer) become the focus of fraction instruction. In this alternative, fractions are used to quantify magnitude values by comparing them to a magnitude value of reference (construed as having the size of 1), in multiplicative terms (Figure 3).

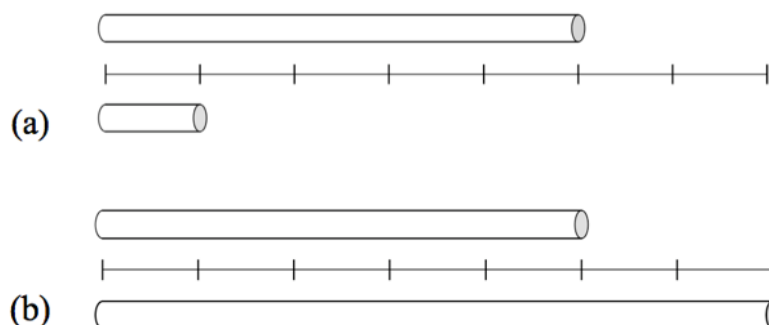


Figure 3. (a) The unit whole (a stick) construed as a specific length (i.e., a magnitude value), and 1/5 of it as another specific length that satisfies the condition of 5 iterations of it being as long as the unit-whole. (b) 7/5 construed as a specific length that is 7 times as long as 1/5 of the length of the unit-whole.

Based on the analyses of these authors, it is possible to picture an alternative image of a unitary fraction to cultivate in students. It becomes a magnitude value that satisfies an iterative condition, regarding a magnitude value of reference, in the following terms: A is $1/n$ as large as B when B is n times as large as A .

Regarding the three characteristics of a viable starting point for instruction, the analyses suggest that opting for problems aimed at orienting students to interpret unitary fractions as

multiplicands would fulfill the third criterion. However, it is unclear whether such problems would fulfill the first two criteria of a viable starting point. We report on initial insights into viability of the approach with regard to the first two criteria.

Methodology and Data Sources

We conducted clinical interviews with sixteen students (ages 8 and 9) who formed the only third-grade classroom in a public school in southern Mexico. An analysis of their notebooks suggested that they had had limited opportunities to learn about fractions with understanding. Instead, their prior instructional experiences seemed to have entailed much of what Anyon (1981) identified as work that most often involves “substantial amounts of rote activity” (p. 203). In addition, students in this classroom performed poorly in the national standardised test known as ENLACE. This group was thus suitable for testing the viability of the approach as a starting point.

Interview Protocol

For the iterative approach to be instructionally useful, we considered that students should, from the outset, have access to ways of reasoning about a core notions that pupils are typically expected to make sense of in the initial phases of fraction instruction, relative size of unitary fractions (NCTM, 2000).

The interview protocol included three problems. In the first, *Milk Carton*, the students were asked to make judgments about the relative capacity of seven different kinds of cups, based on how many cups of each kind could be filled with the milk contained in a carton. The actual cups were not shown to them (Figure 4). We asked students to make comparisons similar to: $1/2$ vs. $1/4$; $1/3$ vs. $1/4$; $1/7$ vs. $1/9$; $1/20$ vs. $1/1$ (formulated “Which cup can hold more milk, plastic or glass one?”).

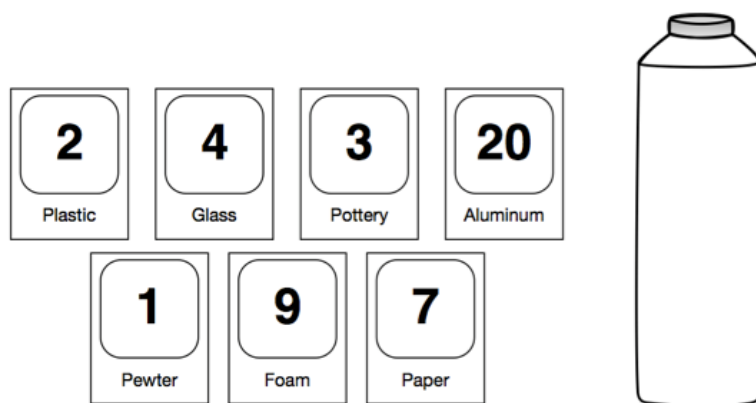


Figure 4. Cards used in the *Milk Carton* problem to show how many cups of each kind could be filled with the milk contained in a carton and image of the milk carton that was used.

Two more similar problems, *Kangaroos*, and *Water Tanks* were introduced and will be reported on during the conference presentation. In these, the students were encouraged to make context-based fraction comparisons such as $1/2$ vs. $1/4$ vs. $1/5$, (*Kangaroos*) and $?/8 = 1/2$, $?/8 = 1/4$, $?/8 = 3/4$ (*Water Tanks*).

Data Coding

The interview data was open coded, following an ongoing formulation of provisional categories (Strauss & Corbin, 1990). We sought to identify similarities and differences in how students construed the problems, and in the mathematical reasoning that emerged.

The evidence we used for determining that a problem had been construed as *experientially real* (first characteristic of a viable starting point) consisted of verbal expressions and gestures that suggested that a student was reasoning about the quantities involved in the problem, and not just about the numbers. For instance, in the *Milk Carton* problem, this type of evidence included students referring to *amounts of milk* or making gestures with their hands indicating the *size of cups*. The *informal ways of reasoning* about fractions (second characteristic) that the students developed in response to the interview problems are outlined in the Results section.

Results

Given the evidence criteria, all of the students construed the three problems as experientially real. The following extract illustrates Marilu reasoning about the size of the plastic cup (Figure 4):

Interviewer: When I serve one cup of milk, how far does the milk carton empty?

Marilu: To here [marking the carton at about the middle].

Interviewer: Why?

Marilu: Because they are this size [gesturing with her fingers the size of a cup]

Gestures referring to the actual size of cups indicated to us that a student like Marilu had interpreted the problem as involving actual quantities. In other words, it indicated that a student was imaginistically involved with the problem at hand and, thus, engaging in personally meaningful mathematical activity. Results from this component of the analysis support considering the proposed starting point for fraction instruction as viable. They suggest that it would be possible to support a group of third grade students, like those we interviewed, to readily construe problems based on the *comparer* approach as experientially real.

We further documented the informal ways of reasoning students developed about *relative size of unitary fractions* and about *basic fraction equivalencies*. Here we report on instructionally relevant differences in how students reasoned about *relative size of unitary fractions*, while solving problems where magnitude values were defined as multiplicands that satisfy an iterative condition (Table 1).

Table 1

Number of Students in Different Categories, According to How They Reasoned about the Relative Size of Unitary Fractions

	Category 1 The bigger the number, the larger the magnitude value	Category 2 Visual evidence	Category 3 Coming to anticipate	Category 4 Anticipating that the more iterations, the smaller the magnitude values
N	2	3	5	6

Students that showed the least sophisticated reasoning (Category 1, N=2) had some intuitions that were consistent with assessing the size of magnitude values relative to how

many iterations of it would be necessary to complete a unit. These students correctly regarded the aluminum cups (twentieths) as being small, and the pewter cups (ones) as being big. However, these intuitions were not sufficiently robust to allow pupils to make sound comparisons of the sizes of magnitude values. They instead followed a different rationale: The bigger the number, the larger the magnitude value.

Category 2 students (N=3), relying on the marks of their estimates (e.g., on the milk carton), made comparisons consistent with the idea that the more iterations of a magnitude value were needed to make as much as a unit, the smaller the magnitude value had to be. However, during the interviews, these students did not come to understand this principle sufficiently well to consider that it would apply in every case.

Category 3 students (N=5) came to anticipate that the more iterations of a magnitude value that were needed to make as much as a unit, the smaller the magnitude value had to be. Finally, the remaining students (Category 4, N=6) readily judged that the more iterations of a magnitude were needed to make as much as a unit, the smaller the magnitude value had to be.

To illustrate, Zaide was one of the students in category 3. She initially anticipated that the plastic cups (halves) would be smaller than the glass cups (fourths). After marking on the carton sensible estimates of where the levels of the milk would be if one plastic cup and one glass cup were served, respectively, she changed her mind and considered that the plastic cups would be bigger. When comparing the glass cups (fourths) and the pottery cups (thirds), she followed a similar path. Finally, she anticipated that the foam cups (ninthths) would hold less milk than the paper cups (sevenths):

Interviewer: Which would be bigger?

Zaide: The foam cups [ninthths].

Interviewer: Bigger?

Zaide: Oh, no, the paper ones [sevenths].

Interviewer: The paper or the foam ones?

Zaide: Paper.

Interviewer: Why the paper ones?

Zaide: Because if they put nine cups it goes down less. And if you put seven it goes down faster.

Interviewer: Which can hold more?

Zaide: The seven ones [paper cups].

Interviewer: The seven ones can hold more?

Zaide: Because they are not many cups.

In the excerpt it is noticeable that Zaide reasoned about the respective capacity of the paper cups based on how the levels of the milk in the carton would change as these cups were served. Her previous estimations seemed to have helped her develop an image that when more cups were to be served, the level of the milk in the carton would drop less with one serving ("it goes down less"). This image allowed her to correctly anticipate that the foam cups (ninthths) would be smaller than the paper cups (sevenths). Importantly, as Zaide progressed through *Kangaroos* and *Water Tanks* problems, she came to readily anticipate the relative comparisons correctly.

With different levels of sophistication, the large majority of the students correctly judged the relative size of the magnitude values involved in the different problems. Even the students who showed the least sophisticated reasoning had some intuitions that were consistent with the just mentioned property of unitary fractions (e.g., regarded the aluminum cups (twentieths) as being small, and the pewter cups (ones) as being big).

Of particular significance were the Category 3 students, whose reasoning became more sophisticated *during* the interview. They started by using a natural number rationale in

comparing the capacity of the cups, and later came to correctly conceive their relative capacity, without having to rely on visual evidence. Their case suggests that problems based on the iterative approach can be a means by which to support students to engage in progressive mathematization (Gravemeijer & Doorman, 1999) in the fraction realm.

Conclusion and Significance

The results indicate that problems based on the iterative approach can be a viable starting point for fraction instruction. Students' informal reasoning during the interviews was consistent with coming to make sense of two core notions of initial fraction instruction—relative size of unitary fractions and basic fraction equivalencies. These findings provide a strong justification for developing instructional sequences that take advantage of the identified starting point.

References

- Anyon, J. (1981). Schools as agencies of social legitimization. *International Journal of Political Education*, 4, 159-218.
- Behr, M. J., Lesh, R., Post, T., & Silver, E. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 91-125). New York: Academic Press.
- Clarke, D. M., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics*, 72, 127-138.
- Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirshner & J. A. Whitson (Eds.), *Situated cognition: Social, semiotic, and psychological perspectives* (pp. 151-232). Mahwah, NJ: Lawrence Erlbaum.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, Holland: Kluwer.
- Gravemeijer, K. (1994). *Developing realistic mathematics education*. Utrecht, The Netherlands: CD-β Press.
- Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, 39, 111-130.
- Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *Journal of Mathematical Behavior*, 26(1), 27-47.
- Kieren, T. E. (1980). The rational number construct—Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125-149). Columbus, OH: ERIC/SMEAC.
- Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Charlotte, NC: Information Age Pub.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Pitkethly, A., & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics*, 30, 5-38.
- Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.
- Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications.
- Streefland, L. (1991). *Fractions in Realistic Mathematics Education*. Dordrecht, The Netherlands: Kluwer.
- Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *Research companion to the principles and standards for school mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.
- Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education*, 30(4), 390-416.

Concept Cartoons as a Way to Elicit Understandings and Encourage Reasoning about Decimals in Year 7

Samone Davidson

Monash University

<smdav9@student.monash.edu>

Mike Askew

Monash University

<Mike.Askew@monash.edu>

This paper is located within the research into encouraging learners to reason mathematically and become engaged with concepts rather than just procedures. It reports on a research in progress examining two techniques to be used at the beginning of a sequence of lessons in order to elicit students' prior knowledge and understanding of topics about to be studied. The research will be conducted with two Year 7 classes of girls as they are introduced to units of work on decimals and fractions. Two techniques will be used to elicit the students' prior understandings: Concept Maps and Concept Cartoons. This paper reports on the ideas behind Concept Cartoons and the development of Concept Cartoons for eliciting understandings of decimals. The paper briefly outlines the research into difficulties with decimals before discussing the rationale and design of the cartoons. The paper is concluded with a discussion of how concept cartoons may or may not elicit different understandings to using concept maps.

Introduction

How students are to learn rich and deep decimal concepts is well known to be problematic: assessing learners' prior understandings and building on these is often advocated as one way of addressing this problem. While pre-testing is one means of eliciting prior understanding, pre-tests can potentially fail to uncover misconceptions as students can solve problems using their procedural knowledge rather than displaying a conceptual understanding or the test design may not address particular misconceptions. Concept Cartoons are a means of assessment that may be able to elicit misunderstandings that pre-tests are unable to. While Concept Cartoons have been successfully implemented into the field of science teaching and while research has been conducted on their use in mathematics, there is a need for further research into their design and implementation. This paper reports on research focused on the development of Concept Cartoons as a means of eliciting prior knowledge of decimals and in particular on the design of two Concept Cartoons. This research builds on the insights from science and the research that has been done in mathematics into understanding decimals.

Teaching mathematics is often defined by a rigid curriculum where each topic or concept is given a particular time span. Teachers feel compelled to teach the skills required by the curriculum and find it difficult to explore each concept thoroughly and acknowledge student's misconceptions and prior knowledge. The use of Concept Cartoons would address this issue as they aim to elicit prior mathematical knowledge from the students. The use of Concept Cartoons within the mathematics classroom is expected to engage students in discussing their prior knowledge and encourage them to challenge their existing beliefs, thus enabling misconceptions to be elicited and addressed.

Concept cartoons are designed to be an effective tool to not only gather meaningful information about a student's prior learning in decimals but also to interest and challenge students. The use of cartoons adds a visual stimulus potentially allowing students greater confidence when conveying their ideas in a group situation. Refining, researching and justifying solutions may also ensure that the students have the opportunity to understand a concept from multiple perspectives. Concept cartoons thus enable students to become more

engaged in their learning with greater focus on the concept presented rather than the procedure required to reach a solution.

This paper considers the design of research to be carried out with Year 7 middle school students and discusses the use of Concept Cartoons to encourage mathematical reasoning and effective engagement in the classroom with respect to decimal fractions. The research will explore and compare Concept Cartoons with Concept Mapping as techniques used to access and build on a students' prior knowledge during the introduction of two mathematical topics in Year 7 (fractions and decimals). Two year 7 classes will be involved in the study. In the first topic one class will trial the use of Concept Maps to elicit prior understandings while the other class will use Concept Cartoons. When the second topic commences the two techniques will be reversed. At the completion of the study each class will have accessed and built on their prior knowledge using concept maps and concept cartoons. As this research is yet to be conducted we focus here on the development of Concept Cartoons for decimals.

The paper begins with a brief overview of the research into understanding decimals as this frames the design of the Concept Cartoons. This is followed by a review of the literature into the effectiveness of Concept Cartoons. These two sections are synthesized in an overview of the development of Concept Cartoons and their proposed use in the classroom. Finally we briefly compare the use of Concept Cartoons with Concept Maps.

Understanding Decimals: Key Issues.

Research that has been conducted into decimals provides clues to common misconceptions that exist when learning takes place. This allows the development of Concept Cartoons to focus on areas which students find difficult to gain a conceptual understanding. Focus has been placed on exploring those misconceptions that are often detected.

'Numbers are undoubtedly central in mathematics but it is impossible to understand what difficulties children meet with numbers if you do not look at them as magnitudes of different sorts, transformations, or relationships.' (Vergnaud, 1979, p. 265).

Students risk misunderstanding the importance of relationships that exist between whole and decimal numbers as whole number place value is often taught without the extension into decimals. Major principles of whole numbers are not always transferred to the decimal system, but equally some principles that apply to whole numbers can be misapplied to decimals. The research into these misunderstandings provides the basis for the Concept Cartoons. Students who realise the importance of understanding all values in a number will ensure that they understand that decimal numbers are part of something bigger in mathematics.

Misconceptions

In researching the literature, the main misconceptions selected for developing into Concept Cartoons are:

- Longer is larger
- Decimal point ignored
- Misunderstanding zero
- Relating decimals to fractions

Longer is larger. Students may consider that the decimal having the greater number of digits is the larger decimal. For example 2.6498 is larger than 2.65 since it possesses a greater number of digits. This misconception is referred to as the ‘longer is larger’ strategy (Steinle, 1998b). This reasoning follows the logic that the more digits a decimal possess’ the larger it is, despite the position of the decimal point, and is likely to arise from the relationship that for whole numbers longer is indeed larger.

Decimal point ignored. Students may also attempt to solve mathematical problems by ignoring the decimal point. When ignoring the decimal point, students considered 0.5 to be half of 10 (Glasgow, 2000). This also occurred when students were presented with decimals such as 0.3658 and 0.65. Students saw these decimals to be 3658 and 65 respectively, thus they consider 0.3658 to be the larger decimal. Those students who ignored the tenths, hundredth etc. reverted back to their knowledge of whole numbers (Steinle, 1998a). This misunderstanding may also contribute to a student’s difficulties in ordering and comparing decimal numbers (Hiebert, 1983).

Misunderstanding zero. Some students find it difficult to assess whether the presence of zero impacts upon the value of the decimal. 0.45 will appear the same as 0.045: once students ignore the zero both decimals appear to be 0.45. On the other hand some students regard equivalent decimals such as 0.40 and .4 to be different values (Hiebert, 1983). A further misconception over zero is in multiplying by 10; students simply added a 0 to the end of the decimal (Hiebert, 1983). Thus 2.65 multiplied by 10 would become 2.650.

Decimals as denominators. The relationship between decimals and fractions causes much confusion for students. When converting between decimals and fractions students may not view a decimal as a part of a whole. Students may recognise 0.4 as $\frac{1}{4}$ and 0.5 as $\frac{1}{5}$. Students may also conclude that 0.4 is bigger than 0.5 since $\frac{1}{4}$ is bigger than $\frac{1}{5}$ (Steinle, 1998a). Evidence by Hiebert suggests that students prior to grade 9 expressed difficulty converting 0.1 to $\frac{1}{10}$ and 0.4 to $\frac{1}{5}$. After this students began to misunderstand the conversion between decimals and fractions confusing $\frac{83}{100}$ for 0.083. (Hiebert, 1983.)

Developing the Concept Cartoons

Why Use Concept Cartoons?

Concept Cartoons have proven effective in creating focused discussions where the presence of misconceptions obtained through previous learning would be uncovered (Kabapinar, 2005). They have been shown to be successful in providing a purpose for scientific investigation by locating concepts in familiar, everyday, situations. This then gives students a familiarity and confidence to explore the situation presented. Concept Cartoons are accessible to the students enabling them to relate to the problem. When students are able to relate to the problem the outcome is seen to be more successful. Whether this extends to mathematics, and decimals in particular, is one of the questions this research will address.

Concept Cartoons aim to elicit a student’s prior understanding of the concept and provide an assessment on which to base further learning. Through prior learning, students have already developed an understanding of mathematical ideas. This previous learning

affects their assimilation of new knowledge. Concept cartoons integrate narrative text with dialogue text to form a visual stimulus to challenge student thinking. The dialogue presented in the cartoons creates the opportunity for students to consider alternative ideas relating to the same topic. Concept cartoons may include more than one view that is correct, thus providing the chance to consider which solutions would be more mathematically acceptable.

In the process of debating Concept Cartoons students are expected to investigate misconceptions of the topic, challenge their understanding and explore acceptable knowledge. In Science Education Concept Cartoons have been shown to enhance student motivation, provide a purpose for the learning, minimise classroom management issues and keep students on task (Kabapinar, 2005). The success of Concept cartoons is found in inviting students to engage in the discussions. Each idea contributed to the discussion is worthy of consideration. Thinking is focused as the students find confidence using another persona to voice their opinions.

Keogh (1999) observed the success of Concept Cartoons when researching groups of students using this strategy. She found that students shifted their thoughts on the knowledge, introduced new ideas, realigned their position and found justification to support their ideas. Conflict was generated in the classroom, which was then investigated as students searched to support an idea. When students agreed on an idea as being correct they then needed to work to justify their position and move to develop a more acceptable scientific idea. Although Keogh based her research on the discipline of Science it could be suggested that a similar situation would exist for mathematics. For learning of mathematics using Concept Cartoons, effective use of this strategy must be combined with correct mathematical epistemology and sound teaching practice.

It is encouraging to find that students become more in control of their own learning when using Concept Cartoons. Research conducted by Kabapinar on a group of 11- and 12-year-olds found that students were not pressured into changing their views to those of their peers as students did not see themselves putting forward the ideas but the views of the characters in the Concept cartoons (Kabapinar, 2005). Concept Cartoons provide a challenge for groups to remain open minded about a solution. They also foster the desire to investigate and research to provide justification to prove their solutions. Students' research and motivation provide an effective stimulus for learning in the classroom. From the range of student ideas the challenge emerges for the group to agree and then discover the same mathematical acceptable solution. The promotion of students having a personal involvement in their lesson engages them to strive to find a consensus that can then be justified. Concept Cartoons thus appear able to turn an explanation provided by the teacher into student driven learning.

Using the Concept Cartoons in Class

The success of Concept Cartoons is dependent on how successfully they are implemented into the classroom. If teachers are not enthusiastic when presenting such a strategy the level of success is limited. (Keogh, 1999). Teachers must carefully monitor the interactions between students in each group to ensure that one student does not dominate the discussion; intervention would then be required to ensure the effective use of the Concept Cartoon. Interaction by the teacher is extremely important and questions exploring the reasons behind a student's choice may be an alert for the presence of a misconception. Over time, students become more confident in discussing their own learning and suggesting ideas to their group which could result in further development of the solution. Students who have the same idea may also display a different approach to discuss their understanding.. This reasoning may then form the basis for a future class discussion (Kabapinar, 2005).

Suggestions for the use of Concept Cartoons within the classroom vary in the amount of individual think time, group discussion and teacher interaction. Concept Cartoons could involve students developing their ideas individually before moving to smaller group discussions. Once this is conducted they may then finalise their ideas on a poster for presentation to the class. Students may also seek ideas as a class first, when beginning Concept Cartoons, to allow the teacher a glimpse of any intervention that may be needed at a later date. Presentation of Concept Cartoons may also be done first as a whole class but then students would be given some time to develop their ideas. A small group discussion would then be held allowing students to finalise and justify their ideas. The focus of these smaller groups would be to ensure a team effort is made to investigate all ideas presented and then students would justify their solution.

Concept Cartoon Compared to Concept Maps?

Students access prior understanding when completing both Concept Maps and Concept Cartoons. Concept Maps encourage brainstorming by students who then endeavour to link key ideas and provide a map showing the connectivity of mathematical concepts. They can be continually updated throughout a topic as students gain new knowledge, thus, Concept Maps become a work in progress. Concept Cartoons encourage a focused discussion between students with the task of justifying mathematical knowledge and eliciting misconceptions. They are aimed at discovering the misconceptions in conceptual understanding that can then be worked on in future lessons. Although concept maps create links between prior knowledge they may not reveal conceptual misunderstandings as these may be masked by a student's ability to follow procedures when finding solutions.

Both techniques will address prior knowledge, however, once this is acknowledged a different path is then taken in moving forward to strengthen mathematical concepts and build new understandings. The extent to which each technique is able to elicit misunderstandings is a key aim of the research.

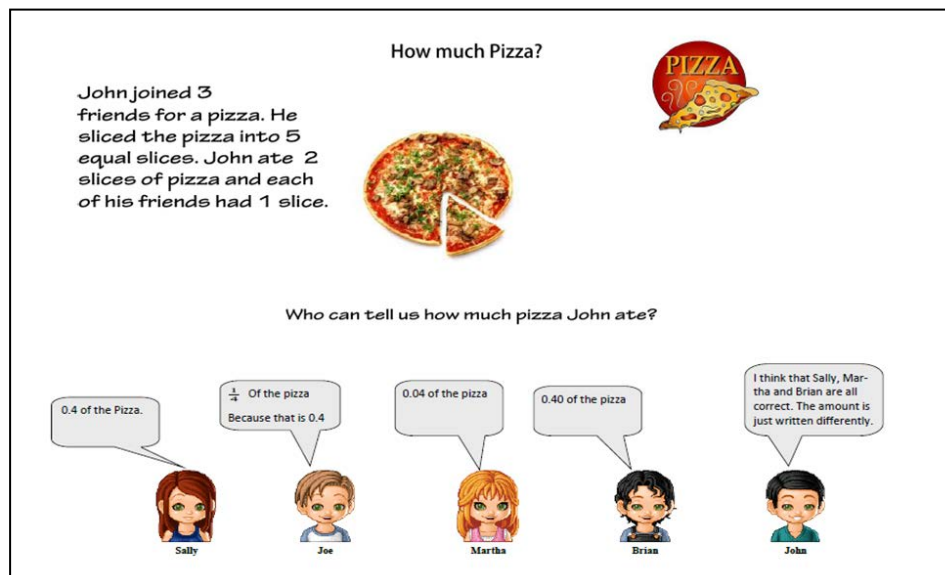
Developing Concept Cartoons for Decimals

Two Concept Cartoons have been developed to address the misconceptions of: longer is larger, decimal point ignored, misunderstanding zero and decimals as denominators.

How Much Pizza?

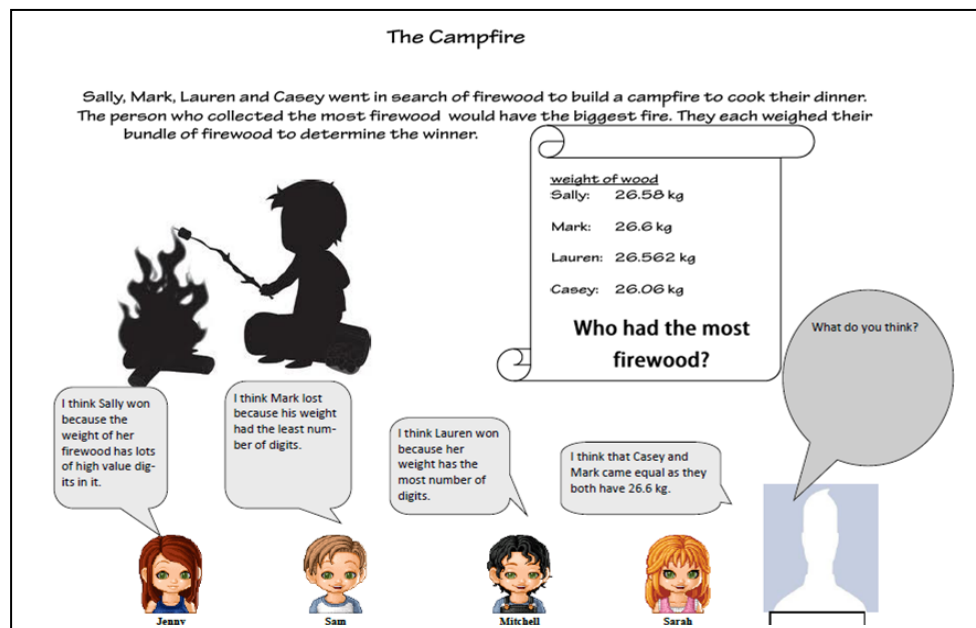
The first Concept Cartoon entitled 'How much Pizza?' explores quantities and decimal representation. Students are asked to choose the correct response when asked how much pizza John ate. Each response is presented as a decimal or fraction. The responses of each of the characters deal with the misconceptions of *misunderstanding zero* and *decimals as denominators*. Sally and Brian have equivalent responses, however, are presented differently with Brian's solution incorporating a zero as the final decimal place value. This highlights a student's understanding of the impact of zero when placed at the end of the number. Martha's response is also focused on the misunderstandings of zero it is placed within a decimal number. Students may ignore the 0 in 0.04 and read this as 0.4. Thus, they may then agree with John.

Joe's response being $\frac{1}{4}$ highlights the difficulties found when converting decimals to fractions. Students may convert 0.4 to $\frac{1}{4}$ by placing the decimal placed digits as the denominator in the fraction. Students may then conclude that Sally and Joe have the same chosen correctly.



The Campfire

The second Concept Cartoon explores the ordering of decimals. This cartoon endeavours to elicit the misconceptions of *longer is larger*, *decimal point ignored* and *misunderstanding zero*. In ‘The Campfire’ students are asked to find the largest decimal number. Decimals of varying lengths are presented.



Classroom Practice

The process of introducing the Concept Cartoons will closely follow the steps below:

1. Give students individual time to choose the idea that they find most mathematically acceptable. Students are to commit to their response of the problem presented.

2. Students will then be placed in a group where they are to go through each idea and choose an appropriate solution. A table (similar to figure 1) will be given to provide the opportunity for the group to sort their ideas.
3. Regrouping will then occur. In the new groups students present their research and justification in the hope of receiving additional feedback.
4. Students join back in their initial groups where they refine, improve (through research) and justify their ideas.
5. Discussion is held by class using the table which students have completed in their groups.

How much Pizza?

Statement	We agree/ Disagree	Proof /Justification
<i>Sally:</i> 0.4 of the pizza		
<i>Joe:</i> $\frac{1}{4}$ of the pizza because that is 0.4		
<i>Martha:</i> 0.04 of the pizza		
<i>Brian:</i> 0.40 of the pizza		
<i>John:</i> I think that Sally, Martha and Brian are all correct. The amount is just written differently.		

Figure 1. Student Explanation worksheet (adapted from Dabell, 2008, p. 35)

While students are working in groups the teacher is given the opportunity to observe students mathematical reasoning and highlight individual misconceptions which are held. For ‘How much Pizza?’ a table would take the form of figure 2.

Student Name	Misunderstanding zero	Decimals as denominator	Additional comments

Figure 2. Teacher misconception observations.

Discussion

Decimals are an integral part of learning mathematics and have a place in the world outside the classroom. They assist in describing part of a whole and allow us to express solutions exactly and approximately. The prior learning that students experience impacts how new knowledge is processed by them. Concept Cartoons have been created to assist in eliciting the student’s prior knowledge of decimals. With the additional information that could be gained regarding the students learning a more effective program may be adopted.

The opportunity for students to become aware of their own learning is provided through the use of Concept Cartoons and Concept Maps, each of which encourage the eliciting of prior knowledge and misconceptions enabling students to focus on their own understanding. The possible advantage of Concept Maps is that prior knowledge may not have been anticipated by the teacher and the extent of knowledge revealed may be greater or less than expected. A disadvantage however is that the absence of misconceptions being revealed in a Concept Map does not necessarily indicate that a learner’s understanding is free of misconceptions. Concept Cartoons have the advantage of directly addressing potential misconceptions, as identified in the research literature, but at the cost of not

enabling learners to reveal the depth or breadth of their understandings. By highlighting and examining these differences in the prior knowledge that each technique elicits it is expected that this research will contribute to helping teachers become aware of the sort of additional support to provide for effective learning to be undertaken.

The effectiveness of using Concept Cartoons to elicit prior knowledge and possible misconceptions in decimals will be the focus of this study. Exploring the most effective way of incorporating these into the classroom practice will assist in gaining greater insight into each students understanding of decimals.

References

- Bell, A., Swan, M, & Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. *Educational studies in Mathematics*, 12, 399-420.
- Dabell, J. (2008). Using Concept Cartoons. *Mathematics Teaching incorporating micromath*, July, 34-36.
- Glasgow, R., Ragan, G, Fields, W, Reys, R., & Wasman, D. (2000). The Decimal Dilemma. *Teaching Children Mathematics* (October), 89-93.
- Hiebert, J., & Wearne, D. (1983). Students' Conceptions of Decimal Numbers. *Annual Meeting of the American Educational Research Association*, 60.
- Hopkins, T. (2007). What is the value of @ ? *Teaching Children Mathematics*, 13(8), 434-437.
- Kabapinar, F. (2005). Effectiveness of Teaching via Concept Cartoons from the Point of View of Constructivist Approach. *Educational Sciences: Theory and Practice.*, 5(1), 135-146.
- Keogh, B., & Naylor, S. (1999). Concept cartoons, teaching and learning in science: an evaluation. *International Journal of Science Research*, 21(4), 431-446.
- Steinle, V., & Stacey, K. (1998a). The incidence of misconceptions of decimal notation amongst students in Grades 5 to 10. In C. Kanes, M. Goos, E. Warren. (Eds). *Teaching mathematics in new times: proceedings of the Twenty First Annual Conference of the Mathematics Education* (Vol. 2, pp. 548-555). Gold Coast, Australia: Mathematics Education Research Group of Australasia.
- Steinle, V., & Stacey, K. (1998b). Students and Decimal Notation: Do they see what we see? *Mathematics: exploring all angles* (pp. 415-422).
- Vergnaud, G. (1979). The Acquisition of Arithmetic Concepts. *Educational studies in Mathematics*, 10(2), 263-274.
- Wearne, D., Hiebert, J. (1986). *Learning Decimals Numbers: A study of knowledge acquisition*: Delaware University, Newark.

Block Model Approach in Problem Solving: Effects on Problem Solving Performance of the Grade V Pupils in Mathematics

Niño Jose P. de Guzman
La Salle Green Hills School
<niojosedeguzman@yahoo.com>

Rene R. Belecina
Philippine Normal University
<rrbelecina@yahoo.com>

The teaching of mathematics involves problem solving skills which prove to be difficult on the part of the pupils due to misrepresentation of the word problems. Oftentimes, pupils tend to represent the phrase “more than” as addition and the word difference as “- “. This paper aims to address the problem solving skills of grade five pupils employing the block model approach which is based on concrete - representation – abstract principle of teaching mathematics.

Since the turn of the century, traditional teaching based on the framework of behaviorism, such as the course in question, is being replaced by inquiry-based teaching, facilitating a constructivist framework of learning. Advocates of the constructivist-teaching paradigm (Draper, 2002), recommend a more student-centered math classroom that deemphasizes rote memorization of isolated skills and facts and emphasizes problem solving and communication. According to Larochelle and Bednarz (1998), a constructivist classroom is rich in conversation. By conversing, the teacher infers the learning level and preparation of the student and coaches the communication so that the learner is able to construct meaning, understanding, and knowledge. Teachers who embrace constructivism reject the transmission model of teaching (Richardson, 1997).

Relating to the constructivist theory of learning, the learner is active and continuously constructs and reconstructs conception of phenomena. The learning is not assessed with separate examination at the end of the course, but assessment methods are integrated into the learning process itself (Tynjala, 1998). The objective of the assessment is to encourage the learning process resulting in the discovery of qualitative changes in the student’s knowledge base. As a result, the course in question would benefit from an assessment method that stresses the application or performance that displays development of metacognition and critical thinking in an authentic and constructive way.

The study in general attempts to examine the effect of the use of block model approach in problem solving of the grade five pupils on problem solving performance in mathematics. Specifically it seeks to answers the following questions:

1. Is there a significant difference between the problem solving performance of the control group and the experimental group after the experiment?
2. Is the effect of the problem solving approach on the problem solving performance moderated by the level of mathematical ability?
3. Is the effect of the problem solving approach on problem solving performance moderated by the type of problem (one - step, two – step, three – step problem)?
4. How do pupils perceive the use of block model approach in problem solving?

Hypotheses

The following null hypotheses were tested in this study

1. There is a no significant difference between the problem solving performance of the control group and the experimental group after the experiment.
2. The effect of problem solving approach on the problem solving performance of the pupils is not moderated by the level of mathematical ability.
3. The effect of the problem solving approach on the problem solving performance of the pupils is not moderated by the type of the problem (i.e. one step, two step, and three step problem).

Research Design

This study employed the Pretest-Posttest Control Group design. The control and the experimental groups were given a pretest, exposed to different treatments and then given a posttest. The control group was taught problem solving using the traditional approach while the experimental group was taught using the block model approach on problem solving. The participants of the study were taken from the ten heterogeneous sections enrolled in the school year 2007 – 2008. Intact groups and group - matching techniques were used to come up with the comparable groups. Fishbowl technique was used as a sampling technique in selecting the control and experimental group. The lessons covered by the study were one step problem solving, two step problem solving and three steps problem solving.

Conceptual Framework

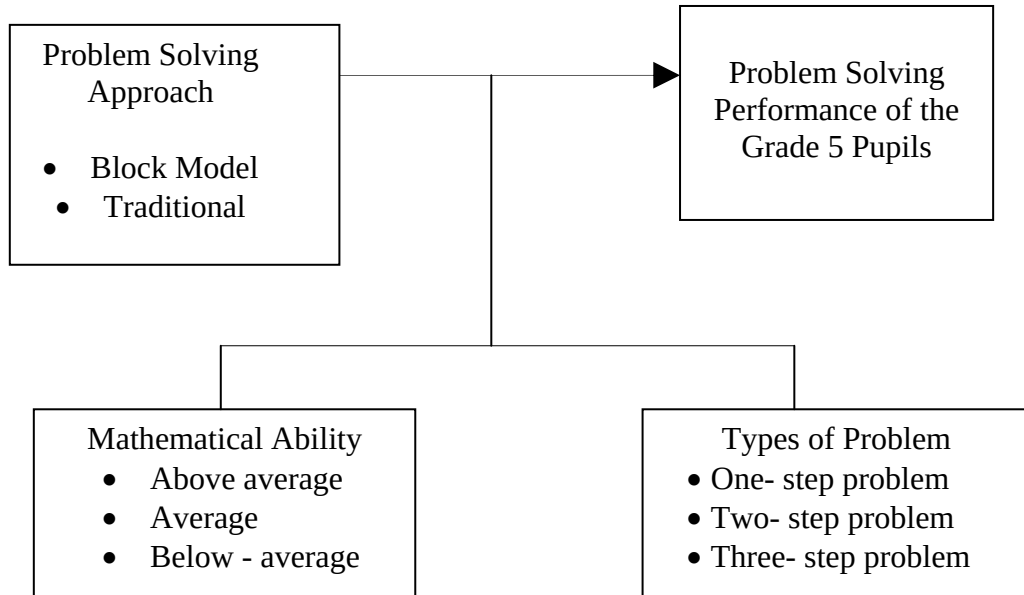


Figure 1. Conceptual Paradigm

Based on the above mentioned theory the conceptual paradigm was constructed. There are two problem solving approaches namely the Block Model Approach treated for the experimental group while the Traditional Method was employed to the control group. The Traditional Method merely employ algorithmic method of solving word problems while the

Block Model Approach employ the concrete- representation – abstract method of solving word problems. The Block Model Approach helps pupils visualize situations because it creates concrete picture of from abstract situation. It may satisfies the pupil's learning through seeing and doing. Finally, it transforms words into recognizable pictures for young minds. The researcher wanted to find out whether the problem solving performance of the Grade 5 pupils was moderated by the problem solving approach. The intervening variables are mathematical ability and types of problem. Under mathematical ability pupils are grouped into above average, average, and below average; for type of problems, it was classified into one - step, two – step, and three – step.

Data Gathering Procedure

The study has three phases. These are; 1) pre – experimental phase; 2) experimental phase; and 3) post – experimental.

Phase 1. Pre – Experimental Phase

An Achievement Test together with the table of specification was prepared. The other instruments used were also prepared. Lesson plans and Activity sheets for both the experimental and control groups. Letter of request were made (Appendices A to F); one was addressed to the Principal's office, the other one is addressed to the Registrar's office through the Office of the Academic Coordinators for the final grades of the respondents during the school year 2006 – 2007. After the thorough evaluation of the Achievement test, the final copy of this test was used as pretest and posttest of the study.

Phase 2. Experimental Phase

The teacher administered the Problem Solving Performance Test a day after the class orientation. The students in the control group went through the conventional way of teaching using the Traditional method. The students in the experimental group were exposed to Block Model Approach. During the experiment, the researcher following the same content coverage handled both groups.

Phase 3. Post-Experimental Period

A posttest was administered to both groups the day after the sessions. The results of the pretest and posttest were analyzed and examined to evaluate or assess the effects of the block model approach on the performance of the students.

Statistical Treatment of the Data

The following statistical tools were used to analyze the data gathered.

- t – test for Dependent Samples. This was used to determine if there is significant difference between the pretest and posttest scores of the experimental group and control group in terms of performance in the test.
- t – test for Independent Samples. This was used to find out if there is significant difference between the posttest mean scores of the experimental group and control group on problem solving performance test.
- Two – Way Analysis of Variance. This was used to determine if there is a significant interaction effect of problem solving approach and the mathematical ability on the problem solving performance of the pupils.

Research Instruments

Problem Solving Performance Test (PSPT)

Stage 1. Preparing the Initial Draft. A table of Specifications for the first draft was constructed to ensure that all content areas were represented in the test (Appendix B₁). The scope and sequence and the instructional objectives in the La Salle Green Hills, Grade School Math Department were the determining factors in the preparation of the table of specifications.

- **Content Validation**
An initial draft of 50 multiple – choice items was made and submitted to the Academic Coordinator and three Mathematics teachers of the La Salle Green Hills for a thorough examination. A table of specifications was prepared to cover all the topics that will be discussed in the duration of the study.
- **Face Validation**
Three Math experts teaching at La Salle Green Hills were involved in the validation of the Problem Solving Performance Test.
- **First Try-out**
The 50 multiple choice test items were tried out to one of the sections of the graduating class of 2007 consisting of 30 pupils. These students were considered for the try – out of the test as they had just learned the items covered in the test few months back that they were in the position to answer the test items.

Stage 2. Item Analysis. After the try-out, the test items were analysed by using the Upper and Lower Index Method. Option analysis was done. Deciding whether to retain or remove the items was based on two ranges. Items with difficulty indices within 0.20 to 0.80 and discrimination indices within 0.30 to 0.80 were retained. From the 50 multiple - choice items, 10 of these were discarded and, 30 items were retained and 10 were subjected to revision.

Stage 3. Second Try – out. After analyzing the results of the first draft, the final version of the 40 item multiple choice problem solving performance was administered to another group of graduating pupils. Another item analysis was done to find out the indices of discrimination and difficulty of the revised test items. Items with difficulty indices within 0.20 to 0.80 and discrimination indices within 0.30 to 0.80 were retained. The same procedure and computation instruments as in the first try – out were used. Appendix C2 shows the option for the second try – out and the analysis of each item as to difficulty and discrimination indices. From 40 multiple choice items, 10 were discarded, 24 were retained and 6 were subjected to revision.

Stage 4. Preparing the Final Draft. After conducting and revising the second try – out the final was ready to serve as pretest and posttest in the study. The final version composed of 30 multiple-choice items can be found in Appendix N.

- **Lesson Plans**
Before the start of the experiment, lesson plans were prepared covering the lessons given during the experimental period, one using the traditional method and the other one using the Block Model Approach. These were submitted to a group of senior Mathematics teachers in La Salle Green Hills. After critiquing, suggestions and recommendations from the Mathematics experts were solicited and observed. The lesson plan can be found from Appendices AB to AQ.

Presentation, Analysis and Interpretation of Data

Comparison between the Problem Solving Performance of the Control Group and the Experimental Group after the Experiment.

Table 1 shows the comparison of the experimental and control groups in terms of problem solving performance after the experiment.

Table 1

Comparison of the Experimental and Control Groups in Terms of Mathematical Ability after the Experiment

Groups	Mean	Std Dev	Mean Difference	df	Computed t-value	Tab value	Interpretation
Experimental	22.71	4.04	3.92	82	3.94	1.99	Significant
Control	18.79	5.03					

It can be viewed from the table the computed t – value of 3.94 is greater than the critical value of 1.99 at 0.05 level of significance. It shows that there is significant difference in the problem solving performance between the control group and the experimental group. It can be also gleaned in the table that the experimental group performed better in terms of problem solving compared to their counter part in control group.

Comparison Between the Pretest Mean Scores of the Control and Experimental

Table 2 shows mean scores on problem solving performance of the experimental group and control group after the experiment when they are grouped according to their mathematical ability.

Table 2

Mean Scores on Problem Solving Performance of the Experimental Group and the Control Group According to their Mathematical Ability after the Experiment

Groups	Mathematical Ability		
	Below Average	Average	Above Average
Control	16.45	19.58	21.71
Experimental	19.62	22.86	27.38

As can be gleaned on the above data, it can be deduced that the mean scores of the pupils belong to the above average group are higher than their counter part in average and below average group. The data also shows that groups in the experimental group perform better than their counter part in the control group. To determine whether problem solving approach on problem solving performance is moderated by the level of mathematical ability, Two Way ANOVA was used. Table 3 shows the summary of the two - way ANOVA for the Interaction Effect of Problem Solving Approach and Mathematical Ability on Problem Solving Performance.

Table 3

Two Way ANOVA for the Interaction Effect of Problem Solving Approach and Mathematical Ability on Problem Solving Performance

Source of Variation	SS	df	MS	Computed F	Tab F	Interpretation
Problem Solving Approach	271.44	1	271.44	19.54	3.96	Significant
Math Ability	401.71	2	200.85	14.46	3.11	Significant
Interaction	27.76	2	13.88	1	3.11	Not significant
Error	1083.51	78	13.90			
Total	1784.42	83				

As can be seen in the table, the main effect yielded significant results. This means that the effect of the problem solving approach on the problem solving performance test is significant. Also, mathematical ability has a significant effect on the problem solving performance as expected. However, the interaction effect of the problem solving approach on mathematical ability on problem solving performance is not significant. On the other hand, the performance of the experimental group which was exposed to block model approach performed better than the control group which used the traditional method. This means that the effect of the problem solving approach on problem solving performance is not moderated by mathematical ability. Thus, with regards to mathematical ability, any problem solving approach can be utilized.

Interaction Effect of Types of Problem and Problem Solving Method on Problem Solving Performance

Table 4 shows the mean scores on problem solving performance of the experimental group and control group after the experiment when they are grouped according to their type of problems.

Table 4

Mean Scores on Problem Solving Performance of the Experimental Group and the Control Group According to Types of Problems after the Experiment

Groups	Type of Problems		
	One - Step	Two - Step	Three - Step
Control	29.92	25.56	25.11
Experimental	34.42	30	30.11

As can be gleaned on the above data, it can be deduced that the mean scores of the pupils who solved one- step are higher than their counter part in average and below average group. The data also shows that groups in the experimental group perform better than their counter part in the control group. To determine whether problem solving approach on problem solving performance is moderated by the types of problems, Two Way ANOVA was used. Table 5 shows the summary of the two - way ANOVA for the Interaction Effect of Problem Solving Approach and Mathematical Ability on Problem Solving Performance.

Table 5

Two Way ANOVA for the Interaction Effect of Problem Solving Approach and Types of Problems on Problem Solving Performance

Source of Variation	SS	df	MS	Computed F	Tab F	Interpretation
Problem Solving Approach	322.02	1	322.02	10.72	4.02	Significant
Type of Problems	288.26	2	144.13	4.8	3.17	Significant
Interaction	0.87	2	0.44	0.01	3.17	Not significant
Error	1621.83	54	30.03			
Total	2232.98	59				

As can be seen in the table, the main effect yielded significant results. This means that the effect of the problem solving approach on the problem solving performance test is significant. Also, type of problems has a significant effect on the problem solving performance. As expected, pupils perform the highest when they are given one – step word problem. Therefore, the number of steps in problem solving affects their performance.

However, the interaction effect of the problem solving approach on type of problems on problem solving performance is not significant. On the other hand, the performance of the experimental group which was exposed to block model approach performed better than the control group which was used the traditional method. This means that the effect of the problem solving approach on problem solving performance is not moderated by types of problem. Thus, regardless of the type of problems any problem solving approach can be utilized.

Summary of Findings

The following summarizes the findings of the study.

1. There is a significant difference between the problem solving performance of the control group and the experimental group after the experiment. Moreover, the experimental group performed better than the control group.
2. The effect of problem solving approach on the problem solving performance of the pupils is not moderated by the level of mathematical ability.
3. The pupils' mathematical ability does not depend on the approach given. However, the performance of the experimental group which uses the block model approach performed better than the control group which was exposed to the traditional method.
4. The effect of problem solving approach on the problem solving performance of the pupils is not moderated by the type of the problem (i.e. one step, two step, and three step problem). However, the performance of the experimental group which uses the block model approach performed better than the control group which was exposed to the traditional method.
5. Regardless of the mathematical ability any teaching approach on the problem solving can be used since there is no interaction effect of problem solving approach and mathematical ability on problem solving performance.
6. Regardless of the types of problems any teaching approach on the problem solving can be used since there is no interaction effect of problem solving approach and the types of problems on problem solving performance.

7. Majority of the respondents found the block model approach in teaching mathematics interesting and useful in solving worded problems. This approach is so helpful that it really make them have an easier time to learn and understand the lessons. Other student-respondent claimed that the block model approach “is very simple, easy to use and it shows how the process happened in word problems.”

Based on the findings from this study, the following conclusions are drawn:

1. The use of block model approach helps the student’s problem solving skills and enhances the retention of concepts learned. Thus, Block Model Approach helps the pupils to performed better in the problem solving performance.
2. Block Model Approach as perceived by the pupils were useful in solving word problems in mathematics and easy to use.

Recommendations

Based on the findings and conclusion on this study, the researcher proposes the following recommendations:

1. Use the Block Model Approach as an alternative approach in teaching the word problem solving in Mathematics.
2. Introduce the Block Model Approach as early as Grade I to master the said approach.
3. Support and fund research to identify programs that successfully tie literacy to content instruction, particularly in Mathematics.
4. Support and fund professional development for teachers regarding the needs of problem solving performance of the pupils.
5. Conduct further research on the relationship between mathematics learning and problem solving.

References

- Draper, R. (2002, March). School mathematics reform, constructivism, and literacy: A case for literacy instruction in the reform-oriented math classroom. *Journal of Adolescent &Adult Literacy*, 45(6), 520.
- Larochelle, M., Bednarz, N., & Garisson, J. (Eds.) (1998). *Constructivism and education*. Cambridge, NY: Cambridge University Press.
- Tynjala, P. (1998, June). Traditional studying for examination versus constructivist learning tasks: Do learning outcomes differ? *Studies in Higher Education*, 23(2), 173.

Constructing and Consolidating Mathematical Entities in the Context of Whole-Class Discussion

Thérèse Dooley
St. Patrick's College, Dublin
<therese.dooley@spd.dcu.ie>

In this paper, the RBC+C framework (Hershkowitz, Schwartz, & Dreyfus, 2001) is used to analyse and describe construction and consolidation of mathematical knowledge by primary pupils in a whole-class setting. I describe a lesson that concerned what is commonly termed the Handshakes problem. One pupil spontaneously established a connection with a related problem in which the class had engaged a month previously. There followed a conversation in which an older construct was consolidated while a new construct emerged - the nature of this intertwined construction and consolidation is discussed.

In research related to mathematical abstraction, a theoretical framework that is receiving considerable attention is 'Abstraction in Context' (AiC) (Schwartz, Dreyfus, & Hershkowitz, 2009). The three epistemic or observable actions identified by Hershkowitz and her colleagues (Hershkowitz, et al., 2001; Schwartz, et al., 2009) as giving a strong indication that mathematical abstraction is taking place are 'recognizing' (R), 'building-with' (B) and 'constructing' (C). R and B-actions are nested within C-actions and indeed C-actions might be nested within more global C-actions. Furthermore RBC actions have been found to be distributed when groups of students collaborate. The phase following construction is consolidation during which there is conscious reuse of the new construct for recognising or building-with purposes. For this reason the model is often termed the RBC+C model where the second C refers to consolidation. In this paper I use the RBC+C model to analyse the construction by primary pupils of mathematical knowledge in the context of whole class discussion. The lesson I describe ('Chess') is commonly known as the 'Handshakes' problem. In one phase of the plenary discussion a pupil spontaneously made a connection with a related problem ('Friendship Notes') with which the class had engaged one month previously. In the ensuing interaction a group of pupils built with understandings developed in the earlier lesson to construct new understandings of Chess. I argue that in this instance consolidation was also distributed. Furthermore the teacher has a particular role to play in drawing attention to earlier constructs while at the same time taking care not to disrupt the flow of conversation.

Abstraction in Context

The AiC framework was first proposed by Hershkowitz, Schwartz, and Dreyfus (2001) and has been verified since in a variety of contexts (Schwartz, Dreyfus, & Hershkowitz, 2009). Schwartz et al. define abstraction as "an activity of vertically reorganizing previous mathematical constructs within mathematics and by mathematical needs so as to lead to a construct that is new to the learner" (2009, p.24). They suggest that the genesis of abstraction passes through a three stage process: (i) the need for a new construct, (ii) construction and (iii) consolidation of the construct. The need arises from the design of the problem or the student's motivation to solve a particular problem. The RBC actions are used to model construction, the second phase of the abstraction process. Recognition occurs when a student realizes that specific prior knowledge is appropriate for the mathematical problem with which he/she is dealing. When 'building-with', the student is not enriched with new, more complex structural knowledge but is using available structural knowledge to

deal with the problem at hand. This stage is evident when he or she is involved in an application task or making a hypothesis or justifying a statement. Constructing, the most significant of the epistemic actions that are constituent of abstraction, is a process of building more complex structures from simpler structures. It involves the reorganization of mathematical elements so that a more refined structure emerges. In order to distinguish between ‘building-with’ and ‘constructing’, it helps if the goals of the particular activities are considered. In constructing, students use a new mathematical structure to attain their goal. In ‘building-with’, a goal is attained by combining existing structures. These three epistemic actions are not linear but nested. In other words, ‘recognizing’ (R) and ‘building with’ (B) do not precede the process of ‘constructing’ (C) but are rather nested within it. Furthermore a construction (or C-action) might have nested within it not only a large number of R- and B- actions but also other C-actions. While the RBC model of abstraction was derived from the study of an individual student (Hershkowitz, et al., 2001) and from one of a pair of students working collaboratively (Dreyfus, Hershkowitz, & Schwartz, 2001), it has since been validated in a variety of contexts. Of particular relevance to this paper is that it has been used to describe a ‘collective abstraction process’ (Hershkowitz, 2009) where different students contribute different parts to the constructing activity. This has been found in the context of a small group of students (Hershkowitz, Hadas, & Dreyfus, 2006) and also in that of whole class discussion (Dooley, 2007).

The third phase of the abstraction process – consolidation – occurs when the learner uses a construct in a flexible manner, for example, when recognising or building-with the structure or using it as a resource in constructing a new mathematical entity (Tsamir & Dreyfus, 2005). The most frequently observed mechanism of consolidation is that of building-with the construct (Dreyfus, Hadas, Hershkowitz, & Schwartz, 2006). As with constructing actions, consolidation has been observed in students working alone on tasks (Monaghan & Ozmantar, 2006) and in pairs of students collaborating on a problem (Tabach, Hershkowitz, & Schwartz, 2006). It can occur in the immediate aftermath of construction (Monaghan & Ozmantar, 2006) and also after a delay of a few months (Tabach, et al., 2006). Furthermore, bearing the characteristics of immediacy, self-evidence, confidence, flexibility and awareness, it is often associated with the language of certitude, e.g., ‘clearly’, ‘obviously’ etc. (Dreyfus and Tsamir, 2004).

Background

I conducted a ‘classroom design experiment’ (Cobb, Gresalfi, & Hodge, 2009) in three different primary schools in Ireland in order to investigate the construction of new mathematical ideas by pupils (Dooley, 2010). Data collected included field notes, audiotapes of whole-class and group interactions, pupils’ written artefacts, digital photographs of blackboard recordings, interviews with teachers and, in two of the schools, pupil diaries and post-lesson interviews with small groups of pupils. Retrospective analysis of data was conducted on micro- (between lessons) and macro-levels (between and after cycles of research in the three classrooms). I developed an analytic framework that comprised four dimensions:

1. Mathematical Principles (the constructs that pupils could be expected to develop by engagement with the task);
2. RBC Epistemic Actions;
3. Vague Language (Hedges and Pronouns); and
4. Teacher Follow-up Moves.

For the purpose of this paper, I will focus mainly to the RBC dimension of the framework. However, some reference will also be made to pupils’ use of vague language as

such language has been found to be integral to RBC+C actions (Dooley, 2011). The lessons I describe took place in the third cycle of research with a class of 31 pupils aged 9 – 10 years.

The Lessons

‘Friendship Notes’ reads as follows:

As part of Friendship Week in Greenville School, each pupil writes a short note to each other pupil in his/her class. Each pupil is given one sheet of paper for each note. How many sheets of paper are needed if there are 5 pupils in a class? 10 pupils? What would be the number for any number of pupils?

The Chess lesson took place four weeks after the Friendships Notes lesson and reads as follows:

In a chess league each participant plays a game of chess with all other participants. How many games will there be if there are 3 participants? 10 participants? 20? Is there a way to find the number of games for any number of participants?

Both Friendship Notes and Chess are characterised by non-reflexivity (that is, no element of a set relates to itself). The main difference between the activities lies in the property of symmetry. ‘Chess’ is symmetrical because if A relates to (‘competes with’) B, then it follows that B relates to A. However, in ‘Friendship Notes’, if A relates to (‘writes to’) B, the reciprocal relationship is not implied. For this reason the function mapping n (the number of people) to y (the number of notes) in the Friendships Notes is $y = n(n-1)$ while in

Chess, $y = \frac{n(n-1)}{2}$ where n represents the number of people and y the number of games. In Friendship Notes a number of pupils constructed the formula with reference to its asymmetric, non-reflexive properties. For example, David described the asymmetric aspect:

David: Because you need em like ... one person would need to give one to the other and the other person would give one back.

Catherine rationalised a solution for $n = 20$ (i.e., 20×19) as follows:

Catherine: Em well you don’t give one to yourself so you take away ... one.

TD: Yeah

Catherine: And then em you multiply them together.

The Chess lesson took place over two days. At the end of the first session (Chess 1), David, conjectured that the number of games for 20 people might be found by multiplying 20 by 19 and halving the product. He built with this strategy in Chess 2 to state a generalised formula. However, he was unable to verify his proposed solution method with reference to the underlying structure of the problem, that is, to give it ‘structural verification’ (Rowland, 1999). Another student, Enda, made a connection with Friendship Notes and this led to a new construction, i.e., structural verification of the formula. The transcript¹ below concerns this C-action.

¹ Transcript conventions are: TD: the researcher/teacher (myself); Ch: a child whose name I was unable to identify in recordings; ... : a short pause; [...]: a pause longer than three seconds; []: lines omitted from transcript because they are extraneous to the substantive content of the lesson.

Turn	Transcription	Pupil Action	Epistemic Action (RBC)	
636	David: Multiply it by the number less ...			Constructing 
637	TD: Huh, huh.			
638	David: [...] and then half it.	David proposed a generalised formula for Chess.	Building-with	
639	Enda: It looks like ... it's pretty much the very same as the friendship cards, it seems kind of like that.	Enda made a connection with 'Friendship Notes'.	Building-with	
640	TD: Right, Enda, do you remember the friendship notes, that's a good thing. Do you remember the friendship notes? Do you remember what you did for the friendship notes? What did you do for the friendship notes? Do you remember the rule? Barry?			
641	Barry: It's kind of the same thing as, eh, you wouldn't have to do themselves so there's going to be one less.	Barry referred to the non-reflexive nature of both activities.	Building-with	
642	TD: Ok, so but the rule ... according to David, when we were doing the friendship notes, [] For example in friendship notes if there were three children than how many notes would there be for three children? ... Do you remember? ... Right, Barry? Does anyone remember how many notes there were for three children in the friendship notes? ... Yeah?			
643	Colin: Em, six.	Colin recalled number of friendship notes for six children.	Recognising	

644	TD: Six but see in the chess game it's only three. So why is it a bit different? Does anyone know why it's a bit different? [...] Myles?			
645	Myles: Em because in chess you will just have to play them, if they played you one time then you have kind of played them once.	Myles referred to symmetric nature of 'Chess'.	Building-with	
646	Ch: Ah!			
647	Myles: In friendship notes you have to play them kind of again so like you give them your note and they will have to ... they will still give you back a note.	Myles referred to the asymmetric nature of 'Friendship Notes'.		
648	TD: Ok, so once you play the game you don't play it back, isn't that what you are saying?			
649	Myles: Yeah.			
650	TD: That's what you are saying. Yes?			
651	Colin: Em, well cos in the friendship notes you have to give two because if there were three you would have to give one to each person ...	Colin referred to the asymmetric nature of 'Friendship Notes'.	Building-with	
652	TD: Hm, hm.			
653	Colin: ... and everyone has to give one to each person, so it's the same as three by two.			
654	TD: Hm, hm.			
655	Colin: Eh, and in chess you only have to play them once even if they challenge you.	Colin referred to symmetric nature of 'Chess'.		
656	TD: Hm, hm.			
657	Colin: So eh ...			
658	TD: And what does that mean for the chess game then? What does it mean for the chess ... rule?			
659	Colin: Eh, you don't ... you don't play them twice.			
660	Ch: Ah!			

661	TD: Ok, so what happens then, what's the rule for the chess? Enda?			
662	Enda: Eh well, I actually definitely agree with David's way by doing the friendship notes, the same way as the friendship notes and halving it ...	Enda made a connection between David's formula and 'Friendship Notes'.	Building-with	
663	TD: Hm, hm.			
664	Enda: ... because all of the things we get in that are half what we get in the chess thing.			
665	TD: Hm, hm.		Building-with	
666	Enda: So I definitely agree with David's way by multiplying by one number less and halving it. I definitely agree with that now.			

Epistemic Actions

Enda's input in turn 639 above is characterised by vagueness (e.g., "... it seems kind of like that."). His use of the adaptor, 'kind of', for example, allows him to make an assertion without fully committing to it and thus shield himself from accusation of error (Rowland, 2000), an important consideration given his large audience. I, for obvious reasons, greet his suggestion with enthusiasm and, following my probing questions, Barry endeavours to explicate the connection. Again he is tentative about the connection between the two problems ("It's kind of the same thing ...") but his description of the non-reflexive nature of both problems is characterised by more certainty (i.e., "you wouldn't have to do themselves so there's going to be one less"). Myles then describes quite confidently the structure of Chess (turn 645). It would seem that his grasp of the asymmetric structure of Friendship Notes (expressed in turn 647) facilitates his understanding of the symmetric structure of Chess. Colin reasoning in turns 651, 653 and 655 is similar. Moreover, in turn 659, he builds-with further by intimating the multiplicative relationship between Chess and Friendship (" [In Chess] you don't play them twice"). According to Monaghan and Ozmantar (2006), consolidation of a previously constructed entity is evidenced by the establishment of interconnections between it and a new construction and by reasoning with these constructions. At the beginning of the constructing action above, Enda was uncertain about the nature of the relationship between Chess and Friendship Notes. However, the reasoning in which Barry, Myles and Colin engaged and their consolidation of the previous construct establishes the connection for him and indeed it is he, Enda, who expresses this relationship in turns 662, 664 and 666. Here his input is marked by certainty and confidence. In turn 664, he finds support for David's suggested method in the list of solutions for Friendship Notes which was available in his diary ("because all of the things we get in that are half what we get in the chess thing"). The error he makes (since the Friendship Notes solutions are double those for Chess!) is obviously a slip. Underpinning the strategy that he states in turn 666 ("multiplying by one number less and halving it") are the non-reflexive and symmetrical aspects of the problem explicated by some of his peers.

Concluding Remarks

According to Tabach, et al. (2006) knowledge constructing and consolidating are dialectical processes in that “new *Constructs* stem from old ones already *Consolidated*, which gain *Consolidating* through the new *Construction*, creating a new abstract entity” (p.255, italics in original). This dialectical process is evident in the transcript above as students move back and forth between consolidation of the Friendship Notes construction and development of a new entity (structural verification of the Chess problem). As they build-with the Friendship Notes construction, they consolidate it further. This is evidenced, in particular, by the generalised manner in which they speak of the construct – in particular the pronoun ‘you’ functions as a ‘vague generalizer’ (Rowland, 2000) suggesting that the pupils are detaching from specific examples to a more generalised description of the structure of the (Friendship Notes) problem. This consolidation, which is used for building-with, is nested within the C-action. Furthermore, it allows the pupils concerned to refer also to Chess in generalities (e.g., “you wouldn’t have to do themselves so there’s going to be one less” and “you don’t play them twice”). There is also evidence that just as constructing can be distributed, so too is consolidation. This is possibly the case because the students ‘reconstruct’ the Friendship Notes construct in order to verify structurally the formula for Chess. The reconstruction is not, as maintained by Monaghan and Ozmantar (2006), the same as consolidation but rather an important part of it. Nonetheless, consolidation of the Friendship Notes structure (as expressed, for example, in turns 641, 647 and 653) is distributed among a few of the class members.

In previous papers I have described how whole-class conversation can be a vehicle for the construction of new mathematical entities. What is critical is the creation of a ‘conjecturing atmosphere’ in which pupils try out, test and modify ideas (Mason, 2008). Over direction by the teacher can stifle such an atmosphere particularly if pupils feel that their ideas are being evaluated. In the interaction above I as teacher-researcher might be accused of ‘over-directing’ children’s contributions. However, the first mention of Friendship Notes was made spontaneously by Enda. Ozmantar and Monaghan (2007) have suggested that “if students are not aware of the importance and necessity of the knowledge artefacts available to them for a new construction, they are unlikely to draw upon them as they (or their attention) are blocked” (p.106). It is for this reason that I broadcast Enda’s contribution to the class and then set sub goals (e.g., in turns 640, 642 and 658). However, in the context of whole-class discussion, such interventions by the teacher have to be used with care since they might serve to disenfranchise other students and curb contributions that they could make to the construction process. Indeed although there were 31 pupils in the class, the transcript above concerns only five of them. In subsequent written reflections, many pupils aligned themselves with David’s solution method. Furthermore, other pupils developed different constructs in that lesson and still others built-with some of the constructs developed in Chess in a related lesson that took place on the following day. This suggests that while some pupils explicitly consolidated and contributed to the construction of new entities there were others in the class who engaged tacitly in the process. Further analysis is required to investigate the extent of this.

References

- Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). A design research perspective on the identity that students are developing in mathematics classrooms. In B. Schwartz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 223 - 243). London and New York: Routledge.

- Dooley, T. (2007). Construction of knowledge by primary pupils: The role of whole-class interaction. In D. Pitta-Pantazi & G. Philippou (Eds.), *Fifth congress of the European Society for Research in Mathematics Education* (pp. 1658 - 1667). Larnaca, Cyprus: CERME.
- Dooley, T. (2010). *The construction of mathematical insight by pupils in whole-class conversation* (Unpublished doctoral dissertation). University of Cambridge, Cambridge.
- Dooley, T. (2011). RBC epistemic actions and the role of vague language. In B. Ubuz (Ed.), *Proceedings of the 35th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 281 - 288). Ankara, Turkey: PME.
- Dreyfus, T., Hadas, N., Hershkowitz, R., & Schwartz, B. (2006). Mechanisms for consolidating knowledge constructs. In N. J., H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 465 - 472). Prague: PME.
- Dreyfus, T., Hershkowitz, R., & Schwartz, B. (2001). Abstraction in context II: The case of peer interaction. *Cognitive Science Quarterly*, 1(3/4), 307 - 368.
- Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *The Journal of Mathematical Behavior*, 23(3), 271 - 300.
- Hershkowitz, R. (2009). Contour lines between a model as a theoretical framework and the same model as methodological tool. In B. Schwartz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 273 - 280). London and New York: Routledge.
- Hershkowitz, R., Hadas, N., & Dreyfus, T. (2006). Diversity in the construction of a group's shared knowledge. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 297 - 304). Prague: PME.
- Hershkowitz, R., Schwartz, B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, 32(2), 195 - 222.
- Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early years* (pp. 57 - 94). New York and London: Lawrence Erlbaum Associates.
- Monaghan, J., & Ozmantar, M. (2006). Abstraction and consolidation. *Educational Studies in Mathematics*, 62, 233 - 258.
- Ozmantar, M., & Monaghan, J. (2007). A dialectical approach to the formation of mathematical abstractions. *Mathematics Education Research Journal*, 19(2), 89 - 112.
- Rowland, T. (1999). 'i' is for induction. *Mathematics Teaching*, 167, 23 - 27.
- Rowland, T. (2000). *The pragmatics of mathematics education: Vagueness in mathematical discourse*. London: Falmer Press.
- Schwartz, B., Dreyfus, T., & Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In B. Schwartz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11 - 41). London and New York: Routledge.
- Tabach, M., Hershkowitz, R., & Schwartz, B. (2006). Constructing and consolidating of algebraic knowledge within dyadic processes: A case study. *Educational Studies in Mathematics*, 63, 235 - 258.
- Tsamir, P., & Dreyfus, T. (2005). How fragile is consolidated knowledge? Ben's comparisons of infinite sets. *Journal of Mathematical Behaviour*, 24, 15 - 38.

Male Students' Perspectives Concerning the Relevance of Mathematics – Pilot Study Findings

Michael Easey

Australian Catholic University

<michael.easey@acu.edu.au>

Elizabeth Warren

Australian Catholic University

<elizabeth.warren@acu.edu.au>

Vince Geiger

Australian Catholic University

<vincent.geiger@acu.edu.au>

A pilot study was conducted with Year 10 males ($N = 154$) preparing to make their senior mathematics subject choice. Survey data revealed that students did not understand the different dimensions of relevance of mathematics. Additionally, a statistically significant difference in the level of agreement concerning relevance was identified between students choosing Mathematics A and those choosing both Mathematics B and C. Students choosing both Mathematics B and C perceived mathematics as relevant for facilitating their career pathway, while Mathematics A aspirants acknowledged the relevance of mathematics was less influential, reporting their choice was guided by their mathematical ability.

For an economy to be productive and competitive, it is reliant upon individuals who are scientifically and technologically literate. The discussion paper, *Towards a 10-year plan for Science, Technology, Engineering and Mathematics (STEM) education and skills in Queensland* (Department of Education Training and the Arts, 2006) reports that, an increasing demand for professional, trade and technical occupations exists in STEM related careers. However, the evident decline in students pursuing high levels of post-compulsory mathematics in Queensland schools has reduced the pool of mathematically capable students undertaking tertiary studies in mathematics or scientific areas. It is therefore necessary to explore and identify what influences students' choice as to whether they participate in high levels of mathematics.

Studies have explored the decline in the number of students participating in mathematics subjects in general, and a variety of findings have been presented. Reasons identified include student disaffection (Nardi & Steward, 2003), students' attitudes (Brown, Brown & Bibby, 2008), and the influence of peers, parents and teachers (Ma, 2000). However, the research literature is beginning to provide increasing evidence that the relevance students' associate with mathematics influences their participation and engagement. In the Australian context, research has been conducted examining the propensity of Year 12 students across ten South Australian schools to study mathematics currently and after completing Year 12 (Brinkworth & Truran, 1998). This study revealed that the *need* for mathematics as a prerequisite to further study was the most noteworthy reason influencing students' current and post-school participation in mathematics (Brinkworth & Truran, 1998). More recently, a study has focused on reasons why capable students were not choosing higher-level senior mathematics (McPhan, Morony, Pegg, Cooksey & Lynch, 2008). This study explored the perspectives of 15-year-old students and identified that one of the explicit influences that affected students' choice was the perception they associated with the *usefulness* of higher-level mathematics (McPhan, Morony, Pegg, Cooksey & Lynch, 2008). These findings echo past research (Brinkworth & Truran, 1998) in that the usefulness of higher-level mathematics was perceived as its value as a prerequisite for entry into post-secondary level courses, especially tertiary courses with a high tertiary entrance score. Additionally, research reveals that the common terms used by students to describe relevance of

mathematics include *need* or *useful* (ref). However, when considering relevance, terms such as *need* or *useful* reveal a more nuanced meaning.

More recent studies have moved beyond looking at the difference between *need* and *usefulness* to consider the relevance of mathematics. Sealey and Noyes (2010) conducted research and reported results from focus group interviews at three contextually different schools located in the United Kingdom. Amongst their findings was the recognition that students from different schools attached different relevance to mathematics. These results provided a framework that categorised relevance as having three different dimensions. The first dimension, acknowledged that mathematics had a useful purpose identified as *practical relevance*. In this sense, students perceived mathematics as a tool used for performing basic calculations or solving simple problems in day-to-day life. The second dimension recognised that mathematics has transferable potential or *process relevance*. In this regard, students perceived that mathematics provided skills that facilitated conceptualising and solving problems in different contexts. The third dimension established that mathematics provided exchange value or *professional relevance*. In this dimension students sensed that mathematics enabled their tertiary trajectory and assisted them to fulfil their career ambitions. Little is known about the relationship between practical, process, and professional dimensions of relevance or the impact they have on subject choice. This paper examines and presents initial findings concerning Year 10 male students' perceptions of relevance as they consider their Years 11 and 12 mathematics subject choice by providing ratings for Likert items and responses to open-ended questions presented in a survey instrument.

Context

This paper reports on an element of a larger study conducted in Queensland that explored the decline in participation in Mathematics B, and Mathematics B and C among male students in Years 11 and 12. The results presented are from the pilot study conducted at Willow College (pseudonym). In Queensland, while studying mathematics is not compulsory in Years 11 and 12, the majority of students participate in either Mathematics A, Mathematics B, or Mathematics B and C. Mathematics A is an increasingly popular subject that provides students with suitable preparation for further study and training in technical trades, hospitality, and administrative roles and enables students to engage in tertiary studies where mathematics requirements are moderate. Mathematics B provides students with preparation for tertiary studies in subjects, especially in the areas of science, engineering, information technology, and mathematics. Its predominant focus is on calculus and algebra utilised in mathematical contexts. Mathematics C is a subject studied in conjunction with Mathematics B and is designed to provide a greater level of preparation for students seeking to engage in tertiary studies where mathematics requirements and its predominant focus is on more abstract mathematical concepts such as complex numbers and matrices (Queensland Studies Authority, 2008).

Method

Participants

Willow College was purposely selected for the pilot study as it shared comparable demographics with the research school, including a male only educational context, a non-selective student enrolment policy, and an ethos based on the vision of the same Christian founder. Furthermore, the Index of Community Socio-Educational Advantage (ICSEA) provided for schools throughout Australia by the Australian Curriculum, Assessment and

Reporting Authority (ACARA), established that Willow College ($M = 1023$) and the research school ($M = 1039$) were statistically similar in terms of the socio-economic standing of their respective communities ($M = 1000$, $SD = 100$). The entire Year 10 cohort ($N = 154$) ranging between the ages of 14 - 16 years ($M = 14.84$) completed the pilot survey instrument, representing a cross-section of mathematical ability, achievement, and experiences.

Pilot Survey Instrument

Two sources were used to develop the pilot survey instrument namely the current research literature and the focus group interviews with staff. First, the literature was examined to identify themes that related to students' participation in mathematics and mathematics subject choice. The themes that emerged were school culture (MacNeil, Prater & Busch, 2009), learning mathematics (Philipp, 2007), mathematics teaching (Stipek, Givvin, Salmon, & MacGyvers, 2001), and perceptions of relevance (Chouinard & Roy, 2008). Second, the survey instrument was informed by the results of focus group interviews conducted at the research school to gather contextual knowledge regarding influences affecting students' participation in Mathematics B, and Mathematics B and C. These interviews were conducted with four groups of purposely selected teachers ($N = 8$) that either taught Year 10 students mathematics ($n = 5$), counselled Year 10 students preparing to make their senior mathematics subject choices ($n = 1$) or had curriculum leadership roles ($n = 2$) at the school. A schedule of questions for each focus group interview was prepared from themes identified in the literature and through collaboration with expert academic staff. The duration of the interview was 45 minutes and responses to questions were audio-recorded and then transcribed for qualitative analysis facilitated by using software, Atlas.ti.

Themes that emerged from analysis of the focus group interviews confirmed concepts and issues identified in the literature. Teachers identified that university prerequisites, teacher pedagogy and teacher experience influenced students' participation in Mathematics B, and Mathematics B and C. However, an analysis of teacher data revealed that based on their perceptions, other elements impacted on students' participation in mathematics, for example commitments outside the school, access to knowledge concerning careers and the unique culture of the school. Together, themes from the literature and focus group interviews combined to inform the construction of the pilot survey instrument.

The instrument consisted of three sections. The first section presented 15 demographic questions inviting the participant to provide their age, cultural background, levels of mathematical achievement and career aspirations. The second section consisted of 47 Likert items distributed across five dimensions. The dimensions used to explore the decline in male students' participation in Mathematics B, and Mathematics B and C were: (1) school culture, 10 items; (2) learning mathematics, 13 items; (3) mathematics teaching, 9 items; (4) the relevance of mathematics, 7 items; and, (5) other elements impacting on students' senior mathematics subject choice, 8 items. Participants evaluated and provided a response to each item according to their level of disagreement or agreement. A symmetrical, five-point rating scale was provided for each item: 1 – Strongly disagreed; 2 – Disagreed; 3 - Unsure; 4 – Agreed; or 5 – Strongly agreed with an item. The third section of the survey invited students to write responses to eight open-ended questions to further explore themes identified in the literature.

Data Collection

The survey was conducted at Willow College in the first week of Term 3, 2011. Six Year 10 mathematics teachers administered the survey to each of their classes. Using the instructions that were provided all students in the cohort completed the survey anonymously in approximately 20 minutes, under examination style conditions. At the completion, teachers returned surveys to the Principal and were collected by the researcher. Overall, a small number of questions in the demographics and second sections were not attempted or were completed with errors by some students and these were accommodated in the statistical analysis.

Analysis and Results

Analysis of the pilot study data was conducted in stages. First, responses from the demographic and second section of the survey were checked for possible errors and missing data and consideration was given to how these issues would be handled in the data entry stage. Second, students' responses were coded by assigning numerical values to each possible response. Third, the data were entered in the Statistical Package for the Social Sciences (SPSS) software to conduct statistical analysis. A Principal Component Analysis (PCA) with orthogonal rotation was conducted on the items associated with each section of the survey (varimax). This statistical technique (factor analysis) was applied to identify whether correlations existed between chosen sets of variables. Figure 1 provides the scree plot that illustrates the distribution of factors associated with items in the relevance dimension.

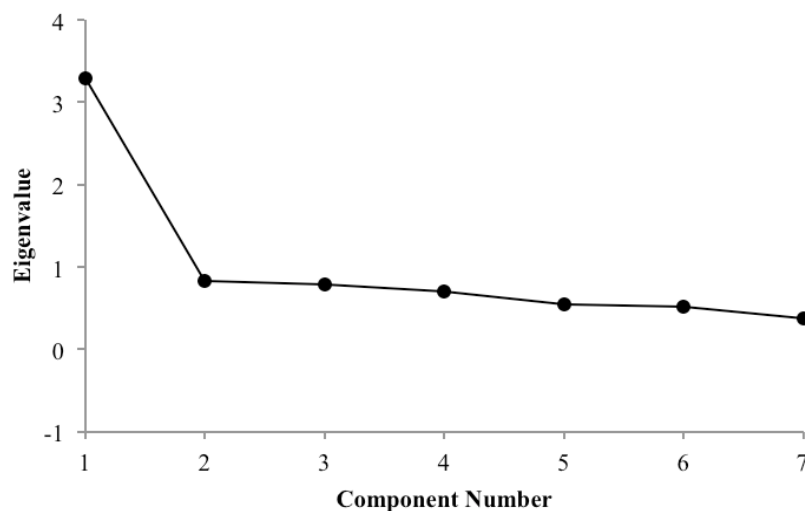


Figure 1. Scree plot presenting eigenvalues for a seven-factor solution.

An initial analysis was run to obtain eigenvalues for each component in the data. Using Kaiser's (1960) criterion, only factors in the scree plot with an eigenvalue greater than one were retained. In this analysis, one factor was identified that accounted for 47% of the total variance. The analysis also provided a rotated factor matrix with loading for all relevance items. Table 1 presents the factor loading after rotation and the associated dimension of relevance for each item.

Table 1.
Dimensions of Relevance, Items and Associated Factor Loadings.

Dimensions	Items	Factor
Practical	I have learnt how mathematics is used in society	0.76
Professional	Careers involving mathematics are highly regarded in society	0.73
Process	Mathematics teaches important problem solving skills	0.70
Professional	Careers involving Mathematics are well paid	0.68
Practical	I use mathematics in my everyday life	0.67
Process	Mathematics assists me in other subjects I study at this College	0.64
Professional	There are many careers that require the use of mathematics	0.62

In order to ascertain the level of agreement in the relevance dimension each student was allocated a mean score for the seven items. Each student's mean score was calculated by aggregating their Likert values for each item in this dimension and dividing this total by the number of items in the dimension. The mean agreement across the participants relevance score was $M = 3.812$. To examine if this mean level of agreement differed across the subject choice of students at Willow College, three distinct groups of students were selected, namely, students who selected Mathematics A, Mathematics B, or Mathematics B and C for Years 11 and 12. The mean score for the subjects Mathematics A ($n = 49$), Mathematics B ($n = 63$), and Mathematics B and C ($n = 24$) were $M = 3.70$, $M = 3.85$ and $M = 4.09$ respectively, and this indicated that students perceptions towards the relevance of mathematics ranged between ambivalence and strong agreement. Figure 2 presents a graph of the mean scores for Mathematics A, Mathematics B, and Mathematics B and C.

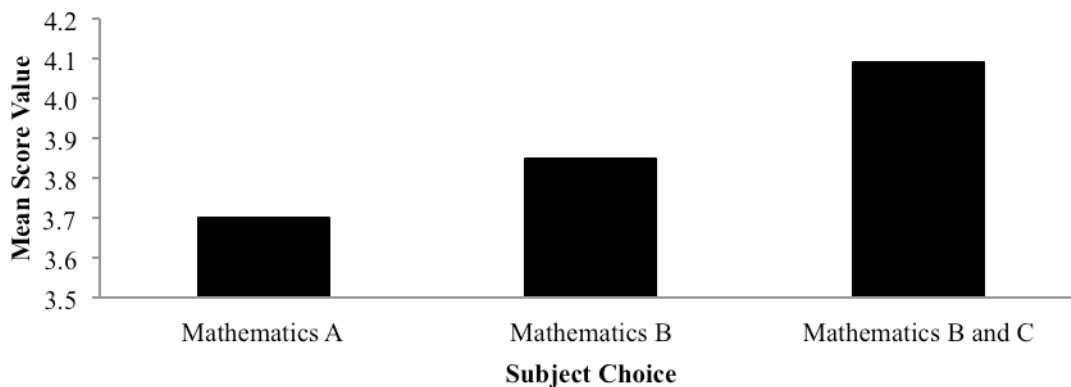


Figure 2. Mean scores calculated from Mathematics A, Mathematics B, and Mathematics B and C students' ratings of relevance items.

A one-way analysis of variance (ANOVA) was performed to ascertain if a significant difference existed between the mean of each group. The post hoc Tukey test revealed a statistically significant difference ($p = 0.029$) between the level of agreement for students seeking to study Mathematics A and those seeking to study Mathematics B and C. This indicated that students seeking to undertake the most demanding level of mathematics placed higher value on the relevance of mathematics as compared to students choosing the less demanding levels of mathematics. An analysis of the written responses from the third section of the survey provided insights into these differences.

Differences between students' perceptions of the relevance of mathematics when choosing to study Mathematics A, and Mathematics B and C was further explored by examining their responses to two open-ended questions, "*What will influence your final decision regarding the mathematics subjects you choose in Years 11 and 12?*" and "*Explain whether you believe mathematics has a critical role in society*" that focused on the notion of relevance. Student's response to each question was entered into Atlas.ti and themes were identified in their responses. Similar themes were grouped to identify key themes in the data. Student's responses were then identified in terms of the key themes, the results of which were aggregated and converted to a percentage of each subject group. Table 2 presents responses to the first question, "*What will influence your final decision regarding the mathematics subjects you choose in Years 11 and 12?*"

Table 2

Themes Identified by Students as Influencing their Senior Mathematics Subject Choice by Intended Subject Level.

Themes	Mathematics A (%)	Mathematics B and C (%)
University prerequisites	4.1	8.3
My intended career path	34.7	54.2
My mathematical ability	36.7	16.7
Advice from others	10.2	8.3
Unsure	4.1	-
Commitments outside the College	2.0	-
No response provided	8.2	12.5

Data in Table 2 indicates that the highest single contributing factor for students intending to study Mathematics B and C was the professional relevance of mathematics as a means of enabling their desired career path. However, for students seeking to study Mathematics A, the professional relevance they ascribed towards mathematics was markedly lower. The factor that had the greatest influence on Mathematics A students subject choice was their perceptions of their mathematical ability.

The second open-ended question, "*Explain whether you believe mathematics has a critical role in society*" invited students to reflect on their perspectives regarding the usage of mathematics beyond the school context. Students' responses were categorised into themes and presented as percentages in Table 3. The data in Table 3 revealed, when considering dimensions of relevance, students seeking to pursue Mathematics A, or Mathematics B and C held similar perceptions concerning its everyday usefulness. More than twice the percentage of students seeking to study Mathematics B and C as compared to Mathematics A appreciated that mathematics was utilised in variety of occupations. Yet, 25% of students seeking to study Mathematics A were alone in their view that mathematics was not critical, and thus they saw no relevance for mathematics in society.

Table 3

Students' Perspectives of Whether Mathematics has a Critical Role in Society by Intended Subject Level.

Themes	Mathematics A (%)	Mathematics B and C (%)
A requirement for a variety of occupations	20.4	45.8
Everyday usefulness	38.3	37.5
Not critical in society	24.5	-
Yes without justification	10.2	12.5
Unsure	2.0	-
No response provided	4.1	-

In summary, data from the pilot study conducted at Willow College revealed a statistically significant difference between students choosing to study Mathematics A, and Mathematics B and C with respect to relevance (see Figure 2). Mathematics B and C students placed a greater emphasis on the importance of mathematics for their chosen career path than those in Mathematics A (Table 2). These students also recognised that mathematics was highly important across many careers in society. Beyond identifying a basic understanding of the usefulness of mathematics, students' choosing Mathematics A (36.7%) were most concerned with whether they possessed the necessary ability to study mathematics.

Discussion and Conclusion

Three tentative conclusions emerged from the data analysed for this dimension of the pilot study. First, there appeared to be some overlap between the three dimensions of relevance as identified by Sealey and Noyes (2010). This was evidenced by the Year 10 cohort ratings of the survey items in the relevance dimension that resulted in one factor that did not delineate between the practical, process and professional dimensions of relevance. While students did not distinguish between the three dimensions, the nuances associated with each dimension were evidenced in their responses to the open ended questions in the survey. It should also be recognised that this outcome may reflect the limited number of questions that were used to explore this dimension.

Second, the relevance of mathematics and its role in subject choice selection was revealed as statistically different for students selecting to study Mathematics B and C as opposed to students choosing to study Mathematics A. The mean level of agreement to this dimension by three groups of students showed that those seeking to undertake more demanding levels of mathematics assigned a higher value towards the relevance of mathematics in contrast to students choosing less demanding levels of mathematics. When invited to respond to the open-ended question regarding factors that would influence their final mathematics subject choice, Mathematics B and C students identified that they would choose this subject because they were aware of its professional relevance, which would contribute to their post-secondary study prerequisites and career pathways. In contrast, Mathematics A students were more concerned with their mathematical ability that mirrors past research findings (Brinkworth & Truran, 1998; McPhan, et.al., 2008). However, whether it is the most noteworthy element influencing students' current and post-school

participation in this subject needs further investigation. Additionally, the results give some insight into the motivation and value that different groups of students associate with studying mathematics. While all students incorporate a dimension of understanding about the future requirement of mathematics for employment, it seems that students studying higher levels of mathematics are more aware of this dimension. Whether this is a consequence of these students choosing careers that have a specific mathematical focus or other influences, such as, home life or in class experiences is beyond the scope of this paper.

Third, for some students, how relevance influences their mathematics subject choice is a secondary concern. Data from the open-ended question, “*What will influence your final decision regarding the mathematics subject you choose in Years 11 and 12?*”, revealed Maths A students were more concerned about their own ability to do mathematics as they choose their subjects than they were about its relevance to everyday life. While it has been suggested that professional relevance is significant in subject choice decision-making (Brinkworth & Truran, 1998), for many students their perceived ability plays an important role. Just how these two influences relate to each other is unclear.

The preliminary findings of this research suggest that there is some positive correlation between relevance and mathematical ability, but whether the relevance of mathematics becomes more transparent as understanding (and achievement) in mathematics increases needs further investigation. This is needed to address the shortfall in students studying higher-level mathematics and entering STEM careers.

References

- Brinkworth, P., & Truran, J. (1998). *A study of the influences on students' reasons for studying or not studying mathematics* (Research report). Adelaide: Flinders University.
- Brown, M., Brown, P., & Bibby, T. (2008). “I would rather die”: Reasons given by 16-year-olds for not continuing their study of mathematics. *Research in Mathematics Education*, 10(1), 3-18.
- Chouinard, R., & Roy, N. (2008). Changes in high-school students' competence belief, utility value and achievement goals in mathematics. *British Journal of Educational Psychology*, 78(31-50).
- Department of Education Training and the Arts. (2006). *Towards a 10-year plan for science, technology, engineering and mathematics (STEM) education and skills in Queensland*.
- Ma, X. (2001). Participation in advanced mathematics: Do expectation and influence of students, peers, teachers, and parents matter? *Contemporary Educational Psychology*, 26(1), 132-146.
- MacNeil, A. J., Prater, D. L., & Busch, S. (2009). The effects of school culture and climate on student achievement. *International Journal of Leadership in Education: Theory and Practice*, 12(1), 73 - 84.
- Nardi, E., & Steward, S. (2003). Is mathematics T.I.R.E.D? A profile of quiet disaffection in the secondary mathematics classroom. *British Education Research Journal*, 29(3), 345-367.
- Philipp, R. A. (2007). Mathematics Teachers' Belief and Affect. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the national council of Teachers of Mathematics* (pp. 257-311): Information Age Publishing.
- Queensland Studies Authority. (2008). *Mathematics C senior syllabus*. Retrieved March 29, 2012, from http://www.qsa.qld.edu.au/downloads/senior/snr_maths_c_08_syll.pdf
- Sealey, P., & Noyes, A. (2010). On the relevance of the mathematics curriculum to young people. *The Curriculum Journal*, 21(3), 239-253.
- Stipek, D. J., Givvin, K. B., Salmon, J. M., & MacGyvers, V. L. (2001). Teachers' beliefs and practices related to mathematics instruction. *Teaching and Teacher Education*, 17(2), 213-226.

Spatial Metaphors of the Number Line

Cris Edmonds-Wathen

RMIT University

<cris.edmondswathen@student.rmit.edu.au>

This paper examines spatial metaphors in the English language associated with the number line, in particular metaphors of direction and motion, and how these are manifested in actual spatial practices associated with number. It considers how these metaphors are culturally influenced, and how the influences of other cultures, such as Arabic, produce inconsistencies that can contribute to confusion in the classroom. It also considers how the metaphors of number vary in some other languages, and how this leads to both different spatial representations of numbers and more challenges in multilingual and multicultural classrooms. In particular, it pays attention to the implications of variety in these metaphors for the mathematics education of Indigenous students in Australia being taught by English speaking teachers.

Spatial metaphors associated with the number line are mixed and culturally specific, particularly in the assumption of a left to right directionality, which can contribute to confusion in the mathematics classroom. An analysis of these common metaphors can help teachers to identify potential confusion in their classrooms, particularly in multicultural and multilingual classrooms, such as those where Indigenous students in Australia are being taught by English speaking teachers. This paper reviews a diverse range of literature, raising critical questions for mathematics educators. In addition to literature from within the mathematics education field, it draws on cognitive science, especially research regarding the links between language and cognition, as well as historical, philosophical and anthropological perspectives. This discussion of the number line is focussed largely on positive numbers, as it is targeted towards the early years of primary mathematics education.

Two prevailing metaphors of numbers are used in mathematics education in the early years of school. Lakoff and Núñez (2000) articulate these metaphors as those used in daily life. The first is that numbers are collections of like objects. The second is that numbers are points on a line.

Both these metaphors include spatiality. The collection view of numbers has volume as a measure of magnitude – numbers are smaller or larger than each other. The number line metaphor introduces *location* and *direction*, but reduces magnitude to a single dimension. In terms of location, numbers have a fixed position on the line with respect to each other and the distance between them represents their magnitude. Although our visual representations of number lines are usually horizontal, we often use vertical terms to describe magnitude, with the larger numbers being higher and the smaller ones being lower (Watson, Partington, Gray, & Mack, 2006).

In terms of direction, the number line *starts* at zero or one and *goes to* infinity, often represented by an arrow head at the end of a written model of the number line (Figure 1). It thus has what Talmy (1996) describes as “fictive motion”, in the sense that we say a road “goes” somewhere, when the road itself does not move and it is us who move along it. Fictive motion is extremely common in language. The fictive motion of the number line is of the type that Talmy labels an *orientation path*: “a continuous linear entity emerging from the front of some object and moving steadily away from it” (1996, p. 217). The origin of the number line is the zero point (or sometimes the point of one). This fictive motion thus introduces temporality as well as spatiality. Numbers can come *before* or *after* each other. A

number that comes *before* another number is closer to the start of the number line than the other number, a number that comes *after* another is further from the start.

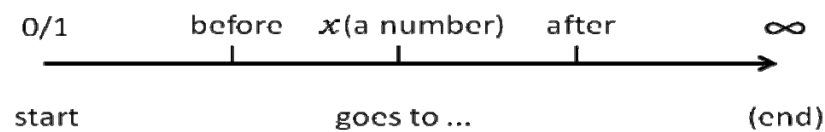


Figure 1. The number line.

Both *zero* and *one* are possible starting points for the number line, as there are cultures that do not have zero in their numbers, a relatively recent mathematical invention. Zero was developed as a solution to a need in written mathematics, specifically to a *spatial* need in written mathematics, as a place holder (Verran, 2001).

Direction of the Number Line

In Australia, number lines are usually drawn from left to right, the same direction as the English writing system. This convention is so ingrained that Lakoff and Núñez include left and right in their definition of the number line metaphor. For example they declare that “point *P* is to the right of point *Q*” maps onto “number *P*’ is greater than number *Q*’ ” (2000: 279). This is the case in the English speaking world, but is not universal.

While Lakoff and Núñez (2000) recognise that ideas from different cultures find their way into mathematics, contributing to its evolution, they also claim that mathematics is independent of culture in the sense that “once mathematical ideas are established in a worldwide mathematical community, their consequences are the same for everyone, regardless of culture” (p. 356). Although Lakoff and Núñez admit diversity in mathematics, they focus more on mathematics as a singular thing and do not contextualise their definitions as culturally specific. Most of the time when most of us think about mathematics, we think of what Barton (2009) calls “near-universal conventional mathematics,” or *Mathematics* as opposed to *mathematics* (Bishop, 1988). However, it is not that case that the consequences of mathematical ideas becoming conventionalised are “the same for everyone.” Our conceptual metaphors are mediated by our languages are well as our cultural experiences. The directionality of the number line is a case in point.

The Mental Number Line

Many people have mental representations of number lines. Some of these are individual and idiosyncratic, others have culturally and historically determined characteristics. Galton (1881) surveyed and documented a variety of ways in which people visualised the layout of numbers. Around a hundred years later, Ernest (1986) conducted a similar survey. Galton claimed that around one man in 30 and one woman in 15 could visualise a stable number line. Ernest found 65 percent of his respondents claimed to be able to do so, a large increase.

The other significant difference between Ernest’s findings and those of Galton was the increase in straight mental number lines. While Galton had found none, Ernest found 60 percent of his total respondents reported straight lines. Ernest reported his findings cautiously as he had only surveyed a small sample. In fact 90 percent of his respondents who did visualise a mental number line had a straight one. Ernest suggested the use of graduated rulers in mathematics instruction as a likely reason for both the increase in mental

number lines and for their straightness. Another study of the idiosyncratic number visualizations of the type found by Galton found that the majority of the number forms reported were oriented left to right (Seron, Pesenti, Noel, Deloche, & Cornet, 1992).

Further evidence that number line features such as straightness and even its linearity are cultural constructs emerged from a study of number-space mappings among the Mundurukú, an Amazonian people who have very few number words. The study investigated how they chose to locate numbers on a line that was provided to them. The Mundurukú were found to represent numbers in a logarithmic fashion rather than a linear scale (Dehaene, Izard, Spelke, & Pica, 2008). Examination of some of the number forms in Galton (1881) and Seron et al. (1992) also suggest a logarithmic representation rather than linear. Dehaene et al. (2008) propose that the logarithmic representation of numbers is more innate than the linear, especially for estimation.

The SNARC Effect

The Spatial–Numerical Association of Response Codes or SNARC effect is described by Wood and Fisher (2008) thus: “We spontaneously associate numbers with space: we think of small numbers as being lower and to the left of us, and larger numbers as being further up and to the right of us” (p. 353). It was observed in experiments about reaction time in accessing information about number parity and magnitude from Arabic numerals (Dehaene, Bossini, & Giraux, 1993). French subjects responded faster to large numerals with their right hand and faster to small numerals with their left hand. This suggested mental number lines running from left to right. The SNARC effect appears to be a literacy effect related to the direction of the writing system. The study found a variation in performance on the task for Iranian subjects (Persian-French biliterate) that depended on the length of their exposure to the left to right writing system used in France. Another study found a reverse SNARC effect acting right to left among Arabic monoliterates (Zebian, 2005). A counter-explanation for the SNARC effect is that it emerges from finger-counting, with the right hand being used as the “counting tool” (Wood & Fischer, 2008). However, this does not explain the reverse SNARC effect. Nor is finger-counting a universal practice that Wood and Fischer claim (e.g. Butterworth, Reeve, & Reynolds, 2011).

Nevertheless, the SNARC effect provides evidence for the left to right direction of the mental number line among cultural groups who practice a left to right writing system. It also demonstrates that this has cognitive effects that extend to the motor system.

Discordance of Number Line with Direction of Writing

In light of the largely unconsciously engrained left to right directionality of the number line among people who write from left to right, it is interesting to examine the discordance of this direction with the direction of much arithmetic calculation.

The origins of this discordance stem from the European adoption of the Hindu numerals and place value system via the Arabic world. The Arabic writing system goes from right to left, and in Arabic numbers are written and read with their lowest power first. So “23” appears the same as in a European script, but it is written first “3” on the right, then “2” to the left of it, and it is read as “three and twenty” (Zebian, 2005).

When the Hindu number system was adopted into Europe by Fibonacci, he kept the order of writing as used by the Arabs. Because European languages such as Latin and English are written and read from left to right, the numbers began to be read from their highest power first (Devlin, 2011). So today in English we say “twenty three” and write the “2” first on the left, and the “3” next on the right.

However, standard methods of written calculation for addition, subtraction and multiplication are conducted from right to left. This is because we perform the operations on the lower powers first, the same as in the Arabic world. We thus need to learn and use a duality of direction of numbers, reading, writing, saying and thinking our numbers from left to right, but operating on them in columns from right to left. In the Arabic world, this duality of direction of numbers does not occur, since direction of reading and writing concords with the direction of calculation.

Cultural and Linguistic Variation in Spatial Representation of Number

The curved and twisted number lines found by Galton (1881) were two dimensional or even three dimensional rather than the one dimension of a straight line. How do the spatial representations of numbers differ for people who do not have or use number words, or who use numbers differently in their grammar? Like the Mundurukú language mentioned above, many Indigenous Australian languages traditionally have very few number words (but see Harris, 1982). Both Warlpiri, an Australian Central Desert language, and Iwaidja, spoken in Croker Island in the Northern Territory, have a “typical” Indigenous Australian number system: “one”, “two” and “more than two”, constructing “three” as “two and one” and “four” as “two and two”. Anindilyakwa, spoken on Groote Island, has words for large numbers, unusually for an Australian language; however they are not much used (Stokes, 1982).

An investigation into the enumeration capabilities of Warlpiri and Anindilyakwa speaking children found that they were able to match and add concrete materials non-verbally but accurately despite having few number words. The researchers claimed that the Warlpiri and Anindilyakwa children “possess the same numerical concepts” as the English speaking comparison group (Butterworth, Reeve, Reynolds, & Lloyd, 2008, p. 13179). However, they also found that the Australian Indigenous language speaking children were more successful with spatial, pattern based approaches to the tasks than with enumeration, compared with English speaking children (Butterworth, Reeve, & Reynolds, 2011). It seems likely that these children do not have a mental number line with unnamed numbers located on it. It is more feasible that the Warlpiri and Anindilyakwa children have a collection concept of number. While the children in the study used two dimensional spatial strategies, these strategies were unique to each task rather than being stable for each individual. Investigations of mental number lines find that they are generally stable for individuals (Galton, 1881; Seron et al., 1992).

In some languages numbers are grammatically more like verbs than nouns (Barton, 2009). How might this affect the visualisation of a number line? Central to the number line metaphor is the idea that numbers are a “thing” that can be located. If numbers are perceived as actions, the mental visualisation of numbers could be very different. In Yoruba, spoken in Nigeria, numbers are nominalised verb phrases (Verran, 2001). Yoruba has a complex multi-base system that facilitates mental computation and traditionally was not written, nor did it use zero. Verran claims that the Yoruba conception of number is that they are “nested” groups: “each number is totally subsumed by its successor ... there is no sense of a linear stretching towards infinity” (p. 203). She argues that large part of the difference in the conception of numbers between English and Yoruba is because of the grammatical roles that they play in each language, noun-like and adjectival in English, and verb-al in Yoruba. Although the Yoruba do not represent numbers as points on in a line, the collection metaphor does not seem adequate either, since it is static, whereas the idea of numbers as verbs is dynamic.

The logarithmic representation of the Mundurukú is still a “line” in the common sense of the word. Other investigations would be required to explore other spatial representations of quantities. Investigations of non-linear spatial representations of numbers would need to be carefully designed so as not to impose researchers’ preconceptions on the outcomes.

The Number Line in the Classroom

Although there is variety in the mental representations of number lines, the “standard” mental number line for people who speak a European language runs straight from left to right. In primary school mathematics classrooms in Australia, as with other places where European languages are used, students encounter physical models of the number line, most of which indeed run left to right. Common physical models of the number line used in primary schools include drawings on the board, strips of cardboard and of course, graduated rulers. As children internalise the left to right reading and writing direction, they also learn that this direction applies to the number line. Simple arithmetic problems are also written on a single line left to right.

Bove (1995) notes the confusion that children can experience when they move from experiences with a standard number line to trying to understand the place value system used in larger numbers: “multi-digit numerals increase in place value from right to left but numbers get larger in a horizontal number line from left to right” (p. 544). There is a lack of agreement between the spatial metaphors used in talking about number and writing them: “a number *higher* than six is actually written *beside and to the right* of it” (Watson, Partington, Gray, & Mack, 2006, p. 24) (emphasis in the original). Stacey, Helme and Steinle (2001) also identify the increase in place value from right to left as a source of confusion for students learning the value of decimals, sometimes leading them to see decimals and fractions as negative numbers as a result of mixing the directions of their spatial metaphors.

When children begin calculations of multi-digit numbers presented in columns, further confusion can occur. When children first encounter this type of calculation, the problems are usually levelled so that children are not required to carry digits across powers and they often attempt these written calculations from left to right. The left to right solution holds until they are required to carry digits, when the problem with the left to right approach becomes evident.

Bove (1995) suggests using a vertical number line commencing with zero at the bottom for the early introduction of numbers. This, she says “would not conflict with subsequent learning. In the vertical number line, numbers *increase* in value as the student moves *up* the line” (p. 544). To put it another way, the spatial and linguistic metaphors associated with the number line would match up with each other instead of contradicting each other. Teaching about the history of the system we use for writing numbers and its adoption from the right to left of Arabic would also help students to identify and understand the discordance.

As we have seen, the mixed spatial metaphors of the number line are a potential source of confusion for English-speaking children in an English-speaking classroom. These metaphors can be even more difficult for children from other linguistic backgrounds in an English-speaking mathematics classroom, such as Indigenous language speaking students in Australia.

Indigenous Students and the Number Line

As we saw above, teachers often mix their metaphors when talking about numbers quite unconsciously, assuming that children will be able to make the links between the various

terms. This can be more difficult for students who are learning English as a Second or Additional Language (Watson, Partington, Gray, & Mack, 2006).

Indigenous language speaking students learning in English often have difficulty grasping the concepts of “before” and “after” in relation to numbers (Edmonds-Wathen, 2011; Graham, 1988). In part, this may be because while they have a collection metaphor for number, they are still developing the “numbers are points on a line” metaphor.

Another reason may be because the semantic scope of these terms in Australian Indigenous languages differs from their scope in English. For example, “it was found in one language that a word was being used for ‘after’ ... [that] could be translated back into English as ‘before’, ‘after’, ‘previously’, ‘following’, etc., depending on the context” (Graham, 1988, p. 129).

In English, the number that comes “after” is “higher” or “greater”, and the number that comes “before” is “lower” or “smaller”. In Warlpiri, the concepts are also linked to height and size but in the inverse relation: The taller tree sprouted first and its smaller companions later, it is taller and older, it came “before”; the “bigger” brother came “first” and his “smaller” brother came “after” him, although the “bigger” brother’s age in years is “greater”. Thus in Warlpiri larger numbers a “larger” number can be seen to come “before” the “smaller” number (Mary Laughren, pers. comm., 11 March 2011).

Research into spatial concepts in Iwaidja had shown that similar to Warlpiri, the concepts of “first”, “in front”, “go before” are combined in one word, *wurdaka*, while “behind”, “later” and “after” are combined in *warrwak*. Iwaidja also has a greater emphasis on *orientation* and implied or actual direction than *location* in spatial descriptions compared to English (Edmonds-Wathen, 2011). This orientation is very rarely described in terms of “left” and “right”. We saw earlier that the type of fictive motion associated with the number line is what Talmy (1996) called an orientation path. Let us explore this metaphor a little more in terms of its implications for the number line when the emphasis is on orientation rather than location. In doing so, I am attempting a “mind-game” of the sort that Barton (2009) played with the mathematical implications of verbal expressions of shape in Navajo. I am not saying that this is exactly how speakers of these Australian Indigenous languages think.

The Path Oriented Approach

From a path oriented¹ approach to the number line, facing direction is always significant. Point on a line are never just scalar points, *they always have direction even if their apparent magnitude is zero*. From this perspective, any given number is facing away from its origin, travelling towards infinity. Larger number are thus before it, in front of it, while small numbers have been left behind, they are coming after it. The line is not going from left to right, or right to left, it is going *forward*.

This also correlates with the idea of increase with time, such that larger numbers began at zero, have been around for longer, getting bigger with time. The big mathematical problem with this type of thinking is that larger (cardinal) numbers have a lower ordinality than smaller numbers. “First” is always contextual, rather than associated with infinity.

The teacher could be more explicit about their use of fictive motion in talking about the number line. The temporal meaning of words such as *wurdaka* “first” and *warrwak* “later” could be emphasised in a journey, such that to reach six, you will have to pass five – it comes “first”, it comes “before”, and then if you keep going you will come to seven, it will come “later”, it will come “after”.

¹ Thanks to Mary Laughren for this phrasing.

Is there a way to map this approach onto the type of number line used in schools? Teachers in Indigenous classrooms are aware that they need to use concrete materials and physical processes in their mathematics teaching, such as number lines. Bove (1995) suggested uses vertical number lines to allay confusion over left and right practices when introducing place value. The vertical number line can also be lain flat in front of the body, stretching away.

Of course, I am not suggesting that simply using a number line pointing away from the body will be sufficient to allay the confusion that Indigenous students experience with the spatial metaphors of number in the classroom. What teachers need however, is an awareness of the metaphors that they are using and of how they may be being interpreted by their students.

Conclusion

We unconsciously absorb the dominant metaphors associated with important concepts from our culture and language. The number line metaphor, which materialises as physical models of number lines within and without classrooms, has spatial associations. The standard number line metaphor in English has both orientation and direction, but location at fixed points is regarded as more important than the fictive journey along it. While possibly we begin with a logarithmic representation of this journey, we learn to divide it into equal sections and name each point as a number. These numbers themselves have no direction. There are different ways of conceptualising spatially quantity and number in different cultures and languages. For some, the number as a collection metaphor may be more highly developed than that of a points on a line. Concepts of ordinality may vary, and for some such as the Warlpiri a “bigger” number may come “before” a “smaller”. Resolving metaphorical contradictions in the mathematical classrooms is no easy matter, yet even in English alone, we learn to mix our mathematical metaphors. For the English speaking teacher, an awareness of these common metaphors in English, and how they differ from those in other languages that might be spoken by their students, can assist them in explicit teaching about how one metaphor maps onto another. An appreciation of the Arabic impact on the directionality of how we write numbers might help us clarify to our students that we use both left to right and right to left in our number manipulations. This is even more important when students come from cultures that commonly do not use the right/left distinction and which focus more on direction and movement than on location.

References

- Barton, B. (2009). *The Language of Mathematics: Telling mathematical tales*. New York: Springer.
- Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education*. Dordrecht: Kluwer Academic Publishers.
- Bove, S. P. (1995). Place value: A vertical perspective. *Teaching Children Mathematics*, 1(9), 542-546.
- Butterworth, B., Reeve, R., & Reynolds, F. (2011). Using mental representations of space when words are unavailable: Studies of enumeration and arithmetic in Indigenous Australia. *Journal of Cross-Cultural Psychology*, 42(4), 630-638.
- Butterworth, B., Reeve, R., Reynolds, F., & Lloyd, D. (2008). Numerical thought with and without words: Evidence from Indigenous Australian children. *PNAS*, 105(35), 13179-13184.
- Dehaene, S., Bossini, S., & Giraux, P. (1993). The mental representation of parity and number magnitude. *Journal of Experimental Psychology*, 122(3), 371-396.
- Dehaene, S., Izard, V., Spelke, E., & Pica, P. (2008). Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures. *Science*, 320, 1217-1220.
- Devlin, K. J. (2011). *The Man of Numbers: Fibonacci's arithmetic revolution*. New York: Walker.

- Edmonds-Wathen, C. (2011). "What comes before? Understanding spatial reference in Iwaidja". In Setati, M., T. Nkambule and L. Goosen (Eds.) *Proceedings of the 21st ICMI Study Conference, "Mathematics Education and Language Diversity"* (pp. 89-97), São Paulo, Brazil, 16–20 September, 2011.
- Ernest, P. (1986). Mental number line images. *Teaching Mathematics and its Application*, 5(1), 1-2.
- Galton, F. (1881). Visualised numerals. *Journal of the Anthropological Institute*, 10, 85-102.
- Graham, B. (1988). Mathematical education and aboriginal children. *Educational Studies in Mathematics*, 19(2), 119-136.
- Harris, J. W. (1982). *Facts and fallacies of Aboriginal number systems*. Series B, m0029743_v_a.pdf, 153-181. Retrieved from http://www.aiatsis.gov.au/lbry/dig_prm/e_access/serial/m0029743_v_a.pdf
- Lakoff, G., & Nunez, R. E. (2000). *Where Mathematics Comes From: how the embodied mind brings mathematics into being*. New York: Basic Books.
- Seron, X., Pesenti, M., Noel, M.-P., Deloche, G., & Cornet, J.-A. (1992). Images of numbers, or "When 98 is upper left and 6 sky blue." *Cognition*, 44, 159-196.
- Stacey, K., Helme, S., & Steinle, V. (2001). Confusions between decimals, fractions and negative numbers: A consequence of the mirror as a conceptual metaphor in three different ways. In M. v. d. Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 217-224). Utrecht: PME.
- Stokes, J. (1982). A description of the mathematical concepts of Groote Eylandt Aborigines. S 40/4. Retrieved from http://www.aiatsis.gov.au/lbry/dig_prm/e_access/serial/m0022785_v_p33to56_a.pdf on 20 September 2008.
- Talmy, L. (1996). Fictive motion in language and "ception." In P. Bloom, M. Peterson, L. Nadel & M. Garrett (Eds.), *Language and Space* (pp. 211-276). Cambridge: MIT Press.
- Verran, H. (2001). *Science and an African logic*. Chicago University of Chicago Press.
- Watson, P., Partington, G., Gray, J., & Mack, L. (2006). *Aboriginal students and numeracy*. Perth: Aboriginal Education and Training Council of Western Australia.
- Wood, G., & Fischer, M. H. (2008). Numbers, space, and action - From finger counting to the mental number line and beyond. *Cortex*, 44(4), 44(4), 353-358.
- Zebian, S. (2005). Linkages between number concepts, spatial thinking, and directionality of writing: The SNARC Effect and the reverse SNARC effect in English and Arabic monoliterates, biliterates, and illiterate Arabic speakers. *Journal of Cognition and Culture*, 5(1-2), 165-190.

Reinventing the Wheel: Historical Perspectives on Theories for Interpreting Discourse Patterns in Mathematics Classrooms

Nerida F. Ellerton
Illinois State University
<ellerton@ilstu.edu>

Pongchawee Vaiyavutjamai
Chiang Mai University
<edipvyvt@chiangmai.ac.th>

M. A. (Ken) Clements
Illinois State University
<clements@ilstu.edu>

Many key aspects of modern theory relating to classroom discourse in mathematics education, especially the theory of didactical situations, have not progressed much further than theories put forward in the 1840s by the North American educator David Page. Aspects of Brousseau's (1997) description of what he called the Topaze effect, are remarkably similar to Page's idea of "drawing out"—a mode of questioning that was likely to remove the cognitive challenge for learners engaged in non-trivial mathematical tasks.

19th-Century Theories on Discourse Factors Influencing Mathematics Learning

Johann Heinrich Pestalozzi on the Development of Concepts

During 1820–1880 the ideas of Johann Heinrich Pestalozzi (1746–1827), a Swiss-German educator, became well known in Western Europe and the United States. Pestalozzi's central thesis was that although a child's intellectual, physical, and moral development (the "head", the "hand", and the "heart") unfolded according to well-defined rules, rather like a plant in the ground, she or he learned best inductively, through meaningful examples. Pestalozzi insisted that the main task of the teacher was to nurture the growing organism so that it could reach its full potential. Attempts to force a child's learning along unnatural paths would be counter-productive (Soëtard, 1981, 1994).

Thus, Pestalozzi maintained, education practices should aim at developing mental and physical strengths through sequential exercises done systematically and in a prescribed order, first at home and then at school. Pestalozzi's theory of elementary education included intellectual, moral, physical, and vocational training, all closely interacting to produce harmonious human development. For Pestalozzi, transmission of concepts, skills and principles from a teacher to a student through words alone was impossible. Teachers needed to introduce students to appropriately-structured sense impressions of what they wanted their students to learn (Soëtard, 1981, 1994). But that did not imply that teachers should not ask questions or make statements in order to help learners advance their thinking.

One of the weaknesses of many interpreters of Pestalozzian writing has been an all-or-nothing mentality. Pestalozzi made it clear that he believed that after children had gained an idea of the objects that were now part of their environments, teachers should name the objects and get students to discuss and compare various aspects of them. This would permit abstraction, so that definitions of components of the objects would become meaningful to the learners. Pestalozzi's ideas can be thought of as developing the constructivist tradition in education, from John Locke, in England, through Jean Jacques Rousseau, in France.

Warren Colburn—Pestalozzi's Disciple in the United States

One of Pestalozzi's chief disciples in the United States was Warren Colburn (1783–1833), America's first major theoretician in the field of mathematics education (Bidwell & Clason, 1970). Colburn's (1821) first publication, *Arithmetic on the Plan of Pestalozzi* took the U.S. education world by storm. In 1822 a new edition, with a different title—*First Lessons in Arithmetic on the Plan of Pestalozzi*—was published, and so was *Arithmetic, Being a Sequel to the First Lessons in Arithmetic* (Colburn, 1822a, 1822b).

Colburn reiterated Pestalozzi's call for instruction in early arithmetic to be linked with the learners' senses. He rejected the prevailing idea that mainly boys should learn arithmetic, and that schools should not attempt to teach mathematical concepts to children less than 10 years of age. For Colburn, as for Pestalozzi, all children—boys and girls—should learn arithmetic from the age of six. The emphasis should be on developing classroom discourse patterns by which learners constructed concepts inductively. Teacher-student discourse should guide students toward the construction of appropriate concepts. Although the teacher's role was not to tell, nevertheless the right questions needed to be asked at the right time.

David Perkins Page's Theorisation of the "Drawing-Out Process"

The theories on classroom discourse of David Perkins Page (1810–1848), probably the most influential North American education theorist of the 19th century, are hardly known today. Page was a young, but nevertheless experienced high-school teacher who, in 1844, became Principal of the State Normal School in Albany, New York. Over the next three years he was outstandingly successful in that capacity, but after a brief illness he died in January 1848, aged just 38. He wrote just one book, *Theory and Practice of Teaching*, but that book became the major 19th century theoretical treatise on teaching and learning for North America and East Asian schools and colleges. What is pertinent for this paper is that Page specifically theorised what we shall call the concept of "cognitive emptying".

Page (1850) powerfully drew attention to how many teachers thought that they were teaching using an inductive form of pedagogy when, in fact, they had developed a very damaging "cognitive-emptying" approach which he referred to as the "drawing-out process" (p. 79). He stated that he had seen this process done so adroitly by a teacher that "a company of visitors would agree that it was wonderful to see how thoroughly the children had been instructed!" (p. 81). This "drawing-out process" involved what lawyers called *leading questions*. It was practised, usually, whenever over-zealous teachers desired to help along their pupils. It could occur when a teacher was questioning an individual student (as in a "recitation" session) or in a session when the teacher was addressing a whole class of students. We illustrate the concept by reproducing two excerpts from Page (1850).

Page's (1850) first example of the "drawing out" or "cognitive-emptying" process.

Teacher: John, what is the number to be divided called?"

John: Um... Um. ...

Teacher: Is it the *dividend*?

John: Yes, Sir. The dividend.

Teacher: Well, what is that which is left after dividing called? –The remainder, is it?"

John: Yes

Teacher: Well, John, of what denomination is the remainder?"

Teacher: Isn't it always the same as the dividend, John?

John: Yes, Sir.

Teacher: Very well, John, what denomination is this dividend? [*points to the work upon the board*] Dollars, is it not?

John: Yes, Sir, dollars.

Teacher: Very well; now what is the remainder? Why dollars, too, isn't it?

John: Oh yes sir, dollars! (pp. 79–80)

Page's (1850) second example of the "drawing out" or "cognitive-emptying" process. A teacher whose school Colburn visited called upon the class to demonstrate what they had studied from a section in Warren Colburn's *First Lessons*. The teacher began by asking where the class should begin in the textbook. Then, the following discourse occurred:

Pupils: On the 80th page, Question 3.

Teacher: Read it, Charles.

Charles: [*Reads*] "A man being asked how many sheep he had, said that he had them in two pastures; in one pasture he had eight; that three-fourth of these were just one-third of what he had in the other. How many were there in the other?"

Teacher: Well, Charles, you must get one-fourth of eight, must you not?

Charles: Yes, Sir.

Teacher: Well, one-fourth of eight is two, isn't it?

Charles: Yes sir, one-fourth of eight is two.

Teacher: Well, three-fourth will be three times two, won't it?

Charles: Yes.

Teacher: Well, three times two are six, eh?

Charles: Yes Sir.

Teacher: Very well. [*a pause ...*] Now the book says that six is one-third of what he had in the other pasture. Doesn't it?

Charles: Yes Sir.

Teacher: Then, if six is one-third, three thirds will be three times six—won't it?

Charles: Yes Sir.

Teacher: And three times six are—eighteen, ain't it?

Charles: Yes Sir!

Teacher: Then he had eighteen sheep in the other pasture, had he?

Charles. Yes Sir.

At this stage of the lesson Page asked if he could request Charles to go through it alone. "Oh yes", said the teacher. "Charles, you may do it again". Charles again read the question and looked up. "Well", said the teacher, "You must first get one-fourth of eight, mustn't you?" "Yes sir". "And, one-fourth of eight is two, isn't it?" "Yes Sir". And so the process went on as before until the final 18 sheep were drawn out, as before. Now the teacher looked around with an air which seemed to say, "Now I suppose you are satisfied".

"Shall I ask Charles to do it again?" said I. The teacher assented. Charles again read the question—and looked up. I waited, and he waited—but the teacher could *not* wait. "Why, Charles", said he impatiently, "You want one-fourth of eight, don't you?" "Yes sir", said Charles, promptly. And I thought best not to insist further" (pp. 82–83). Page admitted that this was an extreme case, but said it was a fair sample of the discourse patterns in that teacher's class during arithmetic lessons. According to Page, this habit of assisting the pupil by asking and answering leading questions was very common. The process of cognitively emptying had "a direct tendency to make the scholar miserably superficial" (p. 83). "For why should he study if he knows from constant experiences that the teacher, by a leading question, will relieve him from all embarrassment?" (p. 83).

Page's "more excellent way". Immediately following the above passage, Page (1850) went on to elaborate a "more excellent way" (p. 84). Drawing from Pestalozzi, he commented that nature taught us that learners would be best served by being taught to depend mainly on their own resources. Page emphasised, though, that it was the teacher's responsibility to check thoroughly whether pupils were learning with *understanding*. It was appropriate for the teacher to give a word of suggestion, and to offer the scholar "a seasonable hint" for that could "save the scholar the needless loss of time". "The inquirer should never be frowned upon" (p. 84), and the "true way is neither to discourage inquiry nor answer the question" (p. 84). Page elaborated on his "more excellent way" but space does not permit this to be discussed in any detail here. Interestingly, Page's theories would have a large effect on education reforms in Japan and Taiwan in the 1880s and 1890s (Wing-kai To, 2011).

The Topaze Effect as a Re-invention of the Wheel

In developing his theory of didactical situations, Guy Brousseau (1997) linked the term "Topaze Effect" to the play *Topaze* by Marcel Pagnol (1928), in which a teacher (Topaze), wanting his student to learn something, led him to the correct answer by a series of leading questions—but, the students effectively learned nothing. The Topaze effect in mathematics education can be described in the following terms: When a teacher wants students to be able to answer a question but perceives that they are unable to obtain that answer by themselves, then the teacher, wishing to involve students in developing the answer without specifically "telling them", asks a series of leading questions aimed at moving the students towards the correct answer. The teacher has devised a way of thinking that she or he wants the students to adopt, and so the questions that the teacher asks are consistent with what the teacher wants the students to think and say. If the students answer a question inappropriately, then the teacher may ask further leading questions until an "appropriate" answer is obtained (Novotna & Hošpesova, 2007).

Most teachers of mathematics are aware of this phenomenon, even if they have never known it as *Topaze*. The leading questions asked by the teacher must be such that students can be expected to be able to answer them correctly. They are usually short, and require brief answers. They address the next small step in the total argument that is in the back of the teacher's mind. Often the Topaze effect is played out, apparently to everyone's satisfaction—the teacher asks cognitively low-level questions, and students answer them (often in chorus, if the teacher permits that). The students are happy when they are able to mouth the correct answers, and the teacher is happy when the students apparently reach the goals set for the lesson.

What is striking for us is how conceptually similar Brousseau's so-called "Topaze effect" is to Page's "drawing-out" process. Indeed, we cannot see any major difference, between the two. Both emphasise how teachers, in their desire to assist learners, are prone to ask sequences of questions that "empty" tasks of cognitive challenge. Although both Brousseau and Page emphasised how teachers did this, Page made the point well over a century before Brousseau did, and the cognitive emptying temptation to teachers was well recognised in North American education literature published in the second half of the 19th century (see, e.g., Cook, 1883; Hart, 1878). Almost certainly, this literature was not known by Brousseau (who made no reference to the writings of Page). Such a lack of knowledge of literature can contribute to a tendency to re-invent the wheel in mathematics education theory. Two of the authors of this paper have previously commented on this tendency at a MERGA conference (Ellerton & Clements, 2005).

Although it is usually assumed that the style of teaching characteristically associated with the term “Topaze effect” is seriously inadequate, and that the students learn little as a result of it, it could be the case that if the teacher has a strong understanding of mathematics, and some appreciation of the cognitive structures of her of his students with respect to the topic under consideration, then the students may be capable of identifying what is in their teacher’s minds and actually learn important methods for tackling problems. In other words, it should not be assumed that a question-answer style of teaching, in which a sequence of low-level questions is asked, is necessarily a poor pedagogical strategy for all teachers and for all students.

Mixed-Method Research is Needed on the Effects of Cognitive Emptying

Almost all mathematics teachers and mathematics educators are aware that the cognitive emptying phenomenon is to be found in many mathematics classrooms. It is also to be found in many interviews that researchers conduct when they seek to uncover the ways students are thinking (about certain concepts, or about problems, etc.). It is also true that often the teachers and the researchers are not aware that others would regard the discourse patterns which they control are examples of cognitive emptying. As Page (1850) observed, it is often almost impossible for a casual observer, or occasional analyst, to identify which components of a discourse feature cognitive emptying and which do not.

Most of the data offered by researchers and scholars in relation to the effects of cognitive emptying (or, equivalently, the Topaze Effect) have been in the form of classroom or interview transcripts—not unlike the transcripts already reported from Page (1850) in this paper. Such transcripts are highly suggestive of the kind of learning that generally is the result of teaching, or interviewing, in which cognitive emptying obviously features strongly. But, often, discourse from mathematics classrooms, or from research interviews, feature cognitive emptying in much more subtle ways than the occasional observer might imagine. Furthermore, the effects of such discourse should not be assumed to be the same on all learners. What is “cognitive emptying” for one learner may not be for another.

We would maintain that there is a need for careful *mixed-methods* research identifying the effects of standard mathematics classroom teaching sequences, and research interview sequences, on learners or interviewees. Undoubtedly, the reporting of transcripts is important, but so too are the quantitative reporting and analyses of pre-teaching (or pre-interview), post-teaching (or post-interview), and retention data. One without the other must convey an incomplete picture. Space does not permit us to elaborate the point here, but the following example might be useful. Vaiyavutjamai (2004) studied the discourse patterns in six ninth-grade algebra classes in Thailand, and these are summarised in Figure 1, which shows 16 graphs generated by analyses of the questions asked by teachers in 16 lessons. The vertical axes are labeled 0, 1, and 2, corresponding to whether a question by a teacher was classified as (a) procedural, or (b) low-level, or (c) high-level. The horizontal axes show the number of questions asked during the lesson. Decisions on the classifications of the levels of the questions were made after these terms had been carefully defined, and after discussion of classifications between Vaiyavutjamai and another independent researcher.

Linear Equations Lesson

Linear Inequations Lesson

Quadratic Equations Lesson

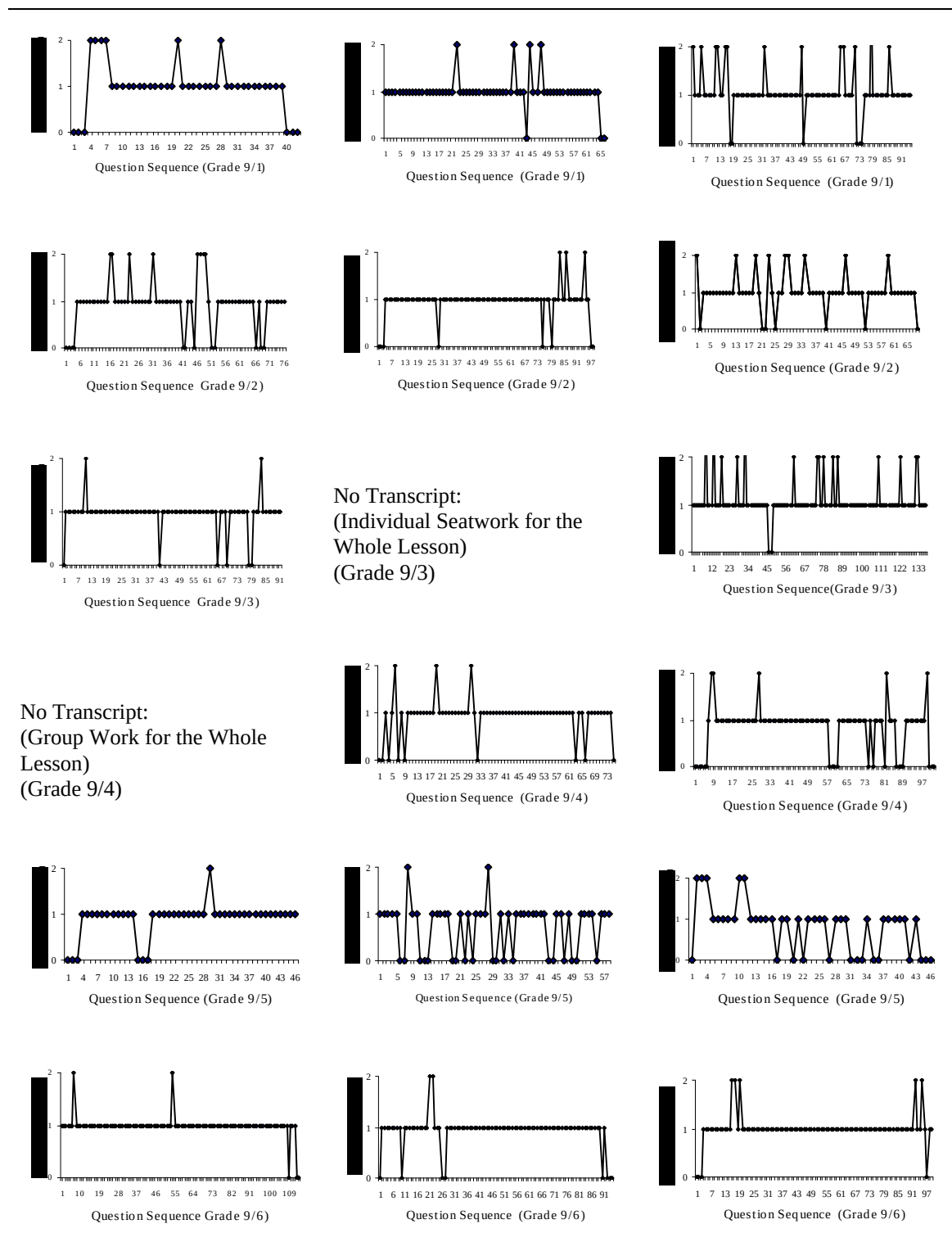


Figure 1. Teachers' patterns of questions asked in six algebra classes (Vaiyavutjamai, 2004).

These classes were in two schools (three classes in each school), and altogether four teachers were involved—whereas one teacher (Teacher A) taught all three classes in one school (these are labelled 9/1, 9/2 and 9/3 in Figure 1), three different teachers (Teachers B, C, D) taught each of the three classes (9/4, 9/5 and 9/6) in the other school. Class 9/1 and 9/4 were high-stream classes, and the other four classes were regarded as medium/low-stream classes. Vaiyavutjamai observed and audiotaped all 16 lessons, each of which was concerned with linear equations, or linear or quadratic inequalities, or quadratic equations. Since the graphs are small an impression of continuity is conveyed. In fact, however, the graphs comprise sets of discrete points (with adjoining points being joined by line intervals to assist interpretation). It can be seen from Figure 1 that each teacher tended to follow up a high-level question with a set of low-level questions.

When teaching linear equations and inequations to each of his three Grade 9 classes, Teacher A asked mainly low-level questions. In his lessons on quadratic equations he asked more high-level questions than the other three teachers did in their lessons on quadratic equations. Yet, even in his lessons on quadratic equations, Teacher A asked many more low-than high-level questions. Most of the questions asked by the three teachers at the second school were low-level, and Teacher C tended to ask more procedural questions than the other three participating teachers.

Vaiyavutjamai's (2004) analyses of the classroom transcripts revealed that all four participating teachers idiosyncratically combined elicitation, exposition and modelling in almost every lesson they taught. Typically, they would ask their classes a high-level question and then use low-level questions to channel their students' thinking toward an answer that they deemed to be acceptable. This elicitation strategy seemed to empty the high-level tasks of cognitive challenge. If teachers needed to teach something quickly (for example, to complete a lesson, or to finish a topic before a test), or if they thought the content was difficult, they tended to move into "expository" mode. During seatwork, students answered exercises by imitating the approaches in the model examples.

Figure 2 shows pre-teaching, post-teaching and retention data on strictly parallel tests for the six classes. It is interesting that the apparently cognitive-emptying discourse patterns in each class generated improvement, even at the retention stage, with both top-stream classes, but with the other classes, the retention mean performances differed by only small amounts from the pre-teaching mean performances.

Concluding Comments

Theories such as the "Topaze Effect" (and the "Jourdain Effect" and the "Dienes Effect", both of which Brousseau (1997) also described) are not theories unless there is the possibility that they can be refuted. In this paper our brief description of Vaiyavutjamai's (2004) study has suggested that both qualitative and quantitative research methods should be combined before such theories can be substantiated.

The paper also suggested that mathematics education researchers need to develop a stronger knowledge of our own history—including knowledge of theories used in earlier times. We cannot afford to keep on re-inventing the wheel.

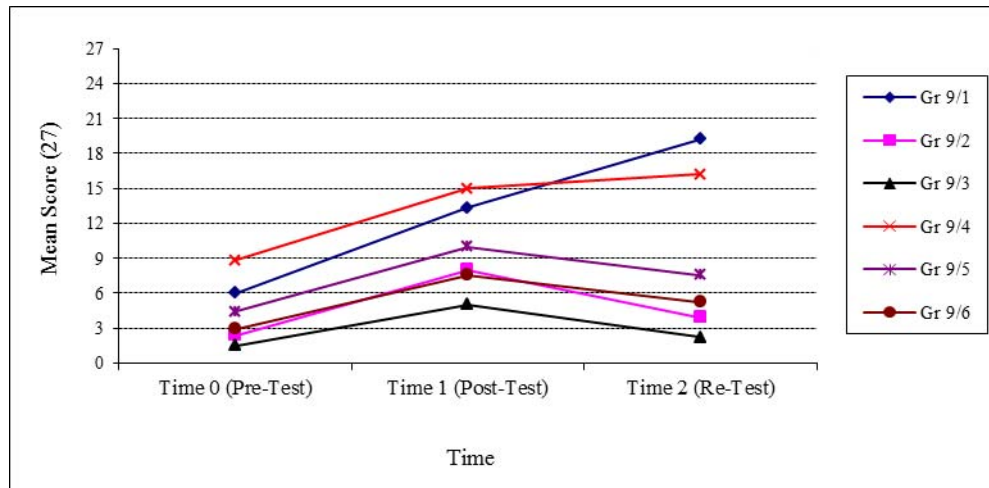


Figure 2. Pre- and post-teaching mean scores of six classes (Vaiyavutjamai, 2004).

References

- Bidwell, J. K., & Clason, R. G. (Eds.). (1970). *Readings in the history of mathematics education*. Washington, D.C.: National Council of Teachers of Mathematics.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics*, Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Colburn, W. (1821). *Arithmetic on the plan of Pestalozzi*. Boston, MA: Cummings & Hilliard.
- Colburn, W. (1822a). *First lessons in arithmetic on the plan of Pestalozzi*. Boston, MA: Cummings & Hilliard.
- Colburn, W. (1822b). *Arithmetic, being a sequel to the First Lessons in Arithmetic*. Boston, MA: Cummings & Hilliard.
- Cook, J. W. (1883). *Methods in written arithmetic*. Chicago: A. Flanagan Co., Publishers
- Ellerton, N. F., & Clements, M. A. (2005). A mathematics education ghost story: Herbartianism and school mathematics. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonagh, R. Pierce & A. Roche (Eds.), *Building connections: Research, theory and practice* (pp. 313-320). Sydney, Australia: Mathematics Education Research Group of Australasia.
- Hart, J. S. (1878). *In the school-room: Or, chapters in the philosophy of education*. Philadelphia, PA: Eldredge & Brother.
- Novotna, J., & Hošpesova, A. (2007). What is the price of Topaze? In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 25-32). Seoul, Korea: International Group for the Psychology of Mathematics Education.
- Page, D. P. (1850). *Theory and practice of teaching: The motives and methods of good school-keeping* (12th ed.). New York, NY: A. S. Barnes & Co.
- Pagnol, M. (1958). *Topaze* (trans. R. Waldinger). Great Neck, NY: Barron's Educational Series.
- Soëtard, M. (1981). *Pestalozzi ou la naissance de l'éducateur. Étude sur l'évolution de la pensée et de l'action du pédagogue suisse (1746-1827)*, Berne, Switzerland: Publications Universitaires Européennes.
- Soëtard, M. (1994). Johann Heinrich Pestalozzi (1746-1827). *Prospects: The Quarterly Review of Comparative Education*, 24(1/2), 297-310.
- Vaiyavutjamai, P. (2004). The effects of teaching on Form 3 students' learning of equations and inequations. In I. P. Cheong, H. S. Harkirat, I. J. Kyeleve, & O. Chukwu (Eds.), *Globalisation trends in science, mathematics, and technical education* (pp. 280-289). Gadong: Universiti Brunei Darussalam.
- Wing-kai To (2011). Bridgewater Normal School and Isawa Shuji's reforms of modern teacher education in Meiji Japan. *Journal of East Asian Cultural Interaction Studies*, 2, 67-86.

Young Children's Metarepresentational Competence in Data Modelling

Lyn English

Queensland University of Technology
<l.english@qut.edu.au>

This paper reports findings from an activity implemented in the final year of a 3-year longitudinal study of data modelling across grades 1-3. The activity engaged children in designing, implementing, and analysing a survey about their new playground. Data modelling involves investigations of meaningful phenomena, deciding what is worthy of attention (identifying complex attributes), and then progressing to organising, structuring, visualising, and representing data. The core components of data modelling addressed here are children's structuring and representing of data, with a focus on their display of metarepresentational competence (diSessa, 2004). Such competence includes students' abilities to invent or design a variety of new representations, explain their creations, understand the role they play, and critique and compare the adequacy of representations. Reported here are the ways in which the children structured and represented their data, the metarepresentational competence displayed, and links between their metarepresentational competence and conceptual competence.

Introduction

Young children are immersed in our data-driven society, with early access to computer technology and daily exposure to the mass media. The need to advance children's statistical reasoning abilities, from the earliest years of schooling, has thus been stressed in recent years (e.g., Langrall, Mooney, Nisbet, & Jones, 2008; Shaughnessy, 2010; Whitin & Whitin, 2011). One approach to enhancing children's statistical abilities is through data modelling (English, 2010; Lehrer & Romberg, 1996; Lehrer & Schauble, 2007).

Data modelling is a developmental process, beginning with young children's inquiries and investigations of meaningful phenomena, progressing to identifying various attributes of the phenomena, and then moving towards organising, structuring, visualising, and representing data (Lehrer & Lesh, 2003). Data modelling should be a fundamental component of early childhood curricula, yet there exists limited research on such modelling and how it can be fostered in the early school years. The majority of the research has been concerned with the secondary and tertiary levels, with the assumption that primary school children are unable to develop their own models and sense-making systems for dealing with complex situations (Greer, Verschaffel, & Mukhopadhyay, 2007).

Recent research, however, has indicated that young children do possess many conceptual resources which, with appropriately designed and implemented learning experiences, can be bootstrapped toward sophisticated forms of reasoning not typically seen in the early grades (e.g., Clements, Sarama, Spitler, Lange, & Wolfe, 2011; English & Watters, 2005; Papic, Mulligan, & Mitchelmore, 2011; Perry & Dockett, 2008). Most research on early mathematics and science learning has been restricted to an analysis of children's actual developmental level, which has failed to illuminate their potential for learning under stimulating conditions that challenge their thinking (Ginsburg, Cannon, Eisenband, & Pappas, 2006; Perry & Dockett, 2008). Data modelling provides rich opportunities to advance young children's statistical development and reveal their capabilities in dealing with challenging tasks.

I now address some core components of data modelling, with a focus on structuring and representing data together with the role of metarepresentational competence in children's creation of data models.

Structuring and Representing Data

Models are typically conveyed as systems of representation, where structuring and displaying data are fundamental; the structure is constructed, not inherent (Lehrer & Schauble, 2007). However, as Lehrer and Schauble indicated, children often have difficulties in imposing structure consistently and often overlook important information that needs to be included in their representations or alternatively, they include redundant information. Providing opportunities for young children to structure and display data in ways that they choose, and to analyse and assess their representations is important in addressing these early difficulties. Yet young children's typical exposure to data structure and displays has been through conventional instruction on standard forms of data representation.

As children construct and display data models, they generate their own forms of inscription. By the first grade, children already have developed a wide repertoire of inscriptions, including common drawings, letters, numerical symbols, and other referents. As they invent and use their own inscriptions, children also develop an "emerging meta-knowledge about inscriptions" (Lehrer & Lesh, 2003). Children's developing inscriptional capacities provide a basis for their mathematical activity. Indeed, inscriptions are mediators of mathematical learning and reasoning; they not only communicate children's mathematical thinking but they also shape it (Lehrer & Lesh, 2003; Olson, 1994). As Lehrer and Schauble (2006) stressed, developing a repertoire of inscriptions, appreciating their qualities and use, revising and manipulating invented inscriptions and representations, and using these to explain or persuade others, are essential for data modelling.

Metarepresentational Competence

Children's use of inscriptions plays an important role in their development of metarepresentational competence (diSessa, 2004; diSessa, Hammer, Sherin, & Kolpakowski, 1991). Such competence includes students' abilities to invent or design a variety of new representations, explain their creations, understand the role they play, and critique and compare the adequacy of representations. The use of "meta" to describe these capabilities, as diSessa's emphasises, is to indicate that no specific representational skills are implied. Unlike the standard representational techniques students might have learned from specific instruction, metarepresentational competence encompasses students' "native capacities" (diSessa, 2004, p. 294) to create and re-create their own forms of representation. Their skills here are more "broadly applicable, more flexible and fluid" (diSessa et al., 1991, p. 118), and are not just confined to a narrow set of instructed representations. In essence, diSessa and Sherin (2000) and colleagues coined the term *metarepresentational* to "describe the full range of capabilities that students (and others) have concerning the construction and use of external representations" (p. 386). Indeed, diSessa's research has shown that students do possess a "deep, rich, and generative" understanding of representations, which seems to exist before instruction and is independent of it (p. 387). It appears students are particularly strong at inventing and modifying representations, which diSessa (2004) refers to as "hyper-richness."

Another issue that has received limited attention with respect to children's metarepresentational competence is the joint development of metarepresentational and

conceptual competence (diSessa, 2004). As diSessa noted, research is limited here and the role of student-created representations in conceptual development is rather complex. Questions needing attention include how certain strengths or limits of metarepresentational competence might advance or hinder conceptual development, and whether metarepresentational competence and conceptual development develop jointly. Specifically, for the third-year activity reported here, I consider how the children structured and represented their data, and how they displayed metarepresentational and conceptual competence.

Methodology

The participants were from an inner-city Australian school. In the first year of the study, three classes of first-grade children (2009, mean age of 6 years 8 months) and their teachers participated. The classes continued into the second year of the study and finally, two classes continued into the third year (mean age of 8 years 8 months, $n=39$). A seventh-grade class (age range of 12-13 years) also participated in one of the activities during the second year and shared their models with one of the grade two classes. For all activities, the children worked in small groups.

Design

A teaching experiment involving multilevel collaboration (at the level of student, teacher, and researcher) was adopted throughout the study, where the developing knowledge of all participants was the focus (Lesh & Kelly, 2000). Such an approach is concerned with the design and implementation of experiences that maximise learning at each level. The teachers' involvement in the research was vital; hence regular professional development meetings were conducted. This paper addresses aspects of the student level of development.

Procedures

Literature, both purposefully created and commercially available, was used as a basis for the problem context in each of the activities implemented across the three years. It is well documented that storytelling provides an effective context for mathematical learning, with children being more motivated to engage in mathematical activities and displaying gains in achievement (van den Heuvel-Panhuizen & van den Boogaard, 2008). For the current activity, the storybook, *Hot Cha Cha* (Nobisso, 1998) was initially read to the children. The activity, *Investigating and Planning Playgrounds*, was then introduced, with the children firstly posing questions that might help them find out more about their classmates' thoughts on their new playground. In their groups, the children then created four survey questions and were to provide four answer options for each question (e.g., one group posed the question, *How long do you spend on each piece of equipment?* with the response options of *30 mins*, *15 mins*, *5 mins*, and *20 mins*). On answering their own questions, each group chose one focus question to which the other groups were to respond. The children were to initially predict how their focus question might be answered by the remaining groups. Each group subsequently analysed all their collected data for their focus question and were to display their findings using their choice of representation. The children were encouraged to represent their findings in more than one way, with no specific direction given. They were supplied with a range of recording material including blank chart paper, 2.5cm squared grid paper, and chart paper displaying a circle shape. The children could use whatever of these materials they liked; no encouragement was given to use any specific recording material. On

completion of the activity, the groups reported back to their class peers on their final data models.

Data Collection and Analysis

In each of the third-grade classrooms, two focus groups (of mixed achievement levels and chosen by the teachers), were videotaped and audiotaped. Altogether there were nine groups of children, five in one class and four in the other, who completed the activity. All artifacts were collected and analysed, along with the transcripts from the video and audio tapes. Iterative refinement cycles for analysis of children's learning (Lesh & Lehrer, 2000) were used, together with constant comparative strategies (Strauss & Corbin, 1990) in which data were coded and examined for patterns and trends.

The analysis of the children's transcripts and artifacts took into consideration: (a) how the children structured and represented their data; (b) the "completeness (shows all relevant information)," and "compactness (better use of space)" diSessa, 2004, p. 313) of their representations; and (c) evidence of conceptual and metarepresentational competence.

Selection of Findings

Prior to addressing some findings, it is worth noting that the teachers had done minimal instruction on statistical representations and had not introduced the children to more difficult representations such as circle graphs or line graphs.

How Children Structured and Represented their Data

Of the nine student groups, seven created two or more representations, with one group creating four representations. Vertical bar graphs using the 2.5 cm squared grid paper and circle graphs were the most popular, with seven groups creating a vertical bar graph and seven, a circle graph, with six groups creating both representations. Beginning with a blank sheet of paper, two groups created a vertical bar graph and one group, a line graph. One group made use of a list of their response options and used tally marks to collate their data prior to representing it. Two groups chose unusual representations, such as the "heart monitor" representation created by James' group. As James explained, "Okay, how are we going to show it our own way?! Let's think, just think. I know, like at the hospital, so like... up here (indicates a line graph similar to a heart rate monitor display). So first we're going to may be do about 3cm" (he places his ruler horizontally across the middle of the paper). Another group member further added: "We're doing like a doctor's sort of thing how they go like that (indicating a rise and fall)...like a doctor does...yes, how they have those lines, so we're sort of doing it with maths though."

Completeness and Compactness of the Representations

To determine the completeness of the representations, consideration was given to children's use of inscriptions, and for compactness with respect to the bar graphs, their use of space. Of the 10 bar graphs created (one group completed two), seven displayed inscriptions on both the vertical and horizontal axes, while the remainder omitted numbering the vertical axis, relying instead on a visual counting of the squares on their grid paper. Six bar graphs featured spacing between the drawn bars, presumably to facilitate reading of the graph, with one group commenting, "I'll leave a space so it is neat." Interestingly, two groups who used the grid paper in a landscape position realised they did not have sufficient squares to represent the data for a couple of their response options and hence chose to

colour two columns for the one response. All bar and line graphs featured colour, with one group debating whether, in creating a new representation, the same colours should be used (“We colour it exactly the same, and then we’re going to colour code exactly as we did on that” [bar graph]; another group member disagreed, however, stating, “We don’t need to.”)

For those groups that created circle graphs, all used inscriptions to indicate the proportion of each response option. These inscriptions included labelling of the option or using a colour key (two instances of the latter), as well as recording the number of each response and/or recording a percentage (five instances of the latter). The children’s use of percent and percentage was an interesting finding, as explained next.

Conceptual and Metarepresentational Competence

There was evidence of development of conceptual and metarepresentational competence as the children created their representations, especially with their circle graphs as they had not been taught how to construct these. The use of a ruler and/or estimation was evident in each of the focus groups, as they tried to represent their data. For example, one group argued over how to estimate a sector for each response option, with one child insisting that “You have to find the middle first. That’s the first thing you actually do.” He then placed his ruler through the centre of the circle and drew a small sector to represent the two “for exercise” responses to the focus question, “Why do you like the equipment you chose?” He explained, “Two will only be like this (drawing a small sector)...cause it’s a very small amount.” He then recorded “2” in the sector he had created. When asked how many “pieces of the pie” they needed, the group quickly replied “four, cause there’s four of them (response options).” They also commented that the four sectors would not be the same size “because if there’s two people, this would have to be a smaller piece to fit two people and a bigger piece to fit nine people in.” In estimating the size of the sector to represent the nine responses of “it’s challenging,” one child claimed that “nine would be half of it” (there was a total of 20 responses to their focus question). After much discussion, the group decided “no, no, no, that nine can’t be that big (half of the circle)” taking into account the frequencies of the other response options (six, three, and two). One child subsequently tried to measure the sectors with his fingers to make the nine sector smaller than the total of the other response options (11), explaining, “Yeah, it actually does have to be a bit smaller.”

Another display of the children’s conceptual and metarepresentational competence occurred when they commented that their second representation would display the same data. After completing a bar graph, one student group decided to construct a circle graph to display their findings from their focus question, “Why do you like the playground?”:

Peter: You just do the same. It’s a pie.

Belinda: So how many people are there...

Kim: But we need to do the biggest one (sector) for the most amount of votes.

Peter: We need to write, we do 12 for the most amount of votes; we take the data that we’ve got from here (bar graph) and we write what it is.

Belinda: Challenging (one of the response options) is the biggest.

Sebastian: So write up “challenging is the biggest” so do it in a pie chart and write 12 percent.” ...

Peter: Yeah, we just take the stuff from here and put it there (on the circle graph).

Sebastian: So do a big thing there and write 12 percent.

What was especially interesting in the children’s apparent links between conceptual and metarepresentational competence was their application of percent and percentages, and their

efforts in trying to represent 0% and 1%. Of the nine grade 2 student focus groups that progressed to the third year, seven tried to apply their awareness of these ideas in creating their representations. The children had not formally studied this topic but were transferring their learning from their experience with the grade 7 class in the second year (e.g., “Do we have to do percentages like the Year 7s did last year?”) and sharing their ideas with those group members who were not involved in the grade 2/7 experience. However, one group who explained that “12% means 12 people” subsequently noted that, “We don’t know how to use it.”

For the five groups who received zero preferences for one of their response options, considerable debate was held on how to display 0% and 1%. For example, one group was experimenting with how to represent their data on their circle graph for the response option to the question, “Why do you like the playground?” The response option, “fits more people than the oval” received zero votes, while the option, “good views” received one preference. After instructing Kim to “Write 12 percent” in one sector of the circle graph (representing 12 responses), Belinda said, “Maybe you could rule a little bit off it...that could be zero percent.” She further noted that the response option of good views “only has one percent...” and “has to be really small, like that small.” This group further struggled with their display of 0%, claiming that there was insufficient space to label the option of “fits more people than the oval.” When their teacher asked, “How can you show 0%?,” Belinda responded that, “You should just rub that out...cause that got nothing.” But then Kim was puzzled by “How would you do zero?” to which Belinda replied “Rub it out, rub it out.”

Another group, however, demonstrated a more advanced understanding of percent, albeit they did not calculate all the percentages correctly. When two group members recommended recording the number of focus question responses (to the question, “Why do you like the Spider Web”) in the circle graph segments they had drawn, Hugh disagreed, saying “Do percent”. When the research assistant queried the group on how they intended determining this, they explained that they knew that the circle graph represents 100% and that of the 20 votes for their focus question, there were two responses that each received five votes, one that received four, and one that scored six. Hugh explained that instead of recording the actual number of focus question responses, percentages should be shown: “That is 100%, so we needed to do 25%; that’s 25 (%), so that should be 24 (%), and that should be 25 (%) and that one should be 26 (%).

Discussion and Concluding Points

The children demonstrated competence in structuring and representing the data they had collected and were readily able to create more than one representation; this was in spite of minimal instruction. They also demonstrated an understanding that their different representations were representing the same data.

The children displayed skills in their use of various inscriptions, labelling vertical and horizontal axes of their bar graphs and the sectors in their circle graphs. Some children’s use of space on their bar graphs, as well as all children’s use of colour on all their representations, suggests they were aware of the importance of making their graphs readily interpretable. Children were also inventive in how they overcame any obstacles in their representational material, such as using two columns to display data for one response option. The innovative representations, such as the “heart rate monitor” graph, also show young children’s creative links with their world and their ability to reason analogically (English, 2004).

The children’s display of conceptual and metarepresentational competence was noteworthy, especially in their efforts in creating their circle graphs, where they invented a

number of ways to determine the placement of their sectors. Their keenness to use the notion of percent and also their debates on how to represent “0%” were surprising findings. Experiencing the models produced by the grade 7 students 12 months previously generated an interest in this notion for the children. Although the majority of children did not actually calculate correct percentages, they nevertheless were aware of the notion of percent and displayed a basic understanding of how it can be used. The sharing of products was a rich learning experience for both grade levels, providing opportunities for appreciating different approaches to dealing with data and for questioning, explaining, and interpreting the data models of others. Consideration should be given to creating such sharing opportunities across grade levels, with research exploring effective activities and the combined learning that takes place.

Further research on the development of children’s metarepresentational competence is also clearly needed, with consideration given to how we can advance their existing “native” competence (diSessa, 2004) here in connection with their conceptual development.

Acknowledgement

This project reported is supported by a three-year Australian Research Council (ARC) Discovery Grant DP0984178 (2009-2011). Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the author and do not necessarily reflect the views of the ARC. I wish to acknowledge the excellent support provided by the senior research assistant, Jo Macri.

References

- Clements, D. H., Sarama, J., Spitler, M. E., Lange, A. A., & Wolfe, C. B. (2011). Mathematics learned by young children in an intervention based on learning trajectories. *Journal for Research in Mathematics Education*, 42(2), 167-198.
- diSessa, A. (2004). Metarepresentation: Native competence and targets for instruction. *Cognition and Instruction*, 22(3), 293-331.
- diSessa, A., Hammer, D., Sherin, B., & Kolpakowski, T. (1991). Inventing graphing: Metarepresentational expertise in children. *Journal of Mathematical Behavior*, 10, 117-160.
- diSessa, A., & Sherin, B. (2000). Meta-representation: an introduction. *Journal of Mathematical Behavior*, 19, 385-398.
- English, L. D. (Ed.). (2004). *Mathematical and Analogical Reasoning of Young Learners*. Mahwah, New Jersey: Lawrence Erlbaum Associates.
- English, L. D. (2010). Young children’s early modelling with data. *Mathematics Education Research Journal*, 22(2), 24-47.
- English, L. D., & Watters, J. J. (2005). Mathematical modelling in third-grade classrooms. *Mathematics Education Research Journal*, 16(3), 59-80.
- Ginsburg, H. P., Cannon, J., Eisenband, J. G., & Pappas, S. (2006). Mathematical thinking and learning. In K. McCartney & D. Phillips (Eds.), *Handbook of early child development* (pp.208-230). Oxford, England: Blackwell.
- Greer, B., Verschaffel, L., & Mukhopadhyay, S. (2007). Modelling for life: Mathematics and children's experience. In W. Blum, W. Henne, & M. Niss (Eds.), *Applications and modelling in mathematics education* (ICMI Study 14, pp. 89-98). Dordrecht: Kluwer.
- Langrall, C., Mooney, E., Nisbet, S., & Jones, G. (2008). Elementary students’ access to powerful mathematical ideas. In L. D. English (Ed.), *Handbook of international research in mathematics education*, (2nd ed.). (pp. 109-135). New York, NY: Routledge.
- Lehrer, R., & Lesh, R. (2003) Mathematical Learning. In W. Reynolds & G. Miller (Eds.), *Comprehensive Handbook of Psychology*, Volume 7 (pp. 357-390). New York: John Wiley.
- Lehrer, R., & Romberg, T. (1996). Exploring children’s data modelling. *Cognition and Instruction*, 14(1), 69-108.
- Lehrer, R., & Schauble, L. (2006). Cultivating model-based reasoning in science education. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 371-386). NY: Cambridge University Press.

- Lehrer, R., & Schauble, L. (2007). Contrasting emerging conceptions of distribution in contexts of error and natural variation. In M. C. Lovett & P. Shah (Eds.), *Thinking with data* (pp. 149-176). NY: Taylor & Francis.
- Lesh, R., & Lehrer, R. (2000). Iterative refinement cycles for videotape analyses of conceptual change. In R. Lesh & A. Kelly (Eds.), *Research design in mathematics and science education* (pp. 665-708). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Lesh, R. A., & Kelly, A. E. (2000). Multi-tiered teaching experiments. In R. A. Lesh & A. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 197-230). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Nobisso, J. (1998). *Hot Cha Cha*. New York: Winslow Press.
- Olson, D. R. (1994). *The world on paper*. Cambridge, UK: Cambridge University Press.
- Papic, M. M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education*, 42(3), 237-268.
- Perry, B., & Dockett, S. (2008). Young children's access to powerful mathematical ideas. In L. D. English (Ed.), *Handbook of international research in mathematics education*, (2nd ed.). NY: Routledge.
- Shaughnessy, M. (2010). *Statistics for all: The flip side of quantitative reasoning*. Retrieved 14 August, 2010, from <http://www.nctm.org/about/content.aspx?id=26327>
- Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Ground theory procedures and techniques*. CA: Sage.
- van den Heuvel-Panhuizen, M., & van den Boogaard, S. (2008). Picture books as an impetus for kindergartners' mathematical thinking. *Mathematical Thinking and Learning: An International Journal*, 10, 341-373.
- Whitin, D. J., & Whitin, P. E. (2011). *Learning to read numbers: Integrating critical literacy*. New York: Routledge

Challenges in Responding to Scaffolding Opportunities in the Mathematics Classroom

Sarah Ferguson
Australian Catholic University
<sarahferguson@exemail.com.au>

This paper reports on a study that explored the use of cognitively challenging tasks with low-attaining mathematics students and in particular, their teachers' attempts at scaffolding. A major finding was that responding appropriately to scaffolding opportunities was challenging for the teachers. In this paper two main factors are discussed which impacted on the teachers' responses to scaffolding opportunities: teacher knowledge of appropriate tools and representations for particular mathematical concepts, and teacher response to student prior knowledge and understanding.

Introduction

Scaffolding holds promise in allowing teachers to assist students without diminishing the cognitive challenge of tasks. Goos, Galbraith and Renshaw (2002) described scaffolding as "mutual adjustment and appropriation of ideas rather than a simple transfer of information and skills from teacher to learner" (p. 195). However, it appears effective scaffolding remains a difficult task for teachers. In their study of scaffolding in classrooms, Bliss, Askew and Macrae (1996) found many instances of "missed scaffolds" (p. 48) where "conditions for scaffolding were present but not noticed by the teacher" (p. 47). In these instances, "pupils are crying out to be scaffolded" (p. 47). It is such instances that I have described as "scaffolding opportunities".

Scaffolding opportunities are similar to teachable moments. Teachable moments have been described as "the set of behaviours within a lesson that indicated that students are ripe for, or receptive to, learning because they express confusion, misunderstanding, uncertainty, struggle, or difficulty with a mathematical problem, concept or procedure" (Arafah, Smerdon & Snow, 2001, p. 3). Muir (2008) described teachable moments as "a teacher's simultaneous act in response to a student's answer, comment or suggestion and is utilized to either address a possible misconception or to enhance conceptual understanding" (p. 362). Effectively dealing with teachable moments as they arise demands teachers have a high level of mathematical knowledge for teaching (Muir, 2008). Responding to teachable moments or scaffolding opportunities is a difficult task. Two key factors emerged in my study that appeared crucial in determining if a scaffolding opportunity was taken or missed. These were teacher knowledge and use of appropriate materials to illustrate concepts, and the teacher responding appropriately to the students' prior knowledge.

Literature

Using appropriate materials and representations has been found to have benefits both cognitively and affectively for students (Sowell, 1989). Sowell's analysis of 60 studies on the use of manipulative materials found that two factors were essential for the use of such materials to be effective in student learning. First, the materials needed to be used over the long-term, such as a full school year or more. Second, teachers needed to be knowledgeable about their use. Boulton-Lewis and Halford (1992) described a distinction between representations that are "concrete embodiments of mathematical concepts and processes" and those that are "inherent in the discipline of mathematics (e.g. number lines and

symbols)” (pp. 1-2). In the present study, materials or representations that were “concrete embodiments” and illustrated concepts were more successful than those that used mathematical symbols (Ferguson & McDonough, 2010).

Successful scaffolding relies on the teacher being able to bridge the students’ current known understanding with understanding that is not yet known. Vygotsky (1978) described this as the “zone of proximal development”. Tasks that fall outside this zone will either be too easy, falling within a students’ known understanding, or too far from the students’ known understanding and therefore too difficult. In order to ensure learning that falls within this zone, teachers need to know their students’ prior knowledge and understanding that may impact on their ability to learn a particular concept. Sullivan, Mousley and Zevenbergen (2006) proposed that teachers should set one core task for the class to work on but that “different students may follow different pathways to the ultimate task, and may be supported or detoured along the way” (p. 120). For students who find the core tasks too difficult for example, Sullivan et al. advised the teacher use “enabling prompts” where “the teacher can explore ways to give the student access to the task without the students being directed towards a particular solution strategy for the original task” (p. 124). This assistance without the teacher being directive resonates with the notion of scaffolding.

In this paper I discuss examples of attempted scaffolding from the mathematics classrooms of two teachers of upper primary children. The responses of two low-attaining students from each class, four in all, will be discussed to explore the challenges of responding effectively to scaffolding opportunities.

Method

This study used a case study research design (Stake, 1995) to build up a rich and detailed description of mathematics teaching and learning in two teachers’ classrooms, particularly focussed on the experiences of four low-attaining students. Case studies offer teachers “real world” examples of teaching and learning. Though the findings of a case study are particularistic they can be recognised in other situations (Stake, 1995). The experiences of the teachers and students in this case study offer teachers the chance to reflect on their own practice. Examining a particular case can also prompt reflection on general areas of complexity in teaching mathematics such as responding to scaffolding opportunities.

Participants

The first teacher, Ms B, had five years of teaching experience, with two years experience in teaching Year 5. Two students in Ms B’s Year 5 class were targeted for data collection. These students, Carl and David, were operating about 12 to 18 months below expected levels in mathematics according to the *Victorian Essential Learning Standards* (Victorian Curriculum and Assessment Authority, 2006). Data regarding David are not discussed in this paper.

The second teacher in this study, Ms L, had eleven years teaching experience. The two target students from Ms L’s class were Sophie, a Year 5 girl and Riley, a Year 5 boy, both operating at about 12 months below expected levels in mathematics according to the relevant curriculum at the time, *Victorian Essential Learning Standards* (Victorian Curriculum and Assessment Authority, 2006).

Data Collection and Analysis

Six sequential mathematics lessons were observed in each class. Each lesson was 80 to 90 minutes long. Data collection included interviews before and after observed lessons with the teachers and audio recordings of each lesson via a mobile recording device worn by the teachers. Data regarding the target students were gathered via lesson observations and a one-to-one interview after each lesson. There were two parts to these interviews. Firstly the interview focussed on the student's feelings about the tasks and their teachers' actions. Secondly a short assessment piece designed by the researcher was given to assess understanding of the concept of major focus within the lesson.

Interviews and lessons were transcribed and detailed lesson observation notes written for each observed lesson. These data were analysed using the NVivo program (QSR International, 2005). Data from a variety of sources such as interviews, observations, and work samples, built up a "rich, thick description" (Merriam, 1998, p. 38) of the teachers' and target students' experiences of their mathematics classrooms. I then "searched for patterns" (Stake, 1995, p. 44), seeking common themes but also recognising instances that differed from such themes in an effort to "come to know the case well" (Stake, 1995, p. 8).

Results and Discussion

The scaffolding opportunities that arose in this study were just some of the many decisions faced by the teachers in the busy world of the classroom. The teachers were dedicated and thorough but responding effectively to scaffolding opportunities was challenging. Some common factors that impacted on the teachers' response to scaffolding opportunities emerged in both classrooms (Ferguson, 2012). In this paper I will discuss two of these factors: knowledge and use of appropriate tools or representations, and responding to student prior knowledge. For each factor I examine instances of both opportunities missed and opportunities taken in both the research classrooms.

Knowledge and Use of Appropriate Tools or Representations

Ms L taught multi-digit multiplication during all six of the observed lessons. Although she talked about the potential positive effect of using manipulative materials or representations, Ms L did not deviate from her lessons emphasising written processes, to use such materials. Ms L was not familiar with how to use squared grid paper to shade regions in a way that could illustrate the distributive properties of multi-digit multiplication. The squared grid paper was a suggestion I made after Ms L asked me about possible representations.

However Ms L shied away from using grid paper by suggesting that using these visual representations could confuse students and in fact, may have confused her. She said:

I might actually show them the other one, drawing the grid ... I was going to show that but I thought that might be too confusing. I think with a lot of kids the visual can confuse them more. I looked at it and thought "Oh I don't know how that's helping me" but kids are different I suppose.

There was evidence from a post-lesson assessment task that using the squared grid paper representation potentially supported Riley's understanding of multi-digit multiplication. This assessment task asked Riley to "Find an answer to 14×16 using a written method and then use the grid paper to draw an array". Riley quickly grasped the idea of using grid paper to show a multiplication expression as an array. He was then able to divide up the larger array into manageable pieces in order to find a total. After showing the outline of the array " 14×16 ", Riley said "It would be much easier if it was 14 times 10 'cause I can go 10, 20,

30, 40 ... I could break it down again 'cause I'm not well [sic] at my 6 times tables. I can break it down again. 1, 2, 3, 4, 5 I can go 5, 10, 15, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75 and then I go down so then I can go, then I know that is 14 so I've got the answer". (See Figure 1) Though he made an error in counting by fives by omitting 20, this strategy was effective and showed a distributive strategy for multi-digit multiplication.



Figure 1. Riley's solution on square grid paper.

I believe Riley's ease at using the grid paper showed that this was potentially a useful representation for him and an effective way to scaffold his learning. If Ms L used representations that, as Boulton-Lewis and Halford (1992) suggested, illustrated the underlying concepts behind these processes, such as the squared grid paper, Riley's understanding could have been scaffolded more effectively. This was a scaffolding opportunity missed as the conditions that Sowell (1989) described as essential for the successful use of materials and representations were missing in that the teacher did not have sound knowledge of the representation and had not experienced long term use of it. This points to the need for teachers of upper primary students to have knowledge of and experience in using the most appropriate representations or materials for complex concepts such as multi-digit multiplication in order to scaffold more effectively.

In the classroom of Ms B the use of representations provided an example of a scaffolding opportunity taken. During all lessons observed, Ms B focussed on fractions and decimal numbers and how they relate to each other. In the same way as Ms L, Ms B asked me what kinds of representations she could use to illustrate decimal numbers. I suggested the decimat, a rectangle divided into a grid of one thousand equal parts with bold lines showing the tenths and the hundredths of the rectangle.

The data from previous lessons and post-lesson interviews revealed that Carl could not name the decimal number that was the same as one half. The day before using the decimat the following exchange was recorded.

- Ms B: What's a half as a decimal?
 Carl: Zero point two, zero point oh two?
 Ms B: How many is half of the tenths? In a whole, how many tenths?
 Carl: Ten.
 Ms B: What's half that?
 Carl: Five.
 Ms B: Five tenths as a decimal? Zero point? ...
 Carl: Zero point zero one?

It was clear from Carl's responses that he really did not know how half could be expressed as a decimal. He also demonstrated a common misconception by offering that "point two or point zero two" might be equivalent to half, seemingly taking the number two from the denominator of one half and applying this to decimal numbers. Carl needed a vehicle to link one half and five tenths for him so that he could see how they are equivalent.

After Ms B introduced the decimat by creating one together on the board, the students began to play a game using this representation. As they were playing, I observed that both

Carl and David were able to link the notation of decimal numbers with the amount they had coloured on their decimat. This was noteworthy progress given their performance during previous lessons. In addition, Carl appeared to be developing his understanding of the decimal equivalent to half.

Carl : [Calling out] Ms B! I've coloured in a half!

Ms B : Good! How many tenths is that?

Carl : Five.

Ms B: Five tenths, yeah, half the tenths.

I am not claiming that Carl now understood how five tenths and one half relate to each other but I believe he was now “on the way” to developing this understanding. This was the first time Carl had used the decimat but this representation appeared to have scaffolded his learning. The decimat representation is an example of a “concrete embodiment” (Boulton-Lewis & Halford, 1992) where the concept is directly represented. Such representations were found in the large study to be beneficial for scaffolding (Ferguson, 2012). By using the decimat representation, Ms B was able to maximise a scaffolding opportunity for Carl.

Responding to Student Prior Knowledge

The decimat task occurred on the final day of lesson observations but during the previous two weeks, Ms B struggled on many occasions to scaffold Carl's learning. The cognitive demand of six out of the eight tasks observed in Ms B's classroom was too high for Carl based on lesson observation and post-lesson interview data about Carl's level of understanding for fractions and decimal numbers. One example is Carl's response to improper fractions. In the first assessment task I asked Carl to draw “three halves”. He hesitated and said “Like three equal bits?” and I said “Mmm. Whatever you think.” Carl drew a circle and divided it into three roughly equal parts. He did not shade any of the parts. He then wrote “3 halves” [sic] beside it. Clearly Carl did not understand improper fractions.

The following day Ms B assigned the class a task called “Fraction Rods” which compared the size of Cuisenaire rods to find the fractional relationships between them. This example is one of the exchanges between Carl and Ms B was recorded during this lesson.

Ms B: If the blue is one and a half, what each of these green ones worth?

Carl: A third

Ms B: A third *of the blue rod* because the blue rod is worth one and a half. How many halves in one and a half?

Carl: Three

Ms B: Yes why?

Carl: Cause ... we just did it then. [Carl's voice is quiet and sulky]

Ms B: No but I'm asking you. How many halves in one and a half?

Carl: I don't know.

Ms B: How many halves in a whole?

Carl: Four

Ms B: How many *halves* in whole?

Carl: Two

Ms B: Have you switched off?

Carl: Yes

Ms B: Switch back on please. How many halves in a whole?

Carl: Two

It is clear from Carl's responses and tone of voice that he was confused about the task. As was shown during the assessment task about improper fractions, asking Carl to

determine one third of three halves was out of Carl's zone of proximal development. Ms B did not act on Carl's obvious confusion by adjusting the task so that it was within his grasp but persisted. This appeared to cause Carl frustration and disengagement, "have you switched off?" "Yes". The task was beyond Carl's current understanding so it is likely that no amount of explanation from Ms B would have been sufficient for Carl to have understood this task. The opportunity to scaffold Carl's learning was missed as Ms B did not respond to his prior understanding or knowledge.

Turning again to Ms L's class, Ms L's conversation with Sophie (Ferguson & McDonough, 2010) revealed responsiveness to Sophie's prior understanding that resulted in a scaffolding opportunity taken.

- Ms L: So now, how can you tell me what 9 times 87 is? What can we keep doing here?
Sophie: Keep adding on 87
Ms L: Until you've added it?
Sophie: Until we've got 9.
Ms L: Okay that will give you the right answer so that's one strategy because addition, multiplication is when we keep adding the same number over and over and over again. So keep adding on for that please.

Ms L invited Sophie to use her preferred strategy, evident from previous lessons, of repeated addition. Ms L responded to the strategy that Sophie had decided to use herself, and responded to Sophie's prior knowledge of multiplication as repeated addition. However, repeated addition is an inefficient strategy when multiplying with larger numbers and it would be desirable for Sophie to develop a more efficient strategy. Ms L scaffolded this development when she returned in a few minutes for the final part to this conversation.

- Ms L: How are we going here Sophie? Okay if you think, this is pretty time consuming isn't it? So let's look at, if it's 9 times 87 do you think perhaps we could use our knowledge, how do we multiply by 10? So 10 times 87 which would be what?
Sophie: 870
Ms L: Okay, spot on. But we only want to multiply 9 times so what do we have to take away from 870 to make it correct?
Sophie: Ahhh... 87?
Ms L: Because we've multiplied one extra. So you do 870 take away 87. See if that will help you. That will be a quicker ... if that's going to help you because that's a quicker way of doing it, isn't it?

Now Ms L offered Sophie an alternative strategy that would be more efficient. Importantly she did so after allowing Sophie to use her preferred strategy, recognising this prior learning. Experiencing an inefficient strategy perhaps added to Sophie's motivation for finding a quicker way. Sophie appreciated the efficiency of the "multiply by ten and take one set" strategy for multiplying by nine, commenting in the post-lesson,

I was confused with trying to find the answers. I got confused trying to find the answer to 87 times 9. Then I used subtraction. If you do 10 times 87 it will make 870. If you minus 87, it gets to 783. If I kept adding 87 to my answer it would've taken a long time. It was Ms L's idea.

Ms L responded in this episode to Sophie's prior knowledge before scaffolding a more sophisticated strategy. The task made it reasonably difficult for Sophie to use her repeated addition strategy but not too difficult in that her initial strategy could still be attempted. In moving from what Sophie knew to something just beyond, Ms L was able to make the most of this scaffolding opportunity.

Practical Implications

This paper has illustrated the challenges for teachers in assisting the learning of low-attaining students by responding effectively to scaffolding opportunities. The two factors addressed here that impacted on the teachers' ability to make the most of scaffolding opportunities can be examined to draw out the implications of these findings for mathematics classrooms.

The first implication is that upper primary teachers need access to high quality materials and representations and that these materials need to be readily available in classrooms. More important than simply having materials available, teachers also need professional learning in using representations and materials to effectively illustrate particular concepts. The teachers in this study asked me about representations for multi-digit multiplication and decimal numbers. This implies that on-going professional development should be provided on appropriate representations and materials for the more complex concepts encountered in upper primary school such as multi-digit multiplication and decimal numbers. As Sowell (1989) suggested, effective use of materials and representations relies on the teacher, both the teacher's familiarity and persistence with the materials and their knowledge of how to use them to effectively scaffold learning. The present study, for example, found that materials and representations that illustrated concepts (Boulton-Lewis & Halford, 1992) were more effective than others when scaffolding the learning of the low-attaining target students (Ferguson & McDonough, 2010). Information is needed for teachers regarding the kinds of materials and representations that illustrate particular concepts.

This study found that missed opportunities for scaffolding where the teacher did not recognise or respond to the students' prior knowledge were crucial. Without tasks that fell within the students' Zone of Proximal Development (Vygotsky, 1978), scaffolding could not take place. Without the task being adjusted in some way, students struggled unproductively.

An initial implication for teachers is to find out what the students' prior knowledge is for the concept to be taught. Armed with this knowledge, tasks can then be adjusted depending on student needs (Sullivan et al., 2006). In the case of the Fraction Rods task described previously, Ms B could have created alternative sets of questions that explored more about proper fractions for students such as Carl who needed consolidation in this area. The task remains essentially the same, comparing rods to find fractional relationships, but is adjusted to provide an "enabling prompt". Examining the use of enabling and extending prompts (Sullivan et al., 2006) might support teachers in planning task adjustments to more effectively respond to students' prior knowledge.

In any lesson there are many decision points for teachers, one of which is when and how to respond to scaffolding opportunities. Responding effectively during a lesson "depends very much on a combination of the teacher's knowledge, beliefs and professional judgement" (Muir, 2008, p. 366). Addressing key factors such as appropriate materials and representations and student prior knowledge before the lesson might assist teachers in responding more effectively to scaffolding opportunities as they arise. This paper illustrated some examples of both scaffolding opportunities missed and opportunities taken. It is hoped that teachers can use these examples to reflect on these issues in their own classrooms.

References

- Arafeh, S., Smerdon, B., & Snow, S. (2001, April). *Learning from teachable moments: Methodological lessons from the secondary analysis of the TIMSS video study*. Paper presented at the annual meeting of the American Educational Research Association, Seattle, WA.

- Bliss, J., Askew, M., & Macrae, S. (1996). Effective teaching and learning: Scaffolding revisited. *Oxford Review of Education*, 22 (1), 37-61.
- Boulton-Lewis, G., & Halford, G. (1992). The processing loads of young children's and teachers' representations of place value and implications for teaching. *Mathematics Education Research Journal*, 4(1), 1-23.
- Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics*, 49, 193-223.
- Ferguson, S. (2012). *Like a bridge: Scaffolding as a means of assisting low-attaining students in mathematics during cognitively challenging tasks*. Unpublished doctoral dissertation, Australian Catholic University, Melbourne.
- Ferguson, S., & McDonough, A. (2010). The impact of two teachers' use of specific scaffolding practices on low-attaining upper primary students. In L. Sparrow, B. Kissane & C. Hurst (Eds.), *Shaping the future of mathematics education* (Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia, pp. 191-198). Fremantle: MERGA.
- Merriam, S. (1998). *Qualitative research and case study applications in education*. San Francisco: Jossey-Bass.
- Muir, T. (2008). "Zero is not a number": Teachable moments and their role in effective teaching of numeracy. In M. Goos, R. Brown & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, Brisbane, pp. 361-367). Adelaide, SA: MERGA.
- QSR International. (2005). *NVivo*. Doncaster, Melbourne.
- Sowell, E. J. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics Education*, 20(5), 498-505.
- Stake, R. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.
- Sullivan, P., Mousley, J., & Zevenbergen, R. (2006). Teacher actions to maximise mathematics learning opportunities in heterogeneous classrooms. *International Journal of Science and Mathematics Education*, 4, 117-143.
- Victorian Curriculum and Assessment Authority. (2006). Victorian Essential Learning Standards. Retrieved February 2, 2010, from vels.vcaa.vic.edu.au/index.html
- Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Using Classroom Episodes to Foster Prospective Teachers' Didactical Knowledge: Issues for Teacher Education¹

Rosa Tomás Ferreira
University of Porto & CMUP
<rferreir@fc.up.pt>

Luís Menezes
CI&DETS & College of Education of Viseu
<menezes@esev.ipv.pt>

Maria Helena Martinho
University of Minho & CIED
<mhm@ie.uminho.pt>

After briefly analysing a classroom episode, we discuss aspects of the teacher's didactical knowledge, namely in its mathematical and instructional dimensions, as reflected in the answers of three prospective teachers to a written assignment based on the episode. We then raise some issues regarding initial teacher education, anchored in the notion of didactical knowledge.

Classroom Episodes and Teachers' Didactical Knowledge

Portugal is currently implementing a new mathematics curriculum for basic education (grades 1 to 9, pupils aged 6 to 14) (ME, 2007). This curriculum introduces some changes in the set of mathematical topics to be taught at those grade levels, and stresses three *transversal skills* – problem solving, reasoning, and communication – assigning them an important status concerning the overarching learning objectives. However, in our opinion, the major reform lies in the orientations given for the dynamics of the mathematics classroom, as there is a strong emphasis on the teacher's and the students' active roles inside a community of learners.

The typical structure of teacher education programs has also changed, following European guidelines, as a result of the Bologna process. We draw our attention to the preparation of mathematics teachers for grades 7 through 12 (pupils aged 12 to 17). Currently, prospective teachers enrol in a master's degree after having earned a bachelor one in a scientific area with a minimum of 120 credit hours in mathematics courses. It is only during the master's degree that they take mathematics education courses and realize a school practicum. However, teacher education programs can differ significantly from institution to institution, namely in terms of courses offered and the student teaching experience. This general structure of teacher preparation has been in place for a few years.

As mathematics educators, we pay special attention to the work around issues of classroom communication and to the teacher's role in that process, both in pre-service and in in-service teacher education contexts (Bishop & Goffree, 1986; Tomás Ferreira, 2005; Martinho & Ponte, 2009; Ruthven, Hofmann, & Mercer, 2011). In fact, if pupils learn to communicate by communicating, teachers also learn managing classroom communication by experiencing and reflecting upon situations that involve various challenges regarding that dimension of their role in the classroom. Analysing and discussing short classroom episodes is an important way to achieve that goal (Bishop & Goffree, 1986, Ruthven et al., 2011; Tomás Ferreira, Martinho, & Menezes, 2011).

¹ This work is supported by national funds through FCT – Fundação para a Ciência e Tecnologia, under the projects *Professional Practices of Teachers of Mathematics* (Grant PTDC/CPE-CED/098931/2008) and PEst-C/MAT/UI0144/2011, and by FEDER funds through COMPETE.

Teachers' didactical knowledge is fundamental for engaging students in meaningful mathematical activity. The notion of didactical knowledge is not consensual, and several authors have elaborated on this idea (Ponte, 1999, in press). In our study, we follow Ponte's (1999) perspective – a teacher's didactical knowledge relates to aspects of practice, being a kind of knowledge “essentially oriented towards the action” (p. 61), and encompassing four dimensions: knowledge of curriculum, knowledge of mathematics, knowledge of students and their learning processes, and knowledge of classroom instructional processes (Ponte & Oliveira, 2002). A teacher's didactical knowledge has a dynamic character, as the experiences and situations of practice the teacher encounters in the classroom contribute to its development and constant reformulation.

Acknowledging the inter-related nature of all dimensions of the didactical knowledge, we focus here on mathematical knowledge and on instructional knowledge (Ponte, in press). In this paper, we briefly analyse one classroom episode – *Rita and Prime Numbers* (Boavida, 2001, adapted from Prince, 1998) – focusing our attention on the teacher's actions and discussing some aspects of the teacher's didactical knowledge that support those actions. We present three prospective teachers' analyses of the episode and discuss aspects of their didactical knowledge that emerge in those analyses. Our main goal is to problematize aspects of the recent general structure and organization of initial teacher education programs, based on the notion of didactical knowledge.

“Rita and Prime Numbers”: A Classroom Episode

The episode *Rita and Prime Numbers*, presented in Figure 1, happens in a classroom corresponding to Portuguese 7th graders (pupils aged 12). The teacher proposed a closed task (to list all prime numbers up to 50) with a low level of cognitive demand. However, building on a student's comment, the teacher's actions quickly rose the cognitive demand, transforming the task from a simple exercise to a task engaging students in higher order thinking processes, such as proving and discussing aspects of elementary logics (such as implications, reciprocals, examples, and counter-examples). Specifically, what did the teacher do? Which instructional actions characterize the course of events in this episode? We now turn to discussing some of the teacher's main decisions in the episode.

A first decision of the teacher was not to immediately validate Rita's idea. Instead, the teacher left the validation process to the students, hoping to have them presenting their finding to the entire class. In fact, a few moments after sharing her finding with the teacher, Rita went to the board and wrote her answer to the initial task; only after did she communicate her finding to her classmates: “The prime numbers, except 2 and 5, end in 1, 3, 7, or 9”. The teacher made a second decision that affected the direction of the discourse: she extended the initial task by building on Rita's finding, challenging the students to check if Rita's idea would work for other prime numbers. The students' reaction was not surprising: they randomly chose several numbers and tested Rita's finding for those numbers. Since they were not able to find any number that could contradict Rita's idea, the class naturally accepted its truthfulness.

Being aware that the proof of a conjecture, and the role of examples in that process, were at stake, the teacher made a third decision which showed to be crucial in the unfolding of events: she wrote on the board “Rita's conjecture: All prime numbers, except 2 and 5, end in 1, 3, 7, or 9”. The writing of the word conjecture was not naïve. Indeed, the teacher knew that this term could be unfamiliar to some students. She rewrote Rita's conjecture in a

way that made it more explicit in its scope, since the episode provides no information about whether Rita had analysed any prime number larger than 50.

Rita's teacher asked her class to find all prime numbers up to 50. After some time, Rita noticed that the prime numbers larger than 5 she had identified so far ended in 1, 3, 7, or 9. She called her teacher to show her this finding. The teacher asked Rita to work with her partner in order to find the best way to share her finding to the class during the collective discussion of the work. Rita wrote on the board all prime numbers smaller than 50 and she read what she had written in her notebook:

Rita: The prime numbers except 2 and 5 end in 1, 3, 7, or 9.

The teacher then asked the class to analyse if the same thing happened with other prime numbers. The students started checking several cases of prime numbers, some of which much larger than 100, and they did not find anyone that would not end in 1, 3, 7, or 9. Shortly, they were strongly convinced that what Rita had found was true for all prime numbers, regardless of having been checked, because all prime numbers that they would check always ended in one of those digits. At this time, the teacher wrote on the board:

Rita's conjecture: All prime numbers, except 2 and 5, end in 1, 3, 7 or 9.

She makes sure the students remember the meaning of conjecture and she challenges them to find a process that allows them to be sure if the conjecture is, indeed, valid for all prime numbers and why that is so. The students try to respond to the challenge and, in this process, they reinforce their conviction that the conjecture is true; yet, their work does not progress. The teacher, then, choosing to work with the whole class, decides to write on the board the numbers from 0 to 9, circling 1, 3, 7, and 9. Almost immediately the students offer several suggestions:

Maria: Teacher, cross out numbers 0 and 5. A prime number larger than 5 cannot end in 0 or 5.

Teacher: Why?

Maria: If it ends in 0 or 5, it is a multiple of 5 and, therefore, it is not prime.

Daniel: You also have to cross out 2, 4, 6, and 8. If it is larger than 2 and is a prime number, it cannot be even!

Bernardo: Of course not. 2 divides...

Teacher: So what?

Bernardo: A prime number can only have two factors.

Rosa: Yeah. If it ends in 2, 4, 6, or 8, that is because it is even and even numbers are multiples of 2.

Rita: We only have 1, 3, 7, and 9 left. Therefore, all prime numbers except 2 and 5 end in the way I found. We're now sure of it.

Inês: Bu the opposite is not true. For example, 21 ends in 1 and it is not prime.

Teacher: Why isn't 21 prime?

Several students: Because 3 divides 21; 3 times 7 is 21.

Bernardo: It has divisors that are different from 1 and 21.

Teacher: So, check if this is true, or not, this that I am going to write on the board: all numbers ending in 1, 3, 7, or 9 are prime.

One can hear several voices saying "it's not true". They mention 21, 27, 33... (...)

Figure 1. A classroom episode.

The teacher's fourth important decision was to open a space for discussing with her students the meaning of conjecture, anchoring the discussion on Rita's finding and engaging the students in proving or disproving a conjecture. This process explicitly involved the students in justifying their own reasoning. Such a challenge was nothing but easy to them. In fact, they did not go further than reinforcing their own ideas by finding more and more examples which, nonetheless, proved nothing. And the teacher made another important decision: to write on the board all ten digits, circling those which corresponded to the unit's digit of a prime number, following Rita's conjecture (i.e., leaving 1, 3, 7, and 9 circled). Based on the episode, the students seemed to have correctly interpreted the teacher's intention when writing all ten digits: they signalled all possible ways of ending a natural number. Thus, using their knowledge of divisibility criteria, the students quickly started an elimination process of those digits. Using questions such as "Why?" and "So what?", the teacher ensured that the students justified their assertions. Inês also made a discovery: "But the opposite is not true. For example, 21 ends in 1 and is not prime". After making sure that

the students understood why 21 was not a prime number, the teacher made another crucial decision: to write on the board the reciprocal of Rita's conjecture, giving an active voice to Inês's finding. And the counter-examples started emerging, easily convincing the students that this new conjecture was not true.

In sum, throughout the episode, the teacher's actions were determinant for the quality of the classroom discourse, raising the cognitive level of the initial task and engaging the students in significant mathematical activity. We must bear in mind that such activity around issues of elementary logics, quite challenging for the students, had not been planned ahead of time. Instead, the teacher took advantage of a teachable moment, realizing the potential of Rita's comment and building on that opportunity to explore with the students some mathematical ideas that were much more demanding than the notion of prime number.

We can identify a few aspects of the teacher's didactical knowledge in the episode *Rita and Prime Numbers*. For example, the teacher listened to the students' contributions to the classroom discourse in a responsive manner (Empson & Jacobs, 2008), valuing all of them as worthy discussing with the whole class, regardless of their correctness or rigorousness in language. The teacher also gave the students the responsibility for proving or refuting both conjectures, orchestrating a productive whole-class discussion (Stein, Engle, Smith & Hughes, 2008) so that there would be a common understanding of conjecture and no answer would be left unjustified. The teacher's actions along the episode were certainly anchored in her mathematical knowledge, which allowed her to recognize a teachable moment triggered by Rita's finding and to build instruction upon it, pushing her students to do mathematics. This is clear in the transformation of the initial task – a task of procedures without connections (Stein & Smith, 1998) – into a task involving processes of proof. Such a new task, given its higher cognitive demand and the fruitful discussion around it, pushed the students to engage in serious mathematical activity.

Aspects of Didactical Knowledge in the Analysis of a Classroom Episode

We have already stressed the importance of discussing, in teacher education contexts, aspects of the teacher's role in managing mathematical communication in the classroom. This is reflected in our own practice as mathematics educators in which the analysis of classroom episodes such as *Rita and Prime Numbers* is a regular activity. The first author is a mathematics educator in a large urban university in northern Portugal. This institution offers a 2-year master's degree which certifies mathematics teachers for grades 7 to 12. The second semester of coursework, immediately before a year-long student teaching experience along with a few courses, encompasses a mathematics education course that addresses issues of classroom dynamics, including the teacher's role in the general management of classroom meaningful discourse as well as the students' participation in that discourse (NCTM, 1991), and the teacher's challenges in orchestrating productive mathematical discussions (Stein et al., 2008) and in their questioning, listening, and responding approaches (Tomás Ferreira, 2005). With such an agenda, the analysis and discussion of classroom episodes is at the core of the prospective teachers' activities. After completing several assignments throughout the semester, varying in nature, degree of complexity, and organization of students' work (individually, in pairs, in small groups), prospective teachers are typically asked to complete a short written, individual, in-class, and final assignment. In the school year of 2010/11, a cohort of 12 prospective teachers was asked to analyse the episode *Rita and Prime Numbers* in their final assignment. Several questions guided this analysis. We focus here on two of those questions: (1) How do you think the teacher should lead the classroom discourse after the last interventions of the students? and (2) Do you

believe Rita's conjecture is proved? If so, why? If not, why? The answers varied significantly both in terms of the mathematical issues underneath the episode and in terms of didactical choices to continue the episode. Next, we briefly analyse the answers of three prospective teachers enrolled in program mentioned above. Since the data gathered for this text is written in Portuguese, instead of illustrating our analyses with excerpts of the prospective teachers' actual work, we resort to our translations of that work.

Júlio completed a bachelor degree in mathematics in the same institution where he is now seeking teacher certification. His suggestion to continue the episode *Rita and Prime Numbers* evidences that he realized that there were two implications involved in the episode, and that one was the reciprocal of the other. Furthermore, he stresses the importance of identifying and distinguishing reciprocal implications and of comprehending the role of examples and counter-examples in the proof and refutation of assertions:

Based on the students' answer, the teacher should tell them that they had shown the assertion was false, through a whole-class discussion, making them understand that it is enough to give an example that does not verify the assertion for this to be invalid. Then, she should ask the students to relate Rita's conjecture with the latter one, questioning them about their difference[s] and truthfulness, in order to conclude the task.

Júlio shows sensitivity towards an important issue of the teaching and learning process: the development of mathematical reasoning, in particular formulating, testing, and proving (or disproving) conjectures. This prospective teacher has no doubts about the truthfulness of Rita's conjecture: "the conjecture is proved, since the students know that all the numbers end on some digit between 0 and 9, and using divisibility criteria, they managed to exclude the even digits and the 5, remaining 1, 3, 7, and 9". He stresses the role of the class discussion and the importance of using students' (prior) knowledge to help them proving Rita's conjecture: "In this way, and using their own knowledge, the students proved Rita's conjecture, through discussion and exchange of ideas".

Júlio's answers to the written assignment suggest that he recognized the mathematical situation underneath the episode, particularly the existence of reciprocal implications and how they might be proved or disproved. He also gave instructional suggestions that are aligned with current recommendations for mathematics teaching (ME, 2007; NCTM, 2000), especially those related to whole-class discussions as means towards the institutionalization of new knowledge (Stein et al., 2008). In his analysis of the episode, we may say that he shows an adequate mathematical and instructional knowledge, in Ponte's (1999) sense.

Carlos has a similar academic background to Júlio's. His analysis of the episode reveals several problems, which may be anchored in an incorrect interpretation of the episode or in a deficient didactical knowledge. Carlos does not seem to understand that, at the end of the episode, the implication being discussed is the reciprocal of Rita's conjecture:

After the students said that it was not true, that all prime numbers end in 1, 3, 7, or 9, the teacher should ask them for explanations. Some [students] mention examples that do not verify the conjecture; yet, the teacher should ask for more examples and have them discussing the reason why they are not prime [numbers]. Afterwards, [the teacher] could build on the fact that 9 is not prime since the conjecture said that all numbers ending in 9 were prime.

Carlos's answer suggests that he pays more attention to having students justifying why a given number is prime or composite, than to having students understanding the meaning of a conjecture, the relationship between reciprocal implications (at the level of 7th graders), and the role of examples and counter-examples in the proof or refutation of conjectures. It is not clear why Carlos suggests the teacher to treat number 9 in a *special* way. He gives no

importance to summarizing the ideas emerged from the whole-class discussion, despite being a crucial step to assign meaning to those ideas (Stein et al., 2008).

Rita's conjecture seems to have been interpreted in two ways. On one hand, Carlos refers that it may be read as "all prime numbers are all [the numbers] that end in 1, 3, 7, and 9, except 2 and 5"; on the other hand, Rita's conjecture may be read as "prime numbers except 2 and 5 end in 1, 3, 7, or 9". This latter interpretation corresponds to what Rita actually says; thus, Carlos's first interpretation is not grounded in the episode. Nonetheless, it is that first interpretation that Carlos considers as having been discussed with the class: "in this lesson, the only thing that was proved was that the first interpretation is not valid". Indeed, Carlos does not believe that Rita's conjecture was proved during the lesson:

The way Rita phrased the conjecture seems to indicate that all prime numbers are all odd numbers except those that end in 5. During the lesson, it became clear that this is not true since 21, 27, 33 are odd numbers ending in 1, 7, and 3, and they are not prime. Rita's conjecture was not proved because prime numbers except 2 and 5 end in 1, 3, 7, 9. What was proved was that the numbers ending in 1, 3, 7, and 9 are not always prime.

As we have seen, Carlos does not correctly interpret Rita's conjecture. In addition, he does not realize that the examples provided by the students (21, 27, and 33) are, indeed, examples that destroy the reciprocal of Rita's conjecture, not the conjecture itself. Thus, Carlos does not see those examples as counter-examples for the reciprocal of Rita's conjecture. Furthermore, it seems that he does not recognize a proof of Rita's conjecture in the teacher's and students' joint work. Unlike Júlio, Carlos shows gaps in his didactical knowledge, both at the mathematical and the instructional level (Ponte, 1999). In his analysis of the episode, he does not give importance to the mathematical discussions that occurred nor to the synthesis of the main ideas that were discussed, two aspects that are quite valued in current orientations for the teaching of mathematics (ME, 2007; NCTM, 2000; Stein et al., 2008). Furthermore, his misunderstanding of the situation described in the episode, in mathematical terms, may have hindered him from making adequate decisions regarding how to possibly continue the episode, i.e., from showing a deeper instructional knowledge. Indeed, as we point out next, a poor knowledge of mathematics seems to be associated to a weakened instructional knowledge.

Though having a short teaching experience, Joana is not a certified mathematics teacher, thus being enrolled in the same program as Júlio and Carlos. Joana's knowledge of mathematics seems to exhibit several weaknesses. For example, she states that

Rita's conjecture was proved and it was incorrect, since the students checked for a large array of numbers, even bigger than 100, thus establishing a degree of certainty in their answers and even finding numbers like, for example '21' which though ending in 1 is not prime since 3 divides 21.

In fact, Joana does not seem to understand the meaning of conjecture nor what it takes to prove (or refute) a conjecture. Though acknowledging the value of the testing procedure, she completely mixes the two reciprocal implications involved in the episode.

Since Joana does not identify Rita's conjecture nor its reciprocal, it is not surprising that her suggestions for continuing the episode make little sense:

The teacher should have let the students reach the conclusion that 'all numbers ending in 1, 3, 7, 9' are not prime and she should not have written on the board and telling the conclusion. Maybe saying the students should conclude or even writing only the sentence 'all numbers ending in 1, 3, 7, or 9 are prime; do you agree?' because, by saying 'So, see if it is true' she is implicitly telling the students that something is wrong.

Joana does not understand the teacher's intentions when writing on the board the two implications, seeing the teacher's actions as offering the students a clue for what would be correct or incorrect.

Both Joana and Carlos show weaknesses in their mathematical knowledge; however, Joana seems to have more problems in this regard than her colleague. Having completed a bachelor degree several years ago and not holding a major in mathematics, Joana has a different academic background from her colleagues', who majored in mathematics the year before enrolling in the master's degree. One could suggest that such differences in background could account for the differences in mathematical knowledge evidenced in the prospective teachers' analyses of the episode. However, Carlos, whose background and grade point average is similar to Júlio's, also shows gaps in his mathematical knowledge. In addition, Joana, unlike her colleagues, did have some teaching experience (besides other experiences in the job marked); nevertheless, her knowledge of instructional processes in the classroom emerges as much weaker than that of Júlio or Carlos.

Raising Issues for Teacher Education

The examples provided support the claim that, when one does not comprehend a mathematical situation, one can hardly make adequate instructional decisions (Kahan, Cooper, & Bethea, 2003). In particular, the orchestration of productive mathematical discussions and the systematization of (new) knowledge, two complex communicative actions, essential aspects of the teacher's role within the current Portuguese curricular orientations, cannot be adequately approached if the teacher's knowledge of the mathematics underneath the teaching situation is not sound (Martinho & Ponte, 2009; Ponte, in press; Tomás Ferreira et al., 2011). In other words, the teacher cannot make adequate instructional decisions (showing appropriate knowledge of classroom instructional processes) if s/he does not hold a vast, solid, and interrelated knowledge of mathematics.

The analysis of a classroom episode in a teacher education context as the one described above allows that prospective teachers understand the situations reported in the episode and search for information as many times as needed, since the episode can be read repeatedly. Very shortly, in their student teaching experience, these prospective teachers will be asked to do something much more demanding: in a classroom context, they will have to immediately comprehend what is at stake and weigh the various options to make adequate instructional decisions. The challenges involved are immense and we believe that the work around classroom episodes, focused on aspects of the teacher's didactical knowledge, will help these future teachers to face those challenges. However, we question ourselves about how the current structure of teacher education programs is helping prospective teachers developing a sound mathematical knowledge that may support the development of a strong didactical knowledge. Since all prospective teachers had a heavy load of mathematics courses in their bachelor degree (in which various techniques of mathematical proof were addressed), we would not expect so many difficulties in understanding basic notions, such as conjectures, examples and counter-examples, nor so many obstacles in comprehending a teaching situation, such as the one described in the episode. The data suggests prospective teachers do not develop adequate conceptions about mathematical proof (in their bachelor degree) which hinders the development of their didactical knowledge (in particular the dimension of instructional knowledge) when enrolled in a teacher certification program. As mathematics educators, how can we deal with this reality? What connections may be made with mathematics instructors so that basic mathematical notions do not constitute a barrier to developing instructional knowledge and taking the most of a teacher education program? What happens in other countries with similar higher education configurations?

References

- Bishop, A. J., & Goffree, F. (1986). Classroom organization and dynamics. In B. Christiansen, A. G. Howson, & M. Otte (Eds.), *Perspectives on mathematics education* (pp. 309-365). Dordrecht: Reidel.
- Boavida, A. (2001). Um olhar sobre o ensino da demonstração em Matemática. *Educação e Matemática*, 63, 11-15.
- Empson, S. B., & Jacobs, V. R. (2008). Learning to listen to children's mathematical thinking. In D. Tirosh & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Vol. 2, pp. 251-281). Rotterdam, the Netherlands: Sense Publishers.
- Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education*, 6(3), 223-252.
- Martinho, M. H., & Ponte, J. P. (2009). Communication in the classroom: Practice and reflection of a mathematics teacher. *Quaderni di Ricerca in Didattica (Matematica)*, Supplemento n.2 al n. 19.
- ME (2007). *Programa de matemática do ensino básico*. Lisboa, Portugal: DGIDC.
- National Council of Teachers of Mathematics. (1991). *Professional standards for school mathematics*. Reston, VA: NCTM.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- Ponte, J. P. (1999). Didáticas específicas e construção do conhecimento profissional. In J. Tavares, A. Pereira, A. Pedro, & H. Sá (Eds.), *Investigar e formar em educação – Proceedings of XI EIEM* (pp. 59-72). Porto, Portugal: SPCE.
- Ponte, J. P. (in press). Estudando o conhecimento e o desenvolvimento profissional do professor de matemática. In N. Planas (Ed.), *Educación matemática: Teoría, crítica y práctica*. Barcelona: Graó.
- Ponte, J. P., & Oliveira, H. (2002). Remar contra a maré: A construção do conhecimento e da identidade profissional na formação inicial. *Revista de Educação*, 11(2), 145-163.
- Ruthven, K., Hofmann, R., & Mercer, N. (2011). A dialogic approach to plenary problem synthesis. *Proceedings of the 35th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 81-88). Ankara, Turkey: PME.
- Stein, M. K., & Smith, M. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School*, 3(4), 268-275.
- Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, 10(4), 313-340.
- Tomás Ferreira, R. A. (2005). *Portuguese student teachers' evolving teaching modes: A modified teacher development experiment*. Doctoral dissertation. Illinois State University, USA.
- Tomás Ferreira, R. A., Martinho, M. H., Menezes, L. (2011). "Teachable Moments": Oportunidades a não perder. *Educação e Matemática*, 115, 39-45.

Interpreting Graphs: Students Developing an Understanding of Covariation

Noleine Fitzallen
University of Tasmania
<noleine.fitzallen@utas.edu.au>

Students' development of an understanding of covariation was the focus of a research project that investigated the way in which 12 Year 5/6 students engaged with the learning environment afforded by the graphing software, *TinkerPlots*. Using data generated from individual interviews, the results demonstrate that upper primary students use their understanding of covariation to draw conclusions about trends in data evidenced in the graphs they created as well as their knowledge of the context. Implications for the Australian Curriculum are considered.

Covariation is used to explore the relationship between two variables and is recognised as an important aspect of statistical reasoning. It is determined by examining graphs of bivariate data that are represented in scatterplots. Scatterplots are an economical way of organising large amounts of data that provide information about two variables that are not necessarily dependent on each other and show the correspondence of the ordination of each variable (Moritz, 2004, p. 228). They also assist in the identification of the relationship between two variables and variation from that relationship as well as being particularly useful for identifying clusters of points and outliers in a distribution of bivariate data.

The characteristics of graphs, such as the mode, scale of an axis, or the variation in the spread of the data can be extracted directly from a graph (Kosslyn, 1989) or from calculations performed by graphing software, such as *TinkerPlots: Dynamic Data Exploration* (Konold & Miller, 2005). Another factor that influences conclusions that are made about graphs is the context of the data (Watson, 2006). An understanding of the context and the nature of the variables of interest may be gained from personal experiences of the context, from information about the data, or from the details embedded in the characteristics of the graphs created from the data. Collectively, the elements of graphs are resources that provide a link between the two-dimensional representation of data and the context of real world measurement situations. Graphing software such as *TinkerPlots*, provides access to the characteristics of graphs and allows data analysis activities to hone in on the story graphs have to tell and focus on how graphs can be used by students to evidence their thinking about the data. To date, few studies have addressed the way in which students construct and evaluate graphs within technological learning environments (Shaughnessy, 2007).

Interpreting Covariation in Graphs

Cartesian graphs, such as scatterplots, depict covariation of two sets of measurements that vary along numerical scales in a two-dimensional space. Scatterplots, in particular, are characterised by data points that correspond to the measures of two variables designated at the same time. Each data point also corresponds to one unit of analysis between the two variables. The values of the two variables may be said to involve some form of relationship, association, function, dependency, or correspondence (Moritz, 2004). In this report, the emphasis for covariation is on the relationships between two variables, which are interpreted as general trends that show the variation of the two variables due to the ordination of the values along each axis of a graph with numerical scales.

Although the use of scatterplots to determine a trend in data has been explored at the middle school level, few studies have focused on primary students. One exception is a study of Year 3, 5, 7, and 9 students that looked specifically at speculative data generation that involved students translating a scatterplot into a verbal statement, interpreting numerical data, reading values from a graph, and interpolating data (Moritz, 2004). Moritz found that students tended to focus on individual data points rather than look globally for the trend in the data and often only considered one of the variables in isolation. Ben-Zvi and Arcavi (2001) also found that Year 7 students focused on particular points of a graph. They noticed, however, that this did not always constrain the students and “served as a basis upon which the students started to see globally” (p. 62).

In 2003, Cobb, McClain, and Gravemeijer set up a classroom experiment that explored 29 Grade 8 students’ understanding of bivariate data, with a particular emphasis on statistical covariation. The part of the investigation that explored the way in which the students used scatterplots to analyse data found that focusing on the density and the shape of data was crucial for the students to interpret scatterplots. They also noted that students “typically reduced scatterplots to lines that signified fixed relationships of covariation rather than conjectured relationships about which the data were distributed” (p. 75).

The Study

The study, from which this paper was taken, employed an educational design research methodology (Akker, Gravemeijer, McKenney, & Nieveen, 2006) within a pragmatist paradigm (Mackenzie & Knipe, 2006). This paradigm was adopted as it positioned the study as being scientifically-based research that could potentially inform and develop evidence-based practice in education. It facilitated the development of a systematic study, whose purpose was to capture the way in which *TinkerPlots* contributed to, supported, and influenced the students’ development of understanding of covariation as they created and interpreted graphs.

The study involved the development of a learning sequence and its subsequent implementation with 12 Year 5/6 students. Over a period of six weeks (45 minute sessions, twice a week), the students worked in pairs to complete a series of assigned tasks that included collecting data, constructing a range of graphs, comparing data sets of varying size, and identifying trends and outliers using *TinkerPlots*. At the end of the learning sequence, each student was interviewed individually using an interview protocol set up in *TinkerPlots*. The interview protocol required the students to explore a data set by constructing graphs and making conclusions and informal inferences about the data seen in the graphs. As the students worked at the computer, they were asked questions that required them to describe their graphs and explain how they made their conclusions about the data from their graphs. The interview protocol was designed to be a culminating performance of understanding. As such, it was open-ended and required the students to draw on and demonstrate their skills and understandings developed throughout the sequence of learning experiences. The data set used in the interview protocol was extracted from the Census@Schools data base (www.abs.gov.au). It was a random sample of 200 Year 5/6 students from across Australia generated from the Census@Schools website. The attributes in the data set were: gender, foot length, height, and belly button height.

As the students worked through the interview protocol with *TinkerPlots*, the authoring software package with an on-screen recording utility, *Adobe Captivate 3* (Adobe Systems Incorporated., 2007), was used to create a video of the students’ activity as they worked on the computer. The *Captivate* files generated showed the screen of the computer as seen from the user’s perspective. The files recorded all movements of the cursor as the students

worked at the computer and provided a chronological record of how each student accessed the features of *TinkerPlots*. *Captivate* also provided audio data of the conversations the students had with the researcher during the interview. The audio data from the files provided evidence of what the students said and the on-screen video data provided evidence of what the students did.

The audio data from the interview videos generated by *Captivate* were transcribed verbatim. The audio transcripts were then synchronised with the on-screen capture video data generated by *Captivate* and descriptors of the students' actions at the computer were added to the transcripts. In the first part of the analysis process, the data excerpts from the audio transcripts and the descriptors of the students' actions were coded according to the four key elements of the *Graphing in EDA Software Environments* framework (Figure 1). After this initial categorisation, the data that were coded for the key elements Understanding Data and Thinking about Data were analysed to determine the way in which the students identified, explained, and interpreted trends and relationships between two attributes in graphs. The aim was to determine the ways in which students develop an understanding of covariation. This paper reports on the results from the second part of the data analysis process.

Categories		Key Elements
<i>Generic Knowledge</i> Speaking the language of data and graphs. Recognising the characteristics of data and graphs. Understanding how to use the features of software and technology	<i>Being creative with data</i>	Reducing data to graphical representations. Summarising data. Constructing different forms of graphs. Describing data from graphs.
	<i>Understanding data</i>	Making sense of data and graphs. Understanding the relationship among tables, graphs, and data. Identifying the messages from the data. Answering questions about the data. Recognising appropriate use of different forms of graphs.
	<i>Thinking about data</i>	Asking questions about the data. Recognising the limitations of the data. Interpreting data. Making causal inferences based on the data. Looking for possible causes of variation. Looking for relationships among variables in the data.

Figure 1. Framework for Graphing in EDA Software Environments (Fitzallen, 2006, p. 206).

Analysis

Kosslyn (1989) contends that to analyse a graph it is necessary to understand the interrelated connections among the constituents of a graph. He asserts that understanding the interrelated connections fosters the interpretation of graphs on three levels. First, the individual elements and their organisation can be described. Second, understanding of the display can be determined by looking at the relationship between the elements of the graph. Lastly, the analysis can extend to the interpretation of the symbols and lines that goes beyond the literal reading of the information. Influenced by the work of Kosslyn, Curcio (1989) considered school students' interpretation of graphs from three perspectives: reading data directly, reading "between" the data, and reading "beyond" the data. These phrases reflect to some extent the increasing demands of the levels suggested by Kosslyn (1989). In 2007, Shaughnessy added to the work of Curcio by suggesting the further need to read "behind" the data. The additional dimension suggested by Shaughnessy (2007) together

with the reading “beyond” the data perspective offered by Curcio, reflects the complexity of graph interpretation at the third level suggested by Kosslyn.

The three levels of graph interpretation highlighted by Kosslyn (1989) and others were used to devise a framework for analysing the students’ responses about covariation and the relationship seen in the data from the graphs they created. The framework devised was based on the Structure of Observed Learning Outcomes (SOLO) model developed by Biggs and Collis (1982). It has three levels that describe the increasing complexity of thinking required when interpreting graphs. At the first level, termed uni-structural, responses employ single elements to describe the relationship seen in the data. These are isolated statements that are not linked together to draw conclusions. This aligns with level 1 of the Kosslyn framework. At the second level, termed multi-structural, responses employ multiple elements to describe the relationship seen in the data. These are offered in the same excerpt of conversation as a string of ideas that are supporting evidence about the relationship seen but are not presented in an interrelated fashion. This aligns with the second level of the Kosslyn framework. At the third level, relational, responses employ multiple elements to describe and justify the relationships seen in the data and make explicit how the elements are interrelated, including elements beyond the literal reading of the data, such as the context. This aligns with the third level of the Kosslyn framework.

Results: Students’ Understanding of Covariation

The students’ statements, descriptions, and justifications about the relationship seen in a graph made during the completion of the interview protocol were analysed to determine the complexity of the responses according the SOLO framework devised reflecting Kosslyn (1989). Of the 12 students interviewed, six students’ responses were uni-structural, three students’ responses were multi-structural, and three students’ responses were relational. Each of the students’ response at each level is described in turn.

Uni-structural Thinking

The results from this study suggest that upper primary students begin with an intuitive appreciation of covariation that is based on their knowledge of and experience with the context of the data. When the students in the study who were operating at this uni-structural level were asked to describe how a relationship identified was evidenced in the graphs they constructed, they were able to offer information from the graphs to support their statements but often their justifications and examples were offered piece-meal without making any connections between the different constituent parts of the graphs and without making connections between the graphs and their knowledge of the context. The students could also identify a trend in a graph and use it to determine the relationship between two attributes but their explanations about how they used the trend to determine the relationship were incomplete. Often their statements involved a declaration that there was or was not a trend evident but little or no justification or reasoning was offered to explain how they made the judgement.

The use of isolated statements was evident when Jake suggested “Umm ... The difference between the tallest, the highest one and the mode is ... 17, I think” as he extracted information from a graph. When making this statement he did not make any connections between this information and the generalised conclusion he made previously for the same graph, “Hmm ... the ... the foot’s not really getting any bigger. It’s staying kind of the same.” Most of the time, his judgements were based on the comparison of individual

values and ranges of data in a graph and he scrutinised the two variables in a scatterplot separately.

Natasha based her judgements about the relationship between two attributes on her knowledge of the context. For each of the covariation graphs she created, she made generalised statements like, “Yeah. ‘Cos the taller you are, like ...there’s usually got like a higher... higher belly button height, if that makes sense?” She was also able to make conjectures about the relationship but was limited to considering one data point at a time and did not make any collective statements to suggest her thinking was shifting beyond being uni-structural. Rory also worked in a similar manner, reporting, “Yeah that’s the height and, well the big people have the big foot, sort of.” Like Natasha, the evidence he offered to support his statement focused on the value of specific data points.

Johnty’s statements about the covariation graphs were uni-structural and for the most part, relied on the comparison of individual data values or were based on the physical location of the data in the graphs. This was evident when he said a relationship existed between the attributes belly button height and height because the dots in the graph were “going up.” He also relied on his knowledge of the context to make his conjectures when he said “The taller you are, usually, the higher your belly button is,” without referring to the graph or the data. In contrast, Kimberley made multiple statements about the graphs that showed she could extract a variety of information from the graphs. This showed she had an appreciation that there was a relationship between the two attributes in a graph but her isolated statements about the data were presented as a list of individual characteristics without connections being made between the statements about the trend or the multiple aspects identified. Her statements included, “Mmm ... well the higher the person is the longer the foot length ... seems” and “this one here’s the highest, and with about 185 but he’s only got a length of 25 cm for the foot.” Kimberley did not attempt to combine her statements to make conclusions.

Like the other students’ responses at the uni-structural level, Natalie identified appropriately from her knowledge of the context that “the smaller you are the smaller your foot length is.” There were, however, instances when Natalie’s responses were beginning to merge into multi-structural. This occurred when she was making a decision about an outlier. She said, “Umm, it’s an outlier because of, like, you wouldn’t think it would be 85 because she’s 124 and I’m taller than her and mine is only 65.” Natalie did not, however, make enough statements at the multi-structural level for her thinking to be considered multi-structural.

Multi-structural Thinking

The students operating at the next level of understanding, multi-structural used multiple constituent parts of their graphs to explain their thinking and to describe the covariation identified. Their responses, however, did not show they had an appreciation of the way in which different elements of the graphs were related and how they could be used to support conjectures and conclusions. They were able to identify regions of the graphs that varied from the trend identified but were unable to explain how the variation informed their overall conclusions. Blaire, for example, when asked about the relationship seen in a covariation graph displaying the attributes belly button height and height, moved immediately to say, “Ok, as your ... of course you would already know that as you get taller your belly button height would get higher.” She then went on to declare that the mean belly button height would be in the middle of a cluster of data in the graph. Although adding the mean to the graph using the function in TinkerPlots corroborated her assumption about the mean, she

could not go further to use the mean to justify her generalised statements about the relationship shown in the graph.

Jessica's also used her knowledge of the context as a starting point for evaluating her graphs, "That yeah, as you grow, your foot length usually grows too." When asked to describe how the graph showed this she went on to say, "Well like there ... the largest person, their foot length is only about 25 cm whereas this person here, they have the largest foot length but they're not the tallest." Jessica was making sense of the data by considering different aspects of the graph but she did not tie the information together successfully to make responses that were relational. Like Blaire and Jessica, Shaun identified variation within the graphs that did not fit with his generalised conclusions but he was unable to make confident statements about the relationship in the graphs. In one exchange with the teacher, Shaun commented on the relationship evident in a graph by saying, "the clump [in the graph] shows most of the people, the higher the foot they have, the higher the belly button to the floor." He also identified specific data points that were outside the main clump and looked at the values of the attributes for those data points. He did not, however, indicate how the additional information influenced his generalised statement.

Relational Thinking

The students at the relational level had developed more sophisticated ways of describing the relationship between two attributes. They did this from both global and local perspectives. They were able to identify a trend in a graph, identify the variation in the graph that did not meet their expectations for a relationship to exist, and use multiple constituent parts of graphs to support their reasoning. They interlaced their explanations with information about specific data points, the variation, and the trend evident, as well as making connections between the information in the graphs and their knowledge of the context to justify their conclusions. James put these ideas together successfully when he stated "that well, like ... when they were getting taller their belly button height was getting bigger." He then went on to explain details in the graph, "And like from around here. I think them ones are like taller, but their belly buttons are only 50 and 60. They'd have to be very short, like to have a belly button that small. Like these ones are all what you'd expect from like, that age. Whereas, like these ones out here ... like aren't."

Mitchell was also able to relate different pieces of information to each other to make his conclusions. This was evident when he said "Yeah I'd say, well, yeah, there's a couple of people that are off it a bit more [referring to the trend identified in the previous graph] but it's pretty much the same as foot length and height. The taller your belly button is the higher you are." Mitchell's ability to make connections to other graphs displaying different attributes supports further that his thinking was relational. William's explanations started with descriptions of individual data points to support his statements about the relationship between two attributes before he moved on to thinking about the data from a broader perspective. For example, he said, "It tells us that that one's the biggest now and it's also the biggest foot length." As he worked, he extended his explanation to include descriptions of regions of data that he considered to be following a trend and then combined this with descriptions of particular data points within the graph that did not fit the trend.

Discussion and Implications for the Curriculum

It is generally assumed that young children's ability to evaluate relationships in data and reflect on supporting evidence is undeveloped (Moritz, 2004). In his study, Moritz found otherwise. The evidence in this study suggests that students in the upper primary years are

able to evaluate the relationship between two attributes and develop an understanding of covariation. At the beginning level of understanding, they rely on their knowledge of the context of the data or individual elements of graphs to make their conclusions. At the next level they start to consider multiple characteristics of graphs but do not always elaborate on the connections among the characteristics or between the graphs and their knowledge of the context. This increases to having the ability to bring together the elements of the graphs and their knowledge of the context effectively at the highest level. The students operating at the highest level of the thinking for covariation were able to make the connections among the constituent parts of the graph, the context of the data, and the message in the data characterised by the trend. These students were able to express their thinking verbally and talked about their graphs from both generalised and local perspectives. It is therefore, imperative that teaching and learning activities provide the opportunity for students to develop strategies for moving back and forth between thinking about what can be seen in the data from the generalised perspective of expressing the relationship between two attributes as the trend seen in the data to the more localised perspective of using the spread of a cluster of data or outliers in a graph to describe how the thinking about the relationship was formulated. This study demonstrated how *TinkerPlots* provided a flexible learning environment that fostered and supported the development of such flexible thinking strategies. Considering that students in this study are able to describe covariation and generalise about a trend evident in the data it may be pertinent to bring the initial introduction of scatterplots into the upper primary years of schooling. This would provide students with the opportunity to establish the notion of covariation, how it is characterised in graphs, and how to reason about covariation well before the introduction of the statistical analysis of correlation and the translation of a trend into an algebraic expression in Year 10.

Development of covariation skills and the use of scatterplots have the potential to facilitate learning outcomes in the Australian Curriculum (ACARA, 2011). In History, for example, when looking at change over time and when exploring the notion of cause and consequence. Although the History curriculum is not explicit about the interpretation of graphs in their outcome statements, it would be beneficial for the History curriculum to elaborate on where graph creation and graph interpretation skills can contribute to the development of outcomes for the History Inquiry Skills strand. Taking into account that the Year 5/6 students in this study demonstrated the ability to describe and generalise about the relationship between two variables, the use of scatterplots to interpret census data to construct arguments about immigration in Year 6, say, would be beneficial. This would lay the foundation for being able to identify trends and draw conclusions about historical events over time, as required in Year 9 and Year 10.

The Science curriculum recommends that graphs are used to make conclusions, interpret data and make inferences; however, the application of graphs for these purposes in the Science curriculum does not align with when the fundamentals of graph creation and graph interpretation are introduced in the Mathematics curriculum. The Science curriculum suggests that students in Year 2 use column graphs yet they do not appear in the Mathematics curriculum until Year 3. Another issue arises in Year 6 when students are expected to compare data and make predictions as well as interpret trends in data. Bearing in mind the Mathematics curriculum does not address the development of the graphing skills required in Science until the secondary years of schooling, it would be worth rethinking where to place the learning about graphing and data analysis in the Mathematics curriculum and its application in other curricula, such as History and Science. Aligning complementary curricula would allow the Mathematics curriculum to coincide with the demands of other curricula, thereby fostering the connections between different learning areas and addressing

the need for all curriculum areas to embrace the cross-curriculum priorities, detailed in the Australian Curriculum (ACARA, 2011).

Acknowledgements

This research is funded by an APA(I) Scholarship associated with an ARC Linkage Project LP0560543 and the industry partner, Department of Education, Tasmania. Thanks to Prof Jane Watson for reading the initial draft and providing valuable feedback.

References

- Adobe Systems Incorporated. (2007). *Adobe Captivate 3*. [Computer software] San Jose, CA: Adobe Systems Incorporated.
- Akker, J., Gravemeijer, K., McKenney, S., & Nieveen, N. (Eds.) (2006). *Educational design research*. New York: Routledge.
- Australian Curriculum, Assessment and Reporting Authority (2011). *The Australian Curriculum*. Retrieved from <http://www.australiancurriculum.edu.au>
- Ben-Zvi, D., & Arcavi, A. (2001). Junior high school students' construction of global views of data and data representations. *Educational Studies in Mathematics*, 45, 35-65.
- Biggs, J. B., & Collis, K. F. (1982). *Evaluating the quality of learning. The SOLO taxonomy*. New York: Academic Press.
- Cobb, P., McClain, K., & Gravemeijer, K. (2003). Learning about statistical covariation. *Cognition and Instruction*, 21(1), 1-78.
- Curcio, F. R. (1989). *Developing graph comprehension*. Reston, VA: National Council of Teachers of Mathematics.
- Fitzallen, N. (2006). A model of students' statistical thinking and reasoning about graphs in an ICT environment. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces* (Proceedings of 29th annual conference of the Mathematics Education Research Group of Australasia, Canberra, pp. 203-210). Sydney: MERGA.
- Konold, C., & Miller, C. D. (2005). *Tinkerplots: Dynamic data exploration*. [Computer software] Emeryville, CA: Key Curriculum Press.
- Kosslyn, S. M. (1989). Understanding charts and graphs. *Applied Cognitive Psychology*, 3, 185-226.
- Mackenzie, N. & Knipe, S. (2006). Research dilemmas: Paradigms, methods and methodology. *Issues In Educational Research*, 16(2), 193-205. Retrieved from <http://www.iier.org.au/iier16/mackenzie.html>
- Moritz, J. (2004). Reasoning about covariation. In D. Ben-Zvi, & J. Garfield, (Eds.), *The challenge of developing statistical literacy, reasoning, and thinking* (pp. 227-255). Dordrecht, Netherlands: Kluwer Academic Publishers.
- Shaughnessy, J. M. (2007). Research on statistics learning and reasoning. In F. K. Lester, Jr. (Ed.), *Second handbook on research on mathematics teaching and learning* (pp. 957-1009). Charlotte, NC: Information Age Publishing.
- Watson, J. M. (2006). *Statistical literacy at school: Growth and goals*. Mahwah, NJ: Lawrence Erlbaum.

Young Pedestrians' Gendering of Mathematics: Australia and Spain

Helen Forgasz
Monash University
<Helen.Forgasz@monash.edu>

Gilah Leder
Monash University
<Gilah.Leder@monash.edu>

Inés M^a Gómez-Chacón
Complutense University
<igomezchacon@mat.ucm.es>

People aged 20-39 were stopped in the streets of Victoria (Australia) and Madrid (Spain) to gauge their views on the gendering of mathematics. The findings suggested that for respondents from both countries, if stereotyped beliefs are held they were more strongly associated with the traditional male stereotype, that is, that males are considered more suited to pursuits in mathematics. However, in general, the Spanish respondents held stronger views than the Australians that mathematics was gender neutral, that is, that it, and related fields, are equally suited to males and females.

Introduction

A glance at recent headlines in the media as well as various websites reveals that gender issues and stereotyping remain topical and in the public eye (e.g., Goldstein, 2012; Topsfield, 2011). The makers of Lego, for example, have pitched their new “Friends” range to girls, the other half of the population (Smith, 2011). Mattel have also developed a “Computer Engineer Barbie doll” (Barbie Doll as Computer Engineer (Geek), n.d.) aimed at encouraging girls to consider this career option.

In the USA, it has been argued that “[D]emocracies are built on public opinion” (The power of public opinion, 2008). For mathematics education, the *Maths multiplies your options* advertising campaign run in Victoria, Australia in the 1980s was aimed at raising parents’ awareness of the importance of mathematics for their daughters. Its success highlighted the positive impact of public exposure on a perturbing, inequitable issue (McAnalley, 1991). Yet in the mathematics education research community scant attention has been paid to the views of the general public on issues associated with mathematics learning. With respect to gender differences in mathematics learning outcomes, explanatory models for these differences typically encompass, among a range of contributing factors, attitudes and beliefs of critical others and societal expectations.

In this paper we build on previously reported findings from surveys of the general public on the gendering of mathematics conducted by Leder and Forgasz (2010, 2011). Data from an expanded Australian sample as well as data gathered in the streets of Madrid, Spain are included. The focus is on findings from participants aged 20-39. This age group formed the largest sub-sample of all participants in both countries, and is of particular interest as they are representative of both Generations X (born 1965 to 1979) and Y (born 1980 to 1994) (McCrindle Research, n.d.), span those who are early in their working lives and those reaching their mid-career, and represent those completing the Organisation for Economic Co-operation and Development’s [OECD] Programme for International Student Assessment [PISA] testing programs in 2000, 2003, and 2006 (but not 2009). In earlier work, Leder and Forgasz (2011) reported that those in the younger age group (under 40) of their sample appeared to hold more traditional gender stereotyped views on mathematics than those aged over 40.

Background Contexts

A sample from Spain, a non-English speaking, but Western country, was included in the present study for several reasons. When PISA results are examined, Australian students' mean scores are typically above the OECD average, while for Spanish students they are typically below the OECD average. Yet gender differences in favour of males are found in both countries. A summary of the mean PISA (2000, 2003, 2006, and 2009) scores for mathematical literacy in both countries is found in Table 1.

Table 1

PISA Results in Mathematical Literacy for Australia and Spain, 2000, 2003, 2006, & 2009

Year	Mean scores: OECD (Participating countries)	Mean scores and stat. sig. gender diffs.: Australia				Mean scores and stat. sig. gender diffs.: Spain			
		All	F	M	Sig.	All	F	M	Sig.
2000	500 (32)	533	527	539	No	479	469	487	Yes
2003	500 (41)	524	522	527	No	485	481	490	Yes
2006	498 (57)	520	513	527	Yes	480	476	484	Yes
2009	493 (65)	514	509	519	Yes	483	474	493	Yes

Australia's legislative history on gender equity dates back to the 1970s. In 1972, it was agreed that the concept of equal pay would be fully implemented by June 30, 1975; Elizabeth Reid's appointment as adviser to the Prime Minister on women's issues in 1973 was the first position of this kind internationally; and the Federal Parliament passed the Sex Discrimination Act in 1984 (Equal Opportunity for Women in the Workplace Agency, n.d.). Similarly, in Spain, gender equity legislation stems from the 1970s when laws were enacted to recognise equal rights and achieve a gender equitable education system to phase out the female social stereotype. The Equal Treatment Directive (Directive 76/207/CEE of Conseil, 9/2/1976) established the principle of equality of treatment for men and women for access to employment, vocational training, promotion, and working conditions. Since 1985, one particular focus has been on equality of educational opportunity. More recently, Spain increased its commitment to gender equity. New legislation was introduced in 2004 and 2008 to promote principles of gender equity in the spheres of private and public life, and to combat gender-based violence (see, for example, <http://www.eurofound.europa.eu/eiro/2006/02/feature/es0602106f.htm>). With respect to schooling, milestone legislation was passed in 2006 to enshrine the concept of co-education into educational policies. This is seen as an important step towards achieving a school model based on gender equity to challenge discrimination, gender stereotypes, and gender hierarchies.

As mathematics is a formal prerequisite for many post school educational and employment options, a high proportion of Australian students study mathematics in their final year of schooling. Ainley, Kos, and Nicholas (2008) estimated that at least 80% of students do so; the situation in Spain is similar in this regard.

With respect to women in tertiary education and in employment, there are remarkable similarities for Australia and Spain. In Australia in 2006, women comprised 54.8% of all tertiary students, and 58% of all Australian women were in the workforce in January, 2008 (Department of Foreign Affairs, n.d.). In 2005-6 in Spain, women reached 54.3% of university enrolments. At 51.9%, women's workforce participation rates have risen sharply in recent years (Cacace, 2009; OECD, 2007). However, in Australia and in Spain males predominate in tertiary enrolments in mathematics dependent courses, such as engineering

and related technologies, and information technology (Department of FaHCSIA, n.d.; Sanchez de Madariaga et al., 2011; UMYC-Unidad de Mujeres y Ciencia, 2008). In Spain, the percentage of women enrolled in Science and Mathematics is 35.1% and in Engineering, 30.4% compared to 76.4 % enrolled in Education (Sanchez de Madariaga et al., 2011).

Theoretical Models Informing the Study

As noted above, many of the early explanatory models for gender differences in mathematics learning outcomes included societal influences and the views of significant others among the contributing factors. There were common elements in these models:

... the emphasis on the social environment, the influence of other significant people in that environment, students' reactions to the cultural and more immediate context in which learning takes place, the cultural and personal values placed on that learning and the inclusion of learner-related affective, as well as cognitive, variables. (Leder, 1992, p. 609)

In addition, when Weiner's (1974) attribution theory for success and failure was applied to mathematics, gender differences were widely reported (see Leder, 1992). More recently, Halpern et al. (2007) concluded that reasons for the overlap and differences in the performance of males and females were multifaceted but that "[e]arly experience, biological constraints, educational policy, and cultural context" (p. 41) might all be involved. Some years ago, Baker and Jones (1993) argued that females may consider mathematics less important for their futures, being "told so in a number of ways by teachers, parents, and friends" (p. 92).

Wigfield and Eccles' (2000) model of achievement motivation, and implicitly of academic success, highlighted the influence that learner-related variables as well as the attitudes of "critical others" at home, at school, and in society at large had on students' learning and behaviour. An early version of this model was described by Eccles et al. (1985), and it continues to be widely used to explain gender differences in achievement and career choice with respect to the STEM fields – science, technology, engineering, and mathematics. The elements of the model also continue to be refined (e.g., Eccles, 2005).

The items included in the administered survey instrument are consistent with the elements of the explanatory models about individuals' social milieus.

The Study

Aims and Methods

The aim of this study was to determine whether 20-39 year olds in the general public hold gender stereotyped views of mathematics and its relevance to career suitability. We were also interested to know if Australians and Spaniards held similar or different views.

In Australia, data were gathered at 12 heavy foot-traffic sites within metropolitan Melbourne, Victorian regional centres, and smaller rural cities. People were stopped in the street and were invited to participate. One morning or afternoon (about four hours) was spent at each location. About 50 surveys were completed at each site. In Spain, participants were drawn from nine sites in the northwest, south, and central areas of Madrid. The percentage of respondents in the northwest was higher than in the other areas. Data collection was conducted one day a week for a two month period; a morning of about three hours was spent at each location.

The Instrument

Leder and Forgasz (2010) described the instrument used in Victoria as follows:

To ensure maximum cooperation from those we stopped in the street, we limited our survey to 15 questions. In addition, however, we asked details about age (under 20, between 20 and 39, between 40 and 59, and over 60), and noted whether the respondents were male or female. Further, as well as the readily code-able responses such as “yes”, “no”, “don’t know”, “boys”, “girls”, “the same”, respondents were encouraged to elaborate and explain the reason for their answer; the comments were manually recorded by those administering the public survey. (p. 331)

The instrument was translated into Spanish for administration in the streets of Madrid. Fourteen of the original items were retained. Question 1 (Q1) in each version of the instrument was contextually bound. In Victoria, we asked: Did you see the recent *You can do maths* campaign on TV? A similar contextually bound question was asked in Spain. The 14 questions, common to the English and Spanish versions, were:

- Q2. When you were at school, did you like learning mathematics?
- Q3. Were you good at mathematics?
- Q4. Has the teaching of mathematics changed since you were at school?
- Q5. Should students study mathematics when it is no longer compulsory?
- Q6. Who are better at mathematics, girls or boys?
- Q7. Do you think this has changed over time?
- Q8. Who do parents believe are better at mathematics, girls or boys?
- Q9. Who do teachers believe are better at mathematics, girls or boys?
- Q10. Do you think studying mathematics is important for getting a job?
- Q11. Is it more important for girls or boys to study mathematics?
- Q12. Who are better at using calculators, girls or boys?
- Q13. Who are better at using computers, girls or boys?
- Q14. Who are more suited to being scientists, girls or boys?
- Q15. Who are more suited to working in the computer industry, girls or boys?

In this paper we report findings for all these items except Q4 and Q7.

Samples

As noted above, findings are reported in this paper from those aged 20-39 who completed the survey. The Victorian sample size was 223 (M=109; F=114); there were 387 (M=163; F=224) in the Spanish sample.

Analyses

The frequency distributions of the responses to the items were examined to determine the views of the Victorian and Spanish participants. Pearson chi-square tests were conducted to identify differences in the responses of the participants from the two countries. Effect sizes (ϕ) for the statistically significant differences were also calculated.

Results and Discussion

Results are reported under two main headings, personal and gender-stereotyped beliefs.

Personal Beliefs

One group of items analysed only dealt with personal beliefs about mathematics. Questions 2 and 3 required participants to respond ‘Yes’, ‘No’, or ‘Average’; the responses

to questions 5 and 10 were ‘Yes’, ‘No’, or ‘Don’t know/Depends’. The results revealed that the majority of Australian and Spanish samples:

- liked mathematics when they were at school (Q2: 57.4% Aus [Australians], 73.7% Span [Spaniards]; $p < .001$, $\phi = .21$)
- believed they were good at mathematics (Q3: 54.7% Aus, 71% Span; $p < .001$, $\phi = .22$);
- believed that students should continue learning mathematics when it is no longer compulsory (Q5: 63.7% Aus, 56.9% Span; ns); and
- believed that studying mathematics was important for getting a job (Q10: 70.9% Aus, 56.9% Span; $p < .01$, $\phi = .14$)

The response distributions for Q2, Q3, and Q10 were significantly different by country, with higher proportions of Spaniards than Australians positive about liking mathematics (Q2) and believing they were good at it (Q3), and a higher proportion of Australians than Spaniards believing that mathematics was important for getting a job (Q10).

Reasons given by participants for their responses to the last two of these items provide insights into why higher proportions of Australians than Spaniards agreed with them. The responses revealed that the Australian sample appeared to place greater emphasis on *the utility of mathematics for everyday life and career* than did the Spanish respondents. Typical examples of the Australian respondents were in relation to two main themes:

1. Mathematics is important/needed for everything:
 - Good discipline - helps with problem solving
 - Every time one needs mathematical skills. Makes the brains faster
 - Vital for lots of stuff - wish I had done it
2. Mathematics is needed in life (e.g., understanding money), for work; some added that it depended on the job while some claimed it is needed in all jobs.
 - I took maths when it was not compulsory but I hated it. Now I need maths. I’m in a management role in hospitality. I need graphs and maths all the time
 - Maths is part of every profession in some way
 - Maths is such a fundamental part of life / relevant to everyday life
 - Must do maths, encompassed in financial literacy

With respect to the first two questions, the participants’ explanations were not helpful in understanding why more Spaniards than Australians liked mathematics or believed they were good at it. Taking into consideration that Australia outperformed Spain in PISA and TIMSS, a possible explanation may be related to the differences in the level of mathematical challenge and content coverage experienced in mathematics classrooms.

Gender-stereotyped Beliefs

The items associated with gender stereotyping are listed in Table 2, together with the response frequencies, Pearson chi-square (by country) results, and effect sizes (ϕ).

The results in Table 2 reveal two fairly consistent patterns.

1. If participants in either country held gender stereotyped beliefs (ie. responded ‘girls’ or ‘boys’), almost invariably the direction of their beliefs was stronger about ‘boys’ than about ‘girls’. For example, in response to Q12 (Who are better at using calculators?), 6.3% of Australians and 5.1% of the Spaniards said ‘girls’, while 14.8% of Australians

and 9.9% of the Spaniards said ‘boys’. The two exceptions to this finding were for Q2 and Q15; slightly fewer Spaniards said ‘boys’ than said ‘girls’.

That it was only among the Spaniards that more people respond ‘girls’ than ‘boys’ is consistent with the second common pattern identified, described next.

2. With one exception (Q11), higher proportions of Spaniards than Australians provided the response, ‘same’ (ie. that there was no difference between girls and boys).

Table 2

Frequency Distributions and Pearson Chi-square Results (by county) for Questions Tapping Gender Stereotyped Beliefs

Question	Responses	Australia (n, %)	Spain (n, %)	Pearson χ^2 , p-level, & ϕ
6. Who are better at maths, girls or boys?	Girls	25 (11.2%)	53 (13.5%)	43.29
	Boys	58 (26.0%)	47 (12.0%)	p<.001
	Same	103 (46.2%)	268 (68.2%)	.27
	Don’t know	37 (16.6%)	25 (6.4%)	
8. Who do parents believe are better at maths, girls or boys?	Girls	32 (14.3%)	31 (7.9%)	32.18
	Boys	52 (23.3%)	42(10.7%)	p<.001
	Same	61 (27.4%)	117 (45.3%)	.23
	Don’t know	78 (35.0%)	141 (36.1%)	
9. Who do teachers believe are better at maths, girls or boys?	Girls	30 (13.5%)	42 (10.7%)	36.48
	Boys	30 (13.5%)	51 (13.0%)	p<.001
	Same	76 (34.1%)	222 (56.8%)	.24
	Don’t know	87 (39%)	76 (19.4%)	
11. Is it more important for girls or boys to study maths?	Girls	4 (1.8%)	5 (1.3%)	10.59
	Boys	9 (4.0%)	5 (1.3%)	p<.05
	Same	204 (91.5%)	352 (90.0%)	.13
	Don’t know	6 (2.7%)	29 (7.4%)	
12. Who are better at using calculators, girls or boys?	Girls	14 (6.3%)	20 (5.1%)	ns
	Boys	33 (14.8%)	39 (9.9%)	
	Same	144 (64.6%)	290 (73.8%)	
	Don’t know	32 (14.3%)	44 (11.2%)	
13. Who are better at using computers, girls or boys?	Girls	11 (4.9%)	3 (0.8%)	12.83
	Boys	74 (33.2%)	123 (31.3%)	p<.01
	Same	120 (53.8%)	241 (61.3%)	.14
	Don’t know	18 (8.1%)	26 (6.6%)	
14. Who are more suited to being scientists, girls or boys?	Girls	18 (8.1%)	33 (8.4%)	39.71
	Boys	46 (20.6%)	18 (4.6%)	p<.001
	Same	146 (65.5%)	311 (79.1%)	.25
	Don’t know	13 (5.8%)	31 (7.9%)	
15. Who are more suited to working in the computer industry, girls or boys?	Girls	7 (3.1%)	5 (1.3%)	38.11
	Boys	85 (38.1%)	67 (17.1%)	p<.001
	Same	121 (54.3%)	295 (73.5%)	.25
	Don’t know	10 (4.5%)	25 (6.4%)	

Seven of the eight items shown in Table 2 had statistically significant differences in response frequencies by country. Effect sizes for these items were all small (ϕ between 0.1 and 0.3). The differences in the response frequencies among the Australians and the Spaniards for all of these items generally reveal that the Spanish participants were more likely than the Australians to respond “same”, and that higher proportions of Australians than Spanish participants responded ‘boys’ to the item.

The quantitative findings reveal that, compared to the Spaniards, 20-39 year old Australians generally held stronger traditionally gender-stereotyped beliefs about mathematics learning (ie. that males are considered better than females at mathematics and more suited to careers in mathematics-related fields). Yet in Spain, as in Australia, males continue to dominate in tertiary mathematics studies, and hence in areas requiring such studies as a prerequisite. The media campaign waged in Australia in the 1980s saw a heightened awareness of the importance of mathematics and an increase in the number of females taking mathematics. The recent emphasis in Spain, in 2008, to promote gender equity in all spheres of society may similarly be encouraging a *perception* that females, too, can do mathematics – perceived by many of the respondents as an easy subject and in which they also claimed to have performed well.

In both groups, not all of those who selected either boys or girls for the items shown in Table 2 gave reasons for their answers. The pool of comments was, arguably, too limited to explain with confidence why this is so. Explanations of those who did comment included:

1. Social factors:
 - Girls – from my experience
 - Girls use maths in more practical ways, e.g., shopping
 - Boys – in my experience
 - People have higher expectations of boys
2. Effort
 - Girls work harder (no comparable statement for boys)
3. Innate factors
 - Girls are just smarter
 - Boys – they have a more analytically and mathematically inclined brain
 - Boys are more intelligent.

The comments parallel the factors typically included in explanatory models of gender differences in mathematics learning, and echo the elements in Weiner's (1974) attribution theory of success and failure.

Final Words

Based on the data gathered from the 20-39 year old citizens of Victoria (Australia) and Madrid (Spain) in this study, the traditional male stereotype is apparently still prevalent. For the majority of items for which gender stereotyped views were held, higher proportions of participants responded that “males” were more suited to studies in mathematics and/or related careers than were “females”. At the same time, more Spaniards than Australians considered there to be no difference between males and females with respect to suitability to study mathematics or work in related fields. Participants' qualitative comments to explain their response choices highlighted factors consistent with early explanatory models for gender differences in mathematics learning outcomes: social/environmental factors, and innate characteristics. The nature-nurture debate, it seems, lives on in the realm of gender issues and mathematics education.

The findings suggest that factors in the social milieu shape individuals' beliefs. Social context cannot be divorced from research on affective factors. Further work in different countries is needed to clarify more precisely which social factors are the more significant contributors to the persistent gendering of mathematics as a male domain.

References

- Ainley, J., Kos, J., & Nicholas, M. (2008). *Participation in science, mathematics and technology in Australian education*. (ACER Research Monograph no. 63). Camberwell, Vic.: ACER. Retrieved from http://www.acer.edu.au/documents/Mono63_MathsSciTechSept08.pdf
- Baker, D. P., & Jones, D. P. (1993). Creating gender equality: Cross-national gender stratification and mathematical performance. *Sociology of Education*, 66(2), 91-103.
- Barbie doll as computer engineer (geek). (n.d.). Retrieved from <http://www.labnol.org/tech/computer-engineer-barbie/12730/>
- Cacace, M. (2009) *Guidelines for gender equality programmes in science. FP7 project. Practising gender equality in sciences (PRAGE)*. ASDO: Italy. Retrieved from <http://www.amites.org/assets/files/publicaciones/guidelines%20prage.pdf>
- Department of Families, Housing, Community Services and Indigenous Affairs [FaHCSIA], Australian Government (n.d.). *Women in Australia, 2009*. Retrieved from <http://www.fahcsia.gov.au/sa/women/pubs/general/womeninaustralia/2009/Pages/chapter2.aspx>
- Department of Foreign Affairs and Trade, Australian Government. (n.d.). *Women—towards equality*. Retrieved from <http://www.dfat.gov.au/facts/women.html>
- Eccles, J. (1985). Model of students' mathematics enrollment decisions. *Educational Studies in Mathematics*, 16(3), 311-314.
- Eccles, J. S. (2005). Studying gender and ethnic differences in participation in math, physical science, and information technology. *New Directions for Child and Adolescent Development*, 110, 7-14.
- Equal Opportunity for Women in the Workplace Agency, Australian Government. (n.d.). *Milestones*. Retrieved from http://www.eowa.gov.au/About_Equal_Opportunity/Where_Are_We_Now/Milestones.asp
- Goldstein, D. (2012, Jan. 4). Forging careers from child's play. *The Age*, 14.
- Halpern, D. F., Benbow, C. P., Geary, D. C., Gur, R. C., Hyde, S. H., & Gernsbacher, M. A. (2007). *Psychological Science in the Public Interest*, 8(1), 1-51.
- Leder, G. C. (1992). Mathematics and gender: Changing perspectives. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.597-622). New York: Macmillan.
- Leder, G. C., & Forgasz, H. J. (2010). I liked it till Pythagoras: The public's views of mathematics. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education*. Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia (pp. 328-335). Fremantle, WA: MERGA.
- Leder, G. C., & Forgasz, H. J. (2011). The public's views on gender and the learning of mathematics: Does age matter? In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices* (pp. 446-454). Adelaide: AAMT and MERGA.
- McAnalley, K. (1991). Encouraging parents to stop pigeon-holing their daughters: The "Maths Multiplies Your Choices" campaign. *Victorian Institute of Educational Research Bulletin*, 66, 29-38.
- McCrindle Research. (n.d.). *New generations at work*. Retrieved from http://www.libraries.vic.gov.au/downloads/Public_Libraries_Unit/newgenerationsatwork.pdf
- OECD (2007). *Babies and bosses - reconciling work and family life. A synthesis of findings for OECD countries. Key outcomes of Spain compared to the OECD average*. Retrieved from http://www.oecd.org/document/57/0,3746,en_2649_34819_39699257_1_1_1_1,00.html
- Sanchez de Madariaga, I, de la Rica, S. & Dolado, J. S. (Coord.) (2011). *Libro Blanco Situación de las mujeres en la Ciencia Española*. UMYC-Unidad de Mujeres y Ciencia. Ministerio de Ciencia e Innovación: Madrid.
- Smith, M. (2011, Dec. 27). Lego makes sure that boys will be boys and girls will be girls. *The Age*, 13.
- The power of public opinion: How much does public opinion matter in policy-making? (2008, Jan. 29). *The Daily*. Retrieved from <http://dailyuw.com/news/2008/jan/29/the-power-of-public-opinion-how-much-does-public/>
- Topsfield, J. (2011, Dec. 14). Gender numbers in VCE maths just don't add up. *The Age*. Retrieved from <http://www.theage.com.au/national/education/gender-numbers-in-vce-maths-just-dont-add-up-20111213-1otdc.html>
- Weiner, B. (1974). *Achievement motivation and attribution theory*. Morristown, N.J.: General Learning Press.
- Wigfield, A., & Eccles, J. S. (2000). Expectancy-value theory of achievement motivation. *Contemporary Educational Psychology*, 25, 68-81.

The Progress of Grade 1 Students Who Participated in an Extending Mathematical Understanding Intervention Program

Ann Gervasoni

Australian Catholic University

<ann.gervasoni@acu.edu.au>

Linda Parish

Australian Catholic University

<linda.parish@acu.edu.au>

Teresa Hadden

Ballarat CEO

<thadden@)ceo.edu.au>

Carole Livesey

ACU & Sale CEO

<clive@acu.edu.au>

Kate Bevan

Sandhurst CEO

<kbeva@acu.edu.au>

Melissa Croswell

ACU & CEOWA

<mcros@acu.edu.au>

Kathie Turkenburg

Ballarat CEO

<kturk@acu.edu.au>

The *Extending Mathematical Understanding* (EMU) Program is a specialised mathematics program that aims to accelerate the learning of Grade 1 students who struggle with learning school mathematics. Forty-two students participated in an EMU Program in 2010 as part of the *Bridging the Numeracy Gap* (BTNG) project. Analysis of students' mathematics knowledge at the beginning of the EMU Program highlighted how diverse was this group of students. The students' mathematics knowledge was assessed again at the beginning of the following year in order to evaluate the effectiveness of the program for accelerating learning. Overall the students made very good progress and their learning was maintained.

Introduction

It is often observed by teachers that some students appear to struggle with learning school mathematics. Indeed, by the beginning of Grade 1 a wide difference in the whole number knowledge of students can be observed (Gervasoni, 2004). Governments and school communities recognise the value of identifying, early in schooling, the students who are not thriving mathematically and in providing them with more intensive instruction. Indeed, providing additional assistance and intervention programs for students who struggle with learning maths in the classroom is recognised as one of the key factors in a whole school approach for improving learning (Fullen, Hill & Crevola, 2006; Gervasoni, Parish, Upton, Hadden, Turkenburg, Bevan, Livesey, Thompson, Croswell & Southwell, 2010). However, many primary schools do not have teachers on staff with specialist knowledge about how to diagnose students' mathematics difficulties and design instruction that will accelerate students' learning. The *Extending Mathematical Understanding* (EMU) Specialist Teacher Course and the EMU Intervention Program for Grade 1 students was designed to achieve this objective. This paper explores the progress of 42 Grade 1 students who participated in an EMU Program in 2010 as part of the *Bridging the Numeracy Gap Pilot* (Gervasoni, Parish, Hadden, Turkenburg, Bevan, Livesey, & Croswell, 2011). The research questions framing the research reported in this paper are:

What are the growth point profiles of Grade 1 students who are mathematically vulnerable and who participated in an EMU program?

What is the progress of Grade 1 students who participated in an EMU program?

The Bridging the Numeracy Gap Project

The *Bridging the Numeracy Gap* Project (Gervasoni et al., 2011) was a collaboration between 44 school communities, the Catholic Education Offices (CEOs) in three dioceses in south-eastern Australia (Ballarat, Sandhurst and Sale), the CEO Western Australia, and Australian Catholic University. It aimed to improve the mathematics learning of students in the early years of schooling. Key approaches used in this project were: one-to-one interview-based mathematics assessment using the *Early Numeracy Interview* and associated framework of growth points (Clarke, Cheeseman, Gervasoni, Gronn, Horne,

McDonough, Montgomery, Roche, Sullivan, Clarke, & Rowley, 2002; Gervasoni, Turkenburg, & Hadden, 2007); and using the *Extending Mathematical Understanding Program* (Gervasoni, 2004) in the second year of formal schooling to provide intensive specialised instruction for children who were mathematically vulnerable.

The *Early Numeracy Interview* (Department of Education Employment and Training, 2001), developed as part of the Early Numeracy Research Project (ENRP, Clarke et al., 2002), is a clinical interview with an associated research-based framework of growth points that describe key stages in the learning of nine mathematics domains. This interview and the growth points were used to obtain the data examined in this paper so it is important that both are well understood. The principles underlying the construction of the growth points were to: describe the development of mathematical knowledge and understanding in the first three years of school in a form and language that was useful for teachers; reflect the findings of relevant international and local research in mathematics (e.g., Steffe, von Glasersfeld, Richards, & Cobb, 1983; Fuson, 1992; Mulligan, 1998; Wright, Martland, & Stafford, 2000; Gould, 2000); reflect, where possible, the structure of mathematics; allow the mathematical knowledge of individuals and groups to be described; and enable a consideration of students who may be mathematically vulnerable.

The growth points form a framework for describing development in nine domains, including the four whole number domains that are the focus of this research: Counting, Place value, Addition and Subtraction, and Multiplication and Division. The processes for validating the growth points, the interview items and the comparative achievement of students in project and reference schools are described in full in Clarke et al. (2002).

To illustrate the nature of the growth points, the following are the growth points for Addition and Subtraction. These emphasise the strategies children use to solve tasks.

1. Counts all to find the total of two collections.
2. Counts on from one number to find the total of two collections.
3. Given subtraction situations, chooses appropriately from strategies including count back, count down to & count up from.
4. Uses basic strategies for solving addition and subtraction problems (doubles, commutativity, adding 10, tens facts, other known facts).
5. Uses derived strategies for solving addition and subtraction problems (near doubles, adding 9, build to next ten, fact families, intuitive strategies).
6. Extending and applying. Given a range of tasks (including multi-digit numbers), can use basic, derived and intuitive strategies as appropriate.

Each growth point represents substantial expansion in knowledge along a path to mathematical understanding (Clarke, 2001). Growth points enable teachers to: identify any children who may be vulnerable in a given domain; identify the zone of proximal development for each child in each domain so instruction may be customised and precise; and identify the diversity of mathematical knowledge in a class. The whole number tasks in the interview take between 15-25 minutes for each student and are administered by the classroom teacher. There are about 40 tasks in total, and given success with a task, the teacher continues with the next tasks in a domain (e.g., Place Value) for as long as the child is successful. Teachers report that the *interview* provided them with insights about students' mathematical knowledge that might otherwise remain hidden (Clarke, 2001). This was a key reason for using the Early Numeracy Interview (*ENI*) for the *Bridging the Numeracy Gap* project.

The Extending Mathematical Understanding (EMU) Program

One key aspect of the approach used in the *Bridging the Numeracy Gap Project* was providing the opportunity for students who were mathematically vulnerable to participate in an EMU program (Gervasoni, 2004). This is a series of lessons specifically designed by a specialist teacher for the purpose of accelerating the most vulnerable students' learning. Groups of three students participate in these lessons for 30-minutes per day, 5 days per week for 50-100 lessons depending on student progress. Each lesson centres on whole number learning with specific focuses on quantity or numerosity (including place value and counting), investigations involving addition, subtraction, multiplication and division problems with an emphasis on the development of reasoning strategies, reflection on learning, and a home task. The specially qualified EMU teachers have completed a course (at Masters level) that includes 36 hours of course work, a minimum of 25 hours of field-based learning, and a program of professional reading.

Identifying Eligible Students for an EMU Intervention Program

All students participating in the *Bridging the Numeracy Gap* pilot were assessed by their classroom teachers at the beginning of 2010 and 2011 using the *Early Numeracy Interview*. This assessment was used by teachers to get to know their students mathematically so that they might refine the curriculum and instruction for class groups and individual students. The assessment was also used to identify any students who were mathematically vulnerable in any domains and who may not fully benefit from the classroom mathematics program. A research-based set of growth points were used for this purpose (Gervasoni, 2004). Students who were identified as vulnerable in any domains were prioritised for an EMU intervention program.

The record sheets completed by teachers during the interviews were analysed by independent coders to determine the growth points reached in each domain. These were entered into an SPSS data base along with information about the background of each student, whether or not they participated in an EMU program, and if so, the total number of EMU lessons. This data was further analysed to explore the mathematical knowledge of the students who participated in an EMU program and their progress across one year.

A total of 136 Grade 1 students from 26 schools in Victoria and Western Australia were prioritised for participation in an EMU Program in 2010 as part of the *Bridging the Numeracy Gap* Project. There were 55 boys (40%) and 81 girls. Five students (4%) were Aboriginal or Torres Strait Islanders, and forty-five students (33%) were from families with Health Care Cards or who received an Educational Maintenance Allowance. Eighteen (13%) had non-English speaking backgrounds, six (4%) had a disability and eight (6%) a severe language deficit. Sixteen children (12%) also participated in *Reading Recovery* (13 in 2010, and 3 in 2009). Of these 136 students, only 42 with the highest priority ratings following their ENI assessment participated in an EMU Program for 10-20 weeks (between 50 and 100 thirty-minute lessons). Due to limited resources it is rare for schools to provide intervention programs for all eligible students. Some schools were able to offer additional support for vulnerable students but not with the regularity required for the EMU Program.

Progress of Grade 1 Students Participating in an EMU Program

Forty-two students had the opportunity to participate in an EMU Program in 2010. They were selected because they were the most mathematically vulnerable students in their grade level on the basis of their ENI assessment and growth point profiles. An analysis of these

students' initial growth points suggests that overall they are a diverse group. Table 1 shows the number of students who were vulnerable in each domain.

Table 1

The Number of Grade 1 Students Participating In an EMU Program Who Were Vulnerable In Each Domain.

	No. of Girls (n=24)	No. of Boys (n=18)	No. of Students (n=42)
Vulnerable in Counting	15 (63%)	14 (78%)	29 (69%)
Vulnerable in Place Value	9 (44%)	8 (38%)	17 (41%)
Vulnerable in Addition and Subtraction	8 (33 %)	9 (50%)	17 (41%)
Vulnerable in Multiplication and Division	16 (67%)	14 (78%)	30 (71%)

The data shows clearly that not all students participating in the EMU Program were vulnerable in every domain. Table 2 shows the number and combinations of domains for which students were vulnerable. Two students were not vulnerable in any domains but the school community prioritised them for the EMU Program because of other assessments.

Table 2

Combinations of Whole Number Domains in Which Students Participating in the EMU Program Were Vulnerable in 2010.

Vulnerable Domains	Counting	Place Value	Add & Sub	Mult & Div	No. of Vulnerable Students	Total Students
0						2 (5%)
1					2	9 (21%)
					0	
					4	
					3	
2					1	13 (31%)
					2	
					7	
					0	
					2	
					1	
3					1	14 (33%)
					4	
					8	
					1	
4					4	4 (10%)

A striking feature of the data presented in Table 2 is the broad range of domains and combinations of domains in which children are vulnerable. Some are vulnerable in only one domain, some in two or three domains, but only four students are vulnerable in all four

domains. Clearly there is no one pattern to describe these children who are mathematically vulnerable. This highlights the complexity of teaching and the need for teachers to be expert at assessing students' knowledge and in designing instruction that enables all students to learn.

In order to show the progress of students participating in an EMU Program, their growth point distributions in 2010 and 2011 for each domain were compared with all Grade 1 students. As illustrative examples, Figures 1 and 2 show the growth point distributions for Addition and Subtraction and Multiplication and Division Strategies.

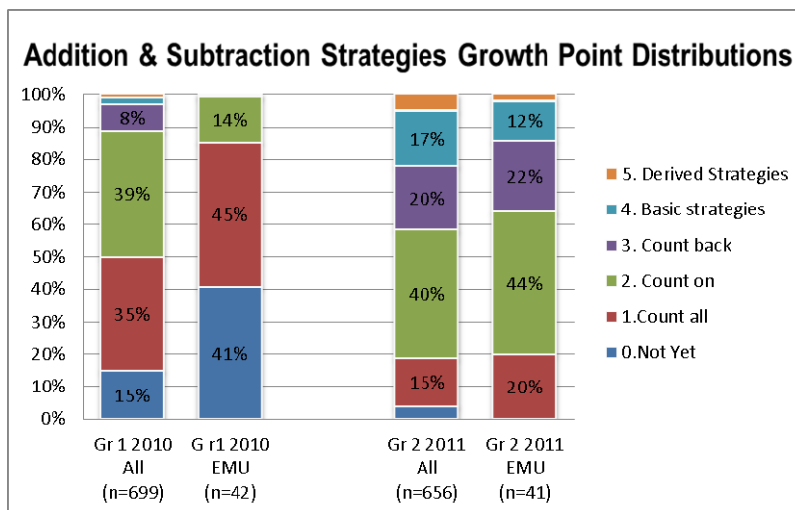


Figure 1. 2010 and 2011 Addition and Subtraction Strategies growth point distributions (beginning of the year) for all Grade 1 students in 2010 and for students who participated in an EMU program in 2010.

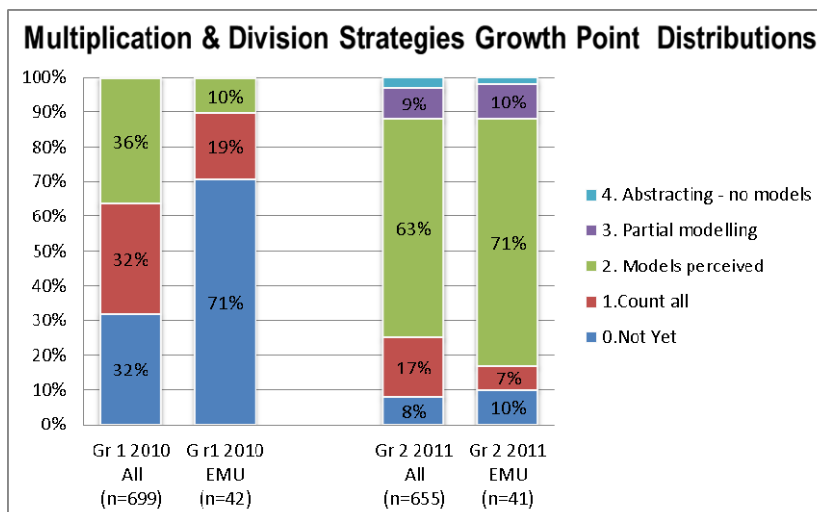


Figure 2. 2010 and 2011 Multiplication and Division growth point distributions (beginning of the year) for all Grade 1

The graphs highlight the growth that occurred across the year for the majority of EMU students. It is striking how similar the growth point distributions are in 2011 for both groups, and this is apparent for all four whole number domains. This suggests that one effect

of the EMU program in 2010 was that learning was accelerated to the point that their growth point distributions mirror the distribution of all students one year later. However, some EMU students remain on the lower growth points despite the fact that others have accelerated and progressed two or three growth points. This is equivalent to two or three years growth compared with mean growth (see Clarke et al., 2002).

An issue highlighted when examining the data for Addition and Subtraction Strategies (Figure 1) is that about 20% of Grade 2 students overall need to *count all* to solve a task. This is a reduction from the previous year but highlights that an important instructional focus for Grade 2 students is developing reasoning based strategies such as doubles, partners for ten and building to ten. An examination of the Multiplication and Division Strategies data (Figure 2) highlights a similar issue. About 25% of beginning Grade 2 students (although fewer in the EMU group) are *counting all* and do not yet use the group structure to assist with solving multiplicative problems. Thus noticing and using the multiplicative structure is a key instructional focus for these students at the beginning of Grade 2, whilst for the majority of others, their instruction needs to initially focus on strategies for solving problems in partial modelling situations.

The progress of EMU students was also analysed by calculating the number of growth points students progressed in each domain. Table 3 shows the findings for Multiplication and Division as an illustrative example. The first column shows the growth point range in 2010, and the remaining columns show progress in 2011. For example, of the 29 students beginning on Growth Point 0 in 2010, 21 had moved to Growth Point 2 by 2011.

Table 3

The Number of EMU Students on Each Multiplication & Division Growth Point in 2011 Compared with Their Growth Point in 2010.

EMU Students Multiplication & Division Growth Point (GP) in 2010	Number of EMU Students on each Multiplication & Division Growth Point in 2011					
	GP0	GP1	GP2	GP3	GP4	Total
GP0	4	2	21	1	1	29
GP1	0	1	6	1	0	8
GP2	0	0	2	2	0	4
Total	4	3	29	4	1	41

The data in Table 3 demonstrates that most students made significant growth over the year. However, 7 of the 41 students remained on their 2010 growth point when they were assessed in 2011. This suggests that some students may require further assistance to enable them to learn mathematics. This finding is apparent for the other domains also.

Discussion and Conclusion

Analysis of the *ENI* data for all Grade 1 students participating in the *Bridging the Numeracy Gap* Pilot highlights the broad range of growth points within this cohort and thus the complexity of classroom teaching. The range of whole number knowledge is extraordinary. It is also clear that there are a number of students on much lower growth points than many of their peers. For example, in contrast to the majority and despite 12 months at school, some Grade 1 students were unable to count a collection of 20 items, or count all to solve addition and subtraction problems, or multiplication and division

problems. This suggests that these students may have difficulty accessing and benefiting from the regular Grade 1 classroom program that typically assumes this knowledge.

Forty-two of the Grade 1 students who were on the lowest growth points had the opportunity to participate in an Extending Mathematical Understanding program designed to accelerate their mathematical learning and assist them to benefit from the classroom program. Analysis of the growth point profiles of these students show that they are a diverse group but that very few were vulnerable in all four whole number domains. Further, some students were vulnerable in only one, two or three domains, and the combinations of domains also varied. Indeed, there was no one pattern to describe these students whole number knowledge or learning needs. The implication of this finding is that their teachers need to be expert at assessing each student's current mathematical knowledge, and at designing highly responsive instruction based on this assessment. It is also important to note that very few of these students had other learning difficulties. Few participated in *Reading Recovery*, had language backgrounds other than English or were assessed with severe language difficulties. Often it is assumed that lower achievers in mathematics have a range of factors affecting their opportunity to thrive in mathematics learning, but this was not typical of many students in this group.

An important focus for this paper was reporting on the progress of students who participated in an EMU program in Grade 1. The students were independently assessed by classroom teachers at the beginning of 2010 and 2011 to provide a measure of their whole number learning across one year. Overall, the forty-two students made very good progress and had achieved significant growth by the beginning of 2011 (and after the extended Christmas holidays). Findings from the *ENRP* (Clarke et al., 2002) suggest that mean growth for Grade 1 students across one year was about one growth point in each domain, and somewhat less in Place Value. The students participating in the EMU program in 2010 progressed between 5 and 11 growth points in the four domains overall. There were some students who did not progress in particular domains, but most students improved two growth points per domain, although this was rare in Place Value. The acceleration of learning is obvious when comparing the growth point distributions of the EMU students when they reached Grade 2 with all Grade 2 students. Indeed, by Grade 2, the growth point distributions of both groups were almost identical compared with one year earlier.

The data analysis also suggests that learning for students who participated in an EMU program is not accelerated for all students in all domains. Further, a small proportion of the students were still mathematically vulnerable the following year. Also clear is that a proportion of the EMU students reached the highest growth points in the distribution.

The EMU program was successful in enabling the majority of participating students to progress their whole number learning beyond what may be anticipated. This is true also for the students who began on the lowest growth points in each domain. The challenge remains to determine why some children make lesser progress than others, and to identify assessment and instruction that may accelerate these students' learning also. It is possible that these students may benefit from specialised instruction beyond Grade 1. It is also important to determine whether the children who participated in an EMU program continue to learn mathematics successfully in subsequent years.

Acknowledgements

The research reported in this paper was funded by the Australian Government as part of the *Bridging the Numeracy Gap in Low SES and Indigenous Communities* Project. The authors acknowledge gratefully the contribution of teachers, parents and students.

References

- Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., Young-Loveridge, J. & Gould, P. (2005). Supporting teachers in the development of young children's mathematical thinking: Three large scale cases. *Mathematics Education Research Journal*. 16(3), 27–57.
- Clarke, D. (2001). Understanding, assessing and developing young children's mathematical thinking: Research as powerful tool for professional growth. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and beyond: Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 9–26). Sydney: MERGA.
- Clarke, D., Cheeseman, J., Gervasoni, A., Gronn, D., Horne, M., McDonough, A., Montgomery, P., Roche, A., Sullivan, P., Clarke, B., & Rowley, G. (2002). *ENRP Final Report*. Melbourne: ACU.
- Department of Education Employment and Training. (2001). *Early numeracy interview booklet*. Melbourne: Department of Education, Employment and Training.
- Fullan, M., Hill, P., & Cr  vola, C. (2006). *Breakthrough*. Thousand Oaks: Corwin Press.
- Fuson, K. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243–275). New York: Macmillan.
- Gervasoni, A. (2004). *Exploring an intervention strategy for six and seven year old children who are vulnerable in learning school mathematics*. Unpublished PhD thesis, La Trobe University, Bundoora.
- Gervasoni, A., Hadden, T., & Turkenburg, K. (2007). Exploring the number knowledge of children to inform the development of a professional learning plan for teachers in the Ballarat diocese as a means of building community capacity. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential Research, Essential Practice: Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (pp. 305–314). Hobart: MERGA.
- Gervasoni, A., Parish, L., Hadden, T., Turkenburg, K., Bevan, K., Livesey, C., & Croswell, M. (2011). Insights about children's understanding of 2-digit and 3-digit numbers. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices* (Proceedings of the 23rd biennial conference of The Australian Association of Mathematics Teachers and the 34th annual conference of the Mathematics Education Research Group of Australasia, Vol 1, pp. 315–323). Alice Springs: MERGA/AAMT.
- Gervasoni, A., Parish, L., Upton, C., Hadden, T., Turkenburg, K., Bevan, K., Livesey, C., Thompson, D., Croswell, M., & Southwell, J. (2010). Bridging the Numeracy Gap for Students in Low SES Communities: The Power of a Whole School Approach. In Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 202–209). Fremantle: MERGA.
- Gould, P. (2000). Count Me In Too: Creating a choir in the swamp. In *Improving numerary learning: What does the research tell us? (Proceedings of the ACER Research Conference 2000, pp. 23–26)*. Melbourne: Australian Council for Educational Research.
- Mulligan, J. (1998). A research-based framework for assessing early multiplication and division. In C. Kanes, M. Goos & E. Warren (Eds.), *Teaching mathematics in new times: Proceedings of the 21st annual conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 404–411). Brisbane: MERGA.
- Steffe, L., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York: Praeger.
- Wright, R., Martland, J., & Stafford, A. (2000). *Early numeracy: Assessment for teaching and intervention*. London: Paul Chapman Publishing.

Auditing the Numeracy Demands of the Australian Curriculum

Merrilyn Goos

The University of Queensland
<m.goos@uq.edu.au>

Shelley Dole

The University of Queensland
<s.dole@uq.edu.au>

Vince Geiger

Australian Catholic University
<vincent.geiger@acu.edu.au>

Numeracy is a general capability to be developed in all learning areas of the Australian Curriculum. We evaluated the numeracy demands of the F-10 curriculum, using a model of numeracy that incorporates mathematical knowledge, dispositions, tools, contexts, and a critical orientation to the use of mathematics. Findings of the history curriculum audit, presented in this paper, highlight the distinction between the numeracy demands and opportunities of the curriculum, and uncover mismatches between claims made about numeracy in the curriculum materials.

The term *numeracy* is used in many English-speaking countries to describe the capacity to deal with quantitative aspects of life. *Quantitative literacy* and *mathematical literacy* are alternative terms that have similar meaning to numeracy. Steen (2001) proposed that the elements of quantitative literacy include: confidence with mathematics; appreciation of the nature and history of mathematics and its significance for understanding issues in the public realm; logical thinking and decision-making; use of mathematics to solve practical everyday problems in different contexts; number sense and symbol sense; reasoning with data; and the ability to draw on a range of prerequisite mathematical knowledge and tools. Many of these elements are also visible in the Programme for International Student Assessment definition of mathematical literacy as:

an individual's capacity to identify and understand the role mathematics plays in the world, to make well-founded judgments, and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (Organisation for Economic Cooperation and Development, 2004, p. 15)

The idea of numeracy is not new. The term first appeared in the Crowther Report of 1959 (Ministry of Education, 1959), and many subsequent reports and investigations in Australia have emphasised the importance of numeracy as a key to social and economic well-being (e.g., Council of Australian Governments, 2008; DETYA, 2000; Vincent, Stephens, & Steinle, 2005). Steen (2001) insists that, for numeracy to be useful to students, it must be learned in multiple contexts and in all school subjects, not just mathematics. Although developing numeracy across the curriculum is a notion that has so far gained little ground in Australia (Thornton & Hogan, 2004), the introduction of the Foundation-Year 10 (F-10) Australian Curriculum may provide new impetus to tackle this important issue.

The Australian Curriculum, currently being developed for the learning areas set out in the *Melbourne Declaration on Educational Goals for Young Australians* (MCEETYA, 2008), identifies numeracy as one of seven general capabilities that apply across all discipline content, not just in mathematics. Version 3.0 of the Australian curriculum (ACARA, 2012a) offers some support for recognising the numeracy demands of different learning areas, for example, by providing a numeracy learning continuum together with icons and filters that link numeracy capabilities to relevant curriculum content. Yet the Australian Curriculum still lacks a theoretically informed model for characterising numeracy, and as a result teachers have little guidance in recognising the numeracy

demands of subjects other than mathematics and in embedding numeracy learning opportunities across the whole curriculum.

This paper reports on the early stages of a project that will implement, evaluate, and refine a rich model of numeracy across the curriculum. The aim of the paper is to demonstrate how the numeracy model can be used to evaluate the numeracy demands and opportunities of learning areas in the F-10 Australian Curriculum. We present an initial numeracy audit of the *Australian Curriculum: History* and compare the findings with claims made about numeracy in the published curriculum documents.

Numeracy Model

Elsewhere we have argued that researchers and educators need to embrace a description of numeracy that recognises the intellectual, affective, contextual, and technological demands of becoming a numerate person in the 21st century (Geiger, Goos, & Dole, 2011a; Goos, Dole, & Geiger, 2011). We developed the model shown in Figure 1 to affirm the value of current definitions of numeracy (e.g., Australian Association of Mathematics Teachers, 1997), while introducing a greater emphasis on tools as mediators of mathematical thinking and action (Sfard & McClain, 2002) and a critical orientation to the ways mathematics is used to support arguments and influence opinions (Jablonka, 2003). Our previous research demonstrated how the model provided teachers with an instrument for planning and reflection, and how it could be used to analyse changes in teachers' classroom practice and personal conceptions of numeracy (Geiger, Goos, & Dole, 2011b; Goos, Geiger, & Dole, 2011). The elements of the model are summarised in Table 1 and elaborated below.

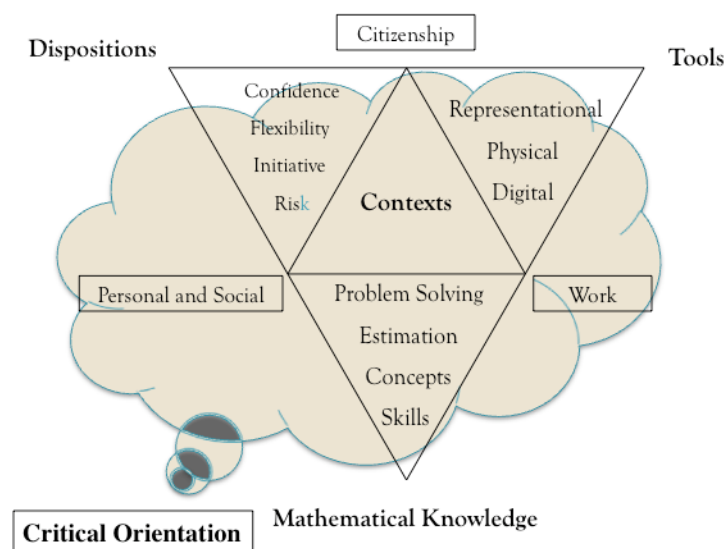


Figure 1. A model for numeracy in the 21st century.

A numerate person requires *mathematical knowledge*. In a numeracy context, mathematical knowledge includes not only concepts and skills, but also problem solving strategies and the ability to make sensible estimations (Zevenbergen, 2004).

A numerate person has *positive dispositions* – a willingness and confidence to engage with tasks, independently and in collaboration with others, and apply their mathematical knowledge flexibly and adaptively. Affective issues have long been held to play a central

role in mathematics learning and teaching (McLeod, 1992), and the importance of developing positive attitudes towards mathematics is emphasised in national and international curriculum documents (e.g., National Council of Teachers of Mathematics, 2000; National Curriculum Board, 2009).

Table 1
Description of Elements of the Numeracy Model

Element	Description
Critical orientation	Use of mathematical information to: make decisions and judgements; add support to arguments; challenge an argument or position.
Contexts	Capacity to use mathematical knowledge in a range of contexts, both within schools and beyond school settings.
Dispositions	Confidence and willingness to use mathematical approaches to engage with life-related tasks; preparedness to make flexible and adaptive use of mathematical knowledge.
Mathematical knowledge	Mathematical concepts and skills; problem solving strategies; estimation capacities.
Tools	Use of material (models, measuring instruments), representational (symbol systems, graphs, maps, diagrams, drawings, tables, ready reckoners) and digital (computers, software, calculators, internet) tools to mediate thinking.

Being numerate involves using *tools*. Sfard and McClain (2002) discuss ways in which symbolic tools and other specially designed artefacts “enable, mediate, and shape mathematical thinking” (p. 154). In school and workplace contexts, tools may be representational (symbol systems, graphs, maps, diagrams, drawings, tables, ready reckoners), physical (models, measuring instruments), and digital (computers, software, calculators, internet) (Noss, Hoyles, & Pozzi, 2000; Zevenbergen, 2004).

Because numeracy is about using mathematics to act in and on the world, people need to be numerate in a range of *contexts* (Steen, 2001). All kinds of occupations require numeracy, and many examples of work-related numeracy are specific to the particular work context (Noss et al., 2000). Informed and critical citizens need to be numerate citizens. Almost every public issue depends on data, projections, and the kind of systematic thinking that is at the heart of numeracy. Different curriculum contexts also have distinctive numeracy demands, so that students need to be numerate across the range of contexts in which their learning takes place at school (Steen, 2001).

This model is grounded in a *critical orientation* to numeracy since numerate people not only know and use efficient methods, they also evaluate the reasonableness of the results obtained and are aware of appropriate and inappropriate uses of mathematical thinking to analyse situations and draw conclusions. In an increasingly complex and information drenched society, numerate citizens need to decide how to evaluate quantitative, spatial or probabilistic information used to support claims made in the media or other contexts. They also need to recognise how mathematical information and practices can be used to persuade, manipulate, disadvantage or shape opinions about social or political issues (Jablonka, 2003).

Curriculum Audit Methodology

The numeracy audit used a similar approach to that employed in our earlier audit of the South Australian Curriculum, Standards and Accountability Framework (see Goos, Geiger, & Dole, 2010). Each member of the research team independently read the full text of the *Australian Curriculum: History* (ACARA, 2012b) and evaluated its numeracy demands by reference to one of the elements of the numeracy model shown in Figure 1: mathematical knowledge, contexts, dispositions, or tools. The team met for a full day to discuss each person's findings and to collectively identify evidence of a critical numeracy orientation in the curriculum document. Evidence addressing each element of the model was sought from the curriculum aims, rationale, description of content structure, statement of general capabilities, statement of links to the Mathematics learning area, and from the F-10 year level descriptions, content descriptions and elaborations. We also consulted the statement of *General Capabilities in the Australian Curriculum* (ACARA, 2012c) to become familiar with the nature and scope of the numeracy general capability and the organising elements of the numeracy learning continuum. The audit is qualitative rather than quantitative in that numeracy-related aspects of the published history curriculum are interpreted in terms of our numeracy model, without the need to assign scores or codes that can be counted. Initial findings of the audit are organised around the elements of our numeracy model.

Numeracy in the History Curriculum

Critical Orientation

The curriculum rationale explains that “history is a disciplined process of inquiry into the past” (ACARA, 2012b, p. 3), and one of the aims of the curriculum is for students to develop a capacity to undertake historical inquiry. The general capabilities section further claims that “critical thinking is essential to the historical inquiry process” (p. 10). The framework for developing students' historical knowledge and understanding is provided by inquiry questions set out for each year level, for example:

- How did Australian society change throughout the twentieth century?
- Who were the people who came to Australia? Why did they come? (Year 6)

It is apparent that a critical orientation to inquiry characterises the methods and procedures of history as it is represented in the Australian Curriculum. Skills used in the process of historical inquiry include some with a mathematical basis, such as chronology. However, the extent to which students are asked to use *mathematical* information to support the process of inquiry is at the discretion of the teacher: the numeracy demands here are not explicit, but rather depend on the learning opportunities that the teacher creates.

Contexts

In the *Australian Curriculum: History*, context is one of the organising devices for teaching the key historical concepts of evidence, continuity and change, cause and effect, significance, perspectives, empathy and contestability. Curriculum contexts become progressively broader throughout the years from Foundation to Year 10, moving from family, friends, and school to the local community and then national and international contexts. There is also scope for content to be taught using specific local contexts that align with students' own lives and interests. The curriculum explicitly identifies contexts for studying history, and it is not surprising that the numeracy demands of these contexts are

not fully elucidated, beyond some examples accompanying the numeracy learning continuum. It is up to the teacher to create opportunities for students to use their mathematical knowledge in a range of historical contexts.

Dispositions

The *Australian Curriculum: History* supports discipline-specific dispositions by encouraging students to develop empathy for others and explore the perspectives, beliefs, and values of different societies and cultures. Activities that assist in cultivating these dispositions may well involve numeracy, but teachers would need to ensure that students also gain confidence in using appropriate mathematical knowledge and skills, and are able to apply these flexibly to investigate historical questions. Students' enjoyment of studying history does necessarily translate into a positive disposition to mathematics, and teachers may need to plan purposefully for numeracy learning opportunities that build mathematical confidence as well as historical empathy.

Mathematical Knowledge

Within the general capabilities section of the *Australian Curriculum: History*, numeracy is described as follows:

Students develop numeracy capability as they learn to organise and interpret historical events and developments. Students learn to analyse numerical data to make meaning of the past, for example to understand cause and effect, and continuity and change. Students learn to use scaled timelines, including those involving negative and positive numbers, as well as calendars and dates to recall information on topics of historical significance and to illustrate the passing of time. (ACARA, 2012b, p. 10)

While this statement seems limited in its reference only to data analysis and time as key aspects of mathematical knowledge that support historical inquiry, the section of the curriculum with links to the Mathematics learning area proposes additional possibilities:

Much of the evidence and reasoning in historical understanding is quantitative: chronology, demography, economic activity, changes in the movement of peoples and in the size and reach of institutions. All of these call for an appreciation of numerical scale and proportion. (ACARA, 2012b, p. 13)

In our analysis of the content descriptions and elaborations we found evidence, in most year levels, of use of mathematical knowledge for chronology and mapping of settlement and movement patterns, such as:

- Sequencing people and historical events, developments and periods in chronological order by developing an annotated timeline;
- Mapping settlement patterns in different regions, noting factors that shaped these patterns (e.g., geographical features, climate, water resources, transport);
- Mapping movement patterns of humans during historical periods (e.g., the movement of humans out of Africa, the transatlantic slave trade).

There is also an emphasis on data representation and interpretation (e.g., investigating the impact of the Industrial Revolution on population growth and distribution), and some evidence of use of measurement concepts (e.g., investigating how the pyramids of Gizeh were built). We characterise these explicit statements found in content descriptions or elaborations as indicating the numeracy *demands* of the curriculum. Numeracy demands are “tagged” by the numeracy icon found in the published curriculum and can be identified by applying the numeracy filter in the online version of the curriculum. However, it is also

possible to recognise numeracy learning *opportunities*, or possibilities for treating the content that may depend on the teacher’s choice of activities and are therefore not identified via this filtering process. Examples are provided in Table 2. The numeracy learning continuum (ACARA, 2012c) also offers examples of mathematical knowledge and skills that can be used in the study of history.

Table 2

Examples of Numeracy Learning Opportunities in the Australian Curriculum: History

History content	Related Mathematics content
Identifying the influence of cultural groups in the community as reflected in architecture and religious buildings (Year 3)	Geometry: make models of three-dimensional objects and describe key features (Year 3)
Identifying the reasons why people migrated to Australian in the 1800s, for example, those dislocated by events such as the Irish Potato Famine (Year 5)	Statistics: construct suitable data displays (e.g., of potato production, migration figures) from given or collected data, (Year 4)
Describing the importance of the River Nile to Egyptian society, such as through inundation and farming (Year 7)	Statistics: interpret and compare a range of data displays (e.g., rainfall) (Year 6)
Investigating the changes in working conditions during the Industrial Revolution, such as longer working hours for low pay and the use of children as a cheap source of labour (Year 9)	Statistics: identify and investigate issues involving numerical data collected from primary and secondary sources (Year 7)

Tools

Maps and timelines are the most common representational tools referred to in the *Australian Curriculum: History*. Maps are used to investigate settlement and movement patterns, and the size and influence of institutions such as the British Empire. Timelines are mentioned in almost every year level because of the importance of chronology in historical inquiry. Both of these tools require understanding of scale and proportion and, as mentioned previously, this is acknowledged in the section of the curriculum where links to the Mathematics learning area are discussed. However, nowhere in the content descriptions or elaborations is the importance of scale highlighted. In particular, timelines are used only to sequence people, events and historical periods rather than to indicate the time between events or their duration. This is surprising, not only because of the mathematical flaws in timelines that are not to scale, but also because such timelines do not allow certain objectives of the history curriculum to be achieved. For example, in Year 7 students learn about a range of societies in the ancient world. They are meant to use a timeline to identify the longevity of each civilisation, but this is not possible without attention to scale. Thus there is inconsistency between the content descriptions throughout the curriculum and the numeracy general capability statement that claims students “learn to use scaled timelines ... to illustrate the passing of time” (ACARA, 2012b, p. 13).

Digital tools are mostly referred to as a means of representing and communicating ideas. Only one instance was found where there was explicit reference to using technology to analyse data: in Year 6, where it is suggested that students should process and record population data showing places of birth of Australia’s people at different times in the past

and today. However, although the numeracy *demands* of the curriculum appear to be under-represented with respect to digital tools, there are many other numeracy learning *opportunities*; for example, students can collect secondary data from websites and create data displays using Excel spreadsheets and charts to interpret historical events or support arguments based on this analysis of sources.

Conclusion

This initial audit of the *Australian Curriculum: History* suggests that history can provide an engaging and meaningful context for developing students' numeracy capabilities, and mathematics can provide analytical tools to support historical inquiry. However, the audit uncovered two issues – one specific to the history curriculum and another that applies to the Australian Curriculum in general. The first concerns the lack of alignment between claims about numeracy development in the different sections of the history curriculum (ACARA, 2012b), and the second is the materials associated with the numeracy learning continuum (ACARA, 2012c). According to some parts of the curriculum, creating timelines to make meaning of the past is meant to help students develop an understanding of *scale and proportion*, and yet the content descriptions focus exclusively on *sequencing* of people and events as the only skill involved in chronology. The second issue highlights the need to distinguish between the numeracy *demands* and numeracy learning *opportunities* in each learning area. While the curriculum explicitly identifies numeracy demands via use of icons and online filters, additional opportunities for developing students' numeracy capabilities are invisible unless one knows how to “see” them. The numeracy learning continuum offers glimpses of such opportunities by identifying mathematical knowledge and skills related to the learning area content, and contextualising these through examples (ACARA, 2012c). But numeracy involves more than mathematical knowledge and contexts: we have argued that tools, dispositions, and a critical orientation are also important. Our theoretical model directs attention to all these aspects of numeracy and provides a lens through which a clearer view of the numeracy learning demands and opportunities of the Australian Curriculum can be gained.

Acknowledgements

This project is funded by Discovery Grant DP120100694 from the Australian Research Council, awarded to Merrilyn Goos, Shelley Dole, Vince Geiger, and Helen Forgasz. We gratefully acknowledge Helen Forgasz's contribution to this paper as a member of the research team.

References

- Australian Association of Mathematics Teachers (1997). *Numeracy = everyone's business. Report of the Numeracy Education Strategy Development Conference*. Adelaide: AAMT.
- Australian Curriculum, Assessment and Reporting Authority (2012a). *The Australian Curriculum v3.0*. Retrieved 24 March 2012 from <http://www.australiancurriculum.edu.au/>
- Australian Curriculum, Assessment and Reporting Authority (2012b). *Australian Curriculum: History*. Retrieved 24 March 2012 from <http://www.australiancurriculum.edu.au/History/Rationale>
- Australian Curriculum, Assessment and Reporting Authority (2012c). *General capabilities in the Australian Curriculum*. Retrieved 24 March 2012 from <http://www.australiancurriculum.edu.au/GeneralCapabilities/Overview/>
- Council of Australian Governments (2008). *National numeracy review report*. Retrieved 23 February 2012 from <http://www.coag.gov.au/reports/index.cfm#numeracy>

- Department of Education, Training and Youth Affairs (2000). *Numeracy, a priority for all: Challenges for Australian schools*. Canberra: Commonwealth Government of Australia.
- Geiger, V., Goos, M., & Dole, S. (2011a). The role of digital technologies in numeracy. In B. Ubuz (Ed.), *Proceedings of the 35th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 385-392). Ankara, Turkey: PME.
- Geiger, V., Goos, M., & Dole, S. (2011b). Trajectories into professional learning in numeracy teaching. In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Traditions and (new) practices* (Proceedings of the 34th annual conference of the Mathematics Education Research Group of Australasia, pp. 297-305). Alice Springs: MERGA.
- Goos, M., Dole, S., & Geiger, V. (2011). Improving numeracy education in rural schools: A professional development approach. *Mathematics Education Research Journal*, 23(2), 129-148.
- Goos, M., Geiger, V., & Dole, S. (2010). Auditing the numeracy demands of the middle years curriculum. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education* (Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia, pp. 210-217). Fremantle: MERGA.
- Goos, M., Geiger, V., & Dole, S. (2011). Teachers' personal conceptions of numeracy. In B. Ubuz (Ed.), *Proceedings of the 35th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 457-464). Ankara, Turkey: PME.
- Jablonka, E. (2003). Mathematical literacy. In A. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Second international handbook of mathematics education* (pp. 75-102). Dordrecht, The Netherlands: Kluwer.
- McLeod, D. (1992). Research on affect in mathematics education: A reconceptualization. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575-596). New York: Macmillan.
- Ministerial Council on Education, Employment, Training and Youth Affairs (MCEETYA). (2008). *Melbourne Declaration on educational goals for young Australians*. Retrieved 5 August 2011 from http://www.curriculum.edu.au/verve/resources/National_Declaration_on_the_Educational_Goals_for_Young_Australians.pdf
- Ministry of Education (1959). *15 to 18: A report of the Central Advisory Council for Education*. London: HMSO.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- National Curriculum Board (2009). *Shape of the Australian curriculum: Mathematics*. Retrieved 12 January 2010 from http://www.acara.edu.au/verve/resources/Australian_Curriculum_-_Maths.pdf
- Noss, R., Hoyles, C., & Pozzi, S. (2000). Working knowledge: Mathematics in use. In A. Bessot & J. Ridgeway (Eds.), *Education for mathematics in the workplace* (pp. 17-35). Dordrecht, The Netherlands: Kluwer.
- Organisation for Economic Cooperation and Development (2004). *Learning for tomorrow's world: First results from PISA 2003*. Paris: OECD.
- Sfard, A., & McClain, K. (2002). Analyzing tools: Perspectives on the role of designed artifacts in mathematics learning. *The Journal of the Learning Sciences*, 11(2&3), 153-161.
- Steen, L. (2001). The case for quantitative literacy. In L. Steen (Ed.), *Mathematics and democracy: The case for quantitative literacy* (pp. 1-22). Princeton, NJ: National Council on Education and the Disciplines.
- Thornton, S., & Hogan, J. (2004). Orientations to numeracy teachers' confidence and disposition to use mathematics across the curriculum. In M. Johnson Hoines & A. Berit Fugelstad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 313-320). Bergen: PME.
- Vincent, J., Stephens, M., & Steinle, V. (2005). *Numeracy research and development initiative 2001-2004 – An overview of numeracy projects*. Canberra: Commonwealth of Australia. Retrieved 28 January 2011 from http://www.dest.gov.au/NR/rdonlyres/C75E96C0-45F9-4495-AF58-DD187874F70D/8470/numeracyoverview_20012004.pdf
- Zevenbergen, R. (2004). Technologising numeracy: Intergenerational differences in working mathematically in new times. *Educational Studies in Mathematics*, 56, 97-117.

Gesture Types for Functions

Sandra Herbert

Deakin University

<sandra.herbert@deakin.edu.au>

This paper reports on the different gesture types employed by twenty-three Year 10 students as they endeavoured to explain their understanding of rate of change associated with the functions resulting from two different computer simulations. These gestures also have application to revealing students' understanding of functions. However, interpretation of gesture is problematic but classification of gestures assisted in the analysis of the video-recorded interviews probing participants' conceptions of rate of change. This paper builds on the classifications reported in previous research. Five additional gesture types are presented, which provide insights into students' thinking about rate of change, and hence functions.

Kelly, Singer, Hicks, and Goldin-Meadow (2002) suggest that the combination of speech and gesture often provides greater insight into children's knowledge and understanding than either words or gestures alone. It was anticipated that the teenage participants of this study may have found it difficult to discuss an abstract mathematical concept (Reynolds & Reeve, 2002). Therefore, consideration of participants' non-verbal communication made available through video-recorded data collection, is vital in revealing the meaning behind their utterances. However, interpretation of gesture is problematic, so classification of gestures may assist in the analysis of this kind of data. Some work has already been done in classifying gestures, especially those related to mathematical concepts (McNeill, 1992; Edwards, 2005; Arzarello & Robutti, 2004; Rasmussen, Stephan & Allen, 2004). This paper extends that work, paying particular attention to the classification of gestures relating to functions.

This paper reports on the different gesture types employed by twenty-three participants as they grapple to explain their understanding of the functions resulting from two different computer simulations. The simulations provided a focus for discussion as the participants attempted to articulate their thinking about the functions and their representations: numeric; graphic; and symbolic. Participants were able to point to specific places on the screen to clarify their explanations. This data provides a rich source of function-related gestures but space considerations limits this discussion to only a few of the most important.

In the sections below previous gesture research is described; details of the interviews and the computer-based simulations are provided; and examples are presented to illustrate the gesture types identified.

Background

Feyereisen and de Lannoy (1991) defined gesture to be "any kind of movement performed during speaking" (p.4). Gestures "may provide a window onto knowledge that is not readily expressed in speech" (Alibali, Bassok, Solomon, Syc, & Goldin-Meadow, 1999, p. 327). Video provides access to this data not available in other forms of data collection. It enables the researcher to take advantage of participants' non-verbal communications, such as sound and images containing facial expressions, tone of voice and gestures, together giving insights into emotions and depth of understanding of concepts (Cope, 2000; Pea, 2006). Pea (2006) claims that video enables the collection of richer and more reliable data about complex social interactions, such as interviews.

A difficulty faced by the researcher is the interpretation of pauses in the audio record. Video enables the researcher to formulate interpretations, such as indications of uncertainty (Reynolds & Reeve, 2002), of these gaps in the audio record. So, participants' representational gestures along with significant pauses, give "a more complete picture of the students' actual understanding of the problem domain" (Reynolds & Reeve, 2002, p.457). Video is especially useful for fast moving and complex events since it can be replayed repeatedly, picking up subtle details and checking interpretations, to ensure accuracy of transcripts and analysis (Pea, 2006). This fine-grained analysis may disclose insights into students' understanding not otherwise available.

Goldin-Meadow (2004) suggests that teachers can and do interpret children's gestures, but do not always take them into account in their teaching. It is interesting to note that Iverson and Goldin-Meadow (1998) report that blind people spontaneously gesture in the same way as sighted people. This suggests that gesture performs the function of assisting gesturers to clarify their own thoughts, so gesture may provide the researcher a window on those thoughts. In this study, the participants are young adolescents who may not be able to express clearly their conceptions of mathematical ideas. Broaders, Wagner Cook, Mitchell, and Goldin-Meadow (2007) assert, "speakers' gestures can reveal knowledge that they have but cannot yet articulate" (p. 539), so a close examination of students' gestures may offer insights into students' conceptions not available from the transcriptions of their utterances, for example, pointing to a particular part of the screen of the computer-based simulations used in this study. When speech adequately conveys a speaker's meaning, gesture only plays a supportive role, but when gestures convey information which differs from the information provided by speech (Goldin-Meadow, 2000) interpretation of gestures becomes especially important. The researcher scrutinising video for students' conceptions needs to be alert for instances where gestures contradict speech, as well as instances where gesture elaborates on speech.

Attention to gesture is a growing research area in the domain of mathematics education research (See for example, Reynolds & Reeve, 2002; Rasmussen, Stephan & Allen, 2004; Arzarello & Robutti, 2004; Arzarello, Robutti, & Bazzini, 2005; Edwards, 2005; Williams & Wake, 2004). Arzarello et al. (2005) in their study of middle-years students' conceptions of "variable" and "function", claim that gestures "play an important role in interpreting the students learning processes" (p.64). Reynolds and Reeve (2002) investigated the significance of two students' gestures in the interpretation of speed-time graphs, and they assert that the expression of difficult concepts, such as functions, may be facilitated by the support of gestures, especially if the language required is specific to a particular domain, for example, mathematics.

Interpretation of gestures informs teachers' and researchers' understanding of students' thinking. However, gestures are sometimes difficult to interpret (Williams & Wake, 2004) but practice in interpretation of gesture can lead to a better assessment of participants' knowledge (Kelly et al., 2002; Gerofsky, 2010). Classification of gestures may assist a researcher's interpretation of participants' gestures. Parrill and Sweetser (2004) propose that

the analyst's claims about gestural meaning can be explicitly laid out, and furthermore, that this enterprise can profit from the use of a framework ... [and] can therefore be used to express both correspondences between physical forms in space (hands, e.g.) and meanings (ideas, e.g) (p.198).

McNeill (1992) proposes a general classification of four types of hand gestures: beat, deictic, iconic and metaphoric. Beat gestures reflect the tempo of speech or emphasise aspects of speech. Gestures are classified as deictic when the participant is pointing at a real item, indicating directions or referring to something previously discussed. Iconic gestures resemble physical phenomena, such as when hands are held to represent the shape of a ball,

and are usually easy to interpret. Metaphoric gestures clearly have some meaning representing an abstract idea, but are more difficult to interpret. However, it may be that interpretation of gestures in specialised domains, such as mathematics, may require an extension of McNeill’s classification to include domain specific gesture types.

Edwards (2005) proposes a revision of McNeil’s iconic classification to include “iconic-physical”, for iconic gestures resembling the physical phenomenon referred to in speech, and “iconic-symbolic”, for iconic gestures referring to mathematical symbols or written processes. In addition, she conjectures that the special role of semiotics in mathematics may need further classifications to facilitate the interpretation of gestures used in explaining mathematical understanding. Arzarello and Robutti (2004) use the term “iconic-symbolic” in the same manner as Edwards, but extend the categorisation further to include “iconic-representational” gestures that are gestures related to graphs. Rasmussen et al. (2004) refer to a number of function gesture types in their study of classroom practices in a class studying first-order differential equations. Images in their paper clearly convey several gesture types and their interpretation in the context of the particular class. They mention gesture classifications for functions: “slope shifting gesture” (p.312); “moving rate gesture”; “slope hand gesture”; “pointing function gesture”; and “pointing moving slope gesture” (p.309).

Awareness of gesture types from previous research provided guidance in the interpretation of participants’ gestures in data initially collected to reveal middle secondary students’ conceptions of rate of change (Herbert & Pierce, 2009). This paper describes additional gesture types related to rate, and hence functions, evident in this data.

Method

Two computer-based simulations were prepared: one in JavaMathWorlds (JMW) simulating two characters walking (Figure 1) (Mathematics Education Researchers Group, 2004); and the other in Geometers’ Sketchpad (GSP) (Key Curriculum Press, 2006) simulating two windows with blinds (Figure 2). These simulations were chosen to provide experientially real-world instances of functions with ease of access to the multiple mathematical representations of the functions: numeric; graphic; and symbolic. It was thought that these twenty-three Year 10 participants may respond differently in different contexts or representations so these two simulations offered contrasting real-world contexts in which functions may be found.

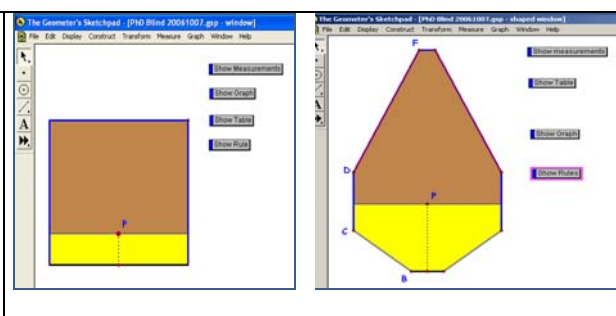
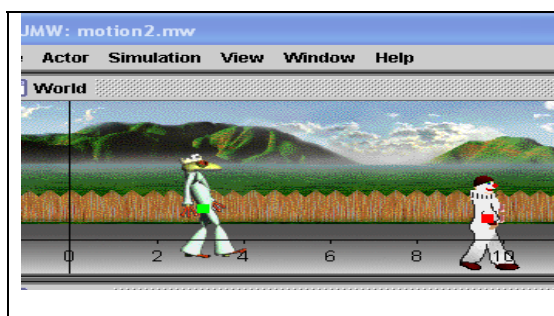


Figure 1. JMW screen with frog & clown walking

Figure 2. GSP screen of diagram of two blinds

Each participant (pseudonyms used) was videoed as they responded to the interviewer who prompted them to discuss the functions seen in the simulations. They were encouraged to explain their reasoning and think aloud as they were presented with different representational forms of function: the simulation, table of values, graph and rule. The video

of each interview was viewed repeatedly to identify gesture episodes which were then isolated into separate clips resulting in approximately twenty to thirty clips per video. Each clip was scrutinised to identify the gesture type shown.

Results

The gesture types noted in the videos combined McNeill's (1992) deictic and metaphoric categories with the refinements of the iconic classification of iconic-representational (Arzarello & Robutti, 2004) and iconic-physical and iconic-symbolic (Edwards, 2005). Also evident were gesture episodes which appeared to correspond to gestures identified by Rasmussen et al. (2004). In addition, further gesture types were observed and five have been described in detail below.

Relationship Gesture

Expression of the relationship between the variables connected in a function is fundamental to a deep understanding of functions. Several participants indicated their awareness of this relationship through the gestures they employed. A typical example of the *relationship gesture* can be seen in Figure 3, where Jo employs deictic gestures combined with a relationship gesture, pointing to particular places on the screen to clarify his explanation of his calculation of constant rate, and then moving his left hand in an upward followed by the horizontal movement of his right hand. This relationship gesture comprising two distinct movements indicates Jo's awareness of the relationship between the variables of area and height and is not just reading values off the graph.

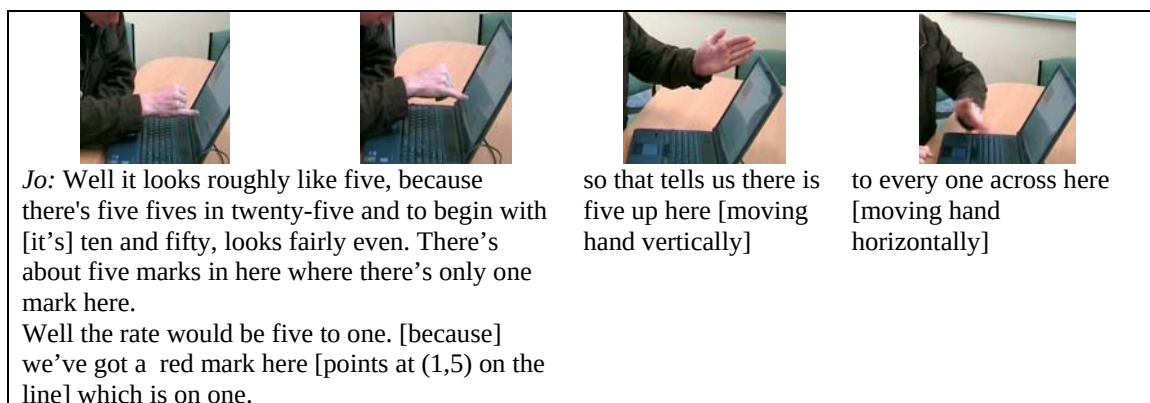


Figure 3. Gestures indicating awareness of relationship between the variables.

Imagined- formula Gesture

In Figure 4, Sue employs gestures to indicate the position of variables in the formula for speed. She is visualising the formula and showing her thinking with the gestures used. This indicates awareness of the need for both distance and time in the calculation of speed, and also expectation that a formula would be supplied to complete the calculation of rate. In this case the participant has remembered the correct formula to use and has experienced substituting values into it.

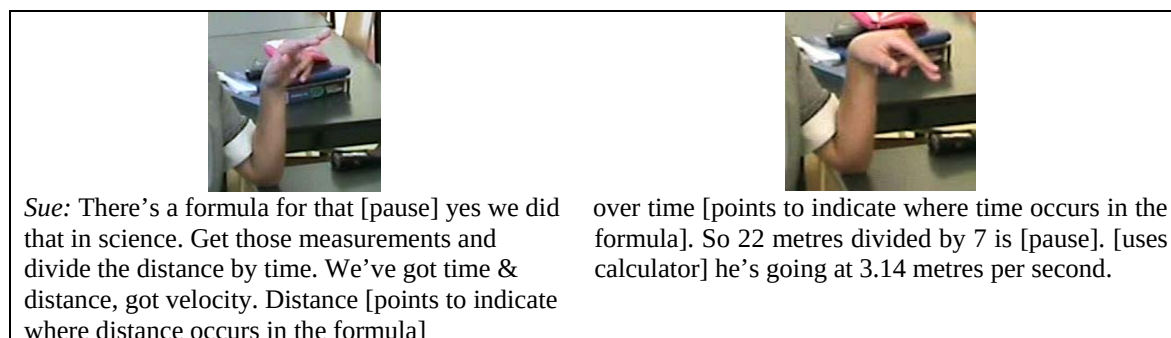


Figure 4. Gestures indicating awareness of the position of variables in formula for speed

Air- graphs Gesture

The *air-graphs gesture* type refers to drawing out the shape of graphs without using a pencil and paper. This can take several different forms. It may be tracing out the graph on a computer screen or drawing with a finger on a desk or drawing in the air. This type of gesture can be seen in Figure 5, where Mimi traces out the shape of the graph in a vertical plane in the air, helping her to explain her reasoning that the speed was not constant.

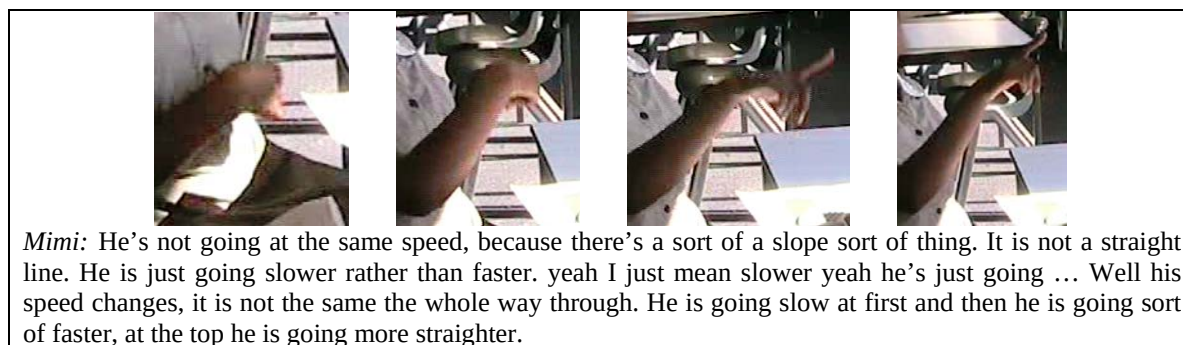


Figure 5. Air-graph gesture

Imaginary- axis Gesture

The *imaginary-axis gesture* type refers to the use of an axis imagined on a table or in the air to support the student's thinking about the change in one of the variables. In Figure 6, Noni is looking at an automated version of the rectangular blind and considering the rate the blind is moving. Noni has been given a timer and the symbolic representation of the linear function.



Figure 6. Imaginary-axis gesture

First Noni establishes the position of the origin by placing her left index finger on the edge of the table. Then she places the index finger on the table to indicate the first position

considered when thinking about the graph, followed by a movement of this finger to further along the imaginary x -axis. This gesture episode suggests that Noni is thinking about the changes in the variable of time in an attempt to express her thinking about the function involved.

Table- difference Gesture

In Figure 7, Verity gestured repeatedly with a curved arch shape with thumb and first finger, sometimes used to indicate a small distance (Herbert & Pierce, 2007), but in this case indicating the difference between values in the table. She holds her fingers in an arched shaped and repeatedly moves it downward to emphasise the common difference coming down the table for a linear function compared to the changing difference for a non-linear function.

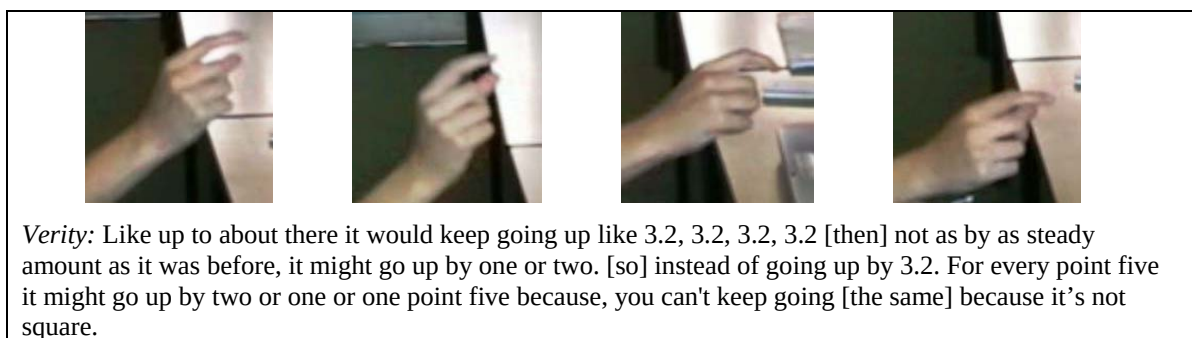


Figure 7. Table-difference gesture

Discussion

In addition to attending to the gesture types described by McNeil (1992), Edwards (2005), Arzarello and Robutti (2004) and Rasmussen et al. (2004) these new gesture types facilitated the interpretation of participants' responses, a combination of verbal and gestures, and supported the analysis of phenomenographic interviews revealing middle secondary students' conceptions of rate of change (Herbert & Pierce, 2009). These additional gesture types also have applicability when considering students' understanding of functions in general.

Whilst the example of the relationship gesture type (Figure 3) shows a participant using one hand, other forms of the relationship gesture might involve the use of both hands to express the relationship (Herbert & Pierce, 2007). The intention of this gesture type classification is to identify gestures which indicate an awareness of a relationship between the function variables. Observation of a relationship gesture informs the teacher or researcher that there is a simultaneous awareness of the changes in two variables and the relationship between them. This is especially important when the student does not possess the words to explain this relationship but can demonstrate their understanding of the relationship by the gestures they employ (Broaders et al., 2007).

Instances of an imagined-formula gesture inform the teacher or researcher that the student understands the position of the variables in a formula and gives an indication of which formula is being imagined. It implies that there is an assumption that mathematics is formula driven and all one requires, when solving a mathematical problem, is the appropriate formula. Employment of this gesture type may suggest inadequate

understanding of the concepts behind the derivation of the formula and further probing may be necessary to reveal the extent of a student's understanding.

The air-graphs gestures are useful in interpreting students' thinking and assessing their understanding of the shape of the graphic representation of a function. Attention paid to air-graphs may inform teachers and researchers of the depth of a student's understanding of the relationship between the representations of functions, for example, when the symbolic representation of a quadratic function is correctly matched with an appropriate air graph the observer can infer that the student is able to transfer some understandings about the function across representations.

Students may employ the table-difference gesture to help explain the manner in which the values in the table differ. This gesture type indicates awareness of the changes in at least one variable. However, if the focus is only on one column, this gesture type may suggest a lack of awareness of the relationship between the variables. A teacher may grasp this teachable moment to discuss the differences in the other column of the table and emphasise the relationship between the columns as a focus on individual columns may result in an inability to connect the columns and hence lack awareness of the relationship between the variables. Similarly, when students display an imaginary-axis gesture, it may imply that they are only focussing on one variable in the function relationship and may not be aware of the necessity to consider both variables, nor understand that the function describes the relationship between the variables.

Conclusion

Gesture classifications supported the detailed analysis of data provided by video-recorded interviews. This analysis afforded increased awareness of participants' understanding of the concepts related to functions embedded in the computer simulations, which provided stimulus for the participants to discuss their mathematical conceptions. These participants, in middle years of secondary schooling, did experience difficulty in expressing their understanding (Reynolds & Reeve, 2002). However, they demonstrated an extensive use of gestures to supplement their utterances (Kelly et al., 2002) in order to explain their thinking about rate of change, and hence functions. The computer simulations gave participants an opportunity to employ deictic gestures to indicate positions on the screen to support and expand their explanations of their mathematical conceptions.

This paper extends or elaborates on existing gesture classifications. The gesture episodes presented illustrate some useful gesture types which, in addition to gesture types described by McNeil (1992), Edwards (2005), Arzarello and Robutti (2004) and Rasmussen et al. (2004), provided insights into participants' thinking not available from their words alone.

Five new gesture types were identified and are presented in this paper. Two gesture types, air-graphs and imaginary-axis gesture types, could be considered as sub-categories of "iconic-representational" (Arzarello & Robutti, 2004) whilst imagined-formula gesture type may be a sub-category of "iconic-symbolic" (Edwards, 2005). The relationship and table-difference gesture types appear to be entirely new classifications.

The evidence presented in this paper highlights the importance of teachers and researchers attending to students' gestures to gain insights into their thinking. Further research is needed to clarify the interpretation of gesture related to mathematical notions.

Acknowledgements

This study is part of the RITEMATHS project, led by Kaye Stacey, Gloria Stillman and Robyn Pierce. The researchers thank the Australian Research Council, our six partner

schools and Texas Instruments for their financial support of this project, and especially the participants.

References

- Alibali, M., Bassok, M., Solomon, K., Syc, S., & Goldin-Meadow, S. (1999). Illuminating mental representations through speech and gesture. *Psychological Science*, 10(4), 327-333.
- Arzarello, F., Robutti, O., & Bazzini, L. (2005). Acting is learning: Focus on the construction of mathematical concepts. *Cambridge Journal of Education*, 35(1), 55-67.
- Arzarello, F. & Robutti, O. (2004). Approaching functions through motion experiments. *Educational Studies in Mathematics* 57(3), 305-308.
- Broaders, S., Wagner Cook, S., Mitchell, Z., & Goldin-Meadow, S. (2007). Making children gesture brings out implicit knowledge and leads to learning *Journal of Experimental Psychology: General*, 136(4), 539-550.
- Cope, C. (2000). *Educationally critical aspects of the experience of learning about the concept of an information system*. Unpublished PhD thesis [On-line]. Accessed 29 January, 2010 from <http://www.ironbark.bendigo.latrobe.edu.au/staff/cope/cope-thesis.pdf>
- Edwards, L. (2005). The role of gestures in mathematical discourse: Remembering and problem solving . In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 135-138). Melbourne: University of Melbourne.
- Feyereisen, P., & de Lannoy, J. (1991). *Gestures and speech: Psychological investigations*. Cambridge, UK: Cambridge University Press.
- Gerofsky, S. (2010). Mathematical learning and gesture: Character viewpoint and observer viewpoint in students' gestured graphs of functions. *Gesture*, 10(2/3), 321-343.
- Goldin-Meadow, S. (2000). Beyond words: The importance of gesture to researchers and learners. *Child Development*, 71(1), 231.
- Goldin-Meadow, S. (2004). Gesture's role in the learning process. *Theory Into Practice*, 43(4), 314-321.
- Herbert, S. & Pierce, R. (2007). Video evidence: What gestures tell us about students' understanding of rate of change. In Watson, J. & Beswick, K. (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Hobart). Adelaide: MERGA.
- Herbert, S. & Pierce, R. (2009). Revealing conceptions of rate of change. In Hunter, R., Bicknell, B., & Burgess, T. (Eds.), *Crossing divides* (Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia). Palmerston North, NZ: MERGA.
- Iverson, J. M., & Goldin-Meadow, S. (1998). Why people gesture when they speak. *Nature*, 396 (6708), 228.
- Kelly, S. D., Singer, M., Hicks, J., & Goldin-Meadow, S. (2002). A helping hand in assessing children's knowledge: Instructing adults to attend to gesture. *Cognition & Instruction*, 20(1), 1-26.
- Key Curriculum Press. (2006). *Geometers' SketchPad* [software]. Accessed 29 January, 2010 from <http://www.keypress.com/x5521.xml>
- McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.
- Mathematics Education Researchers Group. (2004). *SimCalc Projects*. [software]. Accessed August 22, 2005, from <http://www.simcalc.umassd.edu/>
- Parrill, F., & Sweetser, E. (2004). What we mean by meaning: Conceptual integration in gesture analysis and transcription. *Gesture*, 4(2), 197-219.
- Pea, R. D. (2006). Video-as-data and digital video manipulation techniques for transforming learning sciences research, education and other cultural practices. In J. Weiss, J. Nolan & P. Trifonas (Eds.), *International handbook of virtual learning environments*. Dordrecht: Kluwer Academic Publishing.
- Rasmussen, C., Stephan, M., & Allen, K. (2004). Classroom mathematical practices and gesturing. *Journal of Mathematical Behavior*, 23(3), 301-323.
- Reynolds, F. & Reeve, R. (2002). Gesture in collaborative mathematics problem-solving. *Journal of Mathematical Behavior*, 20(4), 447-460.
- Williams, J. S., & Wake, G. D. (2004). Metaphors and cultural models afford communication repairs of breakdowns between mathematical discourses. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of 28th Conference of the International Group for the Psychology of Mathematics Education*. Bergen, Norway: Bergen University College.

Evaluating Middle Years Students' Proportional Reasoning

Annette Hilton
The University of Queensland
<a.hilton@uq.edu.au>

Shelley Dole
The University of Queensland
<s.dole@uq.edu.au>

Mia O'Brien
The University of Queensland
<mia.obrien@uq.edu.au>

Geoff Hilton
The University of Queensland
<g.hilton@uq.edu.au>

Merrilyn Goos
The University of Queensland
<m.goos@uq.edu.au>

Proportional reasoning is a key aspect of numeracy that is not always developed naturally by students. Understanding the types of proportional reasoning that students apply to different problem types is a useful first step to identifying ways to support teachers and students to develop proportional reasoning in the classroom. This paper describes the development of a diagnostic instrument that aims to identify situations in which students can apply proportional reasoning and the types of reasoning they use.

Proportional reasoning is mathematical reasoning involving a sense of co-variation and multiple comparisons (Lesh, Post, & Behr, 1988). It requires multiplicative and relational thinking, which do not always develop naturally in students (Sowder et al., 1998). Lesh et al. (1988) described proportional reasoning as the cornerstone of higher level areas of mathematics, such as algebra, and the capstone of elementary concepts, such as arithmetic, number, and measurement. The importance of proportional reasoning goes beyond the mathematics classroom; its importance in other areas of study, for example, science, economics, and demographics has been widely recognized (Akatugba & Wallace, 2009; Boyer, Levine, & Huttenlocher, 2008). Perhaps more importantly, proportional reasoning is an essential aspect of everyday applications of numeracy. According to Ahl, Moore, and Dixon (1992), proportional reasoning is a “pervasive activity that transcends topical barriers in adult life” (p. 81) and yet, it has been estimated that more than half of the adult population are not proportional thinkers (Lamon, 1999).

Until recently, little was known about students' development or application of proportional reasoning (Lamon, 2005). Establishing the types and accuracy of reasoning used by students in situations of ratio and proportion can assist teachers in selecting teaching strategies and learning activities to target students' learning needs and strengthen their ability to reason proportionally (Misailidou & Williams, 2003). This paper describes the development of a diagnostic instrument designed to identify the types of reasoning used by students in a range of proportional and non-proportional situations and presents results from the pilot instrument to illustrate its use in diagnosing students' strategies and understanding.

Rationale

Several researchers have highlighted the importance of establishing children's proportional reasoning skills (e.g., Bright, Joyner, & Wallis, 2003; Misailidou & Williams, 2003; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). Methods that can be used to determine students' understanding include interviews, pen-and-paper tests (open-ended or multiple choice items), and concept mapping (Tüysüz, 2009). According to Wiggins and McTighe (1998), assessment must require students to explain or defend their

answer. While interviews provide a powerful method of establishing students' understanding, this is impractical in many situations, for example, in research situations where large numbers of students are involved. Tamir (1989) found that asking students for justifications when answering multiple choice questions is a sensitive and effective means of assessing students' understanding and argued that such an approach addresses some limitations commonly associated with traditional multiple-choice questions. To account for these arguments, Treagust (1995) recommended the use of two-tiered multiple-choice instruments as an alternative to individual interviews as a means of obtaining information about the reasoning of groups of students. The first tier of a two-tiered item consists of a multiple-choice question, with two to four choices. The second tier requires students to choose from four reasons to justify or explain their answer to the first tier question. To be considered correct, students must answer both levels of the question correctly. The data generated provide an insight into the common reasoning strategies employed by classes or year level cohorts.

For a number of years, science education researchers have designed and successfully used two-tiered multiple-choice instruments to diagnose students' understanding in a number of topics, as a means of informing pedagogical strategies (Chandrasegaran, Treagust, & Mocerino, 2008; Haslam & Treagust, 1987; Özmen, 2008; Tan & Treagust, 1999; Treagust, 1995, 2006; Tüysüz, 2009). O'Keefe and O'Donoghue (2011) used a two-tiered instrument based on Treagust's work to investigate the effectiveness of an intervention strategy in lower secondary mathematics classrooms; however, they did not provide examples of the items used. To date, such instruments have not been widely used for investigating students' understanding in mathematics.

Method

Instrument Design

The design of many two-tiered items is based on findings reported in research literature about students' common alternative conceptions, or the errors and difficulties they commonly encounter in a particular topic, concept, or reasoning situation. In the case of the instrument described here, research literature and findings from a previous study by members of the research team informed both the choice of problem type and the nature of the reasoning responses in the second tier of each item.

A detailed review of the literature in this area revealed varied descriptions of proportional reasoning problem types and the circumstances in which students typically use reasoning strategies correctly and incorrectly. A brief overview is provided here. Lamon (1993) identified four types of proportion problems: well-chunked or well-known measures (e.g., relationships that are commonly used rates, such as speed); part-part-whole (e.g., ratio problems in which two complementary parts are compared with each other or the whole); associated sets (rate situations in which the relationship between quantities is defined within the question); and stretchers and shrinkers (growth or scale problems). Lamon's categories formed the basis of the item types, however, Lesh et al. (1988) described several problem types that are neglected in textbooks, instruction, and research (e.g., problems involving between or within representation translations and missing value problems). Further, Van Dooren et al. (2005) suggested that often, students rely on proportional reasoning in circumstances that do not require it (e.g., constant, linear, and additive situations). Following a similar line of reasoning, Bright et al. (2003) stated that in order to assess

students' ability to reason proportionally, it is important to provide situations in which students can correctly and incorrectly use multiplicative and additive thinking.

The items on the instrument were chosen to account for these arguments and suggestions and were of the following types:

- Non-proportional (constant or additive situations)
- Part-part-whole, missing value problems
- Scale (one- and two-dimensional)
- Rate (well-chunked measures, between representation translation)
- Relative - absolute (associated sets, within and between representation translation)
- Inverse proportion (well-chunked measures, associated sets)

Nine of the twelve items were based on items from the *Keeping it in Proportion* test (KIIP), used in a study by Dole, Clarke, Wright, and Hilton (2007). In that study, over 800 students (Years 5-9) completed the KIIP test, which employed short response questions that required students to show their working and provide written explanations of their calculated answers. These problems were re-worded to present a scenario and make a statement in the first tier, to which students responded *True* or *False*. To ensure that the instrument described here included items that targeted constant and additive non-proportional situations and two-dimensional scale, three additional items were based on problem types used by Bright et al. (2003) and Van Dooren et al. (2005).

The responses in the second tier of each item were based on the common strategies used by students in situations of proportion described in the literature. For example, multiplicative thinking is the critical factor for the comparison of quantities in proportion-related tasks. In contrast, additive reasoning involves considering the sums or differences in quantities (Bright et al., 2003). Students often find it difficult to discern which reasoning to use in a particular situation. Other difficulties associated with proportional situations and the errors commonly made by students that have been identified by researchers include

- an inability to discern when to use proportional reasoning and difficulty in identifying multiplicative or relative relationships (Van De Walle, Karp, & Bay-Williams, 2010);
- a tendency to approach proportional situations additively instead of multiplicatively (Cramer & Post, 1993; Lamon, 1993; Misailidou & Williams, 2003; Tourniaire & Pulos, 1985);
- the use of multiplicative approaches unnecessarily, for example, when additive strategies should be used (Van Dooren, De Bock, & Verschaffel, 2010);
- inappropriate use of algorithms, such as cross multiplication (Nabors, 2003); and
- incorrect build-up/pattern building (Lamon, 1993; Misailidou & Williams, 2003).

In the instrument described here, second tier responses were used to identify students' correct and incorrect application of additive and multiplicative thinking, their recognition of absolute and relative situations, their ability to distinguish situations of proportion from non-proportion, and the strategies they used. In addition to the literature around students' difficulties in situations of proportion, the findings of the study by Dole et al. (2007) in which the KIIP test were used informed the wording of some options in the second tier of the items.

To illustrate the two-tier items, an example is provided. In the item, students were asked to compare two washing powders and to respond to the statement “Powder A is the better value”. The students were provided with a graphic that portrayed the following information:

Powder A comes in 1 kg containers that cost \$4 for 20 loads of washing.

Powder B comes in 1.5 kg containers that cost \$6.50 for 30 loads of washing.

The second tier responses were

1. Washing powder A costs the least.
2. Washing powder B costs a little bit more but you get 10 more loads of washing.
3. The cost per load of washing is less.
4. Both washing powders are the same value.

This problem requires students to use relative thinking in a situation involving an associated set. Option 1 indicates that the student has used absolute thinking, comparing only price. Option 2 shows that the students have considered only the number of loads and not the price. Option 3 is the most accurate response and shows that students have used relative thinking. Option 4 suggests that students are considering the mass and number of loads but not the cost in their reasoning.

Administration of the Pilot Instrument

Prior to administering the instrument to the pilot group, the items were reviewed and revised by members of the research team (who have extensive mathematics teaching experience in primary and/or secondary schools) and a sample of teachers. The pilot instrument was administered to 140 Year 5 and 6 students (11-12 years old) in composite classes in two primary schools. Three classes of students from each school completed the instrument. Prior to administration of the instrument, the school principals were briefed on the purpose and administration procedure for the instrument. The teachers were provided with written instruction regarding the its administration.

The students’ responses were coded and the percentage of students who responded to each alternative was calculated. The results of each class, school and the whole group were compared to determine whether the results were consistent or whether there were anomalies in the data. The results were similar for each group.

The combinations of responses to the first and second tiers allow identification of the students’ reasoning in each item. Tan and Treagust (1999) suggested that when interpreting such data for the purpose of gaining an understanding of students’ misconceptions, it is reasonable to consider those response combinations that exist for at least 10 per cent of students. The incorrect responses for which the percentage of students was greater than 10 per cent were further investigated to determine the types of reasoning used by the students.

Results

The data for the full sample of 140 students, shown in Table 1, show that the majority of students were able to answer Item 1 and just over half answered Item 7 correctly, however, for a number of items, a high percentage of students chose an incorrect combination.

Table 1
Percentage of Year 5 and 6 Students (n = 140) Selecting Each Response Combination

Item	Response							
	TA	FA	TB	FB	TC	FC	TD	FD
1	13.6	2.1	2.1	0.7	3.6	70.0*	3.6	0.7
2	5.0	5.0	8.6	45.7*	21.4	5.0	5.7	2.9
3	3.6	47.9	12.1	9.3	6.4	13.6	5.0*	2.9
4	1.4	9.3	3.6	32.9	14.3*	10.7	2.9	23.6
5	12.1	5.7	0.7	34.3*	1.4	6.4	2.1	33.6
6	15.7	1.4	4.3	42.9	27.9*	2.1	0.7	2.1
7	5.0	17.1	0.0	52.1*	6.4	8.6	5.0	6.4
8	2.9	23.6*	2.9	10.0	2.9	14.3	37.9	10.7
9	58.6	9.3	3.6	10.7	4.3	0.0	0.7	12.9*
10	4.3	2.1*	78.6	2.1	7.1	0.7	5.0	0.0
11	33.6	13.6	20.0*	7.1	4.3	10.0	0.7	7.1
12	0.7	21.4	4.3	13.6	19.3*	4.3	5.7	29.3

Notes. *indicates the correct response combination for the item. TA, FA, etc. denote first and second tier response combinations.

Examination of the combinations of incorrect responses selected by at least 10 per cent of the students allowed identification of the reasoning used in each situation. In some circumstances, the reasoning used indicated that the students had some level of qualitative understanding of the situation or relationship. The strategies used are shown in Table 2.

Table 2
Common Incorrect Strategies Used by Students

Strategy or error	Choice	%
<i>Some qualitative understanding of relationship</i>		
Understands need to compare part with whole	Item 3 (FC)	13.9
Qualitative understanding of speed-time relationship	Item 5 (FD)	33.6
Qualitative understanding of inverse relationship	Item 8 (D)	48.6
Some evidence of qualitative understanding of linear scale	Item 9 (FB)	10.7
Qualitative understanding of part-part-whole	Item 7 (FA)	17.1
<i>Making absolute comparison instead of relative comparison</i>		
Considering one value without considering the total	Item 3 (TB, FA)	60.0
Considering one value without considering a related value	Item 6 (TA)	15.7
	Item 12 (FA, FB)	35.0
Reading scale as absolute, not considering value of units	Item 9 (TA)	58.6

<i>Using multiplicative strategies inappropriately</i>		
Applying a multiplicative strategy to a constant situation	Item 1 (TA)	13.6
Applying a multiplicative strategy to an additive situation	Item 2 (TC)	21.4
Treating an inverse situation as if it involved direct proportion	Item 5 (TA)	12.1
Incorrect reasoning but recognizes inverse situation	Item 8 (FC)	14.3
No recognition of impact of changing two dimensions on area	Item 10 (TB)	78.6
<i>Using additive strategies in proportional situation</i>		
Increases two quantities by the same amount (direct)	Item 4 (FB, FD)	56.5
Increases two quantities by the same amount (inverse)	Item 8 (FB)	10.0
<i>Erroneous calculation / no calculation</i>		
Inaccurately uses multiplicative reasoning	Item 4 (FC)	10.7
Not calculating a value/estimating	Item 6 (FB)	42.9
<i>Faulty understanding of visual representations</i>		
Interprets graph accurately but chooses irrelevant response	Item 11 (A)	47.2
Misinterprets a distance-time graph	Item 11 (FC)	10.0
Misinterprets relative amounts in pictorial representation	Item 12 (FD)	29.3

Discussion and Conclusion

The results of the pilot instrument showed that it was useful for identifying the students' reasoning and the areas in which they applied incorrect reasoning strategies. The findings also aligned with the findings in the research regarding common errors and problems encountered by students in situations of proportion. As indicated by the data in Table 1, the students' success on the items varied. While many students answered some items correctly, the majority of students found particular items challenging, for example, Item 3 (requiring relative thinking) and Item 10 (requiring understanding of two-dimensional enlargement).

A major problem for students appears to be discriminating non-proportional from proportional situations. For example, for both non-proportional situations (Items 1 and 2), the majority of students who chose incorrectly used multiplicative thinking. Similarly, in proportional situations (Items 4 and 8), the majority of incorrect responses indicated the use of additive approaches. In addition to inappropriately using additive and multiplicative strategies, many students used absolute comparison in situations requiring relative thinking. For example, in several items, they only considered one variable when it was necessary to compare two parts or one part to the whole (Items 3, 6, 9).

It is important to note that in several items (Items 3, 5, 7, 8, 9), a number of students responded in ways that indicated a level of qualitative understanding of the relationship involved. This perhaps suggests that more targeted work with such students might assist them to develop a deeper understanding.

The data from the pilot instrument are for students in Years 5 and 6. As the final instrument is intended for use across Years 5 to 9, the items were designed to allow discrimination across year levels. This was found to be the case and may account for the low scores on some of the pilot items. For example, situations of inverse proportion (Items 5 & 8) are not usually a focus for students in primary years. Based on the data and feedback from the pilot, the original 12 items were retained for the final instrument. The items were

arranged randomly so that no consecutive items were of the same problem type. Although some minor rewording was applied to make the meaning of the question clearer, the items and the second tier responses were retained.

The proportional reasoning two-tiered instrument described in this paper is a new approach to assessing students' proportional reasoning. The data generated from two-tiered instruments allows students' common errors or partial understanding to be identified, which informs researchers and teachers about the areas that teachers might target to enhance students' proportional reasoning. For example, those items that indicate students' use of absolute comparisons when they should make relative comparisons inform teachers that this is an area of weakness in their students. Such data may also be used to inform the design of teacher professional development to promote teachers' understanding of the different types of proportional reasoning and classroom strategies to address their students' needs.

The instrument described in this paper may be used as a pre-test and post-test to allow the researchers and teachers to track students' progress and to identify changes in their proportional reasoning to inform teachers' future curriculum planning and practice. In light of the importance of proportional reasoning development and the difficulties students encounter when reasoning proportionally, the various applications of this instrument have the potential to benefit teachers, curriculum planners, researchers and teacher educators.

References

- Ahl, V. A., Moore, C. F., & Dixon, J. A. (1992). Development of intuitive and numerical proportional reasoning. *Cognitive Development*, 7, 81-108.
- Akatugba, A. H., & Wallace, J. (2009). An integrative perspective on students' proportional reasoning in high school physics in a West African context. *International Journal of Science Education*, 31(11), 1473-1493.
- Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology*, 44(5), 1478-1490.
- Bright, G. W., Joyner, J. M., & Wallis, C. (2003). Assessing proportional thinking. *Mathematics Teaching in the Middle School*, 9(3), 166-172.
- Chandrasegaran, A. L., Treagust, D. F., & Mocerino, M. (2008). An evaluation of a teaching intervention to promote students' ability to use multiple levels of representation when describing and explaining chemical reactions. *Research in Science Education*, 38, 237-248.
- Cramer, K., & Post, T. (1993). Making connections: A case for proportionality. *Arithmetic Teacher*, 60(6), 342-346.
- Dole, S., Wright, T., Clarke, D., & Hilton, G. (2007). Making connections science and mathematics: The MCSAM Project. In U. Cheah, Y. Wahyudi, R. Devadason, K. Ng, J. Chavez, & D. Mangao (Eds.). *Redefining Learning Culture for Sustainability*. Second International Conference on Science and Mathematics Education, Penang, Malaysia, (pp. 184-194). 13-16 November 2007.
- Haslam, F., & Treagust, D. F. (1987). Diagnosing secondary students' misconceptions of photosynthesis and respiration in plants using a two-tier multiple choice instrument. *Journal of Biological Education*, 21, 203-211.
- Lamon, S. J. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education*, 24(1), 41-61.
- Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah, NJ: Lawrence Erlbaum.
- Lamon, S. J. (2005). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah, NJ: Lawrence Erlbaum.
- Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: Lawrence Erlbaum & National Council of Teachers of Mathematics.
- Misailidou, C., & Williams, J. (2003). Diagnostic assessment of children's proportional reasoning. *Journal of Mathematical Behavior*, 22, 335-368.
- Nabors, W. (2003). From fractions to proportional reasoning: A cognitive schemes of operation approach. *Journal of Mathematical Behavior*, 22(2003), 133-179.

- O'Keefe, L., & O'Donoghue, J. (2011). The use of evidence based research on mathematics textbooks to increase student conceptual understanding. *International Journal for Cross-Disciplinary Subjects in Education (IJCDSE)*, 2(1), 304-311.
- Özmen, H. (2008). The influence of computer-assisted instruction on students' conceptual understanding of chemical bonding and attitude toward chemistry: A case study for Turkey. *Computers and Education*, 51(1), 423-438.
- Sowder, J., Aarmstrong, B., Lamon, S., Simon, M., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education*, 1, 127-155.
- Tamir P. (1989). Some issues related to the use of justifications to multiple-choice answers, *Journal of Biological Education*, 23, 285-292.
- Tan, K.-C. D., & Treagust, D. F. (1999). Evaluating students' understanding of chemical bonding. *School Science Review*, 81, 75-83.
- Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational Studies in Mathematics*, 16, 181-204.
- Treagust, D. F. (1995). Diagnostic assessment of students' science knowledge. In S. M. Glynn & R. Duit (Eds.), *Learning science in the schools: Research reforming practice* (pp. 327-346). Mahwah, NJ: Lawrence Erlbaum.
- Treagust, D. F. (2006). *Diagnostic assessment in science as a means to improving teaching, learning, and retention*. Paper presented at the UniServe Science Symposium Proceedings Assessment in Science Teaching and Learning, Sydney, NSW, Australia.
- Tüysüz, C. (2009). Development of two-tier diagnostic instrument and assess students' understanding in chemistry *Scientific Research and Essay*, 4(6), 626-631.
- Van De Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2010). *Elementary and middle school mathematics: Teaching developmentally*. Boston, MA: Allyn & Bacon.
- Van Dooren, W., De Bock, D., & Verschaffel, L. (2010). From addition to multiplication ... and back: The development of students' additive and multiplicative reasoning skills. *Cognition and Instruction*, 28(3), 360-381.
- Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2005). Not everything is proportional: Effects of age and problem type on propensities for overgeneralisation. *Cognition and Instruction*, 23(1), 57-86.
- Wiggins, G. & McTighe, J. (1998). *Understanding by design*. Alexandria, VA: Association for Supervision and Curriculum Development.

Singapore Students' Performance on Australian and Singapore Assessment Items

Siew Yin Ho

*Research Institute for Professional Practice,
Learning & Education, Charles Sturt
University*
<sho@csu.edu.au>

Tom Lowrie

*Research Institute for Professional Practice,
Learning & Education, Charles Sturt
University*
<tlowrie@csu.edu.au>

This study describes Singapore students' (N=607) performance on a recently developed Mathematics Processing Instrument (MPI). The MPI comprised tasks sourced from Australia's NAPLAN and Singapore's PSLE. In addition, the MPI had a corresponding question which encouraged students to describe how they solved the respective tasks. In particular, the investigation considers two tasks the cohort found difficult to solve—a relatively complex task from the Singaporean item bank and a novel task from the Australian tasks.

Introduction

The digital age has profoundly transformed the world around us. Tech-savvy citizens not only possess cognitive and emotional capacities, they engage with (and demand) a digital capacity which is used to navigate and make sense of the landscape of new online media (Kerr, 2011; Ong, 2011). Since online media consists of a myriad of spatial information in the form of diagrams, graphics and non-graphics, expectations for our capacity to represent, manipulate and decode information has increased and is still increasing (Lowrie & Diezmann, 2007). Interpreting and decoding spatial information has not only been recognised as essential in everyday life, this ability has become increasingly important in education contexts (Åberg-Bengtsson, 1999; Lowrie & Diezmann, 2011). In the Singapore primary mathematics classrooms, students, at an early age, are already required to make sense of spatial information in various learning contexts including graphic and non-graphic information in textbooks, the representation of information and the reinforcement of mathematics concepts (via digital and electronic forms).

The ability to make sense of spatial information is an important skill in concept acquisition (Ahsen, 1981, 1989) and aids in the process of problem solving (see Presmeg, 2008). Problem solving plays a key role in the Singapore primary mathematics curriculum (Ministry of Education, Curriculum Planning and Development Division, 2007). In particular, students are given both graphic and non-graphic problems to solve in the mathematics classrooms. Understanding the spatial information and relationships among the elements in the problem is an essential component in problem solving. It is obvious that spatial information can be found in graphic items. Spatial information can too be found in non-graphic items, that is, spatial relationships exist between the elements in the problem. In order to begin solving the problem, students are required to not only understand the relationship among elements in the problem, they also need to be able to interpret how these elements are related to each other spatially. It is important to note that processing information in non-graphic items is high as graphics are not provided in the problem to show the relationships among elements in the problem.

The (Re)presentation and Assessment of Mathematics

At a time when citizens require new forms of numeracy of function effectively in society, changes to assessment practices are occurring in mathematics classrooms. New levels of accountability are being placed on school systems (Lowrie & Diezmann, 2009), with national assessments of students' mathematics performance commonplace. Students performance, even at a primary school level, is reported to describe and rank the performance of individual schools (e.g., in Australia) or to select and apportion students for the transition to high school (e.g., Singapore). Not only do such national assessments have a different intent, the way in which mathematics concepts are presented and represented are quite different.

Indeed, both Australia and Singapore have national assessments for primary-age going students—NAPLAN (National Assessment Program – Literacy and Numeracy) in Australia and PSLE (Primary School Leaving Examination) in Singapore. The National Assessment Program – Literacy and Numeracy (NAPLAN) is one of the assessments under Australia's National Assessment Program which measures and determines whether or not Years 3 (age 8-9), 5 (age 10-11), 7 (age 12-13), and 9 (age 14-15) Australian students are meeting important educational outcomes. The NAPLAN tests "skills that are essential for every child to progress through school and life, such as reading, writing, spelling and numeracy" (Australian Curriculum, Assessment and Reporting Authority, 2011, n.p.). It was implemented nationwide in Australia since 2008 and undertaken annually in the second full week in the month of May.

The primary national assessment in Singapore takes the form of a high-stake examination, the Primary School Leaving Examination (PSLE). It was implemented in 1960 with the primary purpose of the examination to allocate placement for students into secondary school based on the score achieved (Tan, Chow, & Goh, 2008). The PSLE is undertaken by Primary 6 Singaporean students (age 11.5-12) at the end of their final year of primary school education. A student can sit for the PSLE as an approved Singapore institution candidate or as a private candidate. The PSLE is conducted in Singapore annually, usually from mid-August to early October.

This investigation is part of a larger cross-cultural study which focuses on students' interpretation of assessment tasks in numeracy and mathematics learning. The study draws attention to cross-cultural dimensions of students' learning by analysing students' mathematics assessment performance and sense-making in high performing but culturally different contexts—Australia and Singapore. Specifically, the larger project aims to: (a) compare the performance of Australian and Singaporean students on tasks sourced from each country's respective national assessment instruments; (b) identify the approaches and strategies students from different cultures employ to solve mathematics tasks; and (c) draw conclusions about the influence cultural and pedagogical practices have on students' approaches to solving these tasks.

To the best of our knowledge, no studies have simultaneously (1) compared the performance and (2) analysed the mathematics processes of students from different countries on mathematics items drawn from the respective countries. To date, most cross-cultural studies of students' performance are drawn from internationally generic instruments (e.g., Third International Maths and Science Study [TIMSS]). Such studies do not consider the different ways mathematics is present and represent within country—we argue this is important in an age where spatial and graphic-rich representations are increasingly used to display information.

Research Design and Methods

The focus of this paper is to report Singapore students' performance on tasks sourced from Australia's NAPLAN and Singapore's PSLE. This paper does not compare results of the Singapore students' performance with their Australian counterparts. Hence only information for the Singapore sample will be discussed in this paper.

The Participants

607 Grade 6 students (aged 11-12) from 5 Singapore schools (three government and two government-aided) took part in this study. There were 320 boys and 287 girls in the sample.

The Test Instrument

Students' ability to decode mathematics tasks were investigated using a *Mathematics Processing Instrument* (MPI) developed and designed by the research team. The instrument comprised 24 items sourced from both the Australia's National Assessment program Literacy and Numeracy (NAPLAN) and Singapore's Primary School Leaving Examination (PSLE), of which there were: (a) 6 graphic items from Australia's National Assessment Program Literacy and Numeracy (NAPLAN), (b) 6 non-graphic items from Australia's National Assessment Program Literacy and Numeracy (NAPLAN), (c) 6 graphic items from past-years Singapore's Primary School Leaving (PSLE) Examination, and (d) 6 non-graphic items from past-years Singapore's Primary School Leaving (PSLE) Examination.

The analysis in this study focuses on two items, namely the Postcard Problem (see Figure 1) and the Plum Problem (see Figure 2). Both items are non-graphic items where there is only text. In order to solve these items, students need to make sense of the spatial information embedded in the text. The Postcard Problem was sourced from Singapore's Primary School Leaving Examination (PSLE) and the Plum Problem was sourced from Australia's National Assessment program Literacy and Numeracy (NAPLAN).

Gilbert and Hazel have some postcards. After Gilbert gives 18 postcards to Hazel, he has 20 postcards more than her. How many more postcards than Hazel does Gilbert have at first?

Figure 1. A non-graphic Singapore item.

Peta has some plums to give to her friends.
If she gives each friend 4 plums, she will have 6 plums left over.
She cannot give each friend 5 plums because she would need 4 more plums.
How many plums does Peta have?

Figure 2. A non-graphic Australian item.

The Test Procedure

Two research staff from the research team attended specified schools in Singapore during their morning classes. The classroom teachers were asked to help administer the Mathematics Processing Instrument along with the research staff. The administration of the Instrument was in two parts. Firstly, students answered the 24 mathematics items as a pencil and paper test. The duration of the paper-and-pencil test was 1 hour. After a short break,

students then filled out the processing instrument. A common set of instructions (for both parts) were read to the students in each school.

Analysis of Data

The conceptual underpinnings for data analyses were drawn from the work of Presmeg (1986a; 1986b). Presmeg (1986a) defined a visual method as one that contains a visual image—either in the mind’s eye or a diagram—which is an essential component of the solution. A non-visual method, on the other hand, does not rely on a visual image. Presmeg also differentiated the different types of visual imagery in her study. For the purpose of our coding of data, we classify visual imagery as either (1) in the mind’s eye; or (2) diagrams or pictures (usually drawn on paper).

The data in the study were coded according to two criteria; (a) whether the answer was correct or incorrect, and (b) type of method of solution used. The type of method solution used was coded using the coding scheme shown in Table 1. The coding scheme incorporates Presmeg (1986b)’s definitions of visual imagery and also includes another method identified; namely, kinaesthetic imagery. Table 1 outlines the coding spectrum.

Table 1
Coding Spectrum

Coding	Method of Solution used
0	Use of a non-visual method
1	Use of a visual method – diagram on paper (pictorial)
2	Use of a visual method – concrete imagery (imagery)
3	Use of a visual method – kinaesthetic imagery (gesture)

A coding of ‘0’ was assigned to students who used a non-visual method. A non-visual method is one that does not have a visual image as an essential part of the method of solution. A coding of ‘1’ was given to method of solution that showed use of a visual method where a diagram drawn on paper plays an essential role in the solution. A coding of ‘2’ was given when the student used concrete imagery as a method of solution, that is, using “picture in the mind”. A coding of ‘3’ was assigned to method of solution that involved kinaesthetic imagery.

Results and Discussion

This section discusses two items that were performed poorly by the Singaporean students; namely, the Postcard Problem and the Plum Problem. Both problems are non-graphic items. The overall percentage correct for these two items was 36.7%, and 39.7% respectively.

The Postcard Problem

The Postcard Problem is an item from the Singapore past-years Primary School Leaving Examination (PSLE). Table 2 shows the performance of students in this item and the percentage of the various solution methods used by the students.

Table 2
Correct and Incorrect Responses by Approach for The Postcard Problem

Coding	Incorrect Response	Correct Response	Total
0	188 (35.8%)	67 (12.8%)	255 (48.6%)
1	115 (21.9%)	154 (29.3%)	269 (51.2%)
2	1 (0.2%)	0 (0%)	1 (0.2%)
3	-	-	-
Total	304 (57.9%)	221 (42.1%)	525 (100%)

Note: 82 students did not answer the second part of the Mathematical Processing Instrument (MPI). Hence, the percentages in Table 2 were computed over the base of 525 students who answered both parts of the MPI.

Only 42.1% of the students were successful in answering the Postcard problem. The majority of the students who gave an incorrect answer chose a nonvisual method of solution (35.8%). Of the students who were successful, 12.8% chose a nonvisual method, while 29.3% chose a visual method (diagram on paper). It is interesting to note that only 1 student worked out a “picture in the head” and she was unsuccessful in answering the item. This suggests that the item could be too complex for students to visualize and hold any imagery in their minds.

Ho (2009, 2010) found that drawing diagrams for non-graphic tasks helped students see the task visually, specifically, how the elements in the task relate to each other. Ho also found that diagrams aid in the transformation of the task into mathematical forms, that is, mathematical forms may be obtained from the diagram to solve the task. In this study, 115 students (21.9%) who drew a diagram to solve the item were unsuccessful in solving the item. Figure 3 shows two common incorrect bar-model diagrams drawn by the students in the study. Bar-model drawings are problem-solving tools commonly taught in the Singapore classrooms to help students visualize and solve problems (see Fong, 1998 and Yeap, 2010). Bar models represent mathematical quantities (known and unknown) and their relationships given in a problem (Ministry of Education, Curriculum Planning and Development Division, 2009).

This finding points to an important implication that the use of a visual method is necessary but not sufficient for successful problem solving. Although studies on Singapore students have found that the bar-modelling approach is especially helpful for the weaker students, who would otherwise be unsuccessful in solving complex problems like the Postcard Problem (Goh, 2009; Poh, 2007), it is important to take note that the diagram needs to be correctly drawn in order to reflect the correct relationships among the mathematical quantities so as to obtain the correct mathematical forms from them.

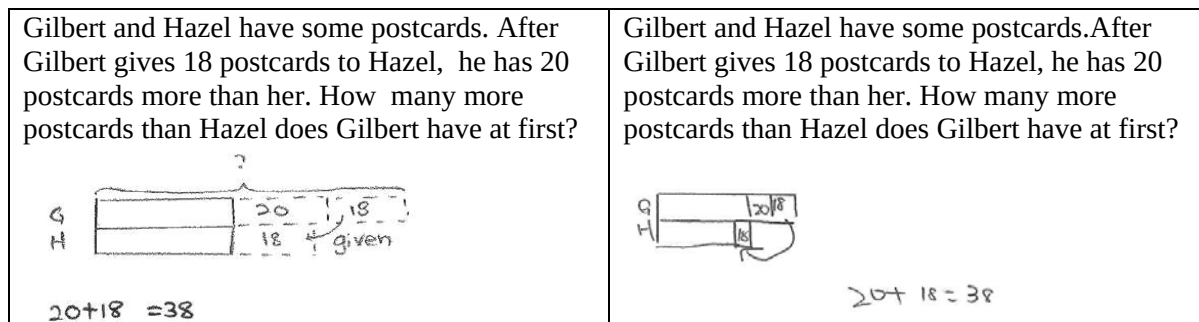


Figure 3. Ashley's (left) and Brenda's (right) solutions for the Postcard Problem.

The Plum Problem

The Plum Problem is an item from Australia's National Assessment Program Literacy and Numeracy (NAPLAN). Table 3 shows the performance of students in this item and the percentage of the various solution methods used by the students. Note that the use of kinaesthetic imagery is not a possible solution method for this item. Also note that 154 students did not answer the second part of the Mathematical Processing Instrument (MPI). Hence, the percentages in Table 3 were computed over the base of 453 students who answered both parts of the MPI.

Around three-fifths of the students were not successful in answering the item. Almost half of the students chose a nonvisual solution method and got the answer wrong. This suggests that students should consider using visual methods when a task is unfamiliar to them. The only visual method used by the students for this item was through drawing diagrams on paper. This suggested that the item could be too complex for them to use concrete imagery to solve the task.

Table 3

Correct and Incorrect Responses by Approach for The Plum Problem

Coding	Incorrect Response	Correct Response	Total
0	220 (48.6%)	134 (29.6%)	354 (78.1%)
1	49 (10.8%)	50 (11.0%)	99 (21.9%)
2	0 (0%)	0 (0%)	0 (0%)
3	-	-	-
Total	269 (59.4%)	184 (40.6%)	453 (100%)

Note: 154 students did not answer the second part of the Mathematical Processing Instrument (MPI). Hence, the percentages in Table 2 were computed over the base of 453 students who answered both parts of the MPI.

The Plum Problem is an unfamiliar task to Singapore students. Singapore students are more familiar with the following task, which is structurally differently from the Plum Problem:

When a number is divided by 3, the remainder is 2.

When the same number is divided by 4, the remainder is also 2.

Find the number.

Kaur (2009) noted that although Singapore students are performing well in international tests such as the Third International Maths and Science Study (TIMSS), they “do not perform well on content that is not an integral part of the local school curriculum” (p. 463). In addition, Singapore students also have “difficulty with items that require them to comprehend concepts or apply knowledge in non-familiar contexts” (p. 463). This is evident in Cedric's solution method (see Figure 4). Cedric used exactly the same method he learnt in school (see Figure 4) to solve the Plum Problem, probably not realising that a different approach of solution method is required.

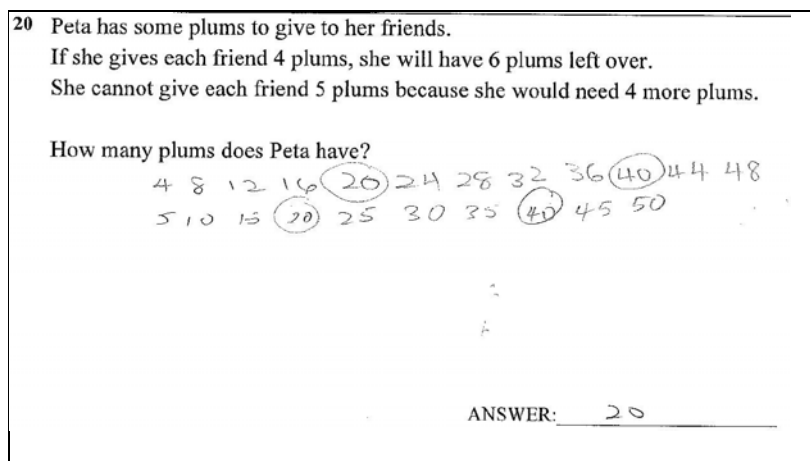


Figure 4. Cedric's solution to Item 20.

Conclusion and Implications

The investigation described the mathematics performance and processing approaches of Grade 6 Singapore students solving mathematics tasks drawn from Singaporean and Australian national assessment instruments. The study particularly focused on items (one from Singapore and one from Australia) that students had most difficulty in solving. For both items, the Singapore students predominately used a non-visual response in their processing approach. Although there has been an extensive body of literature (since the seminal work of Krutetskii, 1976) that suggests that non-visual methods are the most efficient, visual methods are certainly appropriate (and recommended) when the problem solver is faced with complex or novel tasks (Ho, 2009; Lowrie & Kay, 2007; Pirie & Kieran, 1992).

Since The Post Card Problem (a complex Singaporean item) and The Plum Problem (a novel Australian item) were items the students found difficult to solve we would have expected most of the students to use visual methods (see Lowrie & Kay, 2007; Presmeg, 1986a). However, non-visual methods were utilised in 48% (Post Card problem) and 78% (Plum Problem) of cases. Ho (2009, 2010) found that students' use of a visual method in their problem solving was influenced by two factors: (1) the novelty of the problem, and (2) students' perception of their teacher's problem-solving preference. In the current investigation, it was noteworthy that the Australian item was predominately solved using non-visual methods—which may reinforce Ho's second point.

Several implications emerge from the study. From a theoretical perspective, we encourage researchers to consider the relationship between task correctness and the methods or approaches used by students to solve tasks. We argue that the Mathematics Processing Instrument provides an opportunity for such analyses whilst also considering the social-cultural influences embedded within task development. From a classroom perspective, we encourage teachers to provide students with opportunities to develop powerful visual representations when engaged in sense making—especially when confronted with tasks that are complex or novel. Moreover, it is critical for teachers to “value” such approaches and not rely on developing analytic procedures.

References

- Åberg-Bengtsson, L. (1999). Dimensions of performance in the interpretation of diagrams, tables and maps: some gender differences in the Swedish scholastic aptitude test. *Journal of Research in Science Teaching*, 36, 565-582.

- Ahsen, A. (1981). Imagery approach in the treatment of learning disability. *Journal of Mental Imagery*, 5, 157-196.
- Ahsen, A. (1989). Guided imagery: The quest for science. *Education*, 110(1), 2-32.
- Australian Curriculum, Assessment and Reporting Authority (2011). *NAPLAN 2011 National assessment program: Literacy and numeracy. Information for parents* [brochure]. NSW: Author.
- Foong, H K. (1998). *Solving challenging mathematical problems: For primary school*. Sydney: Kingsford
- Goh, S. P. (2009). *Primary Five pupils' difficulties in using the model method for solving complex relational word problems*. Unpublished master's dissertation, National Institute of Education, Nanyang Technological University, Singapore.
- Ho, S. Y. (2009). *Visualization in primary school mathematics: Its roles and processes in mathematical problem solving*. Unpublished doctoral dissertation, National Institute of Education, Nanyang Technological University, Singapore.
- Ho, S.Y. (2010, January/February). Seeing the value of visualization. *SingTeach [E-Magazine]*, 22. Available at: <http://singteach.nie.edu.sg/math-ed/190.html>
- Kaur, B. (2009). Performance of Singapore students in trends in International Mathematics and Science Studies (TIMSS). In K. Y. Wong, P. Y. Lee, B. Kaur, P. Y. Foong and S. F. Ng (Eds.). *Mathematics Education: The Singapore Journey*, pp. 439-463.
- Kerr, C. (2011, August 6-7). Digital forums being skewed by cyberslackers. *The Weekend Australian*, p. 3.
- Krutetskii, V.A. (1976). *The psychology of mathematics abilities in schoolchildren*. Chicago: University of Chicago Press.
- Lowrie, T. & Diezmann, C. M. (2007). Solving graphics problems: Student performance in junior grades. *Journal of educational research*, 100, pp. 369-377.
- Lowrie, T. & Diezmann, C. M. (2011). Solving graphics tasks: Gender differences in middle-school students. *Learning and Instruction*, 21, pp. 109-125.
- Lowrie, T., & Kay, R. (2001). Task representation: The relationship between visual and nonvisual solution methods and problem difficulty in elementary school mathematics. *Journal of Educational Research*, 94(4), 248-253.
- Ministry of Education, Curriculum Planning and Development Division. (2007). *Mathematics Syllabus Primary*. Retrieved from the Singapore Ministry of Education website: <http://www.moe.edu.sg/education/syllabuses/sciences/files/maths-primary-2007.pdf>
- Ministry of Education, Curriculum Planning and Development Division (2009). *The Singapore model method for learning mathematics*. Singapore: EPB Pan Pacific.
- Ong, A. (2011, October 18). Singaporeans need Digital Quotient as well as IQ and EQ: Lam Pin Min. *The Straits Times*. [online]
- Pirie, S. E. B., & Kieren, T.E. (1992). Watching Sandy's understanding grow. *Journal of Mathematical Behaviour*, 11, 243-257.
- Poh, B. K. (2007). *Model method: Primary Three pupils' ability to use models for representing and solving word problems*. Unpublished master dissertation, National Institute of Education, Nanyang Technological University, Singapore.
- Presmeg, N. C. (1986a). Visualisation and mathematical giftedness. *Educational Studies in Mathematics*, 17(3), 297-311.
- Presmeg, N. C. (1986b). Visualisation in high school mathematics. *For the Learning of Mathematics*, 6(3), 42-46.
- Presmeg, N. C. (2008). Spatial abilities research as a foundation for visualization in teaching and learning mathematics. In N. C. Presmeg & P. C. Clarkson (Eds.), *Critical issues in mathematics education: Major contributions of Alan Bishop* (pp. 83-95). New York: Springer.
- Tan, Y. K., Chow, H. K., & Goh, C. (2008). *Examinations in Singapore: Change and Continuity (1891-2007)*. Singapore: World Scientific Publishing Co. Pte. Ltd.
- Wong, K.Y., & Kaur, B. (2011). Introduction: Assessment matters. In B. Kaur and K. Y. Wong (Eds.). *Association of Mathematics Educators Yearbook 2011: Assessment in the mathematics classroom*, pp. 1-15.
- Yeap, B. H. (2010). *Bar Modelling: A problem-solving tool*. Singapore: Marshall Cavendish Education.

Developing Teacher Understanding of Early Algebraic Concepts Using Lesson Study

Jodie Hunter
Massey University
<j.hunter1@massey.ac.nz>

This paper reports on the use of lesson study as a professional development tool. In particular the paper focuses on the way in which the teachers increased their understanding of how tasks, classroom activity and teacher actions scaffolded student learning of early algebraic reasoning of equivalence and the commutative principle. Teacher voice is used to illustrate how lesson study cycles caused the teachers to reflect and review their own understandings of early algebraic concepts and how their students considered the concepts.

Introduction

Algebra is often provided as a reason for both the difficulties individuals encounter learning and making sense of mathematics, and the disaffection many people hold towards it. Given the position algebra holds in the educational and economical future of all individuals, Knuth and his colleagues (Knuth, Stephens, McNeil, & Alibabi, 2006) describe a growing consensus between researchers and educators that algebra be introduced at a much younger age with a focus on the integration of teaching and learning arithmetic and algebra in classrooms. This emphasis is confirmed in policy documents which describe a unified curricula strand (e.g., Department for Education and Employment, 1999; National Council of Teachers of Mathematics 2000). Teachers, within this changing context are required to find ways to make algebra accessible to all their students, through the use of rich learning tasks in environments which provide all students opportunities to learn algebra with rich, conceptual understanding (Chazan, 1996). The focus of this paper is on how a group of teachers used lesson study to explore how some designed tasks could be used to better support student development of key early algebraic concepts.

Teachers have a key role in reforming classroom practice and activities which integrate arithmetic and algebra. But, we know that for many this poses considerable challenges; they may not have understandings of how to make links between arithmetic and algebra, nor may they have had experience constructing and using rich connected types of integrated (arithmetic/algebra) problems. As Blanton and Kaput (2003) suggest, they may not have developed their algebra 'ears and eyes' when working with the patterns and relationship in number which promote rich connected conceptual understandings. Blanton and Kaput suggest a remedy for this situation could be a form of professional development in which opportunities are structured so teachers identify numerical patterns and relationships which connect to early algebraic reasoning. In this paper lesson study was used as a form of professional development to facilitate a group of teachers enhanced algebraic 'eyes and ears'. The aim of this paper is to explore how professional development in the form of lesson study supported the teachers to 'notice' opportunities for developing student's early algebraic reasoning. The questions asked in this study were: How did the use of lesson study support teachers to comprehend how their students understood the key concepts of equivalence and the commutative principle, and; how did the repeated cycles in lesson study facilitate the teachers to identify the challenges involved in students' constructing conceptual understandings of equivalence and the commutative principle.

Lesson Study

Lesson study, developed in Japan, is one form of professional development which aims to increase teachers' knowledge about mathematics, knowledge about ways of teaching mathematics, and knowledge about the ways in which learners engage with and make sense of mathematics (Fernandez & Yoshida, 2004). In the lesson study format a group of teachers collaboratively plan a lesson (termed the 'study lesson') over a series of meetings. A cycle is developed. The 'study lesson' is taught by a team member, observed by other members with particular focus on student responses. Then in subsequent meetings the observed 'study lesson' is discussed, analyzed, reconstructed in line with student responses, then re-taught to a different group of students. This cycle may be repeated or different lessons developed. In this paper lesson study was used as a form of continuing professional development (CPD), which focused on enhancing specific aspects of teacher knowledge but with a particular emphasis on student learning.

Developing Early Algebraic Reasoning

Students constructing rich conceptual understandings of algebraic reasoning takes a long time and requires that their attention is placed on the inter-related connections across all other types of mathematics, and particularly arithmetic (Blanton & Kaput, 2005). The students' intuitive knowledge of patterns and numerical reasoning are used to provide a foundation for transition to early algebraic thinking (Carpenter, Franke, & Levi 2003). Carpenter and his colleagues explain that for students to justify, and generalise their mathematical reasoning about the properties of numbers they also need to be provided with opportunities to make conjectures in the classroom environment. Research studies investigating young children's development of early algebraic reasoning covers a wide field including those which focus on classroom practices which scaffold student justification and generalisations. However, in this paper to explore how lesson study supported a group of teachers to develop understanding of how their students constructed early algebraic reasoning the focus is narrowed to two areas of early algebra, equivalence (equality) and the commutative principle. The next section makes a brief examination of the literature related to equivalence, the commutative property and lesson study as a professional development process.

Equivalence

Developing understanding of equality is a concept fundamental to algebraic reasoning. Kieran (1981) in her seminal studies illustrated that many elementary school students have an inadequate understanding of the equal sign. Other studies (e.g., Carpenter et al., 2003; Knuth et al., 2006) concur. The difficulties these students encounter are caused because they view the equal sign as an indicator of an operator rather than a symbol of a mathematically equivalent operation. This limits the strategies they have available to solve equivalence problems and in later years symbolic equations (Knuth et al.). To address this problem, teachers need to be aware of how many students view the equal sign and construct and use activities in the classroom which expand student understandings of the equal sign and ensure that the misconceptions are identified and addressed. A range of successful classroom interventions (e.g., Carpenter, et al., 2005; Molina, Castro, & Castro, 2009) which enriched student understanding of equivalence have included non-standard representations such as true and false number sentences and balance scale representations.

Commutative Principle

Opportunities to explore the properties of numbers and operations provide a rich platform for developing algebraic reasoning. However, many exploratory studies (e.g., Anthony & Walshaw 2002, Warren 2001) illustrate that elementary students often have limited classroom experiences in exploring the properties of numbers and operations. As a result the students lack understanding of the operational laws and are unable to construct correct generalisations of the commutativity principle. Anthony and Walshaw illustrated that many students generalised the commutative nature of addition and multiplication, but over-generalised the relationship to include subtraction and division. They showed that while some students could explain the commutative property they could not construct generalised statements nor use materials to model their conjectures. However, studies by Blanton and Kaput (2003) and Carpenter and his colleagues (2003) provided clear evidence that when young children are provided with opportunities in the classroom they learn to construct and justify generalisations about the fundamental structure and properties of numbers. Importantly, these studies demonstrated that when classroom activity targeted students' numerical reasoning they explored, constructed and validated conjectures using appropriate generalisations and justifications

Theoretical Framework

The theoretical framing of this paper is based within a socio-cultural perspective. In this view the processes of teaching and learning hold a reciprocal relationship. The teaching is integrally connected to student learning as manifested through the changing competencies and disposition of the students. In turn, the teachers' professional development is interrelated and identified through evidence of their actions in the classroom, and changes in their professional competencies and attitudes.

Methodology

This paper reports on episodes drawn from a larger study which involved a year-long continuing professional development classroom-based intervention. The participants included two separate groups of elementary teachers (one group from England the other from the Channel Islands). The sample was an opportunistic one of teachers who wanted to extend understandings of ways to facilitate young students' development of early algebraic reasoning. This paper specifically reports on one section of the larger study. In this section the teachers engaged in lesson study for the first time although the Beaumont School teachers had engaged in a paired collaborative observation approach the previous school year, teachers at Hillview School had no experience using collaborative approaches to planning or teaching. The schools were a mixture of rural and suburban contexts and the students came from a range of socio-economic and ethnic backgrounds. The teachers had varying levels of experience.

In the lesson study process used in this study each group of teachers worked as a professional learning community within their own school. Over-arching aims relevant to each school were established immediately. These collaboratively agreed goals broadly established that the teachers wanted to develop creative and flexible problem solvers. Then all members of the research team (the teacher groups and researchers) planned an area of focus for the study lessons. The foci corresponded to mathematical concepts their students had difficulties with or those which the teachers felt less confident about teaching. Through collaborative activity 'study lessons' were planned and taught in one classroom and observed by the research group. In-depth analysis and discussion followed observations of

the study lesson and subsequent iterations as it was re-planned, re-taught and re-observed in different classrooms as part of the lesson study cycle.

The lesson study cycles in the two settings differed. At Hillview the teachers wanted to address how their students over-generalised the commutative principle to include subtraction and division. A lesson study cycle was devised which included lessons designed to facilitate student understanding and justification of the commutative property with a focus on the use of representations to model conjectures and justify reasoning. The students were given the following statement made by a student in an earlier lesson: 'If you have two numbers and you are adding them it does not matter which number you add first the answer will still be the same.' The students worked in small groups of four and explored with equipment whether the statement held when applied to the different operations. They were required to model their reasoning with equipment, as well as represent it verbally, symbolically and solve problems which involved multiple operations. At Beaumont the study lesson cycles aimed to develop students' skill at solving multi-step word problems and part of the focus was placed on the equal sign. The students were asked to make a specific number using a number sentence which was then represented as equivalent to another number sentence (for example $45 = 20 \times 2 + 5 = 20 + 25 = 45 - 0$) and included some incorrect multi-step equations.

Data gathering included detailed field notes, video and audio records of the planning meetings and classroom lessons and artefacts. The video and audio recordings were wholly transcribed and through an iterative process using a grounded approach, patterns, and themes were identified. The on-going and retrospective data analysis supported the development and construction of case studies of the two study groups. Evidence was triangulated using classroom observations, artefacts and analytical discussion.

Findings

The first section outlines how the use of the lesson study process facilitated the teachers to notice key aspects of early algebraic reasoning and included both planned opportunities for student learning and spontaneous opportunities which arose as tasks were enacted in classrooms.

Developing Understanding of Student Approaches to the Tasks

Discussion and analysis of student responses in the study group illustrated that opportunities to closely observe student responses during the lesson provided a foundation for the teachers to build understandings of how students approach tasks which challenge their understandings of the commutative principle. The teachers expressed surprise that many students began with the use of counter-examples to show that the commutative property did not apply to subtraction. For example Ellen commented:

Iris and her partner were looking at subtraction without even being prompted to do it because they said straight away "it doesn't work for subtraction but it is working for addition.

Similarly, in further discussion another teacher noted that the students initiated their investigation with the development of a counter-example:

They were doing it with the subtraction. They did four minus one equals three and one minus four and Lauren said "so that is subtraction done then, that doesn't work" and she did it for one, if it doesn't work, it doesn't work whereas she then said "actually five times three and three times five works hmm". Then they did something with twos and then she said "does it only work with twos though". So then they tried with a different number. That was Lauren who said that so she had got the

idea that if with one it didn't work, she just discarded that straight away and went straight onto the next one.

Through their observations in the lesson study they had observed how the students intuitively realised that a counter-example disproved the conjecture. However, the students would explore further with other numbers if the conjecture appeared to be correct; in that situation they were not satisfied that one example proved a conjecture.

Developing Understanding of the Role of Materials in Sense-making

Lesson study provided opportunities to develop teacher knowledge of how students could justify their conjectures using materials. For example, in a study group discussion it was evident that a teacher lacked understanding of how the children could justify their conjectures through use of an array. During the observed lesson two groups of students justified an explanation that multiplication was commutative through use of an array. The teacher did not use their explanation to extend the other students' reasoning. Then when another student had difficulties articulating the same concept the teacher stopped her explanation. In the post lesson discussion the researcher stated what the student was explaining:

Researcher: What Andrea was trying to say but she couldn't quite articulate it was if you just kept making it longer it could be any number because you could just keep adding on and it is still the same amount multiplied by the same amount.

Monica: I was conscious of the time, the bell was going to go and I wasn't sure of what she was trying to say from where I was standing.

The teacher's response illustrates that she did not understand how an array supported the explanation nor could she build on and extend the students' explanations of the commutative property. This was reiterated during further analysis in the follow-up discussion. As the other teachers explained and analyzed the student responses the teacher clarified her own actions:

Monica: I didn't know what he was saying about the two numbers.

Melissa: He was moving the rows.

Ellen: Yes, he was saying to turn them around.

Melissa: He moved the rows, he said look you don't have to...

The teacher explains from her own point of confusion her response:

Monica: I thought actually that might have confused everybody else.

Ellen: But he knew what he meant so he could explain it the other way.

At this point Monica acknowledged that because she was confused by what the representation showed she assumed many students would also be.

The teachers also became aware of how important it is that students have access to equipment to scaffold their understanding. After observing that a group of students encountered difficulties investigating whether division was commutative Ellen commented:

When they had the pegs in front of them then they could argue it but they couldn't argue it just on paper. They needed to be able to see the five pegs and they can't divide them amongst the ten people and quite a few groups were like that.

The teachers also observed the way in which the students used equipment to link to real-life situations to model their reasoning:

I think it was Iris and Andrea, they were talking about the objects and they suddenly became sweets, “if we have got three sweets we can't divide them between seven people” so they were then jumping ahead and moving that relationship on, that was good. I think it was the resources that prompted that.

During further discussion the teachers illustrated how they now understood how physical representations supported students to work at higher levels of generality:

Monica: I think even with John if he hadn't seen it on the grid [referring to an array constructed on a pegboard] he probably wouldn't have got it as quickly as he did.

Melissa: Because he had really got it in his head, hadn't he? Because he wasn't even really sure if six times four what it equaled, he just knew that it was the same.

Monica: Originally he was convinced that it didn't work so it was only after Sridatta disagreed and showed him it on the grid.

Melissa: The fascinating part is he didn't even work out what the answer was. It didn't matter, it was irrelevant [indicates turning array with hands].

However, the teachers were surprised at the difficulties students had using equipment to model and justify conjectures. During the first lesson cycle their attention was drawn to how the procedural use of symbols dominated how the students responded to tasks:

Melissa: The thing is them trying to use them as symbols and they got fixated on the idea, like that group over, they even had the scissors as an equals sign

Ellen: And using the blocks to try and create the numbers... I think making it explicit that the objects are representative of a proper number and that they are not to then start creating equations out of them. We don't want to see them as numbers but as objects.

Again the next lesson cycle drew their attention to the student attempts to use materials as symbols:

Melissa: I think that's the same thing again, they wanted to start putting in the signs and symbols...so they had three colours then a white peg, then one peg and then a white peg and then four pegs and she said “we've put that peg to mean add” so they were doing the same thing. It's like they need to have the symbols there rather than just having like their array as a justification.

Developing Knowledge of How Understanding of the Commutative and Equal Sign is Constructed Over Time and through Specific Teacher Actions

The study group discussion provided many opportunities for the teachers to reflect on what happened in the study lessons and identify missed teaching opportunities. For example in one section of a lesson the discussion focused on examining the structure of multiplication operations and the teacher shifted the children from the general to the specific by guiding them to solve the equations to show the answers were the same. In the analytical discussion with colleagues she recognized that by directing the students towards answers rather than the general structure of multiplication some students became focused on specific equations rather than generalized understandings of the commutative nature of multiplication:

Monica: I shot myself in the foot because I did that because I knew that some of them hadn't got it so I wanted to show them that actually you know you could tell if you worked them out separately. You could ascertain they had the same answer but then it kind of made other people get stuck at that stage.

In the continuing conversation she saw how her actions caused many students to use procedural rather than conceptual understandings.

The lesson cycles also provided a foundation for the teachers to recognise the need to press students beyond specific examples to generalised reasoning. After a second lesson cycle a teacher observed:

Melissa: It is almost as though because they had chosen numbers that were simple enough that they knew that two times four made eight so they weren't looking at them as an array, they were looking at them as if that is the numbers that we are dealing with.

This statement led to further discussion in the group, of actions they could use to scaffold the students towards more generalised reasoning.

The post-lesson discussions also provided evidence of the teachers' growing ability to notice student misconceptions of early algebraic reasoning beyond that of the lesson foci. One example occurred when the teachers discussed the difficulties the students had in representing the commutative principle as a number sentence (for example $6 + 5 = 5 + 6$).

Ellen: They seem to find it really hard to write one continuous number sentence.

In response Monica drew the groups' attention to the on-going difficulties the students had with the equal sign as a concept of equivalence:

They are still not understanding the proper meaning of the equal sign or perhaps they are but when it comes to applying it in a context then they're not.

In the teacher discussions evidence was provided that they became aware that constructing understanding of equivalence is a lengthy and difficult process which requires a press from the teacher and a lot of student discussion and exploration:

Zara: We still had to keep coming back to that, that the two sides of the equation had to balance. How much time we have done that, and even given that they had done that in the first part of the lesson. They don't seem to see that as the same.

Within this discussion on-going analysis of the observation and how the activities caused students to think about equivalence led to further analysis and reflection from another teacher.

Rebecca: I think maybe because we historically present children with a lot of things with the answer just being one box that sort of one where they had to look maybe provoked that thinking a little bit more. You know at the beginning where they said something, something equals and then the next child does equals, I don't know, when I look at it now I think it is a fantastic activity and a fantastic assessment...but maybe they are just seeing and the next one, and the next one, and now it's my turn and they don't actually see the equal sign whereas this question here and that one here in particular really made them think about the idea of balance.

In this statement the teacher has voiced her growing awareness of why students develop misconceptions around the equal sign and the importance of considering how teacher actions coupled with rich tasks structure how students make links between arithmetical and algebraic reasoning.

Discussion and Conclusions

The use of teacher voice in the study group discussions sheds light on the many learning opportunities the teachers encountered as they observed and listened to student activity during the lesson study. Clearly they observed the pivotal role the teacher had, in the study lessons, in making links between the early arithmetical and algebraic reasoning and pressing the students towards situations of generality. Their algebra 'eyes and ears' (Blanton & Kaput, 2003) became more attuned to recognizing common misconceptions as the teachers worked together in the lesson cycles. They also developed cognizance of the need to better match their actions to the classroom discussions and activity.

Of importance in this study was the teachers' recognition of the many challenges they face in developing students' rich connected learning about equivalence and the commutative principle. As previous researchers (e.g., Carpenter et al., 2003; Blanton & Kaput, 2005)

note, students have common misconceptions but for the teachers recognition of these caused reflective and analytic discussions. The findings of this paper suggest that lesson study has considerable promise as a learning tool for teachers to support them reforming their practices to integrate arithmetic and algebra.

References

- Anthony, G., & Walshaw, M. (2002). *Students' conjectures and justifications: Report of a probe study carried for the National Education Monitoring Project*. Palmerston North: Massey University
- Blanton, M., & Kaput, J. (2003). Developing elementary teachers "algebra eyes and ears". *Teaching Children Mathematics*, 10(2), 70-77.
- Blanton, M., & Kaput, J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, 36, 412-446.
- Carpenter, T., Franke, M., & Levi L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.
- Carpenter, T., Levi, L., Berman, P., & Pligge, M. (2005). Developing algebraic reasoning in the elementary school. In T. Romberg, T. Carpenter, & F. Dremock (Eds.), *Understanding mathematics and science matters* (pp. 81-98). Mahwah, NJ: Lawrence Erlbaum.
- Chazan, D. (1996). Algebra for all students? *Journal of Mathematical Behaviour*, 15, 455-477.
- Department for Education and Employment (DfEE). (1999). *National Numeracy Strategy: Framework for Teaching Mathematics from Reception to Year 6*. Cambridge: CUP.
- Fernandez, C., & Yoshida, M. (2004). *Lesson Study: A Japanese approach to improving mathematics teaching and learning*. Mahwah, New Jersey: Lawrence Erlbaum Associates.
- Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12, 317-326.
- Knuth, E., Stephens, A., McNeil, N., & Alibali, M. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, 37(4), 297-312.
- Molina, M., Castro, E., & Castro, E. (2009). Elementary students understanding of the equal sign in number sentences. *Electronic Journal of Research in Educational Psychology*, 7(1), 341-368.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Warren, E. (2001). Algebraic understanding and the importance of number sense. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 399-406). Utrecht: PME.

Designing Opportunities for Prospective Teachers to Facilitate Mathematics Discussions in Classrooms

Roberta Hunter
Massey University
<r.hunter@massey.ac.nz>

Glenda Anthony
Massey University
<g.j.anthony@massey.ac.nz>

How prospective teachers can best be prepared to teach effectively in mathematics classrooms is a topic of current concern. In this paper, we describe our exploration of ways in which prospective teachers were supported to translate what they learnt in mathematics methods classes into pedagogical practice. We illustrate how the use of discourse routines, enacted in iterative cycles of guided rehearsals, disrupted previous beliefs about teaching and learning mathematics and led to them more confidently respond in pedagogically appropriate ways.

Introduction

Over the past decade, New Zealand, like many other countries in the Western world, has undergone significant reform in mathematics education. Consistently reform efforts have centred on teacher professional development as a way to improve the quality of teachers' mathematical understandings and pedagogical practices. For example, the New Zealand Numeracy Development Project (NDP) (Higgins & Parsons, 2009) aimed to enhance teacher knowledge of student numerical knowledge and strategy levels while also promoting a New Zealand wide vision of mathematics classrooms in which pedagogical instruction centred on students' sense-making through interactive discussions. Facilitating such rich mathematical discussions requires significant pedagogical skills which studies (e.g., Lampert, 2001; Sherin, 2002) reveal even able teachers may struggle with. Therefore if proficient teachers exhibit difficulties enacting the pedagogical practices promoted in reform programmes how then can novice prospective teachers learn these complex pedagogical actions so that they are enacted in classrooms with confidence? In this paper we address ways in which discourse routines may be employed as both practical and intellectual tools in instructional activities designed to scaffold prospective teachers to learn to respond in pedagogically appropriate ways to the complexity and uncertainty of mathematics discussion.

Conceptual Framework

In the next section we will look at current research literature which describes changes being enacted in pre-service programmes to better support prospective teachers to translate what they learn in their mathematics methods classes into practice. The second section will outline research findings on the role of productive mathematical discussions in classrooms and the pedagogical strategies teachers use to enact them.

Pedagogies of Enactment

In recent times there has been increased discussion and research on ways prospective teachers can be prepared to teach effectively in mathematics classrooms. This discussion has been prompted by a growing disquiet that pre-service teaching programmes may have limited influence on prospective teacher's classroom practice (Kazemi, Franke, & Lampert, 2009). Hammerness and colleagues (2005) attribute this situation to problems encountered

in enactment and complexity. They explain that prospective teachers in methods classes are given a vision of appropriate teaching and learning but developing the required pedagogical skills is left to be learnt in the interactive situation of the field where the complexity of the situation may be over-whelming. To address this situation Kazemi and her colleagues draw on what Grossman and McDonald (2008) term pedagogies of enactment to illustrate through approximations of practice how prospective mathematics teachers can be supported to learn the ‘what and how’ of teaching but also learn ‘to do’ interactive mathematics teaching.

Pedagogies of enactment suggest it is not sufficient to equip prospective teachers with knowledge of the importance of mathematical discussions or the discourse strategies teacher use to enact them—they also need to know how to do things in practice and be able to enact them in interactive ways (Ghousseini, 2009). Ghousseini argues that effective development of prospective teacher’s pedagogical skills requires “teacher education pedagogy that is grounded both in *action* and in *reflection*” (p. 149) and which develops a disposition of inquiry. Ghousseini illustrated the positive outcomes afforded a group of prospective secondary teachers in a mathematics methods course when they were provided with opportunities to practise parts of interactive teaching. To reduce many of the complexities of enactment, modelling and rehearsing was used. The teacher educator modelled a mathematics lesson with them in the role of students and as it proceeded it was carefully deconstructed and reflectively analysed to make visible the many discourse moves. The lesson was followed by designed opportunities for them to engage in rehearsal of discourse moves using fictional classroom scripts. The rehearsals called upon them to role play and make decisions about applying the discourse moves in responsive ways which progressed the collective mathematical thinking within productive mathematics discussions.

Productive Classroom Mathematics Discussions

Discussions are an essential teaching and learning component of current New Zealand mathematics classrooms although in this context mathematical discussions differ markedly from those of traditional classrooms. Whereas in more traditional settings the teacher’s key role is to deliver instructions and information then quiz for correct responses, the advocated focus of discussion is now on students’ thinking and reasoning rather than correct answers (Nathan & Knuth, 2003). The teacher takes the role of facilitator and engages the class in sustained reasoning about the students’ current mathematical constructions (Chapin, O’Connor, & Anderson, 2003). Through these reasoned mathematical discussions teachers lead students to engage in a range of complex mathematical practices (Lampert, 2001). These include such practices as making conjectures, explaining, justifying, and representing possible problem solutions, and opening them up for other members of the community to analyse and critique and make connections. Significant teaching skills are required to ensure students are able to use these disciplined means of reasoning while at the same time maintaining fidelity with mathematics as a discipline (Boerst, Sleep, Ball, & Bass, 2011). Teachers are called upon to interactively make spontaneous judgements related to the reasoning and at the same time respond in ways which are both respectful and maintain focus at depth on key mathematical learning.

Teachers use a range of different practical and pedagogical strategies to manage successful facilitation of productive mathematics discussions. These include an explicit focus on the development of classroom participation structures and enacted sociocultural and mathematical norms (Sullivan, Zevenbergen, & Mousley, 2002). For example, Hunter (2008) describes the use of a Communication and Participation Framework which teachers used to scaffold a group of diverse learners to reach carefully reasoned mathematically

understandings. Other studies (e.g., Boerst, Sleep, Ball, & Bass, 2011; Kazemi, Franke, & Lampert, 2009) outline the positive outcomes when teachers use specified practices to reduce the degree of in-the-moment responses teachers must make. Smith and Stein (2011) describe the practices “as skilful improvisation” (p. 7) and outline how their use allows teachers to anticipate, plan for, and sequence student contributions in ways which further the lesson’s mathematical agenda. These key practices include *anticipating* possible student responses, then *monitoring* them as they are constructed, *selecting* students to present their responses, *sequencing* them and *connecting* them to key mathematical concepts. Another set of studies (e.g., Chapin, O’Connor, & Anderson, 2003) outline how teachers use a set of talk moves to ensure that the mathematical discussion is productive. These include teacher and student revoicing, teacher initiated requests that a student repeats another student’s contribution, elicitation of student reasoning, teacher request to add on and teacher wait time. Ghouseini (2009) describes these talk moves as “purposeful and disciplined *discourse routines*” (p. 149) which teachers establish and orchestrate to support mathematical learning.

Method

The participants were prospective teachers enrolled in a one-year post-graduate pre-service primary teaching course. The 22 participants in this study self-selected into the mathematics education option within the Classroom Inquiry paper. The overall aim of the paper was to engage prospective teachers in a process of ‘teaching as inquiry’. The participants had completed the first semester mathematics methods course and were enrolled and part way through their second mathematics methods course. They had also completed nine weeks of teaching experience over two practicum placements. The two participating teacher educators, both experienced classroom mathematics teachers and teacher educators, co-taught the Classroom Inquiry mathematics option, and the first author taught the methods courses with the participants.

The focus of the first methods course was placed on students developing understanding of relevant mathematical content knowledge. In the second methods course the focus shifted towards developing the students’ classroom pedagogical knowledge. As part of this course the students discussed and analysed research papers which specifically studied and described teachers’ use of talk-moves (Chapin, O’Connor, & Anderson, 2003), the five practices (Smith & Stein, 2011) and they were also introduced to the Communication and Participation Framework (Hunter, 2008).

In the Classroom Inquiry mathematics option the enactment of the instructional intervention aimed to provide the prospective teachers with opportunities to practise and rehearse key aspects of managing mathematical discussions in productive ways. Within the university setting the prospective teachers engaged in a range of instructional activities. These included providing them with opportunities to predict a range of possible solution problems for a rich task in a peer group and using the task to teach and manage a discussion with a small group of their peers while being monitored by the teacher educators. Other activities included them reading and analysing different scripts and discussing how these related to teacher use of the five practices (Smith & Stein, 2011) and talk-moves (Chapin, O’Connor, & Anderson, 2003). The foci of the instructional activities then shifted to a school setting. The prospective teachers in pairs were each allocated groups of six 9 and 10 year old students. In four iterative cycles they were required to plan mini lessons based on rich problems provided by the teacher educators. As part of the university rehearsal phase they were required to devise a plan based on a protocol that used the five practices of group discussion (Smith & Stein, 2011) and included questions to incorporate to enact specific talk

moves. In the classroom enactment phase involving the four lessons one of the pair taught the lesson and the other video-recorded it. Following each lesson the video record was reviewed by the pair and the talk-moves discussed and analysed. The assessment for the prospective teachers' paper required them to prepare a video presentation of their learning from their experiences as an inquiry teacher.

The complete cycle—from rehearsal to classroom enactment and reflection— generated multiple forms of data. These included the prospective teacher's planning documentation, video records of lessons, and a final video compilation of their learning journey. They also kept a reflective journal in which they were required to record regular entries. The journal entries, in particular, provided the data for this paper.

Results

The initial journal entries illustrated the dilemma the prospective teachers (PT) encountered between their recent practicum experiences and the theory and practice vision they were encountering within the context of the mathematics methods course and the Classroom Inquiry mathematics option. As one stated:

PT 1: I was at a stage of teaching where I was confident in my abilities and what I was “doing” in the classroom, but I had not questioned what an effective classroom looked like and what learning actually was and I was not sure how teachers and learners could be partners.

A central theme of these journal entries showed that the prospective teachers (PT) were grappling with their past experience as students in mathematics classrooms in what was described by one PT as “taken for granted pedagogical perspectives”. Their previously experienced quiet and orderly classrooms had taught them that the teacher held a key position as imparter and validator of the knowledge:

PT 17: My own experiences of learning at school and university had “taught” me that learning was about receiving and regurgitating information. Successful learning was quiet and orderly, with the teacher maintaining control; the teacher's role was the boss.

Others, while acknowledging the benefits which could accrue through inquiring into pedagogical practices that could facilitate productive discourse between students and students and teacher at the same time described their own feelings of trepidation or inadequacy. One talked about enacting such practices as “daunting”, another described his own learning process “as a voyage with detours, speed bumps and maybe the odd shipwreck”. In their reflective statements many PTs noted that this was a difficult journey which required a shift in disposition towards that of inquiry into their own pedagogical practices that was frequently marked by acceptance of the need to change. As one stated:

PT 5: One issue I will need to think about is teacher change—a teacher needs to be open-minded and also willing to make mistakes—like me many teachers have difficulties with this factor.

A clear focus in the initial set of reflections was that the PTs were seeing themselves as learners—learners who faced difficulties and challenges associated with their early experiences of considering and enacting interactive mathematics teaching.

Beginning to Notice and Respond

The influence of previous mathematics learning situations continued to thread through all the PTs second set of journal entries recorded after the first teaching session in school. These earlier schooling experiences and previous practicum experiences continued

to benchmark their experiences during this first teaching session in school and shape the prospective teachers post lesson reflections. For example, one PT questioned her role in previous mathematics teaching on practicum:

PT 11: As a teacher of mathematics throughout my practicums I believed I was providing a rich lesson because we had materials to use, great warm up games and the students were getting the right answers but where was the mathematical discussion? What I notice now is that I was giving students too much support, too much guidance towards just getting the right answer.

In the second journal entry prospective teachers' commentary moved to include a direct focus on their role and obligation to create productive opportunities for children's learning. Many PTs identified the difficulties they had in their first school session with allowing students space and time to reason and take responsibility for their own learning "when you are just itching to steer them in the right direction". A common theme was their need to be in control:

PT 23: A challenge for me is my desire to rescue students when they are going down the wrong path or are stuck in their thinking. If students are to take responsibility for their own learning, explore the problem, and make sense of it...I must allow them the space to make mistakes, have incorrect answers and work through their thinking.

All PTs recognised the importance of mathematical discussions for sense-making. They noted how it supported the students to deepen their mathematical understandings and also how it gave them access to each child's sense-making:

PT 19: I am developing my ability to notice what the students are saying and using their discussions and justifications...through noticing I get an understanding into where the children are at and can use this in my planning and assessment.

However, the complexities involved in enacting productive mathematics discussions continued to concern them. Some talked about the challenge of having all students engage and participate while others described their need to further develop good questioning skills.

PT 2: My thinking about mathematics discussions is challenged by the degree of 'with-it-ness' required for a teacher to perform the role of moderator and orchestrator well. In order to develop into this type of role I need to confront my tendency to drop back into direct teaching and the propensity to want to rescue children by providing too much overt guidance if I see them struggling to adapt to the role of questioner and protagonist at first. I recognise that both of these instincts are rooted in my early learning experiences, in which permission to make mistakes was not extended from teacher to students. One learned to do maths by rote and that mistakes were not considered to be part of the learning process.

The PTs acknowledged the value of the rehearsal phase with the instructional activities in the university setting. While they felt this phase supported their growing ability to plan for interactions using talk-moves, they recognised that to use them well, required time and practice in a more authentic setting.

PT 10: I have had a chance to try out some of my own talk moves during lectures with my classmates, and really see how difficult it is. I still need more time to practise my talk moves and have some more time to ask good questions. I still find it difficult to anticipate some of the 'incorrect' answers and strategies that children come up with but I think a lot of this will be through experience and understanding how children think.

Evident in the second set of entries was growing awareness of themselves in the facilitation role. They recognised the tools they had available in the form of the five practices and the talk moves but knew they were novices in their use of them.

Building Confidence to Notice and Respond

A significant shift is evident in the third set of reflective entries. For all PTs there was a growing recognition of their own strengths as facilitators and partners in students' learning. The iterative cycle of rehearsals had provided them with tools to use in the 'here and now' of classroom interactions. Clearly evident is their recognition of the importance of listening and noticing and how they could access student reasoning through questioning:

PT 23: I notice that my ability to listen and notice student's learning is proportional to the degree to which I use questions to revoice, sequence and in the process make student's thinking visible. I appreciate that these practices are a large part of active listening and that I need to make (for the time being) conscious efforts to use talk moves intervention in order to direct and focus learning on the big math ideas and concepts and to recognise when students are using them.

Although repeated practice had supported them to make the talk moves part of their developing repertoire of pedagogical skills, many described a need to purposely remember to use them "until they become second nature and natural":

PT 10: I have found the ability to listen and notice student learning through the talk moves is not that difficult but for me it has to be done deliberately and consciously, for now at least.

The video records were important reflective tools which provided the PTs with "more detailed monitoring of the students' learning". This included insight into not only the students' correct reasoning but also their misconceptions, which emerged in the interactions. Through use of the records the PTs were able to analyse their strengths and weaknesses and establish their next learning goals:

PT 3: In my teaching session I touched upon the surface of my students' reasoning of the problem. After watching my video over and over it was evident that I was not doing enough revoicing, having students repeat other students' contributions and wait time. So that became my next deliberate focus in my plans and I included questions.

Many PTs described confidence in their skills to anticipate and monitor student reasoning but outlined how linking the different students' reasoning to big mathematical ideas in coherent ways challenged them. However, they acknowledged the importance of this next step:

PT 1: Two of the practices which need more work are sequencing of strategies and the connections..This signifies the important way in which I build connections with the strategic ways that students are working out the problem as well as the big mathematical idea. It also can help take the students understanding to another level by exposing them to a more abstract and multiple ways of thinking and understanding..

The third set of journal entries illustrated that the PTs had developed a vision of productive discourse and their role as a facilitator and were confident that with focused practice they could develop the pedagogical skills to enact it.

Becoming Confident to Notice and Respond

The final set of journal entries, completed after four school visits, demonstrated both PTs awareness of and growth in capability to facilitate productive discourse in practice. They now recognised how rich discussion supported them to listen, notice, and respond to students' reasoning and how discussions supported their students developing rich and deep understandings. They described how the many opportunities to rehearse (both in the university and school setting) had provided "practice in increasing our

automaticity of using talk moves”. They had “become more natural” and served as an important tool to assist them to manage the flow of mathematical talk:

PT 19: Although we had been taught about discourse based instruction in [mathematics methods] class it is not until I tried it out for myself that I had begun to learn the skills...my ability to think quickly has improved and I am able to deal with things going ‘off the script’ more easily. This has improved my ability to respond to students’ contributions and I am sharper when making observations and able to assimilate the unexpected.

Many PTs acknowledged the key role of effective questioning as a tool which was “helpful in navigating the uncertainties of interpreting students’ thinking” and as a way for all students to access the mathematical reasoning:

PT 10: The use of effective questioning has also been an area of learning, particularly to ensure that students’ learning becomes visible to me (for assessment) and to other children (for learning).

Although all PTs were cognisant of the pedagogical skills they had gained during the inquiry they indicated that they knew they were on a learning journey and this needed to continue:

PT 19: The next learning step for me is to continue to use mathematical discourse so it becomes second nature. By continuing the use of effective questioning, talk moves, and the five practices is going to enable me to gain clarity. Within this context a community of learners develop and students become very much a main contributor of their own learning. I need to ensure that the lessons lend themselves to the big mathematical idea that we want students to learn and make sense of.

In the final set of journal entries the PTs’ key focus shifted more towards what they considered was important for their students as learners of mathematics. Their journal reflections demonstrated that they as teachers had learnt that rich mathematical understandings emerged through carefully facilitated mathematics discussions which focused on key mathematical ideas. Most importantly, because talk moves had become part of their pedagogical repertoire they were able to better manage more complex situations as they arose.

Discussion and Conclusions

This paper has illustrated how innovative prospective teacher education programmes can through use of what Kazemi and her colleagues (2009) term approximations of practice, better equip prospective teachers ‘to do’ interactive mathematics teaching. The evidence provided, supports Ghousseini’s (2009) contention that it is not sufficient for prospective teachers to be told about the importance of mathematical discourse or the pedagogical strategies teachers use—they also need to know how to enact them, in structured ways which provide them with success.

The opportunities to plan, rehearse and reflect on their facilitation of mathematics discussions afforded the prospective teachers space to reconstruct their views on teaching and learning mathematics. Rehearsing and reflecting on the actions had affordances which supported them to break from the traditional frames of reference described by Nathan and Knuth (2003) and engage in actions which they would not have habitually used. The integration of the five practices (Smith & Stein, 2011) and the talk moves (Chapin, O’Connor, & Anderson, 2003) provided scripted scaffolds that the novices needed to develop as part of their pedagogical repertoire. In this process, the iterative rehearsals afforded them a reduction in the complexity of mathematical interactions. This resulted in

them being better able to notice and respond in ways that enhanced collective mathematical reasoning. Of importance in the process was the grounding of the teacher education pedagogy in both action and in reflection (Ghousseini, 2009). The approximation of practices enacted in the rehearsals and the reflective analysis of the video records and recordings in the journals supported the prospective teachers to assume a stance of inquiry through which they gained knowledge of the importance of productive mathematical discourse as a tool to deepen the students' mathematical understandings.

References

- Boerst, T., Sleep, L., Ball, D. L., & Bass, H. (2011). Preparing teachers to lead mathematics discussion. *Teachers College Record*, 113(12), <http://www.tcrecord.org> ID Number: 16496.
- Chapin, S. H., O'Connor, C., & Anderson, N. (2003). *Classroom discussions: Using math talk to help students learn* (Grades 1-6). Sausalito, CA: Math Solutions Publications.
- Ghousseini, H. (2009). Designing opportunities to learn to lead classroom mathematics discussions in pre-service teacher education: focusing on enactment. *AMTE Monograph 6. Scholarly Practices and Inquiry in the Preparation of Mathematics Teachers*, 147-158.
- Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal*, 45(1), 184-205.
- Hammerness, K., Darling-Hammond, L., Bransford, J., Berliner, D., Cochran-Smith, M., McDonald, M., & Zeichner, K. (2005). How teachers learn and develop. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world. What teachers should learn and be able to do* (pp. 358-388). San Francisco, CA: Jossey-Bass Educational Series.
- Higgins, J., & Parsons, R. (2009). A successful professional development models in mathematics. *Journal of Teacher Education*, 60(3), 231-242.
- Hunter, R. (2008). Facilitating communities of mathematical inquiry. In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 31-39). Brisbane: MERGA.
- Kazemi, E., Franke, M., & Lampert, M. (2009). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides* (Proceedings of the 32nd annual conference of the Mathematics Education Research group of Australasia, pp. 11-29). Wellington: MERGA.
- Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven: Yale University.
- Nathan, M. J., & Knuth, E. J. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and Instruction*, 21(2), 175-207.
- Sherin, M. (2002). A balancing act: Developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*, 5, 205-233.
- Smith, M. S. & Stein, M. K. (2011). *Five practices for orchestrating productive mathematics discussions*. Reston, VA: NCTM.
- Sullivan, P., Zevenbergen, R., & Mousley, J. (2002). Contexts in mathematics teaching: Snakes or ladders. In B. Barton, K. Irwin, M. Pfannkuch, & M. Thomas (Eds.), *Mathematics education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group, pp 649-656). Sydney: MERGA.

Professional Learning for Teaching Assistants and its Effect on Classroom Roles

Chris Hurst
Curtin University
<c.hurst@curtin.edu.au>

Len Sparrow
Curtin University
<l.sparrow@curtin.edu.au>

The Swan Valley Cluster of schools for the Make It Count project identified the professional learning of teachers and teaching assistants as a key factor in improving numeracy outcomes for urban Indigenous children. Initial training for assistants began in late 2010 and took the form of workshops based on a modified First Steps in Mathematics Number program. It was continued in 2011 and led to a pilot program in training assistants to plan for targeted mathematics learning for individuals and small groups of children. This paper reports on the success of the pilot with regard to the improved confidence and ability of the assistants to assume greater responsibility for teaching, as well their development as integral members of professional learning communities.

Introduction

The Swan Valley Cluster of schools was formed in 2009 as part of the Make It Count project that aims to improve numeracy outcomes for urban Indigenous Australian children. The cluster identified a number of factors that were believed might contribute to better outcomes for children. These were to raise expectations of and for children, improve parent involvement, increase student engagement, provide professional learning for teachers and support staff and to target specific aspects of curriculum that were seen as requiring support (Hurst & Sparrow, 2010). The final two points, particularly the one related to professional learning, are the subject of this paper. The notion of targeting specific curriculum areas is necessarily included as this was the specific focus of the professional learning that was implemented during late 2010, 2011 and which is continuing in 2012. This paper discusses the professional learning of Education Assistants (EAs) and Aboriginal & Islander Education Officers (AIEOs) as part of a pilot program conducted during 2011.

Background

Several aspects of teaching are the focus of the research on which this paper is based. They are pedagogical content knowledge and mathematical content knowledge, confidence in teaching mathematics and the development of professional learning communities (PLCs).

Pedagogical Content Knowledge and Mathematical Content Knowledge

Schulman (1986, p. 9) coined the term ‘pedagogical content knowledge’ (PCK) and he considered it to comprise “. . . the most useful ways of representing and formulating the subject that make it comprehensible to others [and] . . . includes an understanding of what makes the learning of specific topics easy or difficult”. Schulman’s work has since been developed by other researchers who have interpreted and developed his ideas. Some (Ball, Thames & Phelps, 2008, p. 392) have seen PCK as “. . . a kind of amalgam of knowledge of content and pedagogy that is central to the knowledge needed for teaching” and “. . . a teacher’s understanding of how to help students understand specific subject matter” (Magnusson, Krajcik & Borko, 1999, p. 96 cited in Ball et al., 2008, p. 394.)

Ball et al. (2008, p. 395) described mathematical knowledge needed for teaching as the “mathematical knowledge needed to carry out the work of teaching mathematics” and that it

necessarily involved understanding the content of the curriculum in order to be able to assist children, respond to their questions, plan for their learning and evaluate their work, among other things. Ball et al. arrived at a model based on Schulman's original work and this comprised two parts: First, 'subject matter knowledge' (akin to 'mathematical content knowledge') consisted of two domains – common content knowledge and specialised content knowledge. Second, 'pedagogical content knowledge' consisted of two domains – knowledge of content and students, and knowledge of content and teaching (Ball et al., 2008). Ultimately, it could be said that effective teaching involves knowing what students know and don't know, knowing what they need to know, and knowing how to help them make progress.

Confidence in Teaching Mathematics

Beswick, Ashman, Callingham and McBain (2011) cautiously suggest that there is a positive relationship between teacher confidence and improved mastery of mathematical knowledge. This is based on the outcome of interventions undertaken with final year pre-service teachers designed to improve their capacities to teach various strands of the primary mathematics curriculum. Beswick et al. (2011, p. 3) cited a study by Graven (2004) which concluded that "teachers' confidence appeared to grow as they increased their mastery of new ideas and practices". Specifically, Graven (2004, p. 206) suggested that "confidence [is] both a product (resulted from teacher learning) and process (an explanation of teacher learning)" and that this involved "... confidence in their ability to access resources to supplement their learning [and] confidence in their identities as professional competent mathematics educators". These ideas are summarised in Figure 1 that was developed from a graphic presented by Beswick at the 2011 Australian Association for Research in Education (AARE) Conference.

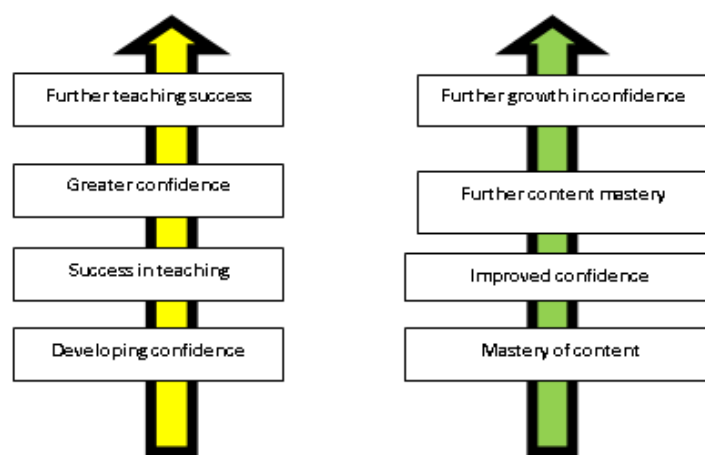


Figure 1. Growth in confidence to teach and mastery of content

Developing Professional Learning Communities

The notion of professional learning communities and their ability to improve outcomes for children have been widely documented in recent times and there is general agreement among various educators as to their characteristics and benefits. DuFour (2004) discusses three key features of PLCs, these being 'ensuring that students learn', 'a culture of collaboration' and 'a focus on results'. These sit well with the framework proposed by Stoll et al. (2005, p. 4) that contains eight features of PLCs, these being "shared values and

vision, collective responsibility for pupils' learning, reflective professional enquiry, collaboration focused on learning . . . professional learning . . . openness, networks and partnerships, inclusive membership, and mutual trust, respect and support". The second of DuFour's points aligns well with the pointers about collaboration, openness, networks, inclusiveness, mutual trust, respect and support. These features are the focus of the research related to PLCs as described in this paper.

Methodology

The study described here sought to answer the following research question:

To what extent does on-going professional learning for EAs and AIEOs enhance their content knowledge for teaching mathematics, their confidence to teach mathematics, and their development as members of professional learning communities?

A group of EAs and AIEOs underwent professional learning in a modified First Steps in Mathematics (FSiM) course in late 2010 and during 2011. Four of them from Valley View Primary School (VVPS) were then selected to form a pilot group, the main criterion for this selection being that VVPS had the services of a Getting It Right Numeracy (GIRN) teacher. During the latter part of 2011, this group worked with the GIRN teacher and underwent further training in planning for teaching Indigenous students, either individually or in small groups. This training was based on the FSiM Professional Judgement Cycle. In December 2011, the four members of the pilot group were interviewed. Data generated from the semi-structured interviews were grouped according to emergent themes. Pseudonyms are used for the school and all participants in the research.

Results and Discussion

All quoted comments were made by interviewees during semi-structured interviews conducted on December 7, 2011 at Valley View Primary School. Five themes emerged from the interview data with no one theme being dominant. Also, there were numerous occasions during interviews that a respondent would say something that could fit into more than one theme. A number of ideas were encompassed in each of the five themes which were:

- Pedagogical content knowledge
- Mathematical content knowledge
- Professional learning community
- Indigenous learners
- Confidence to teach mathematics

Pedagogical Content Knowledge

The main ideas expressed as part of this theme were increased skill at identifying children's needs and asking questions, making adjustments to teaching to meet needs, and becoming more comfortable with the language of mathematics. All interviewees felt that they had learned a lot about teaching mathematics to Indigenous children. Narelle described how she had learned to ask key questions about identifying what children knew and what they needed to know and indicated that she had begun to develop in the children an ability to pose questions for themselves rather than rely on the teacher. She also mentioned the significance of focusing on the language embedded in the tasks and problems at hand.

I've learned about children as individuals . . . that they learn at different rates and I think I'm becoming more confident with the language that I use . . . breaking things down . . . talking about and working out what sort of number sentence it is . . . picking out the key words of what they know about the sentence.

A lot of them would just read the sentence and then wait for me to prompt them to an answer . . . [now] they have to understand what they're trying to work out and how they are going to go about it. That's quite exciting for me . . . that's probably the main thing that I've learned – it's a new method of approaching what we're trying to solve, having them think for themselves more. (Narelle, EA)

Figure 2 shows an example of a worksheet that Narelle uses with her students and shows how she has been able to help them pose questions and recognize key words.

20 children were eating lunch.
Half went to play.
A quarter stayed in to finish their work?
A quarter went home.
How many children went to play? How many children stayed in to finish their work? How many children went home?

What do I know?	What do I need to know?	What strategies can I use to help me solve it?
20 children half went to play a quarter stayed in to finish their work? quarter home	HM went to play? HM worked? HM went home. Half = 2 parts a quarter = 1 part	$20 \div 2 = 10$ half $20 \div 4 = 5$ quarter $20 \div 4 = 5$ quarter

Figure 2. Student worksheet focusing on questioning and recognition of key information

Sharon indicated that she had become more able at adjusting her teaching to meet specific needs and recognizing exact points of intervention needed with children.

It's been good training because you can go, 'Oh they're not up to that', and we have to go back and make sure that they can count or . . . you can go back a little bit and get another tool or some concrete things for them to see the difference . . . doing the diagram or the pictures of the birds in the trees before you move on to the word sentence. (Sharon, EA)

These sentiments were echoed by Frances whose comments reflect an awareness of the need to build on understanding in small stages, which had been a central theme of the professional learning sessions.

We all have a set plan for what has to be covered but because the group that I had was a bit slower, we had to go at their pace, we couldn't jump ahead because if they missed those basics they've got to learn, they're not gonna get the next ones. (Frances, AIEO)

Mathematical Content Knowledge

The main ideas expressed as part of this theme included knowing the content and being able to break it into manageable segments for children, knowing precise points of understanding and what needs to be learned next, using appropriate resources and representations, and knowing and using mathematical terminology. Sharon spoke specifically of how she was better able to select appropriate resources.

The PDs that we did were of great value . . . being able to see how [things like] matchsticks could be used and being able to use the bead sticks that we made a couple of weeks ago . . . they are a handy tool. You get things and go 'Oh, we could use that for some purpose'. The PD has helped us do that. (Sharon, EA)

Sharon also spoke specifically of knowing more mathematics and knowing how to talk about it with children.

The way I was taught maths was totally different to the 'new' ways and just the terminology and things, you know, things like 'part part whole' – you become more fluent with different things . . . we're using the terminology that Terri [GIRN teacher] uses so there is parity. (Sharon, EA)

Narelle made similar comments and agreed that she wouldn't have used definitive mathematics terms such as 'decade' a year ago. She made mention of her ability to use

diagnostic tests and recording sheets and showed some examples of how she had used her newly gained knowledge to identify specific needs of children,

I've been taking copies of things like this [diagnostic record sheet] to see where they're at, regularly, and then building on that . . . they still struggle with the change of decades. (Narelle, EA)

Similarly, Cassie spoke of her increased knowledge and her ability to use it to identify points of error and misunderstanding in children's thinking. She presented a diagnostic recording sheet that she uses (Figure 3). It was evident that these aides are thinking and operating more like teachers since their exposure to professional learning. Cassie stated that she knew much more mathematics than previously and that she felt more confident in knowing what had to be taught to help children understand.

It's about identifying how they're thinking because we use a lot of written language as well so they can understand how they're working it out and we can identify where they're going wrong. I struggled in the past to know where they're not understanding and I would teach the basic things over and over whereas this has helped me to look at other options for how they learn things and shown me other ways of doing things. (Cassie, EA)

Names	Calculate Strategies Demonstrated							
	Counting On By 1s	Counting Back By 1s	Use equivalent groups to count	Use Doubling-1 digit number $\times +$ or -1	Use partners to 10 - Compatible Numbers	Count forwards and backwards in 10s to solve - Front end loading	Partitioning to solve 2 digit $+$ and $-$ problems with informal jottings	Use a calculator or to $+$ and $-$ in correct order
Marni	✓	✓	D	D	✓	•	•	
David	✓	✓	✓	✓	✓	D	D	
Dylan H.	✓	✓	✓	✓	✓	✓	✓	
Georgia	✓	✓	✓	✓	✓	•	•	
Andy	✓	•	✓	✓	✓	•	•	

Figure 3. Diagnostic recording sheet used by Cassie (EA)

Professional Learning Communities

The main ideas expressed as part of this theme included equity in relationships with teachers, genuine team work and a sense of harmony, and pre-planning and post-lesson reviews with the GIRN teacher. Frances made the following comment that indicates the development of genuine teamwork with her teacher.

When we're in the class as a whole, when we're doing something together, they [children] know that they can put their hands up and either me or Roslyn [class teacher] will give them exactly the same answer - we're kind of in harmony. We're using the same approaches. (Frances, AIEO)

All interviewees were of the view that they were valued as equals by the GIRN teacher and class teachers. Cassie noted that she would meet with those teachers on a weekly basis and that she had found the help and advice invaluable.

I was definitely valued as an equal member of staff . . . I've always struggled with maths and I've not gone into as much depth and I haven't been able to try new stuff and been able to put as much time into it. This has really changed my thinking of maths . . . seeking help made me feel silly before [but] having the planning meetings to feed off - she [Terri] constantly checks how you're going and looks at your planning. . . it opens your mind up to a lot more and makes you feel more comfortable with it. (Cassie, EA)

The following exchange between the researcher (interviewer) and Narelle (EA) indicates the level of confidence shown by the aides in seeking help as well as the quality of the relationship.

INT – So you know that this [planning document] is available and you know which bit to use to solve a problem?

NARELLE – Yeah and if I'm not sure, I go to Terri [GIRN teacher] and say 'Help me'.

INT – But you're quite happy to go and ask?

NARELLE – Oh yeah [emphatically] I always ask them . . . I touch base with them about where I'm at because I don't want them [children] to be in a spot where they're not succeeding.

Indigenous Learners

The main ideas expressed as part of this theme included the building of relationships through empathy and connection, building of learner confidence, and the breaking down of a fear of failure. The most significant comments were made by Frances (AIEO) in describing the work she had done with a Year Seven boy, Eric, and the dramatic improvement in his achievement over three months. It is suggested that this improvement resulted from the regular contact he had had with Frances and the steady building up of a relationship between child and aide that was enhanced by the growth in Frances' level of knowledge and confidence. In August when Frances began to work with him, Eric was achieving at a very basic level with poor reading, writing and number skills.

He didn't even know how to write the numbers so I had to practice with him. This was some basic stuff, right at the start of it. Even when he was filling the sheet in, he noticed the pattern [for doubling] – he was following the pattern but he didn't recognize the numbers. He was just copying [the pattern] – he wasn't doing the actual work – he didn't have the understanding. (Frances, AIEO)

Frances also showed some work samples (Figure 4) that demonstrate the growth Eric has made and she commented about his increased application and confidence.

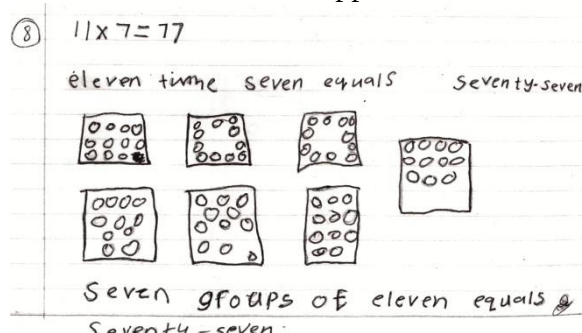


Figure 4. Sample of Eric's work after 3 months of intervention

- Yeah, can't stop him now ... doubles, doubles plus one ... this was the Year Seven who couldn't read or ... you know struggling, always gettin' kicked out of class. He's come a long way. He's much happier ... before, we had to go off looking for him – he'd get lost between the Year Seven classroom and the C Block [location of the Maths activity room]. He'd be wandering around ... he'd go into the toilet block, go to the next toilet block, and we'd be lookin' 'round for Eric – couldn't find him anywhere. Now when it's Maths, he's the first one in there. (Frances, AIEO)

Frances also felt that the relationships built up between aides and teachers and the Indigenous students were helping a great deal.

There's been a big improvement – they've wanted to be there so their attendance has improved because we're focusing on that group [Year 6/7] and them as individuals. (Frances, AIEO)

Narelle expressed similar sentiments about relationship building and how this was helping the Indigenous children.

You just build relationships all the time and you just get to know the children more and they feel confident and they trust you and with these little extra things that I learn, I feel more confident. So as

time goes on, you improve and you see them improve and it's constantly reassuring them that they're doing a great job ... have a go, don't be afraid to say the wrong thing – that's constantly reassuring, building the confidence up, no matter what, if they get it wrong ... building of the trust and relationship is so important – you can't function without that can you? (Narelle, EA)

Sharon also made mention of the on-going value of the relationships being built in that children with whom she works for mathematics, are approaching her more regularly when they want assistance for a variety of reasons as they feel that there is no stigma attached to seeking help.

Confidence to Teach Mathematics

All interviewees were emphatic about having greater confidence to teach mathematics and the main ideas expressed included confidence being derived from knowing more content, from knowing how to help children, and having success in teaching. Each interviewee was presented with the graphic shown in Figure 1 and all agreed that it accurately represented their own situations. That is, that mastery of mathematics content led to greater confidence, which in turn led to further mastery and then greater confidence. They believed that this linked well to the other element in that, as they became more confident, they had more teaching success, which led in turn to greater confidence and further teaching success. Since the professional learning that the aides underwent was focused on increasing the levels of pedagogical and mathematical content knowledge and ways to use this knowledge in planning, this indicated a measure of success for the professional learning intervention. The following comments underline this point.

I'm getting the confidence as I go along and I'm still learning, myself, every day and I always take notes about how they [teachers] go about teaching so I need to be mimicking what they do. I really do enjoy what we do and it's so rewarding to see when they get it. (Narelle, EA)

I actually now enjoy teaching maths whereas before I used to fear it. I struggled to find ways to teach it because I haven't got a maths brain ... I'd put up a wall or barrier and I wouldn't seek help, like I said. Now, with all this, I'd like to continue with this [learning]. It's fantastic, I'm learning it and I can teach it and it's making me feel more confident. Usually, with Maths, I'm petrified of it, but this has been the best thing that I've done (Cassie, EA)

Conclusions and Implications

Several conclusions can be drawn from the pilot phase of this on-going study. Generally, they indicate that involving the EAs and AIEOs in targeted professional learning has been successful on a number of fronts. As a result of the professional learning, it could be said that the EAs and AIEOs:

- Increased their pedagogical and mathematical content knowledge and applied it to their roles
- Became more knowledgeable about helping Indigenous learners and more confident in their ability to do so
- Became more effective and equal members of professional learning communities in planning and implementing targeted learning for Indigenous children
- Began to think, talk, and work like teachers

The success of the pilot study described in this paper has prompted the Swan Valley Cluster of the Make It Count project to expand the professional learning for teacher assistants (EAs and AIEOs) to involve over forty assistants in four schools during 2012. This program has already begun and will continue throughout 2012.

References

- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59, 389-407.
- Beswick, K., Ashman, D., Callingham, R. & McBain, D. (2011). Teachers' and pre-service teachers' confidence to teach primary school mathematics. Paper presented at the 2011 conference of the Australian Association for Research in Education, *Researching Across Boundaries*, Hobart, 2011. J. Wright (Ed.), ISSN 1324-9320
- DuFour, R. (2004). What is a professional learning community? *Educational Leadership*, 61(8), 6-11.
- Graven, M. (2004). Investing mathematics teacher learning within an in-service community of practice: The centrality of confidence. *Educational Studies in Mathematics*, 57, 177-211.
- Hurst, C., & Sparrow, L. (2010). The mathematical needs of urban Indigenous primary children: a Western Australian snapshot. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education. Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia*. Fremantle, WA: MERGA.
- Schulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15, 4-14.
- Stoll, L., Bolam, R., McMahon, A., Thomas, S., Wallace, M., Greenwood, A., & Hawkey, K. (2005). *Creating and sustaining effective professional learning communities*. DfES Report RR637. Bristol, UK: University of Bristol.

Curriculum Leadership: Reforming and Reshaping Successful Practice in Remote and Regional Indigenous Education

Robyn Jorgensen (Zevenbergen)

Griffith University

<r.jorgensen@griffith.edu.au>

The importance of leadership in changing schools and building quality programs is the focus of this paper. While leadership is often seen as a management aspect of school life, the role of leadership in curriculum change may be quite different from that of school leadership vis-a-vis the principal. In small remote schools where there are many factors impacting on school reform, this paper explores curriculum leadership where the schools successfully performed against numeracy benchmarks. Features of curriculum leadership are drawn from these cases to develop a framework for considering effective leadership in remote Indigenous contexts.

Leadership in schools comes in many forms and extends beyond the role of the formal leadership roles, such as Principals and management structures. The principalship is often perceived to be a position synonymous with school leadership. The role is integral in leading and managing schools and may provide vision for the school. Many managerial tasks are part of the principal's brief in terms of overall management and accountability within schools, so some aspects of leadership may be taken by other members of the school community. Leadership comes in many forms and through a range of roles both formal and informal. Curriculum leadership may fall into the ambit of the principal but may also be part of a devolved or distributed model of leadership where a key teacher may assume a role in leading curriculum innovation. As such, curriculum leadership may be an amorphous role within the structure of the school and fall to a person or group of people who assume responsibility for curriculum as a whole or for a particular curriculum area such as mathematics/numeracy. This paper explores the ways in which curriculum leadership was enacted across a number of schools in the Northern Territory. The overall study that is the basis of this paper was to examine the practices at nineteen schools across the Northern Territory that had been successful in literacy and numeracy in the national testing scheme in 2010, and the results published in 2011. This paper draws on the numeracy practices around curriculum leadership in these schools.

Curriculum Leadership

Within the literature on school leadership, there is some consensus that there are a number of key principles that are core to successful school leadership. Leithwood et al (2006) suggest that these include: "building vision and setting direction; understanding and developing people; redesigning the organisation; and managing the teaching and learning program". The importance of the principal in providing vision and strategic direction for the school is critical to the overall direction of the school. In many small schools, the principal may take on the role of leader and manager as well as curriculum leader. The principal may have particular leadership style within the school and share leadership and roles across the school through to being quite autocratic in decision making. The complexity of schools, systems and personalities shape the possibilities of curriculum leadership.

Principals exert an influence on student outcomes (broadly conceived) through a focus on teaching and learning which is driven by their own values and vision, an agreed school vision, elements of transformational leadership, and increasing school capacity, across four dimensions (personal,

professional, organisational, and community), taking into account and working within the school context, and using evidence-based monitoring, and critical reflection to lead to change and transformation. (Drysedale, Goode & Gurr, p.699).

Undeniably, the principal has a key role in the overall school culture and direction. In their study of leadership and principals, Walters et al (2003) reported the strong relationship between the impact of the leadership and student achievement. The principal was key in terms of quality curriculum but inclusive of the overall school environment and support mechanisms within the school. These elements are clearly important in terms of the culture of the school, but for this paper, I would move away from the role of the person to the characteristics of curriculum leadership so that these may apply to the principal or equally to another staff member who is taking on this role.

The notion of distributed leadership closely aligns with the devolution of curriculum leadership to key staff members within the school community. In some cases, the devolution of curriculum leadership within the school may be a strategic decision that aligns with the vision of the school to be inclusive of the strengths of the staff (and possibly students). Alternatively, the move to involve staff in curriculum leadership may be more of an ad hoc or opportunistic decision where a particular staff member may have an interest in the area. As such, Gronn (2002) has argued that the social context of the school must be considered when investigating the impact and roles of leadership within a particular site.

Policies, such as national testing or curriculum, can be contentious and open to interpretation. In the rollout of a curriculum reform in Queensland, it was found that principals and teachers had very different approaches in how they interpreted and enacted the curriculum (Niesche & Jorgensen, 2010). Hayes et al (2004) have argued that there is a need for productive leadership to be considered in how different interpretations of policy are not only possible, but of considerable value, as the contexts in which policies are implemented can vary considerably. More specifically, in schools where there are challenging circumstances, Harris (2002) reported that it was important to have devolved models of leadership so that staff could build positive relationships across the school and where staff were empowered to lead.

It is in the context of National Testing and the outcomes of these tests, that this paper explores curriculum leadership. There were a number of schools identified in the Northern Territory that were successful when compared with similar schools, on their literacy and/or numeracy test results. Identifying the successful practices at these schools was part of a larger study. One of the elements of success was the leadership around numeracy (and literacy) curriculum. It is within this very broad conceptualisation of leadership, that the enactment of numeracy reforms aimed at improving student performance was studied. The contexts of the schools were remarkably different and hence shaped the possibilities of reform and leadership.

Approach

The study was part of a research consultancy for the Northern Territory Department of Education and Training (NT DET). NT DET nominated the nineteen schools as having performed well on National Assessment Program for Literacy and Numeracy (NAPLAN), either scoring above similar schools or having improved their scores over time. The schools were in Darwin Region, Alice Springs Region or in remote sites across the Territory. The study was to document the practices at these sites to understand what was happening at these schools in order to build stories of success to share with other teachers in the NT.

Ten sites were visited and for the others, being in very remote (and difficult to access) locations, staff members were interviewed by telephone. In the sites visited, lessons were

observed and interviews conducted with staff involved with the literacy and numeracy practices at the school. Across all sites, documents and photographs of the key practices contributing to success were collected. The stories were those from the schools. It was not the intent of this study to show if there were a correlation between the practices and the scores as this was not possible within the timeframe and budget. What was sought were vignettes from the schools that documented their practices that could be seen to contribute to success.

All schools were government schools but as the schools varied in size and composition, as well as age cohorts, the number of participants varied for each school. The principles for interviewing were to include the principal to gain a sense of the vision of the school and the overall context; staff involved in the teaching of numeracy and curriculum development; any support staff involved in the reforms at the schools, and any Indigenous education workers associated with the curriculum reforms. Most site visits were for a full day but in some of the larger schools, two days were spent at the school. Two researchers visited each site.

Interviews were shaped to the role of the personnel but were targeted to identify the context of the work; the practices undertaken at the school; the way/s in which reforms had been developed, implemented and evaluated; the roles of people involved in the curriculum innovations; and the perceived benefits of the programs at the school. The telephone interviews were recorded as a record of summary. The data presented in this paper have been drawn from the summary records and then created as a narrative around the discussion points. This paper explores the practices of numeracy leadership in diverse settings. From the cases involved in this study, key principles that were drawn from the study will be highlighted.

Contexts and Curriculum Leadership

Initially considering the data, the responses will be clustered into two main groups. These groups are based on the size of the schools as the practices in the larger, regional centres were markedly different from the small schools in the more remote contexts. These latter schools were usually two to four teachers with the principal in a teaching/principal role. All schools, except two – one in Darwin and one in a remote setting -- had large numbers of Indigenous students. In the regional settings these may have had between 35%-50% Indigenous students whereas the remote, smaller schools (except one) were predominantly all Indigenous students.

Devolved Leadership

The predominant model for leadership was that of devolved leadership where Principals had involved other staff in key decision-making related to the numeracy programs within the school. In one school, the Deputy led the development of a common approach across the school in terms of both curriculum and assessment. She developed systems that were accessible to all staff so that student achievement was transparent. Also, she ensured that successes by teachers and students were part of the public space within the school. Having transparent processes that all teachers were expected to participate in had become part of the school numeracy program. This process had taken nearly three years to develop, implement and embed across the school.

Deputy: When I commenced at the school there was no numeracy program. Teachers were able to do anything they liked. I came from Victoria where core programs across the school were part of the school culture. So, I looked at the numeracy programs we had, and in consultation with the teachers

we chose to use one program that we then did professional development around. Also, there was no common testing program for numeracy across the school, so I looked at the available tests and made some decisions about what we needed. All teachers now use these tests and the school results are recorded on a common database. All teachers have access to this so they can see where their students are, and where they were the year before. Growth is important to us as a school. I draw on the expertise of the teachers in terms of the practices we use to support the school program. One teacher in particular is our model teacher and she is used to exemplify good practice to the others.

What was important was that these views were reciprocated in the interviews with the teachers and other staff at the school, indicating that the practices had been embedded within the school and had not remained at the level of rhetoric. This devolution of curriculum leadership was evident in both numeracy and literacy reforms at the schools but it was very much shaped by the context of the school.

Regional Schools

The larger schools with more than 300 students were based in either the Darwin or Alice Springs regions. These larger schools had a full-time designated principal and at least a deputy principal who may have been full-time or involved in minimal teaching. In these schools, the deputy principal typically led curriculum reform. The models employed at the 4 schools were largely devolved leadership where the Deputy acted in a leader/co-ordinator model and worked with other staff members to support and build the numeracy program. In all schools, there was a key person who used data to inform actions and build appropriate scaffolding programs. This might have been the Deputy but also included a key teacher who had an interest in co-ordinating data. The data included test results from NAPLAN and enabled schools to monitor how they were progressing and areas of weakness in the school numeracy programs. All schools, except one, also used other data to inform their numeracy education practices. The one school that relied heavily on NAPLAN data was a predominantly middle-class school with a very small (almost negligible) Indigenous student cohort.

Principal: At this school, I decided to continue the work of the previous principal as he had worked tirelessly to ensure quality program. As the Deputy had worked closely with him, she has taken over the reins of the numeracy (and literacy) program. She knew what had happened in the past and has been able to build on that. Also, my background is secondary school so she is expert in primary, particularly early years so she knows what has to happen in those formative years. She is much more expert than I am on this so I am happy for her to continue the great work that has been strongly embedded in the school and in the knowledge of the teachers.

Remote Schools

The situation was quite different in the remote schools where there were frequently only two teachers and the principal had a teaching-principal role. There was little scope for devolution of leadership as there were few people for the sharing of load and ownership. In the cases in this study, the principals had been very successful in working with their teachers to develop numeracy programs of which the staff assumed ownership. In other cases, the lack of expertise/interest in numeracy/mathematics was a challenge in some of the smaller schools, particularly those at the primary level.

Teaching Principal: For me, I am not that good or interested in numeracy so it has been difficult to work out a good program for the school. Professional development is hard to access so I have had to rely on someone [regional curriculum consultant] coming in to help us. I have really found the conversations around numeracy from the consultant to be very valuable in helping to build our numeracy program.

Teaching/Principal: I have found it difficult to balance the role of principal and all the paperwork we have to do for that, and then to develop a maths program. We have an enthusiastic teacher who has an interest in maths, so she has taken over the coordination of our numeracy program. That will make a difference to have something organized, and a sense of good equipment that can be used properly. I am more than happy for her to take on this role. I am sure she is going to do a great job.

The number of teachers/staff in a school impacts on the possibilities of devolved leadership. This is further exacerbated by the fact that many of the teachers in remote contexts are from interstate and are new to the career of teaching. Building a numeracy curriculum that aligns with the needs of the school, expertise of the staff, and complies with statutory requirements was an important consideration of the work of the schools.

Issues Around Curriculum Leadership

There were some salient aspects of curriculum leadership that shaped the possibilities for leadership and curriculum reform. These are discussed in this section. A common theme across all schools was the need for consistency in the numeracy programs and transparency so that students and communities were aware of what was constitutive of learning.

Principal: I don't think it really matters what mathematics program we use in the school so long as we are explicit and consistent. What is always a problem is when there is no common or shared program and teachers do whatever they like. It does not work. So if the teachers are all teaching the same way and content is building on each other, then there is scope for learning and growth.

Use of Commercial Programs

Across most sites, there was a tendency for schools to rely on some sort of commercial program for their numeracy program. As noted in earlier comments, there was a common experience across many of the sites where they sought to find commercially available resources – teaching programs, resources and/or assessment tools – that could be used as the numeracy program. The role of the curriculum leaders involved the selection of resources and their implementation at the school level.

Sustainability

Perhaps one of the biggest challenges in schools, and particularly remote schools where there is a high turnover of teachers/staff, is sustainability. Often remote principals are on an upward career path and feel they have to make their mark at a school. Many of the comments raised by participants in this study highlighted this issue and is captured in the following comment:

Teacher: New principals come into these schools and need to make their mark so that they can be moved to a new location probably in Alice or Darwin and get a promotion. They have to change what was here even if it was working. Perhaps it would be productive for the system to recognize that it could be a good thing for a new principal to recognize the good work of the predecessor and continue to work with and refine the earlier work.

This results in a constant change in curriculum as new principals come to the school with their ideas. Having consistency in programs was identified by most of the participants in the study yet, most of them also commented on the constant change in provision of numeracy programs within schools. This made it challenging, particularly in remote settings, where the community had a constant change of staff and, with that, a constant change in programs. However, in the regional settings, it was more likely that the staff were staying in the region for extended periods, thus making for longer stays for the teachers and staff, and less impact on change.

Teacher: It was good when the new principal came in and said we were staying with the program. We had spent a lot of time building resources and on professional development so it was good to continue with it. The whole school is on board so to build it and make it even better was very good.

The situation in the remote schools was slightly different from the above comment. In the remote settings, many factors impact on sustainability and curriculum leadership. In particular, the upward mobility of potential leaders means that they often want/need to be able to demonstrate leadership and hence change programs when they arrive at a remote school. This was particularly noted by principals who had remained in communities for extended periods of time.

Principal: Each time a new principal comes to the school they want to make the school look like what they think is best. The community is sick of changing and they don't know what is going on.

Principal: I have worked in remote areas for a long time and I know the communities are tired of change - every time a new principal comes in. I decided I would keep the numeracy program we had here before and refine it. It was pretty standard stuff so it was not a real issue.

Despite best efforts to build sustainability into numeracy programs, there is the potential for change with the changeover of new staff. In one site, the principal had worked hard with the staff to build a shared curriculum for numeracy (and literacy) in which all teachers had an interest and commitment. She had also worked extensively with the community so that they were aware of the numeracy program. This had included open nights at the school, sending information home to the families, meeting with people from the community and developing an after school homework facility. Indigenous education workers had been employed and trained in the numeracy program and were able to act as conduits between the school and community. However, during a term's leave, a relieving principal was placed in the school and had revised all the programs so that when the substantive principal returned to the school, many of the programs had been changed. She felt demoralized that a temporary person would have created such changes, and as a consequence, did not feel energized to revisit and return to the programs that had been developed earlier. In her interview she expressed disappointment to have thought she had built in many strategies for sustainability only to find that the programs were changed so much in a 10-week term by a temporary principal.

In considering sustainability and curriculum leadership in this context, it would appear that there is some tension between sustainability and the accreditation process where upwardly mobile leaders are implicitly encouraged to create new learning environments that demonstrate their leadership capacity. This engenders a climate of change and hence risks the stability in programs that may offer greater potential for learning among remote communities. Curriculum leadership may be considered around notions of change, but it may also need to be considered as a skill to recognize quality practice and build on existing practices.

High Expectations

Teachers, staff and community need to have high expectations of the students as well as themselves. This was evident in all schools in this study, and the numeracy curriculum was expected to have high expectations of learning – of staff and students.

Principal: I have encouraged teachers to model what they expect of the students. They have to be back in their classrooms before the bell rings so that they are ready to take the students inside. This helps the students know the hidden curriculum of school.

There needed to be high expectations of the students in terms of what they could learn mathematically. This was reflected in the reforms.

Principal: We needed to ensure that the teachers had high expectations of the students. They needed to make sure that the [numeracy] program reflected high levels of learning rather than low standards and basic skills that they had been doing for years.

Teacher: We have built the numeracy curriculum around expected levels of numeracy. The children do the same content that you would see in a school in Alice or Darwin.

While these comments indicate the staff at the schools saw the importance of high expectations, there was a lot of commentary about the challenges of moving staff away from deficit models of learning to models that reflected high expectations. This was an important component of curriculum leadership.

Curriculum Leadership and Community

In many remote locations, the school is part of a community. Relationships between staff, curriculum and community are critical to success. For communities to feel that they know what is happening at the school and feel some sense of connection with the programs, curriculum leaders worked with community to develop the programs and to keep community members informed on the on-going implementation of the programs. Some schools had worked very closely with their communities to develop numeracy programs that built the necessary knowledge and skills inherent in school curriculum policy documents as well as working with the needs and interests of the community.

Principal/Teacher: When I started here, I spent a lot of time talking to the community about what they wanted and what I would like to do. This helped me know where we needed to go with our programs. We have times where we show the families to help their children with some of the maths work we are doing. They commented on how much they appreciated not only knowing what the kids did, but also how they were doing their maths.

VET Teacher: We talked with the people in community and they wanted their kids to be able to get jobs and do things. We started “The Shed” with all the tools they would need to build things. They have had to do a lot of maths just to design The Shed, but also now when they build things, there is a lot of maths they need – in the design and budget for example. The students (secondary) love it and the younger ones can’t wait to get to do things in the shed. It is very practical and the community can see the value in the students learning how to use tools. So they are getting the practical knowledge they need but also it is strongly linked to many mathematical ideas. It works brilliantly for these kids who are very practical in their thinking.

Some schools had created social functions to attract families to the school. This is a critical role in many communities where there are few opportunities for social gatherings as there are no venues for the events found in regional and urban settings. As such, the school can become the hub of the social life of a community.

Being able to operate a school flexibly as the principal immediately above outlined meant that he had scope to shape the timetable. In this case the school operated on longer days from Monday through to Thursday and then only in the morning of Friday. The students could go home and rest, and the school could be prepared for the evening event.

Conclusion

Curriculum leadership is critical to the development of quality and appropriate numeracy programs. The context of the school will have influence on the practices of leaders and their success in curriculum reform, implementation and evaluation. In this study, larger schools were able to adopt a devolved leadership model for curriculum but this was restricted in smaller schools due to the number of staff in those schools. Smaller schools were based predominantly in communities and curriculum leadership in these sites included leadership within the community so that Indigenous families were informed and shared the

vision of the numeracy curriculum. Numeracy curriculum leadership in all schools focused on developing coherent and shared curriculum and assessment practices across the school.

Curriculum leaders had a number of challenges in their capacity of leaders as they built quality programs. Sustainability of numeracy programs within schools was under threat for a range of reasons. Perhaps most significantly was that new leaders come into schools wanting to shape programs in order to demonstrate leadership. In terms of sustainability, and in the community interests, it may be more prudent for curriculum leaders to recognize the strengths of existing programs and maintain and build on existing programs. This model of curriculum leadership is somewhat incongruent with perceptions of leadership and the need to change as demonstration of leadership.

References

- Bezzina, C. (2012) "The road less travelled": Professional communities in secondary schools. *Theory into Practice*, 45(2), 159–167
- Drysdale, Goode, & Gurr, (2009) An Australian model of successful school leadership: Moving from success to sustainability. *Journal of Educational Administration*, 47 (6), 697-708
- Gronn, P. (2002) Distributed Leadership. In K. Leithwood, P. Hallinger, K. Seashore-Louis, G. Furman-Brown, P. Gronn, W. Mulford, & K. Riley (Eds.) *Second International Handbook of Educational Leadership and Administration*, Dordrecht: Kluwer
- Leithwood, K., Day, C., Sammons, P., Harris, A. & Hopkins, D. (2006), *Seven Strong Claims about Successful School Leadership*, National College of School Leadership, Nottingham.
- Niesche, R. & Jorgensen, R. (2010). Curriculum reform in remote areas: The need for productive leadership. *Journal of Education Administration*. 48 (1),.
- Walters, T. W., Marzano, R. J. & McNulty, B. (2003) *Balanced leadership: What 30 years of research tell about the effect of leadership on student achievement: A Working paper*. McRel.

Digital Games for Learning Mathematics: Possibilities and Limitations

Robyn Jorgensen (Zevenbergen)
Griffith University
<r.jorgensen@griffith.edu.au>

Tom Lowrie
Charles Sturt University
<tlowrie@csu.edu.au>

Drawing from Gee's learning principles developed from the digital games environment, we provide a critical analysis of the difference between using these principles in a literacy environment as opposed to a mathematical environment. Using stimulated recall, primary school-aged students played with a number of contemporary digital games. Feedback was sought. This was compared with the descriptions provided by experienced adult gamers. Both players provided insights into the cognitive process used by gamers when engaging with games. Collectively, these sources allow us to propose that the learning principles may restrict deep learning processes for mathematical learning.

Drawing on a number of sources from a large, three-year project, this paper explores the possibilities of digital games to offer engaging, and deep learning, environments for digital natives. Digital natives are described by Prensky (2001) as the generations who have been raised in digital worlds and have developed learning styles and dispositions that have been shaped by these digital worlds. His contention is that young people are differently disposed to learning, problem solving, and feedback because of their exposure to digital media. These dispositions are different from previous generations (digital immigrants) who were raised in print-based worlds. These different worlds have created very different experiences that ultimately shape the habitus of those raised in these worlds. Prensky argued that many of the young people entering schools and work have dispositions that are at loggerheads with the practices of education and work. Furthering this work, Gee (2003) has examined the digital games environments to explore the principles used by the gaming industry to engage players in games. As a highly competitive industry where millions of dollars can be spent on developing games, the industry has designed games that engage players for extended periods of time. Gee's principles have been used to justify reforms in education that will engage the digital natives as they enter schools. Gee and his advocates have argued that the current practices in school are failing to cater for learners (digital natives) and that the thirty-six principles used in games designs could radically shape new learning environments that cater for learning and learners in the digital era. This paper draws on this body of work to examine the potential of games to create opportunities for learning mathematics but also to explore the possibility of new forms of mathematical learning and mathematical understandings to evolve from these worlds.

Digital Games: New Possibilities

A catalyst for this study was the work undertaken by Lowrie (2005) where he worked with eight-year old students using the Pokemon environment. In this work, Lowrie found that the children worked well beyond the experiences being provided in the standard school curriculum in terms of spatial representation and visualisation. This work highlighted the possibilities of the digital games environment for enhancing mathematical learning and understandings that were beyond the realms of standard pencil-and-paper representations. At the other end of the digital native continuum, Jorgensen's (Jorgensen (Zevenbergen), 2009; Zevenbergen & Zevenbergen, 2008) work with young workers found that there were

very different dispositions towards and using digital tools in the workplace and those young workers deferred cognitive labour to digital tools so as to enable them to move quickly through their nominated tasks. These dispositions were not recognised, and often not valued, by employers (or teachers). To this end, the findings reported in this paper draw from a three-year project that sought to clearly identify the possibilities of digital environments, namely games, to enhance mathematical learning and create new mathematical dispositions (or *habitus*) for young people (digital natives).

Gee's (2003) analysis of digital games has created thirty-six learning principles. Drawing on three discourses (situated cognition, new literacy studies, and connectionism), Gee provides a comprehensive account of the possibilities of games to create exciting and engaging learning opportunities. Primarily, Gee focuses on how the games environment allows for new forms of literacy learning and engagement in literacy texts. The literacy demands of these digital worlds are substantially different from the linear text models of the printed media which has dominated literacy since the industrial revolution. It is beyond the scope of the limited page length here to outline each of these principles. Fundamental to Gee's principles are the notions that gamers identify with the game and develop an identity (and affinity) with the game that aids engagement. Once in the game, the player then is further engaged through its underlying structures where there is a progression through the game from simple activities that progressively increase in difficulty. As the player engages with these increasing complexities, he/she is strongly scaffolded through a range of design principles including low-failure, where failure is not public so that there is encouragement to engage. The game is also structured so that skills learnt in one level will be used and extended in subsequent levels. These principles, Gee argues, need to be embedded in school learning. It is beyond the scope of this paper to expand and explain all thirty-six principles. We have elected to focus on four principles to illustrate how they were enacted by the participants and their links to new learnings in mathematics.

Method: Video Stimulated Recall

Drawing on the literature using stimulated recall (Lyle, 2003; Zanting, Verloop, & Vermunt, 2001). Stimulated recall is an "introspective method that can be used to elicit people's thought processes and strategies when carrying out an activity or task" (Henderson & Tallman, 2006, p. 55) and utilises a support mechanism to enhance recall (Gass & Mackey, 2000; Henderson & Tallman, 2006). In this study, participants were given a Nintendo DS and '*The Legend of Zelda: Phantom Hourglass*' game and were asked to play the game as much as possible over a two-week period. The participants were then videotaped, for approximately 30 minutes, at the natural progression in the game. Thus, the game-playing episode was at a novel point that was 'new' to the participants. Two camera angles were utilised, one focused on the two screens of the Nintendo DS and the other was a front view of the participant. These two video files were edited using Studiocode software to create video footage that presented both the screen view and the front view simultaneously. Semi-structured interviews were conducted with the participants after the game playing episode to elicit their reactions to the experience and to encourage the participants to describe their game playing patterns over the duration of the activity.

Within a week, the participants undertook a stimulated recall session which involved them watching the video footage of their game playing episode and answering a pre-designed set of semi-structured and open-ended questions based on particular points and moments in the game that were significant (as identified by an expert who had previously played the game and to elicit participant thinking of a particular action). The games were then played back to the players who described their thinking as they navigated through the

game. Split screen video recording was employed so that there was one camera recording the videotape being played to the player and another camera focused on the player. Coordinating the two images was critical for analysis to coordinate the points in the game with the player's comments. In total, seven players were involved in the stimulated recall with two games being played. Three boys and four girls played the games and their ages ranged from 10 years to 14 years. The stimulated recall participants were selected through an expression of interest sent out through the University system. The expression of interest sought to identify students in middle school who were willing to play specific hand-held games and be prepared to participate in the three-week study.

Recognising that the players were young and may not be able to articulate their thinking, older gamers were also asked to play the same games and develop descriptions (written) of their processes as they navigated through the games (Jorgensen & Lowrie, 2011). Some triangulation between novices and experts enriched the descriptions of the thought processes. While the experienced gamers' descriptions were richer in terms of their detail and capacity to write for an audience, there was considerable synergy in the processes used by both players in how they navigated through the games, and their processes used to solve problems/obstacles as they navigated through the worlds.

Prior to this very detailed (and labour-intensive) part of the project, a large survey (n=428) was administered across two states of Australia to identify the more popular games to use for this part of the project (Lowrie & Jorgensen, 2011). In this paper we draw on the stimulated recall for one of the games used - *The Legend of Zelda: Phantom Hourglass*'. This game was chosen as it was an *Adventure* game, which was a popular game type amongst survey participants (Lowrie & Jorgensen, 2011). The *Legend of Zelda* game challenged the player to navigate around various environments in order to locate a lost person. Like many adventure games, the game play had scaffolded activities which had to be completed before moving to the next phase of the game. The game was also selected because it was a game used on hand-held consoles and therefore could be played in various contexts and settings, and it facilitated curriculum-based mathematics ideas (e.g., problem solving, two-and-three dimensional space, and position).

Learning Principles: Opportunities and Constraints

In the remainder of the paper, we analyse four learning principles – for the opportunities that it provided for mathematical learning and how they constricted the learning opportunities. In this latter analysis, we premise the notion of “constraint” on the current reforms in mathematics education that promote deep and complex learning over rote-and-drill learning. While we entered this project with the hope that the games environment would enable new ways of learning mathematics that were engaging and rich (based on our reading of Gee and Prensky cited earlier in the paper), our data have challenged this assumption. Both the stimulated recall with young players and the descriptions provided by experienced gamers have illustrated a very narrow approach to game playing and one which is the antithesis of reform pedagogy.

In the following section, we draw on the data from the interview with one student, Zane an 11-year-old boy from a regional city in Australia. Zane played *The Legend of Zelda: Phantom Hourglass*' game and was a relatively accomplished player. He regularly played digital games on both hand-held consoles and larger, fixed consoles such as an Xbox360 or a PC. Our aim in this section is to link the data with the most appropriate learning principle described by Gee to provide illustrative examples of the learning principles, their application to mathematics at a generic level, and how the principles were observed within the digital games stimulated recall. The purpose of these data is to provide illustrations from

one student. The extracts could be applied to other learning principles and other players. However, for the purposes of this paper we are seeking to illustrate the experiences and processes used by the player in terms of possibilities for mathematics learning and mathematical concepts/processes. If Gee's principles can be applied to mathematics as they have been done in literacy education, then we contend that there is a radical possibility for new learnings. We use four principles to explore the possibilities of and for learning mathematics: 13) Ongoing Learning Principle; 15) Probing Principle; 12) Practice Principle and 6) "Psychosocial Moratorium" Principle. Within the limited confines of a conference paper, we have elected to use these principles as they are quite diverse in their propositions and as such serve as illustrative examples of both our approach to studying the phenomenon as well as illustrating how the principle was lived out in this game and with this gamer. An outline of each principle is drawn from Gee's descriptions in the literature, . We then provide our interpretation of how such a principle would be lived out in the mathematics classroom and give examples from the stimulated recall transcript to illustrate how this gamer (Zane) interacts with the game. In selecting these extracts, we seek to demonstrate how the principle is lived out when framed within a mathematical context and to note the affordances and constraints of such principles as they apply to mathematics.

13) Ongoing Learning Principle

The distinction between the learner and the master is vague, since learners, thanks to the operation of the "regime of competency" principle listed next, must, at higher and higher levels, undo their routinized mastery to adapt to new or changed conditions. There are cycles of new learning, automatization, undoing automatization, and new re-organized automatization.

Within the context of mathematics education, such a principle is useful in terms of expansive learning where the student will revise previous learnings and concepts and modify these in light of new learnings. A simple example of this is as the students develop proficiency with addition facts, they develop a schema for addition being an increasing process but as this proficiency becomes an automatic process, negative numbers are introduced and the increasing schema must be modified to accommodate the learning. The concept for addition is substantially revised to incorporate new experiences. Learning occurs on a particular trajectory where principles can be intuited and developed into more complex principles as further learning experiences are provided. The scaffolding to enable on-going learning supports the current learning but in subtle ways also expands learning to include anomalies so as to build new, more complex understandings.

In this extract, the player has been working his way through the world and has had to devise a strategy where the statues can be manipulated so that he progresses through this level. He is aware that there is a problem that needs to be addressed in order to move forward. Initially, he had devised a strategy where he thought that if the statues were activated but this was not successful. He realised he needed a different strategy. Earlier learning has created some traces in his game memory of how to get objects, in this case the statues, to be manipulated. While the strategy involved some trial and error to see what might work, there was some progressive refinement of strategies that would enable progress.

Q: So, we're just going to pause it there; we just want to ask you can you tell us about what you were thinking as you completed that last section that we just watched?

Zane: How do you turn the statues around?

Q: Yeah.

Zane: Before I worked out that you could turn them around, I thought you only needed to activate the three and then, the door would open, but then I worked out that you could turn them around.

Q: How did you work that out? What did you do?

Zane: Well I just tapped them, yeah.

The highly visual aspects of the learning environment scaffold the gamer to play and continue to play in the game in order to build new understandings of the game and how to progress through the worlds in which he plays.

15) Probing Principle

Learning is a cycle of probing the world (doing something); reflecting in and on this action and, on this basis, forming a hypothesis; reprobating the world to test this hypothesis; and then accepting or rethinking the hypothesis.

In Lowrie's work with Pokemon, this principle was very evident. Young children created visual maps of the worlds that extended beyond the parameters of the small visual screens. These worlds were also three-dimensional. As such, they were far more sophisticated than is possible in the traditional formats of paper environments. To create these complex visualisations, players create mental images and then control the Pokemon to move into these spaces and to test the validity of these mental maps. This creation is made possible through the games environment but not in traditional school environments.

In the following extract from the interview with Zane, the student has been navigating his way around the world, but making a few errors. Rather than simply using a trial-and-error method, he has referred to the second screen on the DS console to check his progress and to monitor the success of that strategy. At first, it appeared that the strategy was based on trial-and-error but appeared to be somewhat more advanced:

Q: So what were you doing here?

Zane: I was mapping a way to get around and I made a few mistakes.

Q: What sort of strategy did you use to map, like how did you know where to go, apart from the top screen, was there anything else that you use there to help you?

Zane: Not really at that area.

Q: Okay. So, how did you know you'd made a mistake too?

Zane: I looked at the top screen and then looked at the bottom and they didn't match.

12) Practice Principle

Learners get lots and lots of practice in a context where the practice is not boring (i.e. in a virtual world that is compelling to learners on their own terms and where the learners experience ongoing success). They spend lots of time on task.

Some of the heaviest criticisms of mathematics teaching and learning is the reliance on drill-and-practice as a pedagogy. Reform pedagogies have attempted to shift the ideology of drill and practice to one where the learner engages with deep learning so that there is a clear shift from procedural thinking/learning to conceptual thinking/learning. The degree of change in approach to teaching mathematics is somewhat circumspect. In his definition of the role of practice, Gee's proposition is quite different from the rote-and-drill practice regimes of many classrooms on two key points. First, he points out that the practice is not boring and that the practice becomes engaging and compelling for the gamer. This principle links with many of his other principles around scaffolding progress. These include the notion that the learning (or failure) is not public so that others are unaware of the mistakes as the gamer moves through the game. He also contends that for the gamer there is low risk in what is being done so that if a mistake is made, then he/she does not make any substantial losses. This is discussed in the following principle. For example, this is evident in where a mistake may be made but the gamer is not sent back to the start of the game, but to the start of a level. The capacity and willingness to practise "lots and lots" as per the definition is contingent on these key considerations. Finally, the practice also means that the game

spends considerable time on task while he or she is practising. This is both physically with the gaming device but also cognitively as he/she practises, makes errors and refines strategies (even if the strategies are trial-and-error). This type of strategy is evident in the first principle (and transcript) discussed in this paper where Zane indicates he just “tapped them” to find out if the statues were a vital clue in moving forward in the game. As found across our cohort of students, the use of these strategies was commonplace, in part, due to the game encouraging some speed in decision making. In most games the imperative of staying alive to combat and defeat enemies requires the gamer to make decisions as he/she moves through the game based on previous experiences and which may not demand too much cognitive labour. Trial-and-error strategies were commonly observed across all players, in part due to the low risk consequences if errors were made.

In terms of the practice principle, this was highly evident in the stimulated recall scenarios with all players. There was a heavy reliance on practice in order to move forward in the game. The practice enabled players to develop their skills and knowledge of the game environment and thus be able to negotiate new levels. Often this practice aligns with trial-and-error methods as the gamer navigates through new scenarios, drawing on previous knowledge but where those learnings may not be effective but provide cues on how to move forward in the game.

Q: So, we’re just going to pause it there; we just want to ask you can you tell us about what you were thinking as you completed that last section that we just watched?

Zane: How do you turn the statues around?

Q: Yeah.

Zane: Before I worked out that you could turn them around, I thought you only needed to activate the three and then, the door would open, but then I worked out that you could turn them around.

Q: How did you work that out? What did you do?

Zane: Well I just tapped them, yeah.

6) *"Psychosocial Moratorium" Principle*

Learners can take risks in a space where real-world consequences are lowered.

As noted in the previous section there is a need for low risk for the player to engage with the game. This low risk is also associated with failures not being a public display. Much of the acknowledgement of success (and failure) in mathematics is quite public. Inadvertently this public display of failure discourages students from taking risks. In the digital games environment, not only is failure private, it is often low risk. The player who is not able to pass a level will only have to go back to the start of the level rather than suffer a substantial setback. In such an environment, the consequences for failing are low and the possibility for risk-taking great.

In the mathematics classroom, this aspect of Gee’s work is quite scant. Mathematics has historically been one of the most hegemonic of the school discipline areas and as such associated with quite high risk failure. It is not uncommon to hear of risk aversion, math phobia, or other similar terms that connote an aversion to mathematics or reforms that try to encourage risk-taking behaviour in mathematics, but overall the practices are largely in contrast to this principle. The attitude of having a try at something and not having to worry about failure is evident throughout the games environment and is exemplified in the comment by Zane:

Q: I notice you had a bit of a.. you-don’t-really-mind-if-you-die type attitude, does that sound about right, like, you don’t mind if you get hit?

Zane: Yeah.

Q: Yeah why was that?

Zane: Because I knew that I could always do it again.

Conclusion: Interesting Trends

We entered this project with the hope that Gee's work would provide some valuable insights into how the games environment may offer potential for new forms of mathematics learning and understandings. We have confirmed many of Gee's strategies are evident in the games, and the strategies are articulated by the gamers. There were many cases where mathematical concepts and processes were being used and many opportunities for new forms of learning. For this paper, however, we have focused on the principles and how they were evidenced by the gamers and their relationship to mathematics teaching and learning.

What we have observed across the case studies of all the players, was that the principles identified by Gee are in evidence. However, what was surprising from the data –consistently across the responses, as evidenced in the quotations provided – was the common theme where players tended to practice many skills and rely on trial-and-error tactics to work their way through worlds. Most of the strategies used did not encourage higher order thinking but were based on speed. Players were not rewarded (or punished) for making errors so there were few consequences for making errors; there was little to no incentive for trying more complex ways of working through worlds. Unlike literacy education where there appears to be quite deep levels of new literacies being made available through the games environment, this is less apparent in numeracy/mathematics education.

References

- Gass, S. M., & Mackey, A. (2000). *Stimulated recall methodology in second language research*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Gee, J. P. (2003). *What video games have to teach us about literacy and learning*. New York: Plagave MacMillan.
- Henderson, L., & Tallman, J. (2006). *Stimulated recall and mental models: Tools for teaching and learning computer information literacy*, Lanham, Maryland: Scarecrow Press, Inc.
- Jorgensen (Zevenbergen), R. (2009). Young workers and their dispositions towards mathematics: Tensions of a mathematical habitus in the retail industry. *Educational Studies in Mathematics*, 76(1), 87-100.
- Jorgensen, R., & Lowrie, T. (2011). Digital games: Creating new opportunities for mathematics learning *Proceedings of the Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. Electronic). Alice Springs: AAMT.
- Lowrie, T. (2005). Problem solving in technology rich contexts: Mathematics sense making in out-of-school environments. *Journal of Mathematical Behavior*, 24, 275–286.
- Lowrie, T., & Jorgensen, R. (2011). Gender differences in students' mathematics games playing. *Computers and Education*, 57(4), 2244-2248.
- Lyle, J. (2003). Stimulated recall: A report of its use in naturalistic research. *British Educational Research Journal*, 29(6), 861-878.
- Prensky, M. (2001). Digital natives, digital immigrants. *On the Horizon*, 9(5), 1-5.
- Zanting, A., Verloop, N., & Vermunt, J. D. (2001). Student teacher eliciting mentors' practical knowledge and comparing it to their own. *Teaching and Teacher Education*, 17, 725-740.
- Zevenbergen, R., & Zevenbergen, K. (2008). The numeracies of boatbuilding: New numeracies shaped by workplace technologies. *International Journal of Science and Mathematics Education*, 7(1), 183-206.

Influences of Metacognitive and Self-Regulated Learning Strategies for Reading on Mathematical Literacy of Adolescents in Australia and Singapore

Berinderjeet Kaur

National Institute of Education, Singapore

<berinderjeet.kaur@nie.edu.sg>

Shaljan Areepattamannil

National Institute of Education, Singapore

<shaljan.a@nie.edu.sg>

This study, drawing on data from the Programme for International Student Assessment (PISA) 2009, explored the influences of metacognitive and self-regulated learning strategies for reading on mathematical literacy of adolescents in Australia and Singapore. Ordinary least squares (OLS) regression analyses revealed the positive influences of metacognitive learning strategies and control strategies for reading on mathematical literacy of adolescents in Australia and Singapore. In contrast, the two components of self-regulated learning strategies for reading—memorization and elaboration—had negative influences on mathematical literacy of adolescents in Australia and Singapore.

Metacognition, self-regulation, and self-regulated learning are the three influential types of cognitive control processes (Schunk, 2008). Metacognition, originally conceptualized by Flavell (1971, 1976, 1979), refers to “any knowledge or cognitive activity that takes as its cognitive object, or that regulates, any aspect of any cognitive activity” (Flavell, Miller, & Miller, 1993, p. 150). In other words, metacognition is “one’s knowledge concerning one’s own cognitive processes” (Flavell, 1976, p. 232). Metacognition is a multifaceted phenomenon, and comprises of two fundamental elements—knowledge of cognition (metacognitive knowledge) and regulation of cognition (metacognitive control or regulation; Brown, 1987; Flavell, 1979). Metacognitive knowledge is multidimensional in nature, and “consists primarily of knowledge or beliefs about what factors or variables interact in what ways to affect the course and outcomes of cognitive enterprises” (Flavell, 1979, p. 907).

There are three distinct, but interrelated, facets of metacognitive knowledge: declarative knowledge, procedural knowledge, and conditional knowledge (Brown, 1987; Jacobs & Paris, 1987). Declarative metacognitive knowledge refers to “one’s knowledge about oneself as a learner, including knowledge about one’s own abilities (strengths and weaknesses)” (Harris, Graham, Brindle, & Sandmel, 2009, p. 133). Procedural metacognitive knowledge refers to “the knowledge needed to carry out procedures, including strategies, in order to apply declarative knowledge and reach goals” (p. 133). Finally, conditional metacognitive knowledge refers to “knowing when, where, and why to use declarative knowledge as well as particular procedures or strategies (procedural knowledge)” (p. 133).

Self-regulated learning, on the other hand, is defined as “the process by which learners personally activate and sustain cognitions, affects, and behaviours that are systematically oriented toward the attainment of learning goals” (Schunk & Zimmerman, 2007, p. 7). Self-regulated learners are “metacognitively, motivationally, and behaviourally active participants in their own learning process” (Zimmerman, 1986, p. 308). The process of self-regulated learning comprises of three distinct cyclical phases: forethought, performance or volitional control, and self-reflection (Zimmerman, 1998, 2000; Zimmerman & Kitsantas, 2005). Each phase includes several important subprocesses (Zimmerman, 1998). The forethought phase includes goal setting, strategic planning, self-efficacy, learning goal orientation, and intrinsic interest in the task. The performance phase includes focusing attention on the learning task, using self-instruction, imagery, and self-monitoring. The self-

reflection phase includes self-evaluation, attributions, and self-reactions. Whereas proactive learners tend to have high quality forethought and performance phase processes, reactive learners tend to rely on post-performance self-reflections to learn (Zimmerman, 2008). Therefore, proactive learners self-regulate more effectively than reactive learners (Zimmerman, 2008).

Proactive learners—self-regulated learners—are strategic learners (Hadwin, 2008). Van Blerkom (2009) sums up the characteristics of self-regulated learners as follows:

Self-regulated learners are actively involved in their own learning, they set goals, develop plans for achieving those goals, monitor their progress, and evaluate their results. They tend to be more motivated, have higher self-efficacy, and focus more on mastering the material than on grades. Self-regulated learners value the learning process itself and set mastery goals, which lead to the use of self-regulated learning strategies. They use more effective learning strategies and also tend to make adjustments when they are having difficulty mastering material. (pp. 4–5)

Therefore, teachers and policymakers alike agree that students need to be self-regulated learners to learn effectively (Boekaerts, 1997). Although a substantial body of research has examined the influences of metacognitive and self-regulated learning strategies for reading and mathematics on reading and mathematics achievement of adolescents (e.g., Özsoy, 2011; Schneider & Artelt, 2010; van der Stel, Veenman, Deelen, & Haenen, 2010), there is a dearth of research on the influences of metacognitive and self-regulated learning strategies for reading on mathematical literacy of adolescents. However, a growing body of research has demonstrated the crucial role that proficiency in the language of test plays in enhancing mathematics achievement among school children (e.g. Abedi & Herman, 2010; Abedi & Lord, 2001; Beal, Adams, & Cohen, 2010; Le, 2011).

In one of such studies, Beal and her colleagues (2010) investigated the relationship between proficiency in the language of test and mathematics performance of 442 Grade 9 students enrolled in Algebra 1 classes in four high schools in the United States. The authors found that students who had to devote cognitive resources to understand a problem presented in English text performed less well in mathematics than their peers who were able to read English well. Prior research on cognitive processes in mathematics problem solving has also documented similar findings—students who must devote substantial cognitive resources to English comprehension will have less capacity available to devote to math problem-solving operations (e.g., Swanson & Jerman, 2006; Walczyk & Griffith-Ross, 2006). In other words, “if working memory must be devoted to low-level operations, there are fewer cognitive resources available to allocate to higher-order problem-solving activities, such as forming an appropriate problem representation, identifying needed information, and checking progress toward the solution” (Beal et al., 2010, p. 60).

Recently, Le (2011), using the PISA 2003 data sought to detect differential item functioning (DIF) in mathematics items related to reading ability among 275,000 adolescents in 41 countries. He found that lower reading ability students were most disadvantaged on open constructed-response items and advantaged on complex multiple-choice, closed-constructed, and multiple-choice items. Further, lower reading ability students were also disadvantaged on items that required understanding, applying or establishing a formula, and items that required selecting relevant information.

Indeed, learning to perform complex tasks in mathematics may rely heavily on academic language proficiency (Kieffer, Lesaux, Rivera, & Francis, 2009). Because facility with the academic language used to characterize, express, and apply mathematical concepts are indispensable for the development of mathematical skills and mastery of mathematics concepts, students who have not had the opportunity to develop the specialized academic language skills may struggle to understand much of the language that is used in mathematics

classrooms and in the mathematics curricular materials (Kieffer et al., 2009). Hence, such linguistically challenged students may perform poorly on assessments of mathematics (Kieffer et al., 2009). Given the pivotal role of academic language proficiency in mediating the learning of mathematics, it is imperative to explore the influences of metacognitive and self-regulated learning strategies for reading on mathematical literacy.

As Swanson, Jerman, and Zheng (2009) opine, “it is hard not to overemphasize the importance of mathematical ability in a society that requires technical competence among its citizens, a competence that in turn draws on high levels of mathematical literacy” (p. 175). While deficiencies in mathematical competence may seriously limit individuals’ educational opportunities (Hanich, Jordan, Kaplan, & Dick, 2001), individuals proficient in mathematics may earn approximately 38% more than their peers who are not proficient in mathematics (Clarke & Shinn, 2004). Hence, the purpose of the present study was to examine the influences of metacognitive and self-regulated learning strategies for reading on mathematical literacy of adolescents in Australia and Singapore. Specifically, the study addressed the following research question: How well do metacognitive and self-regulated learning strategies for reading predict mathematical literacy among adolescents in Australia and Singapore? Australia was one of the top performing Western countries, which took part in the Programme for International Student Assessment (PISA) 2009, and Singapore was one of the top performing East Asian countries. However, there were statistically significant differences between Australian and Singaporean adolescents in terms of their achievement in reading and mathematics (OECD, 2010). Hence, these two countries were chosen for comparison in the present study.

Method

Data

Data for the study were drawn from the 2009 Programme for International Student Assessment (PISA). PISA is a collaborative initiative of member countries of the Organization for Economic Cooperation and Development (OECD) that is aimed at assessing the knowledge and life skills of 15-year-old students as they approach the end of their compulsory period of schooling. It is a policy-oriented assessment program, designed and guided by an international steering committee to provide regular data that pertain to the most pressing policy issues confronting educational administrators and policy makers around the world. The PISA 2009 data include measures of student proficiency in reading, mathematics, and science; however, reading was the major domain in PISA 2009, assessed with a large and comprehensive set of test items, whereas mathematics and science were minor domains. The Australian sample comprised of 14251 students from 353 schools and the Singaporean sample comprised of 5283 students from 171 schools.

Measures

Mathematical literacy was the outcome measure in the study. This variable was based on 35 test items and was reported on a continuous scale as a set of five plausible values for each student. The plausible values were random elements from the set of scores that could be attributed to each student and their variation helped capture the measurement error at the individual student level (OECD, 2011). The International Database (IDB) Analyser, a plug-in for SPSS, was used to combine the five plausible values.

Metacognitive learning strategies for reading were measured using the PISA 2009 indices: (a) metacognitive strategies-understanding and remembering, and (b) metacognitive

strategies-summarizing. In the PISA 2009 assessment, students were asked to report the usefulness of a number of metacognitive strategies for understanding, remembering, and summarizing the text (11 items). Students' ordering of the metacognitive strategies in reading was compared with the reading experts' agreed order of the metacognitive strategies in reading. A score was assigned to each student based on the reading experts' agreed order of the items (see OECD 2010, 2011). Higher scores on both indices indicate a preference for metacognitive strategies in reading.

Self-regulated learning strategies for reading were measured using the PISA 2009 indices: (a) use of memorization strategies (4 items); (b) use of elaboration strategies (4 items); and (c) use of control strategies (5 items). These items were rated on a 4-point Likert scale ranging from 1 (*almost never*) to 4 (*almost always*). Cronbach's alpha was used to measure the internal consistency of the three indices: use of memorization strategies ($\alpha = .76$, $.74$ in Australia and Singapore, respectively), use of elaboration strategies ($\alpha = .79$, $.77$, respectively), and use of control strategies ($\alpha = .84$, $.76$, respectively).

In addition to these measures, student demographic variables, gender (0 = male, 1 = female) and socio-economic status (SES), were included in the study. The PISA 2009 index of economic, social, and cultural status (ESCS), an index of SES derived from parental occupations, parental education, and home possessions (see OECD, 2011), was used as an SES measure in the current study.

Results

The descriptive statistics and the correlation between outcome variable and predictor variables are presented in Tables 1 and 2. Metacognitive learning strategies for reading—understanding and remembering and summarizing—were positively correlated with the mathematical literacy of adolescents in Australia and Singapore. Similarly, two of the self-regulated learning strategies for reading—use of elaboration and control strategies—were also positively correlated with the mathematical literacy of adolescents in Australia and Singapore. The use of memorization strategies was positively correlated with mathematical literacy in Australia, whereas it was negatively correlated with mathematical literacy in Singapore.

Two separate ordinary least squares (OLS) regression analyses were conducted to address the purpose of the study (see Table 3). Mathematical literacy was the dependent variable. Metacognitive learning strategies for reading—understanding and remembering and summarizing—and self-regulated learning strategies for reading—use of memorization, elaboration, and control strategies—were the independent variables. Student demographic variables, such as gender and SES, were also included in the regression model. The overall regression models predicting mathematical literacy from metacognitive learning strategies for reading, self-regulated learning strategies for reading, and student demographic variables were statistically significant in Australia, Adjusted $R^2 = .32$, $F(7, 13264) = 819.16$, $p < .001$, 95% CI $[.30, .33]$, $f^2 = .47$; and Singapore, Adjusted $R^2 = .31$, $F(7, 5203) = 333.43$, $p < .001$, 95% CI $[.29, .33]$, $f^2 = .45$.

Table 1
Descriptive Statistics

	Australia		Singapore	
	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>
Mathematical literacy	519.49	90.77	563.38	103.42
Gender	.52	.50	.49	.50
Economic, social, and cultural status (ESCS)	.35	.75	-.43	.80
Metacognition—summarizing	-.08	1.03	.17	1.01
Metacognition—remembering and understanding	.03	1.01	.05	.95
Memorization strategies	-.05	1.00	.06	.91
Elaboration strategies	-.13	1.01	.24	.92
Control strategies	.08	1.10	.30	.93

Table 2
Bivariate Correlations

	Australia	Singapore
	<i>Mathematical Literacy</i>	<i>Mathematical Literacy</i>
Gender	-.07	-.03
Economic, social, and cultural status (ESCS)	.37	.38
Metacognition—summarizing	.38	.40
Metacognition—remembering and understanding	.33	.29
Memorization strategies	.06	-.15
Elaboration strategies	.13	.04
Control strategies	.32	.23

Note. All coefficients are statistically significant, $p < 0.05$.

Table 3
Ordinary Least Squares (OLS) Regression Analyses Predicting Mathematical Literacy

	Australia		Singapore	
	<i>B</i>	<i>SE</i>	<i>B</i>	<i>SE</i>
Constant	521.80	2.33	578.94	2.19
Gender	-28.57	1.95	-17.09	2.58
Economic, social, and cultural status (ESCS)	30.58	1.29	37.53	2.08
Metacognition—summarizing	21.37	1.17	27.38	1.53
Metacognition—remembering and understanding	10.62	1.07	11.07	1.41
Memorization strategies	-13.59	1.29	-18.33	1.42
Elaboration strategies	-3.40	1.09	-6.74	1.78
Control strategies	24.04	1.25	21.06	2.15

Note. All coefficients are statistically significant, $p < 0.05$.

Metacognitive learning strategies for reading, understanding and remembering ($B = 21.37$, $p < .05$; $B = 27.38$, $p < .05$, Australia and Singapore, respectively) and summarizing ($B = 10.62$, $p < .05$; $B = 11.07$, $p < .05$, respectively) were statistically significant positive predictors of mathematical literacy in Australia and Singapore. Adolescents in Australia and

Singapore who had higher scores for metacognition in reading scored statistically significantly higher in mathematics than their peers in who had lower scores for metacognition in reading. Similarly, one of the self-regulated learning strategies for reading, use of control strategies, was a statistically significant positive predictor of mathematical literacy in both Australia ($B = 24.04, p < .05$) and Singapore ($B = 21.06, p < .05$). Adolescents in Australia and Singapore who frequently used control strategies in reading scored statistically significantly higher in mathematics than their peers who infrequently used control strategies in reading. In contrast, the other two self-regulated learning strategies for reading, use of memorization strategies ($B = -13.59, p < .05$; $B = -18.33, p < .05$, Australia and Singapore, respectively) and use of elaboration strategies ($B = -3.40, p < .05$; $B = -6.74, p < .05$, Australia and Singapore, respectively), were statistically significant negative predictors of mathematical literacy in both Australia and Singapore. Adolescents in Australia and Singapore who frequently used memorization and elaboration strategies in reading scored statistically significantly lower in mathematics than their peers who infrequently used memorization and elaboration strategies in reading.

Finally, student demographic variables, gender and SES, were also predictive of mathematical literacy in Australia and Singapore. The Australian and Singaporean female students scored statistically significantly lower than their male counterparts in mathematics. The ESCS, a measure of SES, was a positive predictor of mathematical literacy in Australia and Singapore. Put another way, Australia and Singapore are leaving some adolescents behind in mathematics, and they tend to be the less privileged.

Discussion

The purpose of the study was to examine the influences of metacognitive and self-regulated learning strategies for reading on mathematical literacy of adolescents in Australia and Singapore. The results of the study suggested the positive influences of metacognitive learning strategies for reading—understanding and remembering and summarizing—on mathematical literacy of both Australian and Singaporean adolescents. These findings suggest the key role that metacognitive skills in reading may play in enhancing the mathematics achievement of adolescents in Australia and Singapore.

Because metacognitive instruction may help to enhance metacognition and learning in a broad range of students (see Veenman, Elshout, & Busato, 1994), classroom teachers in Australia and Singapore may need to tailor their classroom teaching accordingly. A successful metacognitive instructional programme, however, may need to abide with three fundamental principles (Veenman, Hout-Wolters, & Afflerbach, 2006): embedding metacognitive instruction in the content matter to ensure connectivity; informing learners about the usefulness of metacognitive activities to make them exert the initial extra effort; and prolonged training to guarantee the smooth and maintained application of metacognitive activity.

The results of the study also revealed that two of the self-regulated learning strategies for reading—use of memorization and elaboration strategies—were negatively associated with mathematical literacy of adolescents in Australia and Singapore. Students who use elaboration strategies tend to think deeply about concepts, and tend to integrate what they learn with other material. Furthermore, they are able to transfer concepts to different situations. However, use of elaboration strategies in reading did not help adolescents in Australia and Singapore to enhance their achievement in mathematics. This finding suggests that elaboration strategies in reading may not be similar to the elaboration strategies used in mathematics because the use of elaboration strategies in mathematics has been found to be positively associated with mathematics achievement. Hence, there is an inverse relationship

between elaboration strategies in reading and mathematical literacy among adolescents in Australia and Singapore. Similarly, the use of memorization strategies in reading as well did not help adolescents in Australia and Singapore to enhance their achievement in mathematics. Students who use memorization learning strategies tend to resort to memorization and rote learning (Aharony, 2006). Although memorization strategy is useful for learning basic and new material, it might not be conducive to a deep understanding of concepts.

In contrast, the use of control strategies was positively associated with mathematical literacy among adolescents in Australia and Singapore. Prior studies have demonstrated that the use of control strategies is associated with positive educational outcomes (e.g., Marsh, Hau, Artelt, Baumert, & Pescha, 2006) and mathematics self-concept (Marsh et al., 2006). However, such studies explored the influence of control strategies in mathematics on mathematics achievement. The findings of the present study pinpoint the critical role that control strategies in reading may play in promoting mathematical literacy among adolescents in Australia and Singapore. “Students who can control their own learning in an effective manner are assumed to set realistic goals, to select learning strategies and techniques appropriate to the demands of particular tasks, to shield themselves from competing intentions, and to maintain motivation when learning” (Marsh et al., 2006, p. 345).

In conclusion, the findings of the study provide empirical support that metacognitive strategies in reading may help enhance adolescents’ achievement in mathematics. Similarly, the findings of the study suggest the critical role that control strategies in reading may play in improving the mathematical literacy of adolescents. However, the negative influence of the use of elaboration strategies in reading on mathematical literacy of adolescents in Australia and Singapore warrant further investigation.

References

- Abedi, J., & Herman, J. L. (2010). Assessing English language learners’ opportunity to learn mathematics: Issues and limitations. *Teachers College Record*, 112, 723–746.
- Abedi, J., & Lord, C. (2001). The language factor in mathematics tests. *Applied Measurement in Education*, 14, 219–234.
- Aharony, N. (2006). The use of deep and surface learning strategies among students learning English as a foreign language in an Internet environment. *British Journal of Educational Psychology*, 75, 851–866.
- Beal, C. R., Adams, N. M., & Cohen, P. R. (2010). Reading proficiency and mathematics problem solving by high school English language learners. *Urban Education*, 44, 58–74.
- Boekaerts, M. (1997). Self-regulated learning: A new concept embraced by researchers, policy makers, educators, teachers, and students. *Learning and Instruction*, 7, 161–186.
- Brown, A. L. (1987). Metacognition, executive control, self-regulation, and other more mysterious mechanisms. In F. E. Weinert & R. H. Kluwe (Eds.), *Metacognition, motivation, and understanding* (pp. 65–116). Hillsdale, New Jersey: Erlbaum.
- Clarke, B., & Shinn, M. R. (2004). A preliminary investigation into the identification and development of early mathematics curriculum-based measurement. *School Psychology Review*, 33, 234–248.
- Flavell, J. H. (1976). Metacognitive aspects of problem solving. In L. B. Resnick (Ed.), *The nature of intelligence* (pp. 231–235). Hillsdale, NJ: Erlbaum.
- Flavell, J. H. (1979). Metacognition and cognitive monitoring: A new area of cognitive-developmental inquiry. *American Psychologist*, 34, 906–911.
- Flavell, J. H., Miller, P. H., & Miller, S. A. (1993). *Cognitive development*. Englewood Cliffs, NJ: Prentice-Hall.
- Hadwin, A. F. (2008). Self-regulated learning. In T. L. Good (Ed.), *21st century education: A reference handbook*. Thousand Oaks, CA: Sage.
- Hanich, L. B., Jordan, N. C., Kaplan, D., & Dick, J. (2001). Performance across different areas of mathematical cognition in children with learning difficulties. *Journal of Educational Psychology*, 93, 615–626.

- Harris, K. R., Graham, S., Brindle, M., & Sandmel, K. (2009). Metacognition and children's writing. In R. J. Sternberg, D. J. Hacker, J. Dunlosky, & A. C. Graesser (Eds.), *Handbook of metacognition in education*. New York: Erlbaum.
- Jacobs, J. E., & Paris, S. G. (1987). Children's metacognition about reading: Issues in definition, measurement, and instruction. *Educational Psychologist*, 22, 255–278.
- Kieffer, M. J., Lesaux, N. K., Rivera, M., & Francis, D. J. (2009). Effectiveness of accommodations for English Language Learners taking large-scale assessments. *Review of Education Research*, 79, 1168–1201.
- Le, L. T. (September, 2011). *Differential performance of poor readers in international mathematics tests by item formats*. Paper presented at the European conference on educational research, Berlin, Germany.
- Marsh, H. W., Hau, K., Artelt, C., Baumert, J., & Peschar, J. L. (2006). OECD's brief self-report measure of educational psychology's most useful affective constructs: Cross-cultural, psychometric comparisons across 25 countries. *International Journal of Testing*, 6, 311–360.
- Organization for Economic Cooperation and Development. (2010). PISA 2009 results: What students know and can do—student performance in reading, mathematics and science. Retrieved from <http://www.oecd.org/dataoecd/10/61/48852548.pdf>
- Organization for Economic Cooperation and Development. (2011). PISA 2009 technical report. Retrieved from http://www.oecd.org/document/19/0,3746,en_2649_35845621_48577747_1_1_1_1,00.html
- Özsoy, G. (2011). An investigation of the relationship between metacognition and mathematics achievement. *Asia Pacific Education Review*, 12, 227–235.
- Schneider, W., & Artelt, C. (2010). Metacognition and mathematics education. *ZDM*, 42, 149–161.
- Schunk, D. H. (2008). Metacognition, self-regulation, and self-regulated learning: Research recommendations. *Educational Psychology Review*, 20, 463–467.
- Schunk, D. H., & Zimmerman, B. J. (2007). Influencing children's self-efficacy and self-regulation of reading and writing through modeling. *Reading & Writing Quarterly*, 23, 7–25.
- Swanson, H. L., & Jerman, O. (2006). Math disabilities: A selective meta-analysis of the literature. *Review of Educational Research*, 76, 249–274.
- Swanson, H. L., Jerman, O., & Zheng, X. (2009). Math disabilities and reading disabilities: Can they be separated? *Journal of Psychoeducational Assessment*, 27, 175–196.
- Van Blerkom, D. L. (2011). *College study skills: Becoming strategic learner* (7th ed.). Boston, A: Wadsworth.
- van der Stel, M., Veenman, M., Deelen, K., & Haenen, J. (2010). The increasing role of metacognitive skills in math: a cross-sectional study from a developmental perspective. *ZDM*, 42, 219–229.
- Veenman, M. V. J., Elshout, J. J., & Busato, V. V. (1994). Metacognitive mediation in learning with computer-based simulations. *Computers in Human Behaviour*, 10, 93–106.
- Veenman, M. V. J., Van Hout-Wolters, B. H. A. M., & Afflerbach, P. (2006). Metacognition and learning: conceptual and methodological considerations. *Metacognition and Learning*, 1, 3–14.
- Walczyk, J. J., & Griffith-Ross, D. A. (2006). Time restriction and the linkage between subcomponent efficiency and algebraic inequality success. *Journal of Educational Psychology*, 98, 617–627.
- Zimmerman, B. J. (1986). Development of self-regulated learning: Which are the key subprocesses? *Contemporary Educational Psychology*, 16, 307–313.
- Zimmerman, B. J. (1998). Academic studying and the development of personal skill: A self-regulatory perspective. *Educational Psychologist*, 33, 73–86.
- Zimmerman, B. J. (2000). Self-efficacy: An essential motive to learn. *Contemporary Educational Psychology*, 25, 82–91.
- Zimmerman, B. J. (2008). Investigating self-regulation and motivation: Historical background, methodological developments, and future prospects. *American Educational Research Journal*, 45, 166–183.
- Zimmerman, B. J., & Kitsantas, A. (2005). Homework practices and academic achievement: The mediating role of self-efficacy and perceived responsibility beliefs. *Contemporary Educational Psychology*, 30, 397–417.

Identifying Stages in a Learning Hierarchy for Use in Formative Assessment – the Example of Line Graphs.

Kaye Stacey
University of Melbourne
<k.stacey@unimelb.edu.au>

Beth Price
University of Melbourne
<b.price@unimelb.edu.au>

Vicki Steinle
University of Melbourne
<v.steinle@unimelb.edu.au>

This paper discusses issues arising in the design of questions to use in an on-line computer-based formative assessment system, focussing on how best to identify the stages of a learning hierarchy for reporting to teachers. Data from several hundred students is used to illustrate how design decisions have been made for a test on interpreting line graphs.

Designing Formative Assessment

‘Smart tests’ (Specific Mathematics Assessments that Reveal Thinking) are now being used by more than 500 teachers through the website www.smartvic.com. As of January 2012, there are tests on 55 topics, nearly all with paired pre-tests and post-tests. The tests are short, completed on-line, and results are immediately available to teachers. The intention is that teachers will use these tests just before teaching a topic to better understand the needs of their own students. A simple example of such use was when a teacher was preparing to teach a geometry unit to his class. In advance, his students took the understanding angle smart test which reported to him that four students had the well-known misconception that angle size is related to the length of the arms. He gave these four students a short tutorial before teaching the new topic, and was pleased that they subsequently managed the geometry unit very well. Further information on smart tests is given by Stacey, Price, Steinle, Chick and Gvozdenko (2009).

The process of creating the smart test system has raised a plethora of design issues, which have been explored through trialling with students and teachers. Issues include the nature of items suitable for on-line use; the content, scope and length of tests; the most useful content to test; the provision of information to teachers about the tests and subsequent teaching; and the method of delivery of tests to students and of student results to teachers. Many teachers have initially requested that the information they receive about their students from the smart tests system should directly link to the local curriculum standards, so that they can use this for summative (end of term) reporting to the school and parents. This implies that if teachers are to see the benefits of using formative assessment with a focus on improving day-to-day teaching, the whole system and its reporting components in particular, must be well-designed to meet their immediate teaching needs.

The aim of the paper is to describe, and illustrate with an example, the processes involved in the creation and validation of stages within a learning hierarchy which are used by the smart test system to provide teachers with information about their students.

Descriptions of Learning that are Useful for Teachers

There are many possibilities for reporting students’ results to teachers. The percentage of questions correct gives teachers a ranking of students and a general idea of how well a topic has been mastered, but gives little information on what aspects need further teaching. At the other extreme, detailed reporting of every response for each individual student has the potential to provide excellent information for subsequent teaching, but there is a mass of data which takes time to interpret and does not help teachers to identify the important underlying ideas. In their own teacher-marked classroom tests, teachers have access to such

full information, but it has always been hard to use this other than to identify problematic items which require repeat instruction.

A different approach, which we have selected for the smart tests, is to describe learning in terms of stages along a learning hierarchy. This approach can be used with qualitative and quantitative data. The PISA study (OECD, 2010) provides an example of stages of learning derived from quantitative analysis. Using item response theory (for PISA, Rasch modelling), a scale of difficulty of the items and proficiency of students is constructed. Levels of achievement are then described by encapsulating the characteristics of the items in different regions of the scale. In PISA, proficiency is described in 6 levels and some of the most revealing country comparisons are made in terms of these levels. The level descriptions for a broad ranging survey such as PISA provide broad goals for teachers to work towards, rather than specific information on any topic. For example, OECD (2010, p.130), “At Level 6 students can conceptualise, generalise and utilise information based on their investigations and modelling of complex problem situations.”

Item response theory is a major example of approaches intended to measure learning. However, many researchers interested in mathematical learning in focussed areas of the curriculum, aim to ‘map’ learning rather than ‘measure’ learning (Stacey & Steinle, 2006). This approach aims to show how ideas in a topic interact, building on each other and also interfering with each other, and to reveal the conceptions and misconceptions that underlie performance. Data to support learning hierarchies have been assembled from a wide range of sources: from written questions designed to reveal student thinking in both large-scale and small-scale studies, from task-based interviews as well as from teaching experiments. The resulting hierarchies have proved their usefulness for teaching by underpinning some significant professional development programs for teachers such as Cognitively Guided Instruction (Franke, Carpenter, Levi, & Fennema, 2001). The strategy is that teachers should know the stages of learning; know where their students are situated; and then select learning activities that will move a student forwards.

The smart tests system has adopted this general approach, describing learning in terms of a hierarchy of topic-specific stages (rather than global stages such as in PISA), reporting student performance according to the stages and then giving suggestions for how teachers can move students from one stage to the next. The descriptions of stages need to be easy for teachers to understand and easily related to teaching actions without the need for an associated professional development program. For each smart test, the report to teachers about the learning hierarchy includes additional information about specific common errors or misconceptions which are automatically diagnosed by programming based on matching patterns in student responses.

Example –Interpreting Line Graphs

The purpose of this paper is to describe the creation of a learning hierarchy for the *interpreting line graphs* smart test. We outline the process that we have used and discuss some of the issues which have arisen. In some smart tests, the initially proposed learning hierarchy is supported by all information and data. *Interpreting line graphs* is a test for which it has been more difficult to design stages which are supported by the data. In fact, none of the hierarchies explored in this paper are of sufficient quality to be provided to teachers. So, up to the time of preparation of this paper, the reporting of students’ results in this test is based only on percentage correct. The creation of this test has also underlined the often observed fact, that apparently simple topics in mathematics have interesting subtleties and hidden depths. As will be shown in the following sections, the iterative process to

identify a satisfactory learning hierarchy involves sharpening the items, reviewing the nature of the stages, and in this case, rethinking what mathematical content is fundamental.

Interpreting line graphs is one of a set of tests on the graphical presentation of data. Currently this set also includes interpreting pictographs, interpreting bar graphs, graphs: choosing the best graph type, map reference and coordinates, and plotting coordinates. There are other related tests involving, for example, scale reading. Interpretation at the higher levels requires a sophisticated degree of statistical literacy, for example, distinguishing the significance of data points and other points on a line graph.

Test creation began with the identification of the content to be tested with an indication of the likely stages of learning. Curcio's 1987 study of graph comprehension in Year 4 and Year 8 students highlighted three stages: first "reading the data" (the capacity to read literally the direct factual information on the graph); second "reading between [or within] the data" (attend to two or more data points on the graph, often for comparison purposes); and third "reading beyond the data" (extend, predict, and infer from the data). More recent work of Shaughnessy, Garfield and Greer (1996) and Shaughnessy (2007) suggests a fourth category termed "reading behind the data" which pays particular attention to the context from which the data arise. In Shaughnessy (2007) the four categories were given more detail, and expanded into eight, with the higher ones associated with deeper interpretation and appreciation of context and variation.

The *interpreting line graphs* test contains 15 items set in four contexts. Some items are multiple choice, some are short constructed response, and all are presented and marked by computer. Figure 1 contains 3 sample items. These items ask students to decide whether it makes sense to join the points in the graph, while in other items (with similar graphs) students are asked to read information from points and interpret it and also to indicate whether points intermediate between two data points have a meaning.

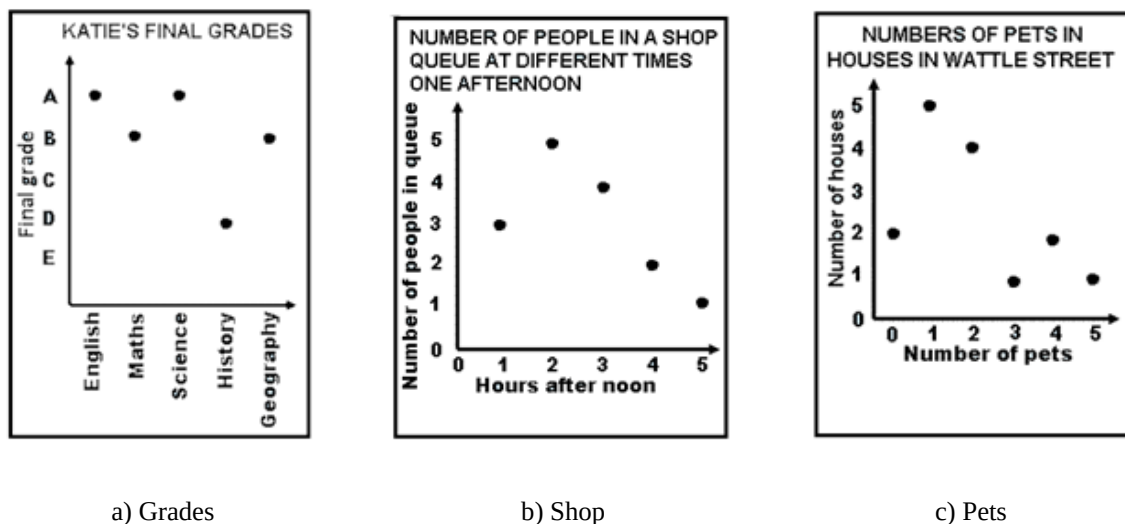


Figure 1. Items about joining points from the *interpreting line graphs* test. In each case, students choose Yes/No from a drop-down box after the statement "It makes sense to join these points"

Data

Data presented in this paper is from on-line users of the smart tests in 2011, so they are the students of volunteer teachers from many schools. The 216 students are from Years 5 to 10, with 59% from Year 8. Data collected before 2011 was used to identify features of the task that are significant for students, improve the items, and trial the early version of the reporting system. Further refinement of the test and reporting is now taking place.

Designing and Confirming the Learning Hierarchy

Version 1 of the Learning Hierarchy

A learning hierarchy can be created empirically from data and/or by postulating a complexity order based on logical analysis, and/or teaching experience and/or using prior research (e.g. in this case, Curcio, 1987; Shaughnessy, Garfield and Greer, 1996 and Shaughnessy, 2007). Stages in a learning hierarchy are confirmed by data if several conditions are met. Ideally, items with similar mathematical characteristics will have similar success rates, and will be completed successfully by the same students. If a learning hierarchy exists, knowledge at one stage is pre-requisite for achieving tasks at a higher stage. This means that students unable to complete items designed to test lower stages will be unlikely to successfully complete items designed to test higher stages. Moreover they should have a low ‘relative risk’ of completing the item successfully, where relative risk is the ratio of probability of success of students at a given stage to the probability of success for all students.

The first attempt to identify a learning hierarchy is shown (with brief examples) in Table 1. It is based on the four categories of Shaughnessy and colleagues, applied specifically to line graphs and expressed in concrete terms. Several items are created for each stage (referred to as, for example, *Stage 1 items*); a hurdle score on a group of items is set (for example, 4 out of 5 correct is the requirement for success on the Stage 1 items); and then students are classified according to their performance on the groups of items (for example, being successful on Stage 1 items followed by not being successful on Stage 2 items leads to a student being classified as Stage 1).

Table 1

Version 1 of the Learning Hierarchy for Interpreting Line Graphs

Stage	Description: Students can	Brief example	Number of items
1	Read information from points on labelled grid lines	Read Katie’s grade in English (Fig. 1a)	5
2	Read information from points between labelled grid lines	Read the weight of a mouse from axis marked in 5 gram intervals	3
3	Interpret the information in a line graph	Identify when a mouse had babies from a graph of daily weight.	2
4	Decide whether it is appropriate to join data points	See 3 items in Figure 1	5

The data from the full set of 15 items was analysed using the stages in Table 1. The first column of Table 2 shows that, of the 216 students, 49 did not reach Stage 1 (so are classified as Stage 0), 28 were classified as Stage 1, 99 as Stage 2, 39 as Stage 3 and only 1 was classified as Stage 4. The entries in the main cells of Table 2 are the average probabilities of success (and in brackets, relative risk) for the items contributing to each stage, for the groups of students who have been classified into each stage.

Table 2

Average Probability of Success (relative risk) on Items for Students Allocated to Each Stage in Version 1 of the Learning Hierarchy

Classification of students	Items in each stage			
	Stage 1 (5 items)	Stage 2 (3 items)	Stage 3 (2 items)	Stage 4 (5 items)
Stage 0 (n = 49)	0.32 (0.40)	0.27 (0.42)	0.18 (0.48)	0.35 (0.88)
Stage 1 (n = 28)	0.91 (1.14)	0.27 (0.43)	0.36 (0.92)	0.45 (1.12)
Stage 2 (n = 99)	0.95 (1.18)	0.84 (1.33)	0.25 (0.64)	0.40 (1.01)
Stage 3 (n = 39)	0.97 (1.24)	0.82 (1.30)	1.00 (2.59)	0.41 (1.02)
Stage 4 (n = 1)	1.00 (1.24)	0.67 (1.06)	1.00 (2.59)	0.80 (2.00)
All students (n = 216)	0.80	0.63	0.39	0.40

Table 2 shows that the data supports Stages 1, 2, and 3 moderately well. For example, inspection of the relative risks in the columns related to the items in Stages 1, 2 and 3 reveals that the cells above the main diagonal are less than 1 and the cells under the main diagonal are greater than 1. Also, in the columns related to Stages 1, 2 and 3 in the upper triangle, the average probabilities of success are low (no more than 0.36). However, we would expect increasing trends down the columns and this is not the case; for example, compared to the Stage 1 students, Stage 2 students have *less success* on the Stage 3 and 4 items. Stage 4 (knowledge of when it is appropriate to join points and the meaning of interpolated points) is especially problematic. The final column in Table 2 shows that students at Stages 1, 2 and 3 all have similar average probabilities of success on these 5 items. In summary, this data does not support these four stages as a learning hierarchy.

Version 2 of the Learning Hierarchy

In an attempt to resolve the issues with the (linear) learning hierarchy in version 1, a branching hierarchy was proposed. The original Stages 1, 2 and 3 in version 1 (reading and interpreting points) remain, but knowledge of when it is appropriate to join points in a graph (the original Stage 4) would not follow on from Stage 3, but rather branch from Stage 1. Table 3 indicates that 5 items were used to determine Stage 1 (the original Stage 1 items) and another 5 items (the original Stage 4 items in version 1) were now used to classify students into Stages 2, 3 and 4 in this new branch of the learning hierarchy. This decision was supported by the use of Statistical Implicative Analysis performed by the CHIC software (Gras 2010) which confirmed that these 5 items about joining points differed from the other items in more than just degree of difficulty.

Table 3 provides the success rates for each of the 10 items for the full cohort of 216 students; the decreasing trend providing initial support for this new branch in the learning hierarchy. The 3 items defining Stage 2 and 3 are in Figure 1. The differences between the success rates of the shop item (71%) where it is appropriate to join the points and the other two items where it is not appropriate (45% and 59%) contributed to the decision to create separate stages.

The two Stage 4 items in version 2 addressed this issue more deeply, by asking for the meanings of interpolated points. Sometimes such points are meaningless but for graphs with continuous data on both axes it is usually possible to interpolate some information. For example, one Stage 4 item involved an altered version of the graph in Figure 1b where the

points were joined and an interpolated point (at 3:30pm) was marked. In this *joined-points shop item*, students were asked to select a suitable meaning for this marked point from this list:

- A) that 35 people were in the shop at 3 pm
- B) that 30 people were in the shop at 3:30 pm
- C) that you could expect about 30 people in the shop at 3:30 pm (Correct)
- D) nothing. This point does not have a meaning.

The other Stage 4 item (the *joined-points pet item*) was derived in a similar way from Figure 1c, with an interpolated point (2.5, 2.5) marked. Again, students were asked to select a suitable meaning for this marked point from a list:

- A) that 2.5 houses had an average of 2.5 pets each
- B) that house 2A had 2.5 pets
- C) nothing. This point does not have a meaning. (Correct)

Table 3

Version 2 of the Learning Hierarchy for Joining Points within Interpreting Line Graphs

Stage	Description: Students can ...	Number of items used	Success rates for each item (n=216)
1	read and interpret data points	5	83%, 75%, 83%, 84%, 76%
2	decide to join points	1	71%
3	reject inappropriate joining of points	2	45%, 59%,
4	interpret interpolated points	2	13%, 12%

Table 4 provides data for the version 2 learning hierarchy in the same way that Table 2 provided data for version 1.

Table 4

Average Probability of Success (relative risk) on Items for Students Allocated to Each Stage in Version 2 of the Learning Hierarchy for Joining Points

Classification of students	Items in each stage			
	Stage 1 (5 items)	Stage 2 (1 item)	Stage 3 (2 items)	Stage 4 (2 items)
Stage 0 (n = 49)	0.32 (0.40)	0.57 (0.81)	0.42 (0.80)	0.17 (1.39)
Stage 1 (n = 42)	0.95 (1.18)	0.00#(0.00)	0.36 (0.68)	0.13 (1.05)
Stage 2 (n = 77)	0.94 (1.17)	1.00 (1.41)	0.38 (0.73)	0.10 (0.83)
Stage 3 (n = 47)	0.96 (1.19)	1.00 (1.41)	1.00 (1.91)	0.09 (0.68)
Stage 4 (n = 1)	1.00 (1.24)	1.00 (1.41)	1.00 (1.91)	1.00 (8.00)
All students (n = 216)	0.80 (1.00)	0.71 (1.00)	0.52 (1.00)	0.13 (1.00)

This is an artefact of having only one Stage 2 item.

While the relative risks in the lower triangle in Table 4 are larger than 1 (as is desirable), not all of those in the upper triangle are less than 1. The relative risk values of 1.39 and 1.05 indicate that Stage 0 and Stage 1 students are more likely to answer the Stage 4 items correctly than average, which does not fit a learning hierarchy. Students are classified at Stage 0 if they did not show adequate skills reading single points, so it is rather surprising that they were more successful than other students on the Stage 4 items. It seems that the

students with higher scores were especially reluctant to say that a point on an interpolated line has no meaning. For example, in the joined-points pet item they were much more likely to choose response A (2.5 houses have an average of 2.5 pets) than response C (This point does not have a meaning). They were keen to show the ability to read the coordinates points, rather than the ability to ‘read behind the data’.

Rethinking the Mathematical Content and the Items

The current test is not producing results that are easy to analyse and to report meaningfully to teachers. We have therefore undertaken a major revision, with several steps. First, some items have been subjected to minor editing, usually to improve clarity. Wording and graphs can often be clarified and extraneous cognitive load can be removed. For example, the unnecessary word ‘queue’ has been removed in the new version of the shop item (shown in Figure 1b). Even though minor edits such as these make little difference to overall success rates, they may help to sharpen the data from individuals, reducing ‘careless errors’ which produce noise in the data. Examining the wrong answers that students have given, also gives clues for minor improvement. For example, in the graph showing daily measurements of the mass of a mouse, many students were out by exactly one day and hence we have clarified the time of day when the mouse was weighed.

The second aspect to the revision is to rethink what should be tested by this test. Mathematics is a highly connected subject, so it is hard to draw strong boundaries around any topic. On reconsidering this test, it seems that one possibility for the difficulty in producing a learning hierarchy is that the test has had too many mathematical facets. The test had also incorporated some scale reading features, but this is not necessary as there is another smart test that looks specifically at reading scales. To avoid these scale features, all points on the new graphs have been placed at the intersection of labelled grid lines.

Reviewing the Fundamental Ideas

A third aspect is to rethink what are the fundamental mathematical ideas for this test. In the first version, we focussed on recognising when to join or not join data points and we asked students to give the meaning (or non-meaning) of data points. However, these items gave unreliable data. We now think that there are two competing considerations for students seeking to make this decision, and we had not taken this well enough into account. The first is ‘Does joining the points make it easier to read the information on the graph by guiding the eye?’ and the second (more mathematically sophisticated, and the only intention of the initial items) is ‘Does it make sense to use this line for interpolation?’ Having considered these issues, it now seems to the authors that it is almost always easier to read a set of data points when they are joined. In effect, we now agree with most students that all of the scattergraphs in Figure 1 are easier to read when the points are joined. So the fundamental issue of joining points is not whether points should be joined, but what additional information (if any) can be gained by looking at points on the interpolating line. This requires ‘reading behind the data’ taking the real context into account.

In the redesigned test, questions are again asked about data points and interpolated points, but focussing on the status of information. For example, in the new joined-points shop item, students will be asked to select whether the statement ‘At 3.30 pm there were about 30 people in the shop’ is true / likely / unlikely / not true / the graph gives no information on this. Overall the new version of the test should produce data that can more readily be reported to teachers on one learning hierarchy, which is better focussed on the real issues of reading and interpreting line graphs.

Conclusion

The creation of a learning hierarchy is an iterative process, cycling between consideration of mathematical goals and potential items, the collection of data, and the analysis both of overall success rates and the examination of student errors in order to gain insight into both student thinking and ways of improving the items.

Reporting results in a learning hierarchy is attractive if the hierarchy description helps teachers focus on the achievements of their students and also what they need to master next. However, as the example above shows, it is not always easy to design tests which fit the empirical and theoretical criteria well. We have not yet trialled our new test, but hope that having built it on the basis of data collected to date, it will have good statistical properties.

On-line testing and computer diagnosis form only a part of assessment of mathematics; although there are new opportunities for item types the possibilities for constructed responses are still limited. On-line testing does, however, through systems such as the smart tests, offer new opportunities to mobilise the results of mathematics education research, making it available to teachers at the point of teaching.

Acknowledgements

We thank Eugene Gvozdenko, Reinhard Oldenburg, Helen Chick, teachers and students, the Australian Research Council and DEECD of Victoria.

References

- Curcio, F. (1987). Comprehension of mathematical relationships expressed in graphs. *Journal for Research in Mathematics Education*, 18, 382-393.
- Franke, M., Carpenter, T., Levi, L., & Fennema, E. (2001). Capturing Teachers' Generative Change: A Follow-up Study of Professional Development in Mathematics. *American Educational Research Journal*, 38(3), 653 – 689.
- Gras, R. (2010): CHIC 5.0 (Software). <http://www.amarkos.gr/en/research/chic/>
- OECD (2010). *PISA 2009 Results: What Students Know and Can Do – Student Performance in Reading, Mathematics and Science (Vol I)*. Paris: Organisation for Economic Cooperation and Development. <http://dx.doi.org/10.1787/9789264091450-en>
- Shaughnessy, J. M., Garfield, J., & Greer, B. (1996). Data handling. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics in education* (pp. 205-237). Dordrecht, The Netherlands: Kluwer.
- Shaughnessy, J. M. (2007) Research on statistical learning and reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp.957-1009). Charlotte, NC: Information Age Publishing.
- Stacey, K. & Steinle, V. (2006). A case of the inapplicability of the Rasch Model: Mapping conceptual learning. *Mathematics Education Research Journal*, 18(2), 77 – 92.
- Stacey, K., Price, B., Steinle, V., Chick, H., Gvozdenko, E. (2009). SMART Assessment for Learning. *Proceedings of International Society for Design and Development in Education Conference*. http://www.isdde.org/isdde/cairns/pdf/papers/isdde09_stacey.pdf

Get Into Vocational Education (GIVE): Motivating Underperforming Students

Gillian Kidman
Queensland University of Technology
<g.kidman@qut.edu.au>

Tom Cooper
Queensland University of Technology
<tj.cooper@qut.edu.au>

David Nutchey
Queensland University of Technology
<d.nutchey@qut.edu.au>

This study explores the effects of a vocational education-based program on academic motivation and engagement of primary school aged children. The *Get Into Vocational Education* (GIVE) program integrated 'construction' and the mathematics, English and science lessons of a Year 4 primary classroom. This paper focuses on investigating the components of the GIVE program that led to student changes in mathematical academic motivation and engagement resulting in outstanding gains in NAPLAN Numeracy results. The components proposed to have contributed to effectiveness of the GIVE program are: teacher and trainer expectations, task mastery and classroom relationships. These findings may be useful to researchers and educators who are interested in enhancing students' mathematical academic motivation.

The rapid development of mining and coal gas liquefaction throughout Queensland will require many thousands of additional trades-people. To address this shortage of skilled workers, Queensland needs to either increase the number of local people with appropriate qualifications or import skilled workers. Educational programs, such as *Get into Vocational Education Program (GIVE)*, are needed to promote interest and provide initial training. Importantly, these programs need to encourage participants from local groups that in the past have not been a major source of such skilled workers. One such group is the underperforming students in local schools, many of whom have a desire to enter employment as apprentices but lack the prerequisite literacy and mathematics and/or have already disengaged from school. This is a problem that is not solely the result of the students' Years 8-10 schooling: Often these students leave primary school without the prerequisite academic and social abilities that will underpin success in secondary school and future employment. This paper reports upon the delivery of the GIVE program in a Year 4 classroom, in which timber construction was selected as a trades-related context through which to target the development of motivation and academic achievement. The shifts in student motivation over the course of the GIVE program are examined and linked to mathematics academic gains, as evidenced in NAPLAN numeracy results.

Academic Motivation

Martin (2001) developed a 'Student Motivation and Engagement Scale' which is based upon literature regarding: student motivation in relation to the pedagogy used by the teacher (Teven & McCroskey, 1997); the teacher-student relationship in the classroom (Kelly & Hansen, 1987); and adult expectations (Dandy & Nettlebeck, 2000). Martin also considered the links between students' motivation and academic engagement: For example, Valentine, Cooper, Battencourt, and DuBios (2002) reported that extra-curricular programs can enhance academic engagement and achievement if the program is related to the students' actual school life. Martin's (2001) scale groups motivation facets into those that reflect enhanced motivation and those that reflect reduced motivation, which Martin refers to as

'boosters' and 'guzzlers' respectively. Martin (2001) identified four guzzlers (anxiety, uncertain control, failure avoidance and self-handicapping) and six boosters (planning, mastery orientation, study management, self-efficacy, value of schooling and persistence). Using LISREL procedures, Martin confirmed a strong factor structure of the scale. Martin validated the scale in terms of literacy and numeracy in an Australian context.

Hoachlander (2008), in the context of high school education, identified four guiding principles in relation to vocational education. Firstly, deliver a school curriculum that motivates and engages the student to succeed in both school and at a vocation or career. Secondly, consider the pedagogical presentation of the literacy, mathematics and science learnings, rather than considering what is to be taught. The academic expectations should not be lowered, in-fact they should be increased as the students participate in an authentic workplace dealing with authentic problems. Thirdly, present a curriculum that appeals to all students, irrespective of their academic ability. The curriculum should not be limiting – it needs to be broad to allow for the selection of apprenticeship trade careers or a university education (e.g., electrician or an electrical engineer). Fourthly, assess student achievement in multiple areas of the curriculum. It was with this type of thinking regarding integrated, vocationally-based contextualisation, that the GIVE program was developed.

The focus of this paper is upon the impact that a construction skills-based program (which promoted integrated and meaningful links between the classroom and the workplace) had upon motivation and academic achievement. The three boosters of self-efficacy, value of schooling and persistence were selected as the facets of motivation to explore, since Martin (2001) suggested these are closely linked to students' motivation to learn.

The GIVE Program

The GIVE program aims to build awareness of, interest in, and familiarity with vocational employment as a future career pathway and opportunity for advancement; enhance literacy, numeracy and science knowledge and achievement; and provide motivation to stay on at school and build towards a productive future. To achieve these aims, the GIVE program was designed as a term-long (nominally 10 week) program tailored to learners' needs by considering the curriculum (as specified by national, state and sector-level authorities) and local industry. The latter consideration stems from the researcher and schools' partnership with local industry, which provided the relevant and authentic opportunities and technical expertise for the vocation-based contextualisation of English, mathematics and science learning. Mathematics topics explored in the construction unit were: number (including place value and fractions), measurement, time and geometry (in particular, the creation of 2D representations of 3D objects).

Method

Sample and procedure. The program was delivered to a Year 4 class of 21 students (aged 8-9 years) who were taught by two teachers during Term 4 of the 2010 school year. Of these 21 students, only the data relating to 16 of the is considered in this paper; the remaining five students entered the school after the Year 3 NAPLAN test and/or left the school before the Year 5 NAPLAN test. Pre-post interview data was collected from both the student and teacher participants. In both cases, a 10-point Likert scale was used to ascertain perceptions of the three factors (the boosters): self-efficacy, value of schooling and persistence. For the students, small focus group discussions were used in which the students were asked to self-report their perceptions (which were verified by their peers). Both teachers were asked to use the same Likert scaled based system to independently rate each

student with regards to the three factors. This process was performed both before and after delivery of the GIVE construction program.

Instruments and analysis. Each student focus group was shown the rating scale. The interviewer explained the usage of the scale, and a sample item “I think I will like the GIVE program” was presented to check for understanding. The students were then presented with a series of three cards, each with a statement relating to the three boosters under investigation (refer to table 1). For each statement, the students were asked to rate themselves using the 10 point Likert scale (‘1’ strongly disagree to ‘10’ strongly agree) and to discuss, and thus verify, their rating with their peers.

For each factor, the class mean was calculated and then used to determine the student self-reported mean. The two teachers’ ratings of each student were compared and the mean calculated for each student; there were no cases of substantial differences of teacher’s perceptions of a student. The individual teacher-perceived student means were themselves averaged to determine the class mean for each of the three factors. The pre-post self-reported and teacher-perceived ratings were compared for changes in motivation and engagement. NAPLAN data from when the participating students were in Year 3 and Year 5 was used to compare changes in each student’s mathematics ability.

Table 1.
Motivation and Engagement Stimuli

Factor	Student statement	Teacher statement
Self-efficacy	If I try hard, I believe I can do my schoolwork well	This student believes that if he/she were to try hard, then he/she do their well in their schoolwork
Value of schooling	Learning at school is important to me	This student believes that learning at school is important
Persistence	If I can’t understand my work, I keep trying until I understand it	If this student doesn’t understand the work, he/she will keep trying until he/she does understand it

Results and Discussion

Observational data revealed significant improvements in a number of areas. Firstly, school attendance for the participating class rose from its normal level of 50-60% to 100% (full attendance) in Week 2 and stayed at 100% well beyond the duration of the program. Secondly, classroom and playground behaviour (which commonly involved repeated negative instances each week and school suspensions) improved to the point where there were only two playground instances of negative behaviour across the period of the program, neither of which were instigated by students participating in the GIVE program.

The mean student self-ratings and the mean teacher-ratings in regards to the three factors of self-efficacy, value of schooling and persistence are shown in Figures 1a-1c. The influences that the changes in these three factors have had upon motivation and in turn academic achievement are discussed in the following sub-sections.

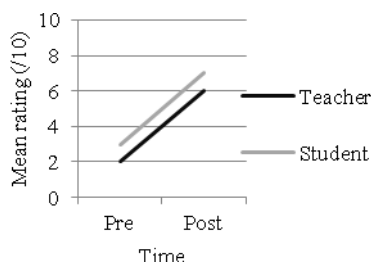


Figure 1a. Self-efficacy

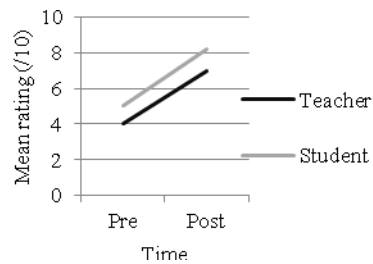


Figure 1b. Value of schooling

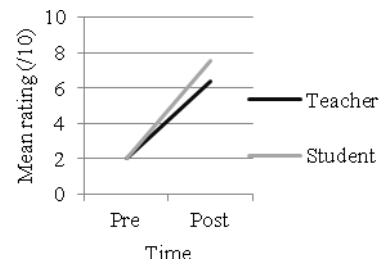


Figure 1c. Persistence

Self-efficacy. The belief in ability and the confidence to succeed at school was initially perceived to be low by both the students and the teachers. The mean rating provided by the students was 3/10, slightly higher than the teacher's perception (2/10). Figure 1a indicates that both the students and teachers perceived a four point improvement in self-efficacy following the GIVE program.

During the program, students were not given the opportunity to quit or not finish an activity. There was an expectation that all students would succeed and complete all tasks, thus showing students that they could indeed succeed at school. At times, frustrations impeded satisfaction when the student set unrealistically high expectations in relation to their construction work:

Yea like, I imagined my tow truck to be really sleek, and smooth running – like on a real truck. But I didn't measure properly. My wheels were crooked. I am not so good at dividing, but still it worked OK. I got to drill and screw new axle holes so I get it now (Larry).

Value of schooling. The usefulness of what is learnt at school was initially perceived to be in the mid-range by the students (mean rating of 5/10). A number of students realized that they needed to be able to read and write, but “once I have got that, why do I have to keep doing it, like you do English all the time, but I can already read and write” (Larry). Another student commented that:

A lot of the stuff you have to do is boring, and I never use it like plus-ing and take-aways. I am not good at that and I don't see why I have to do it. They have calculators to do it (Penny).

The teachers also felt the students had little value for school work “as they don't see its relevance. If it does not relate specifically to what they did yesterday or today, then forget it!” (Teacher 1). However, almost nine weeks later, the mean student perception of the value of schooling had shifted considerably to 8.2/10. Many students commented on the link between mathematics and everyday life: “I need to be able to make good measurements or my truck will look dumb. I don't want everyone to know I can't measure, so I have to work at it in class” (Mitchell). Larry was aware of his changes as he said “last time I thought school was boring and not got anything to do with me, but now I can see it has”.

Persistence. Initially, the drive to keep trying if at first unsuccessful was a difficult notion for the students to understand. The pre-program mean student-reported persistence rating was quite low (2.1/10). This was not surprising, considering many students rated themselves at 1/10 “because I don't start my work, so how can I keep trying?” (Michael). The teachers also rated the students as having very low persistence initially, due to apathy. However, the GIVE program provided many challenges for the students, and as the students

became more willing to engage in an activity, they all seemed to develop some form of persistence.

It is harder to keep trying in maths 'cause I don't know all my times tables. I know what I want to do, like times 'n stuff, but I cannot do it without knowing times tables. We used to be allowed to use a calculator, but now we can't, so I am learning them [times tables] 'cause I need to figure out how much wood to ask for my trinket box (Louise).

Persistence was the factor which shifted the most over the duration of the program. Students' mean self-rating of perception shifted by a rating of 4.4 points, and the teacher's perception of student persistence increased by 5.5 points. By the conclusion of the GIVE program, all students were self-starters on any given academic activity in the classroom: They had the ability to attempt an activity, and modify their strategy if they perceived they were not doing it properly.

Academic achievement. The participating students' NAPLAN results were used as the basis for commenting upon academic achievement. Figure 2 shows the NAPLAN growth of the Year 4 class (Year 3 data was collected 16 months prior to, and Year 5 from 8 months after, the GIVE program). It should be noted that between the two NAPLAN tests the school did not implement any targeted interventions other than the GIVE program.

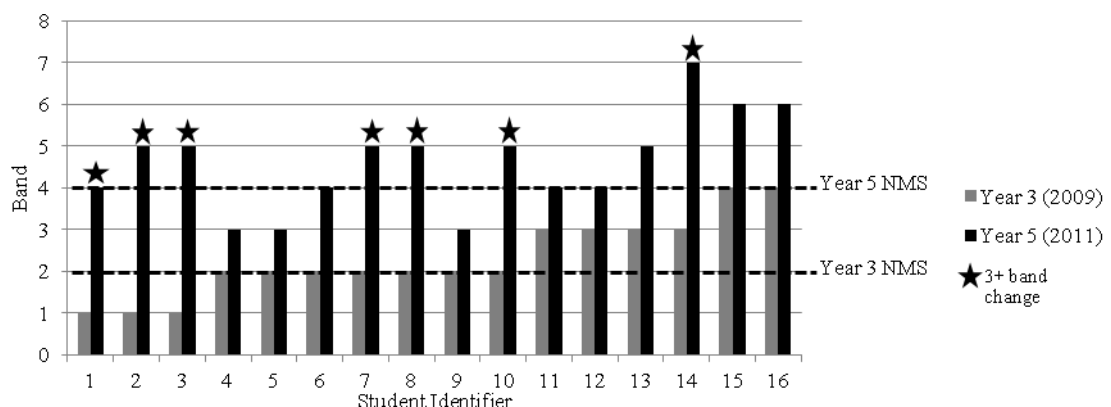


Figure 2. NAPLAN numeracy data

The changes from Year 3 to Year 5 indicate substantial growth for the majority of the 16 students participating in the study. In 2009, three students were below the National Minimum Standard (NMS) (18.75%); seven students were performing at the NMS (43.75%), and six students performed above NMS (37.5%). The year 5 results showed marked improvement: the three students who previously below the NMS were now performing at or above the NMS. In fact, most students experienced personal growths of 2 or more bands, with almost half jumping 3 or more bands.

Implications

Like Hoachlander's (2008) identification of principles for high school vocational education, this study has led to the identification of three practical implications drawn from the GIVE experience. These implications are proposed to be crucial to increasing motivation and engagement and are linked to mathematical academic growth. In the following sub-sections, each implication is discussed in turn, including vignettes to illustrate the implication in action.

Teacher and trainer expectations. At no time were the students permitted to opt out or not complete a task. It was clear to the students that the teachers and trainers had high expectations. In particular, all students were expected to master the skills – both in the classroom and in the workshop. The following vignette illustrates the impact of teacher expectations being understood by the student. With the students knowing what was expected of them at all times, and when combined with the knowledge that they had the power to positively impact on their school lives, and life in general, it is proposed that teacher and trainer expectations impacted positively on academic motivation. Students of all abilities became more motivated to engage with class work as they had the confidence, or self-efficacy, to believe in themselves and their abilities. The students understood that the teachers and trainers expected them to be capable and responsible at all times.

Vignette 1 – The camera man. A number of students were asked to take on additional responsibilities based upon student ability e.g. to be a team leader (Foreperson). However one role was deliberately given (by the Researcher) to the most disruptive and violent boy in the class – that of reporter and camera man. This boy, Michael, was given unlimited access to the video camera, much to the envy of his peers. After two days, the teacher began to suggest that Michael also needed to do the class work or he would not be ready to go to the workshop as he would not have his plans and measurements done to make the tow truck. As Michael understood the expectations of being the cameraman, as well as a student in the class, he reluctantly attempted the class activities. By the end of week one, Michael was achieving accurate measurements, and beginning to assist other students who were struggling – especially in the workshop. He discontinued his disruptive violent behaviours. Over time he became adept at managing a high standard in his class work and construction work, whilst capturing interesting video footage and interviews. Michael's post test for numeracy was 29/30, a marked improvement on his pre-test of 2/30.

Persisting for success. Closely related to the high expectations of the teachers and trainers was an emphasis on mastering tasks and the promotion of success. Students knew it was OK not to master a task on the first attempt provided they gave it another go. The students saw their teachers make mistakes in the workshop, but they also saw the teachers try and try again. This 'persistence' spread to the classroom and even had students requesting to take extra paper home to have another go for homework, or to stay in at break time to have another go, or to finish an activity. The class became a close knit learning community as the students grappled with mathematical concepts well beyond their years. The following vignette illustrates the impact of persistence.

Vignette 2 – Tow truck speed. The construction of tow truck required accurate measurement and technical drawing skills. The completed tow truck was then used to develop science inquiry and number skills. The students used balloons to create balloon powered tow trucks and extended this activity to conducting speed trials of the trucks. To calculate speed they used the formulae $\text{Speed} = \text{Distance} / \text{Time}$. The notion of formulae and substitution was discussed. Using calculators a few students were able to calculate speeds and plot performance graphs. Using a form of peer tutoring, most teams were able to develop good understanding of calculating speed, plotting and comparing a number of tow truck speeds. Louise commented upon the notion of 'time trials' in a car race to get top position at the race start. Other students were able to relate to this real world link and one student wondered if he could look up swimming race times from the Olympics and then calculate how fast the swimmer was going.

The use of formulae and variable substitution, along with the ability to apply the learnings in the real world, was enabled by mastering practical hands on skills in the workshop and using the product in the classroom to develop theoretical concepts. The students knew that they needed to be accomplished in both the workshop and classroom to succeed, and that they had multiple opportunities to master skills. Using activities that were challenging but achievable, that linked the workshop with the classroom clearly enhanced the student perception of the value of schooling. It is proposed that task mastery and peer relationships impacted positively on academic motivation and engagement to grapple with mathematical and scientific concepts well above the Year 4 level.

Classroom relationships. The usual classroom relationship of teacher and student quickly changed. In the following vignette, Michael (M - the cameraman) was one of the first to notice this and he made queries about the changes with the researcher (R) as he set up his camera equipment one morning:

Vignette 3: Changed relationships. (M) You know, I don't get red cards (detention notices) from Miss any more. I think she likes me now. (R) Why do you think she disliked you before? (M) Well, I always messed around and now I don't. I like this sort of school. I see I can do it and need my maths and reading 'n stuff. Like, she is a different person when she goes into the workshop, not telling me what to do and stuff. It's like she is too busy to pick on me 'cause she is learning her skills too. (R) Do you think you might not be making her agro now, so she doesn't have anything to give you a red card for? (M) Maybe, I'm getting things done. But I used to get in trouble for wandering around, and now I don't, but I wander around just the same but with the camera. (R) Perhaps you have a purpose to your wandering now. Like you have a purpose to do the class work? (M) Yea, maybe. Might ask her. Can I interview her? (R) Remember our deal? You can interview anyone who is doing what they should be doing. Is Miss doing the right thing? (M) Yea, she's learning. Michael, did the interview with the teacher, and asked: "Miss, you are learning to drill for the first time, so you are like us – students! I like this idea, you being like a student too. Do you like this?" The teacher explained that "yes, I am a student now, but I am always a student. I am always learning – it's what makes me a human being. Are you always learning Michael?" Michael replied that he "wasn't always learning – I only learn when I do maths and skills and stuff".

Michael, like the other students, responded well to the GIVE program and began to review what it meant to be a student. Relationships between the students themselves improved, as well as those between the students and the adults in the program (teachers, trainers, Principal and researcher). With positive relationships in the classroom, parents reported the students were keen to get to school, keen to attack homework in the afternoon. It is proposed that classroom relationships impact positively on academic motivation. Students saw teachers in a new light, teachers wasted less time on discipline matters, and students began to see the relevance of their schooling.

Conclusion

This paper reports on a study that investigated the motivation and engagement of underperforming Year 4 students. The students undertook a term long integration of mathematics, English and science via the vocational education-based context of 'construction'. Within a very short time, students became motivated to engage with their class work. The data from the program indicates student gains on key components of

motivation over the course of the school term and up to 12 months beyond. Three implications were identified: (a) that teacher and trainer expectations impact positively on mathematical academic motivation; (b) that persistence impacts positively on mathematical academic motivation and engagement enabling students to perform well above the expected level, and (c) that classroom relationships impact positively on mathematical academic motivation.

In summary, this paper has demonstrated how a vocationally-based program can be used in a primary school setting to motivate learners, engage them in the curriculum and enhance their academic skills, whilst simultaneously promoting their interest and awareness regarding trades-based employment. These findings are significant for researchers and educators who wish to enhance student's mathematical academic motivation.

Acknowledgements

The Authors would like to acknowledge RioTinto (Alcan) for their support of the GIVE Program. We would also like to thank Kerry Whittaker (GAGAL CEO) and GAGAL trainers, as well as the teachers and students of the participating primary school for their wonderful collaboration and efforts.

References

- Dandy, J. & Nettlebeck, T. (2000). The model student? An investigation of Chinese Australian students' academic achievement, studying, and causal attributions for academic success and failure. *Australian Psychologist*, 35, 208-215.
- Hoachlander, G. (2008). Bringing industry to the classroom. *Educational Leadership*, 65(8), 22-27.
- Kelly, A. J. & Hansen, D. J. (1987). Social interactions and adjustment. In V.B. Can Hasselt & M. Hersen (Eds.), *Handbook of adolescent psychology* (pp. 131-146). New York: Pergamon Press.
- Martin, A. (2001). The Student Motivation Scale: A tool for measuring and enhancing motivation. *Australian Journal of Guidance and Counselling*, 11, 1-20.
- Sticht, T. (2002). *Functional Context Education: A scientific, evidence-based approach for adult literacy education*. New Brunswick, Canada: National Literacy Database.
- Sticht, T., Armstrong, W., Hickey, D., & Caylor, J. (1987). *Cast-off youth: Policy and training methods from the military experience*. New York: Praeger.
- Stone, J., Alfeld, C., Pearson, D., Lewis, M., & Jensen, S. (2006). *Building academic skills in context: Testing the value of enhanced math learning in CTE*. Columbus: National Dissemination Center for Career and Technical Education, Ohio State University.
- Teven, J. J. & McCroskey, J. C. (1997). The relationship of perceived caring with student learning and teacher evaluation. *Communication Education*, 46, 1-9.
- Valentine, J. C., Cooper, H., Battencourt, B. A., & DuBios, D. L. (2002). Out-of-school activities and academic achievements: the mediating role of self beliefs. *Educational Psychologist*, 37, 245-256.

Mathematical Proficiency and the Sustainability of Participation: A New Ball Game through a Poststructuralist Lens

Mary Klein
James Cook University
<mary.klein@jcu.edu.au>

In Australia we face a serious problem in that over the last twenty years the quality of students' mathematical knowledge and abilities has "deteriorated to a dangerous level" (Brown, 2009, p. 3). Too few students want to study further mathematics and are "unlikely to continue its study voluntarily" (Commonwealth of Australia, 2008) or pursue careers where high levels of mathematical proficiency are needed. In this paper I make use of the poststructuralist notion that 'proficiency' is a state of being daily constituted in classroom practice to (a) at a theoretical level, rethink how it might be ignited and sustained, (b) analyse contemporary interactional strategies that commonly though unknowingly obviate expressions of proficiency and (c) through a combined psychological/poststructuralist lens, nominate three (3) tentative indicators of instructional practice necessary for students to achieve and maintain a state of being 'proficient' as defined in the Australian curriculum: mathematics (Australian Curriculum Assessment and Reporting Authority [ACARA], 2010). It is hoped that an additional, poststructuralist reading of the complexity and tenuous productivity of the learning process might interrupt commonly held assumptions that currently inform research on proficiency in mathematics.

For contemporary mathematics education in Australia proficiency is key in postmodern times (ACARA, 2010), as basic skills are necessary but not sufficient for sustainable engagement and achievement (Luke, 2010). Resnick (2010), in the Wallace Foundation Distinguished Lecture Series, made the chilling claim: "The evidence is now pretty clear. We seem to have figured out how to teach the 'basics' to just about everyone...but we are deeply unsuccessful at our 21st century agenda of moving beyond basic competencies to proficiencies" (p. 183). The emphasis on proficiency, a state of being proficient, introduces an ontological dimension to mathematics education, not yet carefully enough delineated and understood; it raises an urgent and pressing question about the nature of the pedagogic processes and strategies that might render each student proficient, that is, in having an appreciation of mathematics and the confidence to creatively use, investigate and communicate mathematical ideas (ACARA, 2010). Clearly, beyond the construction of content knowledge, qualitatively different instructional strategies are needed that *mobilise* students as active and engaged doers and users of mathematics, sustaining and extending interest and confident engagement in mathematical tasks and investigation in, and after, schooling. It may be useful to try to think again about the sorts of interactional strategies that might have some effect in enabling and enticing students to creatively use and apply constructed knowledge.

As always in contemplating pedagogic intervention great care is needed; as Foucault (1982) said "everything is dangerous" (p. 31). Foucault does not mean that everything is bad, but it could be that, despite our best intentions, despite recourse to inquiry based practice and active engagement, we have been careless in assuming too much about how learning happens and sustains itself across contexts. In the 21st century, and in the interests of having students build and sustain effective mathematical practices that enable "logical reasoning, analytical thought processes and problem solving skills" (ACARA, 2010, p. 126) our attention might turn to the learning *process*, and the extent to which it enables and imbues learner agency. It might be timely to value-add contemporary understandings of learning with poststructuralist concepts that talk to how it is that learners are constituted as

proficient or not in discursive practices, and how it is that so many turn away from mathematics at an early age. That is, in the operation of the mathematics education discourse, learners may strive to establish themselves as proficient but may not be successful (or they might be ‘successful’, but only in terms of outdated notions of proficiency). How the teacher orchestrates the production of knowledge can restrict the field of operation of the students (Foucault, in Dreyfus & Rabinow, 1982); I make this point because it is our understanding and appreciation of the field of operation that requires research and investigation. The mobilisation of conceptual understanding, procedural fluency, strategic competence and adaptive reasoning (Kilpatrick et al., 2006, p. 5) depends on students’ constituted sense of themselves as capable and valued in the construction and creative application of mathematical ideas. Throughout the schooling process and beyond, students must come to know themselves as valued constructors and users of mathematical ideas.

In this paper I want to argue that learners can only *be* as proficient as the operation of the mathematics education discourse allows; in not acknowledging this we compromise learning opportunity and give ourselves permission to ignore the urgency of further research on the necessary attributes of a learning process that mobilises the construction and application of mathematical knowledge. It is in the learning process that mathematical futures are made; but perhaps in somewhat more complex ways than we have previously imagined. Could it be that as teachers and researchers we have for too long found refuge in overly simplistic notions of learners that assume rational thought and autonomous action; we have allowed ourselves to be complacent with models of learning that imagine knowledge constructed has a more or less linear relationship to application. Through bringing both psychological and poststructuralist concepts to bear on one instance of classroom mathematics, I attempt to show how in this instance not only is the students’ construction of powerful mathematical knowledge compromised, but so too their right to enlightened participation in a learning process (which is always constitutive, sometimes in ways we would not wish).

Reading Classroom Practice through an Additional Lens

One couldn’t imagine an analysis of classroom practice in mathematics without an appeal to psychology; over the past 40 years or so, psychological and sociological understandings of how students learn mathematics have grown enormously. It is now considered important that students are engaged in a “learning process that involves making connections, identifying patterns, and organising previously unrelated bits of knowledge, behaviour and action into new patterned wholes” (Cambourne, cited in Killen, 2007, p. 3). The focus is on the learner *constructing* knowledge rather than absorbing it; the process of interaction with the content is most important, because the learning process includes opportunities for students to think and reason mathematically, leading to the construction of analytical thought processes and problem solving skills (ACARA, 2010). However, although the rhetoric surrounding mathematics education includes notions of sense making and active engagement, in practice many learners find themselves served up an emaciated form of mathematics in learning conditions that merely pretend to entice. Learning mathematics too often ends up being experienced as one continual exercise where learners are asked to slot in responses or follow procedures dictated by the teacher or test. Learners in mathematics are like the ball persons in a tennis match; if ever they are to get in on the real action, in a position where they can strategise and use an innovative game plan, they must first of all learn to recognise themselves as legitimate players. Having agency means they have a constituted sense that they can participate competently enough and even go

beyond outmoded tactics to forge a new game plan. Such conditions are not yet regularly available to learners of mathematics, although recourse to humanist assumptions about rational, autonomous action affords educators the licence to assume that they are.

Methodologically, poststructuralism attends to power relationships in discourse and discursive practices (the learning process), and the constitutive effect of these power relations. Mathematics education is a discursive field in which the discourses of mathematics and education come together as discursive practices (group work, marking with ticks and crosses, the teacher asking questions) that structure learning experiences in mathematics. The way in which mathematics education is played out in any context affects the extent to which learners can establish and recognise themselves as mathematically proficient. The learner of mathematics, who manages to establish him/herself as proficient as recently defined (ACARA, 2010), owes this positioning partly at least to interactional practices that nourished his/her initiative and participation in constructing and creatively using mathematical ideas. That is, a state of being proficient (or not) is daily constituted, partly at least, in classroom interaction, and it affects participation post schooling. A poststructuralist analysis, then, is interested in *how* different enactments of the mathematics education discourse operate to support or suppress learners' engagement in the learning processes of mathematics. I suggest that educators and researchers may not fully appreciate the alienating effects on learners of instructional practices that deny them a genuine voice and the opportunity to make sense in personally meaningful ways. As Walshaw and Anthony (2008, p. 535) stated: "the development of thinking depends not so much on the frequency of exchange structures but on the extent to which students are regarded as active epistemic agents. Developing students' thinking also enhances the view that students hold of themselves as mathematics learners and doers".

In taking seriously this notion (Walshaw & Anthony, 2008) of the importance of positioning learners as "active epistemic agents" I think back to my own classroom teaching days when I saw knowledge and the learner as uncomplicated and absolute. Nowadays, through a poststructuralist lens, I appreciate that this is indeed not the case; I have come to see how a process of subjectification overwrites the construction and stockpiling of knowledge, influencing the students' present and future engagement in mathematics. I have come to appreciate how the view a teacher holds of learners and learning mathematics crucially affects interactional patterns and ultimately students' quality of engagement/proficiency. In the table below humanist notions of the learner (from psychology and sociology), learning and proficiency are compared with poststructuralist theorisations.

In this paper I make the argument that contemporary humanist notions of rational, autonomous learners continue to frame discursive practices that can prejudice students' learning in mathematics. An analogy that may or may not be useful is to think of students constructing what I call *mathematical muscle*; from a psychological perspective these muscles (conceptual understanding) are nourished by the mathematics practices (Ball, 2003) and supported by skills at procedural fluency, strategic thinking and adaptive reasoning (the connecting tendons). However, for mobility it matters *how* these are nourished and exercised, their lifeblood is to be found in meaningful and valued participation in making mathematical and participatory sense; to the extent that this does not happen the mathematical lifeblood thickens and congeals, compromising present and future participation in mathematics. In summary, a student's appreciation of mathematics as a discipline and future place in mathematical activity is influenced by how the learning of mathematics is experienced in school; nothing new here, except that in this case I make special reference to relationships of power which constitute that experience. Proficiency,

regardless of the student's level of mathematical understanding, is sensed (or not) in active participation rather than achieved as a *fait accompli*.

Table 1
Humanist and Poststructuralist Notions of the Individual

	HUMANIST	POSTSTRUCTURALIST
LEARNER	Rational, coherent, autonomous.	Constituted in discourses such as mathematics education through one's own and others' acts of speaking and writing.
LEARNING	Learning mathematics is about constructing & applying knowledge.	Learning is rhizomatic, rather than linear, a process of not only constructing mathematical ideas and practices, but also of gradually establishing oneself as proficient in the discursive practices.
PROFICIENCY	A personal attribute	Constituted in participation, a form of agency

Engagement Does Not Ensure Proficiency

Immediately below is an extract (Hardy, 2004, pp. 110-111) made up of interview scenes where the teacher comments on her classroom practices. The children's desks are arranged in blocks of six and each child has two sets of cards, both numbered from 0 to 9, in front of them. They hold up cards to show their answers to the questions asked of them" (p. 110). My intention in this section of the paper is to make visible how the teacher's best efforts can be read as compromising "the 21st century agenda of moving beyond basic competencies to proficiency" (Resnick, 2010). Specifically, in adhering to humanist notions of rational, autonomous learners the teacher makes dangerous assumptions that render invisible the possibility that:

- The students are not actually engaged in doing mathematics at all, and
- Learning is not necessarily liberating. In this case the students are learning (that is, *they come to know*) their place as optional extras in the articulation of mathematical knowledge (a poststructuralist reading). Proficiency is narrowly constructed as recall, giving the students no opportunity to establish themselves as proficient as defined, for example, in the Australian curriculum (ACARA, 2010).

Teacher (to researcher): A few children don't put their hands up. They try to hide, but that's the idea. There is no hiding place. You encourage them as long as you give them positive feedback. Even if they get it wrong, they are not scared to give an answer.

Teacher (in classroom): show me a multiple of five bigger than 75...Is that a multiple of 5 though, Michael? It's bigger than 75 but check it's a multiple of 5...

Well done, Sarah!

Teacher: Show me three threes...

Three threes? Check again please, Lauren.

Check please, Joe. You are looking at someone else's. Don't just look at someone else's. If you're not sure get your fingers and count in lots of three. Let's do it together (chanting) three, six, nine. You should be showing me nine there.

Teacher (to researcher): Some children don't have instant recall of three threes but I've given them a method to work it out. "Get your fingers and count in threes". So as long as they do regular counting in threes and they've got that pattern, they have got a method or strategy that we've talked through

together to help them through that. They are not stood in queues waiting to get a book marked; they are getting instant feedback. They are not scared to get an answer wrong. They're having a go, they are risking things, and you don't gain anything unless you have a few risks and that's what they are doing.

Teacher (in classroom): Have a quick check of that one, Misha. You should be showing me twelve.

Teacher (to researcher): It really works. We've seen it work. The children are motivated. The children want to learn. You never have to tell children "Are you messing around?" they're not. They are trying. They might not be succeeding but they are trying. They really love the pace. Children don't like sitting for 20, 30 minutes on one task especially if they are struggling on it. This doesn't allow that. The children have to find answers. They work together. They help each other but they are also pushing forward. The task is changing all the time. As long as you stay focused on target, most lessons you achieve eighty percent of children come out learning something that they didn't go in knowing and that's a wonderful experience and encourages you to go on further [End of classroom example, taken from Hardy, 2004, pp. 110-111].

A Two-pronged Analysis of the Learning Process

In casting a combined psychological and poststructuralist lens over this snapshot of learning it becomes clear that here mathematics education operates to render some learning practices possible while others are patently unthinkable (Britzman, 2003). In this classroom, on this day, there is an interlude where pre-established knowledge is tested; communication of what students know is transmitted through two sets of cards with numbers on them (Hardy, 2004). From a psychological perspective one is concerned that there seems to be little opportunity for students to actively engage in logical reasoning, develop analytical thought processes or problem solve (ACARA, 2010). Although the teaching is meant to be a two-way process where "pupils are expected to play an active part" (Department for Education and Employment [DfEE], 1999, p. 14) their role is restricted to answering questions. Problematic as this is from a cognitive, psychological perspective, a poststructuralist lens reveals even further complexity. Since classroom experience is constitutive of future experiences with mathematics, it is imperative that the 'active part' learners do play is not only furnished with rich and robust mathematical understandings, but also with the authority to participate in ways that are idiosyncratically reassuring and transformative. That is, 'how' the learners are able to actively participate can support or not the sustainability of participation in mathematics. While making sense of the mathematical content is a cognitive, intellectual affair, sensing competence in participation is an unconscious, embodied and constitutive 'sense' relative to identity which always influences the psychological. Although some of the children in this classroom may eventually answer the endless questions and even pass the exams, because they have no 'authorship' of the complex webs of understanding at the heart of mathematics nor of robust and rigorous demonstrations of proficiency (ACARA, 2010), their flirtation with mathematics is likely to be short lived. Students want to be legitimate participants, not sidelined, in the "game of truth" (Foucault, in Bernauer & Rasmussen, 1987, p. 1) that is school mathematics; their identity and future innovative participation in mathematics related tasks depend on it.

However, teachers worldwide persevere with the above interactional patterns because they subscribe to (have been constituted through) humanist views of the learner and absolutist notions of knowledge; they do not recognise the complexity of the productivity of the learning process, its production of conceptual knowledge and a constituted knowing about one's positioning as proficient, or not. The humanist teacher, as in the excerpt above, essentialises students as uncomplicated, either "motivated" or not, and "wanting to learn", or not wanting to learn. The "wanting to hide" is taken to be a personal failing, easily

overcome through kindness and fellowship in teaching interaction. Within humanism, learning is a personal act where one fulfils one's inner potential, gradually leading to a state of self-actualisation. The teacher is a facilitator, making learning personalised. Note how the teacher addresses the students in demonstrations of fellowship: "Well done Sarah", "Check again please, Lauren", "Check please, Joe...You should be showing a nine there", "Have a quick check, Misha, you should be showing me 12". The teacher is very happy with the learning opportunity she has orchestrated for her students; "It really works", she says, "We've seen it work. The children are motivated. The children want to learn. They are trying. They might not be succeeding but they are trying. They really love the pace. As long as you stay focused on target, most lessons you achieve eighty percent of children come out learning something that they didn't go in knowing and that's a wonderful experience and encourages you to go on further". Herein, of course, lies the rub; students, because they are denied any form of meaningful participation learn (they are constituted to know) that mathematics is not for them, and just as worrying is the fact that this teacher is going to continue this sort of interactional pattern because it is considered to work. Yet the students' authorship or creation of mathematical ideas is sidelined as the teacher concentrates on orchestrating what is considered to be a co-operative, supportive environment; little mathematics is engaged in by students as the teacher provides the correct answers where they do not already know it. A combined psychological and poststructuralist framing of this practice sees as problematic the discursive construction of proficiency as recall, and the allied constitution of students as receivers, not generators, of knowledge.

Tentative Conditions of a Practice Supporting Proficiency

From a two-pronged perspective I have argued that students' competent and generative use of mathematics beyond school depends upon the operation of the mathematics education discourse; the operation of the discourse is influenced by the teacher's, and the wider socio-cultural context's, constituted beliefs about learning in mathematics. It is not that some students are essentially proficient and others not, but that humanist based instructional practices operate on and sustain this assumption. Piaget's child development through stages, Vygotsky's (1978) social interaction as a key force in the development of mind, and Lave's (Lave & Wenger, 1991) 'situating' learning in socially supportive contexts are premised on the rational, autonomous learner of mathematics. While each of these has contributed to research in mathematics education, they have not specifically recognised how learners themselves, as well as mathematical proficiency, are *produced* in teaching-learning interaction. Learners are produced in relationships of power, and should be able to recognise themselves as authoritative (in the sense of having *authorship* of ideas and practices) and competent in the intersecting and competing discourses of mathematics and education. It becomes a nonsense for teachers to imagine that they can 'gift' mathematics to their students; learner-centred means the learning process, where an energy for mathematics is experienced and released in interaction, is the hub of sustainability as it is constitutive. Teaching mathematics should not be about sugar-coating disparate bits of knowledge or the learning process, but providing those conditions for learning that gradually build (constitute) the mathematical energy and muscle for full participation in the 'game' of doing mathematics (Foucault, 1987). To build muscle (knowledge), energy (engagement in mathematical practices) and a desire to actually play the *game* (Foucault, 1987) three tentative, interdependent conditions, informed by psychology and poststructuralism, would seem to suggest themselves:

- The provision of robust and related mathematical knowledge, where student learn
- to speak and write the language of mathematics

- to communicate mathematically
- to develop understanding and fluency through, and for, reasoning and problem solving
- to appreciate the pattern and order of mathematics

Interactional relationships which centre the learner and the mathematics:

- Students come to know mathematics as a method of reasoning, a way of figuring out a certain kind of system and structure in the world (Ministerial Council on Education, Employment, Training and Youth Affairs [MCEETYA], 2008), and
- Students author, initiate, sense-making streams. They sense that they are respected and valued as participants in doing and using mathematical ideas.
- The teacher provides the cognitive, social, cultural space for learners to establish themselves as proficient in participation (regardless of, but taking cognisance of, the level of the mathematics employed).

A classroom and broader social culture which recognises:

- Mathematical knowledge and the learner as always *in process*, growing.
- That students' appreciation of and participation in mathematics now and in the future is nascent in classroom patterns of interaction (because they are constitutive).

Conclusion

Devlin (2000, p. 254) states: "The key to be able to do mathematics is wanting to". I would add that the 'wanting' is constituted. Unfortunately, as recognised two decades ago, although children come to school enthusiastic and eager to learn mathematics, they "leave school with quite negative attitudes" (Australian National Council, 1990, p. 31) and do not want to have anything to do with mathematics once they get out of the school gate. The finger of course turns to the quality of classroom teaching (Masters, 2009) where observations of classroom practice (Luke, 2010) pick up on the pitfalls of having students struggle over too many worksheets, copying off the board, assessment based solely on recall and activity-based *busy work*. While, yes, these do inhibit the construction of robust knowledge and flexible thinking strategies, they also do nothing to nourish the students' recognition of themselves as idiosyncratically competent with/in mathematics. That is, students involved exclusively in these sorts of routines are not able to demonstrate what proficiency demands: competence *with* the mathematics and *in* participation in its contemporary discursive strategies and practices (ACARA, 2010).

Traditional forms of practice, like my previous teaching and the classroom example (Hardy, 2004) above, are held firmly in place by humanist assumptions of rational, autonomous students and absolute knowledge. Any bit of knowledge growth is thought to be worthwhile, and students who do not take up the offer are seen to be not able or unmotivated. Another reading of the situation suggests that students are in an untenable situation; the tasks they are given in the name of mathematics often incur no opportunity for mathematical reasoning at all. These tasks are not representative of the types of participation that build and sustain interest and enjoyment in doing and using mathematics; these tasks are not compelling because they do not author a learning process that follows idiosyncratic impulse. Igniting and sustaining proficiency is dependent on interactional patterns that deliver the very best knowledge, encourage learners to recognise themselves as proficient and mathematics as worth doing. Thus the mathematics, the learner and meaningful participation are equally important and at the centre of mathematics education; each is

constitutive of the other, ideally in ways that mutually energise and mobilise for sustainability into the future.

References

- Australian Curriculum, Assessment and Reporting Authority. (2010). *Australian curriculum: Mathematics*. Canberra, ACT: ACARA
- Australian Education Council. (1990). *A national statement on mathematics for Australian schools*. Carlton, Victoria: Curriculum Council.
- Ball, D. (2003). *Mathematical proficiency for all students: Towards a strategic research and development program in mathematics education*. Santa Monica, CA: RAND.
- Bernauer, J. & Rasmussen, D. (1987). *The final Foucault*. Cambridge: MIT Press.
- Britzman, D. (2003). *Practice makes practice: A critical study of learning to teach*. Albany, NY: State University of New York Press.
- Brown, G. (2009) *Review of education in mathematics, data science and quantitative disciplines*. Report to the Group of Eight Universities. Canberra: The Group of Eight.
- Commonwealth of Australia (2008). *National numeracy review report*. Canberra, Australia: Department of Education, Employment and Work Relations:
- Department for Education and Employment. (DfEE) (1999). *National numeracy strategy: Framework for teaching mathematics from reception to year 6*. Sudbury: DfEE Publications.
- Dreyfus, H. & Rabinow, P. (1982). *Michel Foucault: Beyond structuralism and hermeneutics*. New York: Harvester Wheatsheaf.
- Devlin, K. (2000). *The maths gene: Why everyone has it but most people don't use it*. London: Phoenix.
- Foucault, M. (1982). On the genealogy of ethics: An overview of work in progress. In H. Dreyfus & P. Rabinow (Eds.), *Michel Foucault: Beyond structuralism and hermeneutics* (pp. 229-252). Chicago: University of Chicago Press.
- Hardy, T. (2004). "There's no hiding place. Foucault's notion of normalisation at work in a mathematics lesson". In M. Walshaw (Ed.), *Mathematics education within the postmodern* (pp. 103-120). USA: Information Age Publishing.
- Killen, R. (2007). *Effective teaching strategies: Lessons from research and practice*. Sydney: Social Science Press.
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, D.C.: National Academy Press.
- Luke, A. (2010). Will the Australian curriculum up the intellectual ante in primary classrooms? *Curriculum Perspectives*, 5(10), 3-5.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.
- Masters, G. N. (2009). *A shared challenge: Improving literacy, numeracy and science learning in Queensland primary schools*. Camberwell, Vic: Australian Council of Educational Research.
- Ministerial Council on Education, Employment, Training and Youth Affairs (MCEETYA) (2008). *Melbourne declaration on educational goals for young Australians*. Canberra, ACT: MCEETYA.
- Resnick, L. (2010). Nested learning: Systems for the thinking curriculum. *Educational Researcher*, 39, 183-197.
- Vygotsky, L. (1978). *Mind in society. The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
- Willoughby, S. (2000). Perspectives on mathematics education. In M. J. Burke & F. Curcio (Eds.), *Learning mathematics for a new century* (pp. 1-15). Reston, VA: National Council of Teachers of Mathematics.
- Walshaw, M. & Anthony, G. (2008). The teacher's role in classroom discourse. *Review of Educational Research*, 78, 516-551.

Developing a Culture of Collaboration

Janeen Lamb

Australian Catholic University
<janeen.lamb@acu.edu.au>

Jana Visnovska

The University of Queensland
<j.visnovska@uq.edu.au>

This paper explores the role of professional learning community and collegial discussion as important supports for developing teacher expertise in the teaching and learning of mathematics in rural and remote regions of Queensland, Australia. The research reported in this paper is from the first year of a longitudinal research and development project. Findings suggest that teachers in rural and remote schools with a small number of mathematics teachers may benefit from access to the mathematics professional learning community of a larger rural school.

It is well recognised that many teachers in rural and remote regions of central Queensland are new graduates. They teach for three years in these regions and then return to the city centres along the coast of Queensland. This practice results in high teacher turnover, insuring a constant flow of new, inexperienced teachers to these regions (Heslop, 2003). Making this situation more complex for the teaching of mathematics in years 8, 9 and 10 is the usual practice that this teaching is often delegated to novice teachers without a mathematics background.

This paper reports on one aspect of a larger project. The larger project aims to develop a set of guiding principles (diSessa & Cobb, 2004) for an effective, longitudinal and collaborative approach to increasing the quality of mathematics teaching and learning in remote regions in Queensland. The vehicle used to enhance year 8, 9 and 10 teachers' understanding of mathematics teaching and learning is through the development of a professional learning community among the rural and remote teachers participating in this project. This approach recognises the problems associated with the tyranny of distance often experienced by teachers in remote regions as it prevents access to a mathematics professional learning community and as a consequence collegial discussion that could guide their developing understanding of the teaching and learning of mathematics. In an effort to meet this aim the research investigates ways of developing and sustaining local teaching capacity for mathematics in years 8, 9 and 10 through the promotion of a professional learning community. The argument presented in this paper is from the theoretical perspective that professional learning communities provide spaces for teacher discussion, and are therefore fundamental to the support provided to recent graduates as they develop expertise through their early teaching experiences.

Literature Review

Professional learning communities, also known as professional teaching communities are well recognised in the literature as providing collegial support for teachers (Cobb, McClain, Lamberg, & Dean, 2003; Gamoran et al., 2003; Grossman, Wineburg, & Woolworth, 2001; Lave & Wenger, 1991; Little, 2004; Stein, Silver, & Smith, 1998; Wenger, 1998). In addition, research literature suggests that effective professional development for teaching mathematics has been identified in both qualitative and quantitative studies as needing the following characteristics; extended duration, collective participation, active learning opportunities, a focus on problems and issues that are close to practice, and attention to the use of tools that are integral to that practice (Borko, 2004; Cohen & Hill, 2000; Garet, Porter, Desimone, Birman, & Yoon, 2001). Moreover it seems

important to introduce two strands of support that are particularly relevant in the initial collaboration with groups of mathematics teachers: building teacher community, and building teachers' mathematical competence.

Building Teacher Community

Teachers' learning in professional development settings is often conceptualised in terms of changes in teachers' participation in a community of practice that the group of participants constitute (Wenger, 1998). The value of this analytical approach is in shifting researchers' attention from individual teacher attributes to the norms, routines, and discourses of the profession that both shape and are shaped by the participating teachers. However, the process of *community genesis* (Gueudet & Trouche, 2012) where the professional learning community grows and develops over a period of time is an accomplishment that requires considerable proactive support (Visnovska, Cobb, & Dean, 2012). It involves development of a joint enterprise, norms of mutual engagement, and a shared technical repertoire (Cobb et al., 2003; Wenger, 1998) that would be conducive to reorganisation of mathematics teachers' practices. Dean (2005) reported that in her collaboration with a group of middle school mathematics teachers in the US, the means of supporting the transformation of the teacher group into a community of practice were not part of the initial program design but the need for them became obvious during the initial professional development sessions. Explicit support for the emergence of a professional community is at the forefront of a number of recent professional development programs (e.g., Borko et al., 2010).

Building Teachers' Mathematical Competence

Strong grounding in mathematical content is a well-established characteristic of effective professional development for mathematics teachers (e.g., Carpenter et al., 2004; Smith, 2001). This is because the forms of envisioned teaching practice require that teachers develop deep understandings of mathematics (Ball & Cohen, 1999) and, more specifically, *mathematical knowledge for teaching* (Ball & Bass, 2003).

Research on teaching mathematics for understanding (Carpenter et al., 2004) indicates that teachers need to build on their students' current reasoning while, at the same time, keep in mind significant mathematical ideas that are the goal of instruction (Ball, 1993). Effective instructional practices emphasise students' opportunities to engage in mathematically challenging tasks, where students are presented with opportunities to communicate their mathematical thinking in classroom discussions (Hiebert & Grouws, 2007).

For teachers who do not typically include classroom discussions in their teaching, leading discussions might seem unpredictable and uncertain (Visnovska, Cobb & Dean, 2012). Their concern may rest with how to give students opportunities to contribute to discussion with pre-prepared data that also led students in a way that promoted the intended mathematical ideas. Elsewhere (Visnovska & Lamb, accepted) we have reported that while some of teachers we collaborate with were not accustomed to detailed planning in advance of their classroom interactions, preferring instead to think on their feet, other teachers in the group engaged in deep pedagogical conversations as they planned the delivery of the lesson. In an effort to build on the existing research literature we investigate in this paper what might have contributed to differences in these teachers' participation in group planning discussions. Using data from teacher interviews at the end of the first year of the project, we pay particular attention to teachers' opportunities for collegial interactions at their school sites.

Theoretical Perspective

This research draws on Millett and Bibby's (2004) model (Figure 1) of teacher change within the context of curriculum change to theorise teacher interactions when participating in detailed planning discussions. Their model illustrates the local context of curriculum change and, in so doing, suggests that the role played by a teacher's professional learning community is vital to effective change. They theorise that the role of the teacher is central to the implementation of curriculum change and so a teacher's "personal agency beliefs" and "beliefs about self-efficacy...and academic self esteem" (p. 5) play a part in determining the success or otherwise of the implementation process. The influence of a teacher's personal and professional identity on curriculum change is also supported elsewhere in the literature where efficacy issues in respect to mathematics teaching are identified as critical factors (Ball & Bass, 2003; Gresalfi & Cobb, 2011; Ma, 1999). The importance of time, talk, expertise, and motivation in providing the sources of support necessary for positive self-efficacy has been identified by Millett, Brown and Askew (2004). It seemed that time, talk, and expertise complement internal motivation, resulting in "deep change" (p. 245).

Millett and Bibby (2004) conceptualise teachers' work as being situated within a specific school culture, or *professional community*, called school culture, that shapes the teachers' capacity to change. They explain that the teacher and the *situation* are located within a wider context that includes influences such as: *policy* developed by governments and various authorities; *professionals* external to education; the *private* or commercial sector; and the general *public* or those who are outside the school but are none-the-less interested in education such as parents and the media.

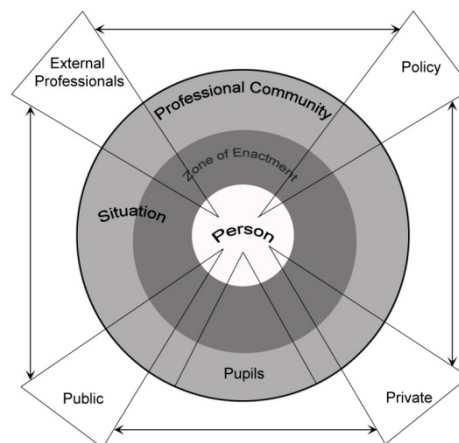


Figure 1. Theoretical model for analysing the context of curriculum change. (Millett & Bibby, 2004, p.3)

Millett and Bibby (2004) draw on Spillane's (1999) earlier work, noting that the impetus for curriculum change often comes from a wider context beyond the school environment that is supported by the professional community stimulating a "zone of enactment" (Millett and Bibby, 2004 p. 4) to which the teacher will respond. Critical to a teacher's response within their *zone of enactment* is the type of support found within their institutional context and whether sharing and critical interrogation of their practice takes place. Positive support includes "rich deliberations" that, when "grounded in practice and supported by resources, [make] curriculum change more likely to be operationalised" (p. 4).

They also noted that external expertise was deemed to be essential to support teachers' learning with respect to new content knowledge and content specific pedagogical knowledge. Heirdsfield, Lamb, & Spry, 2010 found that another factor critical to developing the motivation that leads to deep change is length of time. Individual PD sessions do not

support change but rather sustained engagement with PD providers over extended periods is needed.

Method

There are eleven teachers from six schools and one regional mathematics advisor participating in the project. The group of teachers met for the first time in March 2011 and then on two subsequent occasions during 2011 for a day of professional development. Two teachers, Tom and Jodie are the focus of this paper. Pseudonyms have been used. They are not qualified mathematics teachers and as a result teach across years 8, 9 and 10. Tom's largest class has 16 students and the smallest 6 students. Across years 8-12 there are 5 other teachers teaching mathematics at Tom's school including new graduates qualified in the teaching of mathematics to year 12. Jodie teaches years 9 and 10 mathematics in a smaller school with 8 and 5 children in her two mathematics classes. There is one other mathematics teacher at her school. The second mathematics teacher is a recent graduate qualified to teach mathematics to year 12. The distance between Tom and Jodie's schools is approximately 5 hours drive.

The data reported in this paper are drawn from interviews conducted with these two experienced teachers at the conclusion of the first year of the project. These interviews were audio recorded, transcribed and analysed for evidence of how the school *situation* supports each teacher's *zone of enactment* regarding detailed planning conversations.

The selection of these teachers reported on in this paper was purposive as their cases were chosen for the capacity to illuminate rather than for representativeness (Stake, 2005). In particular, the cases reported in this paper were selected because of the way these teachers are influenced by their school situation and as a consequence the *zones of enactment* that operate in their schools.

Results and Discussion

In both schools there are formal and informal meetings opportunities for the teachers to discuss their teaching and learning of mathematics. Both experienced teachers were asked the same questions. The question regarding informal meeting opportunities that could stimulate the teachers' zone of enactment was, In the last month have you gone to anyone for advise, a question or a concern or just to talk about something that has to do with your teaching of mathematics?

Tom: I am actually located in a room where the majority of the maths teachers are... So I'm in a kind of in a staff room that shares four of the teachers, so all the time, like there's a lot of collegial support about. Lots of questions, is something working or not working in class, and sharing of ideas... If we find something that is working well in one of our classes it is guaranteed to probably do alright in the same age group in the other class so we touch base quite a bit... I guess we informally do it more than several times daily.

Jodie's response to this question was in complete contrast to Tom's response.

Jodie: No, not really because although we do have another mathematics trained teacher in the school, he is only first year. So he doesn't, you cannot say he is experienced. He has never taught high school maths so I cannot really ask him anything. So no, I don't really get to ask anyone. I feel a bit isolated in that aspect.

Researcher: OK. So were there any issues or questions that you wanted to ask someone but because of the situation it just did not seem like an option?

Jodie: No, it is very rarely I need to. I think as I have become more experienced I do not need to ask you know as many questions anymore sort of thing. Actually in my first year when I first started I had a fellow senior math teacher here. So, no, I don't really have many questions or issues anymore.

Tom and Jodie were also asked about informal opportunities to observe other mathematics lessons in their school. Michael explained that this is possible but he does formal observations/collaborative teaching on a regular basis. When Jodie was asked about informal observation of the other mathematics teacher in the school she explained that she observes, "Just what I see when I walk past the classroom." Worthy of note is that the other mathematics teacher does come in to observe Jodie's class from time to time. When asked if she talked about the lesson or the things that she did and why with her colleague she replied: "No, not really. No, I would not say that."

The different situations in these two schools regarding informal opportunities to interact would suggest that the zone of enactment for both teachers and their colleagues is very different. As an experienced teacher Tom is able to induct new teachers into the school culture or situation that promotes collaborative discussion. However it would seem that experience of such collaborative culture has not been part of Jodie's school situation and therefore she does not have the same level of appreciation for discussions as Tom.

Both teachers have formal opportunities to interact with colleagues.

Tom: ...our HOD for Science, Math, IT and Business is the person that shares that meeting. We present the planning that we have, any assessment items, check the assessment calendar and how that fits in with what we are doing with our kids at the moment. We are doing a lot of analysis of students and building up class profiles with those students and where they sit within the group, how their academic results are and our response to that as teachers so one of the documents that we will be developing for the next meeting is each class doing a brief to the HOD on our students.

In addition to these formal monthly meetings Tom is also able to work with another mathematics teachers during tutorial sessions that are formally timetabled by the school and are designed to work with targeted groups of students. He explained that there were tutorials for students needing extension, support and motivation. Tom worked with the students needing support and he often joined with the tutorial group organised to provide motivation for learning of mathematics.

Tom: There are a couple of ways with these tutor support groups. There is an opportunity for me to get in and actually work with other teachers in these tutorial groups ... I guess lots of ideas, lots of how did you go, how did you do this in your group or I have tried this, what did you try, how did that work. So I do lots of reflective conversations about what I am doing and what they are doing to try to see, you know, benchmark in what they are doing. And some of the "I can do it differently to how they did it". Dealing with the one on one when I am in the same class at the same time and one of us dominates the teaching ...and the other one is supporting. So we get an opportunity to be in the driving seat and the others are watching and then when the other person is in the driving seat you watch so you kind of analyse on the spot what they are doing differently and how it is working and watching the students. How they are doing that, how they are engaging differently, whether they are seeing similar things or different things when I am in control of the class. So that has been a really good experience to be able to do that with this group.

Tom's descriptions are typical of those reported in the research literature where teacher collaboration within a school situation promotes thinking about the different ways that children engage with mathematical thinking (Borko, 2004; Cohen & Hill, 2000; Garet et al., 2001).

In contrast, Jodie, in the smaller school, does not seem to have the same opportunities for collaboration. Jodie described one avenue for teachers to seek advice.

[At staff meetings] we regularly do demonstrations or talk to each other about what is going on... Like, they [other teachers] come up in conversations in staff rooms or at our staff meeting. If someone wants some advice they can easily go and see the other teachers.

For Jodie it seems that the only time she would engage in a discussion is when she needs advice. Being an experienced teacher, she does not feel she needs to seek advice. Her perspective is very different from Tom's. Tom considers collegial discussions as valuable learning for the group of mathematics teachers. The depth of discussions indicates that teachers in Tom's school are constantly stimulating each other's zone of enactment.

The size of the school and therefore the number of mathematics teachers may be making a difference to the situation. For Jodie and her colleague, it would seem that they are unable to have the same detailed discussions about their teaching that are possible in a slightly larger rural school where Tom teaches. In addition, it seems that the general discussions that Jodie has with her colleagues do not lend themselves to enhancing the teachers' zone of enactment regarding their teaching of mathematics in the same way that Tom's discussions with his colleagues do. When asked about observations Jodie explained that a formal process operates in the school where each semester she observes one of the other maths teacher's lessons and he observes one of her lessons. She explained:

Yes, I had a lesson observation in one of [other maths teacher] classes at the start of the year and he has been into one of my maths classes too. As an official observation... Yeah, it comes from admin so all teachers need to do it. We are going and observing another teacher and then just give them feedback on their teaching practices. So you know, their behaviour management, their planning for the day, for the lesson. You know. Are they catering for all students? We've actually got a sheet that we fill out.

These different situations provide very different opportunities for teacher observation, feedback and reflection. In Jodie's case the school situation provides opportunities to discuss broad topics but unfortunately it does not provide opportunities that can promote detailed discussion about the planning for teaching and learning of mathematics. It would seem that the teachers do not know how to interact at this level. Therefore the zone of enactment is not stimulated by her or her colleague. The situation at Tom's school provides ample opportunities for these vital conversations to occur. For example Michael said:

We are looking at the tasks for those students with tutorial support twice a week... So there is a bit of a discussion around different ways to come up with a solution to a particular problem... So we do that assessment. We do that with a regular assessment. We look at an assessment after it has been marked to see how they performed and whether we need to take it or change it for next year.

Mathematics specific discussions like the ones described above can stimulate each teacher's zone of enactment and in so doing promote ongoing development of the school situation as well as curriculum leadership among the teachers.

When Jodie was asked about her interactions with her colleagues and whether these interactions have influenced her thinking or planning she responded by saying: "No, not really, I think at this stage in my teaching career I think I have sort of seen it all before." This comment suggests that without opportunities for robust discussions it is easier to become satisfied with a specific teaching approach. While this teaching approach may be the most appropriate for the students in Jodie's class, the situation is not conducive to ongoing improvement of teaching and learning of mathematics within her school.

Conclusion

The results reported in this paper would suggest that because bigger rural and remote towns have a larger number of mathematics teachers they may well support greater collegial collaboration. Certainly in Tom's case he feels very well supported by the situation that is

promoted in his school. He has both formal and informal opportunities to meet with his colleagues and discuss his teaching and to learn from his colleagues. It seems that the informal opportunities are supported by the sharing of a staff room with other mathematics teachers. It would also seem that detailed discussions on the planning for and reflecting on teaching is a feature of this school mathematics team. It is reasonable to argue that this situation is leading to supporting and enhancing Tom's zone of enactment and contributes to his participation patterns in PD sessions. These results certainly support the literature more generally and specifically Millett and Bibby's (2004) theory on stimulating the teachers' zone of enactment.

While large schools might not always have a culture that supports mathematics teachers, very small schools such as Jodie's would almost always provide fewer collaborative learning opportunities. It is thus to be expected that some teachers attending ongoing regional PD sessions would need to be initially supported to engage in collaborative conversations within the PD sessions. The differences in teachers' readiness to collaboratively plan and discuss teaching and learning of mathematics have certainly been present in this project, where only the teachers from some of the larger schools fully engaged (Visnovska & Lamb, 2012 accepted).

Researchers and PD providers in rural and remote regions need to consider that some teachers have opportunities to engage in detailed discussion and are very practiced at this process while other teachers, both new and experienced, may have not had exposure to this level of interaction. It would seem prudent for researchers and PD providers to make an effort to promote professional learning communities beyond the school setting in rural communities and to build a professional learning community of subject area teachers across schools. This may be achieved through the use of resources such as Skype Professional, a low-cost software application that allows for group videoconferencing from different sites.

References

- Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, 93(4), 373-397.
- Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton, AB: CMESG/GCEDM.
- Ball, D. L., & Cohen, D. (1999). Developing practice, developing practitioners: Towards a practice-based theory of professional education. In G. Sykes & L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). San Francisco: Jossey-Bass.
- Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher*, 33(8), 3-15.
- Borko, H., Koellner, K. A., Jacobs, J. K., Roberts, S. A., Baldinger, E., & Risley, R. (2010, May). *Preparing instructional leaders to facilitate mathematics professional development*. Paper presented at the annual meeting of the American Educational Research Association, Denver, CO.
- Carpenter, T. P., Blanton, M. L., Cobb, P., Franke, M., Kaput, J. J., & McClain, K. (2004). Scaling up innovative practices in mathematics and science. Retrieved 2006, from <http://www.wcer.wisc.edu/NCISLA/publications/reports/NCISLAREport1.pdf>
- Cobb, P., McClain, K., Lamberg, T., & Dean, C. (2003). Situating teachers' instructional practices in the institutional setting of the school and school district. *Educational Researcher*, 32(6), 13-24.
- Cohen, D. K., & Hill, H. C. (2000). Instructional policy and classroom performance: The mathematics reform in California. *Teachers College Record*, 102, 294-343.
- Dean, C. (2005). *An analysis of the emergence and concurrent learning of a professional teaching community*. Unpublished dissertation, Vanderbilt University, Nashville, TN.
- diSessa, A. A., & Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *Journal of the Learning Sciences*, 13(1), 77-103.

- Gamoran, A., Anderson, C. W., Quiroz, P. A., Secada, W. G., Williams, T., & Ashman, S. (2003). *Transforming teaching in math and science: How schools and districts can support change*. New York: Teachers College.
- Garet, M., S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal*, 38, 915-945.
- Gresalfi, M. S. & Cobb, P. (2011). Negotiating identities for mathematics teaching in the context of professional development. *Journal for Research In Mathematics Education*, 42(3), 270-304.
- Grossman, P., Wineburg, S., & Woolworth, S. (2001). Toward a theory of teacher community. *The Teachers College Record*, 103(6), 942-1012.
- Gueudet, G., & Trouche, L. (2012). Communities, documents and professional geneses: Interrelated stories. In G. Gueudet, B. Pepin & L. Trouche (Eds.), *From text to 'lived' resources: Mathematics curriculum materials and teacher development* (pp. 305-322). Dordrecht: Springer.
- Heirdsfield, A., Lamb, J., & Spry, G. (2010). Leading learning within a PLC: Implementing new mathematics content. *The Montana Mathematics Enthusiast*, 17(1), 93-112.
- Hiebert, J., & Grouws, D. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371-404). Reston, VA: NCTM.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.
- Little, J. W. (2004). *Organizational learning and leadership: The practice(s) of teachers "working together"*. Paper presented at the Investigating the practice of school improvement: Theory, methodology, and relevance, Bellagio, Italy.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum
- Millett, A., & Bibby, T. (2004). The context for change. In A. Millett, M. Brown & M. Askew (Eds.), *Primary mathematics and the developing professional* (pp. 1-17). Netherlands: Kluwer Academic Publishers.
- Millett, A., Brown, M., & Askew, M. (2004). Drawing conclusions. In A. Millett, M. Brown & M. Askew (Eds.), *Primary mathematics and the developing professional* (pp. 245-255). Netherlands: Kluwer Academic Publishers.
- Smith, M. S. (2001). *Practice-based professional development for teachers of mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Spillane, J. (1999). External reform initiatives and teachers' efforts to reconstruct their practice: The mediating role of teachers' zones of enactment. *Journal of Curriculum Studies*, 31(2), 143-175.
- Stake, R. (2005). Qualitative case studies. In N. Denzin & Y. Lincoln (Eds.), *The sage handbook of qualitative research* (Third ed.). Thousand Oaks, California: Sage Publications, Inc.
- Stein, M. K., Silver, E. A., & Smith, M. S. (1998). Mathematics reform and teacher development: A community of practice perspective. In J. G. Greeno & S. V. Goldman (Eds.), *Thinking practices in mathematics and science learning* (pp. 17-52). Mahwah, NJ: Lawrence Erlbaum.
- Visnovska, J., Cobb, P., & Dean, C. (2012). Mathematics teachers as instructional designers: What does it take? In G. Gueudet, B. Pepin & L. Trouche (Eds.), *From text to 'lived' resources: Mathematics curriculum materials and teacher development* (pp. 323-341). Dordrecht: Springer.
- Visnovska, J., & Lamb, J. (accepted). *Planning for building models of situations: What is involved?* Paper presented at the International Congress of Mathematics Education, Seoul, South Korea.
- Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge: Cambridge University.

K-2 *Make it Count* Students' Views of Mathematics¹

Gilah Leder
Monash University
<Gilah.Leder@monash.edu>

Helen Forgasz
Monash University
<Helen.Forgasz@monash.edu>

Data from an attitude survey administered to students in grades K–2 from four schools participating in the *Make it Count* project are reported in this paper. Few differences were found in the attitudes and beliefs of the Indigenous and non-Indigenous participants. The relevance of these findings for students' longer term mathematics performance is also considered.

Introduction and Background Information

Reports that Australian Indigenous² students, on average, perform well below their non-Indigenous peers on traditional measures of achievement are prevalent. “At both Year 4 and Year 8 in TIMSS³ 2007, non-Indigenous students scored at a substantially higher level than Indigenous students – 91 score points at Year 4 and 70 score points at Year 8” (Thomson, 2010, p. 79). Examination of the Australian 2009 Programme for International Student Assessment [PISA] results also reveals a substantial difference between the average performance of Indigenous and non-Indigenous students on the mathematical literacy assessment component (Thomson, De Bortoli, Nicholas, Hillman, & Buckley (2011, p. 189).

An excerpt from a report published under the auspices of the Queensland government draws on data from the National Assessment Program Literacy and Numeracy [NAPLAN] tests to compare the performance of Indigenous and non-Indigenous students. Although the distribution of Indigenous students varies by state, the Queensland data offer a useful guide to the national situation.

An analysis of the 2008 NAPLAN data indicates that there is a gap at Year 3. The gap for reading on average is 62 and for numeracy it is 54. The gap persists through all year levels. ... The difference in mean scale scores between Indigenous and non-Indigenous students approximates to two years of schooling for reading (Years 5 and 7; Years 7 and 9) and for numeracy (Years 7 and 9). (Department of Education and Training, Queensland government, n.d.)

Inspection of other NAPLAN data, for example performance on items common to the 2009 NAPLAN Year 3 and Year 5 tests, yields a similar picture. Australian Aboriginal or Torres Strait Islander students in Year 5 performed, on average, at approximately the level of the national Australian average for students in Year 3 for six of the 10 items (and slightly better than, but nevertheless well below, the national Australian average for students in grade 3 for the remaining four items).

Presenting performance differences between selected groups of students – evocatively termed ‘gap gazing’ (Gutiérrez, 2012) – can be counter-productive and reinforce stereotyping. “In its most simplistic form, this approach points out there is a problem but fails to offer a solution.... (T)hat it is the analytic lens itself that is the problem, not just the

¹ We gratefully acknowledge the support of Christine Payne and her colleagues for their assistance in the collection of these data

² In this context, “Indigenous” refers to students who identify as either Australian Aboriginal or Torres Strait Islanders

³ Trends in International Mathematics and Science Study

absence of a proposed solution” (Gutiérrez, 2012, p. 31) might be a further consideration. The latter point is addressed in some detail by Jorgensen and Perso (2012) who argue:

the assessed curriculum through NAP (the National Assessment Program of which NAPLAN is an essential part) is not equitable for all Australian students. This type of assessment can privilege select groups of student whilst marginalizing others ... largely due to the fact that these tests require literacy in the dominant language and consequently are linguistically biased in spite of the best efforts of producers to ensure otherwise. (p. 119)

In the Australian context, multiple explanations have been proposed for the reported differences in performance between Indigenous and non-Indigenous student groups (e.g., De Bortoli & Thomson, 2010; Jorgensen & Perso, 2012). Contextual factors regularly invoked to explain between, and indeed within, group differences and cited as possible target areas for fruitful interventions commonly include the home and school learning environment, students’ learning strategies and preferences, and affective factors such as students’ attitudes and beliefs. With respect to the last set, De Bortoli and Thomson (2010) reported that “Indigenous students who believe in their academic ability and who can confidently complete mathematics tasks are more likely to perform at a higher level on the mathematical literacy scale” (p. 88). However, according to Hughes and Hughes (2010), “Indigenous students themselves often have confused expectations.... Even in mainstream schools, differences in expectations of Indigenous attendance and performance are picked up by the students” (p. 17). Low expectations, they and many others (e.g., Sarra, 2011) argue, can be self-fulfilling and lead to low educational outcomes.

The Study

We report data from an attitude survey administered to students in grades K⁴–2 from schools participating in the *Make it Count* [MiC] project (Make it Count⁵, n.d.). The instrument administered was designed to explore whether or not differences are found in the attitudes to mathematics of young Indigenous students and their non-Indigenous classmates. Given the age of the sample, only a limited number of items were included in the survey instrument. Relevant information was read out to the students, additional teacher support to complete the survey was given as needed, but all written responses were made by the students themselves. Core components of the survey are described later, as part of the presentation of results.

Aims

For the K-2 students in the study, the specific questions listed below capture the aims:

- What are the views of Indigenous students about mathematics and related issues?
- Are there differences in the attitudes of Indigenous and non-Indigenous students?
- Can any inferences be drawn from the attitudes of Indigenous students about mathematics and related issues and their performance in mathematics?

⁴ Kinder. The term Prep is used instead in some Australian states

⁵ *Make It Count* is a project of The Australian Association of Mathematics Teachers (AAMT) Inc., funded by the Australian Government as part of the *Closing the Gap* initiative.

Theoretical Model Informing the Study

Reference has already been made to factors found to influence mathematics learning outcomes for students – both Indigenous and non-Indigenous. These comprise both personal and environmental factors. Representative of the set typically considered important is Rothman and McMillan's (2003) listing in their longitudinal survey of Australian Youth: "variables believed to be important influences on achievement in literacy and numeracy ... (include) variables relating to students (gender, Indigenous background, language background, home location), their parents (education level, occupation, birthplace), attitudes toward school, aspirations and self-concept" (p. v). The elements included mirror those found in models of gender differences in mathematics learning (see, e.g., Leder, 1992; Wigfield & Eccles, 2000).

The Sample

The sample comprised 321 students in grades K-2 at four schools. Of these, 89 (46 boys and 43 girls) were Indigenous and 205 (95 boys and 110 girls) were non-Indigenous. Indigeneity and Grade level were not recorded for, respectively, 26 and three students – leaving a useable pool of 292 students. Sample details, by grade, are shown in Table 1.

Table 1

<i>K-2 Students Who Completed an Attitude/Beliefs-about -Mathematics Survey.</i>				
Grade	Kinder	Grade 1	Grade 2	Total
Indigenous	31 (30%)	28 (33%)	29 (28%)	88 (30%)
Non Indigenous	72 (70%)	56 (67%)	76 (72%)	204 (70%)
Total	103	84	105	292

The participating schools were involved in *Make it Count* [MiC], a "four-year project to develop an evidence base of practices that improve Indigenous students' learning in mathematics and numeracy" (Make it Count, n.d). Explicit reference is made on the website of each of these schools to special programs available to Indigenous students and their parents. For example: "our school is committed to working with Aboriginal parents and community members in developing personalised plans for our Aboriginal students"⁶. To ensure that approximately one-third of the sample comprised Indigenous students, only a representative selection of the schools' non-Indigenous K-2 students were involved.

The Instrument

The survey was administered towards the end of the 2011 school year and comprised both multiple choice and open-ended items. Information sought from the students included their names, whether they were male or female, whether they were Indigenous, and whether they were in Kinder, Grade 1 or Grade 2. Core items covered: students' self report of their mathematics achievement; liking of mathematics – their own and that of significant "others" in their environment; liking of reading; perceptions about mathematics; its long term usefulness; and perceptions of mathematics lessons. Students' expectations of being able to solve two mathematics problems were also sought.

"Colour in the face that shows how you feel" was the response format for the multiple choice items. Five faces were shown ranging from smiling broadly to frowning deeply.

⁶ Reference not provided, to retain the school's anonymity

Responses were coded from 1-5 with 1 = deeply frowning face to 5 = broadly smiling face. Because of the limited group sizes, particularly of the Indigenous sample, responses were recoded into three groups comprising categories which represented responses such as: like/like very much; neutral; dislike/dislike very much. The content of the survey can be inferred from the presentation of the results. As already indicated, some of the students completed the survey with teacher assistance.

Analyses

The frequency distributions of the responses to the items were examined for the full sample, by Indigeneity and, when appropriate, by grade level. Pearson chi-square tests were conducted to examine possible differences in group responses. Effect sizes (ϕ) for the statistically significant differences were also calculated. Because of the relatively small sample sizes of the various subgroups, differences found are often most appropriately interpreted as indicative of a trend.

Results

To conform with paper length constraints, reporting of results is limited to selected but representative items. The results are presented under two headings: beliefs about mathematics, and performance in mathematics. The data are presented for the full sample, unless the content of the question makes inclusion of the Kinder group inappropriate.

Beliefs about Mathematics

Liking of reading and mathematics – “this year”

The students’ responses to this question are shown separately for Indigenous and non-Indigenous students in Tables 2 and 3 respectively.

As can be seen in Tables 2 and 3, the majority of students, both Indigenous and non-Indigenous, liked mathematics and reading. The high proportions of Grade 2 students (Indigenous: 93%; non-Indigenous: 87%; overall: 89%) who indicated that they liked mathematics or liked it very much is noteworthy. The differences by grade level in the liking of mathematics were statistically significant for the (larger sample of) non-Indigenous students ($\chi^2_4 = 14.33$, $p < .01$, $\phi = .27$), but not for the Indigenous students. The differences in the liking of reading by grade level were not statistically significant for the group as a whole ($\chi^2_4 = 8.98$, $p = .06$), nor for Indigenous or non-Indigenous students.

Table 2

Indigenous Students (in %): Liking of Reading and Mathematics by Grade Level

Rating	Activity	Mathematics			Reading		
		Kinder	Grade 1	Grade 2	Kinder	Grade 1	Grade 2
Like very much/ like		74%	75%	93%	74%	96%	83%
Neutral		7%	11%	3%	16%	4%	7%
Dislike/ Dislike very much		19%	14%	3%	10%	–	10%
Total Number of Students		31	28	29	31	27	29

Table 3

Non-Indigenous Students (in %): Liking of Reading and Mathematics by Grade Level

Rating	Activity	Mathematics			Reading		
		Kinder	Grade 1	Grade 2	Kinder	Grade 1	Grade 2
Like very much/ like		68%	90%	87%	74%	88%	86%
Neutral		10%	7%	4%	14%	5%	7%
Dislike/ Dislike very much		23%	4%	9%	13%	7%	8%
Total Number of Students		71	56	76	72	56	76

How much does your best friend like mathematics?

There were no appreciable differences in the responses of the two groups to this item, with about 80% of students in Grades 1 and 2 in both groups indicating that their best friend liked mathematics, or liked it a lot.

Circle the words (12 words are shown) which best show how you feel when you are doing mathematics

There were no obvious differences in the responses of the two groups – Indigenous and non-Indigenous students. In both groups a higher proportion circled more positive than negative words. For example, just over 70% and 60% of students in both groups circled “happy” and “clever” respectively, compared with just over 10% and around 20% who circled, respectively, “don’t care” and “bored”. Words about which the two groups differed included “don’t understand” and “excited”, with a higher proportion of Indigenous students circling both these words: negative and positive descriptors respectively.

Is mathematics important for grown-ups?

The differences in the rating by grade level of the importance of mathematics for adults for the sample as a whole was not statistically significant ($\chi^2_4 = 9.02$, $p = .06$). About two-thirds of both groups considered mathematics to be important for adults (Indigenous group: 65%; non-Indigenous group: 67%). Beliefs about the importance of mathematics for adults seemed to decrease with grade level for the Indigenous but not for the non-Indigenous group – see Table 4. At Grade 2, a relatively high proportion (one-third) of Indigenous students seemed uncertain about the long term importance of mathematics. Differences in the rating by grade level were statistically significant for the non-Indigenous group ($\chi^2_4 = 13.57$, $p < .01$, $\phi = .26$) but not for the smaller group of Indigenous students.

Table 4

The Importance of Mathematics for Adults (entries in %)

Is mathematics important for grown-ups?	Indigenous			Non-Indigenous		
	Grade	K	Gr 1	Gr 2	K	Gr 1
Yes	71%	69%	55%	67%	60%	71%
No	13%	19%	10%	22%	7%	9%
Don't know	16%	12%	35%	12%	33%	20%
Total number of students	31	26	29	69	55	76

How will you use mathematics when you are grown up?

The responses from both groups were fairly limited. This was understandable given their age. As noted in the frequency data on the children’s views of the importance of mathematics for adults, there also appeared to be a difference in the responses from the

Indigenous and non-Indigenous students in grade 2. A few students were unsure (2 Indigenous, 6 non-Indigenous) how they would use mathematics when they grew up; two male Indigenous students said they would not use mathematics (e.g., “No, no”). Both groups identified work as one place they would use mathematics (4 Indigenous, 7 non-Indigenous). Sample responses:

- To work out my money if I’m a shopkeeper (Indigenous, male)
- To figure out how much stuff to sell (non-Indigenous, female)
- If I work in an office I might use it a lot (non-Indigenous, male)

In general, however, the non-Indigenous students identified a wider range of ways that they might use mathematics when they grew up. Money was a common theme, as was for further study, and for counting/calculating or measurement. Examples included:

- To help me do my homework (male)
- Count money, count, calculate numbers (male)
- To learn more stuff and be very smart (female)
- Buy food, pay bills, to know the clock time (female)

Other representative responses from Indigenous students included:

- Do it on your own (male)
- Sometimes (female)
- When I’m bored (female)

Beliefs about Mathematics Performance

How good were you at mathematics last year?

For this item, the responses from the students in Kinder are not included. Of the remaining students, those in Grades 1 and 2, the majority thought they were excellent or good at mathematics (Indigenous sample: 84%; non-Indigenous sample: 82%). Few students considered themselves to have been below average or weak: 2% of Indigenous students and 6% of non-Indigenous students. There were no statistically significant differences by Indigeneity or by grade level.

How sure are you that you can do this question? You do not have to work out the answer.

Two questions were shown: the first would require missing numbers to be filled in; the second would require students to draw the same number of spots on the second wing of a butterfly and “write the number facts to match”).

Again, the responses to these items from the students in Kinder were excluded. No appreciable differences in the responses of Indigenous and non-Indigenous students were found. A high proportion (>90%) of both groups indicated that they could do the first and easier problem; a slightly lower proportion (≈80%) that they could do the second problem.

Tell us what happens when you are doing mathematics in school.

The students were asked to describe what happens when they are doing mathematics in school. It was clearly evident that they engage in a broad range of activities at all grade levels. They wrote about: counting activities; playing games; using blocks, playdough, the whiteboard, worksheets, and books; cutting, pasting, and tracing; painting and drawing, reading and writing; and activities involving the four operations. Indications of enjoyment accompanied several of the descriptions. There was no discernible difference in the descriptions proffered by Indigenous and non-Indigenous students.

Summary of Findings for Attitudes and Beliefs about Mathematics

All of the K-2 students from whom data were gathered attended schools involved in the *Make It Count* intervention program in 2011. Positive attitudes to mathematics were evident, particularly among the grade 2 students. Neither the quantitative nor the qualitative data revealed any clear patterns of response differences among Indigenous or non-Indigenous students. However, among the grade 2 students there were some subtle differences, in both the quantitative and qualitative items, about the role that mathematics was thought to play for adults. It seemed that Indigenous students were not as aware as non-Indigenous students of the variety of ways mathematics might impact on adult life.

Relevance to Mathematics Performance: A Glimpse into the Future?

Practical considerations prevented an achievement test being administered to the students. Whether the similarity in attitudes and perceptions about mathematics of the Indigenous and non-Indigenous groups will be accompanied by parity in the groups' scores in the 2012 Year 3 NAPLAN numeracy test deserves close attention. Until then, currently available Year 3 NAPLAN scores from the participating schools serve as a crude measure of the numeracy performance of students closest in age (and school experience) to those involved in the survey. Relevant data for 2008-2011 are shown in Table 5.

Table 5

Year 3 NAPLAN Mean Numeracy Scores for Four Schools, 2008-2011

School A		School B		School C ¹		School D	
Mean	Difference ²	Mean	Difference	Mean	Difference	Mean	Difference
2008 (National mean = 397)							
348	49	311	86	364	33	382	15
2009 (National mean = 394)							
363	31	299	95	342	52	372	22
2010 (National mean = 395)							
343	52	310	85	359	36	382	13
2011 (National mean = 398)							
363	35	349	49	329	69	413	-15

¹ This school participated in the MiC program for the first time in 2011

² Mean national NAPLAN score – mean school NAPLAN score

The data in Table 5 reveal that at three of the schools (A, B, & D), the gap between the Year 3 students' mean NAPLAN score and the national mean score was less in 2011 than in 2008 and 2010. The students in Year 3 at these schools had participated in the MiC program in previous year(s). The NAPLAN achievement data do not distinguish between the performance of Indigenous and non-Indigenous students. Thus, we cannot tell whether the improvement in performance is greater for one or other of these groups. School C joined the MiC program in 2011 and does not yet have a sustained period of MiC linked activities. Its NAPLAN scores do not reflect the pattern of improvement shown for the other three schools. Students who were in Year 3 in 2011 had not been exposed to the MiC program. Although NAPLAN data are only one (partial) measure of achievement and participation in

the MiC program may not fully explain the improvement in the NAPLAN scores for schools A, B, and D, the changes in the scores are, nonetheless, noteworthy.

Final Comment

An important, and positive, finding from the present study was that few differences were found in the attitudes and beliefs of the Indigenous and non-Indigenous K-2 students from the four schools participating in the MiC project. The perceived applicability of mathematics in the world of adults was, however, a provocative exception – a dimension worthy of further focus. The similarities are particularly notable given that previous research findings reveal that there is a positive correlation between attitudes/beliefs and achievement, and that national NAPLAN data reveal a large gap between the performance of Indigenous and non-Indigenous students at each grade level. The aim of the MiC is to find evidence of teaching and learning strategies that work with Indigenous students. Data from the present study suggests that the work done in the participating schools may be well on the way to meeting this aim. It will indeed be interesting to see if the 2012 NAPLAN data for the grade 3 students from these schools – in grade 2 at the time the data reported in this paper were gathered – will reflect the positive indicators noted in the present study.

References

- Department of Education and Training, Queensland government (n.d.). *Closing the gap. Education Strategy*. Retrieved from <http://education.qld.gov.au/schools/indigenous/pdfs/closing-gap.pdf>
- De Bortoli, L., & Thomson, S. (2010). *Contextual factors that influence the achievement of Australia's Indigenous students: Results from PISA 2000–2006*. Camberwell, Australia: Australian Council for Educational Research.
- Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in discourse for mathematics education. Theories, practices, and policies* (pp. 17-33). Dordrecht, The Netherlands: Springer.
- Hughes, H., & Hughes, M. (2010). *Indigenous Education 2010. CIS Policy Monograph 110*. Retrieved from <http://www.cis.org.au/images/stories/policy-monographs/pm-110.pdf>
- Jorgensen, R., & Perso, T. (2012). Equity and the Australian curriculum: mathematics. In B. Atweh, M. Goos, R. Jorgensen, & D. Siemon, (Eds.), *Engaging the Australian national curriculum: Mathematics – perspectives from the field* (pp. 115-133). Online Publication: Mathematics Education Research Group of Australasia. Retrieved from <http://www.merga.net.au/sites/default/files/editor/books/1/Chapter%206%20Jorgensen.pdf>
- Leder, G. C. (1992). Mathematics and gender: Changing perspectives. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 597-622). New York: Macmillan.
- Make it count (n.d.) *Numeracy, mathematics and Indigenous learners (website)*: <http://makeitcount.aamt.edu.au/>
- Rothman, S., & McMillan, J (2003). *Influences on achievement in literacy and numeracy. Research report 36*. Camberwell, Australia: Australian Council for Educational Research. Retrieved from www.acer.edu.au/documents/LSAY_lsay36.pdf
- Sarra, C. (2011). *Time for a high expectations relationship between Indigenous and non-Indigenous Australia*. Keynote address presented at the *Strong Start Bright Futures Conference*, Darwin. Retrieved from <http://chrissarra.wordpress.com/2011/10/19/time-for-a-high-expectations-relationship-between-indigenous-and-non-indigenous-australia/>
- Thomson, S. (2010). *Mathematics learning: What TIMSS and PISA can tell us about what counts for all Australian students*. Paper presented at the ACER Research conference: Teaching mathematics? Make it count. Retrieved from http://research.acer.edu.au/research_conference/RC2010/17august/6/
- Thomson, S., De Bortoli, L., Nicholas, M., Hillman, K., & Buckley, S. (2011). *Challenges for Australian education: Results from PISA 2009*. Retrieved from <http://www.acer.edu.au/documents/PISA-2009-Report.pdf>
- Wigfield, A., & Eccles, J. S. (2000). Expectancy–value theory of achievement motivation. *Contemporary Educational Psychology*, 25, 68–81.

Variation and Mathematics Pedagogy

Allen Leung

Hong Kong Baptist University

<aylleung@hkbu.edu.hk>

This discussion paper put forwards variation as a theme to structure mathematical experience and mathematics pedagogy. Patterns of variation from Marton's Theory of Variation are understood and developed as types of variation interaction that enhance mathematical understanding. An idea of a discernment unit comprising mutually supporting variation interactions is proposed and used as a building block for a mathematics pedagogy that resembles a developmental sequence from coarse idea to precise definition. Classification of plane figures is used as a pedagogical example for illustration.

Mathematical Experiences and Variation

What is mathematics teaching and learning? I take the perspective that teaching and learning of mathematics is about providing learners opportunities to experience mathematics and to create (new) mathematical experiences. How does a learner experience a mathematical “thing”? For examples, memorize a formula, execute an algorithm, write down a string of mathematical symbols, prove a proposition, recognize a pattern, etc. Are all these indications that a learner has experienced mathematics? These activities are no doubt related to mathematics, but are they critical enough to ensure the occurrence of a genuine mathematical experience? To address this question, I may ask the metaphysical question of what mathematics is, but such a discussion would diverse away from pedagogy. Rather, I take a more pragmatic path to ask how to empower a learner to experience mathematics.

Consider an experience as a bridge to connect a person to an observable phenomenon (directly or indirectly), then concerning pedagogy, this would mean how to make an epistemic connection between a learner and an object of learning (the “thing” to be learnt). In the mathematics classroom, this connectedness could be achieved, for example, through dynamic feedback processes that occur between a learner and an object of learning under a designed usage of tools like a ruler, grid paper, or even sophisticated ICT teaching and learning environments. In practice, tools or manipulatives are potential mediators for mathematical experience (cf. Maschietto & Trouche, 2010).

Resnick (1997) advocated that mathematics is a “science of pattern”. Pattern can be interpreted as an emerging invariant structure when a phenomenon is undergoing changes or *variation*. Leung (2010) described mathematical experience as “the discernment of invariant pattern concerning numbers and/or shapes and the re-production or re-presentation of that pattern.” Variation is about what changes, what stays constant and what the underlying rule is.

“To become aware of what is constant in the flux of nature and life is the first step in abstract thinking The conception of constancy in change provides the first guarantee of meaningful actions.” (Wilhelm., 1973, p.23)

In phenomenology, one of the hermeneutic rules is “seek out structural or invariant features of the phenomena”, furthermore,

“The probing activity of investigation is called *variational method*... Variations “possibilize”

phenomena. Variations thus are devices that seek the invariants in variants and also seek to determine the limits of a phenomenon.” (Ihde, 1986, pp. 39-40)

Discernment, variation and simultaneity are the central concepts in the phenomenographic research approach in which learning and awareness are interpreted under a theoretical framework of variation (Marton & Booth, 1997; Marton, Runesson, & Tsui, 2004). The basic ideas behind this approach can be captured in a nutshell by the following:

“As we always act in relation to situations as we see them, effective actions spring from effective ways of seeing. Seeing a situation in a certain way amounts to discerning those aspects which are critical for engaging in effective action and taking all of them into consideration (focusing on them) at the same time. In order to discern a certain aspect, one must have experienced variation in those aspects. There is no discernment without variation. The only way we can prepare for the un-definable variation in the future is by experiencing variation in the present and by having experienced variation in the past.” (The abstract of a seminar given by Ference Marton at The University of Hong Kong, 21 Nov 2006)

According to Marton’s Theory of Variation, discernment of critical features occurs under systematic interaction between a learner and the thing to be learnt, and variation is the agent that generates such interaction (Marton, Runesson, & Tsui, 2004). Local variation in different aspects of a phenomenon unveils the invariant structure of the whole phenomenon. Invariants are critical features that define or generalize a phenomenon. This matches nicely with what doing mathematics is about, for a major aim of mathematical activity is to separate out invariant patterns while different mathematical entities are varying, and subsequently to generalize, classify, categorize, symbolize, axiomatize and operationalize these patterns.

Dienes (1963) attributed the abstraction and the generalization processes in mathematical thinking by what he called the perceptual variability principle and the mathematical variability principle:

“The perceptual variability principle stated that to abstract a mathematical structure effectively, one must meet it in a number of different situations to perceive its purely structural properties. The mathematical variability principle stated that as every mathematical concept involved essential variables, *all* these mathematical variables need to be varied if the full generality of the mathematical concept is to be achieved.” (Dienes, 1963, p.158)

Sieving out invariants using variation is thus an essence of experiencing mathematics.

A mathematics pedagogy that is rooted in variation is one that purposefully provides students with means to experience variation through, for example, concrete tools, multiple representation of a concept, strategically designed exercise, etc. in order to create a *mathematically rich learning environment* (Leung, 2010) that allows students to discern invariant. The root of variation is to observe relationships produced between the parts and the whole while focus of attention changes, and to discover what stays unchanged among these relationships. Mathematically speaking, this is liken to discern relationship among contextual variables; that is, possible aspects that can vary in a mathematical situation. Pedagogically, this means the teacher needs to decide when to focus on which variable(s) in order to bring about the intended learning outcome. To probe into how to create variation interactions that foster discernment, I will discuss types of possible variation interaction and using them to develop a model of mathematics pedagogy based on variation that reflects a process of mathematics knowledge acquisition.

Types of Variation Interaction

In Marton' Theory of Variation, four patterns of variation were proposed: contrast, generalization, separation and fusion. They form the kernel for discernment under variation. Leung (2008) used these patterns to develop a lens of variation to interpret explorations in dynamic geometry environments. These patterns of variation are fundamental elements used to organize a variation experience and they generate interactions between learners and the object of learning. In this paper, rather than patterns, I consider them as types of *variation interaction* that could build up mathematical experience.

A variation interaction is a strategic use of variation to interact with a mathematics learning environment in order to bring about discernment of mathematical structure.

This strategic use can be teacher-designed by or learner-initiated. They are understood as follows.

Contrast is to discern whether something satisfies a certain condition or not, that is, whether something “is” or “isn’t”. Thus contrast seeks to distinguish different and unlike things. On the one hand, to comprehend a mathematical idea, one often resorts to finding counter-examples in order to discern the critical features of the idea. On the other hand, a mathematical concept can be represented in multiple ways, by contrasting them; one can seek to discern traces of invariant features behind the concept. A major activity in doing mathematics is classification which aims to arrive at invariant concepts and contrast is a classification activity.

Separation is the *awareness of critical features and/or dimensions of variation*. A dimension of variation is an *emerging* feature of a phenomenon which can take on different “values” while some aspects of the phenomenon are varying; it may or may not be an invariant feature depending on the focus of attention and what are varying. For example, when analyzing the nature of sunlight using suitable dispersive prism, *color* can become a dimension of variation which takes on values “red”, “green”, “blue”, etc with the orientation of a prism as a varying aspect. In particular, when an aspect of a phenomenon is varying while the other aspects are being kept fixed, critical features of this aspect of the phenomenon may be discerned and a dimension of variation may be separated out. Mathematically, it is liken to conceiving variables in a physical phenomenon, or to seeing the information encoded in the cross-sections of a solid shape while the cutting plane is varying in a systematic way. Separation is an awareness of part-whole relationship awakened by a systematic refined contrast obtained by purposely varying or not varying certain aspects aiming to differentiate the invariant parts from a whole.

Generalization is a variation interaction that is inductive in nature. When the same invariant pattern appears in different situations under contrast and separation, this pattern may be de-contextualized. Generalization is a purposeful contrast to explore whether an observed pattern can occur while certain aspects vary. It is a verification and conjecture-making activity checking the general validity of a separated out pattern which is often a goal of mathematical exploration.

Fusion integrates critical features or dimensions of variation into a whole under simultaneous co-variation. By fusing the separated-out critical features or dimensions of variation together, a whole concept may appear. It is liken to perceiving the graph of a mathematical function as a representation of a relationship between variable x and variable y . By contrasting critical features and dimensions of variation, fusion sculpts meaning and concept when parts of a whole vary in interconnected ways.

Thus in a nutshell, these types of variation interactions are basically different foci of simultaneous contrast.

Discernment Unit

In a pedagogical situation, these four types of variation interaction act together in a concerted way to bring about discernment. Consider the example of classifying geometrical plane figures as an illustration. A collection of different types of plane figures is given to students. Teacher can design suitable activities asking students to visually sort these figures into groups. The design should then focus on a visual intuitive process of contrast and generalization on the number of sides of the figures, the size and orientation of the figures, and the shapes of the angles, etc. That is, cycles of activity contrasting the different visual features and generalizing by sorting these visual features. These features may become critical features or dimensions of variation, and consequently intuitive types of figures can be separated out and classified. For example, figures with the same number of sides or angles, figures with pairs of equal opposite sides, etc. The activities should be designed in a way such that students are sensitized to becoming aware of critical features of the figures and are given the opportunity of becoming simultaneously aware of the variety of figures (Figure 1).

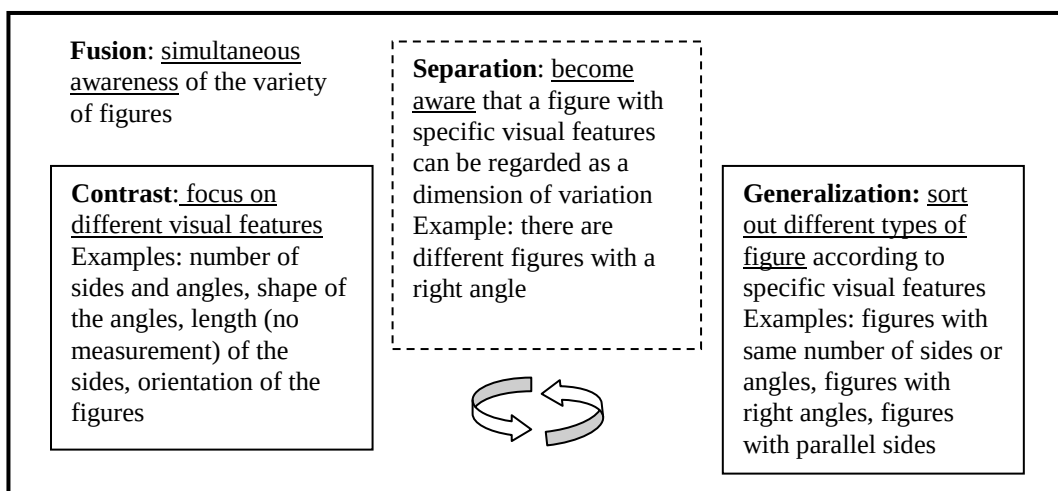


Figure 1. A visual intuitive classification of plane figures utilizing the four types of variation interactions. The circular arrows and the dotted rectangle indicate a mutually enhancing interaction between contrast and generalization is at work to bring about awareness of dimensions of variation and/or critical features

In light of this, I propose an idea of a *discernment unit* which stands for a unit of a pedagogical process driven by these four types of variation interaction. It is a convolution of contrast and generalization driven by separation fused together by simultaneous awareness of critical features (Figure 2). Convolution here means combining the mutually enhancing interaction between contrast and generalization to produce an invariant. It is driven by separation in the sense of becoming aware of different dimensions of variation and/or of critical features via some variation strategies. This discernment unit will be the fundamental building block for a model of mathematics pedagogy based on variation which I will discuss in the next session.

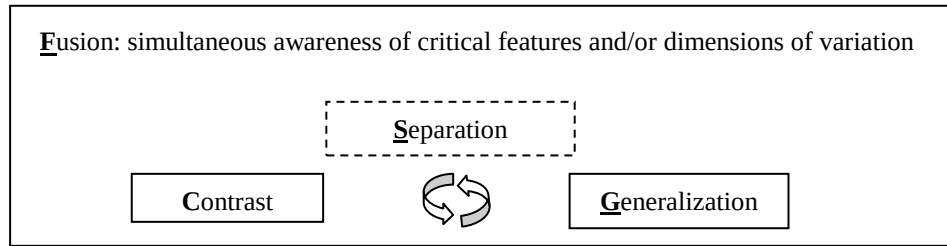


Figure 2. A discernment unit driven by types of variation interaction

Classification of Plane Figures

I have illustrated the use of a discernment unit to frame a mathematical inquiry on visually classifying plane figures. Different focus of attention gives different classification and the types of classification depend on what aspects of a plane figure are being varied. In this sense, a discernment unit is like a function of what's being varied. Let us consider two other possible ways to classify plane figures and see what the discernment unit could generate in these cases. The classification by visual intuition described in the previous section will be labeled as *Discernment Unit One*.

Discernment Unit Two: Classification of Plane Figures by Properties

Students are asked to classify plane figures according to the figures' properties. Students need to separate out these properties through contrast and generalization experiences that involve activities like measuring length of sides and angles, making a list of the figures, and finding (sufficient) conditions to define a figure. These properties can be regarded as dimensions of variation with different figures as values. For example, four right angles is a dimension of variation with squares and rectangles as values, two pairs of opposite equal sides is a dimension of variation with all types of parallelograms as values. The classification is then a fusion of properties that can be used to distinguish different types of figures (Figure 3)

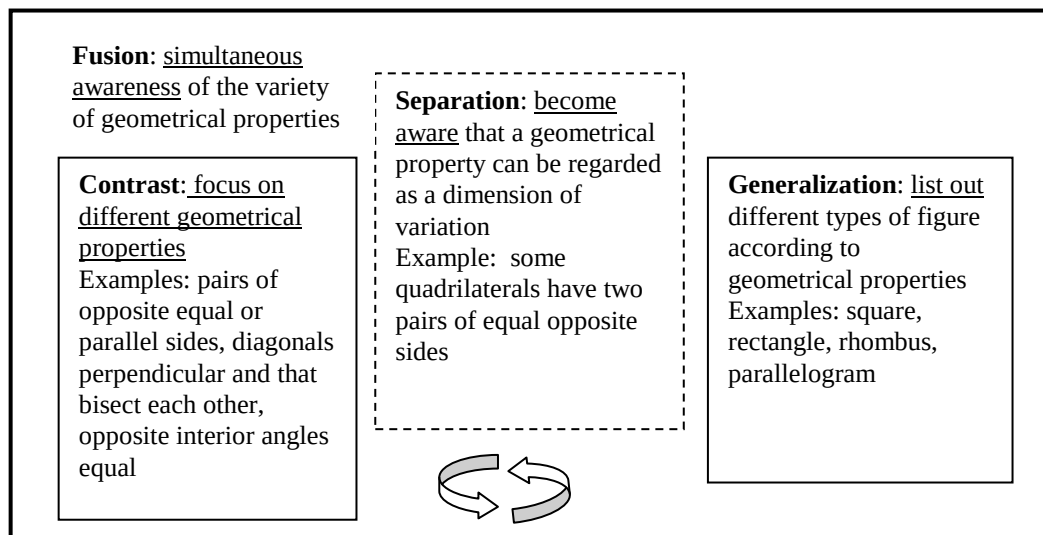


Figure 3. A discernment unit for classification of plane figures by properties

Discernment Box Three: Classification of Plane Figures by Relationships between Properties

Students are expected to use relationships between properties to classify plane figures with activities like looking for counter examples (contrast) to refute the statement “if the diagonals of a quadrilateral are perpendicular to each other, then it must be a square” and investigating the relationship between angles and sides to arrive at the generalization “if a quadrilateral has equal opposite interior angles, then it must be a parallelogram”. The dimensions of variation being separated out in Discernment Unit Two (geometrical properties) now become possible values of a dimension of variation which is a refined concept of a figure characterized by equivalent properties. For example, for a parallelogram, “two pairs of equal opposite sides” is equivalent to “two pairs of parallel opposite sides”. Thus the concept “parallelogram” is formed as a fusion of properties and their relationships to each other (Figure 4).

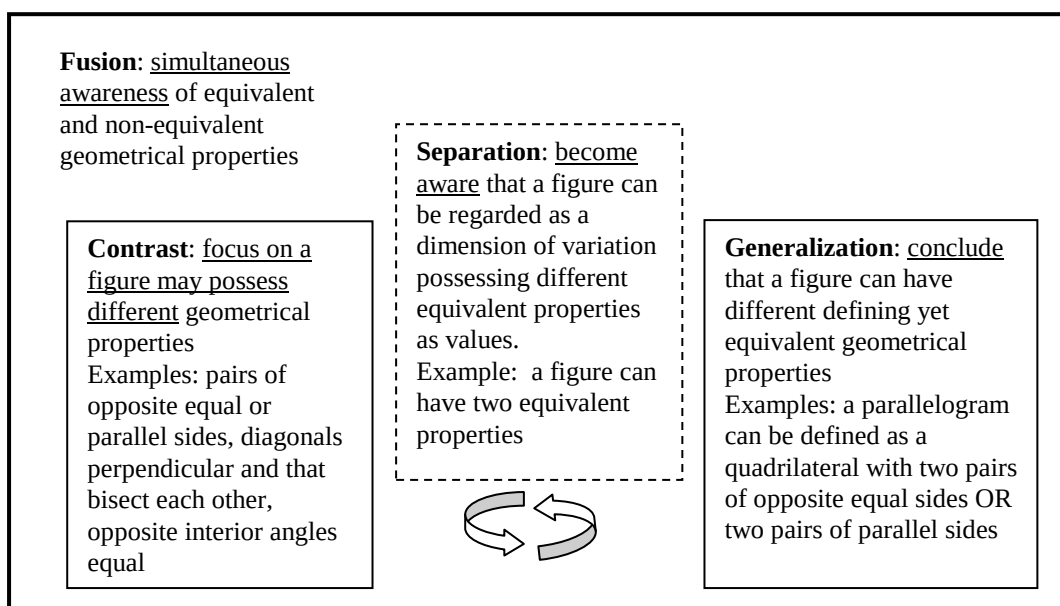


Figure 4. A discernment unit for Classification of plane figures by relationships between properties

A Model of Mathematics Pedagogy Based on Variation

Discernment Units One, Two and Three form a hierarchy of gradual refined concepts of plane figures starting from the intuitive primitive (visual), to the refined (properties) and finally to the fine-grained (relationship between properties). Teachers who take a constructivist's path like guided-reinvention in Freudenthal's mathematisation framework where mathematics knowledge is acquired through a refinement process of doing suitable realistic mathematical activities (Freudenthal, 1973, 1991) can utilize this hierarchy to design a pedagogical sequence on the understanding of plane figures (Figure 5).

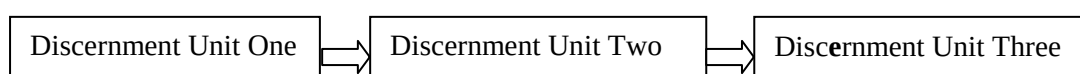


Figure 5. A pedagogical time sequence on the understanding of plane figures

This process of “mathematical understanding” is sequenced by a chain of variation

interactions where simultaneity and focus of attention play critical roles. Notice that the process consists of a sequence of variation interactions that is increasing in sophisticated levels of contrast (visual perception, properties, inter-relating properties) reflecting the evolution of an idea starting from a primitive stage to reaching a more formal mathematical stage. The arrows in Figure 5 refer to *shifts of attention* (cf. Mason, 2008) that may cause a refinement of understanding. The first arrow indicates a shift of focus from visual features to geometrical properties and the second arrow from geometrical properties to a figure, and consequently the related dimension of variation changes from “figure” to “property” and back to “figure” again (see Figures 2, 3 and 4). Aside from synchronic simultaneity when students focus on different aspects of plane figure at the same time, diachronic simultaneity plays the critical role of connection of variation experiences gained in previous and present discernment units. In particular, following such a sequence, learning tasks can be designed capitalizing the variation interactions to develop a mathematical concept progressively from coarse description to precise definition. This would empower students with a rich mathematical experience as discussed in the beginning of the paper. Figure 6 presents such a model of mathematics pedagogy based on variation.

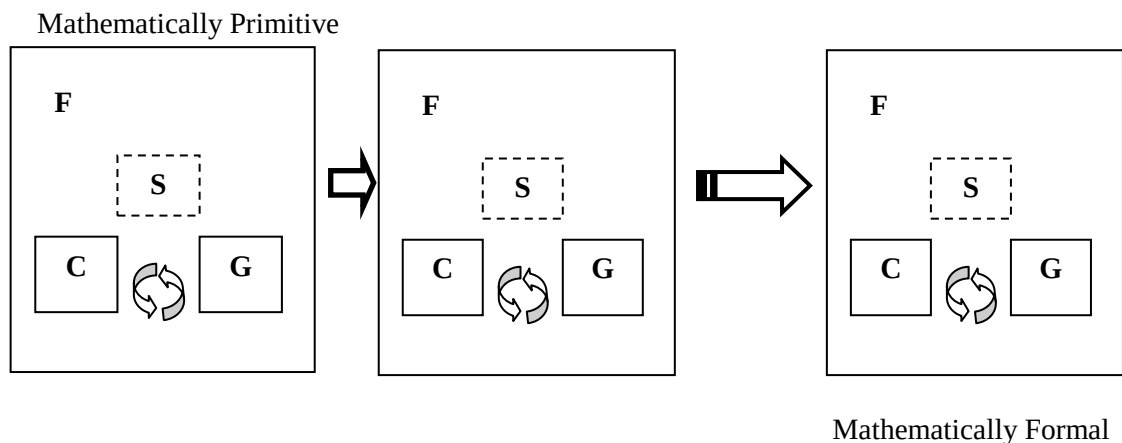


Figure 6. A model of mathematics pedagogy based on variation. **F**(Fusion), **C**(Contrast), **S**(Separation), **G**(Generalization)

Each of the above *discernment units* represents a developing mathematical concept that is fused together by a process of contrast and generalization driven by separation. The sequence represents a process of refinement of the mathematical concept, from primitive to progressively formal and mathematical. This pedagogical model can be seen as a process of doing mathematics and acquiring mathematical knowledge. It is rooted in variation as doing mathematics is basically a variation activity seeking invariant structures in the midst of changes. A learner needs to experience the evolution of a mathematics idea to fully comprehend it, and this pedagogical model can create such an experience.

I hope this discussion opens a window to view variation as a powerful agent to generate mathematical knowledge and stimulates further research and practices using variation as a pedagogical tool in the mathematics classroom.

References

- Dienes, Z. P. (1963). *An experimental study of mathematics-learning*. London: Hutchinson Educational.
- Freudenthal, H. (1991). *Revisiting Mathematics Education*. Dordrecht, the Netherlands: Reidel.
- Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht, the Netherlands: Reidel.

- Leung, A. (2010). Empowering learning with rich mathematical experience: reflections on a primary lesson on area and perimeter, *International Journal for Mathematics Teaching and Learning* [e-Journal]. Retrieved April 1, 2010, from <http://www.cimt.plymouth.ac.uk/journal/leung.pdf>
- Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning*, 13,135-157.
- Marton, F. and Booth, S. (1997). *Learning and Awareness*. New Jersey: Lawrence Erlbaum Associates, INC, Publishers.
- Marton, F., Runesson, U. and Tsui, A.B.M. (2004). The space of learning. In F. Marton & A.B.M. Tsui (Eds), *Classroom Discourse and the Space of Learning* (pp.3-40). New Jersey: Lawrence Erlbaum Associates, INC, Publishers.
- Maschietto, M. & Trouche, L. (2010). Mathematics learning and tools from theoretical, historical and practical points of view: the productive notion of mathematics laboratories. *ZDM - The International Journal on Mathematics Education*, 42, 33-47.
- Mason, J. (2008). Being mathematical with and in front of learners: attention, awareness, and attitude as sources of differences between teacher educators, teachers and learners. In T. Wood and B. Jaworski, *International handbook of mathematics teacher education* (Vol. 4, pp. 31-56). Rotterdam, the Netherlands: Sense Publishers.
- Resnick, M. (1997). *Mathematics as a science of patterns*. Oxford: Clarendon Press.
- Wilhelm, H. (1973.) *Eight Lectures on the I Ching*. Bollingen Series LXII, Princeton University Press.

Supporting Secondary Novices' Efforts to Implement Student- and Discourse-Centred Pedagogical Practices

Gary Lewis
Illinois State University
<glewis@ilstu.edu>

Novice secondary mathematics teachers attempting teaching consonant with NCTM (1991) Professional Standards for Teaching Mathematics experience stresses related to those attempts. Foremost among those stresses are challenges while orchestrating student-centred, whole-class discussions. Such discussions can create uncertainty and stress as novices consider students' mathematical ideas, ask questions to push the class' thinking forward, and choose between multiple competing directions that such discussions may take. Novices' self-reports suggest social resources, particularly teacher learning communities, offer promising support for attempting such teaching.

Because U.S. novice teachers carry the same workload as colleagues, some researchers suspect novices may be too stressed to learn efficiently from their experiences (e.g., Borko & Putnam, 1996). Efforts to enact teaching practices consonant with the NCTM (1991) *Professional standards for teaching mathematics*, or NCTM Standards-based teaching (SBT), may add to those challenges. Because novices are often most familiar with "teaching as telling," efforts to set "telling" aside to attempt SBT may increase novices' feelings of uncertainty (Smith, 1996) and heighten their stresses. The literature is peppered with examples of novices struggling to enact SBT (e.g., Van Zoest & Bohl, 2000) often due to a lack of resources (e.g., Wilcox, Schram, Lappan, & Lanier, 1991). McLaughlin (1993) identified teacher learning communities as a prerequisite to successfully reform teaching. So identifying and building up resources to alleviate SBT-related stresses appear central to educators' efforts to reform pedagogy.

The stress and coping research literature talks about methods that groups of individual use to alleviate stress (Dunham, 1992). Kyriacou (1980) identified three general categories of methods York teachers used in dealing with work-related stresses, based on the methods used in alleviating stresses. The first involved talking to others about problems and feelings and seeking support from them. The second consisted of different ways of dealing with the sources of stress. The third consisted of out of school activities to relax, entertain, or distract from the sources of stress.

This study focused on six novices' teaching stresses related to attempts at SBT and coping resources employed. This paper focus on how novices' coping resources mediated stresses related to SBT.

Background and Definitions

The researcher situated this study in novices' classrooms as an observer-researcher to determine whether novices taught in ways consonant with the NCTM (1991) Standards and to begin discussions of novices' stresses and coping resources with actual classroom events. In this study novice is defined as teachers with less than three years of classroom teaching experience (Berliner, 1988). Based on Kyriacou and Sutcliffe's (1978) definition, teacher stress is defined as a teacher's perception that an aspect of the job is demanding or induces negative affect; the definition is operationalised as a teacher's description of an event or situation as challenging or producing negative affect. Teacher words indicating stress

included stressful, difficult, hard, etc. Words indicating negative affect included frustrate, disappoint, annoy, etc.

There is a great deal of discrepancy in the psychology literature over how to define coping and associated resources for doing so (Dunham, 1992). In this study, coping resources are defined as means of avoiding, alleviating, or eliminating teaching stress, based on Lazarus' (1966; 1976) general definition of coping resources; that definition is operationalised as personal and/or teaching resources novices reported using to alleviate stresses.

SBT is defined by the six characteristics of high quality teaching described in NCTM (1991) *Standards*, namely Standard 1: Worthwhile mathematical tasks, Standard 2: Teacher's role in discourse, Standard 3: Students' role in discourse, Standard 4: Tools for enhancing discourse, Standard 5: Learning environment, and Standard 6: Analysis of teaching and learning. Because teachers have more control over their interactions with students than with the curriculum and technology available at their schools, the definition of SBT was operationalised as teaching non-trivially qualifying in at least two of the three Standards involving teacher-student interactions, namely Standards 2, 3, and 5.

Research Questions

Framed by the aforementioned definitions, this study explored the challenges and coping resources novice secondary mathematics teachers experienced. It addressed the following question:

- Which coping resources do novice teachers utilise to mediate the teaching stresses deriving from the different teaching stresses that they experience?

To answer this question, the study explores teaching stresses and coping resources teachers reported, mainly focusing on coping resources most closely related to novices' SBT attempts.

Methods

Participants were solicited from a progressive, NCTM (1991) Standards-oriented secondary mathematics teacher education program. The senior-year content instructor helped generate a list of recent graduates showing promise at attempting SBT, given their class participation, student teaching observations, and mentor teacher's pedagogical orientation. Using an instrument taken directly from NCTM Teaching Standards 2, 3, and 5 (and their corresponding bullet points), 11 prospective participants' teaching was observed. Six secondary novices, Ms Grant, Ms Riley, Mr Jones, Ms Boone, Ms Wells, and Ms Price, made non-trivial attempts at SBT and chose to participate. The participants' certified teaching experiences ranged from 1.25 to 2.75 years. The classes they taught were either from a spiraling teacher-created curriculum, for Mr Jones and Ms Boone, to the more standard U.S. mathematical course sequence: Algebra, Geometry, Advanced Algebra, Precalculus, and Calculus. Only one teacher self-identified as African American; the others self-identified as non-Hispanic Caucasians. Their teaching contexts were urban, rural, and rural-suburban (see Lewis, 2007; 2008 for more details).

An example of a non-trivial SBT attempt comes from Ms Grant's screening observation. She had students working to solve systems of equations using several different methods (i.e., tables, graphs, and algebraic manipulations). She had students talk about various ways they solved those systems and had them question each other; as they did so, she organised the information into a table. Ms Grant and her students engaged in several activities during the lesson that reached the protocol's threshold for SBT attempts. For example, Ms Grant

acted as a facilitator for this conversation, asking why their solutions worked or whether students understood and agreed with other students' explanations or had questions for their peers (Standard 2). Finally, some students shared their own ideas during this task or asked questions of their peers which helped them understand the other student's strategy or exposed a flaw in a solution (Standard 3). Because this event and others qualified her in at least one aspect at least two of the three NCTM (1991) *Standards* highlighted in the observation protocol, Ms Grant was invited to participate in the study.

Participants were expected to complete a preliminary audiotaped interview, a minimum of three videotaped observations in a class which novices described as constructively talkative, one videotaped observation of another class novices described as involving very different challenges, and a final audiotaped interview. The novices noted surprising, unexpected, and challenging events in a teaching log during observations. Melding the researcher's observations with the teaching logs, video clips of three salient, representative, and potentially stressful events were prepared to view and discuss at the final interview, giving priority to those related to SBT attempts. During the interviews, the researcher asked novices about teaching challenges and coping resources. Information about coping resources was gleaned from the interviews.

Data Analysis

The transcribed interview data was analyzed using a semi-open coding method (Glaser & Strauss, 1967) to refine the existing teacher stress categories in the literature. Each stress category was then evaluated to determine whether it appeared salient, or a *Main* stress, based on five criteria (Lewis, 2007; 2008). *Repetition within* indicates a participant mentioned a stress category more than once during a single interview. *Repetition between* indicates a participant mentioned a stress category during both interviews. *Detail* describes the use of detailed descriptions (i.e., at least three descriptive sentences in one passage) of a teaching stress category. *Magnitude* refers to using a magnitude word (such as *so* stressful, *very* frustrating, etc.) at least once in reference to a stress category. *Top* indicates that the teacher described the category as a top stress (e.g., my *biggest* challenge), which also met the Magnitude criterion simultaneously. If a stress category met at least three of these five criteria, it was labeled as a Main stress; otherwise, it is merely a *mentioned* stress.

Because this study focuses more on teacher preparation and professional development, it seemed more important to focus on types of resources for coping rather than on general methods in the literature (Kyriacou, 1980), so the coping resources were sorted into four categories to help educators understand the supports novices may already have and may need more of, namely Collective, Social, Self, and Physical. *Collective* refers to resources that involved social interactions with groups of at least three people meeting regularly for a specific, non-trivial purpose. *Social* describes resources that involve other people but did not meet all three criteria for Collective. *Self* included using one's own creative ideas, pedagogical strategies, management strategies, or intrapsychic methods to alleviate stress. *Physical* involves the use of curricular materials, the Internet, other teachers' lesson plans, etc. to alleviate teaching stress.

In judging how successful these novices' coping resources were at alleviating their stresses, coders look for whether participants talked about the stress alleviation or about the resources using strongly positive ("helpful" or "cool"), moderately positive ("pretty good," "okay," or it works "sometimes"), or negative ("not really helpful" or "it still happens," respectively) language. If the teacher used strongly positive language to describe the stress, the stress was judge as successfully alleviated. If the language was moderately positive, the stress was somewhat successfully alleviated. And if the language was negative (or if the

teacher avoided the stress altogether, as in “I bailed”) the stress was judged as not successfully (i.e., meaningfully) alleviated in that instance for the novice.

Each stress was also associated with an NCTM (1991) Teaching Standard(s) when applicable. Stress categories that contained at least one reference to attempts at SBT were identified as SBT-related. Each stress was also associated with a coping resource they novice reported it appeared to alleviate, either using each teacher’s explanations (55.1%) or by researcher judgment (44.9%).

Results

Novices reported many stresses in the 22 categories that arose from the data analysis. 12 of those categories were judged to be Main stresses for them. They also linked many stresses overall to SBT, but only 7 of the 22 categories were judged as *strongly* related to attempts at SBT (i.e., at least 50% of passages in the category appeared related to SBT attempts). Few stresses related to SBT attempts were Main stresses; yet one Main category of teaching stress novices frequently mentioned related to SBT attempts was *Managing classroom discussions*, which included novices’ attempts to engage students in mathematical discussions, to elicit students’ mathematical thinking, to engage them in scrutinising and justifying or refuting mathematical conjectures, and to act in a facilitative role during class discussions. For example, Mr Jones felt challenged during a class discussion, because students did not understand why the angle-side-angle condition was sufficient to imply congruence. And he was frustrated that few ideas in the conversation came from the students.

Mr Jones: I think that the students did a good job of talking about how the sides related... [But] I think that I did a little bit more talking than I should have ... [And] it looked like there wasn’t enough wait time for them to actually think ... It was kind of ‘Here, I’m going to tell you what’s another reason why this works.’ ... I was frustrated with myself. And I think the students were frustrated ... And the students were not getting it. And I only heard from 3 students, so it wasn’t necessarily engaging to the majority of students ... And they maybe didn’t even see a connection ... It just was frustrating that we didn’t get to have the conversation that I necessarily wanted to, so it didn’t go [as] I had planned ... It was a little frustrating that our conversation wasn’t as meaningful as it should have been ... I wanted them to explain, just as we did for that criterion, exactly how the sides and angles related to each other, so that you couldn’t make one side longer [or shorter]. You couldn’t change the angle, because ... all the sides and angles had a relationship [Post, 4.29-6.6].

So while Mr Jones may not have felt like the conversation went exactly how he envisioned it, the stress that he reported here resulted from his efforts to engage students in a conversation about mathematical concepts where the ideas were primarily coming from the students. He felt like he had ended up talking more than he wanted to and that the students had not participated as actively as he would have liked.

Novices reported the most important resources for dealing with SBT-related stresses were Social or Collective (i.e., socially-oriented coping resources). Overall, socially-oriented coping resources accounted for 50.0% to 68.2% of the resource passages for each novice; moreover, all six of the novices in this sample reported that their top coping resources were socially-oriented.

For example, for the stress category of Managing classroom discussions, 5 of the 11 coping resources novices reported for dealing with those challenges were Social; the remaining 6 were Self. The Social coping resources that novices reported were judged to be somewhat effective overall (since four were judged as somewhat effective and one was effective), while the Self resources were judged as ineffective overall (since three were judged as somewhat effective and three were ineffective) at helping them cope with those stresses.

The next quote demonstrates one example of an effective coping resource related to Managing classroom discussions. Specifically, Mr Jones talks about how he seeks help from mentors and colleagues when he has difficulties with his teaching.

Mr Jones: I may look at the communication standard for grades 9-12. And I want to make sure that during a lesson at least that I consider “how to organise and consolidate their mathematical thinking through communication”. So these things actually happen every day without me looking in [the NCTM Standards], but it’s just another way for me to make a mental note ... that this idea is communicated very well ... between [the students] ... or we’re talking together ... This is definitely occurring every day with connections ... So these things occur every day, especially representation, different ways to represent the same idea. So to make sure when I look at this, I need to think to myself when I plan, “Okay, this is one way right now to plan. This is one way to set up the problem or the idea, but there’s got to be more. So either I talk to my mentor teacher or someone else. “What are other ideas how I can approach this or how students will think about this?” [Pre, 5.9-5.36]

So Mr Jones makes it clear that he talks to his mentor teacher and others to get ideas about how to approach topics in different ways or with help anticipating how the students will think about those topics initially.

The second quote provides an example of an ineffective strategy for dealing with difficulties related to Managing classroom discussions. In this passage, Ms Riley relates how she struggled to get students to understand why they needed to complete the square during a whole-class discussion in which students played an active role.

Ms Riley: Well, I was challenged by the fact that I did not think that they were getting [why you need to complete the square]. And I was trying to come up with other ways to explain myself. And I was having a lot of trouble ... And they’re good at telling me when they don’t get it. They shake their head, “No, don’t go on.” ... I don’t know if it ever did [get resolved] ... I just dropped it, because I couldn’t handle it anymore... I think how it resolved itself was I got to the point where I said, like in my head, “We do not have enough time to tinker with [this]... Forget it... Just divide this number by 2 and square it. Just do it.” [Ms Riley laughs.] ... I feel like we got so off course with that, so out of what I wanted to do. I mean it was ruined from there (Post, 3.15-4.12).

Ms Riley simply decided to end the discussion when she was struggling with helping her students understand why they would need to complete the square, rather than continue or modify those attempts. Later in our interviews, she said no other teachers had attempted to help their students understand this topic, so she would simply help them memorise and work easy problems and move forward, rather than helping them understand it.

Taken together, these quotes illustrate how these novices’ Social resources (such as peers, colleagues, friends, administrators, etc.) were more effective at helping them navigate their challenges with mathematical discussions than their Self resources (or their own knowledge, experiences, creative ideas, etc.).

Looking at the big picture, three novices, Mr Jones, Ms Boone, and Ms Grant, were making more effective attempts at SBT than the other three novices in this study. In other words, they made more frequent attempts than their peers at SBT, and they also judged those attempts as successful or modified their lessons for greater effectiveness the following year, rather than discounting or abandoning those strategies and attempts. Of those three teachers, two, Ms Boone and Mr Jones, described a teacher learning community in which they actively participated; both of those teachers taught at the same high school. They valued social connections at their respective schools which they described in ways that led me to classify them at times as Social and at other times as Collective resources.

For example, Ms Boone mentioned that she consults her colleagues, including Ms Hyde and Ms Knowles, when she needs help with content or with planning ideas.

Ms Boone: We generally are very collaborative. So a lot of what we come up with comes from us talking it out and creating our own problems, which is pretty nice ... That is definitely one of my best

resources is my co-workers ... I go to Ms Hyde a lot because we teach the same classes. It's just easy to talk to her. So ideally, the grand plan was to meet once every one to two weeks or so. We've been more on like a 3-week schedule. Pretty much whenever a unit starts, we'll get together and talk ... [When I get stuck, I] definitely ask somebody ... Either Ms Hyde or Ms Knowles usually ... I'm pretty much comfortable going to anyone. I've gone to Mr Jones a couple of times to chat with him ... You can get ideas from anyone ... [Pre, 7.43-8.33].

So Ms Boone had a large number of people that she could call on. She could talk to Ms Hyde, Ms Knowles, Mr Jones [who also participated in this study], and many other people at her school. She mentioned that the teachers in her department are very collaborative; they work together to create their own problems and, as she had previously explained, their curriculum. Since they both taught at the same high school, perhaps it was not surprising that Mr Jones shared coping resources similar to the ones that Ms Boone did.

The third teacher, Ms Grant, described an extensive social network from which she drew selectively depending on the type of problem she was facing. In particular, she had only one or two resources for attempting SBT.¹ Her department chair was valuable for pedagogical questions, while the school's technology coordinator was helpful with strategies that contributed to making lessons involving technology, including a few related to SBT, more effective.

Ms Grant: And then supports among teachers—if I tell the people in the math department, that's difficult, because they all have their own idea about what a math classroom should look like. So I'm not necessarily comfortable saying some of my concerns or challenges, unless I know that they're like a universal challenge, because ... they won't be understanding or compassionate to the students the way I'm thinking of it ... But pretty much every weekend, I hang out with teachers. And that's our conversation; it's education ... They're mostly classmates or friends, ... my husband. My mom's an elementary teacher ... I go to [colleagues] for different reasons. One teacher I might go to to talk about technology issues ... Or if we have ... or had common students, we can talk about that. Topic wise, some people are really like [district] pacing guide oriented, so I have to just feel out who can maybe help me with it ... I'm not flipping out, because I haven't finished what I was supposed to finish by the end of the school year ... [My most effective supports?] If I was still at [the university], it would be my teaching class ... I really like talking with [my methods instructors] ... I'll talk to the principal about my challenges ... But probably my department chair is my best support, because he needs support from me, too, a lot. We're good at bouncing things off each other [Post, 22.26-24.18].

So Ms Grant felt like she had to be careful who she asked for help and with what issue. In the past, she sought help from her university instructors, but those resources were not easily accessed when teaching. It appeared the only colleague who shared similar pedagogical views was her department chair. So she described having limited resources requiring strategic use.

Ms Grant's social skills were quite polished. While she appeared on the surface to be coping well with her challenges, she was the only teacher in the sample who mentioned that she was considering a career change (in this case, switching to teach English rather than mathematics). She described teaching lower-level, lower-track math courses as boring, because her students often were not interested in engaging in the types of activities that she attempted to create and because she was limited as to how much time she could spend on any one topic. She felt like it would be easier to hold such discussions if she were teaching English.

Ms Grant: I think the biggest thing that might push me out of teaching math, because I know what it's like to teach English. And I know what social studies teachers do ... I'll help students be problem

¹ Note. The other teachers, Ms Wells, Ms Price, and Ms Riley also reported social resources. Ms Wells, in particular, reported participating in a teacher learning community. But these three teachers did not specifically and clearly relate their Social and/or Collective resources to their attempts at SBT.

solvers ... I want to make them, not like math, but be successful at math when they maybe haven't in the past. And I'm just discouraged by how dull it is for me sometimes ... Teaching it can be very, very dull. And I can get into it, but it's so not authentic, I think, for the students, "It doesn't make sense for her to be all excited about solving for y." And it drives me crazy that 4 weeks later, they still want me to solve for y and show the subtract 4 on both sides and then divide by 3, as if it was a linear equation. That's what frustrates me the most probably ... So if I could teach solving linear equations for a month, or something, or two months. And I could do it in a lot of ways, a lot of interesting ways, creative ... then I wouldn't mind. But because I'm limited to this amount of time and these students, all I can keep saying is you know, "Now subtract 3 on both sides." So that's probably what would push me out ... And that's why I'm looking always [Post, 19.34-20.20].

This passage suggests that novice teachers without organised social resources, even when socially capable (as in the case of Ms Grant), struggle with stresses relating to implementing the type of teaching that they envision in their classrooms, particularly when time limits their opportunities to engage in teaching that meets their ideals. This may result in greater stress for these teachers than for novices in more supportive settings, particularly those lacking the support of an effective teacher learning community.

In summary, while these teachers reported many new stresses, several of which related to SBT attempts, only one of those stresses that related to their SBT attempts, namely Managing classroom discussions, was judged as both salient and frequently mentioned among their teaching stresses overall. Novices with access to a teacher learning community appeared to be less stressed than their counterparts, even when those teachers appeared to have extensive social networks.

Educational and Scientific Importance of the Study

Many teacher educators believe NCTM's (1991) *Standards* outline a powerful pedagogy for K–12 school mathematics. Opportunities to enact such reforms may be hindered by high levels of stress novice teachers experience. Attempting SBT interacts with novices' existing stresses in complex ways. These interactions may limit novices' opportunities to learn from their experiences about teaching, in particular, when implementing SBT practices. The literature describes the discourse advocated by the NCTM *Standards* as stressful for teachers to orchestrate, because they lead them in uncertain directions and press on their content and pedagogical content knowledge in unique ways (Smith, 1996). Novices, in particular, appear to have difficulty navigating such conversations while enforcing classroom norms of behavior (e.g., Van Zoest & Bohl, 2002). Those norms of behavior often differ decidedly from those in more traditional classrooms.

While this study found innovative teaching in a wide variety of educational contexts, there was evidence that those teachers with personal commitments aligned with SBT or who were in contexts most supportive of SBT were making the most effective attempts at such teaching. But this study also found that novices appear to persevere in the most ambitious attempts to implement SBT in contexts such as teacher learning communities that actively and effectively support and promote such teaching.

If we, as teacher educators, educational administrators, and colleagues, wish to support novices in such conversations, teacher learning communities appear to offer supports that effectively sustain novices who embrace and persevere in attempting to enact elements of the NCTM (1991) Standards.

References

- Berliner, D. C. (1988). *The development of expertise in pedagogy*. Paper presented at the annual conference of the American Educational Research Association, New Orleans, LA.

- Borko, H., & Putnam, R. (1996). Learning to teach. In R. Calfee & D. Berliner (Eds.), *Handbook of educational psychology* (pp. 673-708). New York: Simon and Schuster Macmillan.
- Dunham, J. (1988). *Stress and teaching* (2nd Edition). Routledge: London.
- Glaser, B., & Strauss, A. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Hawthorne, NY: Aldine de Gruyter.
- Kyriacou, C. (1980). Coping action and organisational stress among school teachers. *Research in Education*, 24, 57-61.
- Kyriacou, C., & Sutcliffe, J. (1978). A model of teacher stress. *Educational Studies*, 4, 1-6.
- Lazarus, R. S. (1966). *Psychological stress and the coping process*. New York: McGraw-Hill.
- Lazarus, R. S. (1976). *Patterns of adjustment*. New York: McGraw-Hill.
- Lewis, G. (2007). Stresses experienced by novice teachers attempting to teach in the spirit of the NCTM Standards. (Doctoral dissertation, Michigan State University, 2007). *Dissertation Abstracts International*, AAT 3282151.
- Lewis, G. (2008). *Novices' stress and coping: Beginning secondary mathematics teachers' attempts to teach in the spirit of the NCTM Standards*. Saarbrücken, Germany: VDM Verlag.
- McLaughlin, M. W. (1993). What matters most in teachers' workplace context? In J. W. Little & M. W. McLaughlin (Eds.), *Teachers' work: Individual, colleagues, and contexts* (pp. 79-103). New York: Teachers College Press.
- National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics*. Reston, VA: NCTM.
- Smith, J. P. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education*, 27(4), 387-402.
- Travers, C., & Cooper, C. (1996). *Teachers under pressure*. New York: Routledge.
- Van Zoest, L., & Bohl, J. (2002). The role of reform curricular materials in an internship: The case of Alice and Gregory. *Journal of Mathematics Teacher Education*, 5(3), 265-288.
- Wilcox, S., Schram, P., Lappan, G., & Lanier, P. (1991). The role of a learning community in changing preservice teachers' knowledge and beliefs about mathematics education. *For the Learning of Mathematics*, 11(3), 31-39.

The Hammer-and-Nail Phenomenon in Mathematics Education

Kien H. Lim

University of Texas at El Paso

<kienlim@utep.edu>

"For a person with a hammer, everything looks like a nail" is a proverb that can be used to highlight the phenomenon that students tend to rely on familiar ideas as opposed to taking time to think about and analyse a problem. Presented in this theoretical paper is the usefulness of the hammer-and-nail metaphor, other related theoretical constructs, pedagogical causes of student impulsive behaviours, and pedagogical suggestions for addressing them.

The behaviour of "doing whatever first comes to mind ... or diving into the first approach that comes to mind" (Watson & Mason, 2007, p. 307) is quite common among students when solving problems in mathematics. Consider, for example, a 9th grader's response to the task of finding the value of b that would make the equation $(b - 1)(b + 4) = 0$ true. As shown in Figure 1, the student multiplied out the given factors and then re-factored the quadratic equation before using the zero-product property. This student's behaviour is reminiscent of an English proverb: *For a person with a hammer, everything looks like a nail*. Metaphorically, the student saw the factored-form equation as a "nail" that compelled him to use the multiplying-out-factors algorithm, commonly known as FOIL, as a "hammer" to obtain the general form.

$$\begin{aligned}(b-1)(b+4) &= 0 \\ b^2 + 4b - b - 4 &= 0 \\ b^2 + 3b - 4 &= 0 \\ (b+4)(b-1) &= 0 \\ b+4=0 & \quad b-1=0 \\ b=-4 & \quad b=1\end{aligned}$$

Figure 1. A 9th grader's solution

This hammer-and-nail proverb can be used as a means to draw educators' attention to this phenomenon. In this paper, the following aspects of the hammer-and-nail metaphor are discussed: (a) its origin and its usefulness, (b) its manifestations among learners of mathematics, (c) related literature, (d) pedagogical causes, and (e) suggestions for teaching.

Origin and Usefulness of the Hammer-and-Nail Metaphor

The hammer-and-nail metaphor was first utilised in the context of conducting scientific research in the 1960's. Maslow (1966) acknowledged that "it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail" (p. 15-16). He distinguished between two types of researchers: *problem-centring* scientists who "choose to work as best they can with important problems" and *method-centring* scientists who restrict "themselves to doing only that which they can do elegantly with the techniques already available" (p. 16). Kaplan (1964) coined the phrase *the law of the instrument* to refer to the individual scientists' tendency to formulate a problem according to her or his expertise.

Exposing the hammer-and-nail disposition can help researchers become aware of their tendency to be confined by the theoretical perspectives and methods they are most familiar with. In the context of solving mathematics problems, making the hammer-and-nail phenomenon explicit can help teachers and students become aware of their tendency to rely on algorithms they know as opposed to taking time to think and analyse a problem.

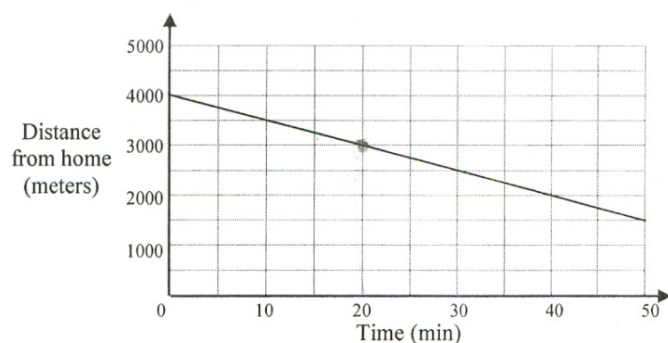
The author used the hammer-and-nail phenomenon as a metaphor to describe students' impulsivity in their problem solving at several professional development workshops for mathematics teachers. The participants generally related very well to this metaphor, especially when they experienced their own impulsivity through solving certain problems in a non-threatening environment. Several presentations on the hammer-and-nail phenomenon have been given at mathematics teacher conferences and have been well received by the audience.

Manifestations of Hammer-and-Nail Behaviours in Mathematics

The hammer-and-nail phenomenon is noticeable when an error is observed to arise from inappropriately applying a mathematical tool (hammer) to solve a mathematical problem (nail). Tools that are commonly used by students include formulas (e.g. quadratic formula), algorithms (e.g., setting up a proportion), strategies (e.g., identifying key words), and intuitions (e.g., multiplication makes bigger).

Treating formulas as “hammers” is common among students, especially when their understanding is superficial. For example, 52% of 307 pre-service K-8 teachers selected “d” as the answer for the question shown in Figure 2. These students interpreted the item as a speed problem (nail), read the distance from the graph (3000m) that corresponds to the 20 minutes, applied the $s = d/t$ formula (hammer), and obtained 150 by dividing 3000 by 20. Only 18% selected the correct answer “b” which is based on understanding speed as ratio of change in distance to change in time. Interestingly, 28% interpreted this task (nail) as a graph-reading activity (hammer) and chose 3000 meters as the answer.

Gina is traveling home from her friend's house. The graph represents a portion of Gina's journey. What is Gina's speed at the 20th minute?



- (a) Approximately 3000 meters
- (b) Approximately 50 meters/min
- (c) Approximately 80 meters/min
- (d) Approximately 150 meters/min

$$s = \frac{d}{t}$$

$$s = \frac{3000}{20}$$

$$= 150$$

Answer: A B C **D**

Figure 2. An inappropriate use of $s = d/t$ formula to find speed

Student overgeneralization of proportionality in solving non-proportional missing-value problems can be interpreted as instantiations of the hammer-and-nail phenomenon. For

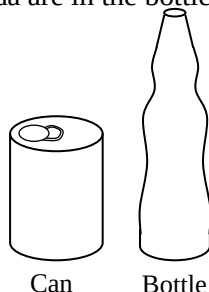
example, 72.1% of Flemish fifth-graders applied a proportional strategy (hammer) to solve the following missing-value problem (nail): “A group of 5 musicians plays a piece of music in 10 minutes. Another group of 35 musicians will play the same piece of music. How long will it take this group to play it?” (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004a, p. 390). The inappropriate use of proportional methods is found to be most prominent in Grade 5 where students in Belgium are taught extensively to reason proportionally. This overgeneralization is also common among pre-service teachers (Cramer, Post, & Currier, 1993; Lim, 2009; Monteiro, 2003).

The phenomenon of indiscriminately applying proportional strategies was observed in an exploratory study involving pre-service elementary and middle school teachers in a course on rational numbers and algebraic reasoning. After being taught ratios and proportions, these prospective teachers performed worse on all three non-proportional items in a post-test but performed better on all four direct-proportional items. A direct-proportional item and an inverse-proportional item are shown in 3a and 3b, respectively. We found that (a) the number of students who chose the correct answer “b” for the direct-proportional item increased from 64% to 78% ($n = 118$); (b) the number of students who chose the correct answer “a” for the inverse-proportional item, on the other hand, dropped from 55% to 42%; and (c) the number of students who chose the incorrect proportional answer “d” increased from 24% to 40%. These results suggest that when students are exposed to a particular solution strategy (hammer), they are more likely to use it even for situations where its use is inappropriate.

Direct-Proportional Item

The ratio of the amount of soda in the can to the amount of soda in the bottle is 4:3. There are 12 fluid ounces of soda in the can, how many fluid ounces of soda are in the bottle?

- (a) 9 fluid ounces
- (b) 11 fluid ounces
- (c) 15 fluid ounces
- (d) 16 fluid ounces
- (e) None of the above



Inverse-Proportional Item

The ratio of the volume of a small glass to the volume of a large glass is 3:5. If it takes 15 small glasses to fill the container, how many large glasses does it take to fill the container?

- (a) 9 glasses
- (b) 13 glasses
- (c) 17 glasses
- (d) 25 glasses
- (e) None of the above

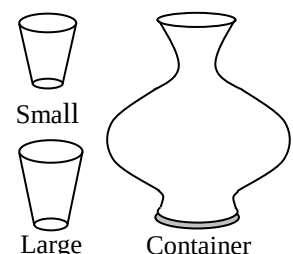


Figure 3. Two math test items used in a pilot study

In this paper, students’ mistakes due to their familiarity with a particular idea or schema are considered manifestations of the hammer-and-nail phenomenon. It is important to note that the hammer-and-nail metaphor is just one of many viable ways of interpreting a situation. Other related perspectives are presented in the ensuing section.

Related Literature

Einstellung Effect. Students’ tendency to approach a problem with conceptual tools that are familiar to them has been noted in the literature since the 1940s. In the famous water jar experiment (Luchins & Luchins, cited in NRC, 2000), after solving many problems using one approach subjects spontaneously used that approach to solve other problems that could have been more easily solved using a different approach. This phenomenon of solving a

given problem in a fixated manner even when a better approach exists is called the *Einstellung effect*.

Spurious Correlation. In mathematics, student hammer-and-nail behaviours can be explained in terms of Ben-Zeev and Star's (2001) *spurious correlation*, which is a two-phase process consisting of: (a) conceiving an association between a problem feature and an algorithm for solving the problem, and (b) using the algorithm upon perceiving the same feature in another problem. In their study, students not only relied on surface-level features in solving problems but "also generate[d] and use[d] correlations between irrelevant surface-level features and solution strategies" (p. 272). Experienced students were also found to be susceptible to this tendency. Although the *Einstellung effect* is the scientific term for this phenomenon, the hammer-and-nail metaphor is less technical and more relatable to mathematics teachers.

Intuitive Rule. Tirosh and Stavy (1999) put forth a set of intuitive rules to account for many alternative conceptions in mathematics and science. For example, students who thought that the heavier the object the faster it falls are said to have relied on the "more A – more B" intuitive rule. Metaphorically, intuitive rules are the hammers that tend to influence students to perceive certain situations as nails.

Dual-System Theories. Several researchers in cognitive psychology have posited that there are two distinct cognitive systems of reasoning. Various names have been used to distinguish the two systems: implicit-explicit (Reber, 1993), associative and rule-based (Sloman, 1996), and System 1 and System 2 (Stanovich & West, 2000). According to Evans (2006), "System 1 processes are rapid, parallel and automatic in nature: only their final product is posted in consciousness" whereas "System 2 thinking is slow and sequential in nature and makes use of the central working memory system" (p. 454). Sloman (1996) points out that the two systems often work cooperatively despite having different goals and specializing in different kinds of tasks. At times, they may each try to generate a response. Although System 1 may have its response overridden by System 2, it "always has its opinion heard, and because of its speed and efficiency, often precedes and thus neutralizes the [System 2] response" (p. 15). Manifestations of the hammer-and-nail phenomenon may be attributed to the domination of System 1 over System 2.

Cognitive Tempo. On an individual-basis level, some students are more likely exhibit impulsive behaviours than other students. Such impulsivity can be considered a *cognitive style*—"a person's typical or habitual mode of problem solving, thinking, perceiving and remembering" (Riding & Indra, 1991, p. 194). In the field of psychology, the term *conceptual tempo* refers to a cognitive style that is along the impulsivity-reflectivity dimension (Kagan, 1965). Kagan, Rosman, Day, Albert, and Phillips (1964) constructed the *Matching Familiar Figures Test* to measure children's cognitive tempo. They classified a child who had an above-the-median response time and a below-the-median accuracy rate as having an impulsive style, and a child with a below-the-median response time and above-the-median accuracy rate as having a reflective style. Nietfeld and Bosma (2003) describe *impulsives* as "individuals who act without much forethought, are spontaneous, and take more risks in everyday activities" (p. 119) and *reflectives* as "more cautious, intent upon correctness or accuracy, and [who] take more time to ponder situations" (p. 119). Students classified as having an impulsive cognitive tempo are considered more likely to exhibit behaviours that are reminiscent of the hammer-and-nail phenomenon.

Impulsive Disposition. Lim, Morera, and Tchoshanov (2009) use the term *impulsive disposition* to refer to a tendency to proceed with an action that comes to mind without analysing the problem situation and without considering the relevance of the anticipated action to the problem situation. Impulsive disposition differs from impulsive tempo in that

the latter is characterized by a fast but inaccurate response whereas the former is characterized by proceeding with an approach that comes to mind without checking its relevance, and not necessarily by how fast an approach comes to mind. Another difference is that impulsive tempo is a cognitive style that is rather stable across time and across situations (Messer & Brodzinsky, 1981), whereas impulsive disposition is a cognitive tendency which depending upon the circumstances may or may not result in an action. Impulsive disposition alludes to a mental tendency that can be modified under favourable learning conditions. In terms of the hammer-and-nail metaphor, students can learn to recognize situations where a particular idea is inappropriate from those where it is.

Knowing-to-Act in the Moment. Spontaneously responding to a situation with the first thing that comes to mind is not necessarily an undesirable way of thinking. Mathematicians often respond automatically to situations with which they are familiar. Mason and Spence (1999) describe this ability as *knowing-to act in the moment*. A person with this ability in relation to a particular tool is considered to have mastered the use of the tool, knows when not to use it, and consequently is able to avoid hammer-and-nail behaviours.

Pedagogical Causes of Hammer-and-Nail Behaviours

The traditional methods of teaching mathematics tend to foster hammer-and-nail behaviours. “The tradition has been to regard ‘mathematics’ as a set of rules for writing symbols on paper, and to regard the ‘teaching’ of mathematics as merely a matter of ‘telling’ students what to write and where to write it, together with supervising some considerable amount of drill and practice” (Davis, 1989, p. 159). Stigler and Hiebert (1999) characterized U.S. teaching as “learning terms and practicing procedures”, as opposed to “structured problem solving” (p. 27) in Japanese teaching. More than three-fifths of U.S. teachers described *skills* as the *main thing* that they wanted their students to learn; “they wanted the students to be able to perform a procedure, solve a particular kind of problem, and so on” (p. 89). Drill and practice of procedures without a conceptual focus will lead to superficial understanding and foster the System 1 mode of operation.

Mathematics curricula are typically organized sequentially, one chapter after another. Textbook problems are designed for students to practice the main ideas in a chapter. Such problems can be regarded as “nails” for students to practice with their newly acquired “hammer”. To help students remember certain facts or procedures, some teachers offer students learning aids such as acronyms (e.g., FOIL), schematic tools (e.g., ratio box to find the missing value from three given values), and key words (e.g., *altogether* means add and *share* means divide). Students typically learn such strategies by associating a feature with a procedure; these associations constitute the bases for hammer-and-nail behaviours. For example, problems that begin with “In a sports car race” tend to cue formulas such as $s = d/t$ or $d = rt$ (Hinsley, Hayes, and Simon, 1977). Such strategies suppress System 2 mode of reasoning because students are relieved from having to grapple with the mathematics they learn and from having to make sense of the problems they try to solve.

Suggestions for Addressing Hammer-and-Nail Behaviours

Addressing the hammer-and-nail phenomenon involves helping students develop a cognitive habit of maintaining *control* over their own mathematical thinking as they solve problems or learn mathematics. To be in control of a situation a student needs to: (a) be aware of his or her impulsive disposition, (b) stop relying solely on *instrumental understanding*—“rules without reason” (Skemp, 1976, p. 20), and (c) develop *relational understanding*—“knowing what to do and why” (p. 20). To help students become cognizant

of their impulsive tendency, superficially-similar-but-structurally-different problems should be used intermittently in class as well as in examinations. To help students overcome their reliance on instrumental understanding, reasoning and sense making such as attendance to meaning of mathematical symbols (Harel, Fuller, & Rabin, 2009) should be emphasized in class. These instructional strategies were used in a study (Lim & Morera, 2010) that was conducted to explore the possibility of helping prospective K-8 teachers overcome their overgeneralization of proportionality within one semester.

Using Superficially-Similar-but-Structurally-Different Problems. After learning how to solve certain problems using a particular idea, students can work on a superficially-similar-but-structurally-different problem that can elicit a conceptual error and thereby allow students to realize the harmfulness of indiscriminately applying a newly learned idea. For example, when students have learned the use of ratio for comparing measures such as steepness of a line or “squareness” of a rectangle, the problem in Figure 4 can be posed. This problem offers students an opportunity to make a mistake, learn from their mistake, and become cognizant of their impulsive tendency. When used as classroom activities, superficially-similar-but-structurally-different problems allow teachers to emphasize quantitative reasoning, to foster attendance to meaning, and to extend student understanding by knowing when an idea is not applicable.

Given the value of the perimeter and the length of each rectangle, determine which rectangle has the greatest width.

	Rectangle A	Rectangle B	Rectangle C
Perimeter	18 meters	28 meters	44 meters
Length	6 meters	10 meters	20 meters
	18 : 6	28 : 10	44 : 20
	3 : 1	2.8 : 1	2.4 : 1
	RECTANGLE A		

Figure 4. A pre-service teacher's incorrect use of ratio to compare widths of rectangles

Emphasizing Reasoning and Sense Making. Awareness of one's impulsivity is useful but is not sufficient to ensure a correct solution: one needs to analyse and visualize the problem situation. For example, in solving a word problem, one needs to make sense of the problem by focusing on quantities—attributes that can be measured or amounts that can be counted—and understanding how those quantities are related, instead of identifying numbers and deciding which operations to use on those numbers. Thompson (1993) defines *quantitative reasoning* as “the analysis of a situation into a quantitative structure—a network of quantities and relationships” (p. 165). One way to foster quantitative reasoning is to challenge students to explain the quantitative structure underlying a problem situation. For example, when students set up a proportion $a/b = c/x$ to solve a missing-value problem (values of three quantities are given to find the value of the fourth quantity), they should provide “reasons in support of claims made about the structural relationships among four quantities” (Lamon, 2007, p. 638). Lim (2009) used non-proportional situations, in the context of burning candles, to emphasize the importance of analysing the problem situation, determining the co-varying quantities, and identifying the invariant relationship. The quantities depicted in a non-proportional missing-value problem may be related by an invariant difference ($a - b = c - x$), an invariant sum ($a + b = c + x$), or an invariant product ($ab = cx$).

Fostering Attendance to Meaning. The hammer-and-nail phenomenon is more likely to manifest among students with superficial understanding. For example, students might use a proportion, $a/b = c/x$, to solve a missing-value problem without knowing what in the context of the problem situation each ratio refers to and why the two ratios should be equal to each other. Such students are said to be reasoning in a *non-referential symbolic* manner—“operating on symbols as if they possess a life of their own” (Harel, Fuller, & Rabin, 2009, p. 2008). This way of thinking often leads to errors. It is therefore important for students to attend to the meanings of the symbols and numbers they encounter in a problem. To foster attendance to meaning in the context of proportional reasoning, students should be challenged to explain what the ratio a/b represents in the context of the problem situation and why it should be equal to the other ratio, c/x . When students develop the habit of attending to meaning, they are more likely to engage in quantitative reasoning and less likely to respond to a problem impulsively.

Concluding Remarks

This paper uses the hammer-and-nail metaphor to highlight the rigidity and impulsiveness that can be found in student mathematical behaviour. The hammer-and-nail phenomenon can be attributed to two plausible causes: the way the human mind works and the way mathematics is traditionally taught in school. To deal with the first factor, we can help students become cognizant of their own impulsive tendency, be more metacognitive, and develop the habit of utilizing System 2 instead of simply letting System 1 dominate. To address the second factor, classroom instruction and assessments should place greater emphasis on relational understanding, analysing, sense-making, and attending to meaning of symbols.

The hammer-and-nail phenomenon deserves the attention of the mathematics education community because it reminds teachers that many students are learning mathematics without engaging in mathematical reasoning and sense making. Awareness of this phenomenon can also help teachers become more sensitive of their own teaching so as to not foster students’ impulsive tendencies. Teacher educators can use the hammer-and-nail phenomenon to highlight the futility of instrumental understanding and to motivate teachers to teach in a manner that fosters relational understanding and analytical reasoning.

References

- Ben-Zeev, T., & Star, J.R. (2001). Spurious correlations in mathematical thinking. *Cognition and Instruction*, 19(3), 253-275.
- Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159–178). New York: Macmillan.
- Davis, R.B. (1989). The culture of mathematics and the culture of schools, *Journal of Mathematical Behaviour*, 8, 143-160.
- Harel, G., Fuller, E. & Rabin, J. (2008). Attention to meaning by algebra teachers. *Journal of Mathematical Behaviour*, 27, 116-127.
- Hinsley, D., Hayes, J.R., & Simon, H.A. (1977). From words to equations. In P. Carpenter & M. Just (Eds.), *Cognitive processes in comprehension* (pp. 89-106). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Kagan, J. (1965). Impulsive and reflective children: Significance of conceptual tempo. In J. D. Krumboltz (Ed.), *Learning and the educational process* (pp. 133–161). Chicago: Rand McNally.
- Kagan, J., Rosman, B., Day, D., Albert, J., & Phillips, W. (1964). Information processing and the child: Significance of analytic and reflective attitudes. *Psychological Monographs*, 78 (No. 1, whole number 578).
- Kaplan, A. (1964). *The conduct of inquiry: Methodology for behavioural science*. San Francisco: Chandler Publishing Co.

- Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Reston, VA: NCTM.
- Lim, K. H. (2009). Burning the candle at just one end: Using nonproportional examples helps students determine when proportional strategies apply. *Mathematics Teaching in the Middle School*, 14, 492–500.
- Lim, K. H., & Morera, O. (2010). Addressing impulsive disposition: Using non-proportional problems to overcome overgeneralization of proportionality. *Online proceedings for the 13th SIGMAA on Research in Undergraduate Mathematics Education Conference*. Raleigh, NC.
- Lim, K. H., Morera, O., & Tchoshanov, M. (2009). Assessing problem-solving dispositions: Likelihood-to-act survey. In S.L. Swars, D.W. Stinson & S. Lemons-Smith (Eds.). *Proceedings of the 31st annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 700-708). Atlanta: Georgia: PME-NA.
- Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics*, 38, 135–161.
- Maslow, A. H. (1966). *The psychology of science: A reconnaissance*. New York: Harper Collins.
- Messer, S. B., & Brodzinsky, D. M. (1981). Three-year stability of reflection–impulsivity in young adolescents. *Developmental Psychology*, 17, 848-850.
- Monteiro, C. (2003). Prospective elementary teachers’ misunderstandings in solving ratio and proportion problems. In N. A. Pateman, B. J. Dougherty and J. T. Zilliox (Eds.) *Proceedings of the 27th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 317–234). Hawaii, USA: PME.
- National Research Council. (2000). *How people learn: Brain, mind, experience, and school*. Washington, DC: National Academic Press.
- Nietfeld, J., & Bosma, A. (2003). Examining the self-regulation of impulsive and reflective response styles on academic tasks. *Journal of Research in Personality*, 37, 118–140.
- Reber, A. S. (1993). *Implicit learning and tacit knowledge*. Cambridge, MA: MIT Press.
- Riding, R., & Indra, C. (1991). Cognitive styles: An overview and integration. *Educational Psychology*, 11, 193–215.
- Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 7, 20–26.
- Sloman, S. A. (1996). The empirical case for two systems of reasoning. *Psychological Bulletin*, 119, 3-22.
- Stanovich, K.E., & West, R.F. (2000). Individual differences in reasoning: Implications for the rationality debate. *Behavioural and Brain Sciences*, 23, 645-726.
- Stigler, J. W., & Hiebert, J. (1999). *The thinking gap: Best ideas from the world’s teachers for improving education in the classroom*. New York: The Free Press.
- Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, 25, 165–208.
- Tirosh, D., & Stavy, S. (1999). Intuitive rules: A way to explain and predict students’ reasoning. *Educational Studies in Mathematics*, 38, 51-66.
- Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2004). Remedying secondary school students’ illusion of linearity: A teaching experiment aiming at conceptual change. *Learning and Instruction*, 14, 485–501.
- Watson, A., & Mason, J. (2007). Taken-as-shared: A review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education*, 10, 205–215.

Impulsive-Analytic Disposition in Mathematical Problem Solving: A Survey and a Mathematics Test

Kien H. Lim

University of Texas at El Paso
<kienlim@utep.edu>

Amy Wagler

University of Texas at El Paso
<awagler2@utep.edu>

The Likelihood-to-Act (LtA) survey and a mathematics test were used in this study to assess students' impulsive-analytic disposition in the context of mathematical problem solving. The results obtained from these two instruments were compared to those obtained using two widely-used scales: Need for Cognition (NFC) and Barratt Impulsivity Scale (BIS). The exhibited correlations of the LtA scores with the NFC, BIS, and a math test provide evidence of the criterion validity of the analytic LtA items, and suggests further revision of the impulsive LtA items to improve the overall measurement validity of the LtA scale. Students LtA scores were found to be marginally correlated to their math scores and correlated to their confidence levels in the math items.

Teachers often see students solve mathematical problems using familiar tools or strategies without seemingly to analyse the problem situation. Watson and Mason (2007) describe this as “doing whatever first comes to mind ... or diving into the first approach that comes to mind” (p. 307). Lim, Tchoshanov, and Morera (2009) started developing an instrument, called the Likelihood-to-Act survey, to assess students' impulsive-analytic disposition. *Impulsive disposition* refers to a tendency to proceed with an action that comes to mind without analysing the problem situation and without considering the relevance of the anticipated action to the problem situation. *Analytic disposition*, on the other hand, refers to a tendency to study the problem situation prior to taking actions. In this paper, scores of students' impulsivity obtained using a recent version of the Likelihood-to-Act survey and a two-version mathematics test are presented and are compared to two other scores obtained using established instruments in the field of psychology.

Theoretical Background

Impulsivity and reflectivity have been contrasted in terms of cognitive tempo or response style. Kagan, Rosman, Day, Albert, and Phillips (1964) constructed the Matching Familiar Figures Test to measure children's cognitive tempo. An impulsive child is one with response time being faster than the median and accuracy rate being lower than the median, whereas a reflective child is one with response time being slower and accuracy rate being higher than the median. In a study on consistency in cognitive responses among adults across academic tasks, Nietfeld and Bosma (2003) found moderate positive correlations for response styles among the three types of tasks they investigated: verbal, mathematical, and spatial. The mathematical tasks used in their study were two-digit addition or subtraction problems arranged in a traditional vertical format. Although such tasks are appropriate for measuring cognitive tempo along a speed-accuracy continuum, they are not appropriate for measuring disposition along an impulsive-analytic continuum. In our study, mathematical tasks are designed to assess whether students respond to the first idea that comes to mind or whether they analyse the problem situation.

Impulsive and analytic responses to a situation can be accounted using the *dual process theories* (Evans, 2003; Stanovich & West, 2000) from cognitive psychology. The basic tenet in these theories is that two modes of cognitive processing are at work. The “intuitive” mode has these characteristics: automatic, fast, preconscious, low effort, associative, parallel

processing, and little variation across individuals. The “analytical” mode has the opposite characteristics: controlled, slow, conscious, high effort, rule-based, serial processing, and greater variation across individuals (Frankish, 2010). Dual process theories are gaining attention in mathematics education recently. Gillard, Van Dooren, Schaeken, & Verschaffel (2009) used them to account for why people fail to solve mathematical tasks they should be able to solve correctly given their mathematical knowledge and skills. Leron (2010) suggested a bridge between intuitive and analytic thinking can deepen student conceptual understanding. We, on the other hand, focus on creating teacher awareness about students’ disposition in solving mathematics problems. This effort initiated the development and testing of a survey instrument.

Instruments for Assessing Constructs related to Impulsiveness

The *Need for Cognition* (NFC) scale is a self-report instrument for measuring one’s “tendency to engage in and enjoy thinking” (Cacioppo & Petty, 1982, p. 116). This instrument consists of 18 Likert-scale items. Presented below are three sample items:

- NFC_6: I find satisfaction in deliberating hard and for long hours.
- NFC_7R: I only think as hard as I have to.
- NFC_14: The notion of thinking abstractly is appealing to me.

The *Barratt Impulsiveness Scale Version 11* (BIS) is a 30-item self-report instrument for assessing impulsivity in one’s behaviours. Patton, Stanford, and Barrett (1995) performed a principal components analysis and found six first order factors. A representative item for each factor is presented below.

- BIS_5: I don’t “pay attention.” (Attentional)
- BIS_6: I have “racing” thoughts. (Cognitive Instability)
- BIS_2: I do things without thinking. (Motor)
- BIS_16: I change jobs. (Perseverance)
- BIS_12R: I am a careful thinker. (Self-control)
- BIS_10R: I save regularly. (Cognitive Complexity).

The current version of the *Likelihood-to-Act* (LtA) survey has 32 items which can be sub-divided into 16 pairs of items, with one impulsive item and one analytic item in each pair. The 16 pairs can be divided equally into four subcategories: algebraic, proportional, fraction, and general. The first four items in the instrument, and their corresponding counterparts, are presented in Figure 1. For example, LtA_1 is an analytic-algebraic item and LtA_17 is an impulsive-algebraic item.

The data reported in this paper is part of a larger project that seeks to develop, test, and refine the LtA instrument. The first version of LtA has nine pairs and was administered to 318 undergraduates, mostly pre-service teachers; the reliabilities of the impulsive and analytic subscales were found to be 0.64 and 0.63 respectively (Lim et al. 2009). The second version with 16 item pairs was administered to 119 pre-service and in-service teachers; the reliabilities for the two subscales are 0.74 and 0.81 (Lim & Morera, 2011). The written work of 92 participants for 6 open-ended math problems were analysed and coded; the coded scores for written responses were found to significantly correlated to both of the LtA subscales (Lim & Mendoza, 2010). Based on the findings, one pair of items was replaced and seven items in the second version were modified to produce the current version of the LtA. The reliabilities of the two subscales in the current version are 0.79 and 0.83. In this paper, the criterion validity of the LtA instrument is investigated by inspecting the correlations between the four subcategories of the LtA instrument and the two established scales: NFC and BIS.

For each item, indicate as honestly as you can how likely you are to act in the manner specified in the statement by circling the number using this scale:

- 1 = Extremely Unlikely
 2 = Unlikely
 3 = Somewhat Unlikely
 4 = Somewhat Likely
 5 = Likely
 6 = Extremely Likely

Start Time:

:

1. $78 + 987x + 654 + 321x = x + 987x + 654 + 321x$ When asked to solve for x , how likely are you to begin by studying the equation and noticing the solution?	1	2	3	4	5	6
2. Given that 4 candy bars cost \$2.15. When asked to find the cost of 40 candy bars, how likely are you to begin by EITHER finding the cost per candy bar OR setting up a proportion?	1	2	3	4	5	6
3. $\frac{3}{4} + \frac{1}{10} + \frac{9}{10}$ When asked to find the answer for the above arithmetic expression without using a calculator, how likely are you to begin by finding the common denominator?	1	2	3	4	5	6
4. When solving a problem in mathematics, how likely are you to read and understand the problem thoroughly before deciding what to do?	1	2	3	4	5	6
17. $90 + 1234n + 567 + 89n = n + 1234n + 567 + 89n$ When asked to solve for n , how likely are you to begin by combining like terms?	1	2	3	4	5	6
18. Given that 6 bottles of mineral water cost \$2.10. When asked to find the cost of 30 bottles, how likely are you to notice the relationship between 30 and 6 and use it to obtain the answer?	1	2	3	4	5	6
19. $\frac{3}{5} + \frac{11}{12} + \frac{1}{12}$ When asked to find the answer for the above arithmetic expression without using a calculator, how likely are you to begin by studying the fractions to see if you can predict the answer?	1	2	3	4	5	6
20. When solving a problem in mathematics, how likely are you to use the first idea that comes to mind?	1	2	3	4	5	6

Figure 1. Sample items in the LtA instrument

In this paper, the results obtained from a mathematics test that contain items designed to elicit impulsive responses are included in the analysis. Frederick (2005) developed a three-item *Cognitive Reflection Test* (see Figure 2) to assess one's "ability or disposition to resist reporting the response that first comes to mind" (p. 35). We expanded his test by including eight multiple-choice items (see Figure 3). For each of the 11 items, a "1" is assigned if students gave an "impulsive" response (e.g. 10¢ for Item 9). Students were expected to indicate their confidence level. We created two structurally-equivalent versions (Version Y and Version Z) to investigate whether a warning that some of the items were "tricky" would

affect students' performance in the second version. Because of page limit constraint this paper does not focus on the effect-of-warning part of the study.

9. A bat and a ball cost \$1.10. The bat costs \$1.00 more than the ball.
How much does the ball cost? Answer: _____ cents
10. If it takes 5 machines 5 minutes to make 5 widgets, how long would it take
100 machines to make 100 widgets? Answer: _____ minutes
11. In a lake, there is a patch of lily pads. Every day, the patch doubles in size.
If it takes 48 days for the patch to cover the entire lake, how long would it
take for the patch to cover half of the lake? Answer: _____ days

Figure 2. The three Cognition Reflection Test items in the Math Test – Version Y

3. Captain Toot was 27 years old when he had 8 years of sailing experience. How old was he when he had 16 years of sailing experience?
- (a) 35 years old
(b) 51 years old
(c) 54 years old
(d) 81 years old
- ☐ I'm certain I'm right
☐ I think I'm right
☐ I've a 50-50 chance of being right
☐ I think I'm wrong
☐ I'm certain I'm wrong
6. The ratio of girls to boys in a Science club at Mesa High School is 2:5. There are 35 students in the Science club. How many members are girls?
- (a) 10
(b) 14
(c) 15
(d) 25
- ☐ I'm certain I'm right
☐ I think I'm right
☐ I've a 50-50 chance of being right
☐ I think I'm wrong
☐ I'm certain I'm wrong

Figure 3. Two multiple-choice items in the Math Test – Version Y

Research Method

A total of 495 undergraduates, mostly pre-service teachers, participated in this study. A convenience sample involving 17 classes was used. Out of 470 participants who specified their program, 29 majored in either math or engineering, 80 are pre-service 4-8 teachers specializing in either math or science, 54 are pre-service 4-8 generalists, and the remaining 307 are pre-service elementary or bilingual or special education teachers. Out of 466 who specified their gender, 72 are males. Within a 100-minute class period, participants took a set of three surveys (NFC, BIS, and LtA), took a version of the 11-item math test, received warning about the items being “tricky”, and took the second version of the test.

The data analysis is based on a sample of 460 participants, with the exclusion of 23 students who had taken the LtA survey before and 12 students who had more than 2 missing entries in the LtA survey. These 460 students took an average of 8.8 minutes to complete the LtA survey, ranging from 3 to 19 minutes.

Results

The NFC and BIS scales were evaluated for construct validity by fitting confirmatory factor models to the items. All items that load to a single factor were retained for further

analysis. For the NFC scale, only items 6, 8, and 18 do not load to a single factor and are eliminated from further analysis. For the BIS scale, 19 items (3, 4, 6, 7, 11, 15, 16, 18, 19, and 21-30) do not load and were eliminated from further analysis. The Pearson correlation between the NFC and BIS sum scores is -0.442 (p-value is <0.0001). The negative correlation between NFC and BIS is expected in that students who enjoy thinking tend to be less impulsive.

Table 1 shows students' mean scores for each subcategory of the LtA instrument. Students' impulsive mean scores are higher than analytic mean scores for the first three subcategories which contain mathematically-specific items. The impulsive mean scores, on the other hand, are lower for the general subcategories. The impulsive mean scores are found to be significantly different from the analytic mean scores for all four subcategories.

Table 1
Results for the Four Subcategories of LtA Items

Subcategories	Impulsive	Analytic	Test Statistics	<i>p</i> -value
Algebraic	4.76	3.57	17.42	<0.0001
Fraction	4.84	3.41	20.93	<0.0001
Proportional	4.33	3.83	8.58	<0.0001
General	4.29	4.52	-3.76	0.0002

Recall that the LtA items are paired so as to measure impulsivity along an impulsive-analytic continuum. Thus, the differences (impulsive minus analytic) could be computed to assess students' impulsivity along this continuum. The impulsive mean scores, analytic mean scores, and impulsive-analytic difference scores for the four subcategories were analysed for any relationship with the NFC and BIS sum scores. Table 2 show the results of the estimated correlations and *p*-values, adjusted for maintaining an overall 5% error rate for the family of inferences using the Hochberg (1988) procedure.

- The analytic LtA scores, for all four subcategories, are positively correlated with the NFC scores and are somewhat negatively correlated with the BIS scores.
- The impulsive LtA scores demonstrate less association with the NFC and BIS sum scores. Interestingly, the NFC sum scores have significant positive correlation with impulsive-proportional and impulsive-general subcategories. The BIS sum score has significant negative correlation with impulsive-general subcategory.
- The impulsive-analytic difference scores are all strongly correlated with the NFC sum score for three subcategories, excluding the proportion subcategory. None of the four subcategories are correlated with the BIS sum scores.

Table 2

Estimated Pearson Correlations between LtA Sub-scores and the Other Two Survey Scores

Subcategories	Need for Cognition		Barratt Impulsiveness Scale	
	Pearson Corr.	<i>p</i> -value*	Pearson Corr.	<i>p</i> -value*
Analytic Items				
Algebraic	0.388	<0.0001	-0.225	0.0011
Fraction	0.291	<0.0001	-0.148	0.0398
Proportional	0.286	<0.0001	-0.115	0.0716
General	0.429	<0.0001	-0.294	<0.0001
Impulsive				
Algebraic	0.118	0.1882	-0.157	0.0669
Fraction	0.072	0.2626	-0.071	0.2626
Proportional	0.283	<0.0001	-0.137	0.1232
General	0.217	0.0040	-0.192	0.0140
Impulsive-Analytic Diff.				
Algebraic	-0.216	0.0050	0.025	0.6987
Fraction	-0.181	0.0258	0.021	0.7426
Proportional	-0.015	0.8206	-0.040	0.5266
General	-0.202	0.0097	0.136	0.1642

**p*-values are adjusted to control the family-wise error rate at 5% (Hochberg, 1988)

The math test items were analysed for any relationship with the NFC and BIS sum scores and the combined (16 pairs) LtA difference scores. Polychoric correlations (Olsson, 1979) are utilized since the correctness scores are not continuous but are scored with a “1” for correct response and a “0” for incorrect response, and the resulting sum scores are the number of correct responses. The polychoric correlations (for correct responses and impulsive responses) or Pearson correlations (for confidence levels) and *p*-values with the Hochberg adjustment are presented in Table 3.

Table 3

Estimated Polychoric Correlations between Scores from Math Tests and the Survey Scores

Survey Scores	Need for Cognition		Barratt Imp. Scale		LtA Difference	
	Corr.	<i>p</i> -value*	Corr.	<i>p</i> -value*	Corr.	<i>p</i> -value*
Correct Responses						
Version Y	0.250	0.0007	-0.103	0.3190	-0.135	0.0524
Version Z	0.232	0.0021	-0.066	0.4224	-0.142	0.0503
Impulsive Responses						
Version Y	-0.225	0.0026	0.111	0.3190	0.196	0.9994
Version Z	-0.175	0.0284	0.051	0.4224	0.171	0.9973
Confidence Level						
Version Y	0.300	<0.0001	-0.229	0.0022	-0.194	0.0039
Version Z	0.235	0.0019	-0.193	0.0136	-0.131	0.0528

**p*-values are adjusted to control the family-wise error rate at 5% (Hochberg, 1988)

Consistent with previous results, the NFC items are correlated with the correctness scores in the math test and negatively correlated with the impulsive-response scores for both versions. The BIS items do not exhibit strong correlations. However, like correctness scores, students' confidence level scores in the math test are positively correlated to NFC sum scores but negatively correlated to BIS sum scores. The LtA paired items are marginally correlated with the correctness score and with the confidence level ratings, but not the impulsive responses to the math test.

Discussion

The results in Table 1 suggest that students tend to respond more impulsively to criterion-specific items like algebra (#1 and #16), proportion (#2 and #17), and fraction (#3 and #18), but more analytically to items with general descriptors (#4 and #20). In terms of dual-process theories, students tend to suspend their "analytical" mode of processing in response to familiar situations (e.g. adding fractions) especially when they have automatic responses (finding common denominators) operating at the "intuitive" level.

The strong positive correlations between LtA-analytic and NFC (see Table 2) suggest that students who agreed with the analytic statements in the LtA survey also tend to agree with the subset of NFC items (excluding items 6, 8, and 18). These results suggest that the LtA analytic items are measuring the same construct as the NFC measures and hence strengthen the criterion validity of the analytic items. The negative correlation between LtA-analytic and BIS suggest that students who agree with the analytic statements in the LtA instrument tend to disagree with the subset of BIS items (1, 2, 5, 8, 9, 10, 12, 13, 14, 17, and 9). The negative correlation was not significant for the proportion subcategory. This lack of statistical significance implies that the set of proportion-related items should be further examined and revised to strengthen the criterion validity.

The correlations for LtA-impulsive items are not as significant as those for LtA-analytic. Three correlations (between proportional and NFC, between general and NFC, and between general and BIS) were significant but they were contrary to what was expected. These unexpected correlations imply that the impulsive LtA items should undergo further refinement in order to improve the criterion validity of the scores.

The correlations between LtA impulsive-analytic difference scores and NFC scores are strong except for the proportion subcategory. This result suggests that the set of paired items for assessing how student react to proportion-related items should be further examined for criterion validity. The correlations between LtA difference scores and BIS scores are not significant. Whereas NFC items are about cognitive activity, BIS items tend to assess impulsivity in a general sense. The connection between impulsive behaviours in general settings and those in educational settings may not be clear. This may explain the lack of relationship between the LtA difference scores and the BIS sum score.

The two versions of the math test have very similar results; this can be taken as an indicator of reliability of the two versions with different but structurally equivalent items. The NFC sum score show evidence of association with all scores related to the math tests. The positive correlation between the NFC score and math correctness score implies that students who score high on the NFC scale also tend to get more math items correct. Conversely, the negative correlation between the NFC and the impulsive-response score implies students with high NFC score tend to choose the impulsive answer less frequently. The positive correlation between the NFC score and the confidence-level score suggest students with high NFC score tend to report high levels of confidence associated with their answers to the math test items.

The relationship between the BIS sum scores with the confidence levels provides evidence that students with high BIS score tends to report low levels of confidence associated with their answers on the math tests. Additionally, it may be inferred from the relationship between the LtA impulsive-analytic difference scores and the math test items and the confidence levels, that students who tend toward impulsive behaviours when approaching cognitive tasks also tend to score lower on the math tests and have less confidence associated with their answers on the math tests. There is no evidence of a relationship between the LtA impulsive-analytic difference scores and the number of impulsive answers on the math tests.

References

- Cacioppo, J. T., & Petty, R. E. (1982). The need for cognition. *Journal of Personality and Social Psychology*, 42, 116-131.
- Evans, J. (2003). In two minds: Dual process accounts of reasoning. *TRENDS in Cognitive Sciences*, 7(10), 454-459.
- Frankish, K. (2010). Dual-process and dual-system theories of reasoning. *Philosophy Compass* 5(10), 914-926.
- Frederick, S. (2005). Cognitive reflection and decision making. *Journal of Economic Perspectives*, 19(4), 25-42.
- Gillard, E., Van Dooren, W., Schaeken, W., & Verschaffel, L. (2009). Dual processes in psychology of mathematics education and cognitive psychology. *Human Development*, 52(2), 95-108.
- Hochberg, Y. (1988). A sharper Bonferroni procedure for multiple tests of significance. *Biometrika*, 75, 800-803.
- Kagan, J., Rosman, B., Day, D., Albert, J., & Phillips, W. (1964). Information processing and the child: Significance of analytic and reflective attitudes. *Psychological Monographs*, 78 (No. 1, whole number 578).
- Leron, U. (2010). The power of natural thinking: Applications of cognitive psychology to mathematics education. In P. Brosnan, D. B. Erchick & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. VI, pp. 9-20). Columbus, OH: The Ohio State University.
- Lim, K. H., & Mendoza, M. (2010). Inferring impulsive-analytic disposition from written responses. In P. Brosnan, D. B. Erchick, D. B. & L. Flevares, L. (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 759-767). Columbus: Ohio State University.
- Lim, K. H., & Morera, O. (2011). Impulsive-analytic disposition: Instrument pilot testing. *Proceedings of the 33rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 284-292). Reno, NV: University of Nevada.
- Lim, K. H., Morera, O., & Tchoshanov, M. (2009). Assessing problem-solving dispositions: Likelihood-to-act survey. In S.L. Swars, D.W. Stinson & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 700-708). Atlanta: Georgia: PME-NA.
- Nietfeld, J., & Bosma, A. (2003). Examining the self-regulation of impulsive and reflective response styles on academic tasks. *Journal of Research in Personality*, 37, 118-140.
- Olsson, U. (1979) Maximum likelihood estimation of the polychoric correlation coefficient. *Psychometrika* 44, 443-460.
- Patton, J. M., Stanford, M. S., & Barratt E. S. (1995). Factor structure of the Barratt Impulsiveness Scale. *Journal of Clinical Psychology*, 51, 768-774.
- Stanovich, K.E., & West, R.F. (2000). Individual differences in reasoning: Implications for the rationality debate. *Behavioural and Brain Sciences*, 23, 645-726.
- Watson, A., & Mason, J. (2007). Taken-as-shared: A review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education*, 10, 205-215.

Teaching Algebra Conceptually: Student Achievement

Chris Linsell

University of Otago

<chris.linsell@otago.ac.nz>

Megan Anakin

University of Otago

<megan.anakin@otago.ac.nz>

Rachel Jones

St Hilda's Collegiate

<rjones@shcs.school.nz>

Donna Smith

St Hilda's Collegiate

<dsmith@shcs.school.nz>

Lynn Tozer

Independent Consultant

<lynn.tozer@gmail.com>

Anna Cox

St Hilda's Collegiate

<acox@shcs.school.nz>

Eric McAuslan

King's High School

<ml@kingshigh.school.nz>

Garry Turner

King's High School

<tr@kingshigh.school.nz>

This paper reports findings from the second year of a two-year study designed to develop approaches to teaching algebra in years 9 and 10. The aim of the research was to explore and develop teaching approaches to assist students to acquire a conceptual understanding of algebra, and to document the impact of these approaches on student outcomes. Previous work (Linsell, 2009) has detailed the strategies that students use to solve equations but there has been no research to date on teaching approaches that make use of these findings. The two main components of this study were a qualitative description of teaching approaches based on videos, teachers' diaries and meeting notes, and a quantitative analysis of student outcomes based on results from a previously developed assessment tool (Linsell, Savell, Johnston, Bell, McAuslan, & Bell, 2006). Approaches to teaching were not uniform between teachers, but all made extensive use of the diagnostic assessment information and recognised that algebraic thinking pervades the entire mathematics curriculum. Assessment of the students at the end of either one or two year's participation in the study showed significant improvements in algebraic strategies and knowledge compared to baseline, though there was little improvement during the second year. Furthermore, the measures of student outcomes displayed significantly higher values for students in this study compared to year 9 and year 10 students in Secondary Numeracy Project (SNP) schools, which are fairly representative of the population.

A large-scale study by Warren (2003) found that the majority of students in Queensland leave primary school with a limited awareness of mathematical structure and of arithmetic operations as general processes. Bednarz, Kieran, and Lee (1996) have shown that algebra in schools is often reduced to rules for transforming and solving equations, and Brekke (2001) has shown that these rules are often meaningless for students. In New Zealand at the secondary school level, the emphasis within the Number and Algebra strand shifts from number towards algebra. The development of algebraic thinking at years 9 and 10 is therefore of great importance. The Best Evidence Synthesis (BES) project (Anthony & Walshaw, 2007) identified outcomes-linked evidence about what works in the secondary school context as a gap in our knowledge.

This project is an extension of a Teaching and Learning Research Initiative (TLRI) study carried out in 2006, and work undertaken in 2007 – 2010, funded by the New Zealand Ministry of Education. It makes use of the diagnostic assessment tool developed in the previous TLRI study (Linsell, et al., 2006) and results about student achievement from the Ministry of Education Secondary Numeracy Project (SNP) study (Linsell, 2010). Previous work detailed the strategies that students use to solve equations and the prerequisite knowledge required for implementing these strategies. However there has been no research to date on teaching approaches that make use of these recent findings other than preliminary

results from the first year of this study (Linsell, Tozer, Anakin, Cox, Jones, McAuslan, Smith, & Turner, 2011).

The current study is based on a conception of algebraic thinking as awareness of mathematical structure, rather than a collection of rules and procedures to be learnt. Mason, Stephens, and Watson (2009, p10) define mathematical structure to be “the identification of general properties which are instantiated in particular situations as relationships between elements or subsets of elements of a set.” With this definition in mind, our study then takes a similar approach to the New Zealand Numeracy Development Projects (NDPs) by separating the algebraic strategies that students use from the prerequisite knowledge required, and by designing learning experiences that address the identified needs of the students. The results from this project help to address the gaps in knowledge about successful approaches to teaching algebra in secondary schools.

Methodology

The research was carried out over two years and the approach used employed both qualitative and quantitative methods. For impact on student outcomes, pre/post data were used as well as comparisons with a representative sample of the population, the year 9 and 10 students from Linsell’s (2010) SNP study. For teaching approaches, case studies were employed. The research questions addressed were:

- How can algebra diagnostic assessment information be used for designing differentiated learning at years 9 and 10?
- What is the impact of the teaching approaches on the most sophisticated strategies students are able to use to solve equations, find relationships between variables, and express generality?
- What is the impact of the teaching approaches on students’ understanding of inverse operations, arithmetic structure, equivalence, acceptance of lack of closure and knowledge of algebraic notation and conventions?

Subjects

The study took place in one secondary school and one year 7 to 13 college, both in New Zealand, with five teachers participating. During 2011 a total of 185 students took part in the study. However, this paper presents results only from 107 year 9 students, and 46 year 10 students who also participated as year 9 students in 2010.

Assessment Tools

An extended version of the diagnostic assessment tools developed by Linsell, et al (2006) was used to document the algebraic thinking of the students. A full description of the assessments and scoring rubrics for students’ strategies for solving equations, understanding of algebraic notation and convention, inverse operations, arithmetic structure, equivalence, and acceptance of lack of closure is given in Linsell (2010). During 2011 additional questions were added to investigate students’ strategies for finding relationships between variables and expressing generality. These additional aspects of students’ algebraic thinking, which are consistent with Trigueros and Ursini’s (2001) model of three uses of variables, were included because the 2010 tool was considered not to encompass the full range of algebraic thinking.

Teaching Approaches

Much of the project was spent designing, trialling, and developing teaching approaches that were specific to the learning needs of the students. As well as considering the most sophisticated strategies that students were able to use, teaching addressed the prerequisite knowledge required by the students. In order to develop teaching approaches Borko's (2004) model of professional development was employed, which involved teachers capturing their lessons on video and sharing selected excerpts with the others in the project. There were regular meetings of the group, in which the experiences of the teachers were used to refine the teaching approaches for the other teachers to try.

Data Analysis of Student Achievement

The measures of student achievement investigated were strategies for solving equations, finding relationships between variables, expressing generality, and knowledge of algebraic notation and conventions, inverse operations, arithmetic structure, equivalence, and acceptance of lack of closure. For each of these measures a score was generated for each student at baseline and end of year. No assumptions were made about normality of data and differences between the baseline and end of year data were tested using the Wilcoxon signed ranks test. Differences between the end of year data and control groups of students at the same year levels in the SNP study were tested using the Mann-Whitney U-test.

Results and Discussion

Teaching Approaches

The teaching approaches developed in the study were not uniform, but were responsive to the particular needs of each class of students and also reflected the teaching styles and beliefs of the teachers involved. Differentiated learning was approached in quite different ways in the two schools. One school used mixed ability classes and created individualised learning for students through use of laptops. The other school streamed classes according to ability and the learning experiences for each class were designed to address the level at which students were working. There was, however, a great deal in common and a consensus was achieved on effective ways to help students learn algebra. Effective teaching approaches involved: establishing what each student understood through assessment, showing students how algebra is everywhere in our lives, promoting algebraic thinking with rich and meaningful contexts, demonstrating the benefits of rigour and correct use of vocabulary, and building a 'toolbox' of knowledge and skills that students could use appropriately and efficiently.

Assessment. The use of diagnostic assessment was considered to be essential so that decisions could be made about next learning steps, especially as students came from a number of contributing schools and therefore had widely differing prior experiences. The purpose was to document the most sophisticated strategies that a student could use and what prerequisite knowledge they had, not to generate an overall score. There was considerable discussion about the arithmetic skills and knowledge that many high school teachers might assume that their students would have learnt at primary school. Consistent with the findings of Warren (2003), the diagnostic assessment revealed that many students did not have a good understanding of arithmetic structure, inverse operations or equivalence. Students' knowledge of algebraic notation and conventions and acceptance of lack of closure were also documented, as well as their strategies for solving equations, expressing generality and

finding relationships between variables. Teachers' approaches with their classes and interventions with individual students throughout the project were guided by the detailed knowledge they had acquired.

Algebra everywhere. A major plank of the approaches taken was to reject the common practice of teaching algebra as isolated units of work once a term or, even worse, once a year. Instead the teachers integrated the teaching of algebraic thinking throughout their programmes. For example, when teaching equivalent fractions such as $\frac{3}{4} = \frac{9}{12}$ and $\frac{55}{25} = \frac{11}{5}$ by multiplying or dividing numerators and denominators by the same factor, this was generalised to $\frac{ac}{ab} = \frac{c}{b}$. The generalisation and the use of algebraic notation to describe the generalisation, was made explicit to the students. A great deal of emphasis was placed on the patterns found when calculating with rational numbers and with rational expressions. Furthermore, the algebraic skills taught were used throughout the year in a variety of contexts and reinforced through regular maintenance.

Rigour and vocabulary. Observation of students' work and discussions with them indicated that very informal written working and a lack of correct mathematical vocabulary were impediments to good mathematical practice and engagement in mathematical discourse. The teachers were particularly careful with correct use of mathematical vocabulary throughout the year. Any terms that were unfamiliar or ambiguous to the students were defined and written into the students' notebooks. Careful board work that modelled correct setting out was used and similar standards were insisted on from the students. Conventions of notation were not assumed but were explicitly taught. Similarly, mathematical identities and laws were made explicit and expressed as generalisations using algebraic notation. The schools ensured that there was consistency from teacher to teacher and from one year level to the next.

The toolbox. When algebraic skills were identified, they were presented to the students as tools to put in their toolboxes. This metaphor was used to promote acceptance and understanding of the skills. These skills included, but were not limited to substitution, manipulating terms, expanding brackets, factorising, and strategies for solving equations. Number skills related to indices, integers, order of operations, basic facts, squares, cubes, and highest common factors were also placed in the students' toolboxes. When solving problems in any context, students were encouraged to select and use tools purposefully. This approach avoided skills being taught in isolation, as students appreciated that these were tools that they would use frequently. Thus the skills became more readily transferrable.

Context. It was regarded as essential that students should perceive algebra as being meaningful and a wide variety of contexts were used to ensure this. Furthermore, teachers interpreted and used the idea of context in a number of different ways. When solving problems in areas such as rational number, students were encouraged to use modelling approaches and set up equations to solve. Science teachers were consulted to ensure that approaches to solving equations and using notation were consistent between mathematics and science lessons. Also, some of the teachers chose to develop thematic units aimed at developing algebraic concepts in holistic ways.

Student Achievement

Strategies for solving equations. Previous work with SNP students (Linsell, 2010) has shown that many students are restricted to very unsophisticated strategies, such as guess and check, and may not even be able to use inverse operations for simple equations of the form

$ax = b$. Furthermore, very few students at the end of years 9 and 10 are able to solve equations formally by carrying out transformations (i.e., by doing the same thing to both sides). Baseline measurements in this project were consistent with these previous findings. Summative assessment at the end of the year, as shown in Figure 1, showed highly significant improvement from baseline in year 9 students' use of strategies for solving equations ($p < 0.001$, Wilcoxon signed ranks test). Furthermore, the year 9 students were significantly more sophisticated in their use of strategies than their counterparts in the SNP study ($p < 0.001$, Mann-Whitney U-test).

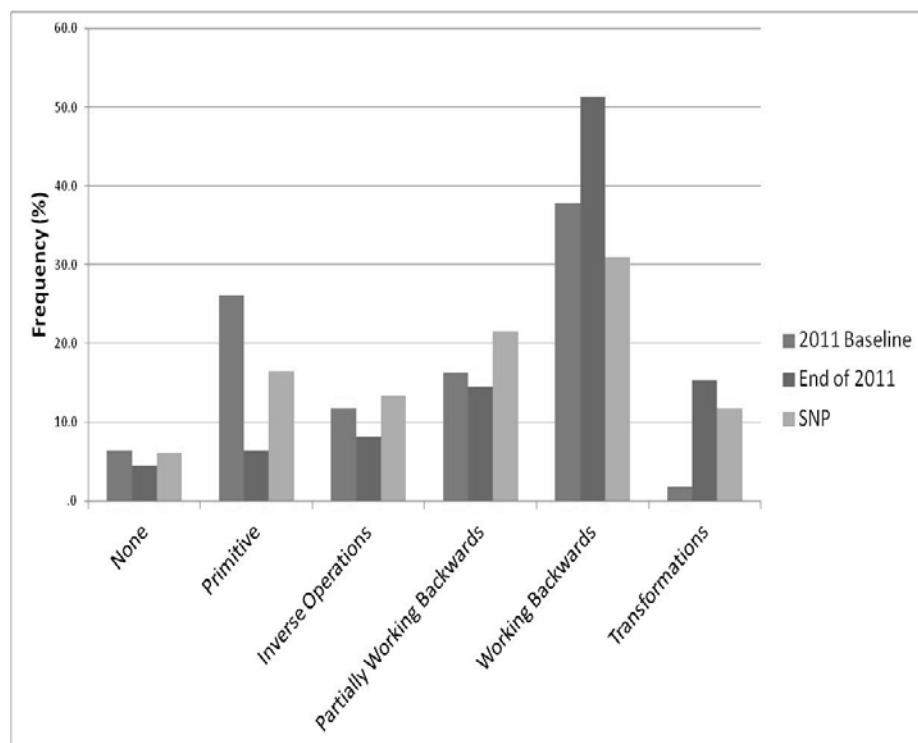


Figure 1. Year 9 students' most sophisticated strategy for solving equations.

Similarly, the year 10 students were significantly more sophisticated in their use of strategies than their counterparts in the SNP study ($p < 0.001$, Mann-Whitney U-test). The major difference between the SNP students and these was that after two years in this project very few year 10 students were using unsophisticated strategies and most were solving equations of forms $ax + b = c$, $\frac{x-a}{b} = c$, etc., by working backwards or by using transformations. However, while 60% of them were able to use the strategy of working backwards, only 26% of them were able to use the most sophisticated strategy of carrying out transformations by doing the same thing to both sides, even at the end of two years' participation. For these year 10 students, a rather interesting picture emerged. Even though there was a highly significant improvement in their use of strategies for solving equations during their first year in the project ($p < 0.001$, Wilcoxon signed ranks test), there was no significant improvement during the second year ($p = 0.24$, Wilcoxon signed ranks test).

Algebraic knowledge. At each assessment point there were four questions in each area that were considered to be prerequisite algebraic knowledge. Each student therefore had a score between zero and four on arithmetic structure, inverse operations, algebraic notation and convention, acceptance of lack of closure, and understanding of equivalence.

For the year 9 students all of the measures showed significant improvements from baseline to end of year assessment ($p < 0.001$, Wilcoxon signed ranks test), except for

knowledge of inverse operations ($p=0.83$). These year 9 students' scores on four of the measures of algebraic knowledge at end of year were significantly higher than their counterparts in the SNP study ($p<0.01$, Mann-Whitney U-test), again with knowledge of inverse operations showing no significant difference.

For the year 10 students only knowledge of notation and convention ($p<0.01$, Wilcoxon signed ranks test) and understanding of equivalence ($p<0.05$) showed significant improvements from start of year to end of year assessment, and knowledge of inverse operations actually showed a decrease ($p=0.001$). However, all of the measures were still significantly higher than in their counterparts in the SNP study ($p<0.001$ for four measures and $p<0.05$ for knowledge of inverse operations, Mann-Whitney U-test). As with the development of strategies for solving equations, most of the improvement in algebraic knowledge for the year 10 students had occurred in their first year in the project.

The apparent lack of improvement over the year in knowledge of inverse operations for the year 9 students, and decreased score for year 10 students is difficult to account for. The questions used in the pre and post tests and in the SNP test appear to be quite equivalent. Furthermore the teachers were very aware of the gaps in their students' knowledge and actively addressed them in their teaching.

Strategies for finding relationships between variables. It is well known that students find it difficult to determine relationships of the form $y = ax + b$ when presented with tables of values (Hoyles & Küchemann, 2001; Trigueros & Ursini, 2001). We therefore included easier questions addressing additive and directly proportional relationships, and also questions that allowed students to find and make use of relationships even if they were not able to express them algebraically. Factor analysis of data from the whole 2011 cohort revealed four factors, which we described as *additive strategies*, *multiplicative strategies*, *use of algebraic conventions*, and *formal use of linear relationships*. Nine questions correlated strongly ($r>0.65$) with one of these factors on both the baseline and summative tests, and were used to indicate the strength of thinking on these factors. Questions correlating strongly with the additive strategies factor could be solved by purely arithmetic additive strategies. Questions correlating strongly with the multiplicative strategies factor could be solved by purely arithmetic multiplicative strategies. Questions correlating strongly with the use of algebraic conventions factor required students to determine relationships of the form $y = x + a$ or $y = ax$ from a table of values. Questions correlating strongly with the factor formal use of linear relationships required students to determine relationships of the form $y = ax + b$ from a table of values, or to use the relationship even if they were not able to express it formally.

For the year 9 students the scores for additive strategies were very high at baseline and showed no significant change at end of year assessment. There were, however, significant improvements in scores for use of multiplicative strategies ($p<0.05$), use of algebraic conventions ($p<0.001$), and formal use of linear relationships ($p<0.001$, Wilcoxon signed ranks test). However, at the end of the year and in spite of the improvement in formal use of linear relationships, only 29% of year 9 students were able to determine a relationship of the form $y = ax + b$ when presented with a table of values.

For the year 10 students there was no significant improvement during the year on any of the scores other than use of algebraic conventions ($p<0.01$, Wilcoxon signed ranks test).

Strategies for expressing generality. It has been known for a long time that students find it difficult to generate algebraic expressions for structured situations and operate on them (Lee, 1987). Most work with high school students has focussed on situations that give rise to linear or quadratic expressions (see, for example, Mason, Graham, & Johnston-

Wilder, 2005), so we decided to also include easier situations that could be described by expressions such as $x + a$ or ax .

At baseline 58% of year 9 students were unable to generate even simple expressions such as $x + a$ or ax . By the end of the year there were significant increases in their ability to generate additive ($p < 0.001$), multiplicative ($p < 0.001$), and linear expressions ($p < 0.001$, Wilcoxon signed ranks test). However, there was no significant change in their ability to operate on generalised numbers to generate non-linear expressions, with only 13% at end of year 9 able to do so.

For the year 10 students there was no significant change in their ability to generate additive, multiplicative or non-linear algebraic expressions, but a significant decrease in their ability to generate linear expressions ($p < 0.001$, Wilcoxon signed ranks test).

Conclusions

This study has demonstrated that achievement in simple algebra can be enhanced by taking a structural perspective that provides high quality diagnostic assessment information to teachers and that integrates algebra into the curriculum, rather than treating it as a separate topic. Through the use of a diagnostic tool which reveals students' algebraic knowledge and strategies, teachers were able to gain significant insights into their students' algebraic world. This information was empowering for the teachers. They placed importance on the diagnostic assessment information to inform them of their students' strengths and weaknesses and in addressing these, they encouraged their students to 'see algebra everywhere' and provided them with relevant and meaningful contexts for exploration.

Many students had poor understandings of arithmetic structure, inverse operations, and equivalence. The teachers in this study addressed these gaps in knowledge to assist the students to use the more sophisticated strategies for solving equations, finding relationships between variables, and expressing generality.

For the year 9 students, the teaching approaches used in this project appear to have improved their prerequisite knowledge and enabled them to progress from being restricted to the least sophisticated strategies. However, even when students have a firm foundation in arithmetic, the most sophisticated algebraic strategies are clearly very difficult to master. Relatively few year 9 students were able to use the most sophisticated strategies of using transformations for solving equations, formal linear relationships for finding relationships between variables, or operating on generalised numbers to generate non-linear expressions. These strategies have in common the requirement for students to operate on the unknowns, generalised numbers or variables.

There was very little progress for the year 10 students during the second year of their participation in the project, in spite of them having made good progress during year 9. The apparent ceiling reached by many students is of great concern.

The teaching approaches used do not appear to have facilitated transition across the 'cognitive gap' between arithmetic and algebra identified by Herscovics and Linchevski (1994). They described this gap as being characterised by "the student's inability to operate with or on the unknown" (p75). It can be argued that the less sophisticated strategies being mastered by the students in this study are essentially arithmetic. Teachers should not underestimate the conceptual leap required for operating on generalised numbers, unknowns and variables. Further research is required on teaching approaches that may facilitate students' transitions to the more sophisticated strategies.

Acknowledgement

This work was supported by a TLRI grant from the New Zealand Council for Educational Research.

References

- Anthony, G., & Walshaw, M. (2007). *Effective Pedagogy in Mathematics/Pangarau Best Evidence Synthesis Iteration (BES)*. Wellington: Ministry of Education.
- Bednarz, N., Kieran, C., & Lee, L. (1996). Approaches to Algebra: Perspectives for Research and Teaching. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching* (pp. 3-12). Dordrecht: Kluwer Academic.
- Borko, H. (2004). Professional Development and Teacher Learning: Mapping the Terrain. *Educational Researcher* 33(8), 3-15.
- Brekke, G. (2001). School algebra: Primarily manipulations of empty symbols on a piece of paper? In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra* (pp. 96-102). Melbourne: University of Melbourne.
- Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27(1), 59-78.
- Hoyles, C., & Küchemann, D. (2001). Tracing development of students' algebraic reasoning over time. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra* (pp. 320-327). Melbourne: University of Melbourne.
- Lee, L. (1987). The status and understanding of generalised algebraic statements by high school students. In J. C. Bergeron, N. Herscovics & C. Kieran (Eds.), *11th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 316-323). Montreal, Canada.
- Linsell, C. (2009). A Hierarchy of Strategies for Solving Linear Equations. In R. Hunter, B. Bicknell & T. Burgess (Eds.), *Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 331-338). Palmerston North: Mathematics Education Research Group of Australasia.
- Linsell, C. (2010). Secondary Numeracy Project Students' Development of Algebraic Knowledge and Strategies. In Ministry of Education (Ed.), *Findings from the New Zealand Numeracy Development Projects 2009* (pp. 100-117). Wellington: Learning Media.
- Linsell, C., Savell, J., Johnston, N., Bell, M., McAuslan, E., & Bell, J. (2006). Early algebraic thinking: Links to numeracy, from http://www.tlri.org.nz/pdfs/9242_summaryreport.pdf
- Linsell, C., Tozer, L., Anakin, M., Cox, A., Jones, R., McAuslan, E., et al. (2011). *Teaching algebra conceptually: Preliminary findings*. Paper presented at the New Zealand Association for Research in Education.
- Mason, J., Graham, A., & Johnston-Wilder, S. (2005). *Developing thinking in algebra*. London: Thousand Oaks.
- Mason, J., Stephens, M., & Watson, A. (2009). Appreciating Mathematical Structure for All. *Mathematics Education Research Journal*, 21(2), 10-32.
- Trigueros, M., & Ursini, S. (2001). Approaching the study of algebra through the concept of variable. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra* (pp. 598-605). Melbourne: University of Melbourne.
- Warren, E. (2003). The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Education Research Journal*, 15(2), 122-137.

Does Knowing More Advanced Mathematics Ensure Effectiveness of Working Towards Demonstrating Specialised Mathematical Content Knowledge of Second-Year Pre-Service Teachers?

Sharyn Livy
Victoria University
<sharyn.livy@live.vu.edu.au>

The theoretical understanding that underpins a teacher's foundation knowledge draws on their common content knowledge (CCK) and influences their mathematics' teaching (Rowland, Turner, Thwaites, & Huckstep, 2009). Teachers who have specialised content knowledge (SCK) demonstrate a unique kind of content knowledge which is more than knowing the content (Ball, Thames, & Phelps, 2008). This study reports on a comparison of two second-year pre-service teachers who had varied mathematical content knowledge (MCK) at the beginning of their Bachelor of Education course. It investigated whether knowing more advanced mathematics or foundation knowledge (Rowland et al., 2009) facilitates working towards demonstrating SCK. The results draw on a qualitative analysis, categorising lesson observation and interview responses using foundation knowledge and connections constructs of the 'Knowledge Quartet' framework (Rowland et al., 2009). Both pre-service teachers experienced course opportunities that consolidated foundation knowledge and demonstrated connections. During their lesson observation they relied on procedural explanations and neither especially demonstrated working towards SCK.

Within a large lecture theatre of pre-service teachers it can be difficult to get to know what is going on in the minds of beginning teachers. All possess a different knowledge of mathematics, mostly gained from their own educational background. At the university where this study took place, many pre-service teachers will gain employment in the Western Metropolitan Region. This region is experiencing a rapid growth in population.

This study endeavours to outline implications that will inform planning and delivery of teacher training for future pre-service teachers and the development of primary mathematics' teachers for local schools and/or beyond. Identifying course experiences that assist pre-service teachers to develop MCK can support their students' mathematical growth. Building on pre-service teachers' foundation knowledge for teaching mathematics is important. "It is fundamental because it underpins all the decisions about which examples or representations to use, connections to make, or how to respond to pupils' ideas" (Rowland et al., 2009, p. 152).

This paper focuses on the experiences and learning related to the developing MCK of two second-year pre-service teachers. The study aims to identify the opportunities and influences that enhanced their MCK during the first and second years of their course. Did knowing more mathematics at the start of their course make a differencing when comparing examples of foundation and connected MCK? These findings will contribute to a larger four-year longitudinal study of 17 pre-service teachers' MCK. The literature highlights a need for cross case analyses of different approaches, programs and settings that affect content knowledge over time, as well as longitudinal studies of learning to teach (Adler, Ball, Krainer, Lin, & Novotna, 2005; Mewborn, 2001).

Literature Review

Research on the study of teacher knowledge is recent when compared to the history of teaching and research on teaching. For the past two decades, research on mathematics

teaching has included a focus on the knowledge teachers use and need for their craft of teaching (Grossman & McDonald, 2008).

Teacher knowledge has been described as complex and existing of many facets. Schulman's (1987) theoretical framework described seven categories which became the foundation for labelling the knowledge base for teaching. Content knowledge is a central feature of his framework and is referred to as the "...amount and organisation of knowledge in the mind of the teacher..." (Schulman, 1987, p. 9).

Many studies have further analysed this complex knowledge. Ma (1999) describes a teacher's deep knowledge of content as Profound Understanding of Fundamental Mathematics, demonstrating breadth, depth, connectedness and thoroughness. Schoenfeld and Kilpatrick (2008) depict teachers' knowledge as similar, with proficient teachers of mathematics knowing multiple methods as well as broad and deep knowledge of mathematics. Effective teachers can draw on a range of mathematical knowledge such as procedural knowledge, procedural fluency, conceptual knowledge and mathematical connections (Ball & Bass, 2003; National Curriculum Board, 2009). These descriptions are examples that describe the unique characteristics of teachers' MCK. Ball and colleagues (2008) refer to this knowledge as SCK which is unique to teaching.

Pre-service teachers bring to the course CCK. For example, CCK is simply when someone is able to calculate an answer, demonstrate competence with the content, or recognise an incorrect answer (Ball et al., 2008). CCK knowledge will vary amongst pre-service teachers depending on their experiences before entering teacher education.

Recent Australian studies have identified weaknesses in pre-service teachers' MCK with many relying on procedural methods (Goos, Smith, & Thornton, 2008). Mewborn's (2001) critique of research found no significant correlation between the number of mathematics courses taken by a teacher and their mathematical knowledge for teaching. Possibly these courses and pre-service teachers' secondary mathematics education focused on procedural or CCK rather than developing SCK. Future pre-service teachers may bring to their course a different breadth of foundation knowledge as Australian schools implement a national curriculum (ACARA, 2012). The introduction of the proficiency strands of Reasoning, Problem Solving, Understanding and Fluency within the Australian Curriculum may deliver more students with a more connected understanding of mathematics.

Framework for MCK

The 'Knowledge Quartet' framework was designed as a means for reflecting on the content of mathematics lessons, assisting teachers to focus on what they know and what they do when teaching (Rowland et al., 2009). It has four main categories describing types of MCK required to teach mathematics well: foundation, connection, transformation and contingency. In this paper the dimensions of foundation and connection were applied when coding interview transcripts of two second-year pre-service teachers:

- Foundation: the knowledge pre-service teachers would bring with them to teacher education acquired from their schooling
- Connection: making connections between procedures and concepts and the cognitive demands of mathematics

Methodology

Context and Participants

The two pre-service teachers selected for this study were completing the Bachelor of Education Course (P-12). At graduation they will have the qualifications to teach in primary schools as well as their discipline specialisations in secondary schools. Most pre-service teachers enrolling in the Bachelor of Education (P-12) course have completed two years of the Victorian Certificate of Education (VCE) as course entry requirements. When completing the final year (Year 12) of secondary school, students may choose to study mathematics: Further Mathematics Units 3 and 4, Mathematical Methods (CAS)¹ Units 3 and 4 or Specialist Mathematics Units 3 and 4 (VCAA, 2010).

The participants in this study, Con and Rose, were part of a larger longitudinal study of 17 pre-service teachers' primary MCK. Two main factors contributed to selection for this study: the pre-service teachers' level of mathematics completed during their secondary education and their school based experiences during the first two-years of their course.

Both pre-service teachers had entered the course with different experiences of mathematics during their secondary schooling. During Year 12, Con completed Specialist Mathematics, which requires students to demonstrate a strong knowledge of mathematics and enables them to take further studies in mathematics or related disciplines when entering university. Con was completing mathematics as a major as part of his course in order to qualify to teach secondary mathematics.

Rose selected Art and Drama as her major for secondary school preparation. She completed and passed VCE, but chose not to continue to study mathematics in Year 12. During Year 11, she passed Mathematical Methods (CAS) Units 1 and 2. This unit is a pre-request for the Mathematical Methods (CAS) Units 3 and 4 that provides students with skills to undertake further study in, for example, science, humanities, economics or medicine (VCAA, 2010).

During their course all pre-service teachers participate in school based experiences known as Project Partnership. They mostly attend Project Partnership each Tuesday and complete full week block experiences. The second criteria for selecting Con and Rose for this study was that their Project Partnership experiences were similar. During first-year Con was in a Year 1 classroom and Rose a Year 1/2 composite classroom. During second-year Con was in a Year 6 classroom and Rose was in a composite Year 5/6 classroom. They both experienced different schools.

Data Collection

During the middle of second-year the pre-service teachers met with the researcher at their placement school for Project Partnership. The researcher observed the pre-service teachers teaching a primary mathematics lesson. As a coincidence both pre-service teachers chose to teach the same topic angles. After their lesson the researcher interviewed the pre-service teachers. The interview questions related to MCK during Project Partnership and course experience.

At the time of their interviews both pre-service teachers had completed different course experiences at university. Rose was interviewed first in May and had completed four education units relating to teaching primary mathematics. During first-year she had

¹ Computer Algebra System

completed Inquiry for Understanding but thought she had only ‘learnt a little mathematics.’ Therefore, during summer school, (before second-year) she enrolled in an elective unit Numeracy and Mathematics (Victoria University, 2007). This unit was designed to assist pre-service teachers who required further understanding of MCK. If required, pre-service teachers usually enrolled in Numeracy and Mathematics during second-year. In particular this unit assisted pre-service teachers who were unable to demonstrate competency in the Mathematics Competency Skills and Knowledge Test for second-year education unit, Reasoning for Problem Solving (Victoria University, 2007).

Con was interviewed one month later but had only completed the first-year education unit relating to primary mathematics. Rose and Con studied their course at different campuses. Therefore, second-year units were completed during different semesters. Rose had completed two education units related to teaching primary mathematics. Con completed these units during second semester of second-year. Therefore, Con had not completed these units at the time of the lesson observation and interview. He had completed three elective units relating to mathematics as part of his course requirement for teaching mathematics in secondary schools. The first unit, Mathematical Foundations 1 was revision of VCE Mathematical Methods; the second two extended his mathematical knowledge.

Data Analysis

This study used qualitative methods to analyse a second-year lesson and interview data of two pre-service teachers’ MCK. Both lessons and interviews were audio taped; these were then transcribed and coded for analysis. Two dimensions of Rowland and colleagues’ (2009) ‘Knowledge Quartet’ framework foundation and connection were used for coding. Findings from lesson observations and interview data relating to MCK, primary mathematical education units of study and Project Partnership experiences during the first and second years of their course are reported below. A selection of reflections and comparison of the two pre-service teachers’ developing foundation and connected knowledge with reference to the ‘Knowledge Quartet’ (Rowland et al., 2009) are reported.

Results

Con’s Reflections on Study and Partnership

Remembering back to the first-year of the course Con was concerned that he did not have ‘*primary maths knowledge*’. He wanted to build on his ‘primary’ MCK. He did not think the first-year primary mathematics unit helped him develop his MCK. Con wanted to learn about what content he should be teaching and said he, “I just wanted help remembering it all.”

He also wanted to know how primary mathematics should be taught. For example, he was concerned about how to multiply. He had forgotten because he was used to using a calculator. To help him practice he chose to tutor a Year 6 and Year 7 student. He now remembers how to do multiplication and has developed a deeper understanding. He also mentioned how you might approach multiplication with larger digits. “A primary teacher should also be able to do bigger times tables, like 11 times 32 and [think] 10 times 32 plus 32 [equals 352].” Con is exploring his connected knowledge of mental strategies using partitioning (distributive law) to solve a multiplication problem.

Con rated his MCK by reflecting on his responses to a practice Mathematical Skills and Knowledge Test he completed during first-year. This test assessed pre-service teachers’

foundation knowledge and understanding of: Space, Number, Common and Decimal Fractions, Measurement, Chance and Data, Structure and Working Mathematically (Victoria University, 2006). When explaining his MCK he said, “I wouldn’t say I am weak with any of them [the topics in the test]. I got 89% or something like that for my practice test in first-year.” By correctly answering these test items, Con is providing evidence that he has brought to the course comprehensive foundation knowledge.

Before the interview, Con had taught a geometry lesson. He drew on his foundation knowledge throughout the lesson as he focused on the aim within his lesson plan: “Assist students to understand the different types of angles that two lines make and to measure angles”. When introducing the lesson he confidently provided definitions, discussing and drawing different angles onto the whiteboard for the students. For example; “If we have an angle that is smaller than a right angle... That is right an acute angle [drew an example onto the board]... Can anyone tell me what the ones are (sic) that are bigger than a 90 degree angle? Or right angle? [Student response: obtuse]... Anything between 90 degrees and 180 [degrees] is called obtuse...”

Con enjoyed teaching mathematics and could rely on his foundation knowledge “I enjoy primary maths [lessons] because it is not a stress to think about it and it is really easy.” His mentor teacher also consolidated his foundation knowledge when planning this lesson. For example, “There are angles, intervals and raise. An interval of a line is so much of a line. A continuous line, the interval would be from point A to point B. A raise is a point A and no end... My mentor told me this and it took a second to learn.”

During primary school Year 5 and Year 6 students would be expected to develop an understanding that angles can be static and dynamic (DEECD, 2006). Con thought primary teachers required more MCK knowledge than their students. He explained that learning about angles is important knowledge required for trigonometry. “I believe it is important for a primary teacher to know up to Year 9 mathematics... for example, like angles have connections to trigonometry, which have connections... and the connections go up...” Con is demonstrating his connected knowledge as he elaborates on how his lesson connects with later learning of mathematics. Secondary school students draw on knowledge of angles when studying trigonometry, for example the angles of triangles.

He also thought it was important to know mathematics because “If a student says, why do I need to learn this? You have the knowledge to explain...They [teachers] should also know all the basic formulas for area, perimeter, trigonometry....fractions, decimals, percentages are really big in primary maths from Year 3/4...I don’t think that teachers at primary school level should have to look it up the day before. They should know it...”

Rose’s Reflections on Study and Partnership

Rose remembered struggling at secondary school during mathematics classes. Mathematics’ textbooks and work sheets provided a negative experience and she was also taught rules, which she forgot. Since starting the course Rose believed her understanding of mathematics had changed dramatically. The four mathematics education units had assisted her to begin to develop connected knowledge. “I am starting to think now. Why do I want to do that? How is that related to anything I have done previously? Seeing the links...I need[ed] someone to explain it clearly to help me.” Rose is developing connections by thinking about breaking down mathematical concepts.

Rose believed doing the course and studying for the test had assisted her mathematical knowledge. “In first year we only learnt a little... I did the practice test...” Realising that she required extra support for developing her MCK, Rose enrolled in the elective unit,

Numeracy and Mathematics. Rose completed a lot of sample Mathematical Skills and Knowledge Tests. She would *'go back and look at how they got the answer'*. Rose sometimes asked her lecturer for help or researched on the internet and completed a mathematical learning log which was an assessment task for Numeracy and Mathematics. "It forced you study, but I would have studied anyway, that made me study more, it made it more clear it was a good assignment it made you learn." During first-year of her course Rose selected course opportunities that assisted her to build her foundation knowledge.

Rose highlighted the areas of mathematics in which she was most confident during second-year. "I feel like we focused more on number because there are so many different areas within number that I think we did... Space I remember as being one of the hardest things for me... I memorised the names of the shapes but now it is fading and I would have to research it again but at least I have all those notes now to go back to. There are so many things and definitions it is hard to remember..."

When planning the lesson Rose used websites and resources to check the names of different angles: acute, right, obtuse, straight, reflex and revolution. "I looked up the definitions and I used them in the lesson. I would have had a rough idea of the definitions for the lesson and now I think I know them off by heart..." In the introduction to her lesson with the students she implemented her revised foundation knowledge of the names of angles. She could correctly draw and label the different angles on the board as she explained these with the students.

Rose also made connections with her foundation knowledge as she began to think about activities, curriculum, knowing her students and recognising the importance of enhancing understanding by not drawing on rote methods or rules. "...the teacher needs to be able to not just use the rules but do hands-on activities... in the course you get to understand how the VELS Levels (DEECD, 2006) work and so you know what to teach certain kids and how to actually put them into groups and find out what they know..."

Discussion and Conclusion

Rowland and colleagues' (2009) 'Knowledge Quartet' framework was suitable for coding pre-service teachers' transcripts of lesson observations and interviews when classifying their foundation and connected knowledge. The findings of this small study suggest both pre-service teachers developed foundation and connected knowledge during the first two years of their course by doing primary mathematics units of study and participating in Project Partnership experiences. The pre-service teachers developed foundation knowledge in conjunction with their connected knowledge. For example, during Project Partnership when teaching students, talking with their mentor or planning lessons. Course experiences at university included: completing practice Mathematical Skills and Knowledge Tests, developing a mathematical learning log as well as completing primary mathematics education units.

At the beginning of their course Con had more advanced mathematics when compared to Rose. However, the practice Mathematical Skills and Knowledge Test during first-year provided an opportunity for both pre-service teacher to identify strengths and weakness relating to their CCK. In particular it assisted Rose in making the decision to enrol in an elective primary mathematics unit. Con identified some gaps in his MCK and chose to tutor mathematics students. These choices would have assisted development of foundation knowledge. A concern is that all pre-service teachers may not be as motivated as the participants in this study, failing to address and implement strategies to consolidate gaps in their MCK.

Con and Rose experienced upper levels of the primary school curriculum during second-year Project Partnership. Teaching older primary students could provide opportunities to practice the foundation knowledge they may have revised for their Mathematical Skills and Knowledge Test. Their foundation knowledge of a topic could be drawn on in the classroom as they made connections when planning, teaching or responding to students' mathematical problems.

Before their interviews both pre-service teachers had taught a similar lesson focusing on geometry and the understanding of the names of different angles. However they both modelled procedural knowledge when introducing their lessons and naming and labelling different angles for the students. Rowland and colleagues (2009) describe this in trainee teachers as an instrumental understanding, excessively relying on procedures. A different approach would have been to allow the students to construct their own angles and sort them into various groups, acute, obtuse etc. This would assist students with a conceptual understanding rather than a rote method and would demonstrate pre-service teachers' connections with knowing the sequence of the topic.

When comparing the two pre-service teachers in this study, knowing more advanced mathematics did not especially promote working towards demonstrating SCK. Mewborn's (2001) critique of research reported similar findings with teachers. During their interviews, both pre-service teachers explained their thinking and provided examples of how they were developing conceptual connections. However, when both pre-service teachers were observed teaching a primary mathematics lesson related to angles they lacked connected knowledge and relied on procedures when introducing the different angle names to their students. They lacked cognitive demands of mathematics for planning and sequencing the lesson (Rowland et al., 2009). Goos and colleagues (2008) also reported that other studies suggest pre-service teachers rely on procedural methods.

From this study, one can conclude that the practice Mathematical Skills and Knowledge Test provided an opportunity for pre-service teachers to assess and address needs relating to gaps in foundation knowledge. Providing opportunities to enrol in an elective numeracy unit assisted development of foundation knowledge. Both pre-service teachers demonstrated that they knew the foundational knowledge for the lesson they taught. A concern was that both pre-service teachers were teaching rote knowledge when introducing their lesson during the second-year of their course. They required further guidance and understanding when designing a lesson for demonstrating connected knowledge of the topic. Making connections is at the heart of teaching and teaching is more than telling students what they need to learn (Rowland et al., 2009). A teacher uses their SCK for identifying mathematical connections when working with students and planning lessons (Ball et al., 2008).

There are limitations to this study as data was only collected from two pre-service teachers. A larger in depth study of 17 pre-service teachers' MCK will use a cross case analysis of various course experiences to test the findings reported here. Further analysis of lessons including those in the final year of their course will provide further detail and will be reported in the larger longitudinal study of pre-service teachers' MCK.

References

- Adler, J., Ball, D. L., Krainer, K., Lin, F.-L., & Novotna, J. (2005). Reflections on an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics*, 60, 359-381.
- Australian Curriculum Assessment and Reporting Authority [ACARA]. (2012). The Australian Curriculum: Mathematics V3.0 [Electronic Version]. Retrieved 18.03.2012 from <http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10>.

- Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematics knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton: AB:CMESG/GCEDM.
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59, 389-407.
- Department of Education and Early Childhood Development [DEECD]. (2006). Mathematics developmental continuum P-10. Retrieved 07.04.2010, 2012, from <http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/space/SP32502P.htm>
- Goos, M., Smith, T., & Thornton, S. (2008). Research on the pre-service education of teachers of mathematics. In H. Forgasz, A. Barkatsas, A. Bishop, B. Clarke, S. Keast, W. T. Seah & P. Sullivan (Eds.), *Research in Mathematics Education in Australasia 2004-2007* (pp. 291-311). Monash University, Australia: Sense Publishers
- Grossman, P. L., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal*, 45(1), 184-205.
- Ma, L. (1999). *Knowing and teaching elementary mathematics. Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, New Jersey: Lawrence Erlbaum Associates, Inc.
- Mewborn, D. S. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Teacher Education & Development*, 3, 28-36.
- National Curriculum Board. (2009). *National mathematics curriculum: Framing paper*. Retrieved 01.07.09, from http://www.ncb.org.au/verve/resources/National_Mathematics_Curriculum_-_Framing_Paper.pdf.
- Rowland, T., Turner, F., Thwaites, A., & Huckstep, P. (2009). *Developing primary mathematics teaching: Reflecting on practice with the knowledge quartet*. London: SAGE Publications Ltd.
- Schoenfeld, A. H., & Kilpatrick, J. (2008). Toward a theory of proficiency in teaching mathematics. In D. Tirosh & T. Wood (Eds.), *The international handbook of mathematics teacher education: Tools and processes in mathematics teacher education* (Vol. 2, pp. 321-354). Rotterdam: Sense Publishers.
- Schulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57(1), 1-22.
- Victoria University. (2006). *Arts, Education and Human Development handbook 2007*. Melbourne: Author.
- Victoria University. (2007). *Arts, Education and Human Development handbook 2008*. Melbourne: Author.
- Victorian Curriculum and Assessment Authority [VCAA]. (2010). *Mathematics: Victorian certificate of education study design*. Retrieved 19.07.2011, from <http://www.vcaa.vic.edu.au/vce/studies/mathematics/cas/casindex.html>.

The Development of an Assessment Tool: Student Knowledge of the Concept of Place Value

Karen Major
University of Auckland
<k.major@auckland.ac.nz>

The importance of student understanding of the concept of place value cannot be underestimated. Place value is a 'gate keeper' in developing mathematical understanding. The purpose of this study was to examine and develop a teacher-made test of place value knowledge. The questions were developed using the progressions from the Number Framework (Bobis, Clarke, Clarke, Thomas, Wright et al., 2005). An exploratory factor analysis was used to evaluate the assessment tool. The analysis of student responses to the test questions revealed a three-factor structure that supported the existing literature on the progression in learning of place value ideas, by identifying the critical key ideas that underlie the concept of place value. The results validated the tool as a test of place value knowledge that could be used to assess the performance of Yrs. 3-9 students.

Introduction

In 2005 a cluster of seven schools formed a professional learning community (PLC) with a common goal: to raise the mathematical achievement of all students within their school community. The schools' numeracy lead teachers began to meet regularly for in-depth discussions on reaching this goal. All these schools had previously participated in a national professional development programme in mathematics (Ministry of Education, 2007). Teacher awareness of the 'gaps' in students' mathematical knowledge began to emerge through participation in this project. Student knowledge of place value was perceived as a barrier to further learning in mathematics. The PLC decided, therefore, to focus on place value. As a first step they wanted to know more about what the students knew about place value, across the seven schools. To this end they asked the author to develop a tool that would provide information about the students in Years 3 to 9 thereby providing a suitable starting point for teaching and learning programmes. This paper reports on the performance and development of the assessment tool that was designed to meet this community's needs (Major, 2011).

Background

Place value is a necessary and fundamental mathematical concept for student success in mathematics and as such is a gatekeeper to further mathematical understanding. The complexity of place value understanding is masked by the simplicity of its modern formulation, which condenses multiple key ideas into an underlying construct. Simply defined as a way of naming or representing numbers, place value provides a structure that allows us to write and say numbers, allowing for both whole numbers and parts of numbers to be represented, and makes mental computation possible. The significance of place value knowledge is evident in our ability to read, write and understand large numbers, carry out complex computations and express the results of our calculations in a variety of forms. Place value is conceptually embedded in any mathematical operation students need to learn, and it is this embedding that makes the teaching and learning of this concept challenging. Research evidence suggests that students can appear to have mastered aspects of place value yet have no real understanding of the concept, being unable to generalise the multiplicative relationships within the place value system (Irwin, 1996; Kamii, 1986; Thomas, 2004).

Research has identified relevant knowledge that is necessary for the development of an understanding of place value. Student knowledge of counting patterns and strategies is markedly related to the knowledge of, and ability to explain, place value (Boulton-Lewis, 1996; Rubin & Russell, 1992; Young-Loveridge, 2008). Young-Loveridge (1998) argued that children can be taught place value when they understand the concept of ones and have constructed number relationships that will support the concept of ten as a unit. Thomas and Ward (2002) found that older children introduced to place value concepts made greater gains than their younger counterparts; perhaps suggesting, like Kamii (1986), that there is a link between age and place value conceptual development. Student progression in the construction of knowledge about place value and the number system has been described as often unpredictable and non-linear (Thomas, 1996). Most of the literature appears to agree that place value concepts should be taught gradually through children developing mental strategies for solving multi-digit addition and subtraction problems (Beishuizen & Anghileri, 1998; Cobb & Wheatley, 1988; Fuson, Wearne, Hiebert, Murray, Human et al., (1997); Kamii, 1986; Thompson, 2000).

A number of key constructs have been identified to describe place value understanding for teaching purposes (Fuson et al., 1997; Jones, Thornton, Putt, Hill, Mogill et al., 1996; Wright & Gould, 2000; Young-Loveridge, 1999). Although these constructs differ in the number of developmental stages they define they do have some common principles that highlight the complexities of the interrelationships between counting, knowledge of number, grouping and partitioning. The most common principle is that children progress through a number of well defined stages as their thinking develops from early counting-in-ones knowledge to a more sophisticated understanding of multi-digit numbers and their use in problem solving. The time children spend using strategies to work out problems facilitates the learning of knowledge, that when mastered becomes automatic. A student may solve the problem $10+4$ many times by 'holding' 10 in their head and counting on four more before the place value understanding of the two parts, consisting of one ten and four ones making one whole number of fourteen becomes an automatic response.

Teacher knowledge of mathematics is an essential component of effective teaching (Anthony & Walshaw, 2007; Ball, Hill & Bass, 2005; Thomas & Ward, 2007; Young-Loveridge & Mills, 2009). Effective teaching of place value requires an understanding of the learning progressions and how to assess against these progressions. The Number Knowledge Framework (Ministry of Education, 2007, p.18) highlights place value by providing a basic progression of place value knowledge detailed in terms of 'grouping' ideas. However, the progression does not include details of the key principles of place value. In fact the apparent clarity and simplicity of the framework belies the complex understandings implied by the learning outcomes.

Assessments tools currently available to New Zealand's teachers do not exclusively test place value knowledge, although many number test items include place value ideas. The literature on the learning and teaching of place value is clear about its importance for learners. Teachers need good information about their students' understanding of place value in order to improve their instructional practices and student outcomes. For this mathematics PLC there was a need for a common assessment tool that was easy to administer, and would give teachers good information about their students understandings from which they could plan for student learning. The purpose of this study, therefore, was to find out how effectively students' knowledge of place value could be assessed by a timed, short answer, written test.

Method

One thousand and forty one students were selected from seven schools, three primary (Yrs 0-6), two full primary (Yrs 0-8), one intermediate (Yrs 7-8) and one college (Yr 9). Six of these schools are in the same geographic area, are low-decile and were members of the Mathematics Cluster. To increase the number of test results available for the factor analysis, one mid-decile primary school that was not a member of the Cluster or in the same geographic area, was invited to participate. Students selected to participate were those attending school on the day of the testing. These participants ranged in age from eight to fourteen years of age. The sample consisted of 50% boys and 50% girls. Ethnic composition of participants by year levels, are shown in Table 1.

Table 1
Ethnicity of Participants by Year Levels

<i>Year Levels</i>	<i>Maori</i>	<i>NZ European</i>	<i>Pacific Islands</i>	<i>Asian</i>	<i>Other</i>
3 - 6	474	257	14	14	18
7 - 8	163	11	4	0	3
9	78	2	2	0	1

A place value test consisting of five sections was devised by the author to assess students' knowledge of place value concepts, based on stages four to eight of the Number Knowledge Framework (Ministry of Education, 2007). Each of the five sections included ten questions derived from that stage. For example, the first section contained questions pertinent to stage four of the Number Knowledge Framework. The items were designed to test the students' early knowledge of place value through groupings within ten and twenty and the number of tens in decades, which required knowledge of counting in ones over a decade, and grouping in and with ten using words and symbols. The design of the items in the following two sections (Stages 5 and 6) extended on this knowledge to groupings within 1000, groupings of tens and hundreds within four digit numbers and rounding whole numbers to the nearest ten, hundred or thousand. These sections also contained several decimal items designed to test students' knowledge of the number of tenths and hundredths in decimals to two places and rounding to the nearest whole number, of decimals with up to two places. The final two sections (Stages 7 and 8) tested students' place value knowledge, again building on the knowledge required for the preceding sections, of decimal fractions. Items were written to reflect the aspects of place value highlighted by the Number Knowledge Framework (Ministry of Education, 2007).

The test was administered to classes via a timed voice-over power point. At face value this test appears to be purely a test of knowledge, as the test is timed and requires students to instantly recall their knowledge. However, students who understand the concepts, rather than having superficial techniques for item types, will be more successful in the test. Students using their knowledge of place value concepts to solve items are more likely to solve the item within the allocated time, than students who try to use a learned procedure (procedural knowledge) to work it out. For example, a student who understands that 8 tens is the same as eighty and that twenty is the same as 2 tens is likely to solve the problem $8 \text{ tens} + 20 =$ more quickly than a student who uses skip counting knowledge 10, 20, 30,

....90, 100. Students who understand the concept of ten as a unit are more likely to give a rapid response.

Students recorded their answers to the test on a tabulated recording sheet. The student responses, marked correct or incorrect, were converted into numerical code for the purpose of a statistical analysis using SPSS software, (SPSS Inc., 1998). To determine whether a factor analysis was appropriate, the assumptions that underlie the procedure were tested. Firstly an alpha coefficient was calculated to test for internal consistency. This ensured that the items were sufficiently related for further exploration of underlying factors to be worthwhile. Alpha was equal to 0.95, indicating that further analysis was warranted. The KMO test of sampling adequacy yielded a value of 0.915. This result indicates that sampling parameters were sufficient for a factor analysis to be conducted. There were 1041 responses to the test, yielding subject to item ratio of greater than 20:1. This means that there were sufficient responses to each item for the results of a factor analysis to be useful. An exploratory factor analysis was undertaken because it was not known how many factors might underlie students' performance on the test. The structure of the Number Framework suggested that all the item types were measures of place value understanding, but how these might be clustered in students' performance was unknown. The exploratory factor analysis enabled the author to find a best-fit solution for the data set.

The data set was non-normal, and entered as category data (1 for a correct answer and 0 for incorrect). For this reason a principal axis factoring (PAF) extraction method was used (Costello & Osbourne, 2005). The scree plot of Eigenvalues was used to establish the number of factors for rotation. The point at which the scree slope flattened suggested that there were three factors present in the data set.

After extraction, a direct oblimin rotation was performed. An oblimin rotation is an oblique rotation that allows correlation between the resulting factors. This was appropriate because the test was constructed to test an underlying concept, and any factors would be expected to correlate. The resulting factor loadings are presented in Table 2.

Factor loadings of 0.32 or greater are commonly regarded as adequate for establishing the existence of a factor (Tabachnick & Fidell, 2001). Items that load greater than 0.32 on more than one factor are cross-loaded. Items strongly loaded on two factors cannot be regarded as distinct. Cross-loaded items are identified in Table 2.

Table 2

Factor Loadings >0.32 Based on a Principal Axis Factoring (PAF) Factor Analysis with an Oblimin Rotation of 50 Items from the Place Value Test (N=1041)

Item	Factor 1	Factor 2	Factor 3
1. 7+10	0.614		
2. 10+?=18	0.664		
3. Write 14 as a number	0.529		
4. 20=13+?	0.371	0.486	
5. 8 tens is the same as what number?	0.629		
6. How many 10s in 38?	0.487		
7. How many ones in 65?	0.335		
8. 5 tens and 2 ones = ?	0.586		
9. 6 tens and 30 = ?	0.463	.409	
10. What number comes next 43,42,41, ?	0.538		
11. Write the number one hundred and forty three	0.692		
12. 604= 4 +?		0.562	
13. How many tens in all of this number? 836		0.507	
14. 50+300+? = 354	0.379	0.494	
15. Round to nearest 10. 598		0.596	
16. How many hundreds in all of this number? 5000		0.477	
17. 20+300+8000+6=		0.580	
18. 697+4= ?		0.580	
19. 596 – 100 =		0.630	
20. 3508 -10=		0.503	
21. How many tens altogether in this number? 3607		0.530	
22. Round to nearest 100. 8574		0.741	
23. How many tenths altogether in four point two three?			0.456
24. How many hundredths in all of thirty point zero six?			0.733
25. Round this number to the nearest whole number. 40.98		0.592	
26. Write the missing number: 830+?=1000		0.633	
27. 0.70 is the same as 7 tens, 7 ones or 7 tenths?		0.547	
28. 4000/10		0.608	
29. 1011 x 8		0.632	
30. 32 x 10		0.669	
31a. Smallest number of 762,123; 89,549; 79,532; 817,300		0.629	
31b. Largest number of 762,123; 89,549; 79,532; 817,300		0.741	
32. Round this number to the nearest tenth – 8.79		0.631	
33. Round this number to nearest 100th – 5.468			0.543
34. How many thousands in all of this number: 814 342		0.590	
35. 0.5+0.001+0.09=			0.469
36. 3.2x 100			0.624
37. 64.2/10			0.707
38. 6.813 + 1/10			0.564
39. Write a number between 0.1 and 0.2			0.407
40. Which decimal fraction has the greatest value? 0.3, 0.299		0.347	
41. Round this number to the nearest hundred – 3254		0.585	
42. Round this number to the nearest hundredth – 9.106		0.623	
43. How many thousandths in all of this number? 4.182			0.320
44. Write these decimal fractions in order, smallest first: 0.29, 0.6, 0.371		0.481	0.692
45. 7.04-0.1=			0.478
46. 0.29 x 10			0.674
47. 48/100			0.761
48. 0.18/10			0.775
49. 17.9 + 1/100			0.652
50. 77 divided by 10			0.518

Results and Discussion

The test was constructed to try to assess a single, underlying concept. The exploratory factor analysis, however, revealed three underlying factors. The pattern that emerged was in line with the literature on place value. An outline of these factors is presented in Figure 1.

Factor 1	Ones and ten-structured concepts using words, digits and symbols – counting by ones and grouping in and with tens
Factor 2	Multi-unit concept – grouping and regrouping of, and within, 3+ digit whole numbers, and rounding to whole number or nearest tenth (2 decimal places)
Factor 3	Extended multi-unit concept – whole numbers and decimals and the relationship between whole numbers and decimals, and simple fractions.

Figure 1. Labels and descriptions of the three identified factors

Factor one is made up of nine items (1-3, 5-8, 10-11) that link to unitary and ten-structure concepts, for example, counting back by one to find the number before, and knowing that two digit numbers are made up of groups of tens and ones. These items include knowledge of the relationship between the number words and digits and the language of operations and their corresponding symbols. The twenty-one questions in factor two (12-13, 15-22, 25-32, 34, 41) are consistent with a multi-unit concept. These questions required students to group and/or regroup whole numbers using part whole concepts and their conceptual knowledge of the base-ten principle – the knowledge that the values of the positions increase or decrease in powers of ten. Factor three (items 23-24, 33, 35-38, 42-43, 45-50) is an extension of the multi-unit concepts involving the relationship between whole numbers and decimals, and their relationship with simple fractions. The analysis identified four items (4, 9, 14, 39) that were cross-loaded on two factors above the factor threshold of 0.32. These items were removed from the test. A further two items (40, 44) were removed as they were questions with multiple parts. Selected response questions or those where subsequent answers were dependent on earlier answers were deleted as they did not differentiate between those students who knew the answers and those that guessed, making these items unreliable.

It was initially assumed that the test items would assess a single underlying concept and therefore items would be grouped onto one factor. The factor analysis showed that most items were indeed closely related. The strong loading of items onto three factors rather than one implies that place value is as complex as the literature suggests.

One factor is made up of whole number items that assess the concepts of counting, and early knowledge of grouping in ones and tens and how these numbers are written. Factor two extends this knowledge into numbers within one thousand and an early knowledge of decimal place value. The third factor contains decimal fraction and fraction items that require a conceptual knowledge of the multiplicative principle of place value. The makeup of these factors is not necessarily surprising. The literature argues that children need a good understanding of whole number place value concepts before moving on to working with decimal numbers (Irwin, 2001; Sowder, 1997; Wearne & Kouba, 2000). The factor analysis supported the literature that children progress through a number of increasingly complex stages as their thinking develops, and identified a structure of three factors through which to measure this development. The structure contained similar principles to those identified by the literature (Fuson et al, 1997; Jones et al, 1996; Wright & Gould, 2000; Young-Loveridge, 1999).

In terms of alignment with the Number Framework (Ministry of Education, 2007) items in factor one match stage four with the exception of one item (11 stage five). Factor two consists of stage five and six items with the addition of one item (41) from stage 7. Most of factor three items are consistent with stages seven and eight, items 23 and 24 were developed from stage six. The test items were presented in the test in Number Framework stages to illustrate the progression of place value knowledge and so that students correct responses within the stages would give an indication of which stage the student was most

likely to be working within. The factor analysis, in clustering the items into the three factors, meant that students could no longer be assigned a single stage for place value against the Number Framework. It would appear to be more useful for teachers to think about next steps for teaching and learning in terms of progress through the ones and ten-structured concept, multi-digit concept and extend multi-digit concept, as presented within the three factors.

Conclusion

The analysis of the place value test used in this study showed that the items did in fact reliably assess the concept of place value. The factor information provided a breakdown of the critical key ideas underlying the concept of place value and supported the existing literature (Fuson et al., 1997; Jones et al., 1996; Wright & Gould, 2000; Young-Loveridge, 1999). The breakdown was consistent with the Number Framework in terms of grouping the stages, for example, factor two reflects a multi-digit concept consistent with stages five and six (Major, 2011). These three factors further specify the progression of understanding suggested by the literature and the Number Framework, and could therefore inform the teaching of place value. Such teaching would approach the concept of place value through the underlying critical key ideas, supporting the development of a robust understanding in students that creates a platform for their further mathematical understanding. This action has the potential to raise student achievement in mathematics – the goal of the Mathematics Cluster.

References

- Anthony, G. & Walshaw, M. (2007). *Effective pedagogy in mathematics/paangarau*. Wellington: Ministry of Education.
- Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 14-22, 43-46. Retrieved from http://www.aft.org/pubs-reports/american_educator/issues/fall2005/BallF05.pdf
- Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., Young-Loveridge, J., & Gould, P. (2005). Supporting teachers in the development of young children's mathematical thinking: Three large scale cases. *Mathematics Education Research Journal*, 16(3), 27-57.
- Beishuizen, M., & Anghileri, J. (1998). Which mental strategies in the early number curriculum? A comparison of British ideas and Dutch views. *British Educational Research Journal*, 24(5), 519-538.
- Boulton-Lewis, G. (1996). Representations of place value knowledge and implications for teaching addition and subtraction. In J. Mulligan & M. Mitchelmore (Eds.), *Children's Number Learning: A Research Monograph of MERGA/AAMT* (pp. 75-88). Adelaide: Australian Association of Mathematics Teachers.
- Cobb, P., & Wheatley, G. (1988). Children's initial understanding of ten. *Focus on Learning Problems in Mathematics*, 10(3), 1-28.
- Costello, A. B., & Osbourne, J. W. (2005). Best practices in exploratory factor analysis: Four recommendations for getting the most from your analysis. *Practical Assessment, Research & Evaluation*, 10(7). Available online: <http://pareonline.net/pdf/v10n7.pdf>.
- Fuson, K., Wearne, D., Hiebert, J., Murray, H., Human, P., Olivier, A., et al. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education*, 28(March), 130-162.
- Irwin, K. (1996). Making sense of decimals. In J. T. Mulligan and M. C. Mitchelmore (Eds.), *Children's number learning* (pp. 243-257). Adelaide: Australian Association of Mathematics Teachers.
- Irwin, K. (2001). Using everyday knowledge of decimals to enhance understanding. *Journal for Research in Mathematics Education*, 32(4), 399-420.
- Jones, G., Thornton, C., Putt, I., Hill, K., Mogill, A., Rich, B., et al. (1996). Multidigit number sense: A framework for instruction and assessment. *Journal for Research in Mathematics Education*, 27(3), 310-336.
- Kamii, C. (1986). Place value: an explanation of its difficulties and education implications for the primary grades. *Journal for Early Childhood Education*, 1(2), 75-86.

- Major, K. (2011). *Place Value: Get it. Got it. Good enough? Developing a tool to improve the teaching and learning of place value*. Unpublished Masters Thesis, University of Auckland, Auckland.
- Ministry of Education. (2007). *Book 1 The number framework*. Wellington, NZ: Learning Media.
- Rubin, A., & Russell, S. (1992). Children's developing concepts of landmarks in the number system. In W. Geeslin & K. Graham (Eds.), *Proceedings of the 16th International Conference for the Psychology of Mathematics Education* (Vol. 3, pp.136-143). Durham, NH: PME.
- Sowder, J. (1997). Place value as the key to teaching decimal operations. *Teaching Children Mathematics*, 3, 448-453.
- SPSS Inc. (1998). *SPSS Base 8.0 for Windows User's Guide*. SPSS Inc., Chicago.
- Tabachnick, B. G., & Fidell, L. S. (2001). *Using multivariate statistics* (4 ed.). Needham Heights, MA: Allyn & Bacon.
- Thomas, N. (1996). Understanding the number system. In J. T. Mulligan & M. C. Mitchelmore (Eds.), *Children's number learning* (pp. 89-106). Adelaide: Australian Association of Mathematics Teachers & MERGA.
- Thomas, G., & Ward, J. (2002). *Exploring issues in mathematics education. An evaluation of the early numeracy project 2001*. Wellington: Learning Media Ltd.
- Thomas, G., & Ward, J. (2007). *What do teachers know about fractions? Findings from the New Zealand Numeracy Development Projects 2006* (pp. 128-138). Wellington: Learning Media.
- Thompson, I. (2000). Teaching place value in the UK: Time for a reappraisal? *Educational Review*, 52(3), 291-299.
- Thomas, N. (2004). *The development of structure in the number system*. Paper presented at the 28th Conference of the International Group for the Psychology of Mathematics Education, Bergen, Norway.
- Wearne, D., & Kouba, V.L. (2000). Rational numbers. In E. A. Silver & P. A. Kenny (Eds.), *Results from the seventh mathematics assessment of the national assessment of educational progress* (pp. 163-191). Reston, VA: National Council of Teachers of Mathematics.
- Wright, R., & Gould, P. (2000). *Reviewing literature relevant to a systematic early numeracy initiative: Bases of CMIT*. Paper presented at the 23rd Conference of the Mathematics Education Research Group of Australasia, Freemantle, Australia.
- Young-Loveridge, J. (1998). *The development of place value understanding*. Paper presented at the Research Seminar on Mathematics Education (Year 0-6 students), Research and Curriculum Divisions, Ministry of Education.
- Young-Loveridge, J. (1999). The acquisition of number. *SET: Research Information for Teachers*, 2, 1-8.
- Young-Loveridge, J. (2008). Analysis of 2007 data from the Numeracy Development Projects: What does the picture show? In *Findings from the New Zealand Numeracy Development Projects 2007*. (pp.18-28, 191-211). Wellington: Ministry of Education.
- Young-Loveridge, J., & Mills, J. (2009). Teaching multi-digit multiplication using array-based materials. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides* (Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia pp. 635-643). Palmerston North, NZ: MERGA

Projects, Puzzles and other Pedagogies: Working with Kids to Solve Local Problems

Margaret Marshman
University of the Sunshine Coast
<mmarshma@usc.edu.au>

Engaging and extending middle years students in mathematics is a continual challenge. One of the aims of the *Australian Curriculum: Mathematics* is to ensure that students are “confident, creative users and communicators of mathematics” (ACARA, 2011). Use of mathematical models and/or problems has been suggested as methods of achieving this aim, and mathematical investigations have been shown to improve student engagement. This paper looks to build on these ideas and combine them with the framework of Knowledge Producing Schools (KPS) (Bigum & Rowan, 2009) to determine whether, when students are working on a community based project of their choice, students become “confident, creative users and communicators of mathematics” (ACARA, 2011).

Introduction

One of the stated aims of the Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2011): is that “mathematics aims to ensure that students are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens”. Another is that students “recognise connections between the areas of mathematics and other disciplines.”

This paper argues that one way to achieve those aims is through the use of mathematical problem solving and modelling, especially through the use of a variation of project-based learning. The proposed research, outlined below, draws on a highly successful approach to project-based learning in which students work on problems that matter to the local community. This is known as Knowledge Producing Schools (KPS) (Bigum, 2002, 2004; Bigum & Rowan 2009; Rowan & Bigum, 2010). Using this approach students are able to produce products and outcomes that are valued by, and have value in the local community. Support is provided by experts, at appropriate moments in the project. A significant outcome of this approach is that students, who were previously disengaged with their work in schools, become engaged. The focus of KPS is on student formulation and finding solutions to community based problems or issues. Arguably, real problems provide students with more opportunities for meaning making, which results in deeper understandings of the discipline (Schoenfeld, 1992; Romberg, 1994).

This paper is based on a current research project where a group of identified gifted and talented students at a regional primary and secondary school in Queensland are undertaking a community project on their local park. This project involves students redesigning the park to include a multicultural theme inclusive of art work that reflects the different cultures from the local community. It will require students to liaise with different community groups and the local council. The students want to include seating and BBQs to encourage a wide range of people to use the park as a meeting place and to make an actual difference in their local community. Despite the categorisation of gifted and talented, I am confident that there is a diversity of talent and expertise (Hong & Page, 2004).

Data collection for this research will include students’ video documentation of their progress, their contributions to a collaborative Wiki and written journal reflections.

Current thinking in mathematics education research is to get students working as mathematicians (Boaler, 2002). Burton (1998-1999) found that mathematicians collaboratively negotiate by exploring problems and actively seeking connections between the real world and mathematics. They also seek connections between the different areas of mathematics and value intuition, insight and the aesthetics of mathematical solutions. “The ability to look at a problem from different angles is crucial.” (Burton, 2001, p. 597).

This study has been broadly located and informed by the use of mathematical investigations, problem solving and mathematical modeling as well as problem-based learning and knowledge producing schools. These approaches and how to learn in mathematics, is considered below.

Mathematics Learning

Mathematics teachers want students to ‘do’ mathematics and ‘to make sense of mathematics’. The question is how this can be achieved while also trying to conform to the aims of ACARA. Schoenfeld (2002) and others maintain that problem solving is the goal of mathematics learning while communication, that is teachers being aware of the students’ thinking, is and always has been an important part. So how can we get our students “to make sense of a real-world use of mathematics, to get them involved in ‘problem formulation, problem solving, and mathematical reasoning?’” (Battista, 1994, p. 463). Problem solving “is not only a goal of learning mathematics but also a major means of doing so” (National Council of Teachers of Mathematics (NCTM), 2000, p. 4). Problem solving tasks also build the discourse in the classroom as students “conjecture, test, and build arguments about a conjecture’s validity ... and ... are encouraged to explore, guess and even make errors “(Battista, 1994, p. 463).

Identity formation is an essential part of learning to be a mathematician, “because learning transforms who we are and what we can do, it is an experience of identity. It is not just an accumulation of skills and information, but a process of becoming ...’ (Wenger, 1998, p. 215). To be a mathematician means belonging to the discourse community (Gee, 2011) so that one can engage in specific ways of thinking, acting and perceiving (Gee, 2011). Burton (1998-1999) identified being a mathematician as participating in collaborative exploration. Boaler, William and Zevenbergen, (2007) discuss the importance of helping secondary students develop an identity that includes being successful at mathematics.

It is also about engaging in a ‘dance of agency’ (Pickering (1995 p. 116) where mathematicians constantly move between their personal or human agency; their initial thoughts and ideas or their extension of established ideas, and the ‘agency of the discipline’ when it is necessary to follow standard mathematical procedures. For Boaler (2002) developing one’s human agency, by using and applying mathematics and participating in mathematical discourse, enables the learner to participate in the dance of agency, develop identities as mathematics learners and a “relationship with the discipline of mathematics” (Boaler, 2002, p. 10).

While problem solving, mathematicians use different representations; (66%) used predominantly visual thinking, (47%) conceptual thinking and (37%) used analytical thinking (Burton, 2001). Ernie, LeDocq, Serros, and Tong (2009) support this by arguing that signature pedagogies for mathematics should focus on “. . . teaching students to use multiple representations to reason about interesting and challenging real-world problems in a student-centred environment” (p. 264). Students need to be assisted to see connections between multiple representations of mathematical ideas, to persevere, to collaborate with

their peers so as to co-construction understanding. They also need to be provided with a dynamic environment where students can learn by doing and experimenting.

Problem Solving

Schoenfeld (1992) observed high school and college students working with unfamiliar problems and noted that “roughly sixty percent of the solution attempts are of the ‘read, make a decision quickly, and pursue that direction come hell or high water’ variety” (p. 61). The students do not change tack even when the method is not working. On the other hand, mathematicians when solving an unfamiliar problem spend time making sense of the problem, then pursue leads, abandon attempts that are not getting anywhere and then solve the problem (Schoenfeld, 1992). To develop this ability Schoenfeld (1992) believes that students need to be taught the “metacognitive aspects of mathematical thinking” (p. 63). Metacognitive ability takes time to develop and includes “assessing one’s own knowledge, formulating a plan of attack, selecting strategies, and monitoring and evaluating progress” (Schoenfeld, 1985 cited in Yimer & Ellerton, 2006, p. 575) It is this metacognitive ability rather than mathematical knowledge that influences a student’s ability to problem solve (Carlson, 1999).

For students to successfully work with these tasks, they may initially need a significant amount of thought and/or mathematics before they can start mathematising. Romberg (1994) describes how to do mathematics and solve problems:

- initially students need to make sense of the task
- formulate the problem and decide on the important variables and relationships between variables;
- determine a model;
- substitute numbers into the variables to find a solution of a numerical model; and finally
- consider the validity of the solution – does it make sense? What effect will minor changes make? This may mean that it is necessary to go around either part or the whole cycle again.

Schoenfeld (1999, 1992) discusses the need for productive beliefs and mathematical dispositions to become good problem solvers. As Boaler (2002) argues, when a mathematician has a problem to solve for which she does not have the necessary knowledge, she will have the mathematical practices previously learnt and have productive relationships with the discipline, meaning she will try a variety of methods to solve the problem.

Problems in the real world are ‘ill-structured’ and so it is necessary to initially formulate them in a well-structured way (Heylighten, 1988). Problem formulation is commonly carried out by the teacher which leaves the student with the task of applying an appropriate algorithm that may be able to be calculated by machine. Taking problem formulation away from students removes a key opportunity for students to engage in mathematical sense making (Battista, 1994).

This can also add extra pressure on the teacher. For the teacher, working with problems or modelling to help students make sense of their world can be much harder than teaching factual information. A focus on pedagogy rather than content is a major shift that needs to occur. As Burkhardt (1988) explains, teachers:

- need to consider the different approaches taken by the students;
- need to decide when to support students with suggestions or questions that will help whilst still allowing the students to be responsible for finding their own solution and this is for each student or group of students in the class; and
- may be put in the potentially uncomfortable position of not knowing all the answers.

Mathematical Investigations

A mathematical investigation is real-life or life-like learning which is: open ended and provides opportunities for students to use multiple pathways to investigate the situation/problem. It may be framed as: a problem to be solved, a question to be answered, a significant task to be completed or an issue to be explored.

A good investigation has multiple entry points allowing students to start at their own level and to design their own pathway (or pathways) through it. Indeed, investigations allow students to undertake activities and thinking that resemble that of the practice of mathematicians, and so they can be viewed as authentic mathematical tasks (Burton, 1998-1999). In this way, investigations allow for the alignment of teaching, learning and assessment.

Boaler (2008) demonstrated that it is possible to engage students in deep mathematical learning using an investigative pedagogy, particularly those students who have been alienated by traditional approaches to mathematics education. Investigations are open-ended questions or problems that are set in a range of contexts. By using investigations that are directly related to the students' lives, mathematics becomes no longer 'useless'. To achieve this, teachers need to provide a socially supportive and intellectually challenging environment in the classroom (Fredricks, Blumenfeld & Paris, 2004) so that students are able to develop strong mathematical identities. When the task is relevant and meaningful most students enjoy a challenge (The Centre for Collaborative Education 2000 cited in Hilton & Hilton, 2005).

When students were asked to design and make a three dimensional model of a swimming pool using a minimum of two different shapes (Marshman & Grootenboer, 2012), many students chose to make creative designs including a heart shaped pool with sloping base, arrangements of three dimensional shapes and pools with spas and tiered seating. This was despite the fact that maintaining the same scale throughout for many was a challenge that meant it took a number of attempts to get their models correct, and the calculation of the area to be tiled was very detailed.

To encourage students to formulate problems the pedagogy needs to change to mathematical activities without one right answer. Increased engagement occurs when students have the opportunity for more ownership of the investigation by having to formulate the problem themselves rather than just being given the necessary algorithms to calculate answers. An example of this is reported by Marshman, Pendergast and Brimmer, (2011) where students were asked to design a middle years' area outside their classroom. At a professional development session prior to beginning the investigation, the teacher had expressed frustration with her class, describing them as "impossible". She felt that they were disengaged, of low ability, badly behaved and could not cope with the work. However, when the researchers visited, the students were excited about the task and could explain what they had to do, how they were going to approach the task and what they were going to put into their middle years' area.

Problem-Based Learning

"Originated in the 1960s at McMaster University Medical School, Canada, Problem Based Learning (PBL) is essentially a collaborative, constructivist, and contextualized learning and teaching approach that uses real-life problems to initiate, motivate and focus knowledge construction" (Ribeiro, 2011). It has been argued that engineering graduates are often unable to work effectively in multi-disciplinary teams and lack the skills to transfer basic knowledge to real-life situations, solve unusual problems in situations, and think

critically and creatively (Ribeiro, 2011). To address this concern, engineering faculties use problem-learning to develop discipline specific knowledge as well as gaining these other skills. PBL engages the learner in authentic activities that simulate professional practice. The tutors support the learning through discourse and scaffold the students' self-directed study.

Modelling as Real World Problem Solving

Galbraith (2011a) argues that when students solve problems in their world they are developing and using mathematical modelling skills. He argues that this empowers students to attempt future problems. The process is messy and cyclical as indicated below.

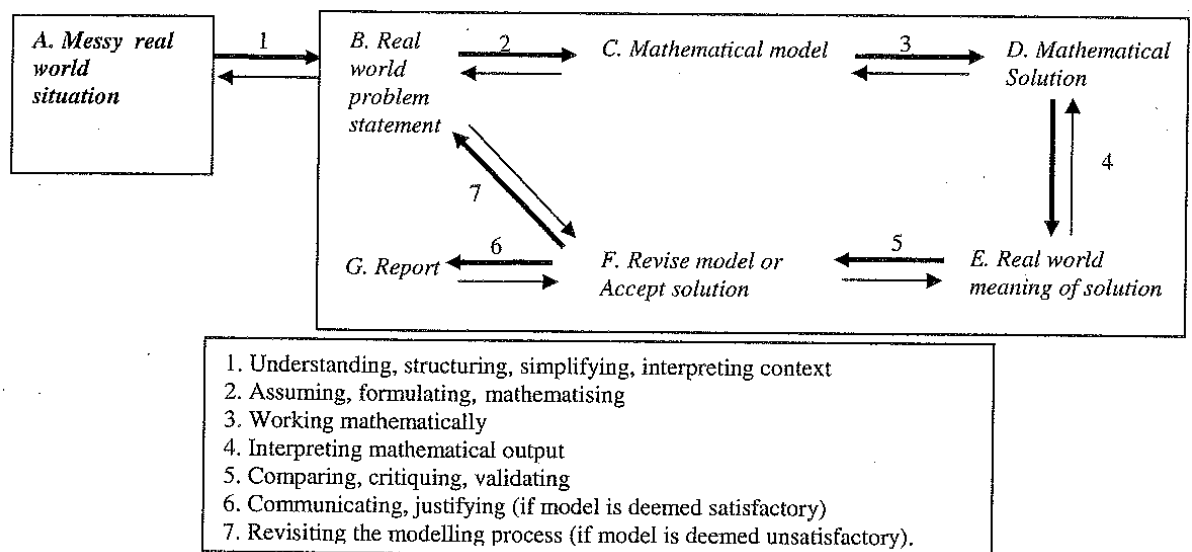


Figure 1. The modelling process (Stillman, Galbraith, Brown & Edwards, 2007)

When comparing “mathematical outcomes with reality” the students must return to the world where the problem is. The important message here is that the process cannot remain solely in the classroom. The biggest challenge students face is “learning how to identify problems, and to formulate related mathematical questions that can be addressed with existing mathematical knowledge” (Galbraith, 2011a).

Knowledge Producing Schools

Scardamalia and Bereiter (1999) argue that schools should be knowledge building organisations,

Community knowledge building, by contrast, is aimed at producing something of value to the community – theories, explanations, problem formulations, interpretations, and so on, which become public property that is helpful in understanding the world and functioning intelligently in it. ...within that group students are contributors to the common good. ... The job of an elementary class that adopts a knowledge-building approach is to construct an understanding of the world as the students know it. (p. 276-7)

Students can function as mathematicians once they are able to participate in the social practices and mathematical discourses. Unlike professional research groups, who are attempting to solve new and specific problems, students in schools are attempting to solve the problems that will help them understand their world. These students when actively attempting to solve a ‘knowledge problem’ are “trying to negotiate a fit between their own

ideas and information obtained from an authoritative source” (Scardamalia & Bereiter, 1999, p. 278). They are trying to construct their knowledge by connecting the new information to what they currently understand.

There is strong evidence that when students are engaged in project work that matters to them, excellent engagement is achieved and if timely access to expertise is provided then the quality of the work is cutting edge. In Knowledge Producing Schools students create products and/or productions for the local community who in turn provide ‘experts’ to work with the children (Bigum & Rowan, 2009).

Conclusion

The proposed research is radically different from all previous attempts to address a problem. It draws on a highly successful approach to project-based learning in which students work on problems that matter in the local community (Bigum, 2002, 2004; Bigum & Rowan 2009; Rowan & Bigum, 2010). With the support of experts at appropriate moments in the conduct of a project, students are able to produce products and outcomes that are valued by and have value in the local community. One of the significant outcomes of this approach is re-engagement with schooling by the students who do such work.

In this research, mathematics will *not* be an upfront focus for students. However, in undertaking this community project there will be instances where mathematical thinking will contribute. These opportunities will be identified by an expert mathematician who will mathematically support the students. Students will invite the mathematician to work with them in the same manner that other experts will be drawn upon to support the project.

While problem solving, mathematical investigations, problem-based learning and mathematical modelling have their place in mathematics classrooms, I argue that they do not support students to develop expertise in “formulating a mathematical problem from a messy real world context” which is unfamiliar (Galbraith, 2011b, p. 931). Arguably, much of the mathematical modelling carried out in schools skims over problem formulation. While some argue that this is accomplished when students interpret their results and evaluate their solution, I believe that this amounts to “stunt mathematics”, that is, giving students kudos for doing the least demanding part of working on a real world problem.

Solving problems in the real world is more than crunching numbers that have been generated by a model developed by someone else. To initially formulate the problem one needs to have a comprehensive understanding of the context in which the problem is set. This may be achieved by visiting the site of the problem, acting out the problem or carefully reading materials associated with the problem (Galbraith, 2011b). Problem based learning is one way to achieve these goals but I argue that a key element in any problem-based approach is genuinely considering the interests and needs of a local community. This occurs in a Knowledge Producing School approach. This approach to working mathematically while resembling what happens in the world outside schooling is untested in the Australian school context and is unique to this research project.

Acknowledgement

This work is being funded by an Internal University Research Grant from the University of the Sunshine Coast.

References

Australian Curriculum, Assessment Reporting Authority (2011) *Australian Curriculum: Mathematics*. Retrieved 30 March from <http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10>

- Battista (1994). Teacher beliefs and the reform movement in mathematics education. *The Phi Delta Kappan* 75(6), 462-470.
- Bigum, C. (2002). The knowledge producing school: beyond IT for IT's sake in schools. *Professional Voice*, 2(2).
- Bigum, C. (2004). Rethinking schools and community: the knowledge producing school. In S. Marshall, W. Taylor & X. Yu (Eds.), *Using Community Informatics to Transform Regions* (pp. 52-66). London: Idea Group Publishing.
- Bigum, C., & Rowan, L. (2009). Renegotiating Knowledge Relationships in Schools. In S. E. Noffke & B. Somekh (Eds.), *The SAGE handbook of Educational Action Research* (pp. 102-109). Los Angeles: Sage.
- Boaler, J. (2002). The development of disciplinary relationships: Knowledge, practice, and identity in mathematics classrooms. *For the Learning of Mathematics*, 22(1), 42-47.
- Boaler, J. (2008). Promoting 'relational equity' and high mathematics achievement through an innovative mixed ability approach. *British Educational Research Journal*, 34(2), 167-194.
- Boaler, J., William, D., & Zevenbergen, R. (2007). The construction of identity in secondary mathematics education. Retrieved 4 February 2008 from <http://www.merga.net.au/documents/RP282007.pdf>
- Burkhardt, H. (1988). Teaching problem solving. In H. Burkhardt, S. Groves, A. Schoenfeld & K. Stacey (Eds.), *Problem Solving: A world view. Proceedings of the problem solving theme group, ICME 5* (pp. 17-42). Nottingham, UK: Shell Centre.
- Burton, L. (1998-1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics*, 37(2) 121-143.
- Burton, L. (2001). Research Mathematicians as Learners and what mathematics education can learn from them. *British Educational Research Journal* 27(5) 589-599.
- Carlson, M.P. (1999) The Mathematical Behavior of Six Successful Mathematics Graduate Students: Influences Leading to Mathematical Success. *Educational Studies in Mathematics*, 40, (3) 237-258.
- Ernie, K., LeDocq, R., Serros, S., & Tong, S. (2009). Mathematical reasoning: Challenging students' beliefs about mathematics. In R. Gurung, N. Chick, & A. Haynie (Eds.), *Exploring signature pedagogies: Approaches to teaching disciplinary habits of mind*. Sterling, VA: Stylus Publishing, LLC. pp. 260 – 279.
- Fredricks, J. A., Blumenfeld, P. C. & Paris, A. H. (2004). School engagement: potential of the concept, state of the evidence. *Review of Educational Research* 74(1) 59-109.
- Galbraith, P. (2011a) Models of Modelling: Is there a first among equals? in *Mathematics: Traditions and [New] Practices*. (Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia). Alice Springs, NT:MERGA.
- Galbraith, P. (2011b) Modelling as real world problem solving: translating rhetoric into action, in *Mathematics: Traditions and [New] Practices*. (Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia). Alice Springs, NT:MERGA.
- Gee, J.P. (2011). *An introduction to discourse analysis: Theory and Method* (3rd ed). NY: Routledge.
- Heylighen, F. (1988) Formulating the problem of problem-formulation. In Trappl, R. (Ed.) *Cybernetics and Systems '88* (pp. 949-957), Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Hilton, G. & Hilton, A. (2005) Higher order thinking. In D. Pendergast & N. Barr, (Eds.), *Teaching middle years rethinking curriculum, pedagogy and assessment* (pp. 196-210) Sydney: Allen & Unwin.
- Hong, L. and Page, S.E. (2004) Groups of diverse problem solvers can outperform groups of high-ability problem solvers, *PNAS* 101(46)16385-16389.
- Marshman, M. & Grootenboer, P. (2012) 'Paper, Scissors, Rock' in Rowan, L. (Ed.). *Transformative approaches to new technologies and student diversity in futures oriented classrooms: Future Proofing Education*. Dordrecht, The Netherlands: Springer.
- Marshman, M., Pendergast, D. & Brimmer, F. (2011) Engaging the middle years in mathematics, in *Mathematics: Traditions and [New] Practices*. (Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia). Alice Springs, NT:MERGA.
- Pickering, A. (1995). *The mangle of practice: Time, agency, and science*. Chicago: University of Chicago Press.
- Ribeiro, L.R.C. (2011). The pros and cons of project-based learning from the teacher's standpoint *Journal of University Teaching and Learning Practice* 8(1) 4.
- Romberg, T.A. (1994). Classroom Instruction that Fosters Mathematical Thinking and Problem Solving: Connections Between Theory and Practice. In Schoenfeld, A. H. *Mathematical Thinking and Problem Solving*, New Jersey: Lawrence Erlbaum Associates.
- Rowan, L., & Bigum, C. (2010). At the Hub of it All: Knowledge Producing Schools as Sites for Educational and Social Innovation. In D. Clandfield & G. Martell (Eds.), *The School as Community Hub: Beyond Education's Iron Cage* (pp. 185-203). Ottawa: Canadian Centre for Policy Alternatives.

- Scardamalia & Bereiter (1999). School as knowledge-building organisations. In D. Keating and C Hertzman (Eds), *Today's children, tomorrow's society: The development health and wealth of nations* (pp.274-289). New York: Guilford.
- Schoenfeld, A. H. (1992) Learning to think mathematically: Problem solving, metacognition and sense-making in mathematics. In D. A. Grouws (Ed.) *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370) New York: MacMillan.
- Schoenfeld, A. (1999). Looking toward the 21st century: challenges of educational theory and practice. *Educational Researcher*, 28(7), 4-14.
- Stillman, G., Galbraith, P., Brown, J., & Edwards, I. (2007). A framework for success in implementing mathematical modelling in the secondary classroom. *Mathematics:Essential Research, Essential Practice Paper* (Proceedings of the 30th Annual Conference of Mathematics Education Research Group of Australia, Hobart, pp. 688-697)Adelaide: MERGA.
- Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. Cambridge: Cambridge University Press.
- Yimer, A. & Ellerton, N.F. (2006). Cognitive and Metacognitive Aspects of Mathematical Problem Solving: An Emerging Model. *Identities, Cultures and Learning Spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, Canberra, pp. 573-582) Adelaide: MERGA.

The Impact of a Professional Learning Intervention Designed to Enhance Year Six Students' Computational Estimation Performance

Paula Mildenhall

Edith Cowan University
<p.mildenhall@ecu.edu.au>

Mark Hackling

Edith Cowan University
<m.hackling@ecu.edu.au>

This paper reports on the analysis of a study of a professional learning intervention focussing on computational estimation. Using a multiple case study design it was possible to describe the impact of the intervention on students' beliefs and computational estimation performance. The study revealed some noteworthy impacts on computational estimation performance and that beliefs were altered to a lesser extent.

Curriculum documents globally are calling for primary mathematics classrooms to be sense making environments with students able to think flexibly when computing (Australian Curriculum and Assessment Reporting Authority, 2011; Department for Education, 2011; National Council of Mathematics Teachers, 2006). Students who are able to interpret and solve number problems with understanding can be described as possessing 'number sense'. Number sense is an emerging term without an established definition. This study used McIntosh, Reys, Reys Bana and Farrell's (1997) definition of number sense:

A person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful and efficient ways of managing numerical situations (p.3)

An essential component of a students' number sense is their ability to estimate in computations – both as a computational choice in its own right and used as a checking tool.

It was of interest to the Researcher, who lectures in primary mathematics education at a tertiary level, to consider how primary school teachers could be better supported to teach computational estimation to Year 6 students. This paper reports on a professional learning intervention designed to improve the teaching of computational estimation and its impact on Year 6 students' beliefs and performance on estimation tasks. This intervention was part of the Researcher's PhD and took place in the school year of 2009. In this paper it outlines how some noteworthy impacts in students' use of estimation as a computational choice occurred. This study was part of a larger research project which also looked at the teachers' beliefs and teaching approaches that were developed as a result of the professional learning intervention.

Conceptual Framework and Theoretical Perspective

Underpinning the study was sociocultural theory, that claims students learn by a process of gradual individualisation rather than the acquisition of rote knowledge (Sfard, 2008). At the forefront of sociocultural theory is the importance of culture in the construction of new knowledge. It is through conversations that new ideas are constructed. The major semiotic tool in sociocultural theory is language and the teacher plays an important role in mediating the different types of classroom discourse.

Effective Teaching and Learning of Computational Estimation and Mathematics

Traditionally computational estimation is differentiated from numerosity and measurement estimation, but with the latest research findings supporting using contexts, such as authentic measurement tasks, this differentiation may be less useful. In the past

computational estimation was often taught in an isolated manner (Dowker, 2003). It has recently been asserted, however, that conceptual estimation be taught within a problem-centred contextual approach (Reys & Reys, 2004). Yoshikawa (1994) argues that where computational estimation is set in a meaningful context, students are more likely to appreciate that estimation can save them computational time and energy. Research has shown that students can use computational estimation strategies. For example, Neill (2006) found that Year 8 students were able to implement a variety of estimation strategies. The use of computational estimation has been impeded by some negative connotations that estimation has in schools. Teachers find it ambiguous and students think that mathematics can only have one exact answer (Yoshikawa, 1994). Changing beliefs may be very important in encouraging the use of computational estimation in classrooms.

The Professional Learning Intervention

The professional learning intervention was underpinned by four principles which were derived from the research literature. The first three were concerned with effective mathematics teaching generally and they were that mathematics teaching is effective when it is active (Franke, Kazemi, & Battey, 2007), metacognitive (McKeachie, Pintrich, Lin, & Smith, 1986), and contextual (Gravemeijer & Terwel, 2000). The fourth principle was that computational estimation as a component of number sense is a valuable computational choice (Reys & Reys, 2004). Various estimation strategies have been identified by mathematics educators (Reys & Reys, 2004) and a detailed taxonomy of these was created by the Researcher (Mildenhall, 2009). The strategies in the taxonomy were, rounding, nice/compatible numbers, front-end loading, benchmarking, range and the sample strategy. These strategies were shared with the teachers in the professional learning intervention. The teachers were then asked to consider how to best teach these strategies to Year 6 students.

Features of effective teacher professional learning were considered in the design of the intervention. These included providing support to the teachers throughout the professional learning program (Ingvarson, Meiers, & Beavis, 2005), encouraging an action research and collaborative approach (Bray, 2002), and creating opportunities for the teachers to reflect on how the planned ideas translated into the classroom (Fennema et al., 1996).

The morning sessions of the professional learning workshops focused on the teachers sharing their personal reflections with the group on their teaching of computational estimation. They then focused on developing their personal PCK of estimation and mathematics. In the afternoon sessions the teachers were presented with suggested teaching activities which were designed to include features of the four principles outlined earlier, and in small groups of colleagues they designed a series of estimation activities that would develop computational estimation. It was also suggested that 'estimation as a checking device' be incorporated in to their mathematics lessons.

Purpose and Research Question

The purpose of this study was to investigate the impact of the professional learning intervention on the teaching and learning of computational estimation. The main focus in this part of the study was the impact of the intervention on the students. The following question was therefore explored: How did the teaching approaches impact on students' beliefs about estimation, their mathematical knowledge and their computational estimation performance?

Methodology

This study used a multiple case study design to generate rich descriptions of the impact of the intervention in specific contexts (Merriam, 1998). Initially there were three individual case studies comprising teachers and their Year 6 students and then a cross-case analysis was conducted. Conducting multiple case studies allowed the researcher to gain a more comprehensive understanding of how the professional learning intervention impacted on the Year 6 students' beliefs and computational estimation performance in different school contexts (Miles & Huberman, 1994).

Participants

Experienced Year 6 teachers from five non-government schools in Perth, Western Australia were invited to take part in the study. Six teachers in total, from four schools agreed to take part in the professional learning program. Three teachers were selected for case studies due to their different levels of engagement in the program (Figure 1.). Bob exhibited a high level of engagement in the program, Wendy was engaged in the professional learning but this engagement was affected somewhat by her school context and Peter exhibited the lowest level of engagement. The individual contexts of the three classrooms were important when evaluating the outcomes. It is noted, for instance, in Figure 1 that only Bob's school organisation allowed him the freedom to not use a textbook. As a result of engaging in the professional learning intervention, the three teachers each used different pedagogies to teach computational estimation. Generally Bob used the estimation tasks set in meaningful contexts and integrated estimation into all of his teaching. He also introduced all of the estimation strategies to the students. Wendy taught the estimation work set in meaningful tasks as a separate topic and also introduced all of the estimation strategies to the students. Peter did not introduce the meaningful tasks until towards the end of the study and did not introduce any estimation strategies other than rounding to the students.

Contextual Factors	Wendy	Bob	Peter
School	Low fee independent	Low fee independent	Low fee independent
Organisation of class	Mixed ability	Top stream of 2 classes	Mixed ability
Curriculum	School text book	Broad outline provided	School text book
ISEA value (mean 1000)	1118	1003	1038
Years of teaching	25	16	9
Number of students in the class	32	25	32
Mean Year 5 NAPLAN score (Australian National Testing)	476	478	474

Figure 1. Individual teachers' contexts.

Instruments

The following data collection instruments were used in this part of the study to capture the students' beliefs and computational estimation performance: the Researcher's audit trail journal, teacher interviews, focus group interviews with six students from the class case study, and students' written work and computational estimation timed pre and post-tests. The computational estimation test was adapted from an existing number sense test, which had been used in internationally recognised research (McIntosh et al., 1997). Two of the six

questions were drawn from other sources, but they were adapted to fit the same format as the number sense test.

Data Collection

The pre-tests, teacher interviews and student focus-group interviews were administered before the study began in February 2009. The study of the teacher professional learning intervention took place over one year and involved the teachers taking part in three one-day workshops. There were four cycles to the research with data collection involving classroom observation, and collection of classroom artefacts taking place during each of these cycles. At the end of the study in November 2009 the post-test and final student focus-group interviews were administered.

Data Analysis

A content analysis was made of the qualitative data that was gathered from the multiple sources of data. Using multiple sources of data increased the internal validity of the study. Using the computer software NVIVO 9 (QSR, 2008) as a data management system, the Researcher undertook a deductive and inductive coding process so that themes which described students' beliefs emerged. Qualitative data concerning students' computational performance was also collected and analysed using the same process. This triangulation supported the findings that emerged from quantitative data analysis and again increased its validity. The quantitative data was collated and analysed using PASW (SPSS inc, 2010). Using this software it was possible to calculate changes in estimation performance on the timed Computational Estimation Test. The analysis focused on the pre- and post-test mean scores for each question and the estimation strategies used by the students. For this analysis it was decided to focus on specific strategies used and after this initial categorisation the specific strategies were grouped into either a reasoned or unreasoned strategy.

Results

Student Beliefs

At the beginning of the study most of the students believed that mathematics was something to be done quickly, was about four operations, addition, subtraction, multiplication and division and one correct answer. Generally this perception did not change by the end of the professional learning intervention. It appears that the professional learning intervention did not impact on the students' general beliefs about mathematics as they perceived the estimation work as something outside normal school mathematics.

When asked questions that specifically focussed on estimation most students showed changes from the beginning to the end of the study. Their understanding of computational estimation was somewhat limited at the beginning of the study, viewing it as a checking device and as a type of guess. They did not perceive that computational estimation could be a choice in its own right when initially deciding how to solve a mathematics problem. At the end of the study all of the students in the three focus groups discussed estimation as an extra activity to normal calculations. Their beliefs about estimation appeared broader and two of the three focus groups students' showed an awareness of the variety of estimation strategies. This change was reflected in the drawing of a concept map at the beginning of the study and at the end. Hannah (a pseudonym) explained at the beginning of the study that she believed estimation to be a "guess instead of finding out the real answer" (Hannah, 10/2/2009). But at the end of the study she produced this concept map shown in Figure 2 below, which shows

that Hannah had an awareness of a variety of computational estimation strategies and some different contexts in which estimation could be useful. This appreciation of how estimation is relevant in certain contexts reflects the socio-cultural view of effective learning taking place in situated activity. The students in the class that spent the least amount of time on the estimation learning activities had the least awareness of strategies; they were only aware of the term rounding.

Most of the students had enjoyed the estimation work. For example, Peter said “ I thought it was fun, especially surf shop ‘cos we were selling things to our friends” (Focus group interview, 18/11/2009). Most of the students found the estimation work as something that made mathematics easier. The exception to this was a few Year 6 boys who were very successful in the normal mathematics lessons. Adam explains:

I am not one for estimating, I normally have a rough idea of what I am after ... I don't like Fermi problems 'cos there is no one particular answer which defines the purpose of mathematics. Mathematics is either black or white.

These boys clung to the notion that maths was about one exact correct answer.

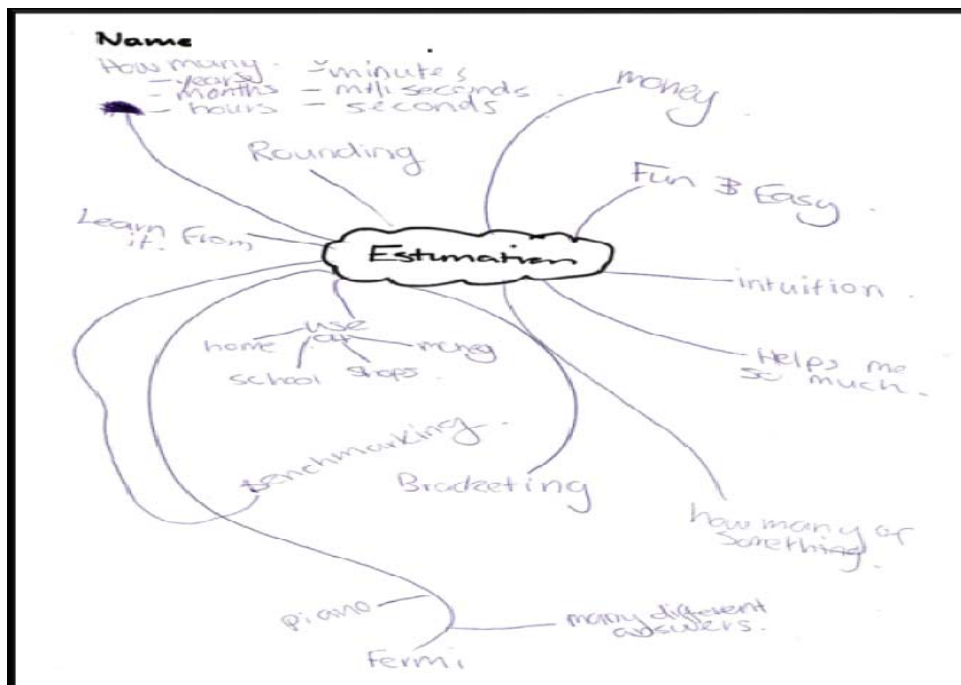


Figure 2. Hannah's post concept map of computational estimation.

Students' Computational Estimation Performance

The professional learning intervention did have an impact on the students' computational estimation performance and their use of reasoned estimation strategies. The qualitative record of the students' work in the classroom showed that students used far more estimation language when solving mathematical problems for which estimation was the main computational choice. There was also a significant improvement in the performance in the computational estimation test. Over the three classes, the students mean pre-test score of 2.94/6 increased to 3.89/6 following the intervention and mean scores increased for all six questions.

In question 4 of the computational estimation test, the students were required to select the best estimate for $45 \times 105 \approx$. Rounding was an important strategy to answer this

question and this strategy was taught explicitly to all three classes and it was observed that this strategy was used extensively when the students solved their problems set in meaningful tasks. This appears to have supported the students' estimation performance as in all three classes more students used a reasoned estimation strategy when comparing the pre and post test responses (Wendy 23% increase, Peter 20% increase and Bob 18 % increase).

At the beginning of the study few students (16%) were able to select the best estimate when adding two fractions. The question from the computational estimation text regarding fractions is shown in Figure 3.

Without calculating the exact answer, circle the best estimate for: $\frac{12}{13} + \frac{7}{8}$

Figure 3. Question 2 concerning the estimation of fractions in the computational estimation test.

At the end of the study 40% of the students were able to select the best estimate in question 2. All three classes used more reasoned estimation strategies in the post-test in comparison with the pre-test to answer this question. The increase by Wendy's students was 36% and Bob's students was 29%. These classes were taught all of the estimation strategies explicitly, including the benchmarking strategy. Peter's students were not taught the benchmarking strategy explicitly and the increase in using reasoned estimation strategies in his class was only 5%. When analysing how many actually used the benchmarking strategy in the post test, 26.7% of Wendy's class used the benchmarking strategy in comparison to 10% in Peter's class.

Wendy also spent time explicitly teaching the front end loading strategy and it was noted that her class results showed a 57% increase in using a reasoned estimation strategy when responding to question 5 (Figure 4) suggesting that explicitly teaching the strategies have a positive impact.

Without calculating the exact answer, circle the best estimate for:

27
38
65
+ 81

Figure4. Question 5 in the computational estimation test.

As the study progressed it became apparent that there may be a correlation between students who used a reasoned estimation strategy to answer the question and those who were more proficient at selecting the best estimate in the computational estimation test. A Pearson product correlation coefficient was computed to assess the relationship between students using reasoned estimation strategies and selecting the best estimates. Increases in students' reasoned estimation strategy use were correlated with increases in estimation performance and there was a strong and significant correlation between the two variables $r = 0.499$, $n = 67$, $p = 0.001$. The question which asked the students to estimate when adding fractions had a particularly strong relationship between students using a reasoned estimation strategy and selecting the best estimate, $\chi^2(1, N = 67) = 34.87$, $p = 0.000$. This suggests

that teaching the benchmarking estimation strategy explicitly enhances students' ability to estimate in computational situations.

Discussion and Conclusion

Inspired by recent research suggesting that beliefs are an important factor in effective learning (Carter & Norwood, 1997), the Researcher decided to make student beliefs a focus of this study. It was hypothesised that, for computational estimation to be used by the students, they needed to perceive that it was a worthwhile component of mathematics. The students in the three case studies generally held a view that mathematics was concerned with one correct answer and this view did not change substantially as a result of the intervention. This finding suggests that the students were exhibiting absolutist beliefs and a non-cultural view of mathematics. For student beliefs to change (Gravemeijer & Terwel, 2000), it may be necessary to have a whole school approach to estimation so that students receive a consistent message that computational estimation is a worthwhile computational choice. This perspective is supported by sociocultural theory that emphasises the social setting of the classroom and the creation of a community of learners (Rogoff, 1995).

However, students' beliefs did change in some respects. At the end of the study their awareness of computational estimation had greatly increased and they had developed positive perceptions of this computational choice. The research data suggests teaching computational estimation using fun and engaging meaningful tasks helped students understand the value of computational estimation.

Overall, the students' computational estimation performance and their use of reasoned estimation strategies were enhanced as a result of their teachers being involved in the professional learning intervention. This finding confirms that Year 6 students are able to understand and use a variety of estimation strategies when solving mathematics problems. The study also suggests that when children are explicitly taught about these strategies it enhances their computational estimation performance. The researcher aims to build on these noteworthy findings and implement elements of the intervention in pre-service teacher education programs, so that computational estimation may take its rightful place in the primary mathematics classroom as a valuable computational strategy.

References

- Australian Curriculum and Assessment Reporting Authority. (2011). The Australian Curriculum. Sydney: Australian Curriculum and Reporting Authority.
- Beswick, K. (2005). The beliefs/practice connection in broadly defined contexts. *Mathematics Education Research Journal*, 17(2), 39-68.
- Bray, J. (2002). Uniting teacher learning: Collaborative inquiry for professional development. *New Directions for Adult and Continuing Education*, 94, 83-92.
- Carter, G., & Norwood, K. (1997). The relationship between teacher and student beliefs about mathematics. *School Science and Mathematics*, 97(2), 62-67.
- Department for Education. (2011). Mathematics National Curriculum. Retrieved 5th January, 2011, from <http://www.education.gov.uk/schools/teachingandlearning/curriculum/primary>
- Dowker, A. (2003). Young children's estimates for addition: The zone of partial knowledge and understanding. In A. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: constructive adaptive expertise* (pp. 243-266). Mahwah: Lawrence Erlbaum Associates.
- Fennema, E., Carpenter, T., Franke, M., Levi, L., Jacobs, V., & Empson, S. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, 27, 403-434.
- Franke, M., Kazemi, E., & Battey, D. (2007). Understanding teaching and classroom practice in mathematics. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225-256). Charlotte, North Carolina: NCTM.

- Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, 32(6), 777-796.
- Hackling, M., Goodrum, D., & Rennie, L. (1999). *A proposal for a collaborative Australian secondary science program*. Perth: Edith Cowan University.
- Ingvarson, L., Meiers, M., & Beavis, A. (2005). Factors affecting the impact of professional development programs on teachers' knowledge, practice, student outcomes and efficacy. *Education Policy Analysis Archives*, 13(10).
- McIntosh, A., Reys, B., Reys, R., Bana, J., & Farrell, B. (1997). *Number sense in school mathematics*. Perth: MASTEC.
- McKeachie, W., Pintrich, P., Lin, Y., & Smith, D. (1986). *Teaching and learning in the college classroom: A review of the research literature*. Ann Arbor, Michigan: National Centre for Research to Improve Postsecondary Teaching and Learning, University of Michigan.
- Merriam, S. (1998). *Qualitative research and case study applications in education*. San Francisco, California: Jossey-Bass.
- Mildenhall, P. (2009). A study of teachers' learning and teaching of computational estimation: Getting started. In B. Kissane, M. Kemp, L. Sparrow, C. Hurst & T. Spencer (Eds.), *Mathematics it's mine: Proceedings of the 22nd biennial conference of The Australian Association of Mathematics Teachers* (pp. 153-158). Fremantle: AAMT.
- Miles, M., & Huberman, M. (1994). *Qualitative data analysis*. Thousand Oaks, California: Sage Publications Ltd.
- National Council of Mathematics Teachers. (2006). *Principles and Standards for School Mathematics*. Retrieved 2nd April, 2008, from <http://www.nctm.org/standards/default.aspx?id=58>
- Neill, A. (2006). Estimation exposed. *Practical Research for Education*, 35, 28-36.
- QSR. (2008). *NVIVO 8*. Melbourne: QSR International Pty Ltd.
- Reys, R., & Reys, B. (2004). Estimation in the mathematics curriculum: A progress report. In A. McIntosh & L. Sparrow (Eds.), *Beyond written computation* (pp. 101-112). Perth, Western Australia: MASTEC.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourse, and mathematizing*. New York: Cambridge University Press.
- Yoshikawa, S. (1994). Similarities and differences-computational estimation in Japan and the United States. In B. Reys & N. Nohda (Eds.), *Computational alternatives for the twenty-first century* (pp. 48-62). Reston: NCTM.

An Exploration into Growing Patterns with Young Australian Indigenous Students

Jodie Miller

Australian Catholic University
<Jodie.Miller@acu.edu.au>

Elizabeth Warren

Australian Catholic University
<Elizabeth.Warren@acu.edu.au>

This paper presents the results from an initial lesson in a series of design experiments focusing on young Indigenous students' understandings of growing patterns. Indigenous students in Year 2 and 3 (n=16) participated in pre lesson activities and a 45 minute lesson on growing patterns. Tentative findings from this study suggest that; (a) Year 2 and 3 Indigenous students are capable of working with growing patterns; (b) contextual artefacts assisted with communication; and (c) gesture played an important two-fold role in the lessons and communication of the mathematics experienced.

This paper explores Year 2 and Year 3 Indigenous students' initial understandings of growing patterns prior to their formal introduction at school. It is conjectured that Indigenous students have an affinity with the notion of pattern (Matthews, Cooper & Baturu, 2007) as an understanding of patterning underpins aspects of Aboriginal culture. An example of this is the construction of their kinship system, Matthews, Cooper & Baturu (2007, p.250) claim, the reason for this is that "their culture contains components that are pattern-based and which may lead to strong abilities to see pattern and structure". Currently, there is little research into how Indigenous students engage with western mathematical patterns, and whether their perceived affinity with pattern assists them in this engagement.

While it is acknowledged that algebraic thinking is the basis of higher levels of mathematics, there remains a persistent belief that young students are not capable of engaging in this type of mathematics (Carraher, Schliemann, Brizuela & Earnest 2006). Subsequent to this notion, teachers do not present complex mathematical tasks, as they feel young students are not ready to learn this type of mathematics and there is a tendency to focus on simple numbers and shapes within lessons (Sun Lee & Ginsburg, 2009). It can be argued that teachers who have engaged in these types of lower contextual teaching actions have limited the accessible knowledge capacity of students by not providing a challenging learning environment which provides opportunity to extend mathematical thinking. In Indigenous contexts, teachers have often provided lower contextual lessons based on skill and drill limiting students access to higher order mathematics (Baturu, Cooper, Michaelson, & Stevenson, 2008; Jorgensen, Grootenboer, Niesche, & Lerman, 2010). Additionally, studies have highlighted that educators have little faith in Indigenous students' mathematical ability (Matthews, Watego, Cooper & Baturu, 2005). There is a developing prospective statement that, in formal education settings with Indigenous students, little algebraic thinking is being developed.

Fundamental to the development of algebraic thinking and concepts, processes and knowledge of mathematics, is the ability to recognise patterns (Cooper & Warren, 2008; Mulligan & Mitchelmore, 2009; Papic, 2007). Research has highlighted that young students can recognise from a range of pattern contexts, the mathematical structure of the patterns. For example, students can identify the structure of repeating patterns as multiplicative, and the structure of growing patterns as functions (Blanton and Kaput, 2004; Cooper & Warren, 2011). In addition, Papic (2007) found that one year after students had engaged in an intervention focussing on creating and interpreting a range of mathematical patterns, these young students were achieving at higher levels in mathematics when compared to students

who had not engaged with these experiences. Early patterning experiences assist students to engage in 'seeing the structure of mathematics', and students experiencing difficulty in learning mathematics do not always recognise pattern and structure (Mulligan, Mitchelmore & Prescott, 2005).

Visual growing patterns are predominately the initial experience students encounter when introduced to formal algebra (Warren & Cooper, 2008). Growing patterns are characterised by the relationship between elements which increase or decrease by a constant difference. Students in the early years experience growing patterns through activities such as coping, continuing, and extending patterns. Eventually, there is a need for the student to see the relationship between the pattern and their position (stage), which can be termed a generalisation. The ability to generalise a growing pattern enables students to predict the pattern beyond terms that are provided (i.e., 10th position, 25th position, 100th position, nth position). As students engage in this function, they begin to explore the concept of co-variation as they reconsider growing patterns as functions, rather than recessive terms. Often this involves students generating a visual representation (drawing) usually involving a mathematical abstraction, this is then recorded in a table, and the relationship is identified between the two data sets (Warren & Cooper, 2008). Research has shown how this often leads to recursive thinking and the relationship between the growing pattern and the term is not identified.

This study aims to explore the following questions:

1. What is young Indigenous students' understanding of growing patterns?
2. How do young Indigenous students communicate this understanding?

Theoretical Framework

The theoretical frameworks underpinning this study are semiotics and Indigenous research perspectives. The learning of mathematics is two-fold; it involves the interpretation of signs, and the construction of mathematical meanings through communication with others (Saenz-Ludlow, 2007). The theoretical perspective of semiotics will be utilised as a lens within this study to interpret the interactions between teacher and students, and between students and context. It also assists in the selection of the types of materials used to represent growing patterns and how they are used in a classroom context.

In conjunction with semiotics, enacting the secondary theoretical perspective of Indigenous research perspectives, places a focus upon building relationships with students in order to facilitate the learning. It is essential to build a trusting environment for the researcher and students to share knowledge, particularly in the light of cultural variances. Acknowledging Indigenous research perspectives frames the research with an emphasis on empowering the participants, and thus facilitating the free transfer of knowledge and reducing the likelihood of the students being inhibited by the presence of the researcher. At the cultural interface, the researcher is conscious of building relationships so that students, IEWs and the researcher can partake in a meaningful exchange. This creates shared space is where empowerment can occur (Denzin & Lincoln, 2008). The implication of this decolonised approach dictates that the study must be viewed within the bounds of the individual community in which the research takes place and not generalised to the broader Indigenous population. In addition, throughout all aspects of the data collection and analysis an Indigenous perspective was provided by continual consultations from two Indigenous Education Officers.

Research Context

Given that the principal component in the study is young Indigenous students, it is essential to analyse students who are demographically located in an Indigenous community. The design research was conducted in one Year 2/3 classroom of an urban Indigenous school in North Queensland. The purpose for selecting the school was that there had already been a relationship established with the students and teachers in this community, as the school is a part of a longitudinal mathematical research project *RoleM* (Representations and oral language experiences in Mathematics) conducted by the second author.

Method

Participants

The students were from an Indigenous College situated in an urban town setting, in Queensland, Australia. Sixteen students (mean age 8.5 years) participated in teaching episodes. The students were in a mixed year level class with seven students in year 2 (4 girls and 3 boys) and nine students in year 3 (6 girls and 3 boys). All students were of Aboriginal or Torres Strait Island descent.

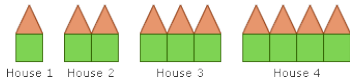
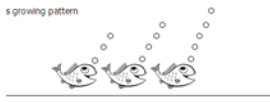
Data Gathering Techniques and Procedures

The methodological approach taken for the study was that of design experiment (Cobb, Confrey, Lehrer, & Schauble, 2010). The design aimed to produce both theoretical analyses and instructional innovations (Cobb, et. al., 2010) with one variation; one of the researchers acted as teacher. During the lessons tasks were modified according to the classroom dialogue and interactions. The design experiment consisted of three teaching episodes; each episode included three 45minute lessons on growing patterns. The teaching episodes took place over a six-month period and they focused on opportunities for the students to draw on their own contextual knowledge to explore growing patterns for the first time. This paper reports on the first lesson of teaching episode 1.

At the commencement of each teaching episode students were given pre lesson activities to determine their current understandings of patterning. All questions were read aloud to the students. Table 1 displays the questions asked to students prior to the commencement of teaching episode 1.

Table 1.

Questions Presented to Students Prior to the Commencement of Teaching Episode 1

Questions presented to students	
1. Draw your own pattern.	
2. Use stickers to create your own pattern.	
3. Copy this growing pattern.	
4. Continue this growing pattern.	
5a. Draw your own growing pattern.	
5b. Explain how your pattern is growing.	

The first lesson of teaching episode 1 was approximately 45 minutes. Lessons were video recorded to capture interactions between the students and researcher/teacher. Two

video cameras were used to capture the lesson, with one focusing on the researcher and the other on the students. During the lesson the researcher and supervisor acted as participant observers. At the conclusion of each lesson the researcher and supervisor reflected on observations to ensure that common perspective of the teaching instructions that occurred and the student responses. Additionally, the researcher interviewed the Indigenous Education Officer (IEO) to ascertain additional information that may not be prevalent to the research due to cultural difference.

Focus of Lesson 1 Teaching Episode 1

The initial part of the lesson reviewed the patterns drawn by the students on the pre lesson activity. Discussion with the students followed, examining the difference between the patterns drawn, particularly the difference between repeating patterns and growing patterns. Students were asked ‘how was their pattern growing?’ and were given opportunities to explain this. Many students provided examples from the environment to describe their initial ideas about growing patterns from the pre-lesson activities. Students were then given the opportunity to copy, extend, and identify the rule of a simple growing pattern using concrete items. The growing patterns provided were non-numeric and drew on environmental contexts. Growing patterns selected for the lesson were visually explicit, meaning that the link between the pattern and the term were easily accessible to students. An example of this was the relationship between kangaroo tails and kangaroo feet. Similar to the example given in past research studies between puppy dog tails and ears to explore co-variational thinking (Blanton & Kaput, 2004).

Data Analysis

The data were analysed in a four-fold process. Firstly, the initial video footage was transcribed to capture students’ verbal responses and for noting emerging themes. Secondly, the evolving data were reanalysed using the construct of semiotic bundles (nodes consisting of signs, gestures, and language exemplifying student learning). In the analysis an emphasis was placed on students’ physical gestures including their manipulation of the concrete objects and their body language. Thirdly, following this analysis the video footage was reviewed with the two Indigenous education officers (IEO). The Indigenous education officers watched the interview and provided feedback about the cultural signs that were displayed within the video. Their input was recorded and then transcribed to match the identified gestures and cultural signs used by the students. Fourthly, given this feedback with regard to cultural signs, the videos were reanalysed with an emphasis placed on students’ physical gestures including their manipulation of the concrete objects and their body language.

Results

Responses to the Questions Presented to Students Prior to Lesson 1

All students participated in the pre lesson questions (n=16). Table 2 presents the frequency of correct responses for questions presented in the pre lesson task. For question 4, six students correctly continued the pattern to the left and three students continued the pattern to the left and to the right (continued both ways).

The 11 students who completed question 1 all drew repeating patterns (i.e., aabbaabb; abababab). Additionally, students exhibited more accuracy in copying growing patterns (question 3) than in continuing growing patterns (question 4) beyond the term given. When

asked to continue the pattern, those students whose responses were considered incorrect, tended to copy the pattern rather than continue the pattern. Four students created a mathematical growing pattern for question 5. The other students who answered the question provided either a repeating pattern (n= 5) or a picture pattern representing growing in an environmental context (n=4). Three students provided no response.

Table 2.

Frequency of Correct Student Responses to Pre-lesson Questions

Question	Frequency (n=16)
1. Draw your own pattern	11
2. Create a repeating pattern using the stickers	12
3. Copy this growing pattern	14
4. Continue this growing pattern	6 (3 both ways)
5. Create a growing pattern	4

Figure 1 displays examples of the three different types of patterns presented by students in question 5.

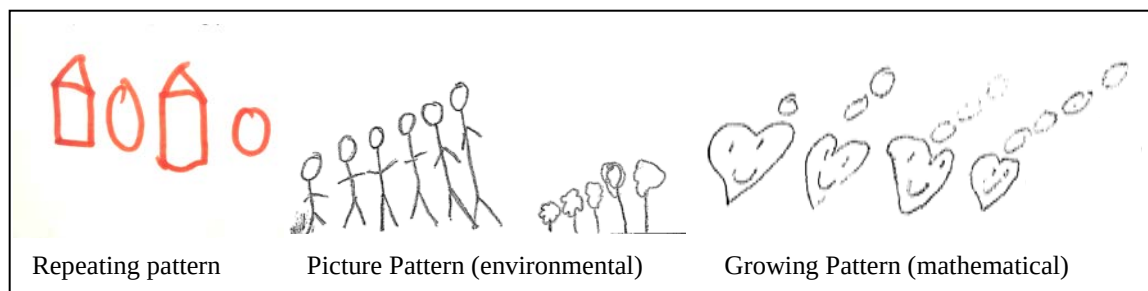


Figure 1. Examples of patterns presented by students for question 5.

Students were then asked how their patterns were growing. Students' responses were categorised into three areas; (a) spatial response – '*It's getting bigger and bigger*' or '*it's getting higher and higher*'; (b) numerical response – '*It is growing with numbers*'; (c) environmental response – '*It is growing with the sun, shade and water*'. This final response was often linked to pictorial patterns as that displayed in figure 1.

Excerpts from Lesson 1: Kangaroo Tails and Feet

During the lesson students were asked if they could predict further terms (if I had 10 kangaroo tails how many feet would I have and identify the relationship between the kangaroo tails and the number of feet. Below is an example of a student predicting beyond the pattern presented to them.

R: If I had 10 kangaroo tails, how many feet would there be?

S3: 20 (called out)

R1: How do you know that?

S3: I just added the same number.

R1: What do you mean?

S3: There were ten tails so I just added another 10 to get the answer.

Additionally, some students were beginning to identify the pattern rule. Below is an excerpt from the lesson.

R: So if I know how many tails I have, how do I work out how many feet there are?

S1: You're doubling it.

R: You're doubling what?

S1: The tails

Discussion with the Indigenous Education Officers

A discussion followed at the end of the lesson with the Indigenous Education Officers (IEO). Both the researcher and the IEOs watched the video recording of the lesson and interactions of the students. Themes that emerged from this discussion were: (a) Students could identify patterns easier when they were using contextual concrete items; (b) students gesture often when discussing the mathematics as they may not have the 'western mathematical language' to explain the concept; and (c) cultural factors contribute to communication in the lesson.

Discussion and Implications

The analysis of the initial teaching episodes has provided three prospective findings about young Indigenous students initial understandings of growing patterns and how they communicate these understandings: (a) students displayed capacity to copy, continue, create, and identify simple growing pattern rules; (b) contextual artefacts assisted with communication; (c) gesture played an important two-fold role in the lessons and communication of the mathematics experienced.

Firstly, students showed aptitude in copying, continuing and creating growing patterns prior to formal teaching. After the first lesson on growing patterns students were beginning to identify rules for simple growing patterns. The setting up of the activities allowed the students to see the structure of the pattern. It was essential that the signs of both variables (pattern and position) were embedded in the visual pattern (Warren, 2005). The use of concrete artefacts represented the growing pattern in such a way that students could attend to both signs and as demonstrated by student 1 begins use co-variation. Using the kangaroo tail and feet assisted this student to begin to see the relationship between the two variables as both position (tail) and pattern (feet) were explicit. Purportedly, by using explicit variables in concrete materials, students were facilitated in making connections with co-variation.

Research has highlighted that young students are indeed capable of thinking functionally (Blanton & Kaput, 2004), however the processes that assist students to 'notice' the relationship between the two variables is still unknown. The recognition of the two variables has been challenging particularly with many older students only identifying the recursive relationship with growing patterns (e.g., Lannin, 2005; Radford, 2006). Further analysis in this study is needed to determine if the explicit nature of the two variables will assist the students to make links to co-variation, identify rules, and generalise simple growing patterns.

Secondly, it became apparent that the students responded positively when the artefacts utilised in the learning activities were related to the students' local environment. The use of contextualised patterns provided opportunity to discuss the pattern in terms of language which was already accessible for students, such as in the case of kangaroo ears and tails. This too was evident through the discussion with the IEOs. The opportunity for the students to engage with the concrete artefact assisted students to discuss the structure of the pattern.

The students manipulated the concrete items and used them during their explanations. Engaging with concrete contextual artefacts acted as a medium between the mathematical structure and the students thinking as they began exploring abstract thinking.

Thirdly, there was a predication toward gesture during the initial lesson on growing pattern. The gesture was two-fold; gesture as embodiment of the task and gesture working with language for communication of mathematical concepts. Gesture as the embodiment of the task requires the students to interact with the artefacts. Both the researcher and the IEO agreed upon the observation that the interaction with the concrete items assisted students to think about the growing pattern. Past research has indicated that the use of the body experiences is seen as strongly related to cognition (Lakoff & Núñez, 2000). This challenges the belief of mathematics that is seen as objective, abstract and disembodied. From a semiotic perspective these physical processes helped students to objectify the task. Secondly, the use of gesture as an adjunct to language for communicating ideas about growing patterns was observed. Many students used gesture to supplement the language used in communicating their ideas about growing patterns. This theme was consistent with literature suggesting that gesture and language play significant roles in the learning of new mathematical concepts. The relationship between the two has been described as ‘unsplittable’ (McNeill, 1992). It was apparent that at times during the lesson, some language was not accessible to the students or the students’ home language was at odds with the mathematics (Goldin-Meadow, 2002). Researchers have concluded when language is not apparent or is mismatched to home language students will use gesture to assist conversation (Goldin-Meadow, 2002). Additionally, gesture may be the first place students display a new thought (Goldin-Meadow, 2002). It was apparent that both of these concepts were the consistent with students in this study. Understanding student gesture in Indigenous contexts and giving opportunities for that is paramount. Teachers must be cognisant of using gesture and providing opportunity for gesture

In conclusion, this paper represents the initial stage of a larger study and further analysis of the subsequent lessons, teaching episodes and interviews will assist in exploring more definitive conclusions. This paper is exploring a phenomenon as opposed to creating a defining stance on Indigenous students. While this study provides some insight into how young Indigenous students engaged in growing pattern tasks, how they reach generalisation remains a pertinent area for analysis.

References

- Baturo, A. R., Cooper, T. J., Michaelson, M. T., & Stevenson, J. (2008). Using national numeracy testing to benefit Indigenous students: Case studies of teachers taking back control of outcomes. In M. Goos, R. Brown, & K. Makar (Eds.), *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 59-66). Brisbane, Australia.
- Blanton, M., & Kaput, J. (2004). Elementary grades students’ capacity for functional thinking. In M. Jonsen Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 135-142). Oslo, Norway.
- Carraher, D., Schliemann, A., Brizuela, B., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, 37(2), 87-115.
- Cobb, P., Confrey, J., Lehrer, R., & Schauble, L. (2010). Design experiments in educational research. *Educational Research*, 32(1), 9-13.
- Cooper, T. J., & Warren, E. (2011). Years 2 to 6 students’ ability to generalise: Models, representations and theory. In J. Cai & E. Knuth (eds.), *Early algebraization: Volume 2 of advances in mathematics Education* (pp. 187-214). Netherlands: Springer.
- Cooper, T. J., & Warren, E. (2008). The effect of different representations on Years 3 to 5 students’ ability to generalise. *ZDM*, 40(1), 23-37. doi:10.1007/s11858-007-0066-8

- Denzin, N.K. & Lincoln, Y.S. (2008). Critical methodologies and Indigenous inquiry. In N.K. Denzin, Y.S. Lincoln, & L.T. Smith (Eds) *Handbook of Critical and Indigenous Methodologies*. (pp. 1-30). London: Sage Publications.
- Goldin-Meadow, S. (2002). Constructing communication by hand. *Cognitive Development*, 17(3-4), 1385-1405. doi:10.1016/S0885-2014(02)00122-3
- Jorgensen, R., Grootenboer, P., Niesche, R., & Lerman, S. (2010). Challenges for teacher education: the mismatch between beliefs and practice in remote Indigenous contexts. *Asia-Pacific Journal of Teacher Education*, 38(2), 161-175.
- Lakoff, G. & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.
- Lannin, J. (2005). Generalization and justification: The challenge of introducing algebraic reasoning Through patterning activities. *Mathematical Thinking and Learning*, 7(3), 231-258.
- Matthews, C., Cooper, T. J., & Baturo, A. (2007). Creating your own symbols: Beginning algebraic thinking with indigenous students. In J. Woo, H. Lew, L. Park, & D. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 249-256). Seoul, South Korea.
- Matthews, C., Watego, W., Cooper, T. J., & Baturo, A. R. (2005). Does mathematics education in Australia devalue Indigenous culture? Indigenous perspectives and non-indigenous reflections. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Proceedings of the 28th annual conference of Mathematics Education Research Group of Australasia*, (Vol. 2, pp. 513-520). Melbourne: Australia.
- McNeill, D. (1992). Gesture and language dialectic. *Gesture*, 1-25.
- Mulligan, J & Mitchelmore, M. (2009). Awareness of Pattern Structure in early mathematical development. *Mathematics Education Research Journal*, 21 (2), 33-49.
- Mulligan, J. T., Mitchelmore, M. C., & Prescott, A. (2005). Case studies of children's development of structure in early mathematics: A two-year longitudinal study. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, p. 1-9). Melbourne: Australia.
- Papic, M. (2007). *Mathematical patterning in early childhood: an intervention study*. Macquarie University: Unpublished thesis.
- Radford, L. (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. In J.L.C.S Alatorre, M. Saiz, A. Mendez, *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp.2-21). Mexico: Merida.
- Saenz-Ludlow, A. (2007). Signs and the process of interpretation: sign as an object and as a process. *Studies in Philosophy and Education*, 26, 205-223.
- Sun Lee, J., & Ginsburg, H. (2009). Early childhood teachers' misconceptions about mathematics education for young children in the United States. *Australasian Journal of Early Childhood*, 34, 37-45.
- Warren E., (2005). Patterns supporting the development of early algebraic thinking. In P. Clarkson, Downton, A., D. Gronn, M. Horne, A. McDonough, R. Pierce and A. Roche (Eds.), *Proceedings of the 28th conference of mathematics education research group of Australasia*, 2, 759-766. Merga: Sydney.
- Warren, E. & Cooper, T. J., (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics*, 67 (2), 171-185.

The Four-Three-Four Model: Drawing on Partitioning, Equivalence, and Unit-Forming in a Quotient Sub-Construct Fraction Task

Annie Mitchell
Australian Catholic University
<annie.mitchell@acu.edu.au>

This paper demonstrates the explanatory power of Kieren's framework for rational number knowing (1988, 1992, 1993, 1995), renamed here the four-three-four model, by describing the different approaches of Grade 6 students to a quotient context task (sharing three or seven custard tarts between five people) using Kieren's terminology of partitioning, equivalence, and unit-forming.

The explanatory power of Kieren's framework for rational number knowing, (Kieren, 1988, 1992, 1993, 1995) has rarely been examined on its own terms in the research literature on children's fraction understandings (see e.g., Mitchell, 2011). This is in contrast to the body of research that uses his five sub-constructs model (1980) consisting of part-whole, measure, quotient, operator, and ratio, which I will refer to as his five-part model. The Rational Number Project researchers (see e.g., Behr, Lesh, Post, & Silver, 1983) adapted his five-part model for their own research. Research on fractions in the Kieren/Rational Number Project research tradition overwhelmingly uses Kieren's five-part model (or minor adaptations thereof) to frame the categorisation of fraction tasks and fraction understanding (see e.g. Lamon, 2007; Pantziara & Philippou, 2012). Kieren's framework for rational number knowing (1988, 1992, 1993, 1995) refigured his earlier model and included *four* sub-constructs (measure, quotient, operator, ratio), *three* underlying concepts (partitioning, equivalence, and unit forming), other named constructs that made up the quotient field, and also a description of learning with *four* levels: ethnomathematic, intuitive, technical-symbolic, and axiomatic deductive. I have renamed Kieren's framework for rational number knowing the four-three-four model to distinguish it from the five-part model more commonly associated with his name.

This four-three-four model refigures the concept of part-whole, and as instruction based on a limited part-whole concept of fractions has been repeatedly shown to restrict children's development of a richer understanding of fractions (see e.g., Lamon 2007), the four-three-four model is worth another look. It would not be possible in one paper to examine the explanatory power of the whole four-three-four model, and so the model is not examined in its entirety here. In the present paper, I use part of the fraction construct aspect of the *four-three-four* model to analyse Grade 6 children's explanations of a quotient sub-construct task.

Review of the Literature

The Four-Three-Four Model

In Kieren's adaptation (1988) of his five-part model of rational number knowledge (1980), the four sub-constructs of measure, quotient, operator, and ratio remained, but were underpinned by three concepts: partitioning, equivalence, and unit-forming (see Figure 1). These sub-constructs and concepts could be engaged with on four levels of understanding. Kieren further elaborated this model in later writing (1992, 1993, 1995). The four-three-four model was not in opposition to Kieren's earlier five-part model (1980), nor to the model used in the Rational Number Project research (Behr et al., 1983; Behr, Harel, Post, & Lesh, 1992). Kieren saw the concepts of partitioning, equivalence and unit-forming as providing a

better categorisation of the underlying fraction concepts that were described more obliquely in the work of the Rational Number Project (see e.g., Behr et al., 1992).

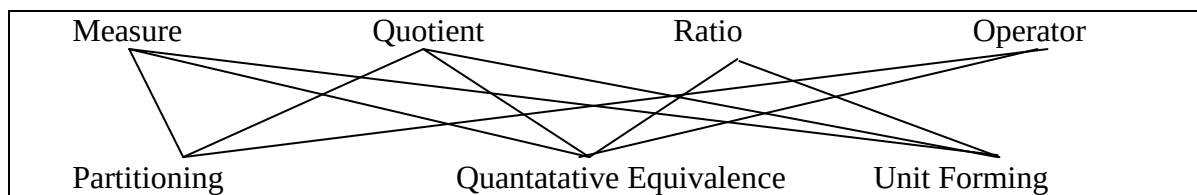


Figure 1. Section from Kieren's framework for rational number knowing (1993)

The description of learning in the four-three-four model has been further refined (see e.g., Pirie & Kieren, 1994), refiguring the four levels (ethnomathematic, intuitive, technical-symbolic, and axiomatic deductive) into eight layers. This aspect of the four-three-four model has been applied outside the fraction context and has stood on its own. The fraction construct aspect of the *four-three-four* model can also be applied independently from an analysis using the levels of understanding. The sub-constructs of Measure and Quotient are the only two in which Kieren suggests all three underlying concepts are being drawn upon (see Figure 1), and so I have sought to report on those two sub-constructs first. In my doctoral study I demonstrated the explanatory power of the four-three-four model with respect to primary school children's explanations of measure sub-construct tasks (Mitchell, 2011). The quotient sub-construct task described in the present paper was used in my doctoral study and is reported in more detail here. The present paper seeks to illustrate the usefulness of the four-three-four model in the description of explanations of quotient sub-construct tasks.

The Quotient Sub-construct

The quotient sub-construct described a context in which sharing between two separate measure spaces took place. For example, three pizzas shared between five people generated equal shares of three fifths: $3 \div 5 = \frac{3}{5}$. Similar tasks have appeared in the research literature (Charles & Nason, 2000; Clarke, Roche, & Mitchell, 2011; Kieren, 1988; Lamon, 1999).

Children's strategies for attempting quotient tasks included the intuitive strategy of repeated halving: cut every pizza in half and deal out all the pieces and see if that works, if not, divide every pizza into quarters and deal out all the pieces, then repeat with eighths if quarters leave a remainder (Pothier & Sawada, 1983). If children then subdivided the remainder, their explanations sounded like "engineering reports" (Kieren, 1988). Children had trouble keeping track of the unit when doing non-equal partitioning (dividing each piece into half and then the leftover half into five), for example, sometimes calling the result "a half and a fifth" instead of a half and a tenth, or six tenths, or three fifths (Lamon, 1999). The Dutch curriculum introduced fractions with a sharing context and used the term French division to describe cutting each pizza into enough pieces for everyone (see e.g., Keijzer & Terwell, 2002); each of the three pizzas would be divided into five parts and a piece from each pizza dealt out to each person, resulting in three one-fifth shares. In a separate Australian study, just less than a third (30.3%) of Grade 6 students could solve the pizzas shared between five people problem (Clarke et al. 2011). Very few of them used the fractions as division concept that three shared between five was three over five or three fifths.

Partitioning

Kieren's description of partitioning was the folding and drawing actions required of children when making equal parts (1995). Stages have been elaborated for learning to partition including repeated halving and using the radius rather than the diameter of a circle to generate thirds and fifths (Pothier & Sawada, 1983).

Equivalence

The concept of equivalence had a specified place in the four-three-four model. In some other frameworks, equivalence understanding had been positioned as early ratio understanding or bundled as order and equivalence (Behr, Wachsmuth, Post, & Lesh, 1984), but it was understood to be an important fraction concept. A strength of the four-three-four model was that it gave equivalence an explicit place.

Unit forming

Unit-forming described the additive nature of fractions. Just as eight could be made of seven and one, or six and two, or five and three, so too fractions could be made from the sum of other non-equal fractional amounts (Kieren, 1995). Kieren called the instructional aspect of unit-forming the combining space and described classroom activities with concrete materials that were used to show fractions as "additively combinable amounts" (1995, p. 32). For example, students were offered the open ended problem: "here are four rectangular pizzas cut in halves, quarters, sixths, and twelfths. Choose some pieces from at least three of these pizzas such that their "sum" is one pizza" (Kieren, 1993).

Methodology

The results presented here were part of a larger study concerning children's explanations of fraction tasks and measurement tasks (Mitchell, 2011). 88 Grade 6 students from three schools in metropolitan Melbourne, took part in a one-to-one task-based interview and their responses to tasks were recorded on a record sheet, audio-taped, and over half were video-taped. Clinical interviews provide insight into children's explanations (Ginsberg, 1997) but cannot describe learning. Smaller studies, such as this one, if conducted meticulously, can provide credibility and insightful analysis, but the results are not generalisable.

The quotient sub-construct was assessed using a sharing situation similar to sharing tasks in the research literature that used pizzas and people (see e.g., Lamon, 1999). Length and area representations were used, but pizzas were not chosen as the context because of the concern that children had preconceived ideas about the number of pieces into which they could be divided. The Sharing Custard Tarts or Liquorice task included four questions presented sequentially with a task card and figurines, and the students could use pen and paper to work out their solutions. All questions were spoken. Question A involved sharing three strips of liquorice between five people. Question B involved sharing three custard tarts between five people: "Five people are sharing three custard tarts equally. The tarts can be cut anywhere. How much of a tart does each person get?" Question C involved sharing seven custard tarts between five people: "Five people are sharing three custard tarts equally. The tarts can be cut anywhere. How much of a tart does each person get?" Question D involved sharing nine strips of liquorice between four people (see Figure 2).

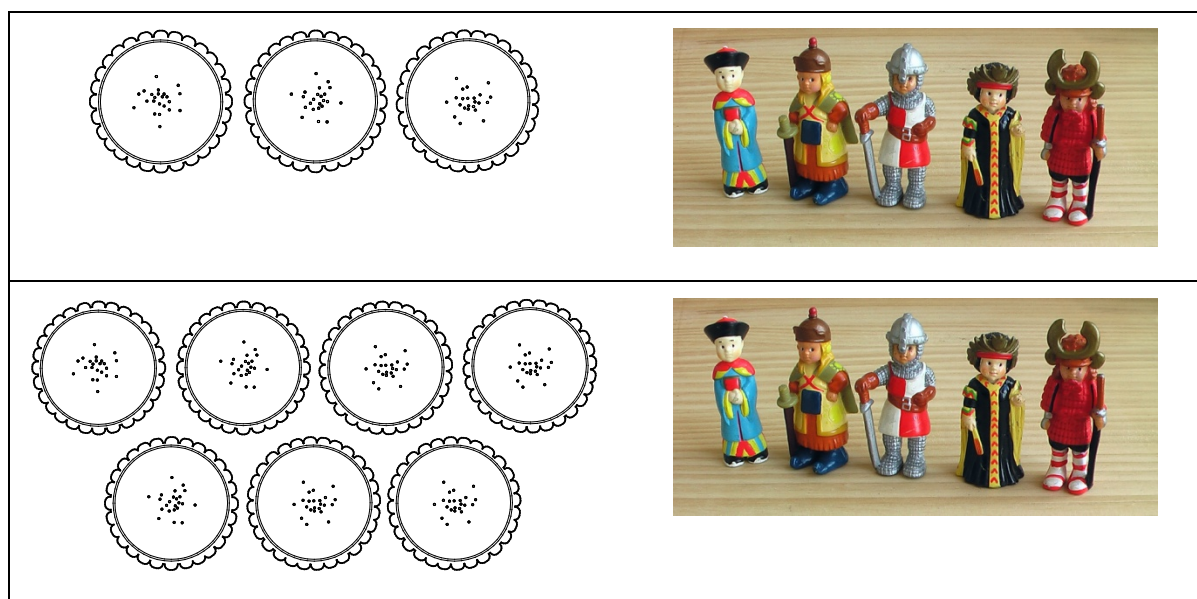


Figure 2. The Sharing Custard Tarts or Liquorice task: Question B and Question C.

Results and Discussion

Students' explanation to two of the questions, Question B (sharing three custard tarts between five people) and Question C (sharing seven custard tarts between five people), are examined here, with respect to how partitioning, equivalence and unit-forming are drawn upon in students' explanations of their answers. In Question B, 26.1% of students successfully made five equal shares (diagram or mentally) and named the share correctly as three fifths, while a further 36.4% could make five fair shares, either as fifths, or a half and a tenth, but were unsuccessful at naming the total share correctly. An improper fraction answer ($\frac{7}{5}$ or $1\frac{2}{5}$) was generated by the problem of five people sharing seven custard tarts (Question C) and 20.5% of the students could successfully share and name the share in this problem. A further 35.2% of the students could create or imagine equal shares but could not name one person's share successfully. In Question C, one student used a fraction as division approach, seven shared between five is seven fifths, and was operating at a level where he did not need to consciously recruit partitioning, equivalence or unit-forming concepts.

Partitioning

Partitioning was evident in the responses of the students to both questions. In Question B, some students used French division to share each custard tarts into five equal parts (see Figure 3). Meg, after a poorly executed first attempt at fifths (crossed out), drew three custard tarts, divided them into fifths, allocated shares, and named each person's share correctly as three fifths. Ebony (see Figure 3) was able to make the partitions and indicate a share, but named those three parts (one shown, the others visualised) incorrectly as three fifteenths.

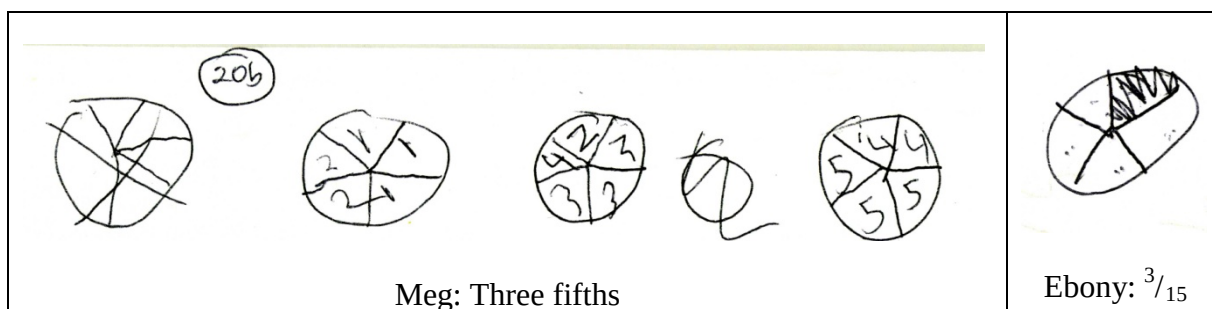


Figure 3. Partitioning using French Division.

Similarly in Question C, some children used French division and divided all seven custard tarts into five parts and some successfully named one person's share as seven fifths or one and two fifths. Some children eliminated the wholes first and then divided each of the last two custard tarts into five, successfully naming this as one and two fifths. As in Question B, some children could do the partitioning but could not successfully name the share.

In Question B, Harry found the partitioning difficult with a circular area diagram. He divided a circle into six parts and after puzzling over more drawings he thought aloud, "there's no way you can cut it up into five pieces". Similarly, Brad (see Figure 4) tried partitioning the three custard tarts into thirds, eighths, and sixths (inaccurately) but could not find a partitioning that would enable fair sharing. For Question C, he partitioned the seven custard tarts into halves and then drew over that to partition them into quarters (see Figure 4). In both Question B and Question C, some children drew on partitioning and used repeated halving but could not create a number of parts that could be dealt out equally.

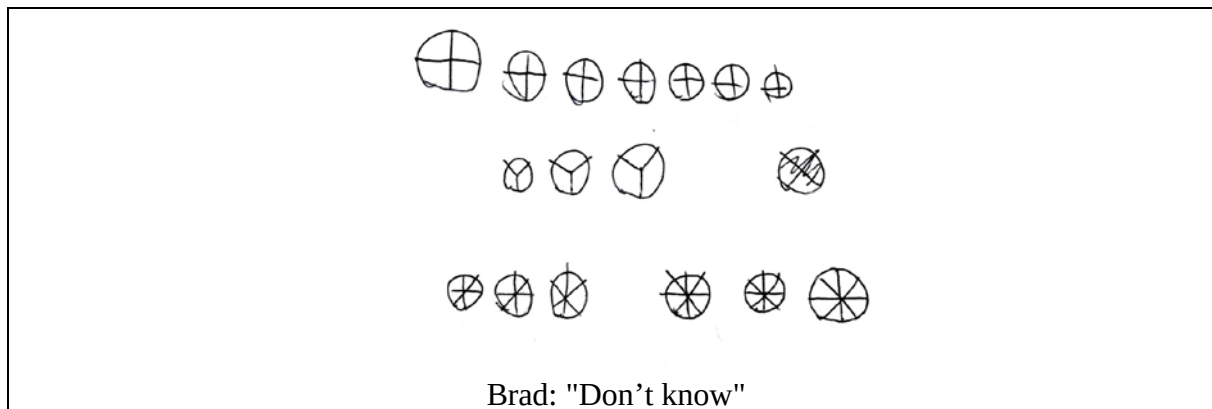


Figure 4. Unsuccessful partitioning in Question B.

Unit Forming

Unit forming was drawn upon in these two questions when students made fair shares by combining a number of non-equal parts. They often also recruited partitioning in the execution of their strategy. Some students who divided all five custard tarts in half then partitioned the leftover half into five and drew on unit-forming to name the combined fair share. Of the 19 students who did this in Question B and made five fair shares out of two unequal parts, only one could name this correctly. Seth drew unit-forming (making a fair shares of two unequal parts) while recruiting partitioning (dividing the leftover half into five equal parts), and then used equivalence ($\frac{1}{2} = \frac{5}{10}$) to name the share six tenths (see Figure 5).

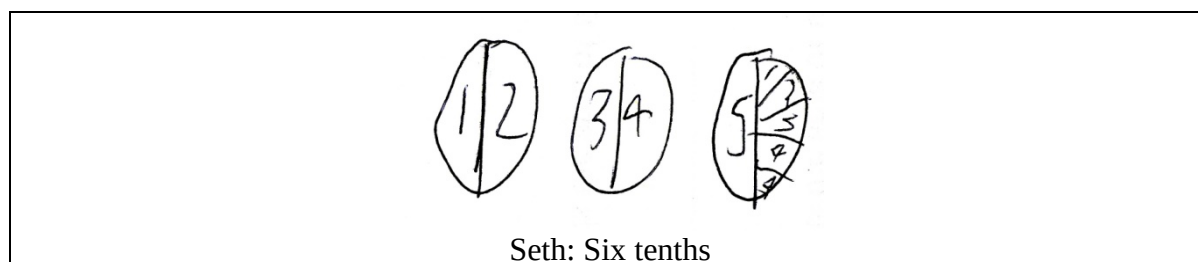


Figure 5. Successful unit forming in Question B.

Some children could describe the share, as a half and a fifth of the extra half, but did not have the equivalence skills to complete the renaming. Other children incorrectly described the share as a half and a fifth, unable to keep track of the unit.

On the other hand, one child, Felix, gave a reasonable estimate using unit-forming, but rather than recruiting equivalence knowledge, he visualising the two pieces combined and then drew on the concept of partitioning to estimate the size of that "part". He combined the half and one piece of the last part divided into five, and described this added together amount as "nearly two thirds": this estimate (0.66) was very close to the exact value (0.6).

In Question C, some students drew on the concept of unit forming, for example, by eliminating the wholes, dividing the last two tarts into thirds, and then the leftover third into five parts. The equivalence knowledge needed to execute the naming of this share, made of combined amounts, was beyond the students who used this approach. However, one student, Courtney, was very logical with her representation (see Figure 6).

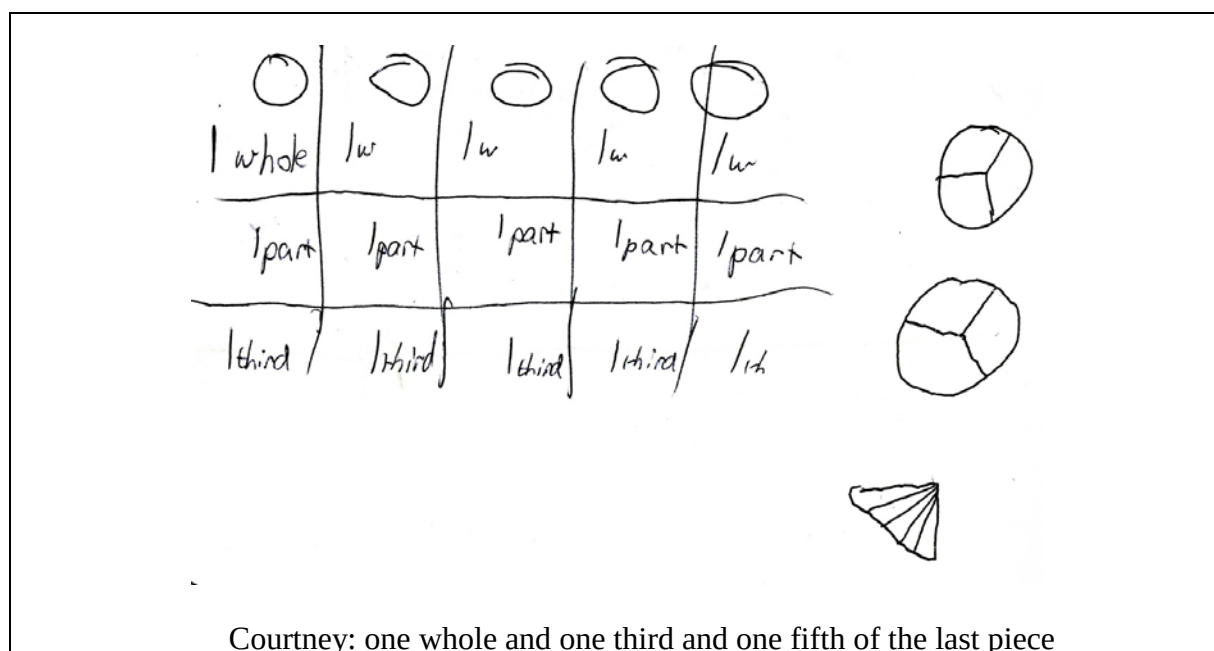


Figure 6. tracking unit-forming: sharing 7 tarts between 5 people.

One child, Ben, drawing on unit forming, made an estimate of his combined amounts, rather than calculate the fair share. After eliminating the wholes, he divided the last two custard tarts into halves. As this gave four parts, not five, he correctly reasoned that the fair share would be one and about three quarters of a half. His reasonable estimate of the non-whole number part, three quarters of a half ($\frac{3}{8}$ or $\frac{15}{40}$) was very close to the precise value

of the part of one of the two leftover tarts ($\frac{2}{5}$ or $\frac{16}{40}$) that was needed to make a fair share when combined with a whole tart.

Equivalence

As described above, students who drew on the concept of unit forming needed to recruit equivalence knowledge to name the overall fair share successfully.

Conclusion

The present study demonstrated that the concepts of partitioning, equivalence, and unit-forming understandings were drawn upon by real students in different approaches to this quotient sub-construct task. The use of the terminology partitioning, equivalence, and unit-forming in describing the students' approaches to this quotient context has enabled a more detailed analysis of students' correct and incorrect responses to the task. The successful and unsuccessful responses described here that drew on partitioning, demonstrate that successful partitioning included imagining, making (diagram or mentally) and naming, sometimes non-contiguous, equal parts. Harry could imagine but not make partitions for fifths. Ebony could make but not name her partitioning when the parts were not contiguous. The results of the present study, particularly the infrequency of drawing on equivalence and the infrequency of using a fractions-as-division approach, demonstrate that the underlying concepts of partitioning, equivalence and unit-forming are still relevant in upper primary school.

A strength of the four-three-four model was its categorisation of unit-forming. This gives prominence to the many correct additive aspects of fraction understanding. Courtney's logical diagram (see Figure 6) is an example of Kieren's engineering reports (1988), and highlights that unit forming is a mathematically correct approach to quotient sub-construct fraction tasks that is not always fully executed. Using the concept of unit-forming, teachers and students could elaborate the restructuring of the whole in more sophisticated terms than a "part-whole" understanding in the five-part model allows. Similarly, the reasonableness of the estimates given by Felix and Ben, can be unpacked with greater depth in terms that more closely resemble their reasoning, by the use of the terminology of partitioning and unit forming. In Charles and Nason's study of Grade 3 children completing similar tasks, 12 strategies were reported but further categorisation under the term unit forming may have underscored the concept being drawn upon in their "repeated sizing strategy" (2000, p. 198).

The adoption of the four-three-four model would not be incompatible with the research already conducted using variations of the five-part model in the Kieren/Rational Number Project tradition. However, elaborating the categories of partitioning, equivalence (both already in use by teachers and students) and unit-forming would broaden teachers' pedagogical repertoire of part-whole instruction, as called for in the research literature on fractions. Equivalence has been a term in use with both teachers and students and so the concept has a recognised place in the curriculum. Extending the classroom vocabulary for both teachers and students, to include unit-forming would extend the explanatory power of the model to the students themselves.

Acknowledgements

The author would like to thank Marj Horne, Doug Clarke, and Anne Roche for the many conversations around these fraction tasks and fraction concepts. The author would also like to thank the anonymous reviewers who gave very insightful comments on this paper.

References

- Behr, M. J., Harel, G., Post, T. R., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York, NY: Macmillan.
- Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-125). New York, NY: Academic Press.
- Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education*, 15, 323-341.
- Charles, K. & Nason, R. (2000). Young children's partitioning strategies. *Educational Studies in Mathematics*, 43, 191-221.
- Clarke, D. M., Roche, A., & Mitchell, A. (2011). One-to-one student interviews provide powerful insights and clear focus for the teaching of fractions in the middle years. In J. Way & J. Bobis (Eds.), *Fractions: Teaching for understanding* (pp. 23-32). Adelaide: The Australian Association of Mathematics Teachers.
- Ginsberg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge, United Kingdom: Cambridge University Press.
- Keijzer, R., & Terwel, J. (2002). Audrey's acquisition of fractions: A case study into the learning of formal mathematics. *Educational Studies in Mathematics*, 47, 53-73.
- Kieren, T. E. (1980). The rational number construct: Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125-149). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.
- Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. A. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Reston, VA: Lawrence Erlbaum.
- Kieren, T. E. (1992). Rational and fractional numbers as mathematical and personal knowledge: Implications for curriculum and instruction. In G. Leinhardt, R. Putnam, & R. A. Hattrop (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 323-371). Hillsdale, NJ: Lawrence Erlbaum.
- Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49-84). Hillsdale, NJ: Lawrence Erlbaum.
- Kieren, T. E. (1995). Creating spaces for learning fractions. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 31-65). Albany: State University of New York Press.
- Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah, NJ: Lawrence Erlbaum.
- Lamon, S. J. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. K. J. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 629-667). Charlotte, NC: Information Age Publishing.
- Pantziara, M. & Philippou, G. (2012). Levels of students' "conception" of fractions. *Educational Studies in Mathematics*, 79, 61-83.
- Mitchell, A. (2011). *Interpreting students' explanations of fraction tasks, and their connections to length and area knowledge*. (Unpublished doctoral dissertation). Australian Catholic University. Fitzroy, Australia.
- Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics*, 26, 165-190.
- Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education*, 14, 307-317.

Virtual Mathematics Education: Using Second Life to Model and Reflect upon the Teaching of Mathematics

Tracey Muir
University of Tasmania
<Tracey.Muir@utas.edu.au>

Two of the challenges faced by mathematics teacher educators involve the issue of pre-service teachers' mathematical content knowledge and the impact this has on their PCK, and the perceived gap between the theoretical knowledge received in their teaching course and the practical knowledge gained in the classroom while on professional experience. This paper examines the use of what is essentially a teacher educator tool or strategy, designed to give pre-service teachers a realistic environment in which to address these two challenges. The study uses data collected from pre-service teachers who participated in virtual lessons using Second Life, post-lesson interviews and a teacher educator survey to document the advantages and disadvantages of using such an approach. The results indicate that the use of Second Life has practical implications for teacher educators as an additional tool for modelling and reflecting upon the teaching of mathematics.

Concerns about pre-service teachers' content knowledge and the impact this has on pedagogical content knowledge (PCK) is widespread in the literature (e.g., Chick, Pham & Baker, 2006; Hill, Ball & Schilling, 2008; Maher & Muir, 2011). Similarly, concerns have been raised about teacher preparation in general, particularly in terms of meeting the challenges associated with bridging the perceived gap between theory and practice (e.g., Lin, 2005). Strategies to overcome these challenges have included the development of frameworks for studying pre-service teachers' content knowledge and PCK, (Ball, Thames, & Phelps, 2008; Chick et al., 2006) and the evaluation of teaching tools and strategies, such as teaching videos (e.g., Star & Strickland, 2008). Second Life offers a virtual environment which simulates 'real-life' situations, and has been used in educational contexts to recreate classrooms, particularly with an emphasis on behaviour management (e.g., Gregory et al., 2011). An extensive literature search, however, has revealed that other than being used as a vehicle for teaching mathematical concepts (e.g., Sweeney, 2008), Second Life has not been utilised to explicitly teach mathematics pedagogy. This paper adds to the limited research in this area through describing a project whereby pre-service teachers participated in two mathematics lessons designed to teach fraction concepts, and were then interviewed and surveyed about their experiences afterwards. Data were also collected from teacher educators to determine their perceptions of the practical uses and implications of the Second Life classroom.

Theoretical Framework

Teacher Content Knowledge and Pedagogical Content Knowledge

There is general consensus in the literature that effective teachers require both well-developed content knowledge and pedagogical content knowledge. Many elementary teachers, however, lack conceptual understanding of mathematics (Ma, 1999; Mewborn, 2001), and pre-service teachers' mathematical knowledge and confidence is of particular concern (e.g., Ball, 1990; Lange & Meaney, 2011). Recent graduate standards have

identified the need for graduates to “know the content and how to teach it” (Educational Services Australia, 2011, p.6) and excellent teachers of mathematics should demonstrate a sound, coherent knowledge of mathematics, appropriate to the students they teach (Australian Association of Mathematics Teachers, 2006). This has direct implications for the teacher educator who is primarily responsible for the preparation of these teachers. Shulman’s (1987) conceptualisation of PCK contributed greatly to current mathematics education research into what teachers need to know about students’ learning with a key aspect of PCK being the identification of students’ misconceptions and the strategies that teachers use to overcome misconceptions (Chick, et al., 2006). There is evidence to suggest that many pre-service teachers exhibit similar misconceptions to their student counterparts (Ryan & McCrae, 2005), and that even practicing teachers tend to overgeneralise their knowledge of whole numbers when working in the domain of rational numbers. In the domain of fractions, which was the topic of the lessons discussed further on in this paper, common misconceptions include misapplication of whole number thinking (e.g., $\frac{1}{8}$ is larger than $\frac{1}{4}$ because 8 is larger than 4), with other difficulties including the understanding of equivalence when using multiple representations and the overuse of an additive strategy when a multiplicative use is required (Ryan & Williams, 2007).

Linking Theory with Practice

It is generally agreed that pre-service teachers view classroom experience as the best teacher (Richardson, 1996), with evidence suggesting that pre-service teachers are not being adequately prepared for the challenges of teaching (Ferry, et al., 2004). Practice-based strategies, such as videos, have been found to be effective in bridging the perceived gap between theory and practice (Lin, 2005; Taylor, 2002) through challenging pre-service teachers’ images of mathematics and mathematics teaching (Taylor, 2002), and in developing their skills in observing teaching (Santagata, Zannoni, & Stigler, 2007). The use of video excerpts in pre-service teacher education is well documented in the literature (e.g., Star & Strickland, 2008), with the main benefit being the opportunity for classes to view the same teaching and learning episode and to share in a discussion of what has taken place. It is not always possible or practical however, to obtain or produce video excerpts that capture specific examples of instruction or situations that teacher educators can use to analyse classroom episodes. Similarly, it is seldom possible to capture video footage of pre-service teachers engaged in the teaching process and then view this footage in a social context. Second Life provides an alternative to locally or commercially produced video excerpts in that the teacher educator can identify, and to some extent influence, the objectives and outcomes of the lesson, it does not require ethical approval or sophisticated recording equipment, it has the advantage of involving the pre-service teachers in the action and is available for immediate viewing after the recording has occurred.

Second Life

Second Life (www.secondlife.com) is a popular, online virtual world where players represent themselves as avatars and interact with each other through conversations and gestures (Gao, Noh & Koehler, 2009). Virtual worlds have been designed to create simulations of real world phenomena to provide environments for experiential learning and training (Eschenbrenner, Fui-Hoon, & Siau, 2008). The benefits noted by educators include the opportunity to conduct activities and tasks in a risk-free environment, enhanced collaboration and communication between participants, including those who are geographically dispersed, and increased engagement (Gao, et al., 2009). Teacher educators

have utilised Second Life in particular to both supplement pre-service teachers' learning and to undertake preparation for professional experience (e.g., Gregory, et al., 2011). In order to address the inconsistencies between pre-service teachers' practical experiences, Gregory and her colleagues designed VirtualPREX in order to provide pre-service teachers with experiences that reflected the complex and diverse nature of a teacher's role in the classroom. VirtualPREX allowed the pre-service teachers to anonymously engage freely in practices, test their strategies, with no long-term effects on 'real' students, with the aim being to better understand themselves as teachers. While an extensive literature search failed to identify any examples of Second Life being used explicitly to teach mathematics pedagogy, it has been used to teach mathematical concepts (e.g., Sweeney, 2008) and for learning about mathematics. Valcke, Van Steenbrugge, and Veeragoudar (2008), for example, utilised Second Life as a mechanism to address their student teachers' attitudes and beliefs about learning mathematics. They created a Second Life environment, called the Fractal Village, in which groups of students worked on weekly tasks centred on the active manipulation of 3D objects. The researchers found that the Second Life environment had the potential to positively influence the student teachers' mathematical dispositions but conceded that further research was necessary in order to more accurately assess the impact.

Methodology

The main aims of the project were: to create a Multi-User Virtual Environment (MUVE or virtual classroom) using Second Life and teacher and student avatars; to provide pre-service teachers with the opportunity to take on the role of teacher and student avatars in a fictional, safe on-line environment; to better prepare pre-service teachers for the challenges of undertaking professional experience placements; to provide an avenue for pre-service teachers to engage in critical discussion and reflective practice and to explore whether or not Second Life could be used to model and reflect upon mathematics teaching and mathematical behaviour.

Virtual Classroom Design

The classroom was designed by an educational developer and consisted of a three stage process involving conceptualisation, finding and purchasing objects, and assembling the objects into a finished product. The researcher met regularly with the developer to discuss the design of the classroom and avatars, and to select objects for inclusion. Objects such as chairs and table, notebooks and pens were purchased using in-world Linden Dollars while other parts of the room such as walls, the roof and windows were constructed using the shape creation tools. Other infrastructure put in place to facilitate and record the sessions were signage and instructions, a specialised 3D navigator control to smoothly pan camera viewpoints, video capture software to record the sessions and video-editing software to produce the final movies. A screen shot of the classroom is shown in Figure 1.

The Avatars

One teacher and seven avatars were created. Initially each of the student avatars had particular behavioural characteristics as the focus was originally on inclusivity and behaviour management, but these were later adapted to describe mathematical behaviours, incorporating common misconceptions as identified in the literature. Each avatar also had a list of actions that he/she was programmed to use. Figure 2 shows a visual representation of one of the student avatars, Matak, who was characterised as being of 'above average' mathematical ability and able to, for example, compare and order fractions, place a variety

of fractions correctly on a number line and explain why. In contrast, Willow and Monty were characterised as being of ‘below average’ mathematical ability and described as showing:

Confusion with ordering fractions – might think, for example, that $\frac{1}{9}$ is larger than $\frac{1}{4}$ because 9 is bigger than 4; would have difficulty with putting fractions in order on a number line; could probably say that $\frac{2}{4}$ is the same as $\frac{1}{2}$ but not consistently; no real understanding of the relative size of a fraction or that a fraction is also a number; if asked to add fractions, would say that $\frac{1}{2}$ and $\frac{3}{4}$ would be $\frac{2}{6}$.



Figure 1. Snapshot of virtual classroom



Figure 2. Matak

Participants and Procedure

Eight pre-service teachers volunteered to take part in the study to trial the virtual classroom. All but one were enrolled as fully on-line students, six were mature-aged and all were in their third or final year of a Bachelor of Education (Primary) course. As we wanted to utilise the voice mechanism, the number of participants for any session was limited to eight to avoid noise congestion and feedback. Prior to each session, each participant was assigned an avatar, given a unique log-in name and password and a description of their avatar in terms of their mathematical ability. An overview of the lesson was emailed to participants, along with instructions for any activities and resources if required. At the beginning of the study, participants were given tutorials in Second Life and provided with guides to assist them with movements, gestures and speaking. Although not all participants were involved in all sessions, for most of the pre-service teachers, the mathematics lessons would have been the seventh and eighth lessons conducted in Second Life. Each lesson ran for about 20-30 minutes, was recorded using ‘machinema’, and was immediately followed up with a debriefing session, which was also recorded.

Data Collection and Analysis

Recordings of the lessons were fully transcribed and manually analysed using codes to identify instances of mathematical misconceptions. The debriefing sessions were also fully transcribed, with the codes relating to evidence of reflection and experiences as a learner. Further evaluation occurred through a follow-up questionnaire for all eight participants (including those who did not participate in the session/s). In addition, seven teacher educators were asked to view the machinema footage and respond to an emailed open-ended questionnaire designed to evaluate its use as a teaching tool. Common themes were identified from the respondents’ answers and are qualitatively described in the next section.

Results and Discussion

The Mathematics Lessons

The focus for the lessons was fractions, with experiences including counting by fractions, using a fraction wall to assist with identifying equivalent fractions and fraction sizes. The role of the teacher avatar in the first lesson was played by the researcher and there were four student avatars of varying mathematical ability. An analysis of the transcript of the first lesson showed that the pre-service teachers were willing to take on the mathematical behaviours of their allocated students as the following excerpts demonstrate (Willow was described as ‘below average’ and Jake as ‘average’):

Miss Carpenter: Can you see the fraction wall there Willow? Do you see the green bits at the top? We can say that’s one whole thing, and if you have a look at your quarters – how many of your quarters equal one of those half green bars?

Willow: Two eighths

Miss Carpenter: Are we looking at eighths or quarters?

Willow: I don’t know, but if I add them together ...

Miss Carpenter: Oh, but if you’re looking at a quarter, what does the bottom number say? There is a one on top, but what number is on the bottom?

Willow: Four?

Miss Carpenter: Good. So see the blue quarters. They are all quarters. How many of them do we need to make one whole thing?

Willow: One, two, three, four

Tr. Excellent So if four will make one whole thing, how many do we need to make half? To make the same as the green half?

Willow: Um, two?

Miss Carpenter: Very good.

Miss Carpenter: This time I’m going to ask you a question and I want you to tell me which fraction is bigger – OK – so Jake, if I said to you which would be bigger – one fifth or one eighth, which would you say is bigger?

Jake: Ah, one eighth?

Miss Carpenter: One eighth? OK, so have a look at the fraction wall, what colour are the fifths?

Jake: Yellow

Miss Carpenter: And what colour are the eighths?

Jake: Blue

Miss Carpenter: Blue, or dark blue so we don’t get confused. So is the yellow bit bigger or smaller than one of the dark blue bits?

Jake: Bigger

Miss Carpenter: OK, very good. So how many of those yellow bits does it take to make a whole one?

Jake: Five

Miss Carpenter: Five, good. And how many of those dark blue eighths does it take to make a whole one?

Jake: Eight

Miss Carpenter: OK, so can you see by looking at that, that we have cut up, like a chocolate bar, into eight little bits and had to share it among eight people that would be the size of the piece they would get. Whereas with the yellow one, you only had to share it among five people so they get a little bigger bit than the people who had to share with eight.

The transcripts show evidence that the pre-service teachers could simulate the behaviour of students who were incorrectly applying whole number thinking to compare the size of fractions and enabled the teacher to model how this would be addressed with the aid of a fraction wall. In the debriefing session afterwards, the pre-service teachers were directly asked to comment on the lesson and in particular how useful they found the fraction wall as a teaching tool. While they generally agreed that it was important to have access to the

concrete aids, one of the pre-service teachers suggested that it could have been more explicitly explained (“that each line is a whole part”), and another suggested that additional models, such as pizza or chocolate, may also have been useful. Responses from the pre-service teachers also indicated that they appreciated the modelling that occurred when addressing children’s incorrect answers:

I think that’s the most important thing because you can really get to see what responses you’re going to get and it’s going to put you on the spot in the real world because you’ve got to think on your feet and it’s a perfect way to sort of do that I think

In the second lesson, one of the pre-service teachers took on the role of Miss Carpenter and some of the student avatars were played by different participants to those who took part in the first lesson. The lesson focus moved on towards adding fractions with different denominators and the transcript provided examples of how the pre-service teacher would teach this concept and address any misconceptions that arose. There were a number of occasions when Miss Carpenter referred to the fraction wall to help make links between equivalent fractions, with the following excerpt showing an example of this:

Look at the fraction wall up there – so if you’ve got one of the green portions and you’re converting it into the blue ones, how many blue portions would you need to make one of the green ones?

There was also evidence, however, to suggest a primarily procedural approach was used when converting fractions to similar denominators. For example:

So all we’ve done is doubled – we’ve multiplied ... and all we’ve simply done is doubled both of those numbers, one and three, to get a two and a six ...so ... whatever we do to the numerator – the top number – we have to do to the denominator – the bottom number – as well.

This was also noted by one of the pre-service teachers who through the questionnaire completed after viewing the footage, questioned whether or not students would actually fully understand equivalence or just “simply find the same fraction size”. Having one of the pre-service teachers take on the role of Miss Carpenter did provide an added opportunity for reflecting upon one’s own performance as a teacher, which was not available in the first lesson. ‘Angela’, who played the role, commented that the “hardest part was that you can’t point to things ... it’s really hard to explain things if you can’t actually demonstrate them, you have to do it all via voice ... and you can’t write anything down on a whiteboard or anything. But it was good.” The follow-up questionnaire produced some more reflective comments, and although most of them related to the restrictions due to the virtual classroom environment, Angela did note that:

I shouldn’t have said ‘no’ to a student’s incorrect answer but rather asked her to explain her answer further, or asked other students to help, or failing all that, explicitly taught the concept.

Evaluation of Second Life as a Teaching Tool

Feedback received from the pre-service teachers through anecdotal, interview and survey data indicated that the main advantages included the opportunity to interact with peers in an on-line environment that was relatively anonymous, “the chance to experiment on computer students, as opposed to real people” and practice with “thinking on your feet”. Most participants agreed that it was possible to recreate an authentic classroom experience through using Second Life, with the following comment being illustrative of the feedback received:

It certainly creates an experience that is closer to authentic than text book scenarios. Practising how you respond to various situations is good practice – whether it is maths or any topic. I like the idea that to a degree you are placed out of your comfort zone as you are not completely sure what will occur, while at the same time you are within the safety of a learning environment. Often the first

practice we get are with real students and you don't have the opportunity to reflect and try again the next session (Rachel, 15/11/11).

The limitations identified included the time involved to master operating the avatar, technical issues associated with slow internet connections, restriction on the use of interactive 'props', and the inability to use spontaneous gestures.

The responses received from the teacher educators indicated that they perceived Second Life to have potential as a teaching tool within their courses. Some of the benefits identified included "it brings out the deficiencies and strengths in mathematical content knowledge as well as pedagogical content knowledge", opportunities for pre-service teachers to reflect upon their use of language and mathematical explanations, increased collaboration and social interaction for on-line students and the provision of a experience closer to reality than is usually provided in a university setting. Some suggestions for use included focusing on demonstrating a particular aspect of teaching and even modelling inappropriate teaching approaches. Like the pre-service teachers, they also perceived the main limitations as the time taken to become comfortable with operating the avatars and restrictions on interacting with the environment.

Conclusions and Implications

Conducting mathematics lessons in Second Life provided pre-service teachers with the opportunity to focus on the teaching of mathematics in a realistic and immersive environment, without having an impact on the learning of real students. It allowed the teacher educator to both role model the teacher's role and act as a facilitator following the sessions to encourage reflection and discussion about the pedagogy involved. Mathematics content knowledge and confidence with teaching mathematics continues to be a concern in teacher education, and Second Life provides an alternative to more traditional approaches, including the use of classroom video footage. The virtual environment can be 'controlled' by the instructor, providing the opportunity for focusing on particular topics, misconceptions, learning difficulties and pedagogical approaches. Feedback from participants indicated that despite some limitations, it had potential for enhancing the practical experience of pre-service teachers.

References

- Australian Association of Mathematics Teachers. (2006). *Standards for excellence in teaching mathematics in Australian schools* [Electronic Version]. Available: <http://www.aamt.edu.au/Activities-and-projects/Standards/Standards-document>.
- Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, 21(2), 132-144.
- Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.
- Chick, H., Pham, T., & Baker, M. (2006). Probing teachers' pedagogical content knowledge: Lessons from the case of the subtraction algorithm. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Educational Research Group of Australasia, pp. 139-146). Adelaide, SA: MERGA.
- Educational Services Australia. (2011). *Accreditation of initial teacher education programs in Australia – Standards and procedures*. Carlton South, Vic: MCEECDYA.
- Eschenbrenner, B., Fui-Hoon Nah, F., & Siau, K. (2008). 3-D virtual worlds in education: Applications, benefits, issues, and opportunities. *Journal of Database Management*, 19(4), 91-110.

- Ferry, B., Kervin, L., Cambourne, B., Turbill, J., Puglisi, S., Jonassen, D., & Hedberg, J. (2004). Online classroom simulation: The next wave for pre-service teacher education? In R. Atkinson, C. McBeath, D. Jonas-Dwyer, & R. Phillips (Eds.), *Beyond the comfort zone*. Proceedings of the 21st ASCILITE Conference (pp. 294-302). Perth, 5-8 December.
- Gao, F., Noh, J.J., & Koehler, M. J. (2009). Comparing role-playing activities in Second Life and face-to-face environments. *Journal of Interactive Learning Research*, 20(4), 423-443.
- Gregory, S., Campbell, M., Knox, V., Dalgarno, B., Reiners, T., & Masters, Y. (2011). Changing directions through VirtualPREX: engaging pre-service teachers in virtual professional experience. In G. Williams, P. Statham, N. Brown, & B. Cleland (Eds.), *Changing demands, changing directions*. Proceedings Ascillite, Hobart, 2011 (pp. 491-501).
- Hill, H., Ball, D.L., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 33(4), 372-400.
- Lange, T., & Meaney, T. (2011). Pre-service teachers learning mathematics from the internet. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices* (Proceedings of the AAMT-MERGA Conference, pp. 438-445). Adelaide, SA: MERGA.
- Lin, P.-J. (2005). Using research-based video-cases to help pre-service teachers conceptualize a contemporary view of mathematics teaching. *International Journal of Science and Mathematics Education*, 3, 351-377.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Maher, N. & Muir, T. (2011). "You just move the decimal point two places": Investigating pre-service teachers' responses to mathematical tasks. Paper presented at the 2011 annual AARE conference, 27 Nov – 1 Dec 2011, Hobart, Tasmania.
- Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Teacher Education and Development*, 3, 28-36.
- Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula, T. J. Buttery & E. Guyton (Eds.), *Handbook of research on teacher education* (2nd ed., pp. 102-119). New York: Simon & Schuster Macmillan.
- Ryan, J. & McCrae, B. (2005). Subject matter knowledge: Errors and misconceptions of beginning pre-service teachers. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 641-648). Melbourne: MERGA.
- Ryan, J. & Williams, J. (2007). *Children's mathematics 4-15: Learning from errors and misconceptions*. New York, NY: Open University Press.
- Santagata, R., Zannoni, C., & Stigler, J. W. (2007). The role of lesson analysis in pre-service teacher education: An empirical investigation of teacher learning from a virtual video-based field experience. *Journal of Mathematics Teacher Education*, 10, 123-140.
- Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57(1), 1-22.
- Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve pre-service mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education*, 11, 107-125.
- Sweeney, B. (2008) Mathematics in a Virtual World: How the immersive environment of Second Life can facilitate the learning of mathematics and other subjects. *Proceedings of Researching Learning in Virtual Environments Conference*, Milton Keynes, 20-21st November 2008.
- Taylor, P. M. (2002). Implementing the standards: Keys to establishing positive professional inertia in preservice mathematics teachers. *School Science and Mathematics*, 102(3), 137-142.
- Valcke, M., Van Steenbrugge, H., & Veeragoudar Harrell, S. (2008). *The impact of Second Life experiences on mathematical beliefs, relationship with mathematics anxiety and perceived learning potential of Second Life for mathematics learning*. Paper presented at the International Conference for the Learning Sciences (ICLS). Utrecht, The Netherlands.

Developing Pedagogical Strategies to Promote Structural Thinking in Early Mathematics

Joanne T. Mulligan

Macquarie University

<joanne.mulligan@mq.edu.au>

Michael C. Mitchelmore

Macquarie University

<mike.mitchelmore@mq.edu.au>

The Pattern and Structure Mathematical Awareness Program (PASMAT) is an early mathematics program designed to promote structural thinking. PASMAT pedagogy removes the structure commonly provided for students in order to challenge them to construct their own, focusing student attention on spatial and numerical patterns and leading them to formulate their own generalisations. In this paper, we give some examples of Kindergarten PASMAT tasks that develop and link the themes of grid patterns, number patterns, multiplication and base ten numeration. Work samples drawn from a recent evaluation study are used to illustrate the range of students' structural development.

In our recent MERGA reports, we described a large-scale evaluation study of the effectiveness of the Pattern and Structure Mathematical Awareness Program (PASMAT) on Kindergarten students' mathematical development (Mulligan, English, Mitchelmore, Welsby, & Crevensten, 2011a, 2011b). This intervention program provided explicit and connected structured tasks over three school terms found to enhance the structural development of patterning and unitising, spatial structuring, base ten and multiplicative reasoning, and mathematical generalisations. This paper illustrates how an emphasis on pattern and structure shaped pedagogical strategies aimed at promoting simple generalisations.

Mason, Stephens and Watson (2009) take *mathematical structure* to mean the identification of general properties that are instantiated in particular situations as relationships between elements or subsets of elements of a set. Their view of structural thinking focuses on an important distinction between simply recognising elements or properties of a relationship, and having a deeper awareness of how those properties are used, explicated and connected. Some pedagogical strategies for promoting such awareness are described in Mason, Drury, and Bills (2007).

Mulligan and Mitchelmore (2009) examine the bases of structural development in early mathematics learning, describing the salient underlying features of pattern and structure common to a range of mathematical concepts. We take *pattern* to mean any predictable regularity in our environment, and *structure* to mean the way the pattern is organised. Most patterns involve number or space, and spatial structuring is critical to establishing relationships between patterns and structures. To quote Warren (2005, p. 305), "Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning".

Much recent research has focussed on children's early learning about mathematical structure. The importance of patterning skills, analogical reasoning and the development of structural thinking has been confirmed in several studies (Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006; English, 2004; Papic, Mulligan, & Mitchelmore, 2011). For example, the Dutch "Curious Minds" project highlights patterning and spatial skills moving beyond early numeracy (van Nes & de Lange, 2007). There is also increasing evidence that early algebraic thinking develops from the ability to see and represent patterns and relationships such as equivalence and functional thinking in early childhood (Warren & Cooper, 2008).

A further line of research has shown that data modelling, a developmental process that begins with young children's inquiries and investigations of meaningful phenomena (Lehrer & Schauble, 2005) also requires children to seek structure and recognise patterns. Preliminary findings of a longitudinal study of data modelling in Grade 1 (English, 2010) indicate that children as young as six years old can successfully collect, represent, interpret, communicate, and argue about the structure of data provided they address familiar themes.

Over the past 10 years, we have accumulated evidence (Mulligan, 2010) that children's responses to a wide variety of mathematical tasks can be reliably classified into the following five levels of structural development:

1. *Prestructural*. Children pick on particular features that appeal to them but are often irrelevant to the underlying mathematical concept.
2. *Emergent*. Children recognise some relevant features, but are unable to organise them appropriately.
3. *Partial structural*. Children recognise most relevant features of the structure, but their representations are inaccurate or incomplete.
4. *Structural*. Children correctly represent the given structure.
5. *Advanced*. Children recognise the generality of the structure.

Our studies led us to conjecture that initial recognition of similarities and differences in mathematical representations plays a critical role in the development of pattern and structure, abstraction and generalisation. The development of multiplicative concepts (including understanding the base ten system, grouping and partitioning) is integral to building structural relationships in early mathematics. Spatial structuring is necessary in visualising and organising these structures (Battista, 1999).

Moreover, we have shown that children tend to respond at the same level on these various tasks (Mulligan & Mitchelmore, 2009). We have taken this finding to be evidence for a general characteristic that we have called Awareness of Mathematical Pattern and Structure (AMPS). We believe that AMPS has two components: not only an understanding of common mathematical structures but also a tendency to look for patterns in new situations. Crucially, children with a high level of AMPS tend to do well at mathematics, while those with a low level of AMPS tend to struggle.

The Pattern and Structure Mathematical Awareness Program (PASMAT)

PASMAT was designed on the assumption that AMPS was not an innate unalterable trait but a characteristic that could be developed through appropriate instruction. Several studies (summarised in Mulligan, 2010) have confirmed this hypothesis from as early as preschool (Papic, Mulligan, & Mitchelmore, 2011).

The program focuses on fundamental processes such as rhythmic and perceptual counting, simple and complex repetitions, growing patterns and functions, unitising and partitioning, grids and arrays, symmetry and transformations, congruence and similarity, and data modelling. For further details, see Mulligan, Mitchelmore, English, and Robertson (2010).

In PASMAT, children are encouraged to seek out and represent pattern and structure across different concepts and to transfer this awareness to other concepts. In other words, the aim is to promote generalisation in early mathematical thinking. This aim is achieved through *pattern-eliciting tasks* that require students to copy or reproduce a model or other representation. In the PASMAT pedagogy, the teacher uses probing questions to highlight important features of their drawings, to compare them with the model or with other children's drawings, and to focus their attention on similarities and differences in crucial aspects of spatial and numerical structure. Tasks are modified and repeated regularly,

reinforcing and extending generalisations and providing links to prior learning. Further details of this pedagogy are to be found in Mulligan (2011).

Some Examples of PASMMap Pattern-Eliciting Tasks

To illustrate PASMMap pedagogy, we have chosen examples to illustrate how awareness of the structure of the rectangular grid is developed in Kindergarten. Some student work samples have been drawn from the evaluation study referred to earlier (Mulligan et al., 2011a, 2011b). In this study, eight classes from four schools in Brisbane and Sydney were taught using PASMMap for their entire Kindergarten year and a further eight classes in the same schools acted as a comparison group. In the PASMMap classes, five high-ability and five low-ability students, as measured by *I Can Do Maths* test (Doig & de Lemos, 2000), were chosen for closer study: students were videotaped and their work samples were collected. We will show some work samples from Heela, a high-ability student, and Lateh, a low-ability student. Both names are synonyms.

Theme 1: Spatial Grid Structure

After a sequence of tasks focused on simple repetition and spatial patterns (Papic et al., 2011), designed to ease children into the program, there is a focus on constructing and analysing simple grids. In the first of these, children are shown a 2×1 grid for a few seconds and then asked to draw it. The teacher then gives them a 2×1 grid and two matching squares and asks how many squares are needed to cover the grid. Different strategies for placing the squares are discussed, and students are also asked to fold the grid to explore the structure. The teacher then asks, “What’s the same?” and “What’s the different?” and students encounter ideas such as counting, shape, sides and vertices, rotation (turning), congruence (same size and shape), and fractions (half). The grid and squares are then removed and children draw the grid from memory in both horizontal and vertical orientations. After sharing and discussing their drawings, the class summarises what they have learnt and looks for links to their earlier tasks (e.g., in the towers they had made from unifix cubes). This may seem a very elementary task, but it is fundamental and many students found it quite challenging (see Figure 1).

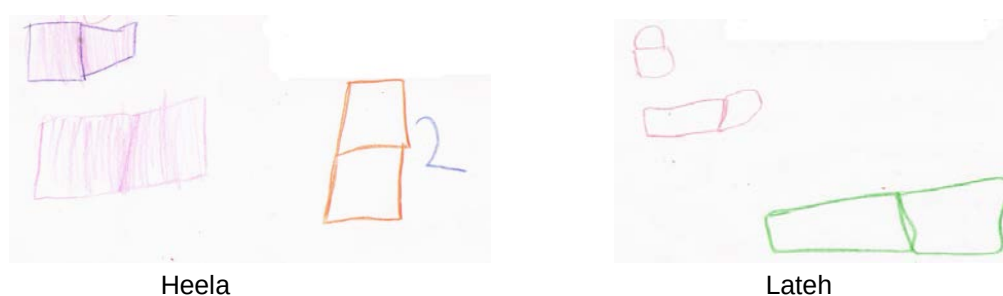
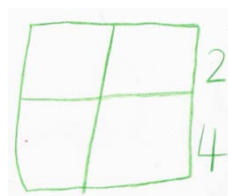


Figure 1. Two contrasting representations of a 2×1 grid.

The next lesson moves on to 2×2 grids (called “windows”), following a similar procedure. Previous ideas are reviewed and extended, and further ideas of rows and columns, clockwise and anticlockwise, vertical and horizontal, diagonals, and even quarters are encountered. The difference between the high- and low-ability children already becomes apparent, and indicates to the teacher how perceptive some children are in terms of recognising structural features while others pay little or no attention to mathematical

features. Figure 1 shows two such contrasting drawings. Heela has already recognised that she does not need to draw separate squares, whereas Lateh is still struggling to draw congruent squares in the standard orientation.



Heela



Lateh

Figure 2. Two contrasting representations of a 2×2 grid.

In subsequent lessons, the task is extended to larger rectangles. By repeatedly looking at what is the same and what is different between a given grid and their drawings, and by seeking generalisations from their observations, children gradually learn that a grid can be drawn using equally spaced, perpendicular lines. Additional tasks include making a sequence of squares of increasing size from 60 square tiles. Each activity reinforces the basic generalisation that we call the ‘spatial structure’ of the grid. This generalisation then finds application in two further themes, discussed next.

Theme 2: Numerical Grid Structure

A related sequence of tasks addresses the numerical structure of a grid. In the first of these, students are given two sets of square grid cards (1×1 , 2×2 , 3×3 , 4×4 and 5×5). After exploring systematic ways in which they can be fitted next to or on top of each other, or in various formations or sequence, the teacher poses the questions, “Can you see a pattern? How many small squares are there on each card? What is the best way to find out?” Students then cut up a second set of grid cards into rows or columns, place the cut-outs on top of the first set of cards, and discuss the numbers of rows or columns and the number of small squares in each. After examining the resulting number pattern (1, 4, 9, 16, 25), the teacher removes all the grid cards and cut-outs and challenges the children to reproduce the visual pattern from memory, first on grid paper and then on plain paper. Discussion of similarities and differences between children’s drawings highlights the crucial fact that a square grid contains the same number of equally sized rows and columns. These ideas are further developed through a sequence of tasks focused on the pattern of squared numbers using grid cards.

In a follow-up task, students are given a 1×1 square and a 2×2 square and asked how many small squares fit on to the larger one. They are then given further 2×2 squares and asked to find the number of small squares in total, thus constructing the sequence 4, 8, 12, ... Finally, they are asked to generalise their findings. Heela had invented a perfectly good means of symbolising her results that closely resembles algebraic notation (Fig 3). In fact, she was treating the task as a functional relationship rather than a simple pattern continuation. Asked what she had learnt from the exercise, she said “I made a pattern so 1 big square is 4 little squares. So it’s 4 for each square. Every time you use the square it’s a four.” Further tasks showed that she had generalised the relationship to all sizes of square and, indeed, any type of rectangle.

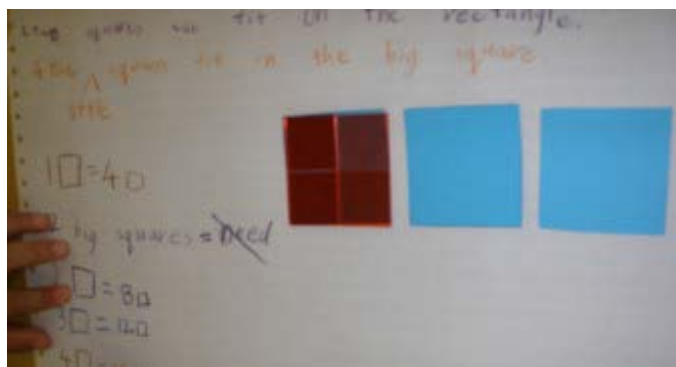


Figure 3. Heela's response to Example Task 1.

Other tasks extend the basic (multiplicative) generalisation to rectangles, often in parallel to Theme 1. For example, in a task set late in Kindergarten, children are asked to relate the number of unit squares needed to cover a rectangle to the size of the unit.

Theme 3: Base Ten Numeration Structure

The generalisations learnt in Themes 1 and 2 can be applied to the structure of the base ten numeration structure, which is essentially multiplicative.

The first PASMAT activities in this theme explore the structure of 10-frames (5×2 grid filled with various numbers of dots) and use them to develop addition and subtraction strategies. The significance of structural understanding was already evident in the way that Heela and Lateh drew an empty 10-frame. Heela had no difficulty sketching 10-frames to represent any 1-digit number or in using them to add or subtract two such numbers. By contrast, Lateh could not draw any 10-frame or use them for addition or subtraction (see Figure 4).

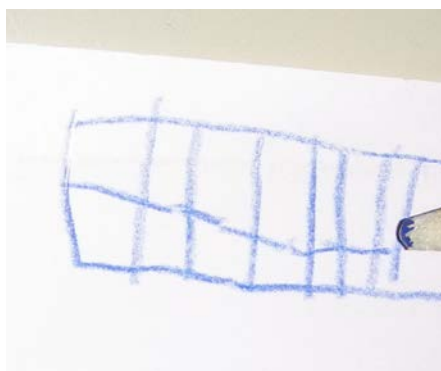


Figure 4. Lateh's drawing of an empty 10-frame.

Another set of tasks focuses on the hundred chart. Building on earlier tasks involving on an empty numeral track, children gradually construct their own chart. They then identify various patterns in the chart, including multiples of 2, 5 and 10; cut the chart into rows and reassemble it; and write the numbers into an empty 10×10 grid.

Following these tasks, children are asked to draw a hundred chart from memory, firstly in an empty 10×10 grid and then freehand. The second task is much more difficult than the first. The teacher frequently prompts them to think about the spatial and numerical structure of the chart, for example “Does it have lots of rows?” and “How many boxes are there?”

During this process, children are encouraged to visualise and develop a plan of how to make their chart and to discuss it with the teacher.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	185	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Heela

1	2	3	4	5
6	7	8	9	10

Lateh

Figure 5. Two drawings of a hundred chart, drawn from memory.

Figure 5 shows how Heela and Lateh completed this task. Heela drew a good approximation to a square, drew the correct number of grid lines inside it, and then filled in multiples of 2 and 5 before completing the chart. Asked to explain her strategy, she said:

We need 100 numbers, so 10 rows and 10 in each row. All numbers on the end in [each] column has to end with the same number. ... It doesn't matter how far I go with the numbers, cause it's the tens column we are adding ... one more ten each time. All the other numbers goes up by 1. I know how to count by all patterns!

Lateh's hundred chart only contained the numbers 1-10, and he claimed it had ten rows and columns. However, his drawing does show considerable structural development: He has learnt to draw a 5×2 grid without drawing separate squares, and he has reproduced the way that numbers are entered both into a 10-frame and a hundred chart. He has applied the structure of the 10-frame to the hundred chart without noticing the differences.

Discussion and Implications

The above examples show the special significance of the rectangular grid structure. Spatially, it is fundamental to early geometry, measurement, and graphical representation. Numerically, it arises whenever a unit is repeated—whether this unit be an object or a set of objects, a single shape or a composite shape, or a measurement unit. And algebraically, it lends itself to an early introduction to functional relationships and symbolisation. An early understanding of this structure is therefore vitally important to children's mathematical development.

Children's lack of understanding of structure can remain hidden when they only face tasks where the structure is provided for them. In all the examples we have shown, drawing from memory has been more revealing. Such tasks also challenge children to rehearse the patterns they have seen and apparently understood at a superficial level, and to use these patterns to gain a deeper understanding of the structure. In this sense, visual memory tasks are especially valuable, both for teachers and students.

The examples also show us once again that young children are capable of much more advanced mathematical thinking than was previously thought. The underlying principle, we

would claim, is the way that PASMMap draws upon children's natural tendencies to look for patterns and then to explore how they are related. By encouraging students to continually seek patterns, to look for similarities and differences, and to form generalisations, they can learn about mathematics as relationships and can abstract and generalise, albeit at a simple level, from an early age.

We believe that the same structural approach could be applied throughout the teaching of mathematics and related areas of learning. We are currently extending our pedagogical approach in a 3-year longitudinal study of mathematics and science learning through novel experiences in data modelling and problem solving.¹ The study tracks three cohorts of students employed in the initial study (Mulligan et al., 2011a, 2011b) through to Grades 2, 3 and 4. In addition, two new cohorts of mathematically able students are being tracked from Kindergarten to Grade 2.

The implications of learning through a structural approach may require fundamental changes to the way that curriculum is conceptualised, structured, interpreted and implemented. The PASMMap approach promotes conceptual and connected knowledge and the development of teaching practices that focus on relational thinking and this may encourage the development of new pedagogical content knowledge (PCK). The Proficiency strands of the new Australian Curriculum–Mathematics (understanding, fluency, problem solving and reasoning) do support the development of mathematics as patterns, relationships and generalisations rather than disconnected concepts and skills (ACARA, 2012). PASMMap could play a major role in supporting the teaching of these proficiencies.

Acknowledgements

The research reported in this paper was supported by Australian Research Council Discovery Projects Grant No. DP0880394, *Reconceptualising early mathematics learning: The fundamental role of pattern and structure*. The authors express thanks to Prof. Lyn English and Dr Coral Kemp; research assistants Susan Daley, Deborah Adams, Nathan Crevensten and Sara Welsby; and teachers, teachers' aides, students, and school communities.

References

- Australian Curriculum, Assessment and Reporting Authority [ACARA] (2012). *The Australian Curriculum: Mathematics* (Version 3.0). Retrieved 30 March 2012 from <http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10>
- Blanton, M., & Kaput, J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, 36, 412-446.
- Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, 37, 87-115.
- English, L. D. (2004). Promoting the development of young children's mathematical and analogical reasoning. In L. D. English (Ed.), *Mathematical and analogical reasoning of young learners*. Mahwah, NJ: Lawrence Erlbaum.
- English, L. D. (2010). Young children's early modelling with data. *Mathematics Education Research Journal*, 22(2), 24-47.
- Lehrer, R. & Schauble, D. (2005). Developing modeling and argument in elementary grades. In T. Romberg & T. Carpenter & F. Dremock (Eds.) *Understanding mathematics and science matters* (pp. 29-53). NJ: Erlbaum.

¹ Australian Research Council Discovery Project No. DP110103586 *Transforming children's mathematical and scientific reasoning*.

- Mason, J., Drury, H., & Bills, L. (2007). Studies in the zone of proximal awareness. In J. Watson & K. Beswick (Eds.), *Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia. Mathematics, Hobart, Vol. 1, pp. 42–58). Hobart: MERGA.
- Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal*, 21(2), 10–32.
- Mulligan, J. T. (2010). The role of representations in young children's structural development of mathematics. *Mediterranean Journal of Mathematics Education*, 9(1), 163-188.
- Mulligan, J. T. (2011). Towards understanding of the origins of children's difficulties in mathematics learning. *Australian Journal of Learning Difficulties (Special Issue)*. 16(1), 19-39.
- Mulligan, J. T. & Mitchelmore, M. C. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal*, 21(2), 33-49.
- Mulligan, J. T., English, L. D., Mitchelmore, M. C., Welsby, S. & Crevensten, N. (2011a). An evaluation of the pattern and structure mathematics awareness program in the early years of schooling. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics: Traditions and [New] Practices* (Proceedings of the joint conference of the Australian Association of Mathematics Teachers and the Mathematics Education Research Group of Australasia, pp.548-556). Alice Springs, NT: AAMT-MERGA.
- Mulligan, J. T., English, L. D., Mitchelmore, M. C., Welsby, S. & Crevensten, N. (2011a). Developing the pattern and structure assessment (PASA) interview to inform early mathematics learning. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics: Traditions and [New] Practices* (Proceedings of the joint conference of the Australian Association of Mathematics Teachers and the Mathematics Education Research Group of Australasia, pp. 1022-1030). Alice Springs, NT: AAMT-MERGA.
- Mulligan, J. T., Mitchelmore, M. C., English, L. D., & Robertson, G. (2010). Implementing a Pattern and Structure Mathematics Awareness Program (PASMAT) in Kindergarten. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education* (Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia, Fremantle, pp. 796-803).
- Mulligan, J. T., Mitchelmore, M. C., & Prescott, A. (2004). Children's development of structure in early mathematics. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 393-400). Bergen, Norway: Program Committee.
- Papic, M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education*, 42(3), 237–268.
- van Nes, F., & de Lange, J. (2007). Mathematics education and neurosciences: Relating spatial structures to the development of spatial sense and number sense. *The Montana Mathematics Enthusiast*, 2(4), 210–229.
- Warren, E. (2005). Young children's ability to generalize the pattern rule for growing patterns. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4, pp. 305-312). Melbourne: PME.
- Warren, E., & Cooper, T. J. (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds thinking. *Education*.

The Influence of Gender, Parents, and Background Variables on Perceived Usefulness of Mathematics among Grade 7 Students in Mozambique

Adelino Murimo

Monash University

<aemur1@student.monash.edu.au>

Data from a larger study were used to identify variables that best predict children's perceived usefulness of mathematics [PUM]. Gender differences in PUM scores were also explored using a sample of 300 grade 7 children and 225 parents from February to May 2011 in Mozambique. Surveys and interviews were used to collect data. Consistent with traditional beliefs it appeared that mathematics is viewed as more useful for boys than for girls. Education of parents, school geolocation, and number of siblings were statistically significant predictors of children's perceived usefulness of mathematics.

In developed countries, research on the factors contributing to gender disparities in mathematics learning outcomes (e.g., achievement, participation rates in high level elective mathematics courses and applied fields, affect, beliefs, and attitudes) has evolved, and measures to ameliorate the problem are well documented (Leder, Forgasz, & Solar, 1996).

In Mozambique, as in many other countries from Sub-Saharan Africa, research on gender and mathematics education has received scant attention (Asimeng-Boahene, 2006). Studies relating gender and mathematics learning conducted in Mozambique focused on secondary school students' performance (Cassy, 2002), and teachers' classroom discourse (Fagilde, 2001). The few studies focusing on primary education examined the portrayal of females and males in mathematics textbooks (Murimo & Forgasz, 2007), and performance in mathematics among grade 6 children (Saito, 2010). However, the studies by the Southern and Eastern Africa Consortium for Monitoring Education Quality [SACMEQ] have reported clear gender differences in mathematics achievement in favour of boys among grade 6 children in Mozambique, and recommendations to conduct research to understand the causes have been put forward (Saito, 2010). Hence, this paper is part of a larger and ongoing study designed to examine the impact of gender, parents, and background factors on primary school children's beliefs and attitudes towards the study of mathematics in Mozambique. In this paper three research questions will be answered:

1. What contribution do gender, education and occupation of parent, geolocation, home language, siblings, and books make to the prediction of children's perceived usefulness of mathematics [PUM]?
2. Which owned economic resource is the most salient predictor of PUM?
3. Are there statistically significant differences in perceived usefulness of mathematics between: (a) girls and boys, (b) mothers and fathers, and (c) parents of daughters and parents of sons?

The answers to these questions are important as they might contribute to the understanding of some of the factors influencing children's motivation and engagement in the study of mathematics in primary schools in Mozambique, and other places with similar contexts. To acknowledge previous studies that served as the basis to answer the research questions of this study, the following literature is reviewed.

Literature Review

Fennema and Sherman (1976) described the notion *perceived usefulness of mathematics* as the “students’ beliefs about the usefulness of mathematics currently and in relationship to their future education, vocation, or other activities” (p. 5). Aiken (1974) acknowledged that not only achievement, but also enjoyment and perceived usefulness of mathematics for the person and for society in general are some of the objectives of mathematics education.

Recently, Luttrell, Callen, Allen, Wood, Deeds, and Richard (2010) identified four dimensions comprising the perceived usefulness of mathematics, namely: *Interest*, *General utility*, *Need for high achievement*, and *Personal cost*. According to Luttrell and colleagues (2010), interest in mathematics learning is related to the importance a person gives to mathematics stemming from a real interest in the subject. General utility is the importance the individual places on understanding mathematics because the person wants to accomplish a variety of goals presently and in the future. Need for high achievement refers to the aspiration a student has on performing very well in mathematics. Personal cost is associated with the subjective estimation of loss suffered by the student as a consequence of learning mathematics at a high level. The perceived loss would include: effort; time spent studying mathematics; entertainments missed out; and negative reactions from peers.

Gender differences in favour of boys have been reported in regard to perceived usefulness of mathematics (e.g., Eccles, 1987; Fennema & Sherman, 1976; Luttrell et al., 2010). Research on perceived usefulness of mathematics is important to education because it has been associated with the type of courses the students elect when mathematics is not compulsory, engagement and achievement in high level mathematics, and career intentions (see Simpkins, Davis-Kean, & Eccles, 2006).

However, on their more recent studies on gendered views about mathematics, Leder and Forgasz (2010, 2011) found that the majority of respondents believed mathematics was important to get a job, and that mathematics was equally useful for females and males, challenging a traditional view that mathematics is more important for males than females.

Method

Study Site

The study was conducted in three districts of Sofala Province in the central region of Mozambique. The districts were selected by the researcher on the basis of geographic location: *Beira* (urban), *Dondo* (rural), and *Buzi* (remote). The districts also represented the three main languages spoken in Sofala Province, namely: *Portuguese*, *Sena*, and *Ndau*. Portuguese is a heritage from the Portuguese colonial master and has been adopted as the medium of instruction in Mozambique. However, the schools were not selected by the researcher as they were determined by the Education, Youth and Technology Services of the respective districts after permission to conduct research in Sofala Province was obtained; three primary schools were selected in *Beira*, one in *Dondo* and one in *Buzi*, and all were government institutions.

Recruitment and Characteristics of the Participants

A convenience sample constituted by 300 grade 7 children (134 boys and 166 girls), 225 parents and other guardians representing the children (118 males and 107 females) participated in the study. The ages of the children ranged from 10 to 16 years, the mean age was 12.9, and the standard deviation was 1.4. The ages of parents ranged from 18 to 67

years. The guardians other than parents were brothers, sisters, and husbands of some girls; despite the oldest girl being 16 years old, some were married.

In order to approach schools and invite children, parents, and school principals to participate in the study, authorizations from relevant institutions were obtained. To meet ethical principles, consent forms were signed by all participants interested in taking part in the study.

Instruments: Surveys and Interviews

Pen-and-paper surveys and interviews were used to collect quantitative and qualitative data respectively. As there is a lack of studies on the same topic in Mozambique, quantitative data were considered important to reveal trends and relationships among variables (Creswell, 2003). In addition, to delve deeper understanding of the data, quantitative data were complemented with more contextualized qualitative data. Qualitative data were collected through one-on-one face-to-face interviews with parents.

The children and parents completed similar surveys. However, items for parents were written to reflect their perspective. Twelve items from the Mathematics Valuing Inventory [MVI] (Luttrell et al., 2010) were selected and translated into Portuguese with the authors' permission. To maintain scale reliability, items were back translated into English with the assistance of a person fluent in both languages. An example of an item from the MVI for children was: 'There are almost no benefits for me to learn mathematics'. The equivalent item for parents was: 'There are almost no benefits for my son/daughter to learn mathematics'. Responses were given on 5-point Likert-type formats ranging from strongly disagree (1) to strongly agree (5); In order to ensure that high scores indicated high levels of perceived usefulness of mathematics [PUM] all negatively worded items were reverse scored.

Apart from items selected from MVI, the children were asked about their gender, education of guardian, occupation of guardian, home language, geolocation, number of siblings, number of books existing in the child's home, and possession of selected economic resources in their homes. In all analyses performed these variables were used as independent variables (IVs), while PUM was used as a dependent variable (DV).

The surveys were completed in the school in the presence of the researcher to avoid collusion. The children completed the surveys first; one month later parents came to school to complete the surveys. In the third month, 10 parents were invited to participate in a one-on-one semi-structured interview. Semi-structured interviews were used to avoid deviating from the topics of interest and to allow additional questions to be asked on the spot. The selection of parents to interview was based on responses to the surveys and willingness to be interviewed. The interview protocol included the following questions: 'Does your child like mathematics?'; 'How good at mathematics is your child?'; 'Do you think mathematics is important for your child?' 'Why?'; 'Do you think mathematics is important to get a job?'; 'Why do you say so?' The inclusion of these questions in this study was inspired by previous research (e.g., Leder & Forgasz, 2010, 2011).

Data Analysis

The quantitative data were analysed using the Statistical Package for the Social Sciences (SPSS) for Windows (version 18), following Pallant's (2009) guidelines. All categorical variables were coded as 'dummy' variables. The data from quantitative variables measured on a continuous scale were inspected for out-of-range values, plausibility of means and standard deviations, and non-violation of the assumptions of parametric statistical

techniques used. Parametric tests were used because they tend to be more robust than the equivalent non-parametric ones when the samples are over 200 (Tabachnick & Fidell, 2007). Multiple regression analyses, ANOVAs, independent-groups t-tests, and paired t-tests were used to answer the research questions when appropriate.

Results and Discussion

To determine whether a model composed of gender, education of parent, occupation of parent, geolocation, number of siblings, home language, and number of books predicted children's PUM scores, a standard multiple regression analysis was applied. PUM scores were entered as a DV, and the categorical variables as IVs (see Table 1)

Table 1

Summary of Standard Multiple Regression Statistics for the Predictor Variables to the Dependent Variable (Perceived usefulness of mathematics) (225 < N < 300).

IVs	B	SE	β	t	p
Gender	0.07	0.07	0.07	1.01	0.307 (NS)
Education of parent	0.05	0.02	0.20	2.97	0.003
Occupation of parent	0.01	0.03	0.01	0.18	0.861 (NS)
Geolocation	0.19	0.05	0.24	3.56	0.000
Siblings	0.13	0.04	0.19	2.95	0.004
Home language	0.06	0.06	0.06	1.09	0.282 (NS)
Books	0.04	0.03	0.03	1.15	0.253 (NS)

Notes. B: Unstandardized coefficients, SE: Standard error, β : beta coefficient, t: t-test statistics, p: significance value, NS. Non-statistically significant at $p < 0.01$ level (Bonferroni adjustment).

It was found that the model as a whole explained 18% of the variance of PUM scores ($R^2=0.18$) and the result was statistically significant ($F(7, 215)=6.8, p<0.001$). Examination of β , t and p values in Table 1, reveals that geolocation ($\beta = 0.24, t = 3.6, p < 0.01$), parent education ($\beta = 0.20, t = 2.97, p < 0.01$), and the number of siblings that a child has ($\beta = 0.19, t = 2.95, p < 0.01$) are statistically significant predictors of PUM scores. Gender of the child, occupation of parent, home language, and number of books were not statistically significant predictors of PUM scores.

After observing that education of parent, geolocation, and siblings were statistically significant predictors of PUM, to identify groups that differed from each other within IVs, one-way ANOVAs were conducted followed by Tukey or Games-Howell post hoc comparisons, as appropriate (Pallant, 2009). It was noted that children whose parents had university education scored a higher PUM mean ($M = 4.04; SD = 0.56$) than the children whose parents had less than grade 6 education ($M = 3.46; SD = 0.41; p < 0.01$) and children whose parents did not state their education ($M = 3.54; SD = 0.47; p < 0.001$). The children from urban schools got a higher mean ($M = 3.79; SD = 0.58$) than the children from rural ($M = 3.47; SD = 0.42; p < 0.001$) and remote region ($M = 3.39; SD = 0.43; p < 0.001$) areas. No statistically significant differences were found in the mean scores of children from rural and remote areas. The children having fewer than three siblings got a higher mean ($M = 3.89; SD = 0.56$) than the children having more than two siblings ($M \leq 3.63; p < 0.01$).

A standard multiple regression analysis was conducted to see whether a model composed of electricity, television, computer, internet, piped water, mathematics textbook,

calculator, and mobile phone predicted PUM scores. The model as a whole explained 11% of PUM variation ($R^2 = 0.11$) and the result was statistically significant, $F(8,289) = 4.5$, $p < 0.001$). However, only electricity ($\beta = 0.28$, $t = 2.1$, $p < 0.001$) was a statistically significant predictor of PUM. When combined with electricity, television was not a significant predictor of PUM due to the high correlation between these two items ($r = 0.91$). When electricity was excluded from the equation, the model explained 10% of PUM variation ($R^2 = 0.098$), and television ($\beta = 0.19$, $t = 3.0$, $p < 0.01$) was the only significant predictor of PUM in the model.

A paired t-test was conducted to determine whether there were statistically significant differences in children's PUM mean scores and parents' PUM mean scores about their children. It was found that the mean for parents ($M = 3.49$, $SD = 0.30$) was statistically significantly lower than that of the children ($M = 3.68$, $SD = 0.56$); $t(216) = 4.5$, $p < 0.001$). This result may reflect the differential experiences that parents had with mathematics in the past and the current experiences that their offspring have.

Independent-groups t-tests were conducted to explore for significant differences in PUM mean scores for boys and girls, mothers and fathers, and for parents of sons and parents of daughters. See Table 2 for means, SDs, and t-test results (t and p values).

Table 2
PUM Mean Scores, Standard Deviations (SD) and t-test Results (225 < N < 300).

Groupings	PUM			t-test results	
	N	Mean	SD	t	p
Girls	164	3.61	0.57	-	NS
Boys	134	3.71	0.53		
Mothers	104	3.46	0.29	-	NS
Fathers	115	3.52	0.30		
Parents of daughters	123	3.48	0.29	-	NS
Parents of sons	96	3.51	0.31		

Note. NS: Mean differences did not reach statistical significance at $p < 0.05$ level.

Although the differences in the means did not reach statistical significance, it can be seen in Table 2 that boys had higher PUM mean scores than girls; fathers scored higher than mothers; and, parents of sons also scored higher than parents of daughters. This trend in the pattern of results suggests that mathematics may be viewed as more useful for boys than for girls.

As indicated earlier, parents were interviewed to explore further their views in regard to the usefulness of mathematics for their sons and daughters. Seven fathers and three mothers accepted invitations to be interviewed. Among other questions, parents were asked whether mathematics is important for their children and for getting a job. Two interview extracts are presented next.

Interview 1 (R: Researcher, FG: Father of a girl)

R: Do you think mathematics is important?

FG: Yes, it is important.

R: Why?

FG: It opens capacity and improves reasoning.

R: Do you think mathematics is important for your daughter?
 FG: Yes, it is important because it opens mind.
 R: Do you think mathematics is important to get a job?
 FG: No, it is not important to get a job.
 R: Why do you think so?
 FG: There are many people out there with a university degree without a job. Well, if you have a job already, then mathematics can help you.
 R: Which profession would you like for your daughter?
 FG: A doctor.
 R: Why?
 FG: Doctors are important. They have employment guaranteed anywhere.

Interview 2 (R: Researcher, FB: Father of a boy)

R: Is your son good at mathematics?
 FB: No, he is not.
 R: Why?
 FB: He hates mathematics. He likes electricity, computers, and football.
 R: Do you think mathematics is important to learn about computers?
 FB: No.
 R: Do you know why your son hates mathematics?
 FB: He is not good at calculations. But I have a daughter who is good at mathematics.
 R: What is her grade?
 FB: She is in Grade 12 this year.
 R: Is she very good at mathematics?
 FB: She is very good at numbers. She explains mathematics to her classmates.
 R: Which profession would you like for her in the future?
 FB: A doctor.
 R: Why a doctor?
 FB: Because a doctor can open her own clinic or to work for several clinics.

The two extracts above are representative of the various views of the parents interviewed. FG believed mathematics is important for his daughter because it improves reasoning, but he did not believe it contributes to having a job. Similarly, FB did not believe mathematics is important to learn about computers. One possible explanation for this is that mathematics does not stand out in most occupations, and members of the general public may not see its role in the development of science. Both parents, as many others interviewed and surveyed, aspired for their daughters to be medical doctors. Explanations for this preference are likely to be related to the context of the study because of the prevalence of disease in Mozambique and the inability of health services to tackle the problem. It is interesting to note that FB believed his daughter was better at mathematics than his son. This observation demonstrates that some parents may hold beliefs that differ from the traditional views of mathematics as a male domain. Leder and Forgasz (2010, 2011) have noted similar changes in public's gender stereotyping of mathematics.

The main reasons parents gave to aspire to an occupation for their children were: the perceived financial costs for training, and the perceived chance to get employment. Fewer parents mentioned the child's aptitude for the occupation (i.e., perceived talent and child interest). Overall, the data revealed that parents and the children themselves expected girls to be teachers, nurses, and physicians, while boys were expected to be police officers and

engineers, supporting the traditional stereotypes that some occupations are viewed as more suited to one gender than the other (e.g., Eccles, 1987). These results seem to support the *Expectancy-Value Model* (Eccles, 1987), in that professions may be selected because of the subjective value of the occupations in terms of the benefit of getting employment. There was no clarity whether expectations of success motivated choices, but some parents believed their children had the aptitude for the occupation.

Conclusions and Recommendations

The aim of this study was to examine the contribution of gender, education and occupation of parent, geolocation, language, siblings, books, and selected economic resources to the prediction of children's PUM. Gender differences in PUM scores were also explored. The main findings of the study are summarized as follows:

1. The children from urban schools, with more educated parents, and with fewer siblings were more likely to report higher PUM scores than the other children.
2. Having electricity, and a TV set at home predicted children's PUM.
3. Although not reaching statistical significance, children and parents were more likely to view mathematics as more useful for boys than for girls, indicating that the traditional view that mathematics is more important for males than for females may still prevail in Mozambique.
4. Interviews conducted with ten parents revealed that mathematics is viewed as important for reasoning and calculations. Most parents did not see how important mathematics is for getting a rewarding job, and how mathematics influences other fields including technology.
5. Most parents interviewed or surveyed expected their daughters to be teachers, nurses, and medical practitioners, and their sons to be police officers and engineers.

In many countries mathematics is considered a 'critical filter' for further education and better paid occupations (Leder, Pehkonen, & Töner, 2002), and to be able to use modern technologies (Schoenfeld, 2002). Based on the findings from this study, it is recommended that to improve the level of PUM by children and parents in Mozambique, as well as in other countries with similar contexts, schools should develop activities to demonstrate the power of mathematics, since most people equate mathematics with calculations. In Mozambique it seems that the gender stereotyping of occupations goes alongside the misunderstanding of what mathematics is all about and how it influences other fields.

Acknowledgements

I would like to express my gratitude to my supervisor Associate Professor Helen Forgasz from Monash University for reading an earlier draft of this paper.

References

- Aiken, L. R. (1974). Two scales of attitude toward mathematics. *Journal for Research in Mathematics Education*, 5(2), 67-71.
- Asimeng-Boahene, L. (2006). Gender inequity in science and mathematics education in Africa: The causes, consequences, and solutions. *Education*, 126(4), 711-728.
- Cassy, B. (2002). *Effect of classroom interaction and gender on mathematics performance and attitudes towards mathematics of secondary pupils in Mozambique*. Unpublished PhD dissertation, University of Witwatersrand, Johannesburg, South Africa.
- Creswell, J. W. (2003). *Research design: Qualitative, quantitative and mixed methods approaches* (2nd ed.) SAGE, Thousand Oaks, London.

- Eccles, J. S. (1987). Gender roles and women's achievement-related decisions. *Psychology of Women Quarterly*, 11 (2), 135-172.
- Fagilde, S. A. M. (2001). *Towards a characterization of communication and gender patterns in secondary mathematics classrooms in Mozambique*. Unpublished PhD dissertation, University of Western Cape, Bellville, South Africa.
- Fennema, E., & Sherman, J. (1976). Fennema-Sherman mathematics attitude scales: Instruments designed to measure attitudes toward the learning of mathematics by females and males. *Journal supplement abstract service*. American Psychological Association (Ms. 1225), Washington, D.C.
- Leder, G. C., & Forgasz, H. J. (2010). I liked it till Pythagoras: The public's views of mathematics. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education*. (Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia, Fremantle, pp. 328-335): Sydney: MERGA.
- Leder, G. C., & Forgasz, H. J. (2011). The public's views on gender and the learning of mathematics: Does age matter? In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices* (pp. 446-454). Adelaide: AAMT and MERGA.
- Leder, G. C., Forgasz, H. J., & Solar, C. (1996). Research and intervention programs in mathematics education: A gendered issue. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 945-985). Dordrecht, The Netherlands: Kluwer.
- Leder, G. C., Pehkonen, E., & Töner, G. (2002). Setting the scene. In G. C. Leder, C. E. Pehkonen & G. Töner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 1-10). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Luttrell, V. R., Callen, B. W., Allen, C. S., Wood, M. D., Deeds, D. G., & Richard, D. C. S. (2010). The mathematics value inventory for general educational students: Development and initial validation. *Educational and Psychological Measurement*, 70(2), 142-160.
- Murimo, A. E., & Forgasz, H. J. (2007). Depictions of females and males in Mozambican and Victorian (Australia) mathematics textbooks. *Pythagoras*, 66, 85-96.
- Pallant, J. (2009). *SPSS survival manual. A step by step guide to data analysis using SPSS* (3rd ed.). NSW: Allen & Unwin.
- Saito, M. (2010). The gender gap in learning: Back to square one. A need to change the focus from access and participation to learning achievement. *IIEP Newsletter*, 28(3), 6-6. Retrieved October 25, 2010, from http://www.iiep.unesco.org/fileadmin/user_upload/Info_Services_Newsletter/pdf/eng/2010/2010_3En.pdf
- Schoenfeld, A. H. (2002). Making mathematics work for all children: Issues of standards, testing and equity. *Educational Researcher*, 31(1), 13-25.
- Simpkins, S. D., Davis-Kean, P. E., & Eccles, J. S. (2006). Math and science motivation: A longitudinal examination of the links between choices and beliefs. *Developmental Psychology*, 42(1), 70-83.
- Tabachnick, B. G., & Fidell, L. S. (2007). *Using multivariate statistics* (5th ed.). Boston, MA: Pearson Education.

Problem Categorisation in Ratio – A Closer Look

Norhuda Musa

Tampines Primary School
<norhuda_musa@moe.edu.sg>

John Malone

Curtin University
<j.malone@curtin.edu.au>

This paper reports further findings from a study on categorisation of typical problems in the learning of ratio originally presented at the 2011 conference. The paper focuses specifically on the seven categories identified; examines strategies used by two classes of Primary 6 students to solve problems, and presents data in the form of student journal writing; interviews; voice recordings and a pupil perception survey. Pre- and post-tests were administered. Outcomes reveal the positive effects of learning ratio through categorisation.

This on-going study was first described at the 2011 *Mathematics Education Research Group of Australasia* (MERGA) conference. That paper examined the potential of categorisation of ratio word problems in order to enhance students' understanding of the mathematics of ratio (Musa & Malone, 2011). Principles such as *Cognitively Guided Instruction* (CGI) and *Case Base Reasoning* (CBR) were tied together to bring about optimal learning. CBR relies heavily on prior knowledge and appears to work well with CGI problem categorisation. In this process, problems are placed in various categories based on the distinctive feature each structure offers. Each category influences the strategies that pupils use to solve problems; hence, these categories are labelled not only based on their structure of questions but also on the concepts used to solve the problems. This current paper examines the seven categories in greater depth and describes the outcomes of the study to date.

Literature Review

Model Method, an innovation in the teaching and learning of mathematics in Singapore, was first introduced at the primary four level in 1983 to address the issue of pupils having great difficulty with word problems (Kho, Yeo & Lim, 2009), and it has become one of the key features of the country's mathematics curriculum. The theoretical basis for the model method comes from the second form of representation of Jerome Bruner's theory (1961), namely the Iconic mode. This mode is pictorial in nature and is in line with the Concrete-Pictorial-Abstract instructional approach. The model method involves drawing rectangular bars drawn in proportion, with known and unknown information or "generators" (Ng, 2004) fitted in. Different types of dotted lines are used to facilitate changes. Ng (2004) defines two stages to learning model construction: developing part-whole concept at primary one and two levels, and then moving on to constructing models based on proportional reasoning at primary three level. In both cases, identifying the generator, abstracting and articulating the technical relationships between parts and the generator are crucial to problem solving via the model method. Yeap (2010) attributed success among average pupils in TIMSS and Grade 6 national test to the use of the model method, which has "no doubt been helping to make challenging mathematics accessible to average students" (p. 89).

This contention is certainly true – to a certain extent. This is because when problems become more complex, construction of models becomes a real challenge for the low and middle-scorer students. Pupils face difficulty in the basic application of model method – deciding on the correct type of models – in particular those involving part-whole, comparative, before and after and working backwards procedures. Furthermore, when not drawn in proportion, the model method fails to reveal the relationships between parts. Goh (2009) reported that despite its effectiveness in solving many challenging arithmetic word problems and its emphasis in classroom teaching and learning, many pupils still experience varying degree of difficulty in using the model method. The difficulties upper primary pupils faced are more apparent as increasing number of complex multiple step arithmetic word problems are attempted, which often require more complex model drawing.

Lo, Watanabe and Cai (2004) stated that Asian textbooks clearly distinguish ratio as a multiplicative relationship between two quantities, complete with examples and exercises to enhance the learning of ratio. However, in Singapore, textbooks do not challenge pupils with problem variety (Ho, Teong & Hedberg, 2004). Hence, relying on textbooks alone is insufficient as a “culture of challenge has developed in Singapore mathematics classroom” (Yeap, 2008, p.1). With a significant proportion of Primary School Leaving Examination (PSLE) items being challenging, it is understood that every pupil is expected to do challenging problems. Prime Minister Goh, in 2004, made the call for teachers to teach less and use the available time to “excite pupils in the learning process” and allow them to learn more. As a result, alternative strategies are encouraged for pupils who are not performing well. Categorisation is a new alternative that offers conceptual understanding of the fundamentals of ratio mathematics.

Middleton and Van den Heuvel-Panhuizen (1995) noted that many pupils have a “high degree of proficiency at arithmetic computation but sometimes with little conceptual understanding”. Yeap (2008, p. 7) concurs with this, stating that the “ability to select the correct operations and perform computations do not always lead to a correct solution”. Categorisation will address the issue of little conceptual understanding as pupils will be dealing with concepts before deciding on the categories. Interestingly, Chick (2010, p. 2) stated that in studies conducted by Mitchelmore, White and McMaster (2007) and Steinhorsdottir and Sriraman (2009) the researchers showed how students’ knowledge can develop with appropriate CGI and identified levels of understanding, including one associated with the difficulties of dealing with non-integer multipliers. It is anticipated that with categorisation, the issues raised can be addressed so that more meaningful learning of ratio can occur.

Method

This study is being carried out in two classes, each of thirty-two Primary 6 pupils in a primary school in Singapore. They form a non-random purposive sampling group. Pupils in the study experience no problem with multiplying and finding equivalent ratios: They were first exposed to ratio in Primary 5; they possess basic knowledge of procedures and are expected to use considerable reasoning in solving problems. It is through this reasoning that pupils are expected to discover the concepts tied to the categories.

Two teachers also participated in this mixed methods approach where quantitative and qualitative data were simultaneously collected and merged. Creswell (2008) believes that the strengths of one data form offset the weaknesses of the other form. A pre-test consisting of seven questions, each from the seven categories, was administered prior to the

commencement of the study and data in the form of MP3 recording, interviews (between teacher and pupils) and journal writing have been carried out at regular intervals throughout the study's duration.

The teachers led the pupils into the categories using the CGI and CBR frameworks and facilitated pupils' learning by scaffolding the learning of the concepts in each category. At the end of the intervention of the seven categories, pupils' are expected to be able to identify the categories on their own. Details on the seven categories follow:

Category 1: Values Assigned

Example: Sam poured lemon juice for her three friends in the ratio 3 : 4 : 8. The smallest share was 250 ml less than the biggest share. How many millilitres of lemon juice did she pour altogether?

Concept: Ratio parts are assigned to values.

Category 2: One quantity remains constant

Example :Three classes were asked to make cards. Class A made $\frac{4}{5}$ of the number of cards made by

Class B. Class C made twice the number of cards made by Class A. Class C made 72 cards more than Class B. How many cards did the 3 classes make altogether?

Concept: One quantity in the ratio remains constant. In this category, three values are compared; A to B and C to A. Pupils are guided to "see" that the value of A should be the same when compared to different quantities like B and C. The idea of reduced ratio is discussed here, as the difference of A is 4 units in the first ratio and 1 unit in the second ratio, comes about because of reduced ratios. 'Generalising' (Ng, 2004, pg 43) will take place at the end of the lesson to ensure pupils are able to identify the pattern (concept) that is consistent of the idea.

Category 3: Constant Difference

Example: A baker makes a total of 120 pies and muffins in the ratio 3 : 5. After he sold an equal number of pies and muffins, the number of pies and muffins left was in the ratio 3 : 8. How many pies and muffins did he sell altogether?

Concept: Difference between parts must remain constant when equal amounts are removed, added.

As pupils become aware that one quantity remained constant in category 2, they are guided through the CBR framework where the old case (category 2) is used to solve the new case (category 3). The number of pies and muffins are changing, but the difference between them remains constant (see Figure 1). The teacher uses Socratic prompts in order to probe thinking at a deep level to bring about that idea. Examples using 'specific numerical ratio' (Chick, 2010) will be added in to promote understanding.

~~P : M~~	~~P : M~~
~~3 : 5~~	~~3 : 8~~
Diff : 2 units	Diff : 5 units
15 : 25	6 : 16
Diff : 10 units	Diff : 10 units

Figure 1.

Following the introduction of Categories 1,2 and 3, pupils were familiar with the CBR approach requiring them to solve particular question types using prior knowledge.

Category 4: Total remains constant

Example: Three boys, Aaron, Ben and Charlie shared the cost of a birthday present for their father. The ratio of Aaron's share to the total of Ben's and Charlie's share was 1:3. The ratio of Ben's share to the total of Aaron's and Charlie's shares was 1 : 5. Charlie paid \$50 more than Ben. Find the total cost of the present.

Concept: Total, when compared to different quantities in a question, remain constant. Here, if the two ratios are compared, pupils will see that sum of each ratio is actually the cost of the present. It is important to note that 1 unit in the first ratio is not equivalent to 1 unit in the second ratio. Pupils will have to find a way to make them equal (see Figure 2). Pupils will then use the parts to solve the problem.

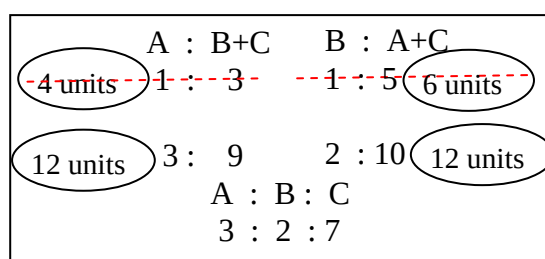


Figure 2.

Category 5: Fractional parts of ratio

Example: Danny, Kendrick and Teddy shared some sweets in the ratio 3 : 6 : 4. Danny kept $\frac{1}{5}$ of his sweets and gave the rest of his sweets to Kendrick and Teddy in the ratio 1 : 3. As a result, Kendrick had 700 sweets more than Teddy. How many sweets did Danny keep for himself?

Concept: Common multiples are used to avoid working with fractions. Pupils will have to take one fifth of 3 units out, which will leave them with a mixed fraction. To avoid this, the idea of common multiples is used. So instead of working with 3 : 6 : 4, pupils will obtain the equivalent ratio 15 : 30 : 20. This way, pupils are able to take one fifth of 15 units, leaving them to work with integers.

Category 6: Ratio of changing quantities

Example: The ratio of the number of Daniel's pens to the number of Eunice's pens was 1 : 5. Then Daniel bought 12 more pens and Eunice bought 17 more pens. The ratio of the number of Daniel's pens to the number of Eunice's pens became 1 : 4. How many pens did Daniel have first?

Concept:

Use of "set" and 'replacement' approaches as all quantities are changing. This category involves the use of algebra. Since pupils have not learnt algebra, the heuristic called 'Sets' is used instead (see Figure 3). Pupils do not realise that by using the 'Set' approach, they are actually solving simultaneous equations.

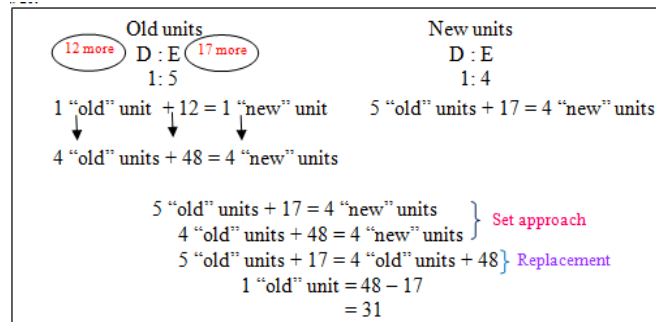


Figure 3.

Category 7: Ratio of a ratio

Example: At a game stall, every child needed 4 tokens to exchange for a prize, while an adult needed 5 tokens. Given that $\frac{2}{3}$ of the people who exchanged their tokens for prizes were children and total of 1092 tokens were collected by the game stall, how many tokens were collected from the adults?

Concept: The “hidden” ratio is multiplied to the “given” ratio. In this category, there are two ratios; one with definite value, “hidden ratio” (number of tokens each person gets) and the other with indefinite value, “given ratio” (the number of people) (Figure 4). The quantities in the two ratios are multiplied together. The pre test was administered again as a post test at the conclusion of the intervention.

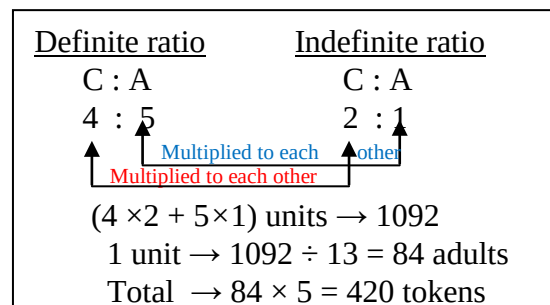


Figure 4.

Findings

Changes in Student Performance

Category	Pre Test (64 pupils)		Post Test (64 pupils)	
Category 1	64	100%	64	100%
Category 2	59	92.1%	64	100%
Category 3	29	45.3%	64	100%
Category 4	24	37.5%	58	90.6%
Category 5	18	28.1 %	60	93.7%
Category 6	15	23.4%	42	65.6%
Category 7	23	35.9%	52	81.2%

Changes in Student Thinking

Interviews: The responses of five of the pupils interviewed are reported here. All mentioned that they had to rely on their own knowledge because their teacher “did very little explaining”. This indirectly “forced” them to read the question for understanding. At times when the problem became too difficult, they admitted that understanding was a problem. Since they were in the culture of having to read the question repeatedly, comprehending the problem became less difficult. Although their solutions were incomplete, they were at least able to transform some of the clues in the problem into simple, correct mathematical forms. Based on their work, it was noticeable that they no longer wrote “incorrect direct translations into number sentence based on random numerical cues (Goh, 2009). This appears to be an excellent way to consolidate the first stage of problem solving: reading and understanding.

Four pupils agreed that there was a positive change in thinking. They found themselves engaged in thinking unlike before where they had little idea of where to begin. They said that categorisation has given them a form of “picture” to help them ‘kick start’ their thinking, although one pupil felt that categorisation did not benefit her. She found it time consuming to identify categories. She preferred to “just see the question and do it” because that had worked well for her.

When asked if it helped that a strategy was attached to a category, three of them agreed that was the case. Although the school had been exposing them to a wide variety of heuristics, they were not able to choose the appropriate one to use. After the categorisation intervention, they found themselves “more aware” of how each strategy was to be used. One of them found it useful only for the last two categories as he “had no idea of what was required” of the question. The other pupil felt that it was unnecessary as she felt that she knew exactly what each problem entailed.

Voice recording through MP3 player:

Transcripts of pupils’ thought processes during pre- and post-tests were compared. It was found that there was a noticeable change in student thinking in the post test; many pupils were able to reason logically and correctly. Hence they were able to categorise questions correctly. Following the introduction of Categories 1, 2 and 3 the thought processes revealed that all students in the two classes (64/64) managed to reason correctly, placing problems in the correct categories and obtaining correct solutions.

Later, 90.6% of the pupils (58/64) managed to identify the category for the given problem in Category 4. Three of the other six pupils identified the category incorrectly, while the rest made some calculation mistakes. In Category 5, pupils’ reasoning was very good, with almost everyone doing well in this category. In the MP3 recording every pupil mentioned the use of ‘lowest common multiple’.

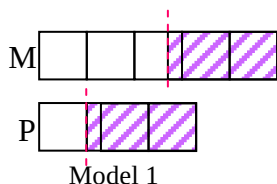
For Category 6, 96.8% (62/64) of the pupils were able to identify the correct category for category 6 problem and 65.6% (42/64) managed to obtain the correct solution. One particular reason for this was because the category involves creating “sets” or equations. Those who managed to get this completely correct used basic algebra to solve this. The rest did it by algebra too, but had difficulties halfway through the working. When MP3 recordings of 12 pupils who failed to get this question correct were played, it was discovered that all of them could identify the correct category and the concept, but were confused when it came to the technical part of algebra – they could not manage when they transposed to the other side of the equation; they were unable to “undo” the operations

(Mason, 1988: Driscoll, 1999 in Ng, 2004). They worked through the equation using their understanding of equivalence and constructed their knowledge based on intuitive problem solving strategies.

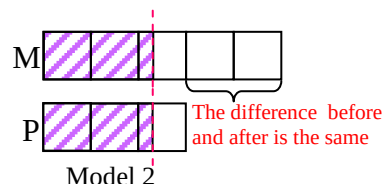
In Category 7, members of one class were able to identify this category as the only one with two ratios that did not refer to the same quantity. The other class reasoned it differently; pupils said that of the two ratios, one of them had a “definite value” and the other an “indefinite value”, and that 1 unit should be multiplied to quantities found in the indefinite ratio. In the MP3 recording of the pre-test, 22 pupils from the first class which mentioned the existence of two ratios referring to different quantities could not find the other ratio $C : A = 4 : 5$. It did not cross their minds that the tokens each received could be written as a ratio. Because of this, they were not able to obtain a solution. Interestingly, 96.8% (31/32) of pupils from the second class were quick to identify the “definite” ratio and managed to pull through the solution quickly and correctly. Different reasons came about from the two classes as the teachers were instructed to let the pupils discover them on their own. More scaffolding was done to the first class to help address the gap between students’ initial ability to understand concepts and the new concepts they had to learn in Category 7.

Strategies Used

In Category 3, pupils who solved using the model method experienced difficulty with the construction of the model. Most left their model drawing as shown in Model 1. They were unable to generate a solution as the method failed to reveal relationships between parts.

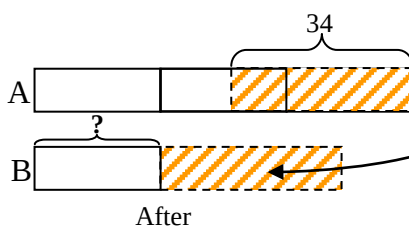
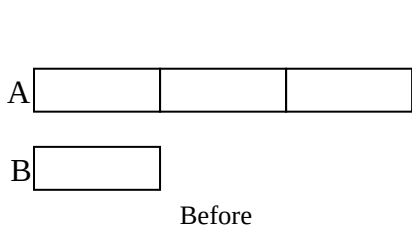


Pupils constructed Model 1 instead of Model 2. By doing so, they were unable to see a that the difference is constant.



These pupils understood the question but they were not able to transform the information in the text into mathematical forms (Newman, 1983). Another mistake commonly made in model drawing is found in Category 4, where a change in situation is involved. Those who did the model method were stuck at the planning stage and were unable to construct the model. They left this question with an incomplete model drawing. An example from a similar problem found in their daily work in class is as follows:

Ratio of Ally’s marbles to Bill’s marbles is 3:1. Ally gave 34 marbles to Bill and found that she had $\frac{4}{7}$ as many marbles as Bill. How many marbles did Bill have at first?



Pupils were able to understand the question and drew the correct diagram but were unable to proceed after that. The solution was left unsolved.

Those who did by categorisation did well for this question (Figure 5). They were able to identify the correct concept which led them to a solution. This approach may seem very abstract, but it *does* help those who have difficulties in constructing model drawings involving change. During one of the classroom sessions, pupils were given this question which did not belong to any specific category. It was given to them to see how they respond to it:

In classes 6A and 6B, the total number of girls was twice the total number of boys. The ratio of boys to girls in 6A was 3 : 4 and the ratio of boys to girls in class 6B was 1 : 6. If there were 8 more girls in class 6A than 6B, find the number of pupils in class 6A.

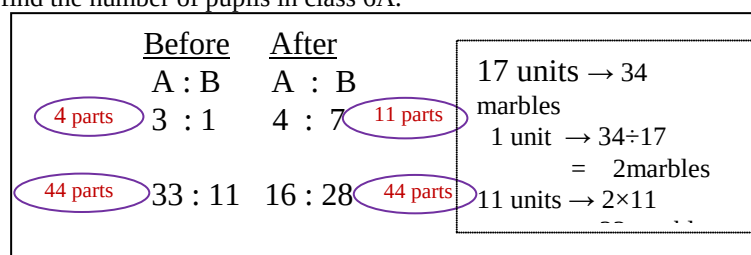


Figure 5.

It was interesting to see the pupils trying to fit this into the seven categories. A group of pupils decided that “This cannot be categorised. We need a new category”. So based on their solution, this particular group worked out the solution and named this category “Conditional Ratio” because according to them, they solved it using the conditions that were given in the question. They adopted the “listing” heuristic. Fig 6 shows their working.

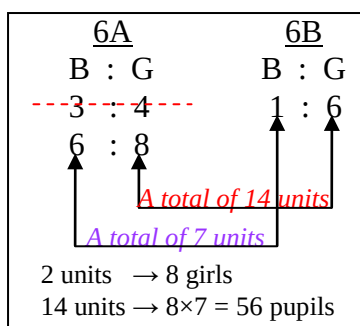


Figure 6.

Journal Writing

In journal writing, pupils were asked to explain briefly to their parents what categorisation was all about. Two pupils took an interesting approach (shown in Figures 7a and 7b). They started a question and varied the categories simply by changing numbers and context. This was impressive and encouraging as it showed that not only had categorisation impacted on these two, but it had also empowered them to create questions based on the categories.

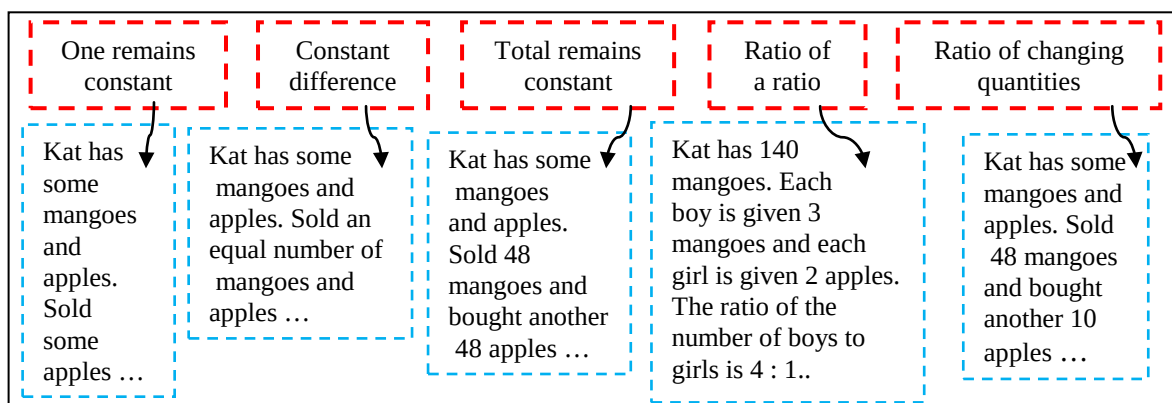


Figure 7a.

When 20 red pens were replaced by 20 blue pens Constant total
 When 20 red pens were replaced by 10 blue pens Ratio of changing quantities
 When 20 red pens and 20 blue pens were removed Constant difference
 When 20 red pens were removed One remains constant

Figure 7b.

Conclusion

Categorisation was formally introduced last year at the Primary Six level. The teachers and pupils responded positively to categorisation. More importantly, categorisation appears to enable pupils to reason logically; analyse mathematical situations and construct logical argument. In order to do this, pupils must first read the problem and understand it. Only then can they devise a plan to solve the problem. Categorisation shows promise for assisting students towards a better understanding of learning the mathematics of ratio.

References

- Chick, H. L.(2010). Aspects of teachers' knowledge for helping students learn about ratio. In L. Sparrow, B. Kissane & C. Hurst (Eds.) *Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia*. (pp. 131-139). Fremantle : MERGA.
- Creswell, J. W. (2008). *Educational research: Planning, conducting and evaluating quantitative and qualitative research* (3rd edn.). Pearson. Merrill Prentice Hall.
- Goh, S. P.(2009). Primary five pupils difficulties in using the model method for solving complex relational word problems. *Unpublished Master of Education thesis* submitted to Nanyang Technological University, Singapore.
- Ho, K. F., Teong, S. K., & Hedberg, J. G. (2004). *Creating problem solving repertoires*. Centre for Research in Pedagogy and Practice, Nanyang Technological University, Singapore.
- Kho, T. H., Yeo, S.M. & Lim, J. (2009). *The Singapore Model Method for Learning Mathematics*. Curriculum Planning and Development Division. Ministry of Education, Singapore : EPB Pan Pacific.
- Lo, J. J., Watanabe, T. & Cai, J (2004). The development of ratio concepts : An Asian Perspective. *Mathematics Teaching in Middle School*, 9(7), 362 – 367.
- Middleton, J. A.& van den Huevel – Panhuizen, M. (1995). *The Ratio Table. Mathematics Teaching in the Middle School*. I (4), 282 – 288.
- Musa, N. & Malone, J. A. (2011). The use of problem categorisation in the learning of ratio. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.) *Proceedings of the 34th annual conference of the Mathematics Education Research Group of Australasia and the Australian Association of Mathematics Teachers*. (pp.565-574). Adelaide: AAMT and MERGA.
- Newman, A. (1983). *The Newman Language of Mathematics Kit – Strategies for Diagnosis and Remediation*. Sydney, Australia: Harcourt Brace Jovanovich Group.
- Ng, S.F. (2004). Developing Algebraic Thinking in Early Grades : Case Study of the Singapore Primary Mathematics Curriculum. *The Mathematics Educator*. 8 (1), 39 – 59.
- Yeap, B. H. (2008). *Challenging Mathematics in Primary School National Examination in Singapore*. www.amt.canberra.edu/icmis16psinyeap.pdf
- Yeap, B. H. (2010). *Bar Modeling. A problem solving tool. From research to practice. An effective Singapore mathematics strategy*. Marshall Cavendish. Singapore

Student Understanding of Large Numbers and Powers: The Effect of Incorporating Historical Ideas

Mala S. Nataraj
The University of Auckland
<snat016@aucklanduni.ac.nz>

Michael O. J. Thomas
The University of Auckland
<moj.thomas@math.auckland.ac.nz>

The value of a consideration of large numbers and exponentiation in primary and early secondary school should not be underrated. In Indian history of mathematics, consistent naming of, and working with large numbers, including powers of ten, appears to have provided the impetus for the development of a place value system. While today's students do not have to create a number system, they do need to understand the structure of numeration in order to develop number sense, quantity sense and operations. We believe that this could be done more beneficially through reflection on large numbers and numbers in the exponential form. What is reported here is part of a research study that involves students' understanding of large numbers and powers before and after a teaching intervention incorporating historical ideas. The results indicate that a carefully constructed framework based on an integration of historical and educational perspectives can assist students to construct a richer understanding of the place value structure.

Introduction

In the current era where very large numbers (and also very small numbers) are used to model copious amounts of scientific and social phenomena, understanding the multiplicative and exponential structure of the decimal numeration system is crucial to the development of quantity and number sense, and multi-digit operations. It also provides a sound preparation for algebra and progress to higher mathematics. However, many students continue to experience difficulties with the place value system, even in secondary school. This may be due to the fact that, as observed by Skemp (1989), while outwardly simple, the place value concept contains a number of complex ideas, with some of them at a high level of abstraction. In this paper we report on a study that turned to the history of mathematics for ideas that might improve students' understanding of a general place value system and show the benefits of such an approach.

For some decades now, it has been argued that the history of mathematics can have a positive influence in teaching and learning (Fauvel & van Maanen, 2000). Arguments are that a historical approach can help to improve perceptions of mathematics, and attitudes to it, raising interest by revealing it as part of human culture. It can show up certain difficulties and conceptual obstacles in the development of mathematics that recur in learning, and how these difficulties were overcome. The theoretical framework for the use of historical analysis in contemporary research is sometimes that *ontogenesis* recapitulates *phylogenesis*; that is, the mathematical development in the individual learner repeats the history of mathematics (Radford, 2000; Sfard, 1995). Although a strict historical parallelism is not considered tenable, one possible pedagogical use of history is to incorporate past conceptual developments and the order in which they arose, into a teacher's knowledge so they can be interwoven with the design of classroom activities (Furinghetti & Radford, 2008; Tzanakis & Arcavi, 2000). The goal of this perspective is to enhance students' understanding in mathematics by looking at the results of a process of historical-cultural development. Briefly, our view is that historical awareness may be beneficial in adopting appropriate teaching strategies for the classroom and also providing a toolkit to understand student

behaviours (Radford, 2000). In this study, in order to develop a didactic framework for a deep understanding of the place value system, a historical-critical methodology (Gallardo, 2001) was adopted. This involved a to and fro movement between history of mathematics texts and related mathematics education writings in order to extract ideas for classroom activities. The historical analysis resulted in the development of an overarching framework that incorporated i) Large numbers including powers of ten, ii) Other ancient numeration systems, iii) Algebraic generalisation and, iv) The compositional structure of the decimal place value system and its generalisation. Teaching sequences for each of these strands were also developed and implemented in the classroom. Due to space constraints, what is reported here is a part of the research study involving students' understanding of large numbers, particularly powers with positive integer exponents.

History of Mathematics

A study of the history of the development of the current numeration system reveals that it originated in India and was then carried to Europe by the Arab mathematicians (Datta & Singh, 2001). It is reasonable to suppose that a study of the historical evolution of this system may hold valuable lessons for today's classroom. This along with the known student difficulties with the place value system provided the impetus for the study. One of the many crucial mathematical developments in India preceding the creation of the present place value system was the consideration of very large numbers, including their representation as powers of ten (Joseph, 2011). What were the types of numbers that were considered in ancient times? A few examples from Indian historical texts are given below:

1. A major milestone in the development of the Hindu-Arabic place value system is a (surprisingly very early) set of number names for powers of ten. In the *Vajasaneyi (Sukla Yajurveda) Samhita* (17.2) (c. 2000 BC) of the Vedas, the following list of arbitrary number names is given in Sanskrit verse: *Eka* (1), *Dasa* (10), *Sata* (10^2), *Sahasra* (10^3), *Ayuta* (10^4), *Niyuta* (10^5), *Prayuta* (10^6), *Arbuda* (10^7), *Nyarbuda* (10^8), *Samudra* (10^9), *Madhya* (10^{10}), *Anta* (10^{11}), *Parardha* (10^{12}) (e.g. Bag & Sarma, 2003; Datta & Singh, 2001). They were aptly called the *dasagunottara samjna* (decuple terms), confirming that there was a definite systematic mode of arrangement in the naming of numbers. The same list of names of powers of ten was then extended to *loka* (10^{19}) (Gupta, 1987).
2. The same list up to *parardha* (one trillion) as in the *Vajanaseyi Samhita* is repeated in the *Pancavimsa Brahmana* (17.14.1.2) with further extensions. The following passage from it will give an idea of the context and the manner in which large numbers were introduced:

...By offering the agnistoma sacrifices, he becomes equal to one who performs a sacrifice of a thousand cows as sacrificial fee. By offering ten of these, he becomes equal to one who performs a sacrifice with ten thousand daksinas (fee). By offering ten of these, he becomes equal to one who sacrifices with a sacrifice of a hundred thousand daksinas... By offering ten of these, he becomes equal to one who sacrifices with a sacrifice of 100 000 million daksinas. By offering ten of these he becomes the cow [one trillion]. (Sen, 1971, p. 141)
3. In the Buddhist work *Lalitavistara* (c. 100 B.C.E), there are examples of series of number names based on the centesimal scale. For example, in a text, the mathematician Arjuna asks how the counting would go beyond *koti* (10^7) on the centesimal scale, and Bodhisattva (Gautama Buddha) replies: Hundred *kotis* are called *ayuta* (10^9), hundred *ayutas* is *niyuta* (10^{11}), hundred *niyutas* is *kankara* (10^{13}),...and so on to *sarvajna* (10^{49}), *vibhutangama* (10^{51}), *tallaksana* (10^{53}). It is to

be noted that there are 23 names from *ayuta* to *tallaksana*. Then follow 8 more such series, starting with 10^{53} and leading to the truly enormous number $10^{53+8 \times 46} = 10^{421}$! (Menninger, 1969).

4. In the Vedic literature, time is reckoned in terms of *yugas* or time cycles. The four *yugas* are *Satya-yuga*, *Treta yuga*, *Dwapara yuga* and *Kali yuga*. According to Hindu cosmology, the time-span of these four *yugas* is said to be 1728000, 1296000, 864000, and 432000 years, respectively, in the ratio 4:3:2:1. The total of these four *yugas* was considered as one *yuga-cycle* or *Mahayuga* and was thus 4320000 years (Srinivasiengar, 1967). Moreover, it is believed that 1000 such *yuga-cycles* comprise one day in the life of *Brahma*, which is 4,320,000,000 years and one day and night period is 8.64 billion years which was further extended to 311×10^{12} . As pointed out by Plofker (2009) time in the astronomical works is bound by cosmological concepts. In one *kalpa* which is 4 320 000 000 years, all celestial objects are considered to complete *integer* number of revolutions about the earth.

It is clear, even from the brief examples above, that large numbers were not only known in early Indian mathematical contexts but formed an integral part of the knowledge and practice of the times.

Large Numbers and Pedagogical Perspectives

In the light of the historical examination of the examples of the use of large numbers in Indian history mentioned above, we may ask what lessons can be learned for today's classroom? As can be seen, some truly enormous numbers were dealt with from a very early time period, and although some of these numbers are literally fantastic, it is significant to note the extent to which these were taken. What is also seen is that the multiplicative (exponential) structure was already present from Vedic times, with number names for powers of ten even in the earliest works. There seems to have been an early awareness of the effect of repeated multiplication. Pedagogically, this implies the value of an early and explicit teaching of powers to create a *foundational base* for a sound grasp of the place value system. However, as in the Mayan civilisation, such numbers in Indian history were mostly employed in a practical context such as astronomy (Plofker, 2009) and time measures (for calendar purposes), according to the needs related to social and cultural practices of the different eras. This suggests that today's students, many of whom are also fascinated by large numbers may be more motivated if a meaningful, realistic context is used. This is not difficult, since large numbers are ubiquitous in today's society, and the concept of exponential growth (Confrey, 1994) is the fundamental idea behind compound interest, computer memory, space distances and is also useful for modelling population growth and spread of disease. This common usage of large numbers suggests that there is a need to understand them, and in this regard students may benefit from *naming*, *reading*, *writing* and *computing* with them (Ronau, 1988) in different contexts. Understanding the *naming convention* (Labinowicz, 1985) for reading large numbers is a prerequisite to grasping them and their structure, and may help students to form a link between the place value chart and the place value system. Students may gain an understanding of the current nomenclature (with digits in clusters of 3) from reading and writing numbers involving million (10^6), billion (10^9), trillion (10^{12}) and so on to decillion (10^{33}). However, the problem is that words like *billion* and *trillion*, and the place value system are based on an

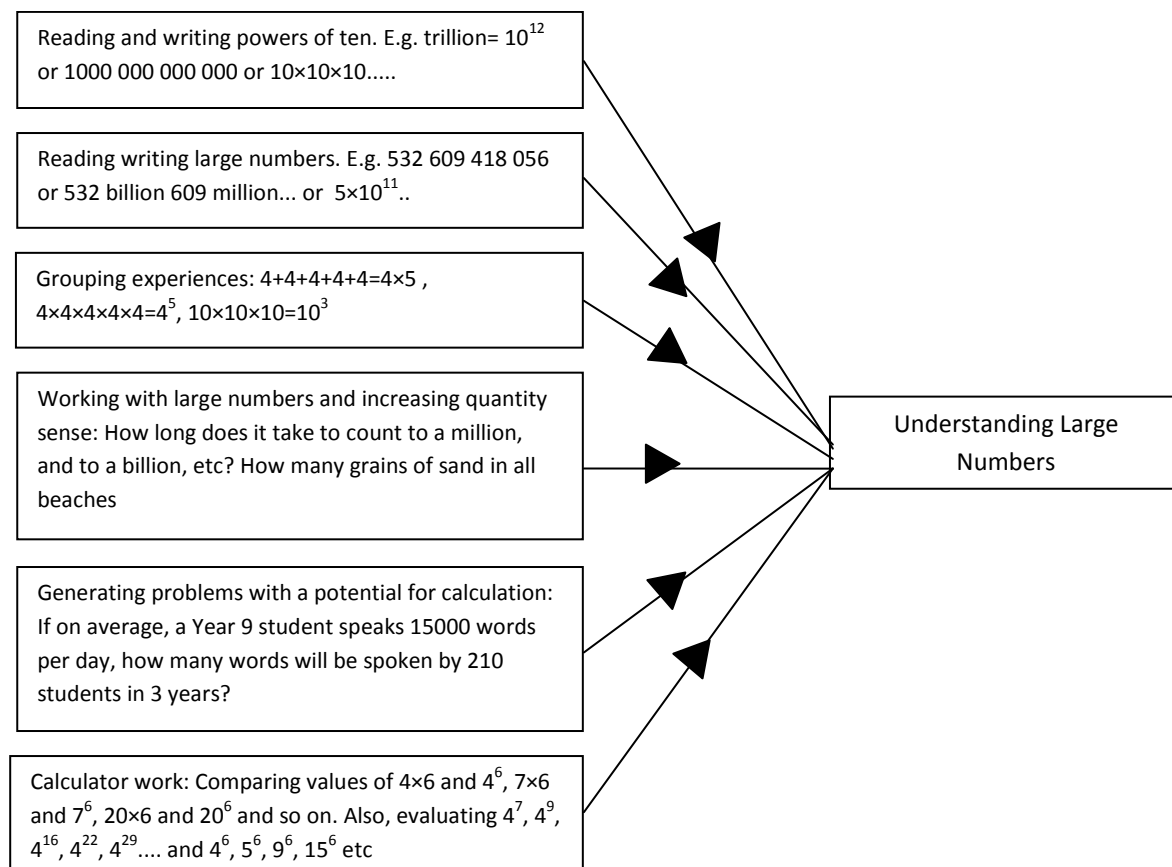


Figure 1. The framework for teaching large numbers and powers.

understanding of *powers* and *exponential growth*, making it difficult to understand one without understanding the other.

This pedagogical *circularity* (Sfard, 1991) or catch-22 situation implies that naming large numbers, and grouping experiences reflecting the embedded nature of repeated multiplication and its shortened notation should go hand in hand. That is, students should be able to move between *multiple representations* (Thomas, 2004, July) making conceptual links between them in order to understand the concept well. Moreover, apart from the nomenclature, it is also important to develop a sense of the size of the numbers and the structure of the powers. This is called *quantity sense* by Wagner and Davis (2010), who define it as a *feel* for amounts and magnitudes. The second historical example given above indicates that groupings of higher order (such as groups of groups of groups) could be employed as a teaching strategy, both to help students understand exponentiation with integer exponents and to develop *quantity sense*. In this study, as a result of viewing the historical analysis within the background of pedagogical research, a framework (see Figure 1) was developed and implemented to teach large numbers and the notion of powers.

The Teaching Method

The second part of the research comprised a case study that involved naming and working with large numbers and then using craft matchsticks to reflect the grouping

structure of the place values/powers. The student participants in this research comprising 26 students (13 years old) members of a Year 9 class in a multicultural state secondary school in Auckland, New Zealand were taught by the first-named researcher. The students were representative of a wide range of socio-economic and cultural backgrounds including Korean, Chinese, Indian, European, Filipino, Maori, Pacific Island and New Zealand European ethnicities. Most students were proficient in English, however six students were attending literacy classes and two students were attending ESOL class (English for Speakers of Other Languages). Students were given a pre-test on large numbers at the start of the year and a post-test in the middle of Term 1 after the intervention. The teaching intervention involved naming and writing large numbers both in words and in different numeral forms. Number names such as *million*, *billion*, *trillion* and so on up to *decillion* and their multiple representations were read and written so as to highlight for students the base ten multiplicative structure, and the sub-base of thousand-unit chunks. The names were written out in words and in the expanded form. For example million was written as $10 \times 10 \times 10 \times 10 \times 10 \times 10$ and as 10^6 . Other examples of numbers such as 7 648 309 105 were read and written out as, for example, 7 billion, 648 million, 309 thousand and 105 and also as

$$7 \times 10^9 + 6 \times 10^8 + 4 \times 10^7 + 8 \times 10^6 + 3 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 0 \times 10^1 + 5 \times 10^0$$

(Initially 10^0 was written as 1). Classroom activity also involved grouping experiences using craft sticks that reflected equal (and unequal) grouping structures (including powers) such as $3+3+3+3$ and $3 \times 3 \times 3 \times 3$, $10+10+10$ and $10 \times 10 \times 10$, and linking these to their abbreviated notation. Further, in order to increase quantity sense, the discussion in class centred around the relative sizes of numbers. The teaching unit also included computing with large numbers and problems such as the following were considered: How far does light travel in a year if the speed of light is approximately 300 000 km/sec? How many grains of sand are there in Muriwai beach if it is given that the length of the beach is approximately 50km, width is 50m and the depth is 4m, and if 1 cm^3 contains about 8000 grains of sand? If a million dollars consisting of \$1000 bills is stacked to a height of 10 cm, how high is a billion dollars, and a trillion dollars? Following these activities, students were given the post-test on large numbers.

Results

The questions in the large number test involved different ways of writing large numbers, students' ability to compute with large numbers, their ability to produce questions that use large numbers with a potential for calculations, exponentiation and the notion of infinity. While all these areas contribute to understanding student thinking about large numbers, in the ensuing analysis, particular attention was paid to students' answers to questions related to exponentiation, due to its importance in gaining a deep understanding of place value structure.

When the participants' answers to the questionnaires were analysed, from the pre-test to the post-test, every student except two improved their score, leading to a significant improvement in the mean score ($\text{Mean}_{\text{pre}} = 2.85$, $\text{Mean}_{\text{post}} = 10.02$, $t=10.46$ and $p<0.0001$).

Evidence of the use and understanding of exponentiation was looked for in both the pre-test and the post-test results. In the event, for the first question, which asked 'What is the largest number that you can say in five seconds?' no student answered in terms of powers in the pre-test. On the other hand, in the post-test, although not specifically asked to do so, 11 out of 26 students gave their answer in an exponential form, with (Student 4) S4 writing

99^{99} , $520\ 50^{150}$, and 368^{50} was written by S8. Some students wrote in terms of a much larger base and exponential numbers, for example: 50250640^{100} (S23), 480910^{80} (S24), 1000000^{1000} (S14), who also wrote this number in words. These examples show that although it is unlikely that the students understood the enormous size of the numbers that they had created, they appeared to be aware that writing the number in the exponential form does rapidly increase the value of the number.

Q4 involved writing one trillion in numerals in two different ways, and in the pre-test only one student was able to write it in a power form, but the answer was incorrect. However in the post-test, 17 students gave the correct answer in two different ways including the power form, namely as 1 000 000 000 000, and 10^{12} . Three students further elaborated that $10^{12} = 10 \times 10 \times 10 \dots$ Qs 9,11,12,13 and 14 that were analysed all involve students' understanding of exponentiation as repeated multiplication, and its generalisation, which was a primary focus of this study. The question facilities and the results in the pre- and post-tests are given in Table 1.

Table 1

Question Facilities on Notation for Repeated Multiplication

Question		% correct in Pre-test (N=26)	% correct in Post- test (N=26)
Q9	What does 7^{11} mean to you?	23.1 (6)	80.8 (21)
Q11	Write what you understand by 2^{96}	19.2 (5)	84.6 (22)
Q12	What does a^7 mean to you?	26.9 (7)	69.2 (18)
Q13	What is the meaning of 2^y ?	3.8 (1)	50 (13)
Q14	Write what you understand by a^y ?	3.8 (1)	50 (13)

Understanding exponentiation as repeated multiplication is a crucial step on the way to number system construction and there was some evidence of numbers written in an expanded form. For Question 9 in the pre-test, 6 out of the 26 students wrote the correct answer, but only 3 of them were able to transfer their understanding of repeated multiplication to the higher powers (Question 11). 50% of the students made the classic error, writing $7 \times 11 = 77$ or a similar answer, while two students did not provide an answer. However, in the post-test, 21 out of the 26 students were successful in giving the correct answer; $7 \times 7 \times 7 \dots$ (11 times). Of these 21 successful students, 19 were able to extend their understanding to higher powers. In terms of errors, in the post-test, it is noteworthy that no student made the error of writing, for example, 7×11 for 7^{11} for any of the questions. However, despite explicit teaching during the intervention, one student (S16) did reverse the numbers, writing "11 to the power of 7". Questions 12 to 14 sought to examine whether students could extend the idea of repeated multiplication to the general form. Qs 13 and 14 were especially difficult for students and many did not attempt these in the pre-test. However, as Table 1 shows, students were more successful in the post-test. What is interesting to note is that for some students, writing the expanded form for large numbers was harder than making generalisations. For example, in Q12 seven students were able to give the expanded form $a \times a \times a \times a \times a \times a$, however, only 5 were able to write that 2^{96} is $2 \times 2 \times 2 \dots$ (96 times). As seen in Table 1, questions involving powers such as 2^{96} , 2^y and a^y appear to have been problematic for students in the pre-test, possibly because the *exponents*, rather than the base either involve a large number or a literal symbol. Given that the main focus of the study was on the *grouping structure* and *notation* of powers, it is noteworthy that many students experienced some success in the questions on exponentiation. Several

students did comment that they found the grouping experiences in class useful in understanding grouping in powers reflecting repeated multiplication.

Discussion and Conclusion

The main purpose of the study was to investigate whether a framework based on an examination of an historical progression of ideas would enhance students' understanding of large numbers, and especially powers of ten. This seems to have been the case. The results appear to substantiate researchers' views (Hewitt, 1998) that, by naming large numbers and working with them, students can construct the naming convention and the base-ten structure of numeration. Further, the evidence of exponentiation in the post-test questions and students' responses, particularly in Q4 tend to confirm Zazkis's (2001) theory that contemplation of large numbers encourages students towards a sense of structure. The teaching module had both an emphasis on operations without actual computation, as proposed by Hewitt (1998), which may help students build abstractions, and also computations with large numbers in support of Wagner and Davis's (2010) view that calculating and connecting to previous experiences may be beneficial to develop a *sense of quantity* involved in large numbers.

Another crucial aspect of understanding place value structure is an awareness of powers as repeated multiplication, leading to the grouping structure (Dienes & Golding, 1971) that reflects this (as seen in the second historical example), and the related quantity sense. The results show a degree of success experienced by students pertaining to the exponentiation concept, and this corroborates Zazkis's hypothesis that consideration of large numbers, particularly powers, can help lead learners to structure sense. Students' experiences in grouping, and working not only in base-ten but also in non-decimal bases may also have contributed to their success in responding to questions that involved a generalisation of powers. The test results illustrate that, working with manipulatives, and employing visualisation aspects are important factors in the early stages of concept development. In addition, working with the different *representations* (including concrete materials), as suggested by many researchers (e.g. Nataraj & Thomas, 2009), may have helped some students to generalise to the meaning of larger numbers such as 2^{96} , and to powers with literal symbols.

What surfaced in the study is that, despite the approach used and the varied practice experiences they had, some students still found grouping in powers confusing and difficult. A key aspect in the construction of place value that was highlighted in the research is the importance of *regrouping* as a *whole unit* at each step in the process and keeping a record of the transformation. Another crucial idea is the establishment of links between manipulatives in the grouped form and their representation in the written form, at every stage in the procedure. What also came to the fore in students' work was the presence of obstacles, similar to those experienced historically. For example, students' way of knowing multiplication appears to sometimes stand in the way of their understanding repeated multiplication and this sometimes resulted in the classic error of writing $2 \times y$ rather than 2^y .

Notwithstanding the above difficulties, the positive results indicate the usefulness of the framework and the related large number teaching sequence and suggest the value of constructing teaching sequences from a combination of historical ideas and educational research. While some students were still developing the exponentiation concept, for most the early naming of, and working with large numbers including powers of ten and its related groupings, appears to have helped them to understand the naming convention in large numbers and the equal grouping structures involved in powers of ten and other bases.

References

- Bag, A. K., & Sarma, S. R. (Eds.). (2003). *The concept of sunya*. New Delhi, India: IGNC, INSA and Aryan Books International.
- Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *Development of multiplicative reasoning in the learning of mathematics* (pp. 291-330). Albany, NY: State University of New York Press.
- Datta, B., & Singh, A. N. (2001). *History of Hindu mathematics* (Vols. 1-2). Bombay, India: Asia Publishing House.
- Dienes, Z. P., & Golding, E. W. (1971). *Sets, numbers and powers*. Harlow, England: The Educational Supply Association and University of London Press.
- Fauvel, J., & van Maanen, J. (Eds.). (2000). *History in mathematics education: The ICMI Study*. Dordrecht, The Netherlands: Kluwer Academic.
- Furinghetti, F., & Radford, L. (2008). Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 626-655). New York: Routledge.
- Gallardo, A. (2001). Historical-epistemological analysis in mathematics education: Two works in the didactics of algebra. In R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds.), *Perspectives on school algebra* (pp. 121-140). Dordrecht, The Netherlands: Kluwer Academic.
- Gupta, R. C. (1987). One, two, three to infinity. *The Mathematics Teacher (India)*, 23(4), 5-12.
- Hewitt, D. (1998). Approaching arithmetic algebraically. *Mathematics Teaching*, 163, 19-29.
- Joseph, G. G. (2011). *The crest of the peacock: Non-European roots of mathematics*. Princeton, NJ: Princeton University Press.
- Labinowicz, E. (1985). *Learning from children: New beginnings for teaching numerical thinking: A Piagetian approach*. Menlo Park, CA: Addison-Wesley.
- Menninger, K. (1969). *Number words and number symbols* (P. Broneer, Trans.). Cambridge, MA: M.I.T Press. (Original work published 1958).
- Nataraj, M. S., & Thomas, M. O. J. (2009). Developing understanding of number system structure from the history of mathematics. *Mathematics Education Research Journal*, 21(2), 96-115.
- Plofker, K. (2009). *Mathematics in India*. Princeton, NJ: Princeton University Press.
- Radford, L. (2000). Historical formation and student understanding of mathematics. In J. Fauvel & J. Van Maanen (Eds.), *History in mathematics education: The ICMI Study* (pp. 143-170). Dordrecht, The Netherlands: Kluwer Academic.
- Ronau, R. (1988). Number sense. *The Mathematics Teacher* 81(6), 437-440.
- Sen, S. N. (1971). Mathematics. In D. M. Bose, S. N. Sen & B. V. Subbarayappa (Eds.), *A concise history of science in India* (pp. 136-209). New Delhi, India: Indian National Science Academy.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.
- Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behaviour*, 14, 15-19.
- Skemp, R. R. (1989). *Mathematics in the primary school*. London: Routledge.
- Srinivasiengar, C. N. (1967). *The history of ancient Indian mathematics*. Calcutta, India: World Press Private.
- Thomas, M. O. J. (2004, July). *Conceptual representations and versatile mathematical thinking*. Paper presented at the Tenth International Congress on Mathematics Education, Roskilde University, Roskilde, Denmark. Retrieved July 15, 2007, from http://www.icme10.dk/proceedings/pages/regular_pdf/RL_Mike_Thomas.pdf.
- Tzanakis, C., & Arcavi, A. (2000). Integrating history of mathematics in the classroom. In J. Fauvel & J. Van Maanen (Eds.), *History in mathematics education: The ICMI Study* (pp. 201-240). Dordrecht, The Netherlands: Kluwer Academic.
- Wagner, D., & Davis, B. (2010). Feeling number: Grounding number sense in a sense of quantity. *Educational Studies in Mathematics*, 74, 39-51.
- Zazkis, R. (2001). From arithmetic to algebra via big numbers. In H. L. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *The Future of the Teaching and Learning of Algebra, Proceedings of the 12th ICMI Study Conference* (Vol. 2, pp. 676-681). Melbourne, Vic., Australia: University of Melbourne.

The Concept of Generalised Number: Valuable Lessons from the History of Algebra

Mala S. Nataraj
The University of Auckland
<snat016@aucklanduni.ac.nz>

Michael O. J. Thomas
The University of Auckland
<moj.thomas@math.auckland.ac.nz>

The importance of understanding the various uses of the literal symbol in algebra, and in particular the idea of generalised number, is well documented in the literature. Many research findings have also reported student difficulties with this vital and central concept. This research study examines the use of a combination of historical and educational research ideas as a way of enhancing students' understanding of generalised number. The results suggest that this approach helped some students to make some generalisations and to understand the difference between specific unknown and generalised number.

Background

Students know that algebra is 'to do with letters' but research has clearly documented that many students have little understanding of what the letters mean and the reason that they are used (Graham & Thomas, 2000; Kieran, 1992; Küchemann, 1981; MacGregor & Stacey, 1997). The concept of variable, in particular, continues to be poorly understood even though it is fundamental in the transition from arithmetic to algebra (Schoenfeld & Arcavi, 1988) and is central to all higher mathematics. Hence, while students need to understand the multi-faceted meaning of the literal symbol as specific unknown, generalised number, and variable, many studies have stressed student difficulties and errors in this area. A view put forward by Radford (1996), and others, is that generalised number is a pre-concept for variable, and he suggests that the ways of thinking associated with generalisation (involving generalised number and variable) and problem solving (involving specific unknown) are "independent and essentially irreducible, structured forms of algebraic thinking" (*ibid*, p. 111). Thus, both generalisation and problem-solving approaches appear to be mutually complementary fields in the didactics of algebra. However, there is often a prevalence of use of letter as specific unknown rather than as a generalised number in schools, possibly due to a focus on examples such as substitution and equation solving in which the letters represent a single specific value. An additional factor that may confuse students is that the term 'variable' is used to refer to letters regardless of whether the actual usage of the letter is as a variable, generalised number, specific unknown or place holder, or has some other use. From the CSMS study, Küchemann (1981) concluded that many students did not progress beyond viewing letters as numerical placeholders, and of the 30-40% who did, the majority interpreted letters as specific unknowns rather than generalised numbers or variables. Another algebraic error that students are prone to make, reported by MacGregor and Stacey (1997) is the misuse of exponential notation; such as writing x^3 instead of $3x$.

In the light of the many difficulties that students face in mathematics, in recent years some researchers have attempted to analyse the history of mathematics in order to inform teaching practice. Educators have asserted (Fauvel & van Maanen, 2000) that the history of mathematics is an excellent resource for teaching and can be of great benefit in enhancing the understanding of mathematics. There are different ways of incorporating lessons from history in the classroom, both explicit and implicit (Harper, 1987) and either way it can bring about a major change in the teacher's approach. A review of historical texts (Datta & Singh, 2001) reveals that the decimal number system with place value and zero, as well as many algebraic ideas, originated in India (Cajori, 1919; Joseph, 2011). For example, Puig

and Rojano (2004) cite how a Mathematical Sign System (MSS), in which the different unknown quantities and their powers are differentiated, was an important step in the development of the algebraic symbolism that had developed by the time of Bhaskara II in the 12th century (or possibly even earlier) in India, and, by Viete, in the 16th century in Europe. This enabled the construction of *general methods* of solutions to equations. One particularly interesting feature of the sign system developed in India was that different *colours* were used to represent various unknowns. Bhaskara II (1150) says: “*yavat-tavat* (so much as), *kalaka* (black), *nilaka* (blue), *pitaka* (yellow), *lobita* (red) and other colours have been taken by the venerable professors as notations for the measures of the unknowns, for the purpose of calculating with them” (Datta & Singh, 2001, Vol. 2, p. 18). Thus Bhaskara II employed abbreviations of the names of the unknown quantities in order to represent them in an equation, such as *ka* for *kalaka* (black) and *ni* for *nilaka* (blue). Although *yavat-tavat* (quoted above) is not a colour its inclusion shows the persistence of an ancient symbol employed long before colours were introduced to denote unknowns.

According to the theory of the structure of attention proposed by Mason (2004) we may focus our attention on the whole, the details, the relationships between the parts, the properties of the whole or the parts, and deductions, becoming more aware of what we notice. He also states (Mason, Graham, & Johnston-Wilder, 2005) that classification is a form of generalisation and asserts that children have an inherent ability to classify objects. Thus in order to detect generalities in arithmetic patterns, he suggests guiding students’ attention towards number patterns by asking questions such as “what is the same about each row?”, “what is different and how is it changing”? This same method of guiding attention was also described by Srinivasan (1989) who advocated the use of ‘*pattern language*’ for number patterns and ‘*design language*’ for shape arrangements to concentrate students thinking on variation and invariants in number and geometrical patterns. His recommended vocabulary centres around *changing*, *not changing*, *changing in the same way* and *changing in different ways* in order to elicit an algebraic expression from students in the form of pattern language, and this was used in this study. Given the above, this research sought to use a combination of historical and mathematics education research ideas to address the question of whether explicitly directing students’ attention to classification, and getting them to write colours in the place of the changing numbers, as in Indian history, would help improve students’ understanding of the concept of generalised number. A framework (see Figure 1) was developed and implemented combining the ideas from the two domains of history and pedagogical research to teach algebraic generalisation.

Method

The research presented here comprised a case study of a class of 26 Year 9 (age 13 years) students at a decile 5 (middle socio-economic level) secondary school in Auckland, New Zealand. The class used in the research represented a wide variety of cultural backgrounds. However, most of the students had their intermediate schooling in New Zealand and hence were proficient in English. The exceptions to this were a Chinese student and a Burmese student who had only recently arrived in the country and hence were taking ESOL classes.

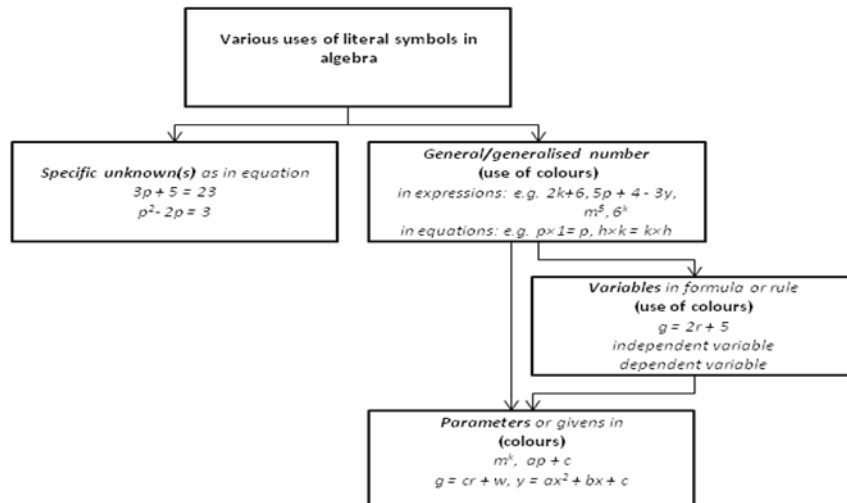


Figure 1. The teaching sequence involving algebraic generalisation.

2. (Eight)³, (six)³, (four)³, (seven)³ Can you generalise this pattern?
3. 4×7, 4×9, 4×12, 4×6.....Generalise the given pattern.
4. Observe what is changing and not changing for the following patterns and then make a generalisation

20+1×15	5×1-2=3
20+2×15	5×2-2=8
20+3×15	5×3-2=13
20+4×15	5×4-2=18
20+5×15	5×5-2=23

5. i) Write down what you understand by $P + 5$ and what are the possible values for P ?
- ii) Write down what you understand by $G - 2 + B$. What are possible values that G and B can take?
6. a) Compare and then explain what you understand by
 - i) $3 \times y$ and ii) $3 \times y = 12$.
 - b) Explain the meaning that you would give y in $3 \times y$ and also in $3 \times y = 12$
7. John has a certain number of sweets in his pocket and to these he adds 3 more. How many does he have altogether now? Can you write this down?
8. Observe what is changing and not changing. Then make a generalisation and simplify.

$9 \times 1 + 8 + 7 \times 1$	$16 \times 1 + 8$
$9 \times 2 + 8 + 7 \times 2$	$16 \times 2 + 8$
$9 \times 3 + 8 + 7 \times 3$	$16 \times 3 + 8$
$9 \times 4 + 8 + 7 \times 4$	$16 \times 4 + 8$

What do you notice about the generalisation in the above columns?

10. What is the meaning of 5×6 ?
11. What is the meaning of $3 \times 3 \times 3 \times 3$?
12. What is the meaning of 5^6 ?
13. Write what you understand by a^7 .
14. What does 2^{96} mean to you?
15. What is the meaning of 2^y ?
16. Write what you understand by a^y .

Figure 2. A representative selection of the algebra test problems.

The teacher, who was the first-named researcher, explained to the students at the beginning of the year and during some of the subsequent lessons what was going to be taught, including what algebra was about and why it was important to their learning. The

teacher and students also discussed why it was important to have a clear understanding of the different uses of letters. Students had intermittent guidance in observation of variation and invariants in number patterns put up on the board, including whole class discussion using vocabulary previously presented (see Figure 3). This vocabulary included words such as changing, not changing, changing in the same way, changing in a different way, expression, equation, specific unknown number, general number, variable, exponent, power, repeated multiplication, symbol, and solving. The students' attention was also guided to variation and invariants in simple number patterns as in $r+6$ and then to other number patterns, such as $9+b-r$ with the use of *colours*. As presented in Figure 3, the changing number was initially shown with a patch of *colour* (for example, red), then via discussion, denoted with the *word* (red) and then shortened to the letter r . After the whole group work, students had to record conclusions in their books and then attempt similar examples on their own. The meaning and notation of exponential forms such as 10^3 , 4^5 , m^4 , 4^q , and r^w were presented too. The students experienced substitution, expansion, collecting terms and simplifying, solving and checking equations, order of operations, pattern generalisation and formulas as part of their normal curriculum. After approximately sixteen one-hour periods on the work, the class spent two such periods answering the 33 algebra questions, not all of which are considered here (see Figure 2 for a representative selection of questions presented in this paper).

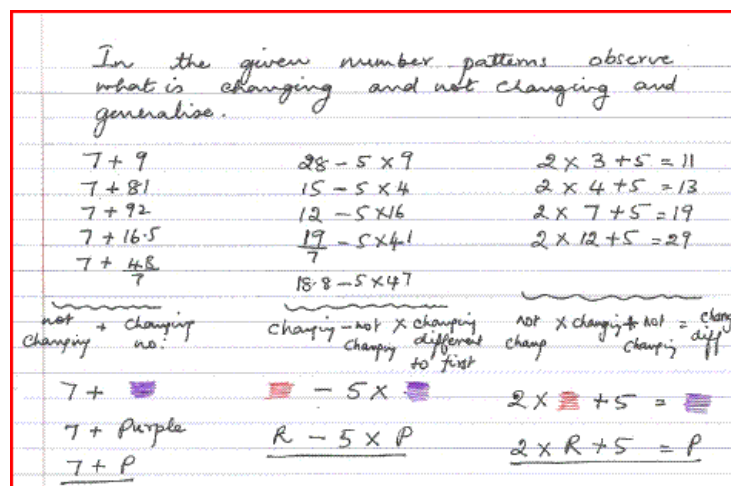


Figure 3. An example of the way in which colours were employed to represent changing numbers.

Results

Questions 1, 2, 3, 4 and 8 were designed to see whether the students could recognise quantities that are changing (ie variation) and distinguish them from those that are not (invariant). Table 1 gives the numbers (and %) of the 26 students that were able to do so for each question (Q1 was similar to Q's 4 and 8).

Table 1

Facilities for Questions on Recognition of Variation and Invariance (N = 26)

Question	1a	1b	2	3	4a	4b	8a	8b	8c
Number Correct	24	19	18	18	22	23	6	24	1
%	92.3	73.1	69.2	69.2	84.6	88.5	23.1	92.3	3.8
Number Wrong	1	2	4	1	2	1	15	1	11
%	3.8	7.7	15.4	3.8	7.7	3.8	57.7	3.8	42.3

With the exception of Q8a), which proved difficult, around 70% or more of the students were able to distinguish variation from invariance and could symbolise the variation using a symbolic literal. In Q8a), most students were able to observe what was changing and what was not changing, but only a few were able to notice that the second general number was changing in the same way as the first general number and so most used a different colour or letter for each rather than the same colour or letter. In Figure 2 we see that S12 has used the same colour for each but has not gone on to symbolise it, while S10 has used the word symbol 'Blue'. However, S2, S3, S4, S7, S11, S13, S19 and S24 were all able to write a correct expression such as $9*y+8+7*y$ and S19 and S24 were able to simplify it, with S24 writing $16B+8$ (see Figure 4), by removing the explicit multiplication signs and adding terms.

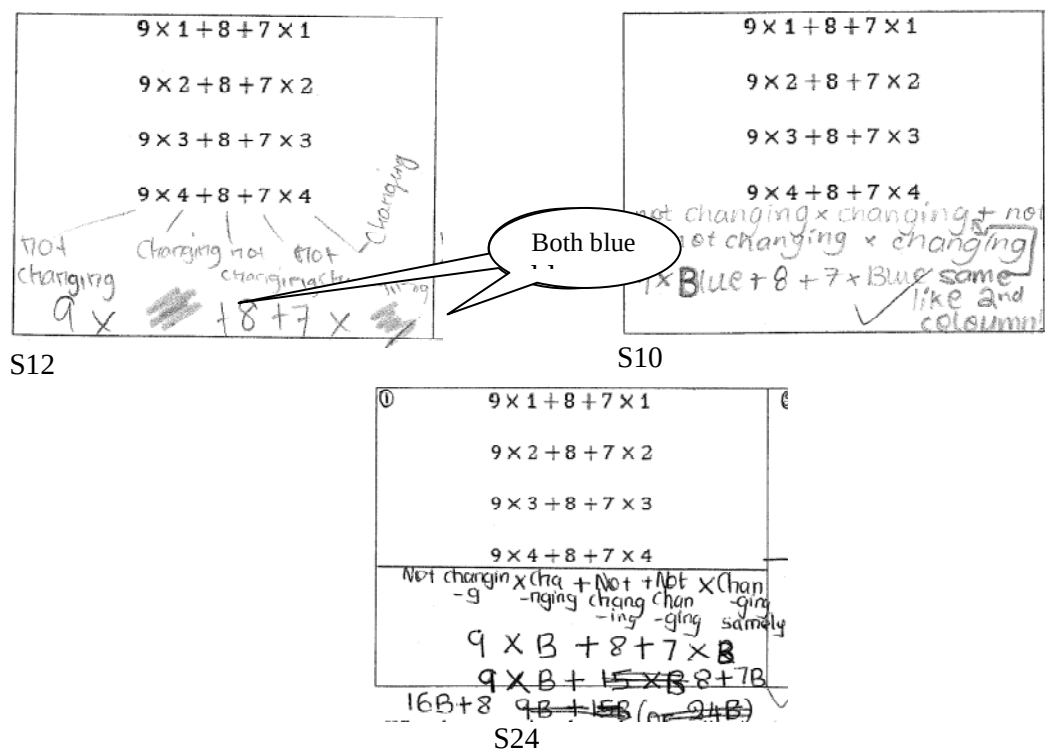


Figure 4. Students' work showing an appreciation of variation and invariance.

Questions 5 and 7 addressed whether students could see or use letter as specific unknown or generalised number. In the event 21 (80.8%) of them did in Q5a) and 69.2% in 5b), with just 2 (7.7%) unable to do so in 5a) and none in 5b). Many of the responses in 5a) showed that they had the idea of a general number and some students also gave possible values for the number (some only gave possible values): S18 "P is a general number it can mean any number like 600 so it could be $600+5$ "; S10 "any number $+5$, unknown number $+5$ "; S13 " $P+5$ is a generalisation, $1+5$ $3+5$ $5+5$ $8+5...$ "; S22 " $P+5$ is an expression, P can be any number so is called a general number". For 5b) student answers demonstrated they understood that the generalised numbers G and B can vary differently, writing: S24 "It is an expression. G and B are changing differently"; and S1 " G can equal any number along with B ". However, this did lead S17 to the common wrong conclusion that G and B can not take the same value: "unknown number $- 2 +$ unknown number. G and B can be any number. But G and B can't be the same number."

In Q7, 69.2% were correct and 19.2% wrong, with most correct answers of the form $S+3$, or similar. Questions 10 through 16 targeted students' understanding of repeated addition and multiplication, and their respective notations. The teaching sequence had involved some discussion on, and practice with matchsticks for the diagrammatic representations of numbers such as 3×4 and 3^4 . Most students were successful in writing the correct answers. As Table 2 shows all the facilities were above 69.2%, with five over 88%. Most correct answers were of the form: a^y is $a \times a \times a \dots y$ times. However, despite the practice, there were some students who were still confused, which shows that this is not an easy concept. For example, S6 and S14 wrote 2^y as y times y , and although S16 was successful in writing the correct answer to Q12 where he had to give the meaning of 5^6 , he was unable to transfer the same understanding to 2^y and made the classic error of giving the answer as 2 times y .

Table 2

Question Facilities on Notation for Repeated Addition and Multiplication (N = 26)

Question	9	10	11	12	13	14	15	16
Number Correct	24	21	25	24	24	23	18	18
%	92.3	80.8	96.2	92.3	92.3	88.5	69.2	69.2
Number Wrong	2	4	1	1	1	2	6	6
%	7.7	15.4	3.8	3.8	3.8	7.7	23.1	23.1

This shows that the transition to algebra is not always easy for students and they continue to need extended discussion and practice for some concepts. Two students, S8 and S18, struggled with these questions, with S18 attempting the questions but getting most of them incorrect, and S8 not attempting to answer questions that involved large numbers and symbolic literals. S4's answers to Q15 and Q16 were interesting. For 2^y in Q15 he wrote "2 multiplied by itself an unknown number of times", and for question 16 (What do you understand by a^y ?) he wrote "an unknown number multiplied an unknown number of times". In Q6a(i) and (ii) 15 (57.7%) of the students were able to distinguish between an expression and an equation. S22 justified her answers here as: " $3 \times y$ is an expression as it doesn't have an answer. $3 \times y = 12$ is an equation as it has an answer and therefore y is a specific unknown number". In a similar vein S21 wrote: i) is an expression and ii) is an expression and an answer. This seems to support the well-known idea that for many students the equals sign signals an answer rather than the notion of equality. Some of the students showed a process-oriented preference and hence tried to solve the equation rather than describing it.

In Q6b) 14 (52.9%) of the students successfully described y in $3 \times y$ as a general number and y in $3 \times y = 12$ as a specific unknown, using varying descriptive terms. Some of their responses included: S24 "in i) y is a general number and it can take any values. In $3 \times y = 12$...it can only take a particular number 4" (see Figure 5); S4 "In the expression $3 \times y$ could be anything like 3×2 but in $3 \times y = 12$ it HAS to equal 4 or it will make no sense. In ii) y is a specific unknown number and it can only take a particular number 4". S3, S7, S11, (see Figure 5), S14 and S20 also saw y in i) as an unknown or a general number and y in ii) as a specific unknown (meaning a particular number). Of the remaining students, S6, S10, S12 and S16 ignored i) and solved ii) successfully. S2, S5, S8, S23 and S25 simply ignored the question altogether. S23 was one of the students who was attending ESOL classes and so it is possible that he was hindered by language difficulties.

S3

in $3 \times y$, y is an unknown number and in $3 \times y = 12$,
 y is a specific unknown number. ✓

S7

In $3 \times y$, y means a general number. In $3 \times y = 12$, y means a specific number. ✓

S11

$3 \times y$ = because y is a general number and take any value
 $3 \times y = 12$ because it's a specific number and only that number.

S24

In $3xy$, y is a general number and it can take any values. In $3xy = 12$, y is a specific unknown number and it can only take a particular number 4. ✓

Figure 5. Students' show appreciation of generalised number and specific unknown. Conclusion

Sfard (1995) maintains that a study of history provides us with opportunities to understand student difficulties and also ways to overcome those difficulties. One such lesson we learn from history is that for the students of today a clear understanding of the notation and concept of letter as variable is difficult, but is nonetheless vital for progress in algebra. In their paper, Puig and Rojano (2004) propose that a study of history of algebra reveals how, historically, substantial progress in algebra (developing general methods of solutions of equations) was only made once a clear terminology for different variables and powers of letters was developed. In Europe this terminology was constructed by Viète (16th century) who developed general methods for solutions of equations, using letters to refer to coefficients in polynomials instead of numbers. However, Indian mathematicians were able to develop general methods for solutions of equations by the time of Brahmagupta (7th century) when a clear notation for different variables and powers of variables existed. It is possible that the decimal number system and notation that arrived in Europe from India via the Arabs played a significant part in that process, since its firm foundation in India, including general methods of operations on zero and integers, seems to have paved the way for a transition to algebra there.

The results of this study show that a method of observing patterns and using colours as symbols to note the variation and invariants was accessible to most students and they were able to classify them with varying degrees of success. This supports Mason's (Mason et al., 2005) contention that observing patterns and classifying/generalising may be inherent in children. Even students who struggled with many of the concepts were able to experience a certain measure of success with questions 1 to 4. In spite of this students need their attention guided to assist with identifying variation. Responses to questions 5, 6 and 7 revealed students conceptual understanding of specific unknown and generalised number, and the ability to distinguish the two, in expression and equation contexts. The fact that around 69% of the students were able recognise them and show understanding of the difference between the two is notable, especially given that Küchemann's (1981) large-scale study reported only 17% of 13 year-old students able to use letter as specific unknown and less than 2% could deal with letter as generalised number. These two views of the literal symbol, as stated by Radford (1996), are critical for progress in algebra. If students are to make sense

of the many uses of letters in their high school algebra learning then they have first to make sense of these fundamental uses as generalised number and specific unknown. This is in agreement with the findings of Puig and Rojano (2004), whose study of the history of algebra identified conceptual understanding of names/letters for different unknowns and powers of unknowns as two vital categories for progress in algebra. In summary, it appears that combining historical ideas with modern didactics may reveal novel approaches to understanding of specific unknown, generalised number, and notation for variables and powers in algebra.

References

- Cajori, F. (1919). *The history of mathematics*. New York: The Macmillan Company.
- Datta, B., & Singh, A. N. (2001). *History of Hindu mathematics* (Vols. 1-2). Bombay, India: Asia Publishing House.
- Fauvel, J., & van Maanen, J. (Eds.). (2000). *History in mathematics education: The ICMI Study*. Dordrecht, The Netherlands: Kluwer Academic.
- Graham, A., & Thomas, M. O. J. (2000). Building a versatile understanding of algebraic variables with a graphic calculator. *Educational Studies in Mathematics*, 41, 265-282.
- Harper, E. (1987). Ghosts of Diophantus. *Educational Studies in Mathematics*, 18, 75-90.
- Joseph, G. G. (2011). *The crest of the peacock: Non-European roots of mathematics*. Princeton, NJ: Princeton University Press.
- Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.
- Küchemann, D. E. (1981). Algebra. In K. M. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102-119). London: John Murray.
- MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics*, 33(1), 1-19.
- Mason, J. (2004, July). *Doing $\neq$ construing and doing + discussing $\neq$ learning: The importance of the structure of attention*. Paper presented at the ICME10 Conference, Copenhagen, Denmark. Retrieved May 20, 2010, from <http://math.unipa.it/~grim/YESS-5/ICME%2010%20Lecture%20Expanded.pdf>.
- Mason, J., Graham, A., & Johnston-Wilder, S. (2005). *Developing thinking in algebra*. London: Sage.
- Puig, L., & Rojano, T. (2004). The history of algebra in mathematics education. In K. Stacey, H. L. Chick & M. Kendal (Eds.), *The future of the teaching and learning of algebra: The 12th ICMI study* (pp. 189-223). Dordrecht, The Netherlands: Kluwer Academic.
- Radford, L. (1996). Some reflections on teaching algebra through generalisation. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 107-111). Dordrecht, The Netherlands: Kluwer Academic.
- Schoenfeld, A. H., & Arcavi, A. (1988). On the meaning of variable. *Mathematics Teacher*, 81(6), 420-427.
- Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behaviour*, 14, 15-19.
- Srinivasan, P. K. (1989). *Algebra for primary school: From class three: Three months correspondence course*. Unpublished paper, Madras, India.

Mathematics Anxiety in Secondary School Students

Lay Keow Ng

National Institute of Education, Singapore

<ng_lay_keow@moe.edu.sg>

The detrimental consequences of mathematics anxiety have repeatedly been evidenced empirically, yet little is known of its influence on secondary school students in Asia. This study thus examined its origins and impact on 294 secondary students in Singapore through interviews and surveys. Results revealed an average anxiety level of 44% and a negative correlation with achievement. Of the top 5 situations that worried students, 4 were test-related. Nonetheless, highly anxious students continued to persevere and enjoy the subject.

Mathematics anxiety has consistently been proven to be debilitating on mathematics achievement (Foong, 1987; Hembree, 1990; Ma, 1999; Preston, 2008). Research has found it to be a learned behaviour, often arising early in one's educational experience and once it lays roots, its damaging effects will last through the school years.

However, research on Asian students is scarce as the majority of the studies are conducted in the United States (Ho et al., 2000). Moreover, most researchers had worked with high school (10th, 11th, and 12th graders) or college students, rather than those in the lower grades (Wigfield & Meece, 1988). Hence, this study seeks to diagnose how fearful secondary students in Singapore are towards mathematics, to examine its effects on their mathematics performance, and to surface causes for their anxieties.

Furthermore, it is interesting to investigate this phenomenon in Singapore, due to its unique education system and Asian culture. As early as primary 4 (grade 4), students are ability-banded by four subjects, including mathematics and in primary 6, mathematics again contributes to their Primary School Leaving Examination (PSLE) score (maximum aggregate of 300). This will then be used to stream students into the express (EXP), normal academic (NA), or the normal technical (NT) course in secondary schools, where the academically-abled students will enter the express stream. At the end of secondary 2 (grade 8), students are ranked, with double merit given to their mathematics results. Those who excel can then opt to take on advanced subjects or are offered more subjects. At secondary 4 or 5, all students will sit for the national examinations, where again, performance in mathematics will determine their post-secondary education paths.

As such, people in Singapore place great emphasis in academic performance, especially on mathematics. As Ho et al. (2000) noted, this familial and societal stress placed on students in Asian countries may produce a big number of students who are extremely apprehensive about mathematics. Hence, this study is to extend the efforts of researching on mathematics anxiety at this part of the world, and to address three questions.

RQ1. How mathematics-anxious are the students in secondary schools?

RQ2. How significant is the relationship between the mathematics achievement scores and the mathematics anxiety scores of the students?

RQ3. What are some causes of mathematics anxiety in secondary schools' students?

Review of Literature

Relevant research started in the 1970s, with Richardson and Suinn (1972, as cited in Preston, 2008, p. 230) as the first to define it as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations". Since then, others have continued

their efforts driven by research evidence that not only does mathematics anxiety inhibit one's ability to perform mathematically (Preston, 2008) but it is highly probable to have originated from classroom experiences too (Newstead, 1998).

Effects of Mathematics Anxiety on Mathematics Performance

Mathematics anxiety has been continually found to correlate negatively with students' mathematics performance, with correlations ranging from $-.11$ to $-.36$ (Ho et al., 2000). Hembree (1990) analysed 151 relevant studies and reported an average correlation of $-.34$ for school students whereas Ma (1999)'s meta-analysis of 26 studies revealed a correlation of $-.27$. Similar results were also reported in Asia, like in China, Taiwan, and Singapore.

Chiu and Henry (1990) conducted a study with 562 children from grade 4 to grade 8 and found the correlation between anxiety and performance to be significant and negative. Interestingly, although three rating scales were used, only one was given to the entire sample. Moreover, it was odd that semester grades were not collected from all the students.

In 1987, Foong carried out an exploratory study with 206 secondary four express female students from a single Singapore school. She reported that both mathematics anxiety and test anxiety correlated negatively with mathematics achievement but the relationship between mathematics anxiety and test anxiety were positive and moderately strong. However, her research was conducted more than two decades ago and the instruments were all self-reported Likert-scaled questionnaires that were unable to locate students' sources of anxiety. Hence there exist this pertinent need to assess the mathematics anxiety level in students and its association with mathematical performance.

Causes of Mathematics Anxiety

The most frequently cited cause is the teacher, identified by Foong (1987) as the main source of students' tension. Highly-tensed students dread presenting solutions in front of their classmates, viewing such situations as threatening (Ashcraft, 2002). Teachers who complained of insufficient instructional time might resort to preparing their students for assessment rather than for understanding. This creates more tension when students encounter unconventional problems or when the mathematics becomes more advanced. Researchers also claimed that anxious teachers spend lesser time teaching mathematics and are more likely to pass their phobia to their students (Hembree, 1990; Ma, 1999).

Like teachers, parents could also pass their dread of mathematics to their children (Hembree, 1990). Parents who are overly-concerned about results end up pressurizing their children, more so in Asian countries. Then there exists this myth that mathematical ability is inborn or hereditary (Godbey, 1997). Others believed that females are weaker in mathematics even though researchers (Hembree, 1990; Ho et al., 2000) have found that though females tend to be more anxious, they are not necessarily weaker in mathematics.

Studies on the impact of teaching methods have been inconclusive. Norwood (1994, as cited in Preston, 2008) argued that traditional methods intensified students' anxieties though he found that college students who were weak in mathematics were more at ease with lecture-based teaching. Newstead (1998) claimed that students were more nervous working in groups and Preston added that as most teachers were recipients of direct instruction, they might not enjoy teaching in the constructivist way.

Next, mathematics has been viewed as an inherently difficult subject. Many students are unable to see its practicality and teachers seldom attempt to make the connections. Foong (1987) explained that due to its cumulative and sequential nature, when students missed out something along the way, it is likely that they may never fully comprehend it.

Furthermore, some students have repeatedly performed poorly, leading to loss of self-confidence and increased tension. There are others who believe that mathematics is a measure of their intelligence (Puteh, 2002) and are embarrassed by their inadequate performance. Lastly, student cohesiveness within a class has been found to have a significant positive correlation with mathematics anxiety level (Taylor & Fraser, 2003). The literature review has underlined the multi-faceted and varied nature of the origins of mathematics anxiety, thus supporting the research aim to diagnose our students' anxieties.

Methodology

This study examined the mathematics anxiety of students from a secondary school in Singapore, which offers all the three courses, the EXP stream (PSLE aggregate of 202 - 227), the NA stream (174-194), and the NT stream (117-154).

Samples

294 students from four Secondary Two EXP classes (155) and four Secondary Three EXP classes (139) participated in a survey that rates their mathematics anxiety level. The Secondary Two EXP classes are named as 2E1, 2E2, 2E3, and 2E4, where 2E1 is considered the most academically-inclined class. The same goes for the Secondary Threes, except that for this particular year, 3E1 and 3E2 were matched academically. The Secondary Ones were excluded as their recent transition from primary to secondary level might have added extra stress on them resulting in a higher than usual anxiety level. Furthermore, it was impossible to retrieve their PSLE scores. The Secondary Fours were also not involved as the fourth class had only 18 students. Moreover, it is possible that their anxiety level will be influenced by the year end national examinations.

Six students (students A to F) were chosen for a face-to-face interview and another six (students G to L) attempted the interview questions in writing. They were selected from my three classes, a Secondary Three EXP class, a Secondary Four EXP class, and a Secondary Four NA class. Students from the two Secondary Four classes did the questionnaire and were selected based on their scores. All the 12 students chosen had high anxiety levels of more than 55%, falling within the 75th percentile. 10 had failed their mathematics last year, whereas two of them (Secondary Threes, A and G) did well.

Instruments

Questionnaire. A questionnaire was designed to measure how anxious the students were and to gather reasons for their worrisome thoughts. The three measurement scales used as reference were the Mathematics Anxiety Rating Scale (MARS), the Mathematics Anxiety Scale (MAS), and the Math Anxiety Questionnaire (MAQ), where MARS and MAS are acknowledged to be the most widely-used instruments (Preston, 2008).

MARS, a 98-item instrument created by Richardson and Suinn in 1972, was the first designed to measure mathematics anxiety. It was however not intended for use with school students and its lengthiness made it impractical too. Hence, researchers preferred the 10-item MAS (Godbey, 1997). More recently, Wigfield and Meece (1988) constructed the MAQ that had seven items to measure the emotionality component and four for the cognitive part, which relates to the worrisome nature about having to do well. As advocated by Alexander and Cobb (1984), mathematics anxiety includes the uneasiness felt when sitting for tests or when receiving test scores. As such items were included in MAQ and it was designed for children from grade 6 to grade 12, making it a suitable reference.

The questionnaire was approved by the school's mathematics level head and reviewed by a mathematics professor in the National Institute of Education. Improvements made included the deletion of uncommon situations in secondary schools, like doing long numerical division. Modifications were also made to terminology to make it more appropriate for local use. The final version had 37 items on a 4-point Likert scale, from zero (not anxious) to three (very anxious) and a free-response question to elicit causes of anxiety. The first 16 statements focused on reactions or thoughts about mathematics and the remaining items were mathematics-related tasks where students rate their anxiety level.

Interview. The semi-structured interview was to have an in-depth understanding of the emotions and perceptions held by highly anxious students. Guiding questions were adapted from two master's theses. Johnson (1997) investigated freshmen's mathematics anxiety and examined the potential of a quiz intervention in reducing anxiety, whereas Thjisse (2002) did a case study on five eighth graders put through an intervention programme, to address both their mathematics achievement and mathematics anxiety. In order to ensure that the questions were manageable and the language used was appropriate, a pilot interview was carried out with a Secondary Four EXP student, of moderate anxiety (50th percentile).

Results and Discussion

RQ1. How mathematics-anxious are the students in secondary schools?

Table 1

Distribution of Students' Mathematics Anxiety Level

Mathematics anxiety, MA (%)	No. of Secondary 2 EXP students	No. of Secondary 3 EXP Students	Total no. of students	Percentage of students
$0 < MA \leq 25$	31	14	45	15.3
$25 < MA \leq 50$	67	73	140	47.6
$50 < MA \leq 75$	47	45	92	31.3
$75 < MA \leq 100$	10	7	17	5.78

The survey attained a high response rate of over 85%. The maximum score was 111, and each raw score was represented as a percentage. The mean anxiety level was 43.8%, with the Secondary Threes (45.1%) slightly higher than the Secondary Twos (42.7%). From *table 1*, almost half of the sample was within the 25% to 50% level of anxiety. Although slightly over one-third had more than 50% level of anxiety, it was somewhat relieving to see that only 5.78% falls in the upper quartile. Survey results indicated that mathematics anxiety seem to distress a considerable group of students, thus justifying the need for similar studies in order to bring more awareness to the education fraternity.

Table 2 showed the top five situations that are most stressful for students, where four were related to mathematics tests. Almost 40% indicated that they are very nervous when waiting to receive their marked papers. This also illuminates the test anxiety felt by our students and the undue stress on them to do well in mathematics (Ho et al., 2000).

Table 2
Scores of the Five Items with the Highest Level of Mathematics Anxiety

Rank	Item Description	Mean Score	No. of students who scored 3
1	Waiting to get back my math test paper.	1.966	114
2	If I am absent from school and I miss a math lesson, I worry that I will fall behind.	1.953	90
3	Waiting to get back a math test in which I expect to do well.	1.941	105
4	Being given a “surprised” math test that I was not told about.	1.938	105
5	Thinking about the final math examination in a math class.	1.936	118

RQ2. How significant is the relationship between the mathematics achievement scores and the mathematics anxiety scores of the students?

A moderately strong negative correlation (-0.571) between achievement and anxiety was obtained, using students’ previous year, end-of-year mathematics examination results. The Secondary Twos (-0.595) reported a slightly stronger correlation than the Secondary Threes (-0.55). These findings coincided with those from relevant studies, indicating that the highly-anxious students are probably not performing in mathematics.

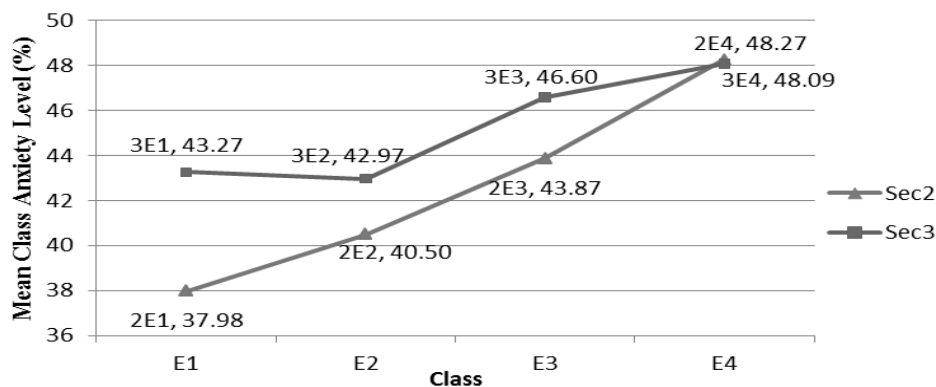


Figure 1. Mathematics anxiety level of each class.

Figure 1 suggested that students in better classes have lower mathematics anxiety level, bearing in mind that 3E1 and 3E2 are matched in academic abilities. Another observation is that the level of anxiety seems to be higher for Secondary Threes rather than Secondary Twos on average. However, more needs to be done to ascertain the relationship between class rankings and the mean class anxiety level.

RQ3. What are some causes of mathematics anxiety in secondary schools’ students?

The causes of mathematics anxiety gathered can be placed in the following categories, in decreasing order of intensity: assessments, personal factors, nature and perception of mathematics, negative experiences, learning environment, and parents’ effects.

- I do not like to know the remaining time left for my exam paper, as my hands will start to tremble and will try to rush through the paper.

- I do not like to receive my ^{exam} paper as I am afraid that I would fail.

Sometimes in an exam, the questions are very difficult and time is running out so I would feel ~~more~~ nervous. Sometimes when revising, also might worry for the test next day.

Figure 2. Mathematics anxiety brought about by tests or examinations.

Assessments. This was the most frequently cited reason (34.7% or 179 out of 516) in item 38, and it tallied with the data in *table 2*. Many were nervous during tests and were fearful of performing badly (*Figure 2*). Others were concerned that their academic outcomes will impede their future. *C* shared that whenever she sits for tests, she will keep checking the time and will be perspiring due to nervousness. *E* was bothered by her constant failure and was worried about the national examinations.

I have a little phobia about learning mathematics. I ~~also~~ never did well in mathematics, thus making me lose confidence time and time again. I cannot handle complex questions even though the working method is actually very simple.

I feel that I can't catch up sometimes and I am particularly weak in this subject.

I'm slow. ~ Student J

Figure 3. Self-induced mathematics anxiety.

Personal Factors. Mathematics anxiety also stems from within the individual (128) (*Figure 3*). For instance, some wrote that their inability to solve problems troubled them and *F* was vexed that she felt stupid during mathematics lessons. Low self-efficacy is also a factor as these people tense up and give up easily. Others who benchmarked their performance with classmates, like *A*, are constantly concerned about their class ranking.

It is very difficult to get it correct, with a many different formulas for a question that can be used. Math requires a lot of common sense and needs a high IQ.

The never ending amount of formulas to be memorised ~ Student I

Too much formula. Too much methods. Just difficult to solve. ~ Student K

Algebra really makes my maths result drop.

Figure 4. Mathematics anxiety related to the nature of mathematics.

Nature of mathematics. Some 73 students find mathematics very computational and difficult (*Figure 4*). Others are troubled by what is known as abstraction anxiety which stems from more abstract mathematics topics taught in the middle grades (Ferguson, 1986).

A number of students believed that algebra is the root cause of their panicky nature, and they are perplexed by the countless symbols and formulae.

Negative experiences. Constant failure in mathematics strikes fear in some students especially those from the weaker classes (67). When *C* was in primary six, she had many mathematics remediation sessions and often had difficulty completing the heavy load of homework. As a result, she often cried while doing mathematics and frequently fell ill.

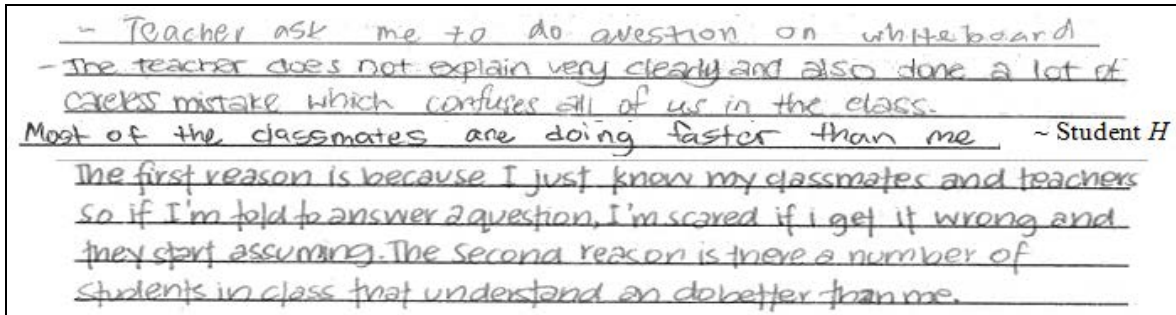


Figure 5. Mathematics anxiety level from teachers, homework and peers

Learning environment. 40 students attributed their fear for mathematics to teachers. Students wrote of teacher actions such as teaching too fast, explaining poorly, or scolding them for doing the questions wrongly. *F* disliked presenting her solutions in class, as she seldom could finish her class work. Homework is an added source of stress (17). *B* admitted that she always copied her classmates' work because she did not know how to do the questions. Seven students wrote that they were hesitant to ask questions as they were worried that their classmates will look down on them (Figure 5). *E* felt especially so when the rest seems to have no issue and was afraid of being blamed for wasting class time.

Parents' effects. Only five linked their anxiety to parents. *A* said that his parents wanted him to do well and to enter a junior college, perhaps because he is the only child.

In consolidation, over a third of the causes were test-related where quite a number were obvious symptoms of test anxiety, either physically, like sweating or shaking, cognitively, like negative self-talk, or emotionally which might lead to low self-esteem and hopelessness, the next highest cited category. This finding is consistent with what have been reported on the positive correlation between mathematics anxiety and test anxiety (Ashcraft, 2002; Foong, 1987). It also illuminated the importance that students attached to the necessity to perform well academically. Other reasons like the abstract nature of mathematics and negative prior experiences were in accord with similar studies on anxiety. Although only 40 explicitly mentioned teachers as their source of discomfort, it is undeniable that teacher actions and decisions are intricately linked to other causes.

What was worth mentioning was that it was encouraging to find that all the 12 students interviewed still like mathematics and would continue trying. Most were certain that spending time on mathematics helps to reduce their stress level. *I* commented that it will definitely boost his confidence, *K* replied "Of course! Practice makes perfect!" and *B* said that she will then improve and will not be looked down on by the class.

Conclusion

This study has informed us of the pervasiveness and roots of mathematics anxiety in secondary school students, an area which clearly needs more attention in, especially in

Singapore. Despite limitations like taking samples from a single school and stream or having the mathematics teacher of the students as their interviewer, it nevertheless underlined the predominant factors that interfere with students' mathematics learning, with the majority been school-related. In view of this, the key implication is for teachers to create safe learning environments and to help students develop positive attitudes, since those who are less anxious are more likely to succeed in mathematics.

References

- Alexander, L. & Cobb, R. (1984, November). *Identification of the dimensions and predictors of math anxiety among college students*. Paper presented at the annual meeting of the Mid-South Educational Research Association, New Orleans, LA. Retrieved from ERIC database. (ED251320)
- Ashcraft, M. H. (2002). Math anxiety: Personal, educational and cognitive consequences. *Current Directions in Psychological Science*, 11(5), 181-185.
- Chiu, L. & Henry, L. (1990). Development and validation of the mathematics anxiety scale for children. *Measurement & Evaluation in Counselling & Development*, 23(3), 121-127.
- Ferguson, R.D. (1986). Abstraction anxiety: A factor of mathematics anxiety. *Journal for Research in Mathematics Education*, 17(2), 145-150.
- Foong, P. Y. (1987). Anxiety and mathematics performance in female secondary school students in Singapore. *Asia Pacific Journal of Education*, 8(2), 22-31.
- Godbey, C. (1997). *Mathematics anxiety and the underprepared student*. Retrieved from ERIC database. (ED426734)
- Hembree, R. (1990). The nature, effects, and relief of mathematics anxiety. *Journal for Research in Mathematics Education*, 21(1), 33-46.
- Ho, H. Z., Senturk, D., Lam, A. G., Zimmer, J. S., Hong, S., Okamoto, Y., Chiu, S. Y., Nakazawa, Y., & Wang, C. P. (2000). The affective and cognitive dimensions of math anxiety: A cross-national study. *Journal for Research in Mathematics Education*, 31(3), 362-379.
- Johnson, C.E. (1997). *Attitude or anxiety: Mathematics disposition of high school algebra students*. Unpublished master's thesis, Wichita State University, Kansas.
- Ma, X. (1999). A meta-analysis of the relationship between anxiety toward mathematics and achievement in mathematics. *Journal for Research in Mathematics Education*, 30(5), 520-540.
- Newstead, K. (1998). Aspects of children's mathematics anxiety. *Educational Studies in Mathematics*, 36(1), 53-71.
- Preston, R. (2008). *Mathematics anxiety: Research and implications for middle school students and teachers*. Conference Proceedings of the Masters in Teaching Program 2006-2008, Olympia, Washington.
- Puteh, M. (2002). *Factors associated with mathematics anxiety*. Malaysia : Universiti Pendidikan Sultan Idris.
- Taylor, B. A., & Fraser, B. J. (2003, April). *The influence of classroom environment on high school students' mathematics anxiety*. Paper presented at the Annual Meeting of American Educational Research Association, Chicago, IL. Retrieved from ERIC database. (ED476644)
- Thijsse, L.J. (2002). *The effects of a structured teaching method on mathematics anxiety and achievement of grade eight learners*. Unpublished master's thesis, University of South Africa, South Africa.
- Wigfield, A. & Meece, J. L. (1988). Math anxiety in elementary and secondary school students. *Journal of Educational Psychology*, 80(2), 210-216.

Students' Summaries of Mathematical Lectures: Comparing the Discourse of Students with the Discourse of Lectures

Magnus Österholm

Umeå University, Sweden & Monash University, Australia

<magnus.osterholm@monash.edu>

This study focuses on a distinction between process- and object-oriented discourses when characterising the discourse of university students' summaries of lectures and examining connections between students' discourse and the discourse of lectures. Results show that students' discourse in general tends to be process-oriented, by their use of active verbs and little use of nominalisations. Students' summaries of process-oriented lectures also tend to be more process-oriented, but the differences between individual students are larger than differences caused by variations of the discourse in the lectures.

Learning, or at least one aspect of learning, can be seen as enculturation into a specific type of discourse. This process can occur through explicit teaching about properties of the discourse, for example as Mercer (1996) describes the teacher's role to give instructions about ways of questioning and arguing. However, the process can also occur through more implicit manners, by seeing a teacher's discourse as a model for students to follow, which for example could be one component when Rodd (2003, p. 15) describes lectures as important for "inducting students into their university's mathematical community".

In this paper, focus is on implicit aspects of mathematical discourse by examining lectures with varying properties of the discourse, which are not explicitly highlighted for the students, and studying students' summaries of these lectures in order to compare the discourse of students and the discourse of the lectures.

What are the central properties of mathematical discourse that students need to adapt to? This is a difficult question since it seems impossible to generally characterize mathematical discourse. For example, Burton and Morgan (2000) show that there is large diversity in the writing in mathematical research articles. The property of mathematical discourse of interest in this paper is a distinction between process- and object-oriented discourses. The reason to focus on aspects of processes and objects is that this distinction can be relevant in several different ways for mathematical discourse, in particular from the following two perspectives: First, when focusing on mathematical learning, aspects of processes and objects are central in several different frameworks (see Tall, Thomas, Davis, Gray, & Simpson, 2000), where a central aspect of learning is the process of objectification or reification (i.e., to create an object from something first seen as a process). Second, when focusing on the readability of texts, the aspect of nominalisation is sometimes discussed, that is, the use of nouns created from other types of words, as in the noun *description*, created from the verb *describe*. Empirical results sometimes show a negative effect on recall (Spyridakis & Isakson, 1998) and on reading speed (Bostian, 1983) for texts that have a more nominal style (i.e., include more nominalisations).

The first perspective above could be seen as focusing on content while the second on form, but it is not necessarily that easy to separate form from content. For example, as Morgan (1999, p. 56) notes concerning: (a) the use of passive voice, that it is "not solely a choice about form but is also a choice about whether to represent or to obscure the role of the agent in the process", and (b) nominalisations, that it is "not simply a different way of expressing the same meaning but actually allows the writer to say new things". In this paper,

I do not focus on differentiating between form and content but I examine more generally properties of discourse, regarding aspects of processes and objects.

Process- and Object-oriented Discourses

A framework of process- and object-oriented discourses (Österholm, 2011a) is used in this study as the basis for the creation of different lectures and for analysing students' summaries of the lectures. *Four aspects* are used that each describes differing degrees of process- and object-oriented discourse. These aspects are labelled and shortly described based on an imaginary shift from process-oriented towards object-oriented discourse:

1. *Direct nominalisation*: A verb is replaced with its corresponding noun, for example when “we use the same symbols” is replaced with “the usage of symbols is the same”.
2. *Indirect nominalisation*: A verb is replaced with its corresponding adjective, for example when “the series converges” is replaced with “the series is convergent”. This aspect is seen as a shift from process-orientation towards object-orientation since the adjective puts focus on a (static) property of an object.
3. *Voice*: An active verb is replaced with its passive counterpart, for example when “we can calculate this” is replaced with “this can be calculated”. This aspect is seen as a shift from process-orientation towards object-orientation since voice is “a choice about whether to represent or to obscure the role of the agent in the process” (Morgan, 1999, p. 56).
4. *Structuralisation*: A process/procedural type of verb is replaced with a structural type of verb (i.e., a form of “to be”), for example when “we do the same as before” is replaced with “it is the same as before”. This change makes an activity/event into a static situation/property, thus seen as a shift from process-orientation towards object-orientation.

It is acknowledged that there can be also other aspects that can be seen as making the discourse more or less process- or object-oriented, but this study focuses on these four aspects, and a plan is to include more aspects in future studies.

Purpose

The study presented in this paper is one part of a pilot study that takes a student perspective on different types of mathematical presentations. The data collection procedure for the whole pilot study included several tasks and activities for students, with the purpose to function as a pilot study for several different foci, for example to examine students' quantitative evaluations of different types of presentations (Österholm, 2011a) and to examine their comments about oral and written presentations (Österholm, 2011b). However, for the purpose of the present paper, focus is on the discourse of students' descriptions of the content of oral presentations (i.e., lectures).

The main purpose of the study presented in this paper is to increase our knowledge about relationships between the mathematical discourse of students and the mathematical discourse of teaching situations, in particular lectures.

This topic is of interest for several reasons, some already highlighted in the introduction. A mismatch between the discourse of students and the discourse of teaching situations can be seen as a potential source of lack of communication and hindrance for students' learning, in particular if properties are not of mathematical necessity, since this can “be one of the sources of disaffection for some students” (Morgan, 1999, p. 55). This issue can be seen as a

generalisation of the notion of readability. Furthermore, it is of interest to examine if and how students adapt to the discourse in teaching situations, since such a (potential for) change of discourse can be seen as an essential part of learning.

More specifically, the present study focuses on the following two research questions:

1. When describing mathematical lectures, is the oral discourse of students mostly process-oriented or object-oriented?
2. Is the discourse of students' oral descriptions of mathematical lectures of the same kind, regarding process- or object-orientation, as the discourse of the lecture?

The first research question is of interest in relation to different types of perspectives and results: On the one hand, scientific discourse in general, but perhaps in particular mathematical discourse, seems to be more object-oriented (Bostian, 1983; Morgan, 1999). Perhaps students have adapted to such properties of discourse, in particular at university level, which is examined in this study. On the other hand, (a) there is evidence that students mainly need to focus on more procedural aspects in mathematics, through imitative reasoning, both in school (Palm, Boesen, & Lithner, 2011) and also at university level (Bergqvist, 2007), which could be seen as connected to process-orientation of discourse, and (b) oral discourse is generally less object-oriented than written discourse, at least when it comes to nominalisations (Einarsson, 1978). These two issues (i.e., a and b) could make it reasonable that students' discourse is more process-oriented in general.

The second research question is of interest in relation to some results I have previously presented (Österholm, 2011a). These results are based on analyses of different parts of the same data material used in the present study, and show that students seem to react to the differences in discourse examined here, regarding process- and object-orientation. In particular, when quantitatively evaluating different presentations, there is a tendency among the students that "process-oriented discourse was preferred for oral presentations, [while] object-oriented discourse was preferred for written presentations" (Österholm, 2011a, p. 367). Therefore, it is of interest to examine whether these types of differences in discourse also affect the students in other ways, in particular concerning their own discourse.

Method

Data Collection

A total of 22 Swedish university students participated in this study, all enrolled in the same calculus course. The students worked in pairs, and first one student watched and listened to a pre-recorded lecture (16-17 minutes long). The student could take notes, in order to imitate an ordinary lecture. After listening to the lecture, the student described the content of the lecture to the other student. Instructions were given that the student who describes the lecture should not show her/his notes but describe as much as possible orally. However, it was allowed to write symbols or draw pictures during the description.

Thereafter, the procedure was repeated but with changed roles, so that the student who first described a lecture then listened to the other student's description. The two lectures covered different content. One lecture was about sequences and included: connections between sequences and functions, different ways to describe sequences (with explicit and recursive formulas), convergence of sequences, and examples of calculating limits for sequences described with explicit and recursive formulas. The other lecture was about Maclaurin polynomials and included: the idea of polynomials as approximations to complicated functions, a derivation of the Maclaurin coefficients for a polynomial of degree five, and an example of calculating an approximate value of the constant e using a Maclaurin

polynomial for e^x . The content of both lectures were scheduled to be covered in the students' calculus course within a few weeks after participating in this study.

The wordings of all lectures were created by first recording and transcribing a spontaneous oral presentation of each topic. Two written versions of each topic were then created: one process-oriented and one object-oriented, where formulations were altered according to the four aspects described above. From these written versions, oral presentations were then recorded, which were used as the lectures in this study. The same person (the author) was being recorded for all different lectures.

Data Analysis

Recordings of students' descriptions were transcribed for further analysis. The focus in this study is on the students' descriptions of the mathematical content presented in the lectures. Therefore, the following parts of the descriptions and conversations between students are excluded from analysis: (a) utterances from the student who are listening to the other student and (b) utterances that only address some kind of level of understanding (e.g., "I think this is the way it is supposed to be") or some practical aspect of either the lecture or own activity (e.g., "he described this really fast" or "let me check my notes").

Each verb, noun, and adjective is analysed by deciding if the word is in a form that is seen as process-oriented (then coded P) or object-oriented (then coded O) according any of the *four aspects*. For example, if the discourse includes the phrase "you can explain this by noting that", this instance of the word *explain* is coded with a P since a possible reformulation is "an explanation of this is noting that", which makes the phrase more object-oriented according to the aspect of direct nominalisation. In this analysis, it is essential that a reformulation is possible, within the specific context where the word is used, which changes the orientation according to a specific aspect, for it to be coded as either P or O. This demand of a potential reformulation is used in order to focus on a type of "choice" that is made between different ways to express oneself, between being more process- or object-oriented. This type of analysis reflects the process when creating the different lectures, where such reformulations were done. Thereby, a direct comparison between properties of students' discourse and of the discourse of the lectures is more reasonable.

A quantitative characterisation of a certain discourse is created by calculating the proportion of occasions coded with a P and with an O, corresponding to how process- and object-oriented the discourse is respectively. Based on this type of quantitative characterisation, each pre-recorded lecture used in this study is either 100% process-oriented or 100% object-oriented. By making this type of quantitative characterisation of students' discourses, it is possible to examine if the discourse is mostly process-oriented or object-oriented (research question 1) and also to compare the discourses between students who listened to different versions of the lectures (research question 2).

This type of characterisation is used in two different ways, in order to give as complete answers to the research questions as possible. First, all coded occasions (all four aspects) are included at once by creating a one-dimensional scale describing a discourse as more process-oriented or more object-oriented. Thereafter, the different aspects are distinguished, in order to examine which aspects are mostly affecting the process/object-orientation of students' discourse. However, it is not always possible to distinguish between all four aspects, since focus in the analysis is on whether a specific part of the discourse can be reformulated into becoming more object-oriented or more process-oriented, and in some situations such reformulations can be done in several ways, which are instances of different aspects. For example, in the example given above, with *explanation* as a direct nominalisation of *explain*, a reformulation could also have been "this can be explained by noting

that”, which makes the phrase more object-oriented according to the aspect of voice, instead of according to the aspect of direct nominalisation. A phrase using the word *explain* can therefore not (always) be coded with a certain aspect, but only as process-oriented. However, it is always possible to code occasions that are more object-oriented with a certain aspect, since the four aspects involve different types of words for object-orientation (nouns, adjectives, verbs in passive form, and forms of *to be*). Furthermore, (some parts of) the aspect of structuralisation can always be distinguished from the other aspects. In particular, two very frequent words are instances of the aspect of structuralisation: forms of *to become* (process-oriented; “att bli” in Swedish) and forms of *to be* (object-oriented; “att vara” in Swedish). In fact, these are the only kind of instances coded as structuralisation in the present study. Therefore, in the present study, instances of structuralisation can always be distinguished from instances of the other aspects.

Statistical analyses and issues of reliability. It would be ideal to characterise the discourse of each individual student, and then to analyse the distribution of these characterisations, in particular when comparing different groups of students. However, due to the pilot study character of this study, the reliability of this type of analysis will be limited, for two reasons. First, students describe quite a short lecture, and their descriptions vary in length between approximately 5 to 15 minutes (and between 19 and 155 occasions are coded for each student). This variation makes at least the shorter descriptions difficult to characterise quantitatively in a reliable manner. Second, only 22 students participated, which can make at least comparisons between subgroups unreliable. Therefore, as a complement to the type of analysis that characterises individual students, an analysis is also performed where the characterisation is done at group level, focusing on groups of students who have listened to different lectures. This analysis consists of adding up all coded occasions within a group of students, and then calculating a quantitative characterisation of this group based on all included occasions. The power of this analysis is not dependent on the number of students but on the number of coded occasions, which are many more, and this type of analysis thus constitutes a better basis for reliable statistical analysis.

Non-parametric methods are used in all statistical analyses, where 0.05 is used as the level of statistical significance. See results section for description of specific methods.

Results

The discourse of individual students tends to be more process-oriented than object-oriented, since all box plots in Figure 1 lie somewhat above 50%. A One-Sample Wilcoxon Signed Rank Test including all students ($N = 22$) also shows that the median of the fraction of process-oriented occasions (60.9%) is significantly different from 50% ($p = 0.000$). Mann-Whitney U Tests show that differences between groups of students who listened to different lectures are not statistically significant, neither when comparing lectures with different discourse ($p = 0.243$) nor lectures with different content ($p = 0.748$).

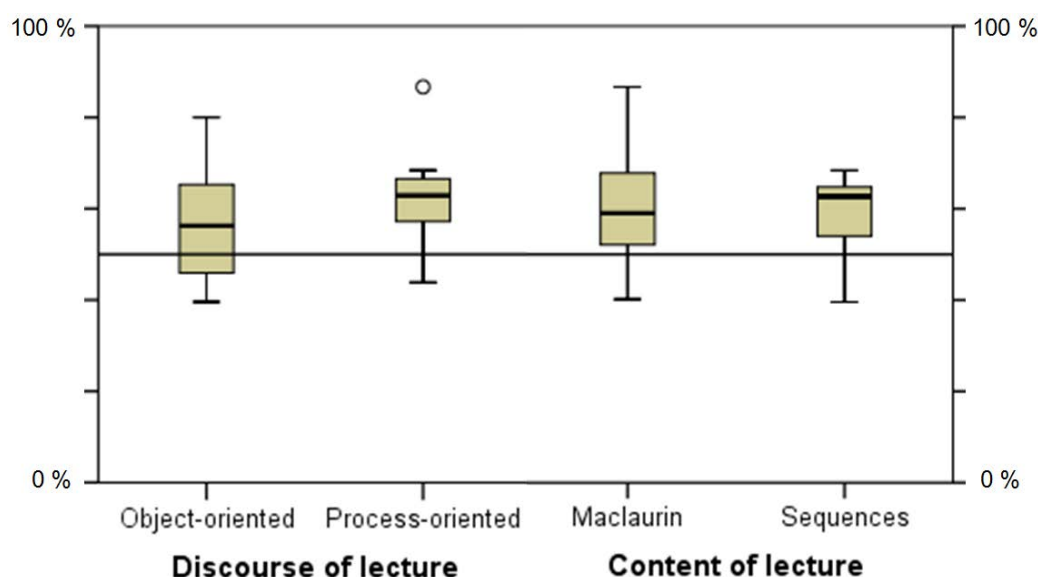


Figure 1. Fraction of process-oriented occasions in individual students' descriptions of different types of lectures.

When not distinguishing between different students, the results in Table 1 below also show that it is more common for the discourse of students to be process-oriented, since 58.2% of all coded occasions are process-oriented. A Binomial Test shows that this fraction is significantly different from 50% ($p = 0.000$). Furthermore, Pearson Chi Square Tests show that descriptions of process-oriented lectures tend to be more process-oriented than descriptions of object-oriented lectures ($p = 0.002$; 61.5% compared to 54.5%) while in general there is no significant difference in the discourse when students describe lectures with different content ($p = 0.458$; 59% and 57.3%).

Table 1

Type of Discourse in Groups of Students' Descriptions of Different Types of Lectures

Lecture	Occasions in students' descriptions		Fraction of process-oriented occasions
	Object-oriented	Process-oriented	
Object-oriented	406 (114 & 292)	487 (414 & 73)	54.5% (78.4% & 20.0%)
Process-oriented	372 (94 & 278)	595 (481 & 114)	61.5% (83.7% & 29.1%)
Maclaurin	397 (116 & 281)	571 (449 & 122)	59.0% (79.5% & 30.3%)
Sequences	381 (92 & 289)	511 (446 & 65)	57.3% (82.9% & 18.4%)
All	778 (208 & 570)	1082 (895 & 187)	58.2% (81.1% & 24.7%)

Note. The two numbers in parentheses refer to occasions of different aspects of process- and object-orientation: the first number refers to the first three aspects and the second number refers to the fourth aspect.

Among all 778 object-oriented occasions, 93 are instances of the first aspect (direct nominalisation), 63 of the second aspect (indirect nominalisation), 52 of the third aspect (voice), and 570 of the fourth aspect (structuralisation). The main distinction can therefore be made between the first three aspects on the one hand and the fourth aspect on the other hand, for which data is given within parentheses in Table 1. Binomial Tests show that

occasions in the first three aspects are more process-oriented (81.1% is significantly different from 50%; $p = 0.000$) while occasions in the fourth aspect are more object-oriented (24.7% is significantly different from 50%; $p = 0.000$). When comparing different lectures, Pearson Chi Square Tests show that:

- Descriptions of process-oriented lectures tend to be more process-oriented than descriptions of object-oriented lectures, both for the first three aspects ($p = 0.026$; 83.7% compared to 78.4%) and also for the fourth aspect ($p = 0.004$; 29.1% compared to 20.0%).
- Descriptions of Maclaurin lectures tend to be more process-oriented than descriptions of lectures about sequences, but only for the fourth aspect ($p = 0.000$; 30.3% compared to 18.4%) while no significant difference exists for the first three aspects ($p = 0.145$; 79.5% and 82.9%).

Conclusions

An answer to the first research question, about characterising students' discourse, includes two parts. First, students' discourse when describing the lectures is in general more process-oriented than object-oriented. However, the variation among individuals is fairly large, where the maximum difference between students is of magnitude 30-40% (see Figure 1) on the scale used in this study. Second, although there is a general tendency for students' discourse to be more process-oriented, there is a difference between different aspects of process- and object-orientation. In particular, the process-orientation of their discourse mainly comes from relatively little use of nominalisations (direct and indirect) and passive voice, while the aspect of structuralisation is more object-oriented in students' discourse, by mainly describing what *is* and not what *becomes*. These results are somewhat contradictory, since students' discourse seems to focus on what happens or what you do (by using mostly active verbs and less nominalisations) but at the same time does not seem to focus on what *becomes* from such events or activities but on what *is* already there. Perhaps this distinction comes from describing your own activities on the one hand (what is being done) and describing the mathematical content on the other hand (what is there and what is true¹). A further analysis of the content of students' descriptions could test this hypothesis.

An answer to the second research question, about connections between students' discourse and properties of the lectures, includes two parts. First, students' discourse tends to follow the discourse of the lecture, regarding the property of process- or object-orientation, also when examining specific aspects of the orientation of discourse. However, this result only appears when not distinguishing between individual students and does not appear when characterising individual students, which might be due to unreliable measures as discussed in the methods section. In addition, the result does not show a *complete* shift in students' discourse when listening to lectures with different discourse but the shift involves 5-9% of the discourse (see Table 1) when examining groups of students and according to the scale used in this study (this is of similar magnitude as the difference between medians shown in Figure 1). Second, an unexpected difference between lectures with different content is observed, but not as clear as for lectures with different discourse. A significant difference is only observed for the aspect of structuralisation, where students' descriptions of the Maclaurin lecture tend to be more process-oriented when compared with the lecture

¹ This can also refer to a more relational, and not operational, perspective on mathematical symbols and concepts, which for example has been discussed and examined for the equal sign (e.g., see Kieran, 1981).

about sequences. Similarly as noted above, a further analysis of the content of students' descriptions could perhaps explain this difference.

In summary, the variation when looking at the process- and object-orientation of discourses of individuals (30-40%) is larger than the shift in students' discourse caused by lectures with different discourse (5-9%). A model of students' discourse can therefore be that the discourse of each individual is a more robust construct, acting as a type of baseline that in a specific situation (e.g., a lecture) can be adjusted to some degree based on the (dominant) discourse of that situation. It is then of interest to examine, more in-depth and longitudinally, situations where the dominant discourse is close to or far away from the discourse of a student, in order to find situations where a shift in discourse is not only an adjustment to a specific situation but if and how an individual's mathematical discourse (i.e., the baseline) also shifts. Using specific aspects of discourse as done in the present study can complement other research that has a similar focus but uses other methods of analysis (e.g., Sfard, 2008). It is also of interest to examine connections between changes in discourse and different measures of students' mathematical knowledge, in order to decide if and how specific properties of discourse, regarding process- and object-orientation, are connected to other aspects of students' mathematical knowledge and learning.

References

- Bergqvist, E. (2007). Types of reasoning required in university exams in mathematics. *Journal of Mathematical Behavior*, 26(4), 348-370.
- Bostian, L. R. (1983). How active, passive and nominal styles affect readability of science writing. *Journalism Quarterly*, 60(4), 635-640, 670.
- Burton, L., & Morgan, C. (2000). Mathematicians writing. *Journal for Research in Mathematics Education*, 31, 429-453.
- Einarsson, J. (1978). *Talad och skriven svenska: Sociolinguistiska studier [Spoken and written Swedish: Sociolinguistic studies]*. Doctoral Dissertation. Lund, Sweden: Ekstrand.
- Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12, 317-326.
- Mercer, N. (1996). The quality of talk in children's collaborative activity in the classroom. *Learning and Instruction*, 6(4), 359-377.
- Morgan, C. (1999). Convention or necessity? The impersonal in mathematical writing. *Proceedings of the British Society for Research into Learning Mathematics*, 19(1), 55-60.
- Palm, T., Boesen, J., & Lithner, J. (2011). Mathematical reasoning requirements in Swedish upper secondary level assessments. *Mathematical Thinking and Learning*, 13(3), 221-246.
- Rodd, M. (2003). Witness as participation: The lecture theater as site for mathematical awe and wonder. *For the Learning of Mathematics*, 23(1), 15-21.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.
- Spyridakis, J. H., & Isakson, C. S. (1998). Nominalizations vs. denominalizations: Do they influence what readers recall? *Journal of Technical Writing and Communication*, 28(2), 163-188.
- Tall, D., Thomas, M., Davis, G., Gray, E., & Simpson, A. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior*, 18(2), 223-241.
- Österholm, M. (2011a). A framework for studying differences between process- and object-oriented discourse. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 367). Ankara, Turkey: PME.
- Österholm, M. (2011b). What aspects of quality do students focus on when evaluating oral and written mathematical presentations? In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics: Traditions and [New] Practices. Proceedings of the AAMT-MERGA conference held in Alice Springs, 3-7 July 2011* (pp. 590-598). Adelaide, Australia: AAMT and MERGA.

Identity and Ethnomathematics Projects in Papua New Guinea

Kay Owens

Charles Sturt University

<kowens@csu.edu.au>

Indigenous students may find mathematics in schools difficult because there is discontinuity between cultural mathematics and school mathematics. One of the reasons for this is that their teacher's identity as a mathematical thinker may not link to their cultural ways of thinking. In Papua New Guinea, there is a subject to assist student teachers to develop their own and hence their students' consonance between cultural and school knowledge. In the subject, student teachers undertake a project to link culture and mathematics. The question for this research is to explore how student-teacher identity as a mathematical thinker is enhanced when they explore the cultural setting of their mathematics. From 239 reports collected over a 10 year period, 60 were analysed to explore the impact of sociocultural contexts on identity as a mathematical thinker. The document analysis informed the argument that such projects encourage teachers to express both culture and school mathematics and to identify with their cultural mathematical ways of thinking and to value these in school education.

Overview

A quote from one of the reports by a student teacher at the University of Goroka (UOG) in Papua New Guinea (PNG) illustrates his beliefs about mathematics and the role of a teacher in identifying and using ethnomathematics (culture-based mathematics).

Why do students find it hard to solve mathematics problem? ... Why is mathematics too hard to comprehend? Is it because mathematics is written using the second language? Or is it that it is not part of the Papua New Guinea culture? These are serious questions parents, teachers, students and even the elites ask. ... To impart what is in the text books to the students is hard for it is not part of their life especially in the society which they come from. (This is) the missing part in the school mathematics curriculum ... (By using) traditional mathematics which is in them which they thought was not mathematics ... only then will the students be able to understand the current mathematics. (Pepeta, 2001, p. 1)

Pepeta was one of the preservice and inservice student teachers (referred to in this paper as teachers) who reported on their projects in a single elective subject *Mathematics, Language and Culture*. Their projects related cultural activities and artefacts to school mathematics curriculum. The document analysis of their reports covered the ecocultural knowledge and values that are at the forefront of the teachers' identities as mathematical thinkers. The implications of this study for Indigenous teacher education are framed in terms of an ecocultural pedagogy. The teacher's culture, language, physical and geographic spaces are recognised as having a continuous impact on spatial and other mathematical thinking (National Research Council Committee on Geography, 2006) through their social identity, responsiveness, and self-regulation (cognitive and affective aspects) (see Figure 1). As a result identity as a mathematical thinker has a sociocultural dimension that affects teaching. Concepts are developed through interactions with other people, through reflection on ideas that at first seem unrelated as may occur through out-of-school and in-school experiences, and through the resultant self-negotiation mediated through practice (Radford, 2006). In the current study, practice is manifest in a defined project.

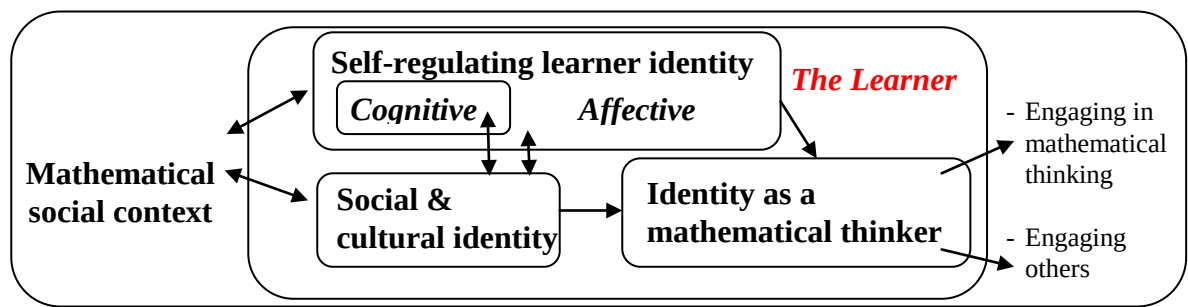


Figure 1. Developing identity as a mathematical thinker (simplified from Owens, 2007/2008).

An Ecocultural Pedagogy of Mathematics

Mathematics is situated in a context whether it is western culture or the long-standing culture of an Indigenous group. For example, variations of spatial concepts as portrayed by sentences and the overall worldview of the Navajo (USA) indicated three basic concepts (a) the essence of movement and the viewing of objects as a slice in time, (b) the sense of volumeness/flatness, and (c) the notion of dimension (but not as an orthogonal grid) (Pinxten, van Dooren, & Harvey, 1983). Based on these basic concepts, Navajo school mathematics expands into a western school topic, topology. However, the recognition of mathematics within the culture requires teachers to make the connections deliberately. In some cases, this is explained in the curriculum documents such as in Alaska with the Yup'ik (Barnhardt, 2007; Lipka & Adams, 2004) and in Northern Territory Australia with the Yolngu (Thornton & Watson-Verran, 1996). The PNG Education Reform recognises that education is best developed from the home culture and community of the students (Litteral, 2001). It is difficult to be specific in the curriculum since PNG has more than 800 language groups having unique logical systems, languages and purposes for mathematical thinking. Thus it is important to establish a teacher identity that incorporates an ecocultural perspective. An ecocultural pedagogy values holistic, integrated knowledge that has changed and accumulated over time and consists of interrelated constructs that continue to develop with thinking, problem solving and experience (Crotty, 1993; Morrison, 2007). The ways of representing cultural resources and maintaining wellbeing are mediated by teachers to reduce the complexity of everyday experiences, respectfully and "without losing site of the rich and dynamic totality of their lives" (González, Moll, & Amanti, 2005, p. 21). A hybridity of this fund of knowledge develops as two-ways or both-ways education (Stanton, 1994) in which there is negotiation of meaning respecting cultural and school mathematics (Ovington, 1994).

Competing values and identities in cultural and school activities require mediating for consonance of knowledge (Bishop, 2002; Lerman, 2001). The mediating tool may be in the mind such as imposed tessellations of triangular figures on plantations of trees (Owens & Kaleva, 2008a). In the current study, the tool used for mediating was a project (described in detail in methodology) to link cultural ways with the school mathematics curriculum and teaching. When connections are smooth, there is a synergy of knowledge and identities (cultural and mathematical) (de Abreu, 2002; Esmonde & Saxe, 2004; Valsiner, 2000). Establishing meaning depends on the ecocultural backgrounds of teachers (Gruenewald & Smith, 2008), no matter how colonised their backgrounds might be (Chinn, 2007). A teacher may value an aspect of knowledge more than another in general or in a specific situation. During transitions from the culturally mediated tools to school mathematics, there is a bridging of the gap to explain practices (de Abreu, 2002). For example, the phrase

centimeter-meter in PNG's creole Tok Pisin was a way of saying "a small unit and a big unit composed of the smaller units" (Owens & Kaleva, 2008a). Language also plays a part when teachers use words from their own languages.

Social identity is the identification with the values, knowledge, beliefs and attitudes of a social group but it is influenced by systemic beliefs that regulate the person (Malmivuori, 2006). This was exemplified in a study of PNG architecture students faced with solving a design problem. They had a pride in their cultural heritage that led them to consider they were good architectural designers (Owens, 1999). Their self-regulation, self-confidence and cultural responsiveness (Owens, 2007/2008, see Figure 1) resulted in unique, beautiful, and balanced paper sculptures that reflected cultural artefacts. Identity is an enactive, dynamic, interactive ever-changing state of being. It is a becoming within a social context (Davis, 1999). The interaction between social context and self is two way and does not occur in a linear fashion but flows back and forth over time and in different aspects of being (Macmillan, 2009).

In PNG, ethnomathematics is barely articulated in the secondary curriculum or textbooks. For this reason, teacher education incorporating the links between language, culture and mathematics will be a predominant way of achieving the stated goal of education to value and maintain cultural knowledge (Papua New Guinea Government, 2003). A similar issue occurs with the Australian Curriculum where culture is mentioned but not elaborated (Australian Curriculum Assessment and Reporting Authority, 2010). The question is how does an ethnomathematics project develop a teacher's identity as an ethnomathematical thinker and prepare a teacher who plans teaching according to an ecocultural pedagogy?

Methodology

UOG has a large collection of 239 reports produced by (preservice and inservice) teachers in *Mathematics, Language and Culture*. The teachers had received a book of readings and lectures on language and ethnomathematics. They were required to describe aspects of their cultures and relate these to the school curriculum by providing lesson plans and examples that illustrated this connection. All reports (held at UOG) were read but 60 reports that gave good details of culture and school mathematics were analysed to see how the teachers identified and related their cultural and school knowledge.

Figure 1 provided a framework for the analysis. Social identity includes the identity as a PNG mathematics teacher within this university and class. The teachers come from many different language groups and all PNG regions (from high forested mountain ranges, coastal plains, river catchments and islands). Many of their parents have minimal school education and speak their own language fluently as do the students. Cultural identity was evident by the students' emotive words when referring to their culture, family, and fellow students, references to themselves as part of the cultural community of practice, the details of their descriptions, and use of their language words.

Identity as a mathematical thinker was evident by their sense of ownership of the cultural and school mathematics as expressed by affective terms, knowledge depth, ability to problem solve through the project such as finding appropriate connections between cultural and school mathematics, and the quality of the written reports. Each of these identifiers has a basis in the self-regulating learner's cognitive and/or affective learning.

The document analysis was part of a larger research project with ethics approval from my Australian University and the host PNG University whose relevant Department had

requested the document analysis and supplied all the documents. The wider project and 15 years of living in Papua New Guinea and an additional 25 years continuing research with Papua New Guinean colleagues has informed my analysis.

Results and Discussion

Cultural identity was strong. Every teacher despite distance from ancestral land or non-use of traditional language proudly gave their language and details of artefacts or cultural activities that involved mathematics. Cultural identity was expressed through the stories in their examples; the affective language that showed pride in the relationships and achievements of their culture, relatives or ancestors.

Dingi (2007) from Gumine District, Simbu Province, showed pride in his cultural heritage and illustrated how the ecocultural context supports not only his cultural identity but his identity with cultural mathematics and hence his identity as a mathematical thinker. He pointed out the dissonance with school mathematics as well as the unconscious tacit knowledge of his people (Frade & Falcão, 2008; Malmivuori, 2006). Dingi explained:

The conditions that people are in determine the type of houses they build. ...The housing traditions ... reflect the social patterns, climate and resources, which exist in a particular area ... an outward sign of community self-reliance and cultural pride. ...The people have been very practical with their knowledge ... but the system of education does not help our students to acknowledge and relate those knowledges to the Western brought school knowledge. However, Ethno mathematics tries to change this perspective. This project looks at investigating the Mathematics that are found in the everyday activities... The people have precise knowledge and understanding of the behavior and characteristics of the materials they use to build their houses ... in consideration with the climatic factors and its durability,... Given the limited range of available building materials, the people exhibit skillful response to the spanning of space ---- the collection of gravitational forces and their transmission to the ground ---- and the need for stability against lateral loads. The basis of the structures of the buildings come as a result of trial-and-error, intuition and from astute observation of structures found in nature (e.g. trees) ... All buildings in my area are constructed by their inhabitants with assistance from close kin who would expect the favor returned. ... Though my ancestors and some of my elderly people did not go to school, they are indeed great mathematicians. In their everyday life, they use mathematics that is taught in the classrooms. In almost all the activities that they do, they unconsciously use mathematics.

The project as a mediating tool engaged Dingi and he was able to identify with both cultural mathematics and scientific thinking (Kahan, 2004) and make connections, thus developing his mathematical identity. Dingi also presented sound examples connected to the syllabus about shapes and measurements related to village houses. "Working on this project gives me greatest insight to see into many ordinary activities that I do and people at home do (that) involve a lot of mathematics that I learn in the classroom." As a tool, the project fulfilled Lebow's (1993) suggestions for quality teaching: (a) it provided a context for learning that encouraged both autonomy and relatedness, (b) the reasons for learning were embedded in the learning itself, and (c) it supported self-regulated learning by providing skills and attitudes that enabled the learner to take increasing responsibility for the restructuring process. Dingi's emotive language and close connection to family activities were linked to increasing self-esteem, even of school dropouts.

I think we use many activities from our local environment in our teaching ... this would at least help the students to realize that in whatever they do they are putting their intelligence into practice... and this would help maintain their self-esteem. At the end, we can expect fewer problems in our society.

In terms of the diagram in Figure 1, his identity as a mathematical thinker is being applied back into society, a feature found in the USA studies mentioned above when culture and

place were taken into account in mathematics education. Bire (2007) also noted the benefits to society

my Mate Mr. (MN) ... has explicitly demonstrated the construction of Bandicoot Trap in a sequential order by using the appropriate materials available from the bush. ... This project is trying to restore and rejuvenate the traditional bandicoot trap which is ... (a part) of our unique culture. ... The hunters are very skilful ... In actual fact, the elderly old people who constructed the trap do not know about western mathematical ideology (sic) but they make use of it. Nevertheless, I as a mathematics student (teacher) from that particular culture can ... figure out the different branches of mathematics that blend in to complement the success of the trap.

To conclude, it (is) now apparent that many of our cultural activities involve mathematical concepts. Inasmuch, the reformed education emphasizes the importance of upholding our cultural heritage and identity; which is important to teach mathematical concepts using cultural activities. ... It is evident that the mathematical ideas which are very abstract are well understood when a teacher/instructor uses analogies from the cultural context in teaching. ... students will learn mathematics and on other hand, they also learn their own cultures and perhaps master the abstract concepts better in their own cultures. Hence, they will ... fit well in their various societies ... there will be a reduction (in) crimes.

Bire noted the hunters' skill and ability to plan carefully; he lamented the use of purchased traps. He expressed his belief in the role that culture plays in learning mathematics. About 95% of teachers identified themselves as mathematics teachers having similar beliefs about mathematics, a recognised cultural identity, a role in restoring culture, and a desire to make mathematics more meaningful to their own students. These common values of promoting culture were supported by National policy (Papua New Guinea Government, 2003). Within the sociocultural context of the Reform curriculum, there is the sociocultural community of practice of the University class itself (Wenger, 1998). The sociocultural contexts impinged on their cognitive and affective learning (Figure 1).

Most teachers (95%) adequately represented school mathematics. About 20% of the teachers provided the steps and ways of thinking required for creating an artefact whether a garden or an object like a wig. For example, Piru (2005), Huli, Southern Highlands, marvelled at the mathematics of his ancient cultural group, and at the explanation that was linked to school mathematics with drawings and superimposed orthogonal axes: to symmetries, parabolas, and calculus. He mixed school mathematics adequately with the customary explanation of wig making in terms of curves, comparisons and measures, order of events, and exactness. He is cognitively and affectively engaged. Onggi (2005) used the making and painting of a door board for a large community building (e.g. a *haus tambaran* for men or a family long house) by the Telefol, Sandaun Province. He had ratios of paint mixtures and geometry topics (e.g. symmetry) linked to both cultural design and school mathematics. The two-way synergy of school and cultural mathematics was evident.

However, teachers frequently situated school mathematics examples in their cultural background but they did not always make a link between the mathematical thinking of the community and school mathematical thinking. Teachers used the cultural artefact as a contextual problem for engaging school students and providing a context for school mathematics (Owens, 2012) rather than providing ethnomathematical thinking required in creating the artefact. This is similar to western school students considering the mathematics found in environmental objects during mathematics trails or scavenger hunts in a park or a zoo (Owens, Pattison, & Lewis, 2003; Smart & Marshall, 2007).

Language is an important component of cultural identity. Bire used Nega language words to describe each kind of material required and gave distances in both Nega and metric language. He called the steps of bandicoot trapping *Klge'vu* an action word. Piru (2005) used $K(ki) = \frac{1}{2}$ cm for the unit in his wig-making. Onggi (2005) used questions on

proportion structured like school mathematics related to mixing paint using his language for colours, grease, and typical containers for measurement. About 30% of teachers used language words for the names of objects used in the activities and/or the counting systems but 10% gave action words or measurement words involved in a mathematical process.

The teachers considered the intuitive, experiential mathematics gained from observation or activity in the village as “building blocks” (Bafimu, 2007) or fundamental analogies for school mathematics. For example, Pepeta (2001), quoted in the opening paragraph of this article, from Wapenamanda, Enga Province, described a connection between the children’s game *bras flaua* (matching cards in the same family) and the collection of like terms in algebra. Teachers generally considered school mathematics as more exact than village mathematics which only had to meet the needs of the village but several reports show this perspective is contested. For example, ropes, sticks and steps were used to measure and ensure the house was sturdy while small units like finger widths ensured a wig or mud-head mask fitted the wearer properly.

Piru’s wig study illustrated how the synergy between culture and school mathematics plus the enthusiasm and ownership generated by these connections resulted in well planned school mathematics and pedagogical content knowledge. Overall the plans were of a high standard. However, a small number (5%) of reports showed a discrepancy between the derived construction of school mathematics from the cultural artefact and the generally accepted school mathematics (cf, Owens & Kaleva, 2008b). The sociocultural context did not always ensure development of school knowledge despite having school textbooks available. For example, after studying the symmetry of different weaving and bag designs, one teacher (whom I have selected not to name) discussed the angle sum of interior angles of polygons: “Since there are six equilateral triangles to the hexagon, the sum of the total interior angles would be $180^{\circ} \times 6 \text{ triangles} = 1080^{\circ}$ ” but this includes angles at the centre of the hexagon which are not regarded as interior angles in school mathematics. This teacher used words such as “subtended” and “interior” without appreciating their meaning. Despite the use of a diagram, the teacher was unable to recognise the error. This suggests that language difficulties resulted in a struggle to review her work well, that is to self-regulate, a critical aspect of identity (Owens, 2007/2008). Nevertheless, these few teachers still reported clearly their mathematical journeys.

Conclusion

The social contexts for the teachers were an ecocultural background and a school system that adapted western mathematics. The ethnomathematics project engaged the teachers and provided them with an opportunity to inquire about their cultural mathematics. They showed a sense of ownership of the knowledge that they constructed from their personal, cultural backgrounds. They had a strong sense of identity with their culture, and their projects encouraged them to reflect upon their cultural heritage and recognise that it was valuable and relevant to school mathematics education. Construction of knowledge arose from resolving the dissonance between curriculum knowledge and the ethnomathematics knowledge embedded in cultural practices. As Presmeg (2002) noted any dissonance is associated with a negative shattering of valuing themselves as agents who can affect their own lives and society’s development. This point was recognised by teachers who were concerned about the dropouts from school.

The teachers adapted an ecocultural pedagogy in planning mathematical problems for school students. There were differences in links to school mathematics between the making

process in terms of the systems used for construction versus the use of the end-product. Teachers valued the abilities or mathematical processes of their ancestors and Elders but their understanding was enhanced by school mathematics. The teachers recognised the importance that an ecocultural pedagogy had in terms of learning and developing their students' sense of worth, and hence to be a sociopolitical force in society. The adequacy of expressions of school mathematics reflected the mediation that occurred during the teachers' projects. However, the sociocultural background of valuing school textbook questions still impacted on the variation in connections between ecocultural and school practices as shown in the planned examples. Nevertheless the project strengthened both social and cultural identity and identity as a mathematical thinker who was able to provide an ecocultural pedagogy for mathematics.

References

- Australian Curriculum Assessment and Reporting Authority. (2010). Australian Curriculum Mathematics
- Bafimu, K. (2007). *The making of yam gardens*. Unpublished report. University of Goroka. Goroka, PNG.
- Barnhardt, R. (2007). Creating a place for Indigeneous knowledge in education: The Alaska Native Knowledge Network. In D. Gruenewald & G. Smith (Eds.), *Place-based education in the global age: Local diversity* (pp. 113-133). New York, NY: Lawrence Erlbaum.
- Bire, J. (2007). *Traditional bandicoot trap*. Unpublished report. University of Goroka. Goroka, PNG.
- Bishop, A. (2002). Mathematical acculturation, cultural conflicts and transitions. In G. De Abreu, A. Bishop & N. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 190-210). Dordrecht, The Netherlands: Kluwer.
- Chinn, P. (2007). Decolonizing methodologies and Indigenous knowledge: The role of culture, place and personal experience in professional development. *Journal of Research in Science Teaching*, 44(9), 1247-1268.
- Crotty, T. (1993). *Constructivist theory unites distance education and teacher education*. River Falls, WI: University of Wisconsin-River Falls.
- Davis, B. (1999). Thinking otherwise and hearing differently: Enactivism and school mathematics In W. F. Pinar (Ed.), *Contemporary curriculum discourses: Twenty years of JCT* (pp. 325-345). New York Lang.
- de Abreu, G. (2002). Towards a cultural psychology perspective on transitions between contexts of mathematical practices. In G. de Abreu, A. Bishop & N. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 170-189). Dordrecht, The Netherlands: Kluwer.
- Dingi, D. (2007). *Construction of houses*. Unpublished report. University of Goroka. Goroka, PNG.
- Esmonde, I., & Saxe, G. (2004). 'Cultural mathematics' in the Oksapmin curriculum: continuities and discontinuities Paper presented at the Proceedings of the 6th international conference on Learning Sciences, Santa Monica, California
- Frade, C., & Falcão, J. T. d. R. (2008). Exploring connections between tacit knowing and situated learning: Perspectives in the context of mathematics education In A. Watson & P. Winbourne (Eds.), *New directions in situated cognition* (pp. 205-232). New York, NY: Springer.
- González, N., Moll, L., & Amanti, C. (Eds.). (2005). *Funds of knowledge: Theorizing practice in households, communities, and classrooms*. Mahwah, NJ: Lawrence Erlbaum.
- Gruenewald, D., & Smith, G. (2008). *Place-based education in a global age: Local diversity*. New York, NY: Lawrence Erlbaum.
- Kahan, S. (2004). Engagement, identity and innovation: Etienne Wenger on communities of practice. *Journal of Association Leadership*
<http://www.asaecenter.org/PublicationsResources/JALArticleDetail.cfm?ItemNumber=16217>.
- Lebow, D. (1993). Constructivist values for instructional systems design: Five principles toward a new mindset. *Educational Technology Research and Development*, 41, 4-16.
- Lerman, S. (2001). Cultural, discursive psychology: A sociocultural approach to studying the teaching and learning of mathematics. *Educational Studies in Mathematics*, 46, 87-113.
- Lipka, J., & Adams, B. (2004). Culturally based math education as a way to improve Alaska native students' math performance. Working Paper 20. Appalachian Collaborative Center for Learning (ED484849).
- Litteral, R. (2001). Language development in Papua New Guinea. *Radical Pedagogy*, 3(1).
- Macmillan, A. (2009). *Numeracy in early childhood*. Sydney, NSW: Oxford University Press.

- Malmivuori, M.-L. (2006). Affect and self-regulation. *Educational Studies in Mathematics*, 63, 146-164. doi: 10.1007/s10649-006-9022-8
- Morrison, D. (2007). E-Learning in higher education: The need for a new pedagogy. In M. Bullen & D. Janes (Eds.), *Making the transition to e-learning* (pp. 104-120). Hershey, PA: Information Science.
- National Research Council Committee on Geography. (2006). *Learning to think spatially: GIS as a support system in the K-12 curriculum*. Washington DC: National Academies Press.
- Onggi, C. (2005). *Deriving academic mathematics from Telefol traditional door board designs*. Unpublished report. University of Goroka. Goroka, PNG.
- Ovington, G. (1994). "Both Ways Education" dominant culture access and minority culture maintenance: An analysis of competing theories. *Journal of Multilingual and Multicultural Development*, 15(1), 29-46.
- Owens, K. (1999). The role of culture and mathematics in a creative design activity in Papua New Guinea. In E. Ogena & E. Golla (Eds.), *8th South-East Asia Conference on Mathematics Education: Technical papers*, (pp. 289-302). Manila, The Philippines: Southeast Asian Mathematical Society.
- Owens, K. (2007/2008). Identity as a mathematical thinker. *Mathematics Teacher Education and Development*, 9, 36-50.
- Owens, K. (2012). Papua New Guinea Indigenous knowledges about mathematical concepts. *Journal of Mathematics and Culture (on-line)*, 6(1), 15-50.
- Owens, K., & Kaleva, W. (2008a). Case studies of mathematical thinking about area in Papua New Guinea. In O. Figueras, J. Cortina, S. Alatorre, T. Rojano & A. Sepúlveda (Eds.), *Annual conference of the International Group for the Psychology of Mathematics Education (PME) and North America chapter of PME, PME32 - PMENAXXX* (Vol. 4, pp. 73-80). Morelia, Mexico: Organising Committee of PME32-PMENAXX.
- Owens, K., & Kaleva, W. (2008b). *Indigenous Papua New Guinea knowledges related to volume and mass*. Paper presented at the International Congress on Mathematics Education, Discussion Group 11 on The Role of Ethnomathematics in Mathematics Education, Monterrey, Mexico.
- Owens, K., Pattison, J., & Lewis, E. (2003). Organising maths trails. *Reflections*, 28(1), 27-30.
- Papua New Guinea Government. (2003). *National curriculum statement*. Port Moresby, PNG: Ministry of Education.
- Pepeta, P. (2001). *Developing mathematics in the traditional/cultural activity*. Unpublished report. University of Goroka. PNG.
- Pinxten, R., van Dooren, I., & Harvey, F. (1983). *The anthropology of space: Explorations into the natural philosophy and semantics of the Navajo*. Philadelphia, Pa: University of Pennsylvania Press.
- Piru, P. (2005). *Mathematics derived from Huli wig construction*. Unpublished report. University of Goroka. Goroka, PNG.
- Presmeg, N. (2002). Shifts in meaning during transitions. In G. De Abreu, A. Bishop & N. Presmeg (Eds.), *Transitions between contexts of mathematical practices*. Dordrecht, The Netherlands: Kluwer.
- Radford, L. (2006). The anthropology of meaning. *Educational Studies in Mathematics*, 61, 39-65. doi: 10.1007/s10649-006-7136-7
- Smart, J., & Marshall, J. (2007). A geometric scavenger hunt. *Science & Children*, 45(2), 36-40.
- Stanton, R. (1994). Mathematics "Both Ways": A mathematics curriculum for Aboriginal teacher education students. *For the Learning of Mathematics*, 14(3), 15-23.
- Thornton, M. B., & Watson-Verran, H. (1996). *Living maths*. Abbotsford, Victoria: Yirrkala Community School and Boulder Valley Films.
- Valsiner, J. (2000). *Culture and human development*. London: Sage.
- Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. New York, NY: Cambridge University Press.

Let's Count: Evaluation of a Pilot Early Mathematics Program in Low Socioeconomic Locations in Australia

Bob Perry
Charles Sturt University
<bperry@csu.edu.au>

Ann Gervasoni
Australian Catholic University
<ann.gervasoni@csu.edu.au>

Sue Dockett
Charles Sturt University
<sdockett@csu.edu.au>

The *Let's Count* pilot early mathematics program was implemented in five early childhood educational contexts across Australia during 2011. The program used specifically formulated materials and workshops to enlist the assistance of early childhood educators to work with parents and other family members of children in their settings to help develop these children's awareness, confidence and skills in early mathematics. The pilot program was evaluated by the authors of this paper using a multi-methods approach. The evaluation was focused on the success of the *Let's Count* program in bringing early childhood educators, parents and other family members together, to enhance children's mathematical engagement, learning outcomes and dispositions.

Let's Count is a new early mathematics program commissioned by The Smith Family to assist parents and other family members to help their young children aged 3-5 years notice, explore and talk about powerful mathematical ideas in ways that develop positive dispositions to learning as well as mathematical knowledge and skills. It relies on parents and other family members providing the opportunities for the children to engage with, talk about, and document the mathematics in their everyday lives, and to extend it in ways that are relevant to them. *Let's Count* is not a mathematics teaching program. However, it does involve early childhood educators in the role of advisers to parents and family members of the children in their settings. This can have the impact of encouraging these educators to consider their own pedagogical approaches in mathematics and add to their repertoire of successful practices.

Let's Count fits well with the *Early Years Learning Framework for Australia* through its use of play and investigation as key pedagogical approaches and through its contribution to Learning Outcome 5: *Children are effective communicators*.

Children bring new mathematical understandings through engaging with problem solving. It is essential that the mathematical ideas with which young children interact are relevant and meaningful in the context of their current lives. Educators require a rich mathematical vocabulary to accurately describe and explain children's mathematical ideas and to support numeracy development. Spatial sense, structure and pattern, number, measurement, data, argumentation, connections and exploring the world mathematically are the powerful mathematical ideas children need to become numerate. (DEEWR, 2009, p. 38)

Background to the *Let's Count* Program

Let's Count is based on current research literature in three important areas:

1. the role of families in young children's learning;
2. play and investigation in young children's mathematics learning; and
3. young children's learning of mathematics.

This section of the paper explores each of these areas briefly in order to establish the research basis for the program.

The Role of Families in Young Children's Learning

Families are important contexts for young children and can have major impacts on their learning opportunities.

Most children grow up in families and, in their early years especially, families have a major impact on children's lives, influencing their developing patterns of thinking, feeling and behaviour. Families also have a critical role in instilling fundamental values in children, which in the vast majority of cases enables them to become ethical and law-abiding citizens. Although children may spend a lot of their time in contexts other than their family, the emotional intensity of families and the durability of kinship ties mean that learning in this context is likely to be extensive and long lasting. (Bowes, Watson, & Pearson, 2009, p. 91)

Children's experiences within their families influence their learning and dispositions towards learning. Resources provided to the child by the family, home routines and environment strongly predict educational and behavioural outcomes for children well into the primary years (Melhuish et al., 2008). For example, living in poverty can restrict families' abilities to provide materials and associated learning opportunities for their children. However, there are many examples of children living in poverty who have achieved positive educational outcomes. Often this is due to particularly strong, positive approaches from family members. In many ways, it is not the nature of the family that makes the difference. Rather, it is what happens within the family that is important (Melhuish et al., 2008).

Despite the recognised importance of family involvement at all levels of their children's education, there is evidence that many families – notably those described as having 'disadvantaged' backgrounds – do not actively participate (Miedel & Reynolds, 1999). Multiple reasons have been advanced for this lack of participation, including a perceived lack of partnership between families and educators (Brooker, 2003).

While the development of positive, fruitful and collaborative partnerships with families can be quite time-consuming for early childhood educators, such partnerships are of great importance in children's learning and underpin the *Early Years Learning Framework for Australia* (DEEWR, 2009).

Learning outcomes are most likely to be achieved when early childhood educators work in partnership with families. Educators recognise that families are children's first and most influential teachers. (DEEWR, 2009, p. 12)

Play and Investigation in Young Children's Mathematics Learning

Much has been written about the importance of children's play in their learning (Shonkoff & Phillips, 2000). The *Early Years Learning Framework for Australia* makes the following comments about the importance of play in young children's learning:

Play provides opportunities for children to learn as they discover, create, improvise and imagine. When children play with other children they create social groups, test out ideas, challenge each other's thinking and build new understandings. Play provides a supportive environment where children can ask questions, solve problems and engage in critical thinking. Play can expand children's thinking and enhance their desire to know and to learn. In these ways, play can promote positive dispositions towards learning. (DEEWR, 2009, p. 15)

The great potential for mathematics in children's play has long been recognised (Lee & Ginsburg, 2007; Perry & Dockett, 2008). However, this potential can only be realised if the mathematics in children's play is noticed, explored and talked about. Having access to a 'knowing adult' is important.

Educators (including parents and teachers) who are effective in promoting children's learning through play often engage in sustained shared thinking, where children and adults

together work through an issue, solve a problem or clarify understandings (Siraj-Blatchford 2009). Sustained shared thinking occurs within the context of responsive, trusting relationships where all show genuine interest as they engage with issues or questions, suggest or clarify explanations, and are provoked to consider alternative explanations, as well as potential consequences of these. When adults engage in sustained shared thinking with children there is potential for all participants to develop confidence in their own abilities as learners as well as develop new understandings.

Young Children's Learning of Mathematics

There has been some reluctance amongst early childhood educators to include mathematics as part of the formal curriculum (Sarama & Clements, 2002; Perry & Dockett, 2008). However, the past decade has seen a significant shift in thinking, with young children celebrated as capable mathematical thinkers and learners (Lee & Ginsburg, 2007; Sarama & Clements, 2002). This is reflected in the Australian Association of Mathematics Teachers and Early Childhood Australia statement that

all children in their early childhood years are capable of accessing powerful mathematical ideas that are both relevant to their current lives and form a critical foundation to their future mathematical and other learning. Children should be given the opportunity to access these ideas through high quality child-centred activities in their homes, communities, prior-to-school settings and schools. (AAMT & ECA, 2006, p. 1)

In a recent survey of early childhood educators in four states of Australia (Hunting, Mousley, & Perry, 2012), there was overwhelming agreement from educators that young children were capable of mathematical activity well before they started school. This result echoes those in many other studies with the general agreement that all young children are capable of accessing powerful mathematical ideas and should be afforded opportunities to access these ideas through high quality child-centred activities in their homes, communities, and prior-to-school settings (Faragher, Brady, Clarke, & Gervasoni, 2008; Lee and Ginsburg, 2007; Perry & Dockett, 2008; Sarama & Clements, 2002).

Let's Count Pilot Program

During the first half of 2011, the *Let's Count* early mathematics program was piloted in five sites experiencing social and economic disadvantage, in five states across Australia. *Let's Count* included a program of two full-day modules of face-to-face professional learning for early childhood educators in each site with approximately two months between each module. The themes and aims of the modules were:

- Module 1: Noticing and exploring everyday opportunities for mathematics
- Module 2: Celebrating mathematics

The first module provided information about *Let's Count* and its theoretical and practical bases. It also included discussions, workshops and activities to prepare the educators for working with parents and other family members from their settings to notice, explore and talk about mathematics using play, games, singing, investigation and intentional teaching.

Between modules, the early childhood educators also completed the following activities, along with others that they suggested during the first professional learning day:

- making a photographic and/or video record of the mathematics being experienced by children in their settings;
- keeping a journal of the ways in which these children use mathematics;
- noting interactions with family members around mathematics in the families;

- preparing a presentation for the second professional learning day that showed children's mathematical experiences in the setting;
- collating their families' documentation of mathematical experiences outside of the setting for presentation on the second professional learning day; and
- writing a learning story about their experience of *Let's Count*.

The second module of professional learning provided opportunities for the early childhood educators to celebrate what they had done in the program and to develop an action plan about how to sustain and extend *Let's Count* in each site. The day concluded with discussions about the evaluation of the program.

Evaluation of the *Let's Count* Pilot Program

The aim of the evaluation was to ascertain the success of the pilot offering of the *Let's Count* program in bringing early childhood educators, parents and other family members together, to enhance children's mathematical engagement, learning outcomes and dispositions. The research questions for the evaluation were:

1. How has *Let's Count* assisted early childhood educators and parents and other family members to enhance children's mathematical engagement, learning outcomes and dispositions?
2. What improvements can be made to the *Let's Count* structures, materials and personnel in order to ensure its sustainability as it is scaled from five sites to all eligible sites in Australia?

The evaluation study used a multi-methods approach in which educators undertook written surveys at the time of the first workshop (Time 1) and directly after the second workshop (Time 2) and individual telephone interviews at Times 1 and 2, and four months after the second workshop (Time 3). Parent interviews were undertaken post-program only. Table 1 reports the number of participants at each of the five pilot sites who completed surveys and/or participated in interviews.

Table 1.

Numbers of Participants in Workshops, Surveys and Interviews

State	Workshop Participants	Survey Time 1	Survey Time 2	Interview Time 1	Interview Time 2	Interview Time 3
NSW	24	19	10	4	4	4
QLD	9	9	3	4	3	2 (plus 3 parents)
SA	18	18	15	4	2	2
TAS	5	4	4	1	1	2
VIC	8	8	7	4	3	1
Totals	64	58	39	17	13	11 (plus 3 parents)

All data from the two rounds of surveys were entered into SPSS for quantitative analysis. For the most part, these analyses consisted of frequency counts and simple statistical techniques such as crosstabs and χ^2 calculations. Qualitative data from the surveys was used in conjunction with interview data to provide a clear picture of the respondents' attitudes to mathematics and mathematics pedagogies.

All the interviews were audio-recorded and transcribed for analysis. An initial coding frame was established for the interview data using a coding procedure based on the principles of grounded theory (Strauss & Corbin, 1998). Each interview was double coded using this coding frame. Codes were then coalesced to generate broad categories of responses. Any discrepancies in coding were resolved through discussion and agreement between the coders. The coded data were analysed using NVivo.

Results and Analysis

In this paper, we report only educator data from three categories:

- noticing and talking about mathematics and making it explicit;
- changing attitudes towards mathematics and growing confidence about the potential of mathematics; and
- engaging families and building relationships with mathematics.

Noticing and Talking About Mathematics and Making It Explicit

Data pertaining to this category were generated through the interviews with educators, with some change discernable over the life of the program. Issues raised in the following statements from educators were consistent across the group of respondents.

It's about recognising what they're [parents] wants are and reassuring them that it's [*Let's Count*] not going to be sitting down at desks learning mathematics. It's basically play based. And also giving them ideas about how they could be following through at home. (Interview Time 1)

For a parent, when you talk about maths it's about number and counting and adding this to this. So it's just sort of broadening their view of what maths actually is and how it can be incorporated into everyday life. It's more from the parent perspective I think things have changed for me primarily. (Interview Time 2)

We started off I made an obstacle course and I with the other staff that we stuck up words on the wall so that as the children were doing or going through the obstacle course we got them to use words like 'over' and 'under' and 'through' and 'next to' and things like that. And simply did that once, it has been a consistent thing in every group since. I always have to now set up the obstacle course and the children and the parents have actually taken on board the language. (Interview Time 3)

The *Let's Count* mantra of 'notice, explore and talk about' mathematics has been embraced successfully by the educators and they can see the message being utilised by at least some of their parents.

Changing Attitudes towards Mathematics and Growing Confidence about the Potential of Mathematics

Educators' confidence and positive attitudes to mathematics impact on the quality of their teaching of mathematics and the mathematical outcomes of the children with whom they interact (Perry & Dockett, 2008). In both iterations of the survey for educators, participants were asked to indicate where they felt they fitted along a continuum from absolutely disliking mathematics (indicated by a score of 1) to feeling that mathematics is the best thing ever (indicated by a score of 10). Difference scores were calculated for 'liking mathematics' for the 37 participants who answered this question in both iterations of the survey. These are reported in Figure 1.

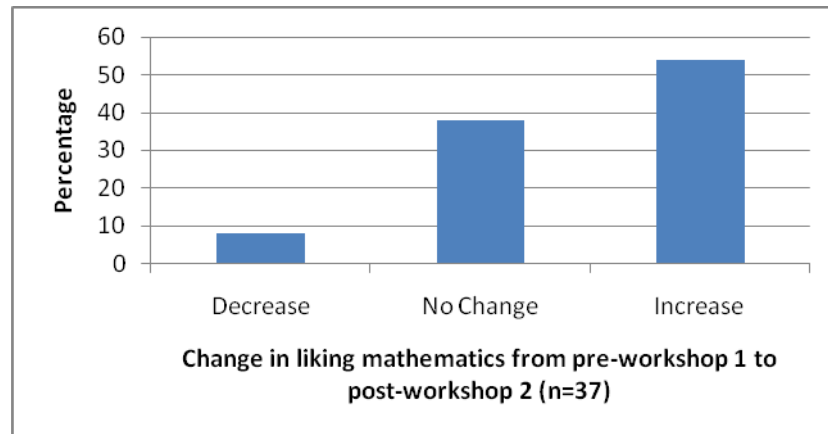


Figure 1. Changes in 'liking' ratings from Survey 1 to Survey 2.

Educators' perceptions about liking mathematics were significantly different over time (Wilcoxon Signed Ranks Test, $z=3.365$, $p<0.005$). From just prior to training in the *Let's Count* program to the completion of the second training workshop, 54% of participants reported an increase in their liking of mathematics, while only 8% reduced their 'liking' score. This suggests that the educators, in general, found the *Let's Count* experiences 'to their liking'.

Educators were asked in each of Surveys 1 and 2 to indicate their agreement with 12 statements regarding their attitudes to mathematics. Cross tabulations were carried out between scores at the first and second implementations of the survey to ascertain the level of movement between AGREE and NOT AGREE across the *Let's Count* period. For all of the 12 statements, the movement was 'positive' in the sense of the educators reporting more positive inclinations towards mathematics. While the numbers of participants who responded to both surveys was relatively small ($n=37$), it is interesting to note that of the 12 statements canvassed, all but three of them showed statistically significant positive movement (at the $p<0.005$ level). Those statements for which statistical significance was demonstrated were:

- At school my friends always came to me for help in mathematics
- I am quite good at mathematics
- I have always done well in mathematics classes
- *I felt the need to study mathematics beyond high school*
- *I have no trouble understanding anything that is based upon mathematics*
- *I do well on tests that require mathematical reasoning*
- Mathematics is a thing of beauty
- It's important for everyone to pass Year 12 mathematics
- *Mathematics does not make me feel inadequate*

(Note that negatively worded statements have been written in the positive, had their coding reversed and are shown in italics.)

Given that some of these statements relate to 'historical' information that should not have changed, it would seem that there may be an overall 'more positive feeling' about mathematics that has taken place during the period of the *Let's Count* program. There are several instances in which data from educator interviews have reinforced this feeling of being more positive about mathematics, seeing the potential to develop mathematical ideas through a play-based program run by parents and, in fact, attributing these to *Let's Count*. For example,

We don't need to be masters of mathematics. Just let's change our attitude towards maths. Let's recognise that mathematics is everywhere all around and it's enjoyable, it's easy. You don't need anything special to do it. (Interview Time 2)

Well, *Let's Count* it just was a very positive experience for me and also the biggest thing for me is that it gave me confidence. The confidence to expand things and to actually initiate conversations, more conversations around mathematics and encourage parents a lot more to use or think about mathematical concepts in everyday life. (Interview Time 2)

It's been just a really positive experience for me. It's given me a lot of confidence to take different learning out there and encourage parents to add something to what they're already learning. (Interview Time 3)

Oh it's made a big difference. I used to hate maths. Maths wasn't my strong point in school but by sitting down and doing this program with the kids I've had a lot of fun doing it. I've been able to break down more complex problems and being able to get the point across to the kids in a simpler format. I think I might have probably a better understanding of things. (Interview Time 3)

Data from both surveys and interviews suggests that the *Let's Count* pilot program has had positive impact on the educators' attitudes to mathematics and the potential they see for families to play and investigate mathematics with their children.

Engaging Families and Building Relationships with Mathematics

During the course of the evaluation, educators consistently shared concerns about how they might engage families with the *Let's Count* program. Concerns varied depending on the nature of the settings and opportunities available for the family engagement. This is consistent with evidence that the successful engagement of families is, in part, linked to the types of educational spaces families encounter and the extent to which they feel comfortable and empowered within these spaces (Miedel & Reynolds, 1999). Educators reported that *Let's Count* provided opportunities for them to partner with parents in ways that were different from their normal interactions. These partnerships may have begun with mathematics through *Let's Count* but were often extended more broadly:

They [parents] became quite excited that this was happening. They were very willing participants. They would come in and ask to swap games and take other games home. They talked about their children's learning periodically, when they dropped them off or picked them up. They've said, even in their written stuff, they didn't know who was having more fun, the parents or the children, when they were doing all these things outside in their own time. (Interview Time 2)

Let's Count has got me thinking more about how to engage parents and get them to do the activities with the child rather than me doing them and them choosing whether or not they want to engage. (Interview Time 2)

I think it has strengthened our relationship with the parents. They understand that we weren't trying to force it down their throats ... They've learnt from the whole process as well. ... It's definitely strengthened the bond that we have with the parents. (Interview Time 3)

Conclusion

Let's Count has provided participants with many opportunities to enhance the mathematical outcomes of children and their families. As well, educators, parents and children have had the opportunity to build or maintain positive dispositions and increased confidence towards mathematics. Educators reported the value of *Let's Count* to their own mathematics education confidence, knowledge and skills and to their abilities to engage with the families from their settings.

Many of the educators reported how they had used some of the methods discussed during *Let's Count* Workshop 1 to engage families in innovative ways beyond the *Let's*

Count program. They were very strong in their praise for the skills they had built under the auspices of *Let's Count*. For almost all of the educators who participated in the pilot program, *Let's Count* seems to have made a positive difference in their approaches to children's mathematics education as well as engagement of the families in this education. This in turn has fostered confidence in their own abilities to assist in the mathematical development of their children. It would seem that the *Let's Count* pilot program has been successful in many ways. The challenge now is to build on this success as the program moves to scaling across Australia.

References

- Australian Association of Mathematics Teachers and Early Childhood Australia (AAMT/ECA). (2006). *Position paper on early childhood mathematics*. Retrieved from http://www.aamt.edu.au/about/policy/earlymaths_a3.pdf
- Bowes, J., Watson, J., & Pearson, E. (2009). Families as a context for children. In J. Bowes & R. Grace (Eds.), *Children, families and communities: Contexts and consequences* (3rd edn.) (pp. 91-110). Melbourne: Oxford University Press.
- Brooker, E. (2003). Learning how to learn: Parental ethnotheories and young children's preparation for school. *International Journal of Early Years Education*, 11(2), 117-128.
- Department of Education, Employment and Workplace Relations (DEEWR), (2009). *Belonging, being and becoming: The early years learning framework for Australia*. Canberra: Commonwealth of Australia. Retrieved from http://www.deewr.gov.au/earlychildhood/policy_agenda/quality/pages/earlyyearslearningframework.aspx
- Faragher, R., Brady, J., Clarke, B. & Gervasoni, A. (2008). Children with Down syndrome learning mathematics: Can they do it? Yes they can! *Australian Primary Mathematics Classroom*. 13(4), 10-15.
- Hunting, R. P., Mousley, J. A., & Perry, B. (2012). A study of rural preschool practitioners' views on young children's mathematical thinking. *Mathematics Education Research Journal*. DOI 10.1007/s13394-011-0030-3.
- Lee, J., & Ginsburg, H. (2007). What is appropriate mathematics education for four-year-olds? Pre-kindergarten teachers' beliefs. *Journal of Early Childhood Research*. 5(1), 2-31.
- Melhuish, E. C., Phan, M. B., Sylva, K., Sammons, P., Siraj-Blatchford, I., & Taggart, B. (2008). Effects of the home learning environment and preschool center experience upon literacy and numeracy development in early primary school. *Journal of Social Issues*, 64(1), 95-114.
- Miedel, W. T., & Reynolds, A. J. (1999). Parent involvement in early intervention for disadvantaged children: Does it matter? *Journal of Social Psychology*, 37(4), 379-402.
- Perry, B., & Dockett, S. (2008). Young children's access to powerful mathematical ideas. In L. D. English (Ed.). *Handbook of international research in mathematics education* (2nd ed) (pp. 75-108). New York: Routledge.
- Sarama, J., & Clements, D. (2002). Building blocks for young children's mathematical development. *Journal of Educational Computing Research*, 27(1 & 2), 93-110
- Shonkoff, J., & Phillips, D. (Eds.) (2000). *From neurons to neighbourhoods: The science of early childhood development*. Washington, DC: National Academy Press.
- Siraj-Blatchford, I. (2009). Quality teaching in the early years. In A. Anning, J. Cullen, & M. Fleer (Eds.), *Early childhood education: Society and culture* (2nd ed.) (pp. 147-157). London: Sage.
- Strauss, A. L., & Corbin, J. (1998). *Basics of qualitative research: Grounded theory procedures and techniques*. Thousand Oaks, CA: Sage.

Trialling a Professional Statistical Literacy Hierarchy for Teachers

Robyn Pierce

University of Melbourne

<r.pierce@unimelb.edu.au>

Helen Chick

University of Tasmania

<Helen.Chick@utas.edu.au>

Jane Watson

University of Tasmania

<Jane.Watson@utas.edu.au>

Michael Dalton

Victorian Curriculum and Assessment Authority

<dalton.michael.m@edumail.vic.gov.au>

Magdalena Les

Victorian Curriculum and Assessment Authority

<les.magdalena.m@edumail.vic.gov.au>

Teachers now require high levels of statistical literacy in order to take advantage of the many statistical reports analysing assessment data that are provided by system authorities. In this report 16 items from the Attitudes and Statistical Literacy Instrument (ASLI) are used with 704 teachers to provide a hierarchical scale of teacher ability to interpret these assessment data. Using Rasch analysis, three levels of ability are identified, related to reading values, comparing values, and analysing a data set. Implications are drawn for professional learning for teachers and for further research.

Introduction

Evidence-based practice is expected of today's principals and teachers. Schools must provide education system authorities with data on a wide variety of variables, including student assessment data. These are analysed and then returned to schools and teachers in the form of statistical reports. Schools are expected to consider these data for planning and decision-making, requiring commensurate statistical skills. This is just one example of the need for statistical literacy in today's society generally, and workplaces more specifically. In this paper we report on research trialling a proposed framework (Figure 1) for professional statistical literacy by considering the Rasch analysis of the responses to survey items based on this framework. First some background is provided and the details of the study explained. Next the results of the Rasch analysis are discussed and finally some conclusions drawn. Our purpose is to determine whether or not there is a hierarchy of understanding for professional statistical literacy, and what areas are difficult for teachers.

Professional Statistical Literacy

A Framework for Professional Statistical Literacy

A number of authors have proposed hierarchies for statistical understanding. Curcio's graph comprehension study (1987) involved relatively elementary data sets, and identified three levels: "reading the data" (directly reading factual information on the graph), "reading between [or within] the data" (attend to multiple points, often for comparison), and "reading beyond the data" (predict and infer from the data). Shaughnessy and others (1996, 2007) posit a further category focusing on context, named "reading behind the data". Watson's three-tiered statistical literacy hierarchy (2006) also emphasised the place of context. Her first tier involves understanding basic terminology, the second tier focuses on understanding language and concepts within the context of wider social issues, and the third tier focuses on challenging and questioning statistical claims. Finally, Gal's proposed statistical knowledge base (2002, p. 10) emphasises knowing why data are needed, having familiarity with basic terms, and understanding how statistical conclusions are reached.

Pierce and Chick's (2011) framework for considering professional statistical literacy draws on and synthesises this background (Figure 1). The teacher must be able to examine data at multiple levels, each more complex than and dependent on the lower levels (as

indicated by the nested circles in Figure 1). The lowest level, *reading values*, involves understanding features such as keys, scale, and graph type, together with the capacity to read specific data points on the graph. The second level, *comparing values*, requires awareness of relative and absolute differences, early informal inference, and low-level statistical tools. Finally, *analysing the data set* involves being able to consider the data as a whole: observing and interpreting variation, trends, and changes with time or other variables; and attending to the significance of results. The framework also acknowledges the role of context, with two key contexts identified as being significant (the surrounding split rectangle in Figure 1). *Professional context* concerns information relevant to the profession and needed to interpret the data set (e.g., meaning of specialist terms such as “band”, “VELS level”); and the *local context* comprises knowledge about the specific data set that is not evident in the data set alone (e.g., knowledge of local school situation). These two context components may overlap, hence the dashed line between them in Figure 1. This framework underpinned both the structure of and items created for the study, in order to assess school principals’ and teachers’ statistical literacy.

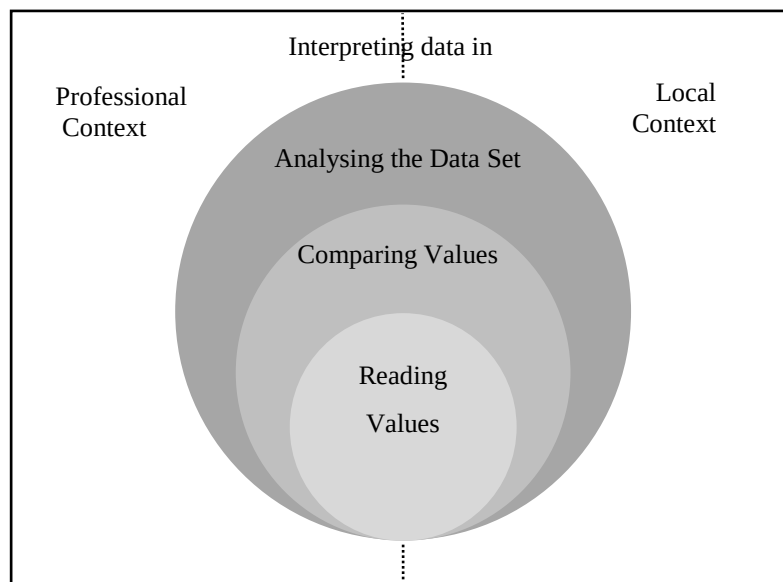


Figure 1. A framework for considering professional statistical literacy (Pierce & Chick, 2011, p. 633).

Tools for Professional Statistical Literacy

The statistical tools required to analyse data are dependent on the form in which the data are supplied. In the present case, the main graphical tool that is used for interpreting data sets provided by systems is the box plot, although some data are supplied in tabular form, with numerical summary statistics. Appropriate interpretation of box plots is hence essential for high-level statistical literacy. Although straightforward definitions of the box plot are found, for example, in the *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority, 2011, p. 53), many variations in format exist, as summarised by Wickham and Stryjewski (2011) in their discussion of the 40-year history of the box plot. It is thus important to read carefully the keys associated with any presentation of box plots. In addition, there is a subtlety associated with the frequency versus density representation of a box plot compared, say, to a histogram. In a histogram area represents frequency of values, whereas in a box plot area represents density of values. Bakker, Biehler, and Konold (2005) presented data on this confusion from students as a reason for postponing the introduction of box plots to the later middle school years. Teachers’ ability

to interpret box plots is hence a major component of assessing their statistical literacy understanding.

The Study

Instrument

This study was conducted with teachers from Victorian government schools, and was Stage 2 of a larger project. Stage 1 had involved the use of pen-and-paper questionnaires and face-to-face focus group discussions with 152 teachers (see Pierce & Chick, 2011). This led to the development of an online questionnaire for Stage 2, the focus of this study. The online Attitudes and Statistical Literacy Instrument (ASLI) took respondents about 20 minutes to complete and included items used in Stage 1 but refined to use the Stage 1 teachers' own language in the case of some items, and to focus on key issues identified in Stage 1. The number of items was reduced from the original pen-and-paper questionnaires to target misconceptions and knowledge gaps identified in Stage 1.

This paper reports data from the 16 items that formed the statistical literacy section of the online ASLI (sample items are shown in Figures 2 and 3). The first 9 items (numbered 1-9 for the purposes of the Rasch analysis and corresponding to items numbered 13-18, including sub-questions, in the actual ASLI) were based on typical reports regarding student achievement that are returned to Victorian schools following analysis of NAPLAN data. These reports were chosen because NAPLAN reports are sent to most schools and the reports of Victorian Certificate of Education data follow a similar format. Some further items (numbered 10-16 in the Rasch analysis, or 19-25 in the ASLI) targeted teachers' understanding of distributions represented by histograms and boxplots. Items numbered 8, 9, 11, and 16 in the Rasch analysis were open response items, whereas all others were multiple-choice items using distractors based on data from the Stage 1 questionnaires and interviews. [For this paper, the Rasch item numbering will predominate, but the ASLI numbering is included to allow cross-referencing with other papers from this data set.]

Data Analysis

For multiple response items the options were coded on a numerical scale where the best or correct answer was allocated the highest score. Open response items were numerically coded by comparing them with scoring rubrics, which allowed partial credit for responses exhibiting partial understanding. Scoring was done primarily by one researcher with a second independently scoring a systematic sample of at least 10% of the responses. Agreement between scorers was at least 80%. Once all of the scoring and coding was complete the data underwent Rasch analysis using *ConQuest* (Australian Council of Educational Research, n.d.) to determine whether, as hypothesised, professional statistical literacy is a unidimensional hierarchical construct.

Participants

The goal for Stage 2 was to gather data from a random sample of 1000 Victorian primary and secondary school teachers (excluding those who had participated in Stage 1 of the project). In anticipation of a 60% school participation rate and then a participation rate of 50% from teachers within schools, 104 randomly selected schools were approached. Sixty percent of principals agreed to their school's participation but in the end only 704 teacher responses were received (217 male, 487 female; 314 primary and 390 secondary teachers). This still provides a sufficiently large sample from which to draw tentative

conclusions about areas of success and difficulty for teachers with respect to professional statistical literacy, and for examining the hierarchical nature of this construct.

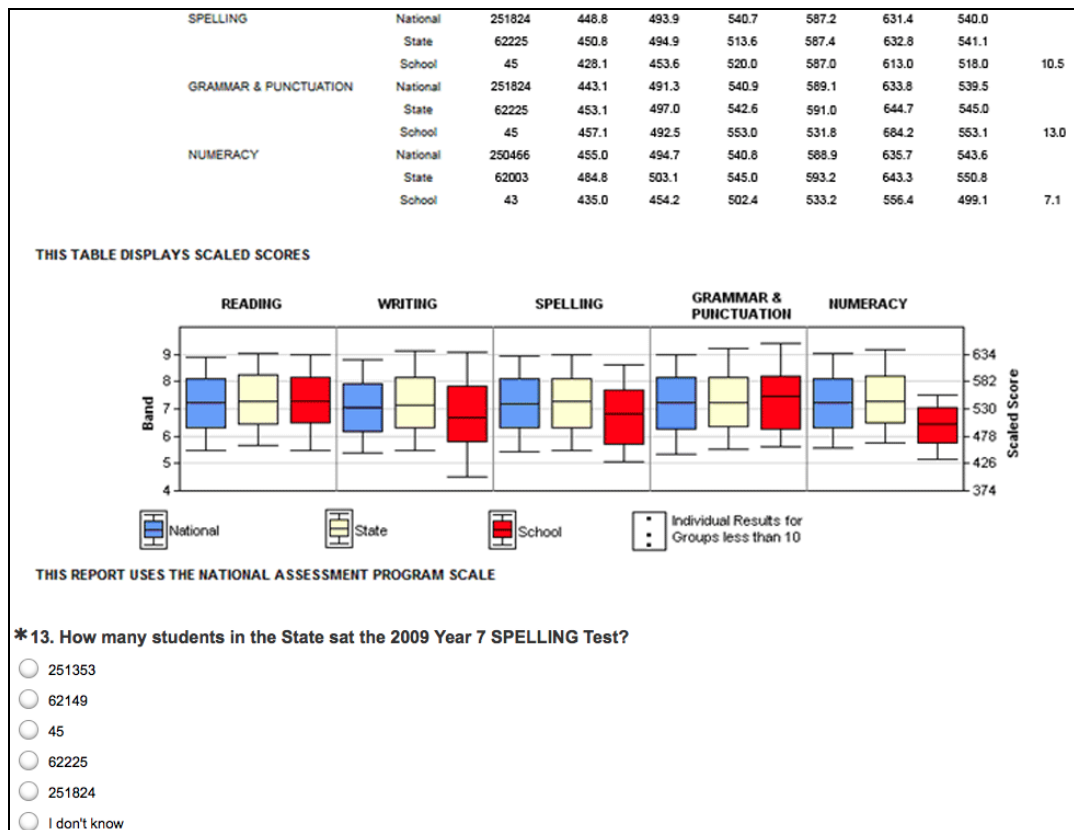


Figure 2. Item 1 (ASLI item 13) from the on-line ASLI.

***15. Look at the 'Writing' and the 'Spelling' results for Victoria College.**

Choose the best response for each of the following statements:

	DEFINITELY TRUE	DEFINITELY FALSE	NOT ENOUGH INFORMATION	I DON'T KNOW
For Writing the median result is higher than the mean result	☐	☐	☐	☐
The spread of the 50th – 90th percentile results is wider in Writing than in Spelling	☐	☐	☐	☐
In the Writing results, fewer students were between the 25th percentile and the median than were between the median and the 75th percentile	☐	☐	☐	☐
Victoria College's Writing results have a greater range than the State results	☐	☐	☐	☐

***16. From the results displayed what would you identify as the school's weakest area? (Choose one only)**

☐ Reading
☐ Writing
☐ Spelling
☐ Grammar & Punctuation
☐ Numeracy

Figure 3. Items 3-7 (ASLI items 15—with sub-items— and 16) from the on-line ASLI.

Results

The Rasch analysis of data from the ASLI showed that all items had individual weighted mean square values between the commonly accepted values of 0.75 and 1.30 (Bond & Fox,

2007) and that the item infit mean square was 1.00 (S.D. 0.124), suggesting that the items form a hierarchical unidimensional scale. An alpha reliability value of 0.81 indicates that the scale also meets the requirements of classical psychometric theory. The Rasch analysis produced the graph of respondent abilities (on the left) and item difficulty (on the right) in Figure 4. The existence of many respondents above logit 0 suggests that the ASLI included many that the teachers found to be easy. This was, in fact, a deliberate part of the design, in order to counter the anxieties with respect to quantitative data exhibited by many teachers in the pilot study.

The right side of Figure 4 provides some details of the easiest and hardest four items, as indicated by the Rasch analysis. These results show that the easiest items were focused on the reading and comparing levels of the framework (the inner circles of Figure 1), and the hardest items focused on the analysing and interpreting data in context levels (the outermost circle, together with the framing context rectangle).

The easiest item, as determined by the Rasch analysis, was an item that required teachers to locate a single datum from a table based on the locating information supplied in the item. The next easiest items involved some form of comparison between or among values. Two of these, Rasch items 3 and 13, involved comparison of two things: in the first case, of two readable data points, and, in the second, of the spreads of two graphs. Since spread involves attention to the extreme values of the graph, this comparison necessarily involves more than two data points, which suggests this item should have been the harder of the two. That this was not the case may be due to the complexity of locating, in a large complex table, the two specific values needed for the comparison in item 3. The remaining easy item, Rasch item 7, involved the comparison of box plots, and asked teachers to identify the school's weakest area from the data presented in Figure 2. The work of Pfannkuch (2006) highlights that a full comparison of box plots involves consideration of the five number summaries of the relevant data sets (i.e., the critical components of the box and whiskers). This suggests that item 7 had the potential to be a difficult item, because of the number of comparisons to be made, both between any two box plots, and across the collection of box plots. However, the school's numeracy box plot (the right-most box plot in Figure 2) is so markedly different from and lower than the others at the macro scale that even a simplistic reading and comparison of the plots could yield the correct answer. It is hypothesised that if the differences had been less "gross" then this item would have been more difficult.

At the other extreme, most of the harder items involved not only analysis of the data set as a whole entity, but attention to context as well. Item 5 addressed the "density vs frequency" misconception associated with box plots, in which the size of a region is taken to be proportional to the number of data points in it (which is true for histograms but not for box plots). To answer item 5 correctly teachers had to understand that the two components of the box in a box plot—determined by the median and the 25th and 75th percentiles—actually contain the same number of data points. This involves understanding how a box plot depicts the whole data set, and knowing the technical meanings of median and percentile.

Items 6 and 9 addressed particular peculiarities of the reports that are supplied to teachers by government agencies. This brings professional context into play, because these characteristics are a consequence of how the agencies choose to depict the data in their reports. Although the key for the box plots in Figure 2 clearly states that the whiskers end at the 10th and 90th percentiles, the results indicate that teachers were misled by the visual image of the box plot and had a tendency to assume that it showed the full range of the data, rather than hiding the results of the top and bottom ten percent of students. Similarly, item 9

asked teachers to explain the differences between two kinds of box plot representation that they receive, neither of which show the full data set. The difficulties with these two items highlight the power of a graphical image to give rise to misleading convictions, in this case associated with ignoring what is not seen.

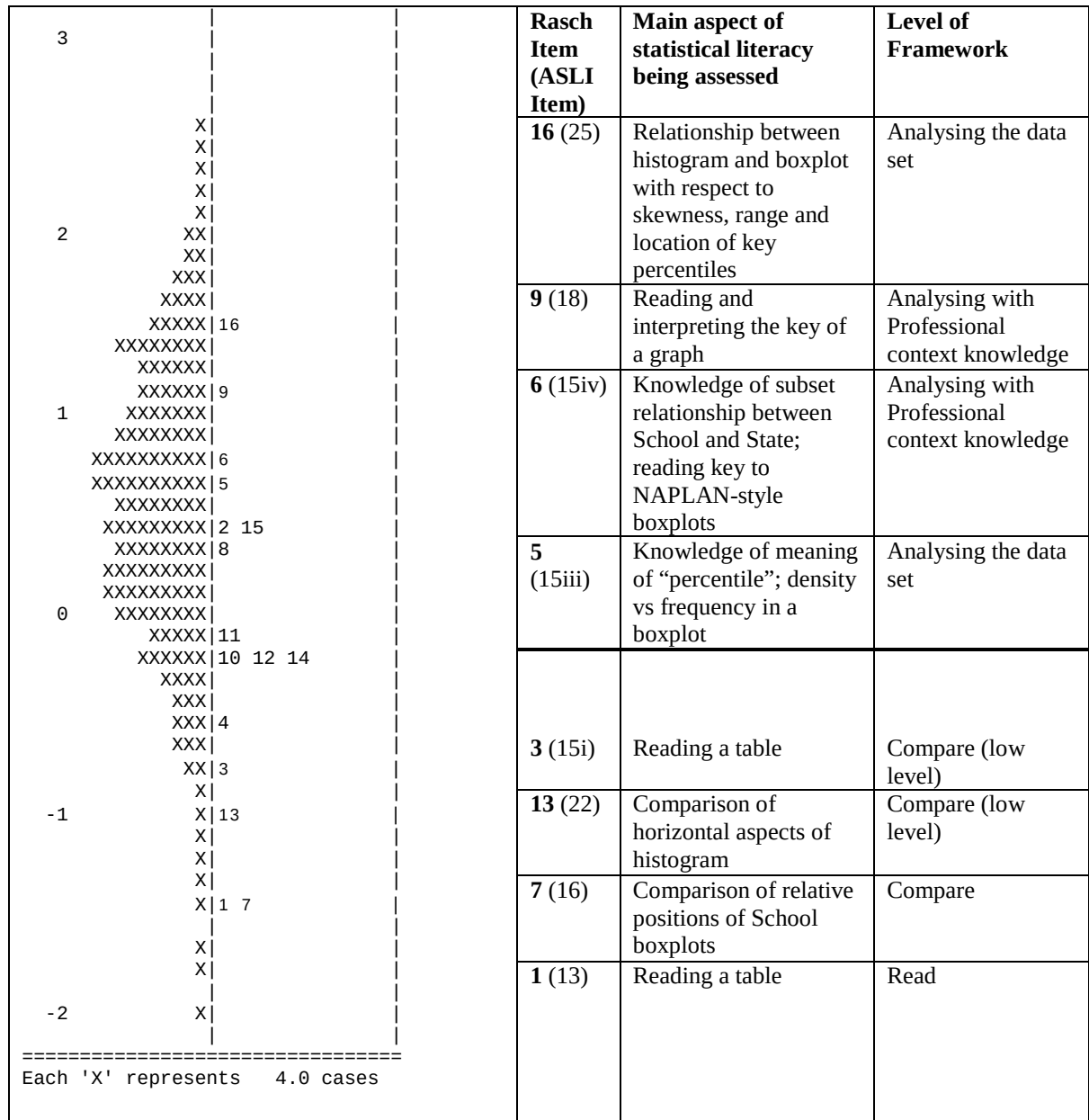


Figure 4. Left: Rasch analysis map of latent distributions and response model parameter estimates. Right: Focus of 4 easiest and 4 hardest items as identified by Rasch analysis

The most difficult item was an open-response item asking teachers to explain why a particular box plot represented some data that had originally been presented in a frequency histogram. In this case, the context was irrelevant (although one was supplied), but, in order to receive full credit, teachers had to identify and link key points on the box plot with the original histogram data. This required extensive analysis of the full data set in order to locate the appropriate percentiles and median. The multiple choice item that preceded this

on the ASLI, item 15, simply required the selection of the appropriate box plot rather than justification for the choice, but was also the equal fifth-hardest item overall.

The items of middle difficulty, according to the Rasch analysis, generally involved comparisons, or relatively straightforward analyses of the data set as a whole. Two of these are worth examining more closely. Item 12 required teachers to determine which of two distributions, displayed as histograms, had the larger mean; however, in order to design data sets that led to an unambiguous distinction in the means, the resulting distributions presented to teachers were quite noticeably distinct. This made it possible to guess correctly based on a superficial examination of the graphs rather than a deep analysis of the range of possible values for the means (which is what the authors needed to consider when designing the item). This may explain why this item was not among the harder items; it seems plausible that a similarly focused item could be “designed” to be quite difficult. Item 2, in contrast, appeared to involve the straightforward reading of data from a graph, but in this case professional contextual knowledge was needed in order to realise that the graph datum could only take whole number values, rather than the apparent continuum of values that were shown on the vertical axis. Most teachers who got this wrong were successful in reading an appropriate fractional value, but did not know that the corresponding variable was discrete. Here the impact of professional context knowledge on professional statistical literacy is particularly evident.

Implications and Conclusions

The results of the Rasch analysis suggest that there is, indeed, a hierarchy for professional statistical literacy that reflects, to a reasonable degree, the different levels posited in the framework in Figure 1. Teachers appeared to have little difficulty with direct data reading of single values and simple comparisons of data values or comparisons among whole data sets where the distinctions were grossly evident. The items that caused greater difficulty were those that required deeper analysis of the whole data set, and attention to the technicalities of the data presentation in this particular professional context. Because of the scope of the survey, which was intended to be used with teachers across a range of school contexts, and hence had to use generic data rather than data from their own specific schools, the role of local context could not be explored.

Although it appears that we can safely assume that teachers can read straightforward data values, the results indicate that teachers do not always have a good understanding of the technicalities of certain kinds of representations. In this case, the extensive use of box plots for presenting data causes difficulties, particularly since the ones used in the reports have certain idiosyncrasies, such as the cut-off percentiles for the whiskers. Although a key was supplied, the basic graph-reading principles of actually reading the key of a graph, seemed sometimes to be neglected by data users. Alternatively, the implications of this information were not fully understood, perhaps because of an assumption about already knowing what was being depicted. In follow-up work with Stage 1 teachers, participants showed they were not fully cognisant of the implications of the fact that the box plots do not show the top and bottom 10% of the data, being noticeably surprised on realising that their strongest and weakest students were, in effect, invisible in the data representation. Similarly, the prevalence of the density versus frequency misconception, observed by Bakker et al. (2005) for school students, also suggests that the value of box plots is being undermined by lack of familiarity or fluency with this graph type. It is not clear that alternative representations might be better, because in any representation there is a need to understand how it depicts the whole data set, and to consider for what is displayed or hidden by a particular representation. Regardless of the representation type, it seems that professional

learning for teachers should focus on gaining fluency with holistic analysis and how whole data sets are depicted.

The professional statistical literacy framework provides a reminder of key issues to be addressed in the teaching of statistics, to ensure that high-level statistical literacy skills, applicable beyond school, are developed. In particular, the results and framework highlight the impact of and need for paying attention to statistical context (e.g., key and scales), along with professional and local contexts in all work with data. De-contextualised skills may result in incomplete interpretations of data, limiting the statistical literacy of the user.

Acknowledgements

We thank the schools and teachers for their involvement in this project. We also acknowledge the contributions of other members of the research team: Sue Buckley (Victorian Department of Education and Early Childhood Development); Ian Gordon, Sue Helme, Roger Wander (University of Melbourne). This research is funded by the Australian Research Council (LP100100388), the VCAA, and the DEECD.

References

- Australian Council for Educational Research. (n.d.). *ConQuest: Generalized item response modelling software*. [Computer software]. Melbourne: Author.
- Australian Curriculum, Assessment and Reporting Authority. (2011). *The Australian Curriculum: Mathematics, Version 1.2, 8 March 2011*. Sydney, NSW: Author.
- Bakker, A., Biehler, R., & Konold, C. (2005). Should young students learn about box plots? In G. Burrill & M. Camden (Eds.), *Curricular Development in Statistics Education: International Association for Statistical Education (IASE) Roundtable, Lund, Sweden, 28 June-3 July 2004* (pp. 163-173). Voorburg, The Netherlands: International Statistical Institute.
- Bond, T.G., & Fox, C.M. (2007). *Applying the Rasch Model: Fundamental measurement in the human sciences*. (2nd ed.). Mahwah, NJ: Lawrence Erlbaum.
- Curcio, F. (1987). Comprehension of mathematical relationships expressed in graphs. *Journal for Research in Mathematics Education*, 18, 382-393.
- Gal, I. (2002). Adults' statistical literacy: Meanings, components, responsibilities. *International Statistical Review*, 70, 1-51.
- Pfannkuch, M. (2006). Comparing box plot distributions: A teacher's reasoning. *Statistics Education Research Journal*, 5(2), 27-45.
- Pierce, R. & Chick, H. (2011). Reacting to quantitative data: Teachers' perceptions of student achievement reports. In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices* (Proceedings of the 34th Annual Conference of the Mathematics Education Research Group of Australasia, Alice Springs, pp. 631-639). Adelaide: AAMT/MERGA.
- Shaughnessy, J. M. (2007). Research on statistical learning and reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 957-1009). Charlotte, NC: Information Age Publishing.
- Shaughnessy, J. M., Garfield, J., & Greer, B. (1996). Data handling. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics in education* (pp. 205-237). Dordrecht, The Netherlands: Kluwer.
- Watson, J. M. (2006). *Statistical literacy at school*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Wickham, H., & Stryjewski, L. (2011). *40 years of box plots*. Manuscript submitted for publication. Retrieved from <http://vita.had.co.nz/papers/boxplots.html>

The Classicist and the Frequentist Approach to Probability within a *TinkerPlots2* Combinatorial Problem

Theodosia Prodromou
The University of New England
<tprodrom@une.edu.au>

This article seeks to address a pedagogical theory of introducing the classicist and the frequentist approach to probability, by investigating important elements in 9th grade students' learning process while working with a *TinkerPlots2* combinatorial problem. Results from this research study indicate that, after the students had seen the systematic construction of the event space via combinatorial analysis, they viewed the sample space as an essential property that regulated the results of the distribution of each sum's theoretical frequency.

Introduction

The biggest leaps forward in the next several decades- in business, and society at large- will come from insights gained through understanding data. It is of paramount importance to acquire the mathematical skills and understandings required to enable citizens to become informed about our world and to understand the data that politicians, advertisers and other advocated are using to promote particular causes that will impact on the future of our planet. This embraces the capacity of not only understanding the underlying messages that the data attempt to reveal but also critically examining the probabilistic statements presented in news media in terms of data and data representations such as charts, tables, and graphs.

These skills and everyday understanding of mathematics might be best described as quantitative literacy (Steen, 1997). Quantitative literacy was defined as "an aggregate of skills, knowledge, beliefs, habits of mind, communication capabilities, and problem-solving skills that people need in order to engage effectively in quantitative situations arising in life and work" (Steen, 2001, p. 7). Statistical literacy is an important part of quantitative literacy skills. As Wallman (1993) suggested in her Presidential Address to the American Statistical association, there is a need to enhance citizens' "ability to understand and critically evaluate statistical results that permeate daily life and to acknowledge the contributions that statistical thinking can make in public and private, professional and personal decisions." (p. 1).

Australian Curriculum and Reporting Authority K-10 (ACARA, 2010) recognizes that the twenty-first century world is information driven, and through Statistics and Probability students can make informed judgments about events involving chance. The role of probability as a central component for statistical investigations has engendered the need for "probability literacy" to deal with a variety of real-world situations that encompass interpretation or generation of probabilistic messages as well as decision-making. Elements of probability-related knowledge and some dispositions that may be the building blocks comprise probability literacy were proposed by (Gal, 2005), but they have received little explicit attention in discussions of people's numeracy and statistical literacy. Gal (2005) listed these elements as follows: (1) several foundational big ideas such as randomness, independence, and variation; (2) Figuring probabilities of events in order to estimate the probabilities of events, (3) Language of chance, that is the terms and phrases related to chance and the various ways to represent and communicate about actual events; (4) Understanding the role and the implications of probabilistic issues and messages in different

contexts and in personal and public discourse; and (5) Critical questions to reflect upon when dealing with probabilities.

The proposed view of probability literacy illustrates a “scope of probability at the school level to reflect the study of random events, the development of appropriate probabilistic intuitions, a basic understanding of language and simple events, an appreciation of distribution and the addressing of misconceptions” (Watson, 2006, p. 127-128). This scope can be seen as consistent with the tendency to place less emphasis on the knowledge pertaining to theoretical probability when addressing important issues relevant to teaching data-based statistics (Moore, 1997). Watson (2006) claims that the aim in statistics teaching is to enhance data handling through an empirical frequency-based approach to probability that is an essential foundation for later work in the study of theoretical probability based on sample spaces. Continuous classroom-based investigations based on frequencies may reinforce the building of an appreciation of a frequency approach to probability when performing trials and comparing favourable outcomes to total outcomes. Watson (2006) also claims that it is not always appropriate to introduce an experiment to calculate relative frequencies before suggesting a theoretical model based on the possible outcomes of a sample space. Ultimately, introducing the environment of theoretical probability where data will be collected should not be left behind or dismissed. Hands-on simulations and simulation software provide students with opportunities to explore the nature of the sample space and probability distributions, particularly in the light of refining understanding of variation.

In order to better understand pedagogical theory, we have developed an experiment that uses Watson’s theory of pedagogy, and will analyse the students’ behaviour using Radford’s (2009) theory of cognition to guide our attention to important elements in the students’ learning processes. Hence, this article examines Watson’s (2006) claim, by having students explore the generation of a sample space within *Tinkerplots2*.

Radford’s (2009) theory of cognition is the knowledge objectification, a theoretical perspective on teaching and learning in which learning is taking place as a social process through which students become progressively conversant with cultural forms of reflection (Radford, 2009). Within the theory of objectification, the distinctive sensuous and artefact-mediated nature of thinking emphasizes the semiotic means of objectification through which knowledge is objectified. The semiotic means of objectification include kinaesthetic actions, gestures, signs (e.g. mathematical symbols, graphs, inscriptions, written and spoken language), and artefacts (e.g. rulers, calculators). Thus, in what follows, in the practical investigation of 9th grade students’ probabilistic thinking, attention will be paid to important elements in the students’ learning processes when students explore the generation of a sample space within *Tinkerplots2*.

Data Collection

The data come from an ongoing research study conducted in a rural secondary school. The data have been collected during regular mathematics lessons. In these lessons the students spent extensive time working in pairs. The researcher interacted continuously with the pairs of students during the pair work phase in order to probe the reasons that might explain their thinking. The data collected included audio recordings of each pair’s voices and video recordings of the screen output on the computer activity using Camtasia software. When students’ body language or facial expression appeared to be indicative of their conceptual evolution, notes were kept. The researcher (Re) prompted students to use the mouse systematically to point to objects on the screen when they reasoned about computer-based phenomena in their attempt to explain their thinking. Plain paper was also available

for students' use in case students needed to explain their thinking in a written form. This article focuses on one pair of students, Rafael (Ra) and George (Ge).

Data Analysis

In the first lesson, students watched an instructional movie that shows how to use *TinkerPlots2* features to build a simulation of rolling two dice. They then built a simulation of rolling four dice where the dice were presented by spinners (top right in Figure 1). They ran the simulation many times and graphed the sum of four dice (bottom in Figure 1).

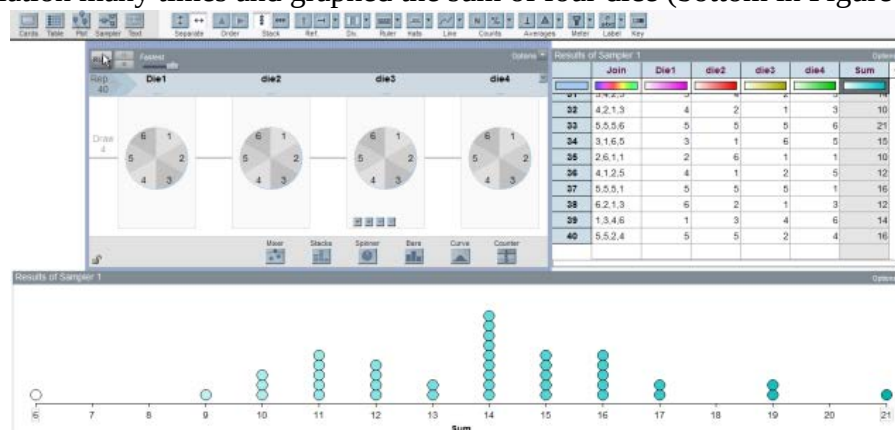


Figure 1. Simulation of rolling four dice in *TinkerPlots2* and the graph the sum of four dice.

In the second lesson, students watched a *TinkerPlots2* movie. *TinkerPlots2* provides a sampler that is essentially a non-conventional form of probability distribution. The *TinkerPlots2* movie showed how to use two counters to generate a sample space of rolling two dice (Figure 2). Each counter had numbers one through six. As the simulation ran, the

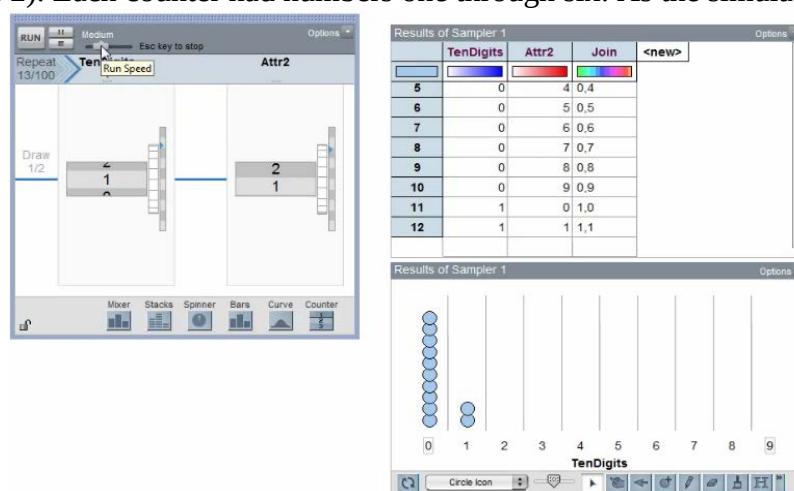


Figure 2. Two counters generate a sample space of rolling two dice

right counter circled though the numbers one through six, but the left counter stayed on one. Then the left counter advanced to two and the right counter circled through one to six again until all the possible outcomes were produced. The students observed the systematic listing of all possible outcomes of rolling two dice and the creation of a graph that shows the sum of two dice and the use of the sample space to calculate theoretical probabilities. The students were asked to use counters to build a sample space of rolling four dice (Figure 3) and then, after answering some questions, to come up with general rules applied.

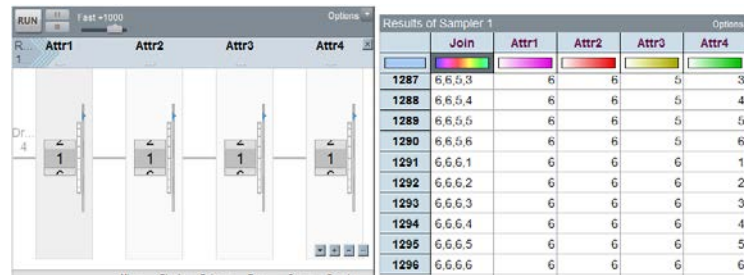


Figure 3. A sample space of rolling four dice

In tune with our theoretical framework, a multi-semiotic data analysis was conducted to investigate students' probabilistic thinking. At the first stage, the audio recordings were fully transcribed and screenshots were incorporated as necessary to make sense of the transcription. The most salient episodes of the activities were selected. Focusing on the selected episodes with the support of the transcript, along with a detailed account of significant actions, gestures, and artefacts used; I study the role of spoken language with the gestures including students' actions when pointing to objects on the computer screen and artefacts.

Results

The students built a simulation of rolling four dice and made a graph showing the sum of four dice (left, in Figure 4). They ran the simulation and observed the possible outcomes of rolling four dice. The researcher spent the rest of the first day discussing with the students the graph that showed the sum of four dice. They run the simulation 80 times.

1. Ra: We see 14 is the most likely sum. (Gesture—uses the mouse to point to the 14th column of the graph on the screen)
2. Ge: Run the simulation again.
3. Ra: This time it's 15. (G— uses the mouse to point to the 15th column of the graph on the screen)
4. Ge: Run it again, 80 times.
5. Ra: Okay, it's 16 now. (G— uses the mouse to point to the 16th column of the graph on the screen)
6. Ge: It's a bit inaccurate.
7. Ra: It's in that general area. 13, 14, 15, 16 but they're not all high values. Like see, 14, 15 taken a drop here (right, Figures 4). (G—uses the thumb and the index finger of his right hand to point to the interval from the 13th to 16th columns and then uses the mouse to point to the 14th and 15th columns of the graph individually).

Rafael changed from using the mouse to point to individual columns of the graph, to using two fingers (the thumb and the index) to indicate and emphasize an interval on the graph and then again he used the mouse to point to individual columns of the graph. Gestures dominated Rafael's respond to the researcher's question about the most likely 4-dice sum or a range of 4-dice sums. The coordinated use of spoken language and the use of gestures served as semiotic means of objectification for Rafael. The boys continued their discussion about the most likely 4-dice sum.

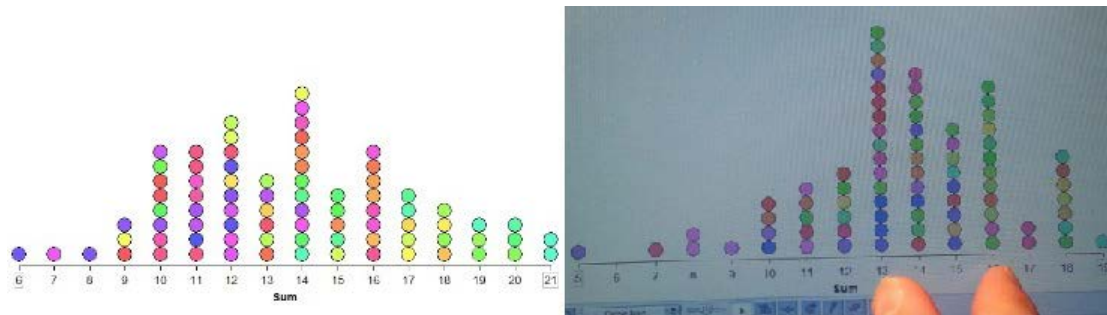


Figure 4. Graphs of the sum of four dice

8. G: Um, probably 14 because the first time it was the highest. (G- uses the mouse to point to the 14th column of the graph on the screen)
9. Ra: It's remained constant.
10. G: Second time wasn't yeah. It's remained constantly fairly high.
11. Ra: It's, um, generally it's usually the same area most likely so that would seem the most likely one that would come up. 13, 14, 15, 16 yeah, but particularly 14's always stayed constant. Like even though it's not the highest here, it's always stayed in the range of the top highest.
12. G: Yeah, it's in the top 3.

The boys reflected on previously observed graphs. Rafael became consciously aware of a region in which the most likely outcomes were included. This region was placed around the 14th column of the graph. Although the 14th column was not always the highest, it was, according to the boys, constantly included in that region. The boys continued to observe features of the graph while running the simulation several times (100 rolls each time).

13. Ra: Yeah, but 13, 14, 15, 16 they always get high. (G-uses the thumb and the index finger of his right hand to point to the interval from 13th to 16th column).
14. G: When 15 was our highest, 15 was the most likely outcome by overall.
15. Ra: If you're viewing in consistency standards, it would be 14. It seems to be the most consistently high, or the most consistently high or most consistent and reliably one to stay around that level or that range.
16. Re: Are you talking about the distribution of the outcomes.
17. Ra: Well it's generally seems to stay between um... 12, 13-ish to about 17. (G-uses the thumb and the index finger of his right hand to point to the interval from 12th to 17th column).

The boys ran the simulation several times increasing the number of rolls from 200 to 250. They both concluded that 14, 15, 16 have the highest consistency. George added:

18. G: That seems to be what's happened. That they are the most likely you're going to get.

The next day (2nd lesson) the researcher drew students' attention to permutations without using any explicit mathematical terminology.

19. R: Which 4-dice sum can be made up by adding different ordered numbers?
20. Ra: It should be ... There is only 4 chances out of the possible outcomes. Like if I was to get 24 I would have to get 4 six's
21. G: That's very unlikely. We didn't get it there (referring to the graph they observed during the previous lesson).
22. Ra: The reason, the range of those numbers (14 to 15 range).... were so high because there's so many dice. You don't need to get much of each dice. Those numbers there allow you to have a wider variety of combinations like um if I... 13, 14, 15 being the highest ones.

The boys were able to appreciate the connection between the outcomes from the previously observed graphs, with the different faces of 4 dice that summed up to those outcomes. While they attended to the shape of the graph, the permutations were not perceptually distinguishable as actual experimental outcomes. When the researcher asked students to list (on paper) those permutations that added up to 15, the boys began to randomly list the possible outcomes for rolling four dice.

23. Ra: If I write it down once, do you want me to write it down every other way I could write it down? I mean if I have, I have $6 - 4 - 3 - 2$ do you want me to also write $3 - 4 - 6 - 2$? Is it important?

24. Re: Yes. The order of the inscriptions is important. Why do you think that the order is important?

25. Ra: I'm thinking, I guess, I'm trying to like answer but I don't know if it's right but it's like it's, they vary...

26. Ra: ... it's just umm, like the fact that it's variable. Like it's varied umm, I think is important because like there is a better chance of getting that because the amount of variables that you get with the equation that equals up to 15....If that sort of makes sense, I don't know, I'm just thinking.

As the previous dialogue shows, Rafael seemed able to understand the uniqueness of permutations (i.e., that $[3,5,6,1]$ and $[5,3,1,6]$ were distinct) but he was unable to articulate why permutations are distinct. When the researcher mentioned that the order of inscriptions is important, Rafael seemed aware of the significance, but was unable to explain it. The researcher instead of impressing upon the students the importance of the ordered outcomes in a combinatorial experiment, asked students to watch a *TinkerPlots* movie which showed how to use two counters to generate a sample space of rolling two dice. The students observed the systematic listing of all possible outcomes of rolling four dice and made the graph that showed the sum of four dice (Figure 5).

27. Ra: That's how I would have worked out the four but I would have never ... it would take someone too long to do by hand anyway. And I suppose they also do this with locks and stuff. Yeah, when you get your bike lock. The 4 combinations 1 to 6. It says how many combinations. I honestly never thought there would be that many. Pretty, amazing!

When Rafael was previously asked to list all the possible outcomes of the sample space

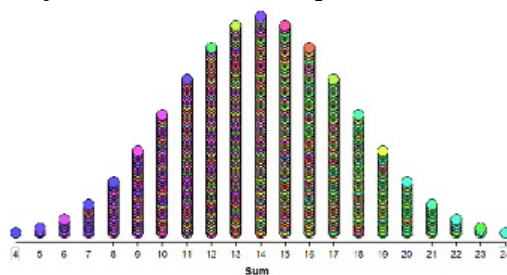


Figure 5. The that showed the theoretical outcomes of the sum of four dice

he had difficulties with constructing combinatorial type outcomes, because he either did not exhaust the sample space or duplicated possibilities. When Rafael watched how *TinkerPlots* features systematically listed all the possible outcomes of rolling four dice using the odometer strategy, he was surprised by the number of possible outcomes. The animated generation of a sample space made it possible for the students to see the use of various representations in solving the combinatorial problem situation including the use of animations, systematic listings, and a table holding the all possible outcomes of rolling 4-

dice; these are semiotic means of objectification. When the students attempted to talk about the graph that showed the theoretical outcomes (as opposed to the histograms generated by running a sample, like they were looking at earlier in the paper):

28. Ra: Look at this peak, being 13, 14, 15 high (G-uses the thumb and the index finger of his right hand to point to the interval from 13th to 16th column). We noticed that before (refers to the graph they observed during the first lesson).
29. G: The difference between these two is that one (the graph created using all the outcomes of the sample space) is a lot more accurate as it's going through every single stage compared to the other one.
30. Ra: This is a lot more thorough, as it goes through every single possibility you could have ... It would probably be really helpful if you knew this, if you were gambling or placing bets... It shows you the possible outcomes if you're playing a game of chance. If you want to know the possible outcomes of the ace coming up. Like you want to know if there is more chance than it won't come up, than it will not. You could do that also, at horses.

The gesture (uses the thumb and the index finger of his right hand to point to the interval from 13th to 16th column) reminded boys of the previously observed graph, so they began comparing the two graphs before the researcher would ask them (the researcher intended to ask such a question). After this interplay, we observe that although R and G had never been taught at school about the classicist and frequentist approaches to probability, the means of objectification helped them to develop understandings about the two approaches to probability. The students expressed a preference for using the classicist approach to conceptualise the probabilistic experiment. They viewed the classicist approach as a "more accurate" process, because it is based on all the possible outcomes of the sample space. The sample space appeared to the students as an essential property that can regulate the results of the graph, which shows the theoretical frequencies of the possible sums of 4-dice. After the students had seen the systematic construction of the event space via combinatorial analysis performed by features of *TinkerPlots2*, both Rafael and George made sense of the role of the event space in relation to the experiment it is said to model. They saw connections between the classicist approach and games of chance and suggested how to apply such a probabilistic approach to the solution of new problems.

Discussion

Our data offer an initial glimpse of the development of probabilistic thinking about combinatorial problems. It shows how the *TinkerPlots2* combinatorial problem implemented the pedagogical objective of enabling Grade 9 students with minimal probabilistic knowledge to understand the generation of the event space of rolling four dice. Students' "spontaneous" perception was successfully transformed through the interaction of students with the *TinkerPlots2* combinatorial problem and the systematic listing of all possible outcomes of rolling four dice. This interaction might be conceptualized as occurring in the zone of proximal development out of which the students constructed new understandings about the relevance of order (permutations) in the construction of the event space. The researcher questioned students as they explored the combinatorial problem, to promote students' combinatorial understanding. The interaction of spoken language with gestures (using fingers to point to the graph, using the mouse to point to the graph on the screen), signs (graphs, written inscriptions, tables) and artefacts (computer animation) led students to modify, or refine some of their original structuring relationships between the semiotic means of objectification, or to consolidate new structuring relationships and

understandings of probabilistic concepts related to both the classicist and frequentist approaches.

In this paper, there is evidence that the students expressed a preference for using the classicist approach to conceptualise the probabilistic experiment. The results should not be interpreted to indicate that activities that enable students to encounter classicist approaches to key principles of combinatorial concepts are sufficient to achieve desired pedagogical objectives.

References

- Australian Curriculum, Assessment and Reporting Authority. (2010). *Australian Curriculum: Mathematics*. Version 1.1. Retrieved March 15, 2011, from <http://www.acara.edu.au>
- Gal, I. (2005). Towards “probability literacy” for all citizens: building blocks and instructional dilemmas. In G. A. Jones (Ed.), *Exploring Probability in School: Challenges for Teaching and Learning* (p. 39-63). New York: Springer.
- Moore, D. S. (1997). New pedagogy and new content: The case of statistics. *International Statistical Review*, 65(2), 123-165.
- Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, 70, 111-126.
- Steen, L. A. (Ed.). (1997). *Why numbers count: Quantitative literacy for tomorrow's America*. New York: College Entrance Examination Board.
- Steen, L. A. (Ed.). (2001). *Mathematics and democracy: The case for quantitative literacy*. Washington DC: Woodrow Wilson National Fellowship Foundation.
- Wallman, K. K. (1993). Enhancing statistics literacy: Enriching our society. *Journal of the American Statistical Association*, 88, No. 421, 1-8.
- TinkerPlots: *Dynamic data exploration* (Version 2.0) [Computer software]. Emeryville: CA: Key Curriculum Press.
- Watson, J. M. (2006). *Statistical literacy at school: Growth and goals*. NJ: Lawrence Erlbaum Associates.

Diffusion of the Mathematics Practical Paradigm in the Teaching of Problem Solving: Theory and Praxis

Khiok Seng Quek

Nanyang Technological University

<khiokseng.quek@nie.edu.sg>

Eng Guan Tay

Nanyang Technological University

<engguan.tay@nie.edu.sg>

Yew Hoong Leong

Nanyang Technological University

<yewhoong.leong@nie.edu.sg>

Tin Lam Toh

Nanyang Technological University

<tinlam.toh@nie.edu.sg>

Jaguthsing Dindyal

Nanyang Technological University

<Jaguthsing.dindyal@nie.edu.sg>

In this paper, we discuss the diffusion (of an innovation) and relate it to our attempt to spread our initial design of a mathematics practical paradigm in the teaching of problem solving.

Mathematics education researchers build innovations to improve mathematics teaching and learning (Roschelle, Tatar, Shechtman & Knudsen, 2008). For innovations to effect the intended changes at scale, they need to be supported by education theory as well as a theory of spreading the innovation. Yet, it was acknowledged that few mathematics education researchers are involved in research on the challenges, demands and outcomes of implementing innovations on a larger scale (Elmore, 1996).

This paper discusses issues related to the diffusion of our initial innovation for the teaching of mathematical problem solving in Singapore schools. This is part of the project MProSE (Mathematical Problem Solving for Everyone).

Briefly, the research project MProSE is an attempt to address the gap between what is enacted as teaching problem solving in Singapore mathematics schools and what is intended in the Singapore mathematics curriculum. Anecdotal evidence suggests that most teachers at the primary school levels have reduced problem solving to the teaching of heuristics, and that problem solving is usually downplayed in view of the national high-stake examinations at the Upper Secondary level (Toh, Quek, Leong, Dindyal & Tay, 2009). For details on the rationale and conceptualization of the earlier phases of the MProSE project, the reader may refer to Toh, Quek and Tay (2008) and will not be elaborated in this paper.

Diffusion of Educational Innovations

In examining the model of diffusion, we draw heavily from Rogers (2003) who is widely acknowledged to hold a seminal position in the *theory of diffusion of innovations*. According to him, diffusion is the “process by which an innovation is communicated through certain channels over time among the members of a social system” (p. 5). Rogers further explained diffusion as one which is dictated by uncertainty reduction behaviour among potential adopters during the introduction of innovations. Innovative practices offer its potential adopters new ways of handling the problems, but the uncertainty as to whether it will be better than the existing method poses an obstacle to the adoption process. To overcome this, potential adopters must seek additional information, particularly from their peers. Rogers proposes five factors that influence the rate of adoption: observability, trialability, compatibility, complexity and relative advantage. In MProSE, observability refers to the degree in which the MProSE innovation is seen by the adopting schools as

producing the expected results; trialability as the degree in which the schools can experiment with the innovation in a limited way; compatibility as the degree to which the schools see the innovation as being in line with their values, experiences, and needs; complexity or, simplicity and ease of use, as the degree to which the schools perceive the innovation as easy to understand and use; and lastly, relative advantage as better than their current teaching of mathematical problem solving

Diffusion occurs fast if only minimal amount of attention is needed for people to embrace the innovation (Larson & Dearing, 2008). However, the literature abounds with examples of many highly effective innovations which did not achieve widespread use. According to Fishman, Blumenfeld, and Krajcik (2004), many effective innovations in Learning Sciences have not spread widely into the classrooms. This is the problem space in which research on diffusion fills. MProSE is a part of this diffusion research that attempts to spread the mathematics practical paradigm design beyond the initial site of innovation.

MProSE as Innovation

Rogers (2003) states that “[a]n innovation is an idea, practice, or object that is perceived as new by an individual ... If an idea seems new to an individual, it is an innovation.” (p. 11) Since mathematical problem solving has been at the heart of the Singapore mathematics curriculum from the 1980s, in what way is MProSE an innovation? While MProSE does not claim to be innovative at the level of theory-generation (for we built on the theoretical cornerstones of Polya and Schoenfeld’s work), the innovation is in the way that schools carry out the teaching of problem solving. The theory has been worked out, as Schoenfeld (2007) pointed out:

That body of research—for details and summary, see Lester (1994) and Schoenfeld (1985, 1992)—was robust and has stood the test of time. It represented significant progress on issues of problem solving, but it also left some very important issues unresolved ... The theory had been worked out; all that needed to be done was the (hard and unglamorous) work of following through in practical terms. (p. 539)

Making Mathematics Problem Solving ‘Practical’

MProSE introduces a practical paradigm for the teaching of mathematical problem solving, analogous to the practical in the teaching of science. In an attempt to ‘make’ the students follow through Polya’s processes of problem solving¹, especially when they were clearly struggling with the problem, MProSE problem solving lessons use a worksheet similar to that used in science practical lessons. The students were instructed to treat the problem solving class as a mathematics ‘practical’ lesson, in the sense that they are asked to work on only one problem just as they would work on only one experiment in the science practical. For details on the complete set of materials for the specialised problem solving lessons, including the mathematics problems used, lesson plans and teacher guides, the reader may refer to Toh, Quek, Leong, Dindyal & Tay (2011a).

Our use of the word ‘practical’ has also a different sense: the meaning of practicability. An attempt to implement any novel curriculum—including one centring on problem solving—must take into account the complexities and realities of classroom practice (Ball, 2000; Lampert, 2001). Teachers need to balance different and sometimes competing goals

¹ Polya’s model is highlighted in the Singapore mathematics curriculum.

of teaching (Wood, Cobb, & Yackel, 1995). Teaching problem solving is thus not seen as an isolated or only goal in this project but is analysed in the realistic context of other interacting and worthy instructional goals (such as keeping to time in teaching) as teachers carry out practice in the classroom (Leong, Dindyal, Toh, Quek, Tay & Lou, 2011). As such, the focus in MProSE involves not only careful framing of the roles of problem solving in the school mathematics curriculum and the ongoing development of teachers to prepare them for the enterprise but also the workability of a heavy emphasis on problem solving in actual classroom practice.

Design Experiment

MProSE uses design experiment as the overarching methodological approach (Brown, 1992; Collins, 1999; Doerr & Lesh, 2003; Middleton, Gorard, Taylor & Bannan-Ritland, 2006; Quek et al, 2011; Wood & Berry, 2003), which has its roots in the field of engineering. Design experiment is adopted by the education research community to address the demands of research in real-life school settings in all its complexity. It is an approach to “the development of theory and method based in the real-time, formative experience of implementing, assessing, and improving classroom practice, classroom research, and classroom learning” (Larson & Dearing, 2008, p. 512).

The methodology of design experiment argues for the application of multiple techniques to study a complex phenomenon in education. The envisaged outcome of MProSE is to produce a workable problem solving design that can be adapted to the setting of mainstream Singapore schools (Quek, Dindyal, Toh, Leong & Tay, 2011). In the process, like product refinements in engineering design experiments, education designs undergo iterative cycles of adjustment to fit local school conditions of implementation. This aspect of design evolution dovetails with a key principle in Rogers’ *Diffusion of Innovations*: reinvention. The principle of reinvention refers to the degree to which the innovation evolves to meet the individual needs of demanding or risk-averse people in the population. As such, in the MProSE design experiment, school leaders and teachers have key roles in the design. The designer-researcher and teachers must collaborate in the entire process. We have identified two key sub-processes in the design experiment to develop the MProSE problem solving product: (1) *refining* in the light of research findings; and (2) *accommodation* to meet the realistic constraints faced in practice. We think it is important in design research to distinguish between the two sub-processes. Figure 1 shows the design-theory-practice troika underlying our design-experiment for MProSE. The initial design had to be adapted for the purpose of diffusion among the mainstream schools. The school has to make changes (for example, curriculum reorganisation and teacher capacity) to accommodate the problem solving design.

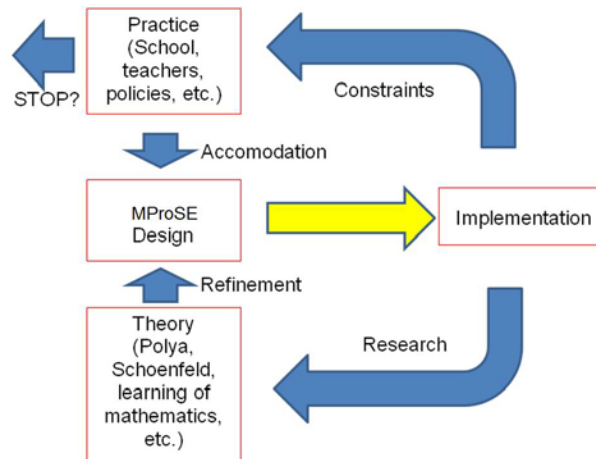


Figure 1. Design-theory-practice troika underlying MProSE.

Diffusion of MProSE

MProSE first implemented the innovation in a school in Singapore which specialises in the teaching and learning of Mathematics and Science (hereafter we shall refer to as MProSE Research School). This school admits students it considers to be high achievers in mathematics or science. MProSE reasoned from a “best-case scenario” perspective that the test bed for the initialisation phase of an innovation should be in a school that would be most conducive for success.

After two years of study in the MProSE Research School, the school adopted the mathematics practical for all Year 8 (age 15 to 16) students. The relevant findings may be summarised as follows:

- the practical paradigm useful to guide them in solving mathematics problems (Toh, Quek, Leong, Dindyal & Tay, 2011b); and
- the students were able to demonstrate the use of Polya’s model in solving mathematics problem . (Toh, Quek, Tay, Leong & Dindyal, 2011).

A problem solving seminar was organised with a view of inviting teachers from other mainstream schools to participate in the diffusion stage of the MProSE design experiment. The seminar disseminated the findings and shared lessons learnt from the MProSE Research School (<http://math.nie.edu.sg/mprose/seminars.aspx>). More than 200 teachers participated in the seminar. Generally, the feedback was positive. Teachers from thirteen schools expressed their initial interest to participate in the MProSE project. Subsequently, the MProSE team obtained written forms of commitment of participation from four schools, roughly spanning across the performance band. The original MProSE mathematics problems were modified to suit the needs of these schools.

Focussing on First Stage of MProSE Diffusion: Adoption

According to Rogers (2003), the first stage in the diffusion process is adoption. Potential to address adoption we consider the five aspects as spelt out by Rogers.

Relative Advantage

Singapore teachers have been successful in preparing their students for the high-stake national examinations. They would be hesitant to adopt the MProSE design if they do not perceive its relative advantages. Through personal communication between the MProSE team and the teachers, we noted that teachers recognized the trend of increasing the number of non-routine problems occurring in the national examinations. They also acknowledged that the usual practice of “routinizing non-routine problems” might not be a sustainable mode of instruction to this end. They were looking for alternative approaches in the teaching problem solving. This is where they think MProSE could fill the gap.

Compatibility

The MProSE is compatibility with the existing Singapore mathematics curriculum. The heart of the Singapore mathematics curriculum is mathematical problem solving, and Polya’s model is highlighted as a model to use. There is thus no conflict at the curricular level between MProSE design and the Singapore mathematics framework.

At the local level, to make the MProSE design more workable for mainstream schools, the original MProSE mathematics problems were modified to render them more compatible with the needs of the schools. For example, in the original MProSE design, Problem A was used to introduce the heuristics of “Using Suitable Numbers” and “Thinking of a Related Problem”

Problem A

Show that the integer n always has the same last digit as its fifth power n^5 .

While Problem A was suitable in the MProSE Research School, it was deemed not so appropriate in the mainstream schools, as they do not emphasize Number Theory in the Lower Secondary school curriculum. Keeping the same objective of introducing heuristics, Problem B was designed to replace Problem A.

Problem B

In 2009, Peter was given a pay rise of 5% and in 2010, he was also given a pay rise of 5%. Is his total pay rise 10% over the two years? Justify your answer.

Problem B was considered to be suitable for mainstream schools since the topic of Percentage is in the school mathematics curriculum.

Complexity

In acknowledging that teachers may find the MProSE design to be complex, we provided the necessary support in building teachers’ capacity to understand the design as well as to implement it. We devote much of the professional development time to building the requisite content knowledge and the matching pedagogy. In particular, during the professional development courses, one of the MProSE researchers discussed explicitly the content knowledge required for the problems and modelled the pedagogy for teaching the problem solving processes. In addition, schools that needed further support in teacher implementation can tap upon the MProSE team’s expertise in observing their teaching in the context of Lesson Studies. During these post-observation meetings with teachers, some of these complexities were discussed and unpacked.

Trialability

There was opportunity within the MProSE design for the collaborating schools to trial the design on a limited scale. For example, in one school, the Head of Department proposed the trialling of MProSE for a selective number of Year 7 classes. In the language of Rogers (2003), the early implementers could serve as early *adopters* of the innovation. Subsequently, the innovation may spread within the school to a stage where the later adopters from the remaining Year 7 classes could buy in and implement the design. This is in line with the vision of the Principal of the same school: that other teachers of the remaining Year 7 classes would implement the design after they have witnessed the tangible benefits of the initial design.

Observability

Observability refers to how visible the *results* of an innovation is. In the case of Singapore teachers, the *results* are often interpreted in terms of student achievement in examinations. More specifically, they may measure the usefulness of the innovation in terms of their student results in these examinations, after attending MProSE lessons. In keeping with the sub-process of accommodation within our design experiment, we recognize this need to consider examination results as an indicator of success of the innovation. In fact, the MProSE team rode on the affordances within the project to answer the following question:

Do the students who participate in the MProSE programme improve their performance in the mathematics achievement tests?

The observability of such results in an educational innovation would aid in the adoption of our MProSE innovation.

Summary

In this paper, we briefly reviewed the theoretical and practical issues regarding the attempt to spread the MProSE design innovation to other schools in the light of Rogers' (2003) theory of diffusion of innovations. In particular, we are at the stage of negotiating adoption with a number of schools which have shown initial interest. Using further Rogers' five aspects to predict adoptability it seems promising. But we anticipate the entire process of diffusion to the social system (Singapore schools) to be a long process fraught with challenges. We take comfort that this investment of time to cultivate ongoing collaborative researcher-practitioner partnership is one of the key enablers of diffusion of innovations. In fact, according to Lemke and Sabelli (2008), "[t]he development of effective partnerships takes 5 – 10 years" (p. 125). Hopefully this project could provide insight into the issues, challenges, and critical success factors in the diffusion of innovations to mainstream schools.

References

- Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000 Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 3-10). Perth: MERGA.
- Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions. *The Journal of the Learning Sciences*, 2, 137-178.

- Collins, A. (1999). The changing infrastructure of education research. In E. C. Langemann & L. S. Shulman (Eds.), *Issues in education research* (pp. 15 - 22), San Francisco, CA: Jossey-Bass.
- Doerr, H. & Lesh, R. (2003). Designing research on teachers' knowledge development. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics Education Research: Innovations, Networking, Opportunities* (pp. 262-269). Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Geelong.
- Elmore, R.F. (1996). Getting to scale with good educational practice. *Harvard Educational Review*, 66(1), 1 – 26.
- Fishman, B., Marx, R. W., Blumenfeld, P., & Krajcik, J. (2004). Creating a framework for research on systemic technology innovation. *The Journal of the Learning Sciences*, 13(10), 43 – 76.
- Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.
- Larson, R. S., & Dearing, J. W. (2008). Design research and the diffusion of innovations. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education* (pp. 511 – 534). Mahwah, NJ: Lawrence Erlbaum Associates.
- Lemke, J. L., & Sabelli, N. H. (2008). Complex systems and educational change: Towards a new research agenda. *Educational Philosophy and Theory*, 40(1), 118 – 129.
- Leong, Y. H., Dindyal, J., Toh, T. L., Quek, K.S., Tay, E. G., & Lou, S. T. (2011). Teacher education for a problem-solving curriculum in Singapore. *ZDM: The International Journal on Mathematics Education*, 43(6-7), 819-831.
- Leong, Y. H., Tay, E. G., Toh, T. L., Quek, K. S., & Dindyal, J. (2011). Reviving Polya's "look Back" in a Singapore school. *Journal of Mathematical Behavior*, 30(3), 181-193.
- Lester, F. K. (1994). Musing about mathematical problem-solving research: 1970-1994. *Journal of Research in Mathematics Education*, 25, 660-676.
- Middleton, J., Gorard, S., Taylor, C. & Bannan-Ritland, B. (2006) The 'compleat' design experiment: from soup to nuts. *Department of Educational Studies Research Paper 2006/05*. University of York.
- Polya, G. (1954). *How to solve it*. Princeton: Princeton University Press.
- Quek, K. S., Dindyal, J., Toh, T. L., Leong, Y. H., & Tay, E. G. (2011). Problem Solving for Everyone: A design experiment. *Journal of the Korea Society of Mathematical Education Series D: Research in Mathematical Education*, 15(1), 31-44.
- Rogers, E. M. (2003). *Diffusion of innovations (5th Ed)*. New York: Simon and Schuster.
- Roschelle, J., Tatar, D., Schechtman, N., & Knudsen, J. (2008). The role of scaling up research in design for and evaluating robustness. *Educational Studies in Mathematics*, 68, 149 – 170.
- Schoenfeld, A. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
- Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, 334-370. New York: Macmillan.
- Schoenfeld, A. (2007). The complexities of assessing teacher knowledge. *Measurement: Interdisciplinary Research and Perspectives*, 5(2/3), 198 – 204.
- Toh, T. L., Quek, K.S., & Tay, E. G. (2008). Mathematical problem solving – a new paradigm. In Vincent, R., Pierce, R., & Dowsey, J. (Eds.) *ConnectEd Maths* (pp. 356 – 365). Melbourne: Mathematical Association of Victoria.
- Toh, T. L., Quek, K. S., Leong, Y. H., Dindyal, J., Tay, E. G. (2009). Assessment in a Problem Solving Curriculum. In Hunter, R., Bicknell, B., & Burgess, T. (Eds.) *MERGA 32 Conference Proceedings* (pp. 686-690). Wellington, New Zealand: Massey University.
- Toh, T. L., Quek, K. S., Leong, Y. H., Dindyal, J., Tay, E. G. (2011a). *Making Mathematics Practical: An Approach to Problem Solving*. Singapore: World Scientific.
- Toh, T. L., Quek, K. S., Leong, Y. H., Dindyal, J., & Tay, E. G. (2011b). Assessing problem solving in the mathematics curriculum: a new approach. In B. Kaur & K. Y. Wong (Eds.) *Assessment in the mathematics classroom* (pp. 33 – 66). Singapore: Association of Mathematics Educators.
- Toh, T. L., Quek, K. S., Tay, E. G., Leong, Y. H., & Dindyal, J. (2011). Enacting a problem solving curriculum in a Singapore school. In L.A. Bragg (Ed.), *Maths is multi-dimensional* (pp. 87 – 94). Melbourne: Mathematical Association of Victoria.
- Wood, T. & Berry, B. (2003). What does "design research" offer mathematics education? *Journal of Mathematics Teacher Education*, 6, 195-199.
- Wood, T., Cobb, P., & Yackel, E. (1995). Reflections on learning and teaching mathematics in elementary school. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp.401-422). Hillsdale, NJ: Lawrence Erlbaum Associates.

Coordinating Known and Unknown Quantities in a Multiplicative Context: Problem Conceptualization, Affordances and Constraints

Ajay Ramful

Mauritius Institute of Education

<a.ramful@mieonline.org>

In line with continuing efforts to explain the demanding nature of multiplicative reasoning among middle-school students, this study explores the fine-grained knowledge elements that two pairs of 7th and 8th graders deployed in their attempt to coordinate the known and unknown quantities in the gear-wheel problem. Failure to conceptualize the multiplicative relation in reverse, mainly due to the numeric feature of the problem parameters and inherent inverse proportional relationship, led the students to use more primitive fallback strategies.

Multiplication and division situations have been analyzed from different orientations in mathematics education (Greer, 1992). For instance, Steffe (1994) analyzed children's multiplying schemes from the perspective of units-coordination. Vergnaud (1988) interpreted multiplication and division situations in terms of dimensional analysis. Thompson (1994) gives accounts of multiplicative reasoning in terms of quantities and quantitative relationships. Schwartz (1988) argues that multiplication and division are referent-transforming operations and distinguishes between extensive and intensive quantities. Nesher (1988) analyzed multiplicative relations in terms of the textual structure of problems. Fischbein et al. (1985) illustrate how implicitly-held models of multiplication and division lead to cognitive conflict with formal algorithmic structures. They assert that "people naturally tend to interpret facts and ideas in terms of structured models that are behaviorally and enactively meaningful" (p. 15). For example, the primitive model for multiplication is repeated addition.

In its basic form, a multiplication/division situation can be regarded from the perspective of a relation, mathematically equivalent to $a_1 \times a_2 = a_3$, where a_1 , a_2 , and a_3 are either integers, rational or real numbers. If, for instance, a_2 is unknown, one has to coordinate the known quantities a_1 and a_3 to determine a_2 . Among others, the context and numeric features of a problem determines how such a multiplicative relation is established by the problem solver. In a previous study (Ramful & Olive, 2008), I analysed how two middle-school students articulate the multiplicative relation $w_1 d_1 = w_2 d_2$ in a balance beam context (where w_1 , w_2 , d_1 and d_2 represent the weights and distances on the two sides of the fulcrum). In the present study I analyse the multiplicative relation $n_s S = n_L L$ when two gears turn synchronously (where S , L , n_s , and n_L represent the number of teeth and number of turns of the two gear wheels). The main aim of this study was to understand how problem conceptualization enables or constrains students in reasoning multiplicatively. In that sense, I looked for those critical knowledge elements which opened solution paths or which hindered students from coordinating the quantities. I referred to the literature on multiplication/division, fraction, ratio, and proportion to identify those constraints that have been shown to affect students' ability to work with multiplicative structures. Vergnaud (1988) points out that:

the complexity of problems depends on the structure of the problem, on the context domain, on the numerical characteristics of the data, and on the presentation; but the meaning and the weight of these factors depend heavily on the cognitive level of the students. (p. 143)

Along the same lines, Kaput & West (1994) point out that there are four broad categories of task variables to take into account in problem solving situations: semantic structure, numerical structure, tools and representation, and the forms of the text. Fischbein et al. (1985) also list a similar set of factors: familiarity of context, quantities involved, size and type of numbers involved, relation between the situation referred to and the appropriate operation, rigidity effects associated with specific operations and intuitive intervening models. They found that the numerical features (e.g. decimals and size of operator and operand) of the data in a problem determined the choice of arithmetic operation as being either multiplication or division. Further, they argue that such intuitive models may often “slow down, divert or even block the solution process when contradictions emerge between the model and the solution algorithm” (p. 14). In addition, Harel, Post, & Behr (1988) showed that changing a divisor from an integer to a decimal may lead respondents to different interpretations of the same problem.

Theoretical Framework

To understand how the participants in this study conceptualized the problem situations, I used Vergnaud’s (1988) idea of concepts-in-action and theorems-in-action. Vergnaud’s theory provides a theoretical framework that permits the articulation between the mathematical problems to be solved, knowledge deployed, schemes, concepts and symbols involved in the solution procedure. Mathematical concepts exist in relation to each other and draw their meaning from a variety of situations. To analyze the complexity of the interrelatedness of concepts, Vergnaud (1988) introduced the theory of conceptual fields. He defines a conceptual field as “a set of situations, the mastering of which requires the mastery of several concepts of different natures” (p. 141). He considers the conceptual field of multiplicative structures as “all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply and divide” (p. 141). This field can be regarded as consisting of a range of concepts and operations including multiplication, division, fraction, ratios, proportions which are referred as concepts-in-action. Apart from mathematical concepts, it also encompasses students’ ideas (both competencies and misunderstanding), “procedures, problems, representations, objects, properties, and relationships that cannot be studied in isolation” (Lamon, 2007, p. 642). Vergnaud characterizes students’ reasoning by symbolizing the different ways in which they articulate mathematical relations. He defines Theorems-in-action as the “mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem” (1988, p. 144). Theorems-in-action are held to be true propositions and may even be flawed. They provide behavioral evidence of the mathematical relations that students may be using. The same mathematical relation may be interpreted through various theorems-in-action. Moreover, the term *affordance* is conventionally referred to the action potential (Bower, 2008) of an ICT-mediated environment. I use the term *affordance* to refer to a representation of a theorem-in-action that opens solution paths in the course of mathematical problem solving.

The Multiplicative Context: The Gear-Wheel Problem

I chose the gear-wheel problem (Lamon, 1999) as a multiplicative context to identify the concepts- and theorems-in-action that the four participants deployed as well as the conflicts

that they encountered. Essentially, this problem consists of coordinating the number of teeth (S and L) and turns (n_s and n_L) as two gears with different number of teeth turn synchronously. This problem shares three features that make it suitable for studying multiplicative reasoning. Firstly, this multiplicative comparison situation is complex enough to probe students' thinking as it requires the coordination of 4 variables (S , L , n_s , n_L) which are in multiplicative relation, i.e., the four variables are related in terms of proportion and not additively. Secondly, it involves an inverse proportion relation between the number of teeth and the number of turns as can be inferred from $n_s S = n_L L$ or $n_s / n_L = L / S$ or $n_s : n_L = L : S$. Thirdly, it can be solved in a number of ways and offers possibilities to identify different theorems-in-action. In fact, the relation, $n_s S = n_L L$ may be interpreted in a myriad of ways (e.g., $n_s / n_L = L / S$, $n_s = (L \times n_L) / S$, $n_s = (L / S) \times n_L$ or in terms of a ratio or proportion), all of which are mathematically equivalent but not necessarily operationally from the perspective of mental processes. Further, the numeric feature of the problem parameters may allow other creative strategies, as the data in this study show.

Method

Data were collected from two pairs of students at grades 7 and 8 in an urban middle school. The grade 7 students (Aileen and Brian) were interviewed in May 2008 and January 2009 while it was only possible to interview the grade 8 students (Jeff and Eric) in May 2008. In some cases, observing the constraints that the students encountered, I interviewed them on the same task on two different days. In the initial phase of the clinical interview (Ginsburg, 1997), I asked them to determine the number of turns that the small gear makes when the large gear makes only one turn and vice-versa (Table 1).

Table 1

Structure of Problems (S and L : number of teeth; n_s and n_L : number of turns)

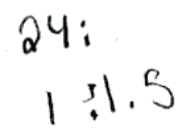
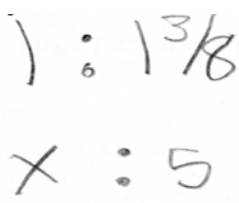
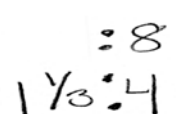
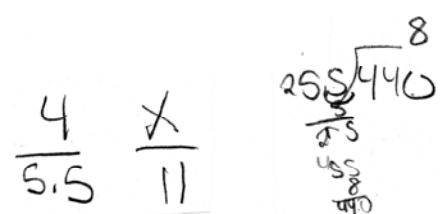
<i>Problem</i>	<i>Structure</i>	<i>Problem</i>	<i>Structure</i>
1	$S = 5, L = 10, n_s = 1, n_L = ?$	7	$S = 8, L = ?, n_s = 4, n_L = 1 \frac{1}{3}$
	$S = 5, L = 10, n_s = ?, n_L = 1$	8	$S = 8, L = 11, n_s = 5, n_L = ?$
2	$S = 8, L = 11, n_s = 1, n_L = ?$	9	$S = 8, L = ?, n_s = 4, n_L = 1 \frac{1}{3}$
	$S = 8, L = 11, n_s = ?, n_L = 1$	10	$S = ?, L = 11, n_s = 5 \frac{1}{2}, n_L = 4$
3	$S = 6, L = ?, n_s = 3, n_L = 1$	11	$S = ?, L = ?, n_s = 1, n_L = 2$
4	$S = ?, L = 24, n_s = 1 \frac{1}{2}, n_L = 1$	12	$S = ?, L = ?, n_s = 1 \frac{1}{2}, n_L = 1$
5	$S = ?, L = 36, n_s = 1 \frac{1}{2}, n_L = 1$	13	$S = ?, L = ?, n_s = 2 \frac{3}{4}, n_L = 1 \frac{1}{4}$
6	$S = ?, L = 20, n_s = 1 \frac{1}{4}, n_L = 1$		

Gradually, the number of turns and teeth were increased and they were asked to coordinate the four parameters in the problem. I encouraged them to work in pairs and allowed them to solve the problem to the point where they were satisfied with their answer. I kept their written work as a trace of their thinking moment-by-moment. I used two video cameras so as to produce a restored view (Hall, 2000). In problems 3 to 10, three parameters were specified and the students had to determine the fourth unknown parameter by articulating the multiplicative relation. Problems 11 to 13 involved the determination of two unknowns, S and L , starting from two known quantities (n_s and n_L).

Results and Discussion

The four students differed in the ways in which they conceptualized and coordinated the number of teeth and number of turns in the gear-wheel problem. I characterize the way in which they conceptualized the problem in terms of the concepts- and theorems-in-action that they used consistently in Tables 2 and 3.

Table 2
Problem Conceptualisation by the 7th Graders

	Concept-in-action		Theorem-in-action
Aileen	She interpreted the number of teeth and number of turns in terms of two ratios (represented in the form of a proportion template). In each of the different situations she generated two ratios, one for the number of turns and one for the number of teeth.	Problem	
		4	
		Problem	
		8	 She subtracted $\frac{3}{8}$ from the 5 turns, compensating additively rather than multiplicatively
		Problem	
		9	
		Problem	
		10	
Brian	He interpreted the problem as multiplication and division situation, and did not explicitly wrote a ratio to coordinate the quantities unlike his interviewer partner Aileen. He focused primarily on number of turns.	Problem	
		4	He interpreted $1\frac{1}{2}$ as 3 units of $\frac{1}{2}$ and constructed one whole as $\frac{2}{3}$ of $1\frac{1}{2}$, i.e., $\frac{2}{3}$ of 24.
		Problem	
		8	He multiplied $1\frac{3}{8} \times 5$ which represents the number of turns that the small gear makes when the large gear makes 5 turns. He did not consider the inverse proportional relation.
		Problem	
		9	He multiplied 4×8 (i.e., 32) by $1\frac{1}{3}$ to obtain $42\frac{2}{3}$ operating $1\frac{1}{3}$ on the known quantity rather than the unknown quantity .

<i>Problem</i>	He gave the following answer: “It will be 44 instead because this go by 4 turns. It started at 11, 11 times 4 is 44. So, if one has 44, this thing is going all around once. Then the other one have to have 5.5 times ... “ and he did not proceed further.
10	

Table 3

Problem Conceptualisation by the 8th Graders

Student	Concept-in-action		Theorem-in-action
Jeff	He interpreted most of the problem as a multiplication and division situation and focused on number of teeth to solve the problem.	<i>Problem</i>	
		4	$S = \frac{L \times n_L}{n_S}$
		<i>Problem</i>	
		8	$n_L = \frac{S \times n_S}{L}$
Eric	He interpreted the problem as a multiplication and division situation and was occasionally constrained by problem context.	<i>Problem</i>	He changed his consistent strategy from working with the number of teeth to the number of turns ($n_S = 4$ and $n_L = \frac{4}{3}$) due to the divisibility relationship between 4 and $\frac{4}{3}$.
		9	
		<i>Problem</i>	
		10	$S : \frac{L \times n_L}{n_S}$
Eric	He interpreted the problem as a multiplication and division situation and was occasionally constrained by problem context.	<i>Problem</i>	He gave the following justification: “Well, I did, I made 24, I made it into a fraction. I did 24 over 1 divided by 3 over 2 which is the same as 1.5. And I got 8 over 2 which is equal to 4”
		4	$\frac{3}{2} \cdot \frac{24}{1} = \frac{8}{2} \quad 4$
		<i>Problem</i>	In the first interview, his response suggests that he had no clear idea what he was doing. In the second interview, he multiplied $1\frac{3}{8}$ by 5 to obtain $\frac{55}{8}$ or $6\frac{7}{8}$ where he worked with the number of turns.
		8	
Eric	He interpreted the problem as a multiplication and division situation and was occasionally constrained by problem context.	<i>Problem</i>	He did not contribute much in the first interview. In second interview, the $\frac{1}{3}$ in $1\frac{1}{3}$ prompted Eric to divide 32 by 3.
		9	
Eric	He interpreted the problem as a multiplication and division situation and was occasionally constrained by problem context.	<i>Problem</i>	Following Jeff’s lead, he attempted to divide 44 by 5.5.
		10	

While Jeff chose to work with the number of teeth, Aileen and Brian preferred to work with the number of turns and these called for different conceptualizations of the same situation. For instance, in problem 8 working with number of turns called for the theorem-in-action $(\frac{8}{11}) \times 5$ while working with the number of teeth prompted Jeff to use the theorem-in-action $\frac{8 \times 5}{11}$. Such differential conceptualizations generated different forms of conflict. Further, I captured the intuitive knowledge elements they deployed in their attempt to conceptualize the problems to open solution paths. For example, in problem 2, before the computation, Eric deduced that the small gear should make less than two turns: "Because I know the little gear had to at least make one turn but it wouldn't be like 2 complete turns." In problem 10, rather than dividing 44 by 5.5, Eric's strategy was to find x such that 5.5 multiplied by x gives 44. Instead of doing division involving decimals (5.5), he preferred to plug in numbers to find which number times 5.5 gives 44.

Constraint 1: Determination of the Unknown Quantity

The gear-wheel problem requires the coordination of known and unknown quantities in the multiplicative relation $n_s S = n_L L$. In some instances, the participants operated the given fraction on the known rather than the unknown. For example, in problem 9, Brian computed $1\frac{1}{3}$ of 32 rather than interpreting $1\frac{1}{3}$ of the unknown as being 32. It could also be observed that often the participants used the denominator of the given fractions as a pointer to determine the unknown quantity. For instance, in problem 4, Aileen used the fraction $\frac{1}{2}$ in $1\frac{1}{2}$ as a pointer to divide 24 by 2.

Constraint 2: The Inverse Proportional Relation between Number of Teeth and Turns

The conflict arising as a result of the inverse proportion between the number of turns and teeth could be observed at different instances of the interviews. Aileen and Brian (the seventh graders) and Eric (the eighth grader) encountered the same conflict in problem 6 ($11x = 8 \times 5$). They all multiplied $1\frac{3}{8}$ by 5 instead of multiplying $\frac{8}{11}$ by 5. They did not take the inverse proportional relationship between number of teeth and turns into consideration at that instant. This inverse proportional relation is analogous to measurement situations where the larger the size of the unit, the smaller the measure.

Constraint 3: Numeric Features of the Data

All four participants spontaneously solved problem 3, involving an integer ratio of turns (i.e., $n_s : n_L = 3:1$). However, Aileen, Brian and Eric were constrained to articulate the multiplicative relation in the other problems due to the non-integer ratio. The data show that students' capability to coordinate the known and unknown quantities in a multiplicative situation (mathematically equivalent to $a_1 \times a_2 = a_3$) is sensitive to the numeric feature of the data. When a_1 and a_3 are factors/multiples of the other, the students could readily find a_2 , as could be observed in problem 3. However, when mixed numbers or improper fractions were involved the students experienced much difficulty. Consequently, estimation strategies were used to determine the unknown in the multiplicative relation. As highlighted in the review of the literature, Kaput & West (1994), Fischbein et al. (1985) and Vergnaud (1988) also reported such influence of numerical characteristics of problem parameters on students' ability to solve problems.

One may solve problem 8 using either number of turns in which case we use the ratio of turns that the small and large gear makes i.e., $1:\frac{8}{11}$ (involving the multiplicative comparison of fractional quantities) or in terms of ratio of teeth 8 : 11 (involving the multiplicative comparison of integer quantities). Jeff solved the problem using the number of teeth and as such worked with integer quantities. On the other hand, Aileen and Brian worked with the number of turns and encountered much difficulty in finding n_L . As is generally the case, problems involving fractional quantities are more demanding than those involving integer quantities. In addition, this study shows that even within fractional quantities the form of the number representation may influence problem solving behavior. The ‘improper fraction’ and ‘mixed number’ interpretation of rational numbers may offer different routes to solve the same problems. For instance, we observed that the same rational number $1\frac{1}{3}$ and $\frac{4}{3}$ cued different resources in problem 7, where the representation $\frac{4}{3}$ (interpreted as 4 thirds) allowed Jeff to observe the proportional relationship between 4 and $\frac{4}{3}$ to reverse his thought process.

In summary, this study shows that the four students solved the same problem using different strategies, deploying different resources and experienced different forms of conflict depending on the knowledge elements available to them at that point in time. The gear-wheel problem was conceptualized as a multiplicative comparison situation and as a ratio/proportion. However, they articulated these multiplicative relations with varying level of facility. The students were at different points in their development of multiplicative reasoning. On a continuum, I would consider Eric and Jeff at the two extremes, with Jeff at the upper end and Aileen and Brian in between. Neither Jeff nor Eric felt the necessity to use a proportion schema like Aileen. However, although she had the proportional schema, she was at times limited in using it. One of the fallback strategies that was observed is that students tend to make intuitive attempts, leaving behind mathematical principles in their attempt to solve problems. For example, in problem 8, to find x in her proportional schema $1:1\frac{3}{8}$ and $x:5$, Aileen subtracted $\frac{3}{8}$ from 5, thereby compensating additively rather than multiplicatively. An important observation from this study is that divisibility relationship among problem parameters is a critical factor that influences problem conceptualization.

In terms of instructional implications, this study suggests that middle-school students need to be given purposeful tasks to work with non-integer ratios and to work interchangeably between ratios and fractions. Similarly, such students need to be given opportunities to work with fractional units e.g., interpreting $1\frac{1}{2}$ as 3 units of $\frac{1}{2}$. More importantly, multiplicative comparison which lies at the basis of ratio and proportion situations, should be fostered especially in terms of fractional quantities.

References

- Bower, M. (2008). Affordance analysis - matching learning tasks with learning technologies. *Educational Media International*, 45(1),3-15.
- Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, 16(1), 3.
- Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. New York: Cambridge University Press.
- Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). New York: Macmillan.

- Hall, R. (2000). Videorecording as Theory. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*. Mahwah, NJ: Erlbaum.
- Harel, G., Post, T., & Behr, M. (1988). *On the textual and semantic structures of mapping rule and multiplicative compare problems*. Paper presented at the Proceedings of the XII International Congress, Psychology of Mathematics Education (PME).
- Kaput, J. J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235–287). Albany: State University of New York Press.
- Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* Mahwah, NJ: Lawrence Erlbaum Associates.
- Lamon, S. J. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 629–667). Charlotte, NC: Information Age Publishing.
- Nesher, P. (1988). Multiplicative school word problems: Theoretical approaches and empirical findings. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 19-40). Reston, VA: Lawrence Erlbaum Associates and National Council of Teachers of Mathematics.
- Ramful, A., & Olive, J. (2008). Reversibility of thought: An instance in multiplicative tasks. *The Journal of Mathematical Behavior*, 27, 138–151.
- Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 41-52). Reston, VA: Lawrence Erlbaum Associates and National Council of Teachers of Mathematics.
- Steffe, L. P. (1994). Children' s multiplying scheme. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3–39). Albany: State University of New York Press.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany: State University of New York Press.
- Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141 –161). Reston, VA: National Council of Teachers of Mathematics. Hillsdale, NJ: Lawrence Erlbaum.

Strategies Used by Students to Compare Two Data Sets

Robyn Reaburn
University of Tasmania
<Robyn.Reaburn@utas.edu.au>

One of the common tasks of inferential statistics is to compare two data sets. Long before formal statistical procedures, however, students can be encouraged to make comparisons between data sets and therefore build up intuitive statistical reasoning. Such tasks also give meaning to the data collection students may do. This study describes the answers given by beginning university students to tasks involving comparing data sets in graphical form, originally designed for students between Grades 3 to 9. The results show that whereas all the students had successfully completed either pre-tertiary mathematics or a bridging mathematics course many had similar difficulties to students of a younger age. In particular, they did not use a measure of centre or proportional reasoning when appropriate.

One of the common tasks in inferential statistics is to compare two data sets. For example, is one group faster than the other group? Does the new drug work better? In the formal procedures of inferential statistics, questions similar to these are often answered by comparing the values of the arithmetic mean of each group while taking into account the value of the standard deviation of each group.

Using less formal means of making comparisons, however, students can compare two data sets by using a measure of centre such as the arithmetic mean or by using proportional reasoning. For students to use a measure of centre they need to know that this statistic is somehow representative of a group (Gal, Rothschild, & Wagner, 1990). Despite the wide spread use of the arithmetic mean (the *average*) in everyday applications, previous research has shown that students often only perceive the arithmetic mean as the learned algorithm. Because these students do not regard the arithmetic mean as a representative number they are generally unsuccessful in using it to make decisions about data (Mokros & Russell, 1995).

Gal, Rothschild, and Wagner (1989) investigated how primary students (Grades 3 and 6) compare two data sets. They found that most of the students in Grade 6 did not use the arithmetic mean in their solutions, even though they were familiar with its calculation. Many of the students used totals even when the data sets were not of equal size. They also found that many of the students in Grade 6 had difficulty in using proportional reasoning. In a later study Gal, Rothschild, and Wagner (1990) found that as students became older their understanding of the characteristics of the arithmetic mean improved but there was still a reluctance to use it as a tool to distinguish between two data sets. Whereas the formula for calculating the arithmetic mean was familiar to 2% of Grade 3, 61% of Grade 6, and 91% of Grade 9 students, the algorithm was applied by only 4%, 14% and 48% of the students respectively. They also did not generally use proportional reasoning or visual comparisons of the given graphical displays to reach their conclusions. Watson and Moritz (1998) also investigated students' thinking in comparing two data sets. In their study, 88 students from Grades 3 to 9 were given a series of tasks that required them to make comparisons between two data sets given in graphical form. Many of the students did not use the arithmetic mean in their conclusions, and those who did (10% of the Grade 6 students and 54% of the Grade 9 students) did not always do so successfully.

Another strategy in such tasks is to use proportional reasoning, which is valid when the groups are not of equal size. Proportional reasoning involves multiplicative reasoning instead of additive reasoning. For example, in answer to the question, "If green paint is

made from one blue and three yellow and I added two more blues, how many more yellow would I need?” many primary students answer that as two have been added to the blue, two will have to be added to the yellow (an additive strategy) instead of realising that for every one blue, three yellows are required (a multiplicative strategy) (Parish, 2010). It has been argued that children are not able to use proportional reasoning under the age of 12, but research has shown that young children are capable of using proportional reasoning if the tasks are selected appropriately (Boyer, Levine & Huttenlocher, 2008). For example, proportional reasoning problems can be as simple as deciding which group has “more” girls if there are two girls in a group of four, or two girls in a group of five (Van de Walle, Karp, & Bay-Williams, 2010).

An ability to carry out an algorithm is not necessarily accompanied by an understanding of the significance of the answer, and previous research has shown that students may mask a lack of understanding by following the required algorithm (Garfield & Ahlgren, 1988). If statistics instructors assume that students already have a familiarity with fundamental ideas such as the arithmetic mean when they do not, students’ future understanding may be compromised.

Method

Participants

The study described here was part of a wider project which, in part, assessed students’ familiarity with statistical reasoning on entry to a tertiary institution (Reaburn, 2011). The sample consisted of 75 tertiary students on entry to a first year introductory statistics unit. All the students had either successfully completed a pre-tertiary mathematics course, or successfully completed an introductory calculus bridging unit before admission to the unit. The students were asked to fill out a questionnaire in the first week of the unit, part of which was based on the tasks in the study by Watson and Moritz (1999). There were four tasks in the section described here, and these were ordered in what was considered to be of increasing difficulty.

The Tasks

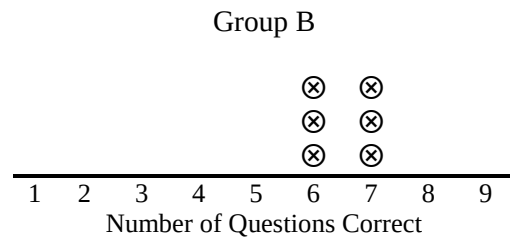
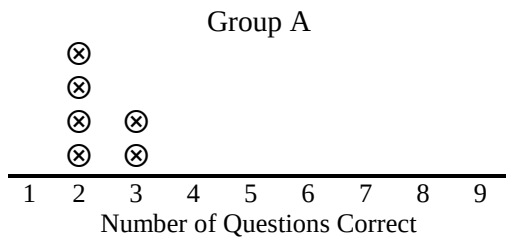
For these tasks the students were to compare the scores of four pairs of groups based on data presented in graphical form (Figure 1). For the first pair of groups (Task 1) the groups were of equal size and all of one group had higher scores than the other. For the second pair (Task 2) the two groups again were of equal size and although there was some overlap in the scores, one group clearly had a higher mean score than the other. In the third pair (Task 3) there were equal numbers in each group and the means, medians and modes were equal, but one group had a wider spread than the other group. For the final pair (Task 4) the group numbers were not equal, and it was expected that students would have to make a judgement using the value of the arithmetic mean or median or by using proportional reasoning. The introduction to these four tasks was.

A tertiary institution is comparing the scores of some tutorial groups on a test of basic statistics facts. The test had nine questions. The scores for two of these tutorial groups are shown in the charts below. Each circle represents one person. Therefore for Group A four people answered two questions correctly, and two people answered three questions correctly.

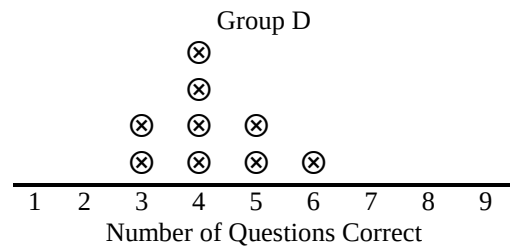
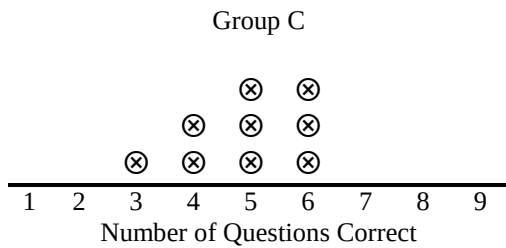
After each pair of graphs the question was asked.

Did the two groups perform equally well, or did one group perform better?
Please give reasons for your answer.

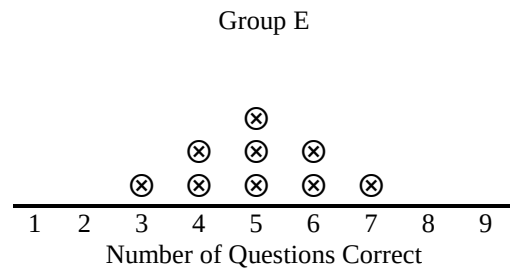
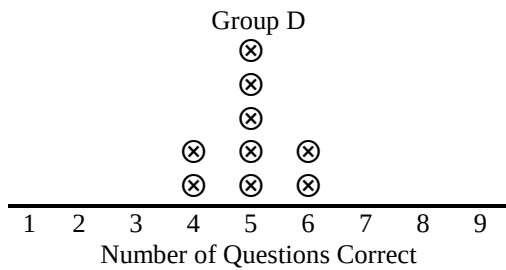
Task 1



Task 2



Task 3



Task 4

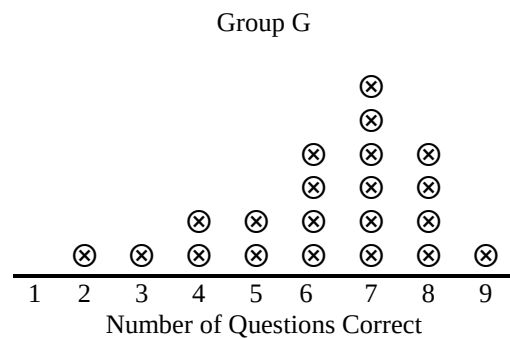
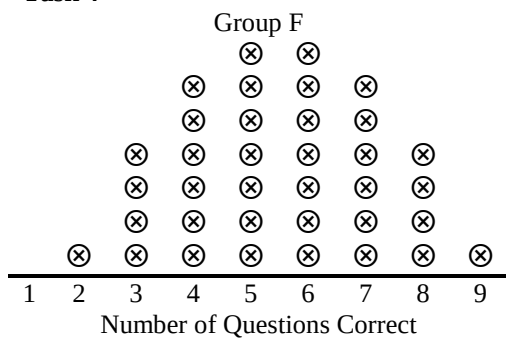


Figure 1. The tasks given to the students. For each pair of graphs, the students were required to state which group “performed better” and why.

Results

Task 1

Table 1 indicates that the most common response to this task was to state that there were “more” correct answers in Group B, with no further justification. The next most common response was to indicate that the entire group in B had higher scores than the entire group in A. Of the students who used the arithmetic mean and/or median, four calculated the

arithmetic mean fully without using an approximation. The other students who answered Group B had performed better stated directly what the scores were, for example “Group A has 2s and 3s and Group B has 6s and 7s.” A small number of students stated that Group A had performed better.

Task 2

Table 2 indicates that again the most common response was to state that there were “more” correct answers. Some students used the arithmetic mean and/or the median to justify their answers and others explicitly compared the scores. Several students used totals in their justifications, half of whom stated that the group sizes were equal.

Table 1

Answers Given by the Students to Task 1

Answer	Number n = 75	Percentage (Rounded)
Group B		
There are “more” correct answers in Group B	23	30
Everyone in Group B had higher scores than Group A	15	21
Arithmetic mean and/or median used	10	13
Used the scores, e.g. “Group A has 2s and 3s and Group B has 6s and 7s.”	15	21
Totals calculated and stated that group sizes are equal	1	1
Group A	4	5
No response	7	9

Task 3

In this task 47% of the students stated that the two groups performed equally well. Approximately half of these used the mean and/or median in their justifications, and the others used the totals. The most common reason given by those who designated that one of the groups performed better was that “more” had higher scores. This reason was used by those who selected both Group E and Group F as the better performer.

Table 2

Answers Given by the Students to Task 2

Answer	Number n = 75	Percentage (Rounded)
Group C		
There are “more” correct answers in Group C	31	41
Arithmetic mean and/or median used	8	11
Used the scores, e.g. “Group C has one 3, two 4s, three 5s, and three 6s. Group D has two 3s, four 4s, two 5s and one 6.”	15	21
Totals calculated and stated that group sizes are equal	3	4
The groups were equal	7	8
Group D	3	4
No response	8	11

Table 3
Answers Given by the Students to Task 3

Answer	Number n = 75	Percentage (Rounded)
Equal		
Arithmetic mean and/or median used	19	25
Totals calculated and stated that group sizes are equal	10	13
No explanation	7	9
Group E		
More consistent	3	4
More got higher scores	13	17
Arithmetic mean and/or median used	2	3
No explanation	6	9
Group F		
More got higher scores	3	4
No explanation	3	4
No response	9	12

Task 4

Table 4 shows that a higher proportion of students used a calculation or estimate of the arithmetic mean and/or median to answer this task than the previous tasks. Of the other students who stated that Group H performed better, 13 explicitly used proportional reasoning, one implied proportional reasoning, and five did not give an explanation. Of the 20 students who stated that Group G performed better seven said that there were more people, therefore giving a “better” score. Seven others did not give an explanation. Six students said the two groups performed equally well and five said that the problem was “not fair” or could not be done.

Table 4
Answers Given by the Students to Task 4

Answer	Number n = 75	Percentage (Rounded)
Group H		
Higher arithmetic mean and/or median	20	27
Proportional reasoning	13	18
“More” got higher scores	1	1
No explanation	5	7
Group G		
More people therefore this group performed better	7	9
More in the higher range	4	5
More balanced	2	3
No explanation	7	9
Equal	6	8
Too hard/cannot be done	4	5
Not fair	1	1
No response	5	7

Discussion

For Task 1 a visual inspection quickly shows that all the scores in Group B are higher than those in Group A. A calculation of the arithmetic mean or median is not required to answer this task and most of the students used strategies that did not require such calculations. In Task 2, where there was some overlap of the scores, a higher proportion of the students used a full calculation or estimates of the arithmetic mean and /or median. A number of students also used totals, which is an acceptable strategy when each group has the same number of scores.

Task 3, judging by the relatively large number of students who gave no explanation, appeared more difficult. The question of what was meant by “better” in this context was deliberately left open, and a small number of students used the criterion of consistency to make their judgments. There were students who used the reasoning of “more with higher scores” to choose both Groups E and F. Of particular concern is the small number of students who incorrectly calculated the arithmetic mean for these groups and therefore came to the conclusion that Group E performed better than Group F. The arithmetic mean, median and mode are all equal in this task, and this should have been apparent using visual inspection. This suggests that these students did not have a conceptual understanding of these statistics. It is also apparent that the students who selected one of the groups as better (apart from when they used the criterion of consistency) were not aware that the arithmetic mean could also be considered as a balancing point, a strategy used successfully by some of the students in Grades 6 and 8 in the study by Mokros and Russell (1995).

Task 4 was deliberately set so that more sophisticated reasoning would be needed in making a judgement than in the previous tasks. An indication of the difficulty is shown by the increased proportion of students who did not give explanations for their answers. Of the 39 students who correctly answered Group H, 20 used the arithmetic mean or the median and 13 used proportional reasoning, either directly or implied. Close to one half of the students did not answer this question correctly. Of particular concern is the number of students who argued that as there were more people in one group then this group performed better. These students did not think to use even the simplest form of proportional reasoning available to young students (Boyer, Levine & Huttenlocher, 2008). All of these students would have been previously exposed to proportional reasoning problems. Also of concern was the number of students who stated that the task could not be done or was not fair. This form of reasoning was used by students of Grade 7 and under in the study by Watson and Moritz (1999) and it is disturbing to find this reasoning used by students who have successfully completed a pre-tertiary mathematics. It is not likely that any of these students were unfamiliar with the algorithm to calculate the arithmetic mean.

This study has found that problems noted in younger students may persist beyond the end of secondary education, even by those who have studied mathematics until the end of their schooling. Some students did not use the mean or median even when this would have helped answer the question (Gal, Rothschild, & Wagner, 1989, Watson & Moritz, 1999). This study also adds to previous research by Mokros and Russell (1995) who found that many students only see the measures of centre as the algorithms used to calculate them, and do not see these statistics as representative numbers. Since this study the researcher has introduced a question early in the introductory statistics unit asking the students to explain the meaning of the statement “The average score for a class test was 74” without explaining how the statistic is calculated. Generally the students have great difficulty in answering.

Some Implications for Instructors

These results show that students can use a variety of strategies to answer problems and can select the quickest strategy in each context. They might not do so, however, if instructors are not open to allowing students to choose and compare different strategies in the classroom. More importantly, this study alerts instructors that students may competently use algorithms but have little or no understanding of the significance of the results of their calculations. Mokros and Russell (1995) have suggested that instructors who rely too heavily on the algorithm in their teaching may, in fact, inhibit students' understanding. Therefore students not only require practice in using algorithms, but also require practice to put the results of these algorithms into a context. The Australian Curriculum (Australian Curriculum Assessment and Reporting Authority [ACARA], 2011) states that students should draw back to back stem-and-leaf plots in Year 9 and draw box plots in Year 10. It is not until Year 10A, however, does the curriculum explicitly state that comparisons between data sets should be made. Perhaps such comparisons should be used in earlier years to help develop statistical reasoning and to help give meaning and interest to the data students are given.

References

- Australian Curriculum, Assessment and Reporting Authority. (2011). *The Australian Curriculum: Mathematics*. Version 2.0, 17 October, 2011. Sydney: ACARA.
- Boyer, T., Levine, C., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong, *Developmental Psychology*, 44(5), 1478-1490.
- Gal, I., Rothschild, K., & Wagner, D. (1989). *Which group is better? The development of statistical reasoning in elementary school children*. Paper presented at the Meeting of the Society for Research in Child Development, Kansas City, MO.
- Gal, I., Rothschild, K., & Wagner, D. (1990, April). *Statistical concepts and statistical reasoning in school children: Convergence or divergence?* Paper presented at the Meeting of the American Educational Research Association, Boston, MA.
- Garfield, J., & Ahlgren, A. (1988). Difficulties in learning basic concepts in probability and statistics: Implications for research. *Journal for Research in Mathematics Education*, 19(1), 44-63.
- Mokros, J., & Russell, S. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education*, 26(1), 20-39.
- Reaburn, R. (2011). *Students' understanding of statistical inference: Implications for teaching* (Doctoral dissertation). Australia: University of Tasmania.
- Van De Walle, J., Karp, K., & Bay-Williams J. (2010). *Elementary and middle school mathematics: Teaching developmentally* (7th Ed.). Boston, MA: Allyn & Bacon.
- Watson, J., & Moritz, J. (1998). The beginning of statistical inference: Comparing two data sets. *Educational Studies in Mathematics*, 37(2), 145-168.

Exploring Student Reflective Practice during a Mathematical Modelling Challenge

Trevor Redmond
A.B. Paterson College
<trd@abpat.qld.edu.au>
Joanne Sheehy
A.B. Paterson College
<jsy@abpat.qld.edu.au>

Raymond Brown
Griffith University
<ray.brown@griffith.edu.au>
Harry Kanasa
Griffith University
<h.kanasa@griffith.edu.au>

This paper seeks to compare the reflective writings of two cohorts of students (Year 4/5 and Year 8/9) participating in mathematical modelling challenges. Whilst the reflections of the younger cohort were results oriented, the older cohort's reflections spoke more to the affective domain, group processes, the use of technology and the acquisition of mathematical knowledge. This study supports the idea that with scaffolding middle years students can engage in reflective practice to develop mathematical modelling skills.

The ability to mathematically model real world problems so they can be solved is a skill that is required in many fields: All fields of science, engineering (Gainsberg, 2006), the social sciences, economics, computer science, politics and so forth (Lesh, 2007). While mathematical modelling has been typically reserved for high school mathematics courses (Galbraith, Blum, Booker & Huntley, 1998) other researchers are finding students in primary and middle school are capable of modelling real world situations beyond simple number and space concepts such as counts and measures (English, 2006).

Lesh and English (2005) describe a mathematics and modelling (M&M) approach which moves beyond mathematics as computation to mathematics as conceptualisation, description and explanation. Modelling also differs from 'word problems', which often only require one or two steps, in that it requires the mathematisation of real world situations in ways that are meaningful to the student (English, 2006). Lesh and Doerr (2003) describe mathematisation as a cyclical process of interpreting the problem, selecting relevant quantities, identifying operations that may lead to new quantities and creating meaningful representations. Teaching from an M&M perspective is in agreement with constructivist notions of learning through requiring students to construct their knowledge and understanding of a situation.

Advocates of the teaching of mathematical modelling (e.g., English & Watters, 2005) in the middle years state a number of benefits of this approach. Boaler (2001) compared students taught traditionally (completing exercises from a text) to students taught through problem solving and mathematical modelling. After three years students taught using the later approach were able to apply their knowledge in a greater variety of situations and also outperformed students instructed traditionally in the UK national examinations on both procedural and conceptual mathematical questions. Students at the later school were also more likely to see the applicability of mathematics to their lives and future jobs.

English (2006) examined the modelling abilities of 10-year-old students in their construction of an answer to the question 'which is the best snack chip?'. Students worked collaboratively over six 40-45 minute lessons to answer this question. Transcripts of audio and video recordings and student created artefacts were interrogated for '...the ways in which the children interpreted and reinterpreted the problem, their approaches to selecting and categorising factors, their mathematisation processes as they quantified factors and transformed quantities, their attention to issues of sampling and representation, and their establishment of group roles (English, 2006; p. 309). English concluded that students in

primary school are capable of engaging in mathematical modelling and do so at quite sophisticated levels. This is in agreement with English and Watter (2005) who found similar results in the mathematical modelling abilities of 8-year-old students. Benefits in addition to those found by Boaler (2001) were students were engaged in social interaction, reflection, critique and higher order thinking as they participated in the activity.

Another advantage in teaching from a M&M perspective is the ability to promote interdisciplinarity (English, 2009). Due to commonalities between mathematics and science Lesh and Zawojewski (2007) propose an approach to mathematical modelling which involves model construction, evaluation and revision. This cycle is similar to that employed by professional scientists and mathematicians (Romberg, Carpenter & Kwako, 2005). This paper seeks to analyse the first day reflections of two cohorts of middle years students who participated in two Modelling Challenges and explores their use of reflective practice to enhance the collaborative aspects of mathematical modelling.

The Context

This study compares two groups of students who participated in two different Modelling Challenges, the first located on the Gold Coast, Australia and the second located in Melbourne, Australia. In both cases the students in the study had the same mentor, that is, a mathematics teacher who scaffolded student participation in the completion of an inquiry based task. The students in the Gold Coast Challenge were in Years 8 and 9 (mean age = 13 years) while the students in the Melbourne Challenge were in Years 4 and 5 (mean age = 10 years). The Years 8/9 group (24 students) was a mixture of boys and girls with students from Australia and Singapore while the Years 4/5 group (15 students) consisted entirely of boys from Australia. To aid readability the students in Years 8/9 will be referred to as the 'older students' and the Years 4/5 students as the 'younger students'. Groups in both locations were randomly assigned to groups of four. A balance of age, school and gender (for the older students) was ensured for all groups. Groups in both cohorts were assigned the same equipment and task.

Students were asked to construct a rotor for a wind powered generator that would maximise electricity generation and construct a tower to support it. All groups were provided with a dynamo; cardboard for the blades of the rotor; balsa wood; glue and pins for the construction of the tower; and a fan for testing. All students engaged in an introductory discussion on wind turbines, the affordances and constraints of wind turbines and the horizontal or vertical orientation of the rotor blades. Students also participated in a teacher led discussion of the variables that might affect the voltage generated.

Students had access to the TI-Nspire graphics calculators and voltage probes. These were used to measure, tabulate and analyse the dependent variable of voltage. Some of the older students had previous experience with the graphics calculators whilst none of the younger students had any previous experience with these calculators. However, after the mentor conducted a 30 minute session with the younger students they had sufficient working knowledge of the calculator to collect, tabulate and graph data.

Students were asked to present a proposal using a poster presentation with the aim of convincing the executives of General Electric (GE) to adopt their wind turbine design in the next wind generator. Students systematically manipulated and recorded the variables of blade area, blade shape, the number of blades and blade pitch to maximise electrical output of the generator. Students were asked to record the changes to the variables and to present them as part of their poster presentation.

Each challenge was conducted over two days with presentations at the end of the second day. After the initial introduction session where the context was introduced and elaborated

by the mentor, the groups of students mostly worked independently asking questions to clarify their understanding of the task or the use of the technology. At the conclusion of the first day students were asked to reflect on their progress during that day by responding to the following questions:

1. What did you like about today?
2. What are you going to do differently tomorrow?
3. Did your group work as effectively as you would have liked? Why or why not?
4. If you could change one thing, what would it be?

In this study we analyse the students' written reflections that were provided to the mentor at the end of the first day of the Challenge. While much of the literature on reflective practice within mathematics focuses on pre-service and in-service teachers (e.g., Graham & Phelps, 2003) many of the precepts can be applied to the students learning of mathematical modelling. For example, Whitton et al., (2004) conceptualises reflective practice as three steps involving (a) Direct experience; (b) Analysis of beliefs, value or knowledge; and (c) Consideration of the options as a function of the analysis. These three steps correlate with the questions asked of students. The purpose of this study is to explore middle years students' reflections on their practice during the Challenge.

The Data

A thematic analysis was conducted using a grounded theory approach (see Table 1).

Table 1

Identified Themes in Students' Written Responses to Each Question

Question	Themes
What did you like about today?	Affective (Fun, challenging/interesting, creative) Working in teams Technology Learning new concepts Results focus
What would you do differently tomorrow?	Time management (individual and team) Results focus Thoughtful inclusion of mathematics and analysis of data
Did your group work as effectively as you had hoped?	Work effectively Team co-operation Efficiency Collaboration Communication
If you could change one thing about today what would it be?	Time management Data collection Output focus Affective Group involvement

The comments of the older students were categorised first to yield a greater variety of themes as there were more older students. Students' responses could be coded into more than one theme. A Year 8/9 student responded with the following reflection to the question 'what did you like about today?'

"I got to meet new friends and we managed to work together in solving the math modelling task. I learnt how to use the graphing calculator to collect the voltage and also learnt more about wind turbines."

This reflection was coded into the themes of *Working in Teams*, *Using Technology* and *Learning New Concepts*. In some instances extra information is provided to qualify the reflection provided by the student. Table 2 shows the coding for question 1.

Table 2

Coding for Question 1, 'What did you like about today?'

Theme	Older	Younger
Affective	10	0
Working in Teams	13	4
Technology (Calculations and Collecting Voltage data, calculating area)	5	2
Learning New Concepts (learning about wind turbines, optimisation)	10	3
Results Focus (designing and constructing Rotors, Tower using the generator, planning the poster)	12	11

An analysis of Table 2 reveals distinct differences in the respective foci of the two cohorts. The older students' written reflections were spread across the themes of *Affective* (fun, challenging/interesting and creative), *Working in Teams*, *Learning New Concepts* and a *Results Focus* (designing and constructing the rotors and tower for the generator, planning for the poster), whilst coding of the younger students responses revealed a *Results Focus* as being a priority. Of lesser importance to the younger students were the themes of *Working in Teams*, *Learning New Concepts* and *Technology*.

Some of the differences may be attributed to the maturity level of the students, and hence their level of cognitive development and also to the group composition of the older and younger students respectively. The younger students appear to be focused on engaging in the task and achieving results, they do not rate the theme *Working in Teams* with the same level of priority as the older students. It is interesting to note the absence of any reflections from the younger students that could be related to the *Affective theme*, a theme of considerable importance to the older students.

The coding for question 2, 'What would you do differently tomorrow?' is shown in Table 3. Both groups identified *Time Management* as a priority to be engaged with at the group level for the next day of the Challenge. The distribution of reflections though was spread between *Time Management* for the *Individual* and *Group Time management*. During the introduction phase of the challenge the mentor did provide a quasi-timeline for both groups of students but left the ultimate timing of the various activities up to the individual group. As such, each group decided on the plan of attack and the length of time spent developing the rotor, the number of iterations of the modelling process and the variables they would investigate. The time and work pattern devoted to the analysis of the data from

each iteration, building the tower and the development of the poster presentations were also determined by each group.

Table 3

Coding for Question 2, 'What would you do differently tomorrow?'

Theme	Older	Younger
Time Management		
Individual (Work Harder to meet objectives/Plan actions /not waste time)	6	3
Team (Leadership)	6	7
Results Focus		
Make the Tower	4	1
Make the poster	7	
Company name and logo	1	1
Work on Rotor	1	
Thoughtful inclusion of Mathematics and analysis of data	3	2

As can be seen in Table 3, the older students identified the theme *Results Focus* as a priority for the second day of the challenge, interestingly for the younger students, while this theme was a priority for the first day of the Challenge, it was not seen as a priority for the second day. Of interest too are the reflections that indicate the need for students to be thoughtful about the data collected from the rotor and about the inclusion of suitable mathematics to support their conclusions. Students in both cohorts, albeit small in number, indicated that this should be a priority for their group's time on the following day.

The mentor in the introduction sessions to each Challenge indicated the need for the students to justify their conclusions and to provide supporting mathematics as part of that justification. The limited number of students who questioned their data and the procedures implemented during that first day suggest that in the exploration of the rotor blade the students' focus in both groups was more on building the rotor and measuring the subsequent voltage generated rather than being thoughtful about the variables and how they might be varied in order to collect reliable data. This could be interpreted that the justification of their findings using mathematics played a secondary role to what they perceived to be more pressing issues.

An analysis of Question 3 (see Table 4) suggests that the majority of students in both cohorts thought that their group did work effectively together. It was interesting to note that while a number of younger students indicated that their group was working effectively they also indicated that there were disagreements of sufficient nature to be mentioned in their reflections. However, as one younger student wrote,

"Yes my group is work[ing] as effectively as I would have hoped even though we did have our disagreements but what's a group without disagreements."

As the members were randomly assigned to the groups, adjusting for the same school and age, prior to that morning, the students did not know each other. However, the majority of students reflected that they were able to come together for the task and make contributions to create an effective group. Not all students in the older cohort felt this way.

Four older students indicated that their team did not function effectively. All these responses were from the same group. It is of interest that while these written reflections were categorised as not working as a team, in two instances the student reflections indicated

that these students, felt that the group was effective even though the group was split in two, boys and girls. A number of reflections from the older students indicated the need for the group to work faster or that the group was wasting time.

Table 4

Coding for Question 3, 'Did your group work as effectively as you had hoped?'

Themes	Older	Younger
Work effectively (divided the tasks up)	5	9a
Complexity of the task inhibited the group members	2	
Team- Co-operation		
Cooperative	8	1
Need to be More Cooperative	1	
Efficiency		
Need to Work Faster	3	1
Wasting Time	1	
Collaboration in the team	3	1
Communication	2	3
Lack of Communication	1	1
Did not work as a team	4	1

^a three groups indicated disagreements.

Table 5 presents the categorisation of students written reflections about Question 4.

Table 5:

Coding for Question 4, 'If you could change one thing about today what would it be?'

Themes	Older	Younger
Time Management (Plan more carefully from the start. More time on rotor design)	10	9
Data Collection (More accurate data collection. Complete Data collection)	3	1
Results Focus (Building the Tower)	2	2
Affective (Change Mindset/Attitude. Not my type of Maths. Challenging and Confusing)	4	0
Group Involvement	4	2

Question 4 'If you could change one thing today, what would it be?' (see Table 5) identified the themes of *Time management* and *Group Involvement* as being the two main issues students from both cohorts would like to change. This is consistent with the reflections recorded for Question 2. There are a small number of reflections in both of the cohorts that prioritised the theme *Results Focus*. Again, this is consistent with Question 1 where a high proportion of the reflections from both cohorts (50% for the older students and 80% for the younger students) identified this theme as being a feature of the first day about which they were satisfied. Once again some students seemed to be contemplating the accuracy of the data and the method used to collect the data. It is also interesting to note that

a small number of student written reflections appear to show concern about the reliability of the data and the efficacy of the method used to collect data.

These reflections also indicated that some students were concerned about their attitude and mindset. However, these students do not seem to have recorded a related reflection in Question 2 indicating that they were prepared to change their mindset for the following day. For example, one of these students wrote:

“If I could change something, I want to change my working attitude. I was a little distracted too. It is very common that anyone can get bored, thus I won’t bear grudges with anyone. I just want to make friends with them; not enemies. So, please understand that I am NOT insulting Australians, I apologise if I have any offensive words here.”

This student’s written reflection for Question 2 was,

“I am going to try to be even more pro-active tomorrow and try to get the group to work more efficiently because of the tight schedule that we have.”

This student’s reflection of ‘what they would change about today’, suggests they were concerned about their attitude, about being a little distracted and wishing to make friends. However, in their reflection about their actions for tomorrow they refer to being “pro-active... and getting the group to work more efficiently”. It would appear that some of their perceived poor attitude might be attributed to the function of the group. It may also have a basis in the cultural differences that may exist in the way the members of the groups from the different countries come to know and do mathematics.

Conclusion

The two cohorts of students were very different in composition. One was comprised of Years 4 and 5 Australian boys, whilst the other was mixed gender groups of Year 8 and 9 students from two different countries. Yet surprisingly their reflections at the end of the first day on the same task share a number of common themes. Both cohorts of students valued working with their peers and had a results focus when faced with a modelling problem. For a small number of students the task was too challenging, but for most of the older students they were happy to accept the challenge and in the words of one older student,

“I liked the interaction and communication among[st] our group. Everyone chipped in and we could get along well. I think the question/task we were asked to do was quite interesting.”

While the novelty of the task was not something that was rated highly by the younger students they focused on the hands-on component which allowed them to try a design and then change that design to see the effect.

“What I like[d] about today was meeting my group and making the blades.”

Mathematical Modelling is about students engaging with a task, properly scaffolded to allow them to achieve the intended aims of the activity without removing any of the complexity, freedom of choices and richness of exploration so that students are able to utilise their mathematics in meaningful ways and to operate as a mathematician could. This study identifies that both cohorts of students, through engagement with this inquiry based task, wrote common reflections that suggested that they were focused on the collaborative aspect of their mathematics.

It can be concluded from these students’ reflections not only was collaboration necessary to engage with the mathematical modelling task, but that this aspect of doing mathematics was valued by the students in both cohorts. Students’ reflections suggest that engagement in group modelling tasks encourages students to value diversity of thinking

without devaluing the quality of the mathematics. For example, a younger student in reflecting on what they would change about the day wrote,

“I would change how we approached the task of the model, because I thought straight away, Oh’ I can do it by myself and I already had it planned in my head, but I didn’t consider my group, but my groups ideas turned out great.”

This paper provides some insights into how engaging students in reflective practices such as providing written responses to open questions about their doing of mathematics can be used to inform the collaborative learning and teaching of mathematics in the middle years of schooling.

References

- Barratt, R. (1998). *Shaping Middle Schooling in Australia: A report of the National Middle Schooling*. Deakin West, ACT: Australian Curriculum Studies.
- Boaler, J. (2001). Mathematical modelling and new theories of learning. *Teaching Mathematics and its Applications*, 30(2), 121-127.
- English, L. D., & Watters J. J. (2005). Mathematical modelling in the third-grade classroom. *Mathematics Education Research Journal*, 16, 59-80.
- English, L. D. (2009). Promoting interdisciplinarity through mathematical modelling. *ZDM Mathematics Education*, 41, 161-181.
- Gainsburg, J. (2006). The mathematical modeling of structural engineers. *Mathematical Thinking and Learning*, 8, 3-36.
- Galbraith, P. L., Blum, W., Booker, G., & Huntley I. D. (1998). *Mathematical modeling: Teaching and assessment in a technology-rich world*. West Sussex, UK: Horwood Publishing.
- Graham, A., & Phelps, R. (2003). Being a teacher: Developing teacher identity and enhancing practice through metacognitive and reflective learning processes, *Australian Journal of Teacher Education*, 27(2), 11-24.
- Lesh, R. (2007). Foundations for the future in engineering and other fields that are heavy users of mathematics, science and technology. In R. Lesh, E. Hamilton, and J. Kaput (Eds), *Models and Modeling as Foundations for the Future of Mathematics Education* (pp. vii-ix). Mahwah, NJ: Lawrence Erlbaum.
- Lesh, R., & Doerr, H. M. (Eds.). (2003). *Beyond constructivism: Models and modeling perspectives on mathematic problem solving, learning and teaching*. Mahwah: Lawrence Erlbaum.
- Lesh, R., & English, L. D. (2005). Trends in the evolution of models and modeling perspectives in mathematical learning and problem solving. In H. Chick and J. Vincent (Eds), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*. University of Melbourne, pp. 192-196.
- Lesh, R., & Zawojewski, J. S. (2005). Problem solving and modeling. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*. Greenwich, UK: Information Age Publishing.
- Romberg, T. A., Carpenter, T. P., & Kwako, J. (2005). Standards based reform and teaching for understanding. In T. A. Romberg, T. P. Carpenter & F. Dremock (Eds.), *Understanding mathematics and science matters*. Mahwah: Lawrence Erlbaum.
- Whitton, D., Sinclair, C., Barker, K., Nanlohy, P., & Nosworthy, M. (2004). *Learning for teaching: Teaching for learning*. Southbank, Victoria: Thomson Learning.

Steps in Developing a Quality Whole Number Place Value Assessment for Years 3-6: Unmasking the “Experts”

Angela Rogers
RMIT University
<angela.rogers@rmit.edu.au>

Upper primary school students’ difficulties with decimal place value are widely recognised due largely to the availability of assessment tools that accurately identify students’ misconceptions in this domain. Yet, many students’ decimal difficulties seem to stem from a lack of understanding of whole number place value for which there are few, if any, comprehensive assessment tools at the upper primary school level. The inevitable result of this is that teachers tend to assume upper primary students have a deeper understanding of place value than is often the case and limit their teaching to fairly routine tasks that do not expose student misconceptions. This paper describes the derivation of a research-based, multi-dimensional Hypothetical Learning Trajectory for whole number place value. It explains the process involved in identifying the key aspects of place value and includes indicative examples of the assessment items used to evaluate this.

The Problem

As a teacher in Victorian Catholic Primary Schools for over seven years, I have experienced many facets of teaching place value. However, while teaching in the upper year levels (Years 3-6) I was constantly dismayed by students’ lack of whole number place value knowledge.

Research documents the difficulties students have grasping decimal place value (Baturu, 1998; Irwin, 1996; Steinle & Stacey, 2003) and provides teachers with assessment tools to accurately identify their misconceptions. While these tools are undoubtedly valuable for this purpose, Irwin (1996), Baturu (1998) and Moloney & Stacey (1997) suggest that many students weak grasp of decimal fractions can be attributed to their shaky foundation in whole number place value.

As a consequence, I set about looking for an appropriate assessment tool to measure students’ level of whole number place value understanding. It quickly became apparent that there was a lack of comprehensive assessments addressing this area for upper primary school students. This exposed the need for a quality place value assessment tool to assist teachers to effectively monitor and scaffold their students’ whole number place value development. It was from these informal observations that the impetus for my research developed.

Whole Number Place Value Research- Unmasking the ‘Experts’

An understanding of place value underpins our capacity to work flexibly within the English written and spoken number system. Yet, despite the unchanging and recursive nature of our base-ten system, it seems some students never manage to fully unravel the hidden code that underlies place value, failing to become true “*place value experts*”. The complexities associated with place value understanding for students have been well documented (Cobb & Wheatley, 1988; Ellemor-Collins & Wright, 2009; Ross, 1989; Steffe, 1992; Thomas, 2004).

Essentially, place value refers to the conventions we use to read, say and record numbers. To be a true *place value expert*, one must comprehend all aspects of our numeration system. This system uses a positional base 10 structure, whereby the value of

each digit is indicated by its relative position in the numeral. For example, in 3456, the position of the digit 3 indicates a count of *thousands*. In contrast, the spoken system (beyond hundreds) involves using the number word followed by the value of that number. For example, 3456 is read as *three* thousand, *four* hundred and fifty-six, therefore the “thousand” names the value of the digit three (Fuson & Briars, 1990). Perhaps one of the most important, yet least understood characteristics of our system is the “10 of these is 1 of those” concept. This relates to the “infinitely extendable” (Thomas, 2004, p. 305) concept that each place value unit is ten times the value of the unit to its right (Fuson, 1990). This is known as the “recursive multiplicative structure” (Thomas, 2004, p. 311) of the base 10 numeration system.

Much research has been conducted into the teaching and learning of place value in primary schools. This research, for the most part, has been associated with the acquisition of two-digit place value knowledge in young students (6-8 years old) (Baroody, 1990; Cobb & Wheatley, 1988; Fuson & Briars, 1990; Ross, 1989; Steffe, 1992). Studies have stemmed from observations of students attempting to make the conceptual leap from a unitary view of number where individual items are counted, to the recognition and understanding of abstract composite units (Steffe, 1992). This conceptual jump is considered to underpin place value knowledge. Yet many researchers seem to incorrectly assume that students who understand two-digit numbers can automatically apply this knowledge to the entire number system. This is disputed by Thomas (2004) and Sinclair, Garin & Tieche-Christinat (1992) who document the significant difficulties students have in comprehending and applying the recursive multiplicative structure of the number system beyond two-digits.

It is widely recognised that multiplicative thinking poses major difficulties for students, with fewer than 10% of Year 4-8 students able to work flexibly with large whole numbers and decimal fractions (Siemon, Breed, Dole, Izard, & Virgona, 2006). Yet a deep understanding of the place value system is intrinsically linked to multiplicative thinking (Thomas, 2004). Irwin’s (1996) work with decimals supports this idea suggesting that “complete understanding (of decimal fractions) requires multiplicative thinking” (p. 418). Thomas’ (2004) provides further evidence of the difficulty students experience, stating that even by Year 6 very few students in his study were able to generalise the multiplicative structure of the number system. This considered, the difficulty students have understanding the multiplicative numeration system beyond two-digits may be underestimated by educators.

Place value is a undoubtedly a difficult concept for students to grasp. Ross (1989) describes how many students in Year 4 and 5 have serious place value misconceptions, but “appear to understand more than they actually do”(p. 50) and Sinclair, et al., (1992) note that often students’ explanations of place value are “trivial or glib and not representative of their knowledge” (p. 194). Ross (1989) explains the gaps in students’ place value knowledge may be a result of teachers using routine instructional tasks and standardised tests which fail to expose students’ place value misconceptions. Thus assumptions that a student is a *place value expert* may be inaccurate.

As Ross (1989) suggests, the incorrect identification of *place value experts* may be a result of the type of assessment items teachers typically rely on to measure whole number place value. One of the most widely used place value assessment tools in Australia is the *Mathematics Online Interview (MOI)* (Department of Education & Early Childhood Development, 2011b). This assessment is designed for use with “students in the first five years of school” (Department of Education & Early Childhood Development, 2011b, p. 3) . However, in the absence of a whole number place value assessment tool that specifically caters for students in year 3-6, teachers may assume that students who reach the ceiling of

the MOI place value section to be *place value experts* requiring little further instruction. It is not my intention to criticise the MOI. It is a completely valid assessment tool for the cohort of students it is designed for. I merely wish to point out that an assessment tool is desperately needed to measure upper primary students' understanding of the whole number place value system, particularly the multiplicative nature of this system.

It is clear that developing a deep understanding of whole number place value is of utmost importance in mathematics education. Studies have suggested the possible implications of poor place value knowledge not only include difficulty in understanding decimal place value (Irwin, 1996), but also difficulty in achieving relational understanding in algorithms (Fuson, 1990), and general problems in developing number sense and accessing mathematics at the secondary school level (Seah & Booker, 2005). As such, the need for teachers to be able to accurately identify true *place value experts* is fundamental to improving place value teaching and learning in the upper years of schooling.

Current Study

The methodology of this study has been significantly influenced by the *Scaffolding Numeracy in the Middle Years* (SNMY) (Siemon, et al., 2006) research project that used a Hypothetical Learning Trajectory (HLT) (Simon, 1995) to develop a research based Learning Assessment Framework for Multiplicative Thinking (LAF). This project will aim to do the same for Whole Number Place Value. In contrast to the SNMY project, this research is of a much smaller scale and will use an embedded single-case design (Yin, 2009). The major aim of the project is to produce a comprehensive research-based whole number Place Value Assessment Tool (PVAT) for use in Year 3-6 classrooms across Australia.

The first phase of this research included a comprehensive review of the literature associated with place value. From this review, a HLT for whole number place value was created. The HLT was then used as a guide to source and design assessment items to create the PVAT. Once initial investigations related to PVAT item readability and difficulty were completed, a pilot case study of the PVAT took place with all year 3-6 students (approximately 280 students) at a large Catholic Primary School in metropolitan Melbourne, Victoria (School A). The results from this pilot are currently being analysed using Rasch analysis (Adams & Khoo, 1996), which is a probabilistic model that measures item difficulty and student achievement on the same logit scale (Siemon, et al., 2006). This analysis will provide invaluable insights into the strength and behaviour of each item and the test as a whole.

The final phase of this research will see the PVAT trialed with all Year 3-6 students (approximately 290 students) at another large Catholic Primary School in metropolitan Melbourne (School B) during May and December 2012. A Rasch analysis will take place using this data and these results will be used to further validate the PVAT and the whole number place value HLT. At both schools, interviews will take place with 10% of the sample group in order to gain qualitative insights into the approach students take when completing the PVAT. The theoretical paradigm that underpins this research study, is informed by Cobb's (1995) emergent perspective that acknowledges the social and psychological elements at play in the construction of meaning.

Derivation of a Multi-dimensional HLT

As defined by Simon (1995), a Hypothetical Learning Trajectory (HLT) is the "the teacher's prediction as to the path by which learning might proceed" (p. 135). It is based on

the idea that most student learning follows a similar path and is described as “hypothetical” because it is only a “expected tendency”(Simon, 1995, p. 135). In the years since Simon introduced the term HLT, researchers have interpreted and developed this construct in many different ways (Clements & Samara, 2004). Teaching Sequences, Learning and Assessment Frameworks and Learning Progressions have been created, and despite differing in their design and theoretical underpinning, all have the common purpose of attempting to map the path students take when coming to understand a particular domain.

In Australia major learning trajectories are exemplified by the identification of Growth Points across different areas of mathematics (Department of Education & Early Childhood Development, 2011a) and the LAF for multiplicative thinking (Siemon, et al., 2006). These frameworks assist teachers to be more informed about finding student’s point of need, and better able to scaffold their mathematical learning (Clark, 2001).

Clements and Samara (2004) describe the three aspects that they consider contribute to a complete HLT- “the learning goal, developmental progressions of thinking and learning, and sequence of instructional tasks”(p. 84). The first two aspects were used to form the basis of the whole number place value HLT described in this paper.

Firstly, a learning goal for the whole number place value HLT was identified. Simon and Tzur (2004) explain a learning goal is “based on knowledge of the students’ current mathematical knowledge”(p. 96). The learning goal for this HLT was to identify the broad sequence of concepts and strategies needed for students from Year 3-6 come to a full understanding of the whole number place value system.

The next aspect of creating the HLT was to investigate the “developmental progressions of thinking and learning”(Clements & Samara, 2004, p. 84) for place value. This process began with a comprehensive review of research literature associated with whole number place value. Many assessment instruments were also reviewed in order to gain an indication of the number and type of items assessing place value, particularly those involving numbers beyond three-digits.

A synthesis of the literature saw the emergence of seven major aspects which appeared to be integral to the development of whole number place value understanding. These aspects were: *Count*, *Make/Represent*, *Name/Record*, *Rename*, *Compare/Order*, *Calculate* and *Estimate*. As these aspects were quite distinct, it was decided that individual HLTs would be created for each aspect.

It was suspected that each of the seven aspects not only related to the overall “big idea” of place value, but in some way linked to each other. The interplay between these aspects would be investigated in more detail using the results of the Rasch analysis later in the project. This information would then be used to create a multi-dimensional whole number place value HLT.

The following is a brief summary of the skills, thinking and literature incorporated into each aspect of the place value HLT (No developmental order implied):

Count: Counting forwards and backwards in place value parts (e.g., 45, 55, 65 is counting using the unit ten). Bridging forwards and backwards over place value segments (e.g., 995 and one more ten requires bridging forwards over hundreds to thousands). (Cobb & Wheatley, 1988; Jones et al., 1996; Steffe, 1992)

Make/Represent: Make, represent or identify the value of a number using a range of materials or models- these may be proportional, non-proportional, canonical and non-canonical. (Baroody, 1990; Jones, et al., 1996; Thomas, 2004)

Name/Record: Read and write a number in words and figures (e.g., 75 is written as ‘seventy-five’). Identify the value of digits in a number (e.g. The value of 3 in 345 is 3 hundreds). Rounding numbers to the nearest place value part.(e.g., Round 2456 to the

nearest thousand). (Jones, et al., 1996; Ross, 1989)

Rename: Recognise and complete partitions and regrouping of numbers. (e.g., 1260 has 126 tens). (DES, 1979; Jones, et al., 1996; Resnick, 1983)

Compare/Order: Compare numbers to determine which is larger or smaller and place them in descending or ascending order (Bednarz & Janvier, 1982).

Calculate: Apply knowledge of place value when completing calculations (e.g., 45 by 10 is 45 tens) (Fuson, 1990; Wearne & Hiebert, 1994)

Estimate: Use knowledge of magnitude of numbers when estimating (e.g., estimate how many oranges fill a classroom 10?100?100 000?) (Siegler & Booth, 2004; Thomas, 2004)

Having identified the seven aspects, empirical evidence was used to order the ideas and strategies within each aspect from least to most difficult. Each aspect was broken into hypothetical “stages”. The accuracy of this hierarchy will be tested using the results from the Rasch analysis. An abridged version of the HLT for the “Rename” aspect is included below in Table One.

Table One

Abridged Version of the Hypothetical Learning Trajectory for the ‘Rename’ Aspect.

Stage	Hypothetical Learning Trajectory for Rename (from least to most difficult)
1	May describe 10 ones is 1 ten and 10 hundreds is 1 thousand but they cannot rename accurately. Asked how many tens in 345, they may answer 4 (Sinclair, et al., 1992).
2	Can recognize and use equivalent representations like 2 hundreds=20 tens= 200 ones (Jones, et al., 1996).
3	Students are able to complete non-standard partitions and regrouping of numbers. (Resnick, 1983). Students may appear to use the multiplicative structure when multiplying and dividing numbers by 10, 100,1000, but may use rote learned rule e.g., adding a zero.
4	Students recognise when it is appropriate to split and combine numbers in place value parts (Jones, et al., 1996). Understand the exponential and reciprocal nature of number system (Baturu, 2000) .
5	Able to apply their base 10 knowledge to problems with other bases (DES, 1979).

Illustrative Examples of Items related to the HLT

Once the seven HLTs were created, these were used as a guide to select, modify and design a range of items to address that particular aspect in the PVAT. This was considered to be parallel to the “sequence of instructional tasks” (p. 84) which Clements and Samara (2004) describe as the third part of the development of a HLT. PVAT items were designed to be in the form of a paper and pen test as this was considered to be both appropriate for the year levels being tested (Year 3-6) and the most time effective assessment for teachers to administer. Items were a combination of multiple choice or short response questions with a mix of partial and full credit questions. At least one item was developed to address each stage of the HLTs. This was to ensure all parts of the HLT could be validated using the results from the PVAT Rasch analysis. Below are two illustrative examples of trial PVAT items which were designed to address Stage One and Two of the aspect “Rename” (See Figures 1 and 2 below).

Item 43b How many tens are in 387?	Item 21b 40 tens are the same as ____ ones?

Figure 1. Stage 1 Rename

Figure 2. Stage 2 Rename

The *Stage 1* item requires students to display their knowledge of renaming the place value parts of a three-digit number. An incorrect response of “8” may be indicative of a student who does not recognise that there are 30 tens in 3 hundreds so 38 tens altogether.

The *Stage 2* item requires students to recognise and use equivalent representations. A student who answers 400 recognises that 40 tens is the same as 400 ones.

Finally, all items were collated into the first form of the PVAT which was trialed at School A.

Where to Next?

To date, the trials of the PVAT have been successfully conducted at School A and a Rasch analysis of the data is currently taking place. This analysis will allow items to be validated or eliminated in preparation for the pilot form of the PVAT which will take place at School B during 2012. It is anticipated that the Rasch results will be used to investigate possible links between the seven aspects and form a multi-dimensional HLT for whole number. This will allow for a greater understanding of the complexities of each aspect of place value and provide teachers with a comprehensive whole number place value teaching sequence.

It is hoped that the PVAT will also allow teachers to more accurately determine Year 3-6 students’ level of whole number place value understanding and as a result determine students’ readiness to move onto decimal place value.

In conclusion, the PVAT will attempt to finally “unmask” students and provide teachers with the accurate diagnostic assessment information they require to challenge their students to truly become *place value experts*.

References

- Adams, R. J., & Khoo, S. T. (1996). *QUEST-Version 2.1* The interactive Test Analysis System.
- Baroody, A. (1990). How and when should place-value concepts and skills be taught? *Journal for Research in Mathematics Education*, 21(4), 281-286.
- Baturo, A. (1998). The implication of multiplicative structure for students’ understanding of decimal-number numeration. In F. Biddulph & K. Carr (Eds.), *People in Mathematics Education (Proceedings of the 20th Annual conference of the Mathematics Education Research Group of Australasia)*, pp. 90-97. Rotorua, NZ: MERGA.
- Baturo, A. (2000). Construction of a numeration model: A theoretical analysis. In J. B. A. Chapman (Eds.), *Mathematics Education Beyond 2000 (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 95-103). Fremantle, WA: MERGA.
- Bednarz, N., & Janvier, B. (1982). The understanding of numeration in primary school. *Educational studies in Mathematics*, 13(1), 33-57.

- Clark, D. (2001). Understanding, assessing and developing young children's mathematical thinking: Research as a powerful tool for professional growth. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *24th Annual Mathematics Education Research Group of Australasia Conference*, pp. 9-26. Sydney, Australia: MERGA.
- Clements, J., & Samara, J. (2004). Learning Trajectories in Mathematics Education. *Mathematical Thinking and Learning*, 6(2), 81-87.
- Cobb, P. (1995). Constructivist, emergent and sociocultural perspectives in the context of developmental research. In D. Towens (Eds.), *17th Annual Meeting North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 3-29). The Ohio State University, Columbus Ohio, USA: ERIC
- Cobb, P., & Wheatley, G. (1988). Children's initial understandings of ten. *Focus on Learning Problems in Mathematics*, 10(3), 1-28.
- Department of Education & Early Childhood Development. (2011a). Early Numeracy Research Project growth points Retrieved November 12 from: <http://www.education.vic.gov.au/studentlearning/teachingresources/maths/enrp/enrplaf.htm#H3N10078>
- Department of Education & Early Childhood Development. (2011b). Mathematics Online Interview Booklet Retrieved 22 December from: <http://www.eduweb.vic.gov.au/edulibrary/public/teachlearn/student/mathscontinuum/onlineinterviewbklet.pdf>
- DES. (1979). Mathematics 5-11: A Handbook of Suggestions *HMI Matters for Discussion* 9. London: HMSO.
- Ellemor-Collins, D., & Wright, R. (2009). Developing conceptual place value: Instructional design for intensive intervention. In R. Hunter, B. Bicknell & T. Burgess (Eds.), *Crossing Divides (Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia*, pp. 169-176). Palmerston, NZ: MERGA.
- Fuson, K. (1990). Issues in Place-Value and Multidigit Addition and Subtraction Learning and Teaching. *Journal for Research in Mathematics Education*, 21(4), 273-280.
- Fuson, K., & Briars, D. (1990). Using a Base-Ten Blocks learning/Teaching approach for First and Second-Grade Place-Value and Multidigit addition and Subtraction. *Journal for Research in Mathematics Education*, 21(3), 180-206.
- Irwin, K. (1996). Making Sense of Decimals. In J. Mulligan & M. Mitchelmore (Eds.), *Children's Number Learning* (pp. 243-257). Adelaide: Australian Association of Mathematics Teachers.
- Jones, G., Thornton, C., Putt, I., Hill, K., Mogill, T., Rich, B., & Van Zoest, L. (1996). Multidigit number sense: a framework for instruction and assessment. *Journal for Research in Mathematics Education*, 27(3), 310-336.
- Moloney, K., & Stacey, K. (1997). Changes with age in students' conception of decimal notation. *Mathematics Education Research Journal*, 9(1), 25-38.
- Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 109-151). New York: Academic Press.
- Ross, S. H. (1989). Parts, wholes and place value: a developmental view. *The Arithmetic Teacher*, 36(6), 47-51.
- Seah, R. T. K., & Booker, G. (2005). Lack of Numeration and Multiplication Conceptual Knowledge in Middle School Students: A Barrier to the Development of High School Mathematics? In B. Bartlett, F. Bryer & D. Roebuck (Eds.), *Stimulating the 'Action' as Participants in Participatory Research* (Vol. 3, pp. 86-98). Nathan, Qld: Griffith University.
- Siegler, R., & Booth, J. (2004). Development of Numerical Estimation in Young Children. *Child Development*, 74(2), 428-444.
- Siemon, D., Breed, M., Dole, S., Izard, J., & Virgona, J. (2006). *Scaffolding Numeracy in the Middle Years-Project Findings, Material and Resources*. Final Report Retrieved 23/11/11 from: www.eduweb.vic.gov.au/edulibrary/public
- Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), 114-145.
- Simon, M. A., & Tzur, R. (2004). Explicating the Role of Mathematical Tasks in Conceptual Learning: An Elaboration of the Hypothetical Learning Trajectory. *Mathematical Thinking and Learning*, 6(2), 91-104.
- Sinclair, A., Garin, A., & Tieche-Christinat, C. (1992). Constructing and Understanding of Place Value in Numerical Notation. *European Journal of Psychology of Education*, VII(3), 191-207.
- Steffe, L. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(259-309).
- Steinle, V., & Stacey, K. (2003). Grade-related trends in the prevalence and persistence of decimal misconceptions. In N. A. Pateman, B. J. Dougherty & J. Zilliox (Eds.), *27th Conference of the International Group for the Psychology of Mathematics Education*, pp. 259-266. Honolulu: PME.

- Thomas, N. (2004). The Development of Structure in the Number System. In M. J. Hoines & A. B. Fuglestad (Eds.), *28th Conference of the International Group for the Psychology of Mathematics Education*, (pp. 305-312). Bergen, Norway: Bergen University College Press.
- Wearne, D., & Hiebert, J. (1994). Place Value and Addition and Subtraction. *The Arithmetic Teacher*, 41(5), 272-274.
- Yin, R. (2009). *Case Study Research Design and Methods* (4th ed.). London: Sage.

On Diagnosis and Intervention: Helping Students with Special Needs Learn Fraction Ideas Involving Decimal Numbers

Rebecca Seah
Griffith University
<rtkseah@gmail.com>

Despite the emphasis that children should have a robust sense of number and a thorough understanding of fraction (National Mathematics Advisory Panel, 2008), many students continue to struggle with these concepts. Booker Diagnostic Assessment Framework (Booker, 2011) can inform decision about teaching that improves students' learning outcomes. Focusing on decimal numbers, this paper reports how effective use of the framework and instructional materials can help three teachers challenge their students' misconceptions with mathematics, leading to improved learning outcomes.

Introduction

Since Shulman's (1986) proposed pedagogical content knowledge, learning to teach necessitates efficient use of three micro-skills: general pedagogical, subject matter and pedagogical content knowledge (Borko & Putnam, 1996). General knowledge refers to the broad principles and strategies of general classroom management and instructional strategies needed to ensure a smooth running of lessons. Subject matter knowledge involves key facts, concepts, principles and explanatory frameworks of a discipline, and the type of evidence used to guide the inquiry. It is influenced by the ways in which teachers think about their subject matter and the choices they make in their teaching (Grossman, Wilson, & Shulman, 1989). Pedagogical content knowledge and beliefs are ways of representing and formulating a subject that make it comprehensible to others (Shulman, 1986). It is subject-specific, influenced by individuals' past and present training, teaching and learning experience (Hashweh, 2005). The inexorably intertwined relations between these three aspects of knowledge ensured that teaching is a complex and lengthy process extending for most teachers well after their initial training (Calderhead & Shorrock, 1997).

Compounding the teaching process is an increasing diversity of needs found within classrooms. There is limited information about how these students, many of them having special needs, learn mathematics (Forgasz et al., 2008). This lack of information is due largely to a clear divide between the research interest of mathematics and special education researchers. The former focuses on deep content and pedagogical knowledge, whereas the latter is dominated by a deficit view of disability, relying on experimental design to study basic skills development, memorisation of facts, and drill and practice. From this branch of research comes the most contentious claim, that proficiency in learning among students with special needs is best achieved through direct and explicit instruction. The efficacy of these instructions have been cited and promoted in a range of research papers and policy documents (Apps & Carter, 2006; Purdie & Ellis, 2005). Direct instruction originated in the late 1970s and was deemed inappropriate for fostering the growth of reasoning and higher order cognitive skills subsequently (Palincsar, 1998). The resurgent popularity can be traced to the policy of inclusive education. With a lack of information from mathematics researchers on how best to teach this cohort, it seems that the only credible source of information is found in meta-analysis report such as those produced by Baker et al. (Baker, Gersten, & Lee, 2002) endorsing direct and explicit instruction.

To endorse any teaching approaches one first need to understand what it is one is subscribing to. The fourth edition of *Designing Effective Mathematics Instruction: A Direct*

Instruction Approach (Stein, Kinder, Silbert, & Carnine, 2006) provided a window into the characteristics of such practice. There are 390 basic arithmetic facts: 100 each for addition, subtraction and multiplication of numbers between zero and nine, and 90 division facts for students to learn. In counting, students were taught to use tally marks and the likely difficulty teachers would encounter is when ‘students write crooked lines or crowd them together’ (*ibid*, p.53). A similar approach was taken to problem solving, where the focus was on recognising key words and word patterns instead of developing the thinking needed for problems solving. Such practice not only undermines mathematics as a discipline, it also neglects the importance of conceptual understanding and fails to acknowledge students’ rights and ability to construct mathematics of their own.

Criticisms about the sole reliance on well-controlled experimental design have come from within the field of special education. However, much of the arguments remain mere rhetoric without any constructive suggestion for improvement. These conflicts in the philosophical and conceptual frameworks have produced an epistemological and ontological stalemate (Gallagher, 2007), or what Kauffman et al. (Kauffman, Nelson, Simpson, & Mock, 2011) termed as ‘permanent fixtures’ of special education. Such a stalemate can no longer provide new insight into how students with special needs learn. Instead, a framework that builds on a well-established mathematical epistemology that addresses students’ mathematical misconceptions is needed. Meanwhile, Norwich and Lewis (2007) argue that students with special needs do not need a qualitatively different pedagogy. Perhaps good teaching that begins with accurate diagnosis of students’ misconceptions, combined with an integration of subject matter knowledge, pedagogical content knowledge, can improve the learning outcomes of these students.

Theoretical Framework

This study posits that knowledge is actively created in social situations (Booker, Bond, Sparrow, & Swan, 2010) and distributed across people, environment and objects (Pea, 1993). Mathematics relies on a system of symbolic representations to express its meaning. These representations, in the form of digits, signs and objects, are not always ‘transparent’ to the learners (Cobb, Yackel, & Wood, 1992) and therefore could not be mastered merely through drill and practice. Instead, any mathematical concept begins with providing full meaning of number words to match instructional representations, and later linking it with the symbols that represent them (Booker et al., 2010). This makes the design of effective instructional representations and learning sequence to bring out the abstract nature of mathematics, and ways to construct understanding vital. Based on Booker Diagnostic Assessment Framework (Booker, 2011), effective diagnosis of students’ difficulties with mathematics begins with designing tasks that relate to mathematical conceptual frameworks and not mere procedural accuracy. Such tasks provide insights into students’ underlying thinking pattern, including what is known, what needs to be known and gaps in a student’s thinking. Once students’ misconceptions are established, appropriate teaching instructions can be developed to mediate learning success. Specifically, instructions on decimal numbers began with an acknowledgement that students’ prior knowledge about whole number would have substantial impacts on their learning about decimal numbers and this is unavoidable (Ni & Zhou, 2005). Instead, the aim of instructional design was to create various opportunities to explicitly confront the conceptual conflicts students’ prior knowledge about whole number has with fraction numbers. These conflicts include knowing that (a) decimal numbers are a composition of whole numbers and fractions broken by the decimal point; (b) its language ends with ‘th’ and reading it differs from reading whole numbers as there is no constant unit that correspond to ones; (c) the 10-1 relationship in decimal numbers begin

with tenths and decrease in quantity towards the right, whereas the same relationship in whole numbers increase in quantity towards the left; (d) a zero is a placeholder and when it is at the extreme right on a decimal number, its value unchanged.

Method

This study was conducted in a high school located in a low socio-economic area in Brisbane, Australia. The participants were three teachers and their Year 8 and 9 students with special needs. Both Ted and Hana are experienced teachers in special needs and English respectively whereas Ann is a beginning teacher with no mathematics training. 12 Year 9 students had been identified by a medical practitioner as having a developmental disability under the Education Queensland guidelines and attended a separate program for Mathematics. 19 students were identified through the school's internal screening process as having a learning difficulty and attended a support program that lasted one term. Using design research method, the study began with diagnostic assessments of students' decimal numbers knowledge. The data collected were then used to design instructional sequence based on Booker et al.'s (2010) connected, conceptual understanding of learning mathematics. The aim was to help teachers teach decimal number place value, renaming and sequencing skills. In particular, instructional games were used to (a) provide a platform where the concept of decimal numbers could be constructed in a non-threatening matter, (b) develop teachers' subject matter and pedagogical knowledge, and (c) create a rich learning environment. Ongoing daily analysis of teachers' lesson presentations and student's participation in classroom activities further shaped and refined subsequent lessons.

A total of 50 lessons, spanning across two terms for the Year 9 class and 5 weeks for the Year 8 classes, were planned, observed, and redesigned. Year 9 class taught by Ted received 24 lessons, Ann of Year 8A 15 lessons and Hana of Year 8B 11 lessons. Each teacher was briefed on the mathematical basis of each learning activity and ways of using the instructional materials. Individually, the teachers decide how to introduce each concept, what examples to provide in order to help students comprehend the concept, and how to present and facilitate the activities. The data presented here include students' performance on *Booker Decimal Fraction Test A* and *B* (Booker, 2011), video recording of the lessons, and teachers' feedback. The tests were administered to all three classes at the beginning and the end of the project. The 10 questions assess students' understanding of basic concept, naming tenths and hundredths, sequencing, ordering, computing and solving word problem. The initial data collected contribute to the lesson design while the results obtained from Test B provide an indication of lesson effectiveness.

Results and Discussion

The graphs presented in Figure 1 provided comparison results for each class for Test A and B. The initial data revealed that most students had difficulty naming decimal numbers involving hundredths. They also could not sequence, order, compute and solve word problems and many similar errors were made in the test. For example, while most students had no difficulty writing 3 and 7 tenths as 3.7 (question 2a), some had trouble understanding 8 tenths as 0.8 (question 2b) and wrote it as 8.0, 8.10 or 80 instead. A majority of students were unable to name decimal fractions involving hundredths especially when an internal zero was involved. Thus, 7 and 2 hundredths was written as 7.2 or 7.200 (question 3a) and 16 hundredths as 1.6 or 16.0 (question 3b).

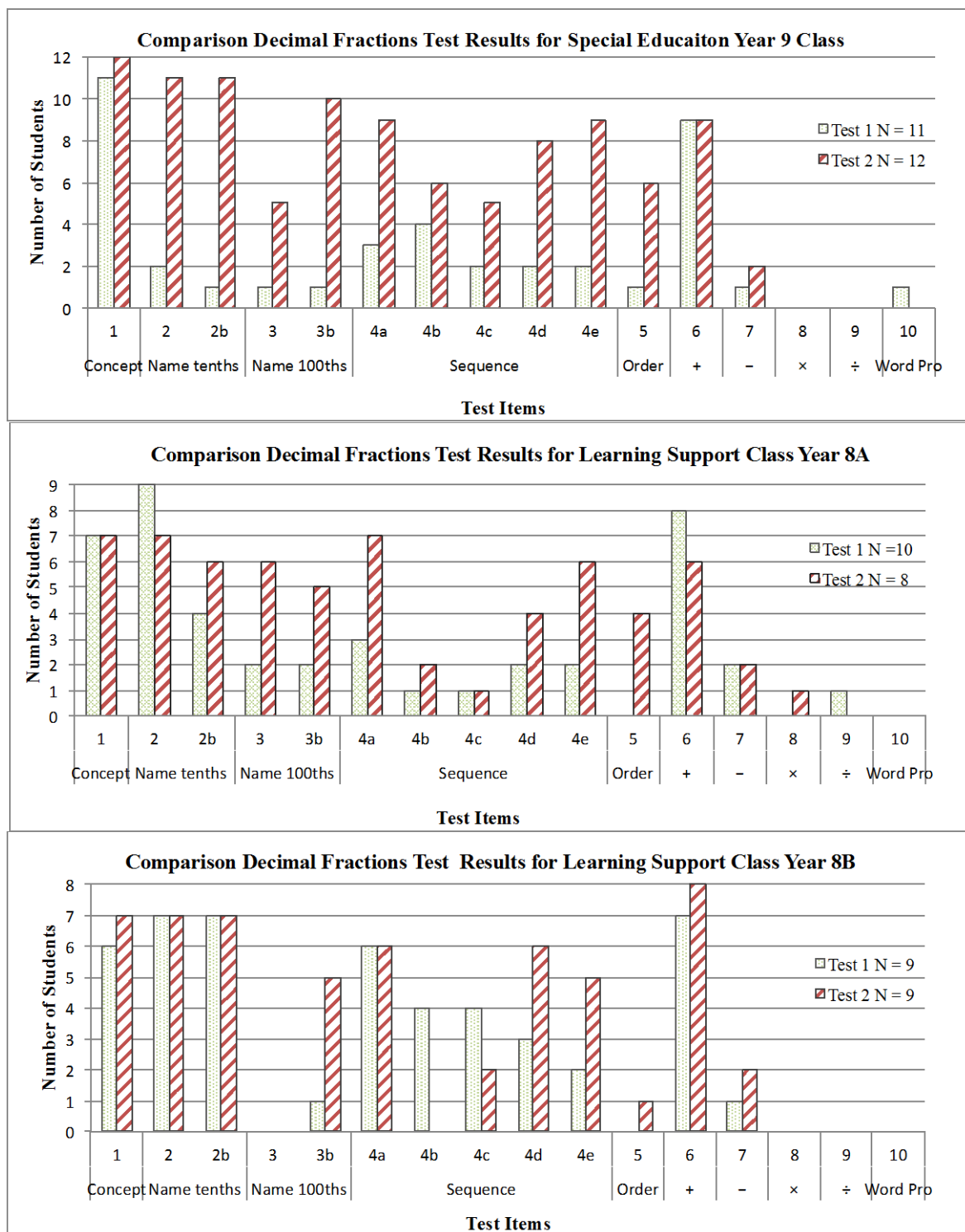


Figure 1. Comparison results of decimal fraction test across three classes.

This sort of error, caused mainly by a lack of decimal fractions place value understanding, was particularly evident when attempting to sequence decimal fractions. When asked to write the numbers that are 3 tenths more from the corresponding set of decimal fractions (question 4), many students wrote: (4a) 8.4 as 11.4, (4b) 9.7 as 9.12, (4c) 6 as 9, (4d) 8 as 0.11, and (4e) 2.9 as 5.9. Ordering decimal fractions from least to greatest (question 5) was the most challenging task apart from computation. A majority of students wrote 3.0, 3.2, 3.04, 3.10, and 3.16 or the reverse, 3.16, 3.10, 3.4, 3.2 and 3.0. It would

appear that these students assumed '16' is bigger than 10, 4 and 2 and responded accordingly. A lack of place value understanding also hindered students' ability to compute with and solve problems involving decimal numbers. The varying teaching durations among the three classes reflect the nature of design research and in part have contributed to large gains made between tests in focus areas. Further, six out of nine students in Year 8B were recent arrivals holding refugee status. The need to adjust to a different form of schooling may have influenced their learning.

Although the results identified students' mathematical difficulties, the 'why' and 'how' questions were not obvious to participating teachers. As such, the instructional design served both to provide the subject matter and the pedagogical content knowledge needed to teach decimal numbers. The subsequent classroom research further enabled the teacher and researcher to comprehend learning from the students' point of view. Drawing on the initial results, the instructional sequence focused on decimal number place value, renaming and sequencing skills. These findings are addressed below.

The Concept of Place Value and Renaming

The introduction of decimal numbers began with finding numbers smaller than one, as represented by the base 10 blocks. Students in all three classes could only thought of 'halving it' or cut it into quarters. Only when the teacher reminded students the 10-1 relationship learned in whole numbers could students comprehend the logical process of breaking 1 one is 10. Once this was established, the ones and tenths and ones and hundredths grids (Figure 2) were used as instructional representations to develop the language and associated symbols. When showed 1 tenth on the tenths grid, many students straightaway said, 'it's one out of ten' but unable to write it as symbols. Teachers in all three classes had to use the ones block with the grids and a place value chart to show the writing for 1 tenth is 0.1.



Figure 2. Using the grids and place value chart to show the difference between 1 tenth and 1 hundredth.

The introduction of 1 hundredth followed similar instructional path of breaking each tenth into ten parts, resulting 100 parts. When asked how to write it as a number, Ann's student Mondie insisted that it should be written as 0.100, with 'two zeros' added at the back. When asked what is the meaning of zero? The range of responses included 'it's a number', 'it's zero' 'it's nothing', 'you can times it' or 'you can add on to other number like twenty'. The concept that zero is a placeholder was discussed when students agreed it was erroneous to write 975021 as 97521. Once that was established, students were then ready to accept that 1 hundredth should be written as 0.01 and not 0.100.

Making sense of numbers is a critical aspect of learning. To have number sense, students need to have full understanding of number concepts such as zero. The other skill is to be able to rename numbers in equivalent forms (Booker et al., 2010). In fractions, it does not restrict to naming 1 third as 2 sixths, 3 ninths, but also knowing that 2 and 3 tenths can be read as 23 tenths and written as $2\frac{3}{10}$ or 2.3. Bingo games were designed where the printed words 2 and 6 tenths (or 26 tenths) can be matched with either the symbol 2.6 or the ones and tenths grid. In this way, decimal numbers are no longer symbols devoid of meaning but words that express their values and can be 'seen' on the grids. Numbers that involves

hundredths provided further opportunities to challenge whole numbers conceptual conflicts and extend students' understanding of place value. Whereas students could guess the answer for game cards that are restricted to ones and tenths, such as 224 hundredths as 2.24, cards that involved hundredths required deeper knowledge of the number system since 30 hundredths does not look quite the same as 0.3.

Given that renaming was new to both the teachers and students, the instructional games provided both the subject matter and pedagogical knowledge to help teachers and students construct the understanding. Expression such as *'Oh gosh, zero point seven is 7 tenths!'* could be heard when students came to the 'discovery' on their own through the games. Teachers too, benefited from the games, as Ann remarked, *'All my life I'm afraid of mathematics. I was learning the game as they were learning the games. I feel much more equipped to lead them on this learning journey and I've a better understanding now.'*

Comparing, Sequencing and Ordering Decimal Numbers

Teaching students to sequence numbers quickly brought to the realisation that they could not count decimal numbers. Many students simply spelled out the words, *'zero point seven, zero point eight, zero point nine'*, which naturally led them to say *'zero point ten'* and wrote 0.8, 0.9, 0.10. In one activity, students have been working on adding 3 tenths more to 2.3, 5.7 and 3.9. These the students completed with ease. However, when asked to add 3 tenths more to the number 4, everyone replied seven. Instead of seeing the numbers as a set, their focus was entirely on the last digit. This realisation helped teachers to change their instructions, reading mathematics as a language such as 2 and five tenths instead of spelling it out so students could determine the number magnitude. In one activity, Ted noticed that his student Sam had great difficulty finding the difference between 6.9 and 7.7. Only when Ted read it as 77 tenths and 69 tenths did Sam able to comprehend the task.

Teacher General, Subject Matter and Pedagogical Content Knowledge

The relationships between general, subject matter and pedagogical content knowledge are intimately linked in the diagnosis and teaching process. Teachers' subject matter knowledge greatly influenced how they perceived students' response and their subsequent instructional choice. For example, Hana of Year 8B asked her students, *'What is a fraction?'* to which Jama replied, *'There's a numerator and a denominator.'* Hana was pleased with the response and did not pursue the issue further. Subsequent lessons and continuing probing quickly revealed that what Jama knew was 'naming' the fraction parts, and not a conceptual understanding of the part/whole relationship. Likewise, when students repeatedly wrote 5.9 and $\frac{5}{9}$ or 1.2 and $\frac{1}{2}$ as examples for fractions, there is a need to probe further to determine if the use of similar digits was coincidental or a misconception of treating the decimal point as the vinculum.

Using instructional representations in the teaching process was not straightforward, as the teachers tended to briefly mention them and focus extensively on using symbols to explain. Cobb et al. (1992) argues that many teachers assume that students will automatically construct the correct internal representations from the materials presented and understood 'the mathematical relationships they are to construct before they have constructed them' (p. 5). What teachers failed to realise is that materials that are thought to be 'transparent' – where their meanings are obvious, is the product of teachers' own understanding. For many students, the meanings are not obvious. In one activity, students were asked to shade decimal numbers next to the corresponding grids. Many students across

all three classes shaded 2 and 4 hundredths for 2.4 on the task sheet. Ted first tried to explain that the tenths is bigger and the hundredths is smaller because of their position on the place value to no avail. It was only when both numbers were represented on the grids that the difference between them was made clear. This experience made Ted realise, '*It wasn't just about teaching a concept from the board, but also how to transfer what's taught to the physical, mental, knowledge of the terminology*'.

The use of questioning technique to elicit students' underlying thinking is another important diagnostic and teaching skill. For Hana, the classroom research helped develop a task-analytic approach to teaching by breaking learning down to teachable steps. She admitted that initially, she thought, '*the program is basic, is easy and the lesson is going to drag*,' but soon appreciated that each sub-construct needed to be carefully introduced and cemented before introducing a new construct. She learned to (a) check for understanding at every step of the lesson, (b) ask many questions at the right level to ensure a high success rate, and (c) build confidence and monitor closely the work being done by each student. Ann too, learned to probe for further understanding. She remarked,

'You ask, "What do you mean by that? Oh ok... so you think it could be..." Whatever the answer is and you know it is wrong but don't say that's wrong. You said, "Oh ok, and how did you get to that? Ok... could you show me how you did that, what processes did you use?" And then they correct themselves... and they beam'.

Prior to the study, all three teachers mentioned managing students' non-compliance as their most challenging task when teaching this cohort. Students in all three classes were quite restless during the initial study. Some questioned why they had to be removed from their regular class. The use of instructional games and representations to help students learn the concept soon eliminated many behavioral problems. Ann remarked,

'No behaviours, the same kids who had been suspended and on the red book. They're committed to their learning. They want to learn.' Hana agreed, 'They're experiencing success and they want to come. They didn't want to go back to their regular class. The lessons just go... I wish it's the same with literacy!'

Conclusion

Students' misconceptions with decimal numbers reported here are not new and have been well documented in the research literature. The value of this data lies in the fact that these students are ascertained to have either a learning difficulty or disability, such as intellectual impairment, and Autistic Spectrum Disorder. The result points to the nature of mathematics and the way it was instructed as a cause of students' difficulty, and not as a result of student deficit. In particular, good teaching that challenges students' misconceptions and guides them to construct their own mathematics, as well as the time allocated to learn a particular topic is of the essence. Booker (2011) maintains that well-focused assessment could reveal students' underlying belief systems and provide guidance to improve teaching. In this case, the results obtained enabled research to design specific activities that highlighted misconceptions and built understanding. More research is clearly needed to extend our understanding of teachers' subject matter and pedagogical content knowledge and students' mathematical thinking at the classroom level to effect change.

References

- Apps, M., & Carter, M. (2006). When all is said and done, more is said than done: Research examining constructivist instruction for students with special needs. *Australasian Journal of Special Education*, 30(1), 21-38.

- Baker, S., Gersten, R., & Lee, D.-S. (2002). A synthesis of empirical research on teaching mathematics to low-achieving students. *The Elementary School Journal*, 103(1), 51.
- Booker, G. (2011). *Building numeracy: Moving from diagnosis to intervention*. Melbourne: Oxford University Press.
- Booker, G., Bond, D., Sparrow, L., & Swan, P. (2010). *Teaching primary mathematics* (4th ed.). Frenchs Forest, NSW: Pearson Prentice Hall.
- Borko, H., & Putnam, R. T. (1996). Learning to teach. In D. C. Berliner & R. Calfee (Eds.), *Handbook of educational psychology* (pp. 673-708). London: Prentice Hall International
- Calderhead, J., & Shorrock, S. B. (1997). *Understanding teacher education*. London: Routledge.
- Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, 23(10), 2-33.
- Forgasz, H., Barkatsas, A., Bishop, A., Clarke, B., Keast, S., Seah, W. T., & Sullivan, P. (Eds.). (2008). *Research in mathematics education in Australasia 2004-2007*. Rotterdam, The Netherlands: Sense Publishers.
- Gallagher, D. (2007). Challenging orthodoxy in special education: On longstanding debates and philosophical divides. In L. Florian (Ed.), *The Sage handbook of special education* (pp. 515-527). London: Sage.
- Grossman, P., Wilson, S. M., & Shulman, L. S. (1989). Teachers of substance: Subject matter knowledge for teaching. In M. C. Reynolds (Ed.), *Knowledge base for the beginning teacher* (pp. 23-36). New York: Pergamon Press.
- Hashweh, M. Z. (2005). Teacher pedagogical constructions: A reconfiguration of pedagogical content knowledge. *Teachers and Teaching*, 11(3), 273-292.
- Kauffman, J. M., Nelson, C. M., Simpson, R. L., & Mock, D. R. (2011). Contemporary Issues. In J. M. Kauffman & D. P. Hallahan (Eds.), *Handbook of special education* (pp. 15-26). New York: Routledge.
- National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*: US Department of Education.
- Ni, Y., & Zhou, Y.-D. (2005). Teaching and Learning Fraction and Rational Numbers: The Origins and Implications of Whole Number Bias. *Educational Psychologist*, 40(1), 27-52.
- Norwich, B., & Lewis, A. (2007). How specialized is teaching children with disabilities and difficulties? *Journal of Curriculum Studies*, 39(2), 127-150.
- Palincsar, A. (1998). Social constructivist perspectives on teaching and learning. In J. Spence, J. Darley & D. Foss (Eds.), *Annual review of psychology (Vol.48, pp. 345-375)*. Palo Alto, CA: Annual Reviews.
- Pea, R. D. (1993). Practices of distributed intelligence and designs for education. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 47-87). New York: Cambridge University Press.
- Purdie, N., & Ellis, L. A. (2005). *A review of the empirical evidence identifying effective interventions and teaching practices for students with learning difficulties in year 4, 5 and 6*. Camberwell, Victoria: ACER.
- Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *The American Educational Research Association*, 15(2), 4-14.
- Stein, M., Kinder, D., Silbert, J., & Carnine, D. (2006). *Designing effective mathematics instruction: A direct instruction approach* (4th ed.). Upper Saddle River, NJ: Pearson Prentice Hall.

Chinese Primary Students' Mathematical Task Types Preferences

Wee Tiong Seah

Monash University, Australia
<WeeTiong.Seah@monash.edu>

Anastasios (Tasos) Barkatsas

Monash University, Australia
<Tasos.Barkatsas@monash.edu>

The number of ethnic Chinese students in schools across Australian cities is small but increasing. It is important to understand how these students socialise into the Australian (mathematics) education system, so that we can better facilitate their education experiences in ways which optimise their potential to learn. This paper reports on part of a larger study which seeks to deepen our knowledge in this area. The research question addressed by this paper is: what are the preferences amongst three types of mathematical tasks of Grade 5 and 6 students from Chongqing, China? Through the administration of a questionnaire to 1109 students, it was found that across the topics of Number and Geometry, contextualised tasks were the most preferred by Chinese students. 'Challenging', 'easy to do' and 'involving a model' were students' reasons for preferring particular task types. The significance of providing all students with a diverse range of task types, at the same time providing them with opportunities to be challenged and to experience success, are emphasized.

Introduction

Australia's diverse demographic and cultural profile is reflected in its schools, with a relatively high proportion of ethnic Chinese students studying in many schools in the capital cities. Most of these ethnic Chinese students (or their parents) arrived from mainland China as migrants (mainland China was in 2010 the third highest source country of immigration in Australia (Australian Bureau of Statistics [ABS], 2011)), or as international students. Amongst international students holding the non-tertiary 'schools' visa in Australia in the years 2008-9, the largest proportion (70.7%) were born in mainland China (ABS, 2011). Also, no other source country of international students has as high a percentage (14.5%) of their own students holding the 'schools' visa (ABS, 2011).

It is thus important to understand how these students socialise into the Australian education system, so that we can better facilitate their education experiences in ways which optimise their potential to learn. Such understandings may also be useful for education systems elsewhere. This paper reports on part of a larger study which investigates these students' socialisation process in the school classroom, specifically, the types of mathematical tasks that are posed on paper which appeal to these students and through which they learn most effectively, and the underlying reasons and values. In order to ensure a large enough sample of student participants, and to more accurately investigate the students' preferences without the influence of the Australian school culture, data were collected from students who were located in mainland China at the time of the study. The research question which is relevant to this paper is: *What are the preferences amongst mathematical tasks of Grade 5 and 6 students from Chongqing, China?*

Mathematical Tasks

In this study, mathematical tasks refer to the "explicitly focused [classroom] experiences [posed on paper] that engage children in developing and consolidating mathematical understanding" (Clarke, 2009, p 6). Given the Chinese educators' belief (e.g. Leung, 2001) that practice in mathematics promotes and consolidates cognitive skills, we regard exercises as mathematical tasks too. The choice of task types – as well as the way a task is

incorporated into a lesson – has been found to regulate the quality of student learning (Kilpatrick, Swafford, & Findell, 2001).

Mathematical task types have been categorised differently in different research studies, such as in the QUASAR study (see Stein, Smith, Henningsen & Silver, 2000). However, given that this study is interested in investigating the types of mathematical tasks used in the Australian classroom that appeal to mainland Chinese students, reference was made to the range of task types documented in the Australian Task Types in Mathematics Learning [TTML] project (Sullivan, Clarke, Clarke, & O'Shea, 2009). Specifically, three types of mathematical tasks have been identified: Type 1 (representational tasks), in which the tasks are designed to exemplify the mathematics through the use of models, representations, tools or explanations; Type 2 (contextualised tasks), in which mathematics is situated within a contextualised practical situation; and Type 3 (open-ended tasks), where each question posed by the tasks has more than one possible correct response.

All three types of mathematical tasks are commonly found in mathematics lessons in Australia, and are included in the annual National Assessment Program – Literacy and Numeracy [NAPLAN]. The following examples for each task type were drawn from the 2010 Year 5 paper.

Task type 1: (Item 25) 3.62 is equal to

(a) $0.3 + 0.6 + 0.2$

(b) $3.0 + 0.6 + 0.2$

(c) $3 + 0.6 + 0.02$

(d) $3 + 0.06 + 0.02$

Task type 2: (Item 36) A meeting is held on the first Tuesday of each month. There was a meeting held on 6 March. What is the date of the April meeting?

Task type 3: (Item 29) Write a number in the box to make this number sentence correct.

$24 + 15 > \square \times 5$

Mathematical Tasks in the Chinese School Curriculum

To what extent are mainland Chinese students familiar with these three types of mathematical tasks? Mainland China occupies a huge landmass (about 1.25 times the area of Australia) whose provinces experience extreme variations of climate and terrain. This, together with the country's emphasis on decentralisation in the current basic education reform exercise, has meant that no two mathematics lessons are taught the same way. However, traditional norms and practices (including Confucianism) continue to play a key role in shaping pedagogical decisions and actions in Chinese schools (Ryan, Kang, Mitchell & Erickson, 2009). The first author's 20-year experience in schools across different locations in mainland China also affirms the characteristics of typically Chinese-style mathematics lessons, such as relatively large classes, frequent testing, and homework.

So, in terms of the mathematics tasks which 'typical' Chinese students can be expected to be engaged in at school, they would include all the three task types introduced in the previous section. Type 3 open-ended tasks were added to the existing task types 1 and 2 in schools (and in new editions of textbooks) throughout China in the latest, 2001 Basic Education Curriculum Reform exercise. The ways in which these tasks are posed to students in mainland China are no different from those we are accustomed to in the 'West' – teacher-posed verbal questions, as well as mathematical tasks which they are expected to answer individually, with a peer, or as part of a group. In addition, the Chinese students can also expect to be assigned mathematical tasks to work on as homework. Another difference between Chinese and Australian primary school students' experiences with mathematical tasks would be the students' personal possession of an assigned mathematics textbook in the Chinese classroom, and its relative absence in Australia at the primary school level (where

textbooks are often used by teachers alone). This has implications to the opportunities students across the two cultures have to engage with (different types of) mathematical tasks in class and at home.

Research Design

The study within which this paper is contextualised has adopted the sequential mixed methods design (Creswell, 2009). This paper reports on the quantitative phase of the larger study, which aims to map the field relating to the preference for and usefulness of different mathematical task types amongst mainland Chinese students. The research method adopted for this phase is the survey questionnaire, translated from the one constructed for the TTML project. This 15-item questionnaire has a mix of Likert type items, ranking exercises, and open-ended questions. In translating the questionnaire to the Chinese language, the contextual information of several items in the TTML version was changed to accommodate the societal realities in mainland China (Seah, Barkatsas, Sullivan, & Li, 2010). Its validation through the process of back-translation also revealed that culturally-different ways of describing phenomena and of teaching act as blindspots in the process of translating (see Seah, Barkatsas, Sullivan, & Li (2010), for examples).

Data were collected from 1109 Grade 5 and 6 students from 15 classes in 3 (state) primary schools in Chongqing, a major municipality with some 31 million people in Southwestern China. Six hundred and nine of these students were in Grade 5, and five hundred were Grade 6 students. Given that the schools were neither private nor key schools, the sample was deemed to be representational of mainland Chinese students.

This paper reports on the findings relevant to one of the research questions of the wider study: *What are the preferences amongst mathematical tasks of Grade 5 and 6 students from Chongqing, China?* The data addressing this research question come from questionnaire items 9 and 11, as shown in Tables 1 and 2 respectively. Item 9 is concerned with the area of Number, whereas Item 11 relates to Geometry

Table 1
Questionnaire Item 9

In this table there are four maths questions that are pretty much the same type of mathematics content asked in different ways.	
We don't want you to work out the answers.	
Put a 1 next to the type of question you like to do most , 2 next to the one you like next best, and 3 next to the type of question you like least :	
9ai An adult cinema ticket costs RMB25, and a child ticket costs RMB12. How much would the tickets cost for 2 adults and 4 children to watch a movie?	
9aii 2 adults and 4 children spent RMB120 on movie tickets. How much might an adult ticket and a child ticket cost?	
9aiii $25 \times 2 + 12 \times 4 =$	
You like to do this type of question (the one you put a 1 against) the most because:	

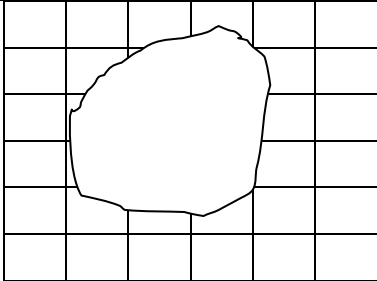
Table 2
Questionnaire Item 11

In this table there are four more maths questions that are pretty much the same type of

mathematics content asked in different ways.

We don't want you to work out the answers.

Put a 1 next to the type of question **you like to do most**, 2 next to the one you like next best, and 3 next to the type of question **you like least**:

11ai Find the area of the following figure.		
11aii If the area of a figure is 10 square units, what might the shape of the figure be?		
11aiii An athletic track is made up of two straight sections and two semi-circles. The straight section is 100m long. What is the area of the athletic track?		

You like to do this type of question (the one you put a 1 against) the most because:

Results

Table 3 summarises the students' ranking of their preferred task types. Each of the 6 mathematics questions across the two questionnaire items has been tagged as task type 1, 2 or 3, in the same way that these were used in the TTML study (see above).

Table 3

Grade 5 and 6 Students' Ranking of Favourite Mathematical Tasks

Item	Task type	Favourite	Like next best	Like the least	Valid n
9ai	2	514 (47.24%)	463 (42.56%)	111 (10.20%)	1088
9aii	3	380 (35.12%)	321 (29.67%)	381 (35.21%)	1082
9aiii	1	377 (37.55%)	286 (28.49%)	341 (33.96%)	1004
11ai	1	247 (22.74%)	311 (28.64%)	528 (48.62%)	1086
11aii	3	335 (31.54%)	395 (37.19%)	332 (31.26%)	1062
11aiii	2	464 (42.73%)	400 (36.83%)	222 (20.44%)	1086

A Friedman test was used to test for statistically significant differences in the ways students rank ordered the three types of mathematical tasks (items 9ai-iii and 11ai-iii). The results are shown in Tables 4 and 5.

Table 4

Friedman Test Results for Student Rank Ordering of Items 9ai – iii

Item	Mean rank

9aiii (task type 1)	2.09
9ai (task type 2)	1.76
9aai (task type 3)	2.14

Table 5

Friedman Test Results for Student Rank Ordering of Items 11ai – iii

Item	Mean rank
11ai (task type 1)	2.26
11aiii (task type 2)	1.75
11aai (task type 3)	1.99

The differences in rankings were statistically significant in both cases: [$\chi^2(2, 1001) = 97.45, p < 0.001$] and [$\chi^2(2, 1058) = 153.44, p < 0.001$] respectively.

Thus, it may be said that in the area of Number (Item 9), Grade 5 and 6 students in Chongqing, China, preferred mathematical tasks in the order of types 2 (contextualised tasks), 1 (representational tasks) and 3 (open-ended tasks), whereas in the area of Geometry (Item 11), the order of preference is task types 2, 3, 1.

Respondents were also asked to provide a reason for the nomination of any question as being the favourite. Given that the responses were not prompted, there was a huge range of what the students offered. Thus, we found the need to regard as similar responses which were closely related. For example, responses like ‘have to think’, ‘difficult mystery’, ‘lots of steps’ and ‘more complex’ were regarded as reflecting ‘challenging’. In this way, the reasons given by the respondents were coded into 9 categories, as shown in Table 6.

Table 6

Codes for Reasons Cited by Respondents in Ranking Exercise

1. Challenging	2. Easy to do
3. Real life scenario	4. Involving a model
5. Multiple solution strategies available	6. Has more than one possible answer
7. Fun	8. Numbers not words
9. Other	

The bar chart in Figure 1 displays the percentage in each coding category (1-9) of student respondents who nominated item 11ai over items 11a ii and 11a iii as their favourite. Bar charts for the other 5 questions may be similarly constructed.

Figure 1 suggests that amongst the students surveyed, those students who ranked representational (type 1) tasks as their favourite Geometry tasks appeared to value them for one of the following three categories of reasons: challenging (coding category 1, 31.65%), easy to do (coding category 2, 42.62%), and fun (coding category 7, 16.46%). The same three categories of reasons were also most commonly cited by the other respondents who rated contextualised (type 2) and open-ended (type 3) tasks as their favourite.

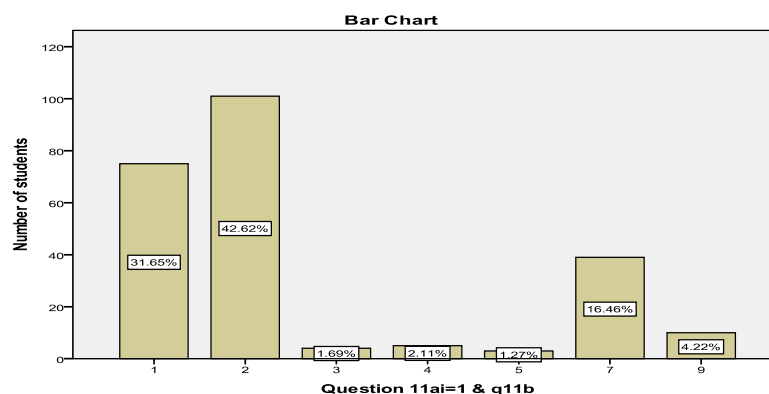


Figure 1. Percentages by coding category for ranking Item 11ai as favourite.

As for the Number questions in Item 9, these three categories of reasons were most commonly cited as well by students who rated open-ended (type 3) tasks as their favourite. Amongst those who rated contextualised (type 2) tasks as their favourite, 2 of these categories ('challenging' and 'easy to do') were most commonly cited. For those who rated representational (type 1) tasks as their favourite, 'easy to do' was most commonly cited.

A polychotomous (or polytomous) logit model was used to investigate the significance of these coding categories. This model is a special class of loglinear models and it is used to model the relationship between one or more dependent categorical variables and a number of independent categorical variables.

When the dependent variable has more than two values, we can construct many odds ratios for the same combination of values of the independent variables. The logit procedure considers the last category of each variable as the reference category. In our case, the category 'fun' (coding category 7) is set to zero, and 9ai=3, 9aii=3 and 9aiii=3 are all set to zero respectively in the corresponding logit models. The last two categories from Table 6 had not been considered because there were less than ten responses in each of these categories. Given the space constraints, the results of the polychotomous logit statistical analysis for the Number question (item 9) only are shown in Table 7.

The design for this test is governed by the following models: constant + q9ai + q19ai * q9b, constant + q9aii + q19aii * q9b, constant + q9aiii + q19aiii * q9b. The first number in each cell is the parameter estimate. The first number within the parenthesis is e^{λ} , followed by the p value (only in the case of statistically significant results). Two cell entries from Table 7 will be discussed; the other cell entries may be interpreted in the same way.

The parameter estimate for 'real life scenario' being the favourite for item 9ai (Code 3, 1st column, 3rd row) is .536. The value of e^{λ} is $e^{.536} = 1.709$. This tells us that based on the model, the students in the study are almost twice (i.e. 1.709 times) more likely to have nominated 'real life scenario' as the reason for selecting item 9ai as a favourite over the same reason being nominated when it is the least liked, compared to nominating 'fun' as the reason for selecting item 9ai as a favourite over it being nominated when it is the least liked. In other words, we can say that the odds of Grade 5 and 6 Chongqing students in the study nominating 'real life scenario' as a reason for Item 9ai being a favourite over it being nominated when it is least liked (amongst the three task types in Item 9a) is 1.709 times the odds of nominating 'fun' as a reason for Item 9ai being a favourite over it being nominated when it is least liked. Nevertheless, these odds are not statistically significant.

Table 7

Parameter Estimate (λ , e^λ) Summary

Q9b (Reasons cited in ranking exercise)	9ai=1 vs 9ai=3	9aii=1 vs 9aii=3	9aiii=1 vs 9aiii=3	9ai=3, 9aii=3 9aiii=3
1:Challenging	-1.853 (.157, .001)	1.200 (3.32, .003)	-1.240 (.289, .001)	0 (1)
2:Easy to do	-.237 (.789, .000)	-2.826 (.059, .000)	1.795 (6.019, .000)	0 (1)
3:Real life scenario	.536 (1.709)	-.256 (.774)	-1.266 (.282)	0 (1)
4:Involving a model	-1.325 (.266, .039)	-2.247 (.106, .006)	1.636 (5.134, .039)	0 (1)
5:Multiple solution strategies	-1.661 (.1899)	1.106 (3.022)	-1.885 (.152)	0 (1)
6:Has more than one possible answer	-3.271 (.0379)	.371 (1.449)	-.418 (.658)	0 (1)
7:Fun	0 (1)	0 (1)	0 (1)	0 (1)

The parameter estimate for ‘easy to do’ being the favourite for item 9aiii (Code 2, 3rd column, 2nd row) is 1.795. The value of e^λ is $e^{1.795} = 6.019$. That is, based on the model, the students are statistically significantly six times more likely to have nominated ‘easy to do’ as the reason for item 9aiii being a favourite over nominating the same reason when the same item is the least liked, compared to nominating ‘fun’ as the reason for Item 9aiii being a favourite over it being nominated when item 9aiii is the least liked.

We can thus postulate what the statistically significant reasons were underlying student choice of favourite task type (over the same task type being least liked) for the topic of Number, relative to the reason of ‘fun’. It appears that regardless of the task type, ‘challenging’, ‘easy to do’, and ‘involving a model’ were statistically significantly the reasons for particular task types to be nominated as favourites, compared to ‘fun’.

Findings

Three types of mathematical tasks were examined with 1109 Grade 5 and 6 students in Chongqing in this research study. The data suggest that for both Number and Geometry items, the Chinese students preferred most to engage with tasks involving contextualised situations. This is different from the responses of their peers in Australian schools, which relate representational tasks as their favourite (Sullivan, Clarke, Clarke, & O’Shea, in press). Thus, Chinese immigrant / foreign students in the Australian mathematics classroom were likely to respond differently to mathematical tasks compared to their Australian peers, preferring Type 2 tasks the questions of which are situated within a practical or real-life context. Perhaps the Chinese students did not prefer the Type 1, representational tasks because the large amount of homework practice they have to complete has led to an affinity for questions which appear to be less monotonous and more interesting.

The mainland Chinese students also did not prefer the Type 3, open-ended tasks overwhelmingly. In the context of Hofstede’s (1997) cultural dimensions, China has a low uncertainty avoidance index, which suggests that the mainland Chinese tend to feel

threatened by events or things which are ambiguous or unknown to them. Open-ended tasks embody such sense of ambiguity and uncertainty. For the students, the competitive assessment system in schools might also contribute to their aversion to mathematical tasks for which they cannot be certain of having found ‘the’ answer. Nevertheless, that the mainland Chinese students’ preferences are different from their Australian peers reinforces the importance of students being exposed to a range of mathematical tasks. It also provides teachers with specific task types to focus on when interacting with particular groups of students in the Australian classroom.

A variety of reasons were offered by the mainland Chinese students for preferring particular task types. Most of these refer to the tasks being challenging or easy, although they also preferred certain tasks because the tasks involved modelling (including the use of drawings and grids). Did the students find the use of a model to be a worthwhile strategy? Were they particularly skilled in deploying this strategy? Or are there other explanations? These questions, as well as those identified in the previous paragraph, are the very things which have informed the next phase of the larger study, involving the interviewing of selected student respondents to shed light on understanding the reasons underlining their task preference. At this stage, the data suggest that teacher valuing of *challenge* and *easiness* can stimulate in both groups of students the types of mathematical tasks they like, thereby promoting or sustaining student engagement. The association of students’ valuing of *challenge* with the mathematical tasks they prefer also complements related studies (Jensen et al, 2012) which link effective education with the presentation of challenging mathematics content in class.

References

- Australian Bureau of Statistics [ABS]. (2011). *Migration 2009-10*. Canberra: Australian Bureau of Statistics.
- Clarke, B. (2009). Using tasks involving models, tools and representations: Insights from a Middle Years mathematics project, *Proceedings of 32nd Annual Conference of the Mathematics Education Research Group of Australasia (MERGA)*, 5 - 9 July, 2009, Mathematics Education Research Group of Australasia Incorporated (MERGA), Palmerston North, New Zealand, pp. 6-9.
- Creswell, J. W. (2009). *Research design: Qualitative, quantitative, and mixed methods approaches* (3rd ed.). Thousand Oaks, CA: Sage Publications.
- Hofstede, G. (1997). *Cultures and organizations: Software of the mind* (Revised ed.). NY: McGraw-Hill.
- Jensen, B., Hunter, A., Sonnemann, J., & Burns, T. (2012). *Catching Up: Learning from the best school systems in East Asia* (No. 2012-3). Victoria, Australia: Grattan Institute.
- Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
- Leung, F. K. S. (2001). In search of an East Asian identity in mathematics education. *Educational Studies in Mathematics*, 47(1), 35-51.
- Ryan, J., Kang, C., Mitchell, I., & Erickson, G. (2009). China's basic education reform: an account of an international collaborative research and development project. *Asia Pacific Journal of Education*, 29(4), 427-441.
- Seah, W. T., Barkatsas, A., Sullivan, P., & Li, Z. (2010). Chinese students' perspectives of effective mathematics learning: An exploratory study. In T. Desmond (Ed.), *The Asian Conference on Education official conference proceedings 2010* (pp. 389-403). Japan: International Academic Forum.
- Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing Standards-based mathematics instruction: A casebook for professional development*. NY: Teachers College Press.
- Sullivan, P., Clarke, D., Clarke, B., & O'Shea, H. (in press). *Teaching with tasks for effective mathematics learning*. Dordrecht, The Netherlands: Springer.
- Sullivan, P., Clarke, D. M., Clarke, B. A., & O'Shea, H. (2009). Exploring the relationship between tasks, teacher actions, and student learning. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 185-192). Thessaloniki, Greece: PME.

Mathematical Modelling for Singapore Primary Classrooms: From a Teacher's Lens

Cynthia Seto

Academy of Singapore Teachers

<cynthia_seto@moe.gov.sg>

Thomas Mary Magdalene

Ponggol Primary School

<thomas_mary_magdalene@moe.edu.sg>

Ng Kit Ee Dawn

National Institute of Education

Nanyang Technological University

<dawn.ng@nie.edu.sg>

Chan Chun Ming Eric

National Institute of Education

Nanyang Technological University

<eric.chan@nie.edu.sg>

Wanty Widjaja

Deakin University

<w.widjaja@deakin.edu.au>

Limited Singapore research indicated a lack of exposure of modelling tasks at primary levels. Teacher reflection is used as a tool in design research cycles exploring the potentials of modelling tasks in a Singapore primary five classroom. Findings reveal that the teacher identified three potentials of a modelling task on children's mathematisation process: the task provided a platform for children to (a) identify variables and form relationships between them, (b) relate school-based math learning to real-world experiences, and (c) justify their mathematical models. Implications on the promotion of modelling tasks at primary schools as well as teacher education are drawn.

Introduction

The Singapore Mathematics Curriculum Framework (MOE, 2007, p.12) identifies problem solving as central in mathematics teaching and learning. One of the goals of mathematics education is the "acquisition and application of mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems" (p.12). Traditionally, problem solving skills in primary schools are taught through teacher demonstration and student practice of various word problems which are mainly prescriptive in nature. Open-ended tasks such as investigations and modelling activities are encouraged in recent years to provide students with opportunities for exploration, experimentation, and discussion of mathematical ideas.

Mathematical modelling refers to a process of representing real world problems mathematically so as to understand and find solutions to the problems (Ang, 2010). Modelling activities are espoused to foster mathematical reasoning processes (English & Watters, 2005) and are hence perceived to be strong platforms for problem solving in the Singapore curriculum. Nonetheless, limited local research indicated a lack of exposure of modelling tasks in Singapore primary mathematics classrooms. Several reasons account for this. Firstly, anecdotal evidence suggests that many teachers have taken mathematical modelling to be synonymous with "the model method" (i.e., the use of bar models to represent the information given in a word problem), a common heuristics applied in problem solving at primary levels in Singapore schools (Chan, 2011). Secondly, Singapore teachers are familiar with application tasks and saw little difference between these and modelling tasks as both types of tasks involve the use of real-world contexts (Ng, 2011). Lastly, mathematics teaching in Singapore classrooms is predominantly prescriptive and teacher-centred. Teachers often hold fixed views of mathematics application as the use of formulae, standard algorithms, and predictable working. Modelling tasks which encourages

multiple mathematical representations of real-world problems pose some challenges to teachers in their comfort zones (Ng, 2011).

This paper is an outcome from a larger study on building teachers' capacity in primary mathematics school-based assessment using *model-eliciting tasks* (see English, 2010) among others. It reports findings on an initial exploration of implementing modelling tasks in a Singapore primary school. The findings are drawn from the teacher's reflections on the modelling task and its potential to promote in-depth mathematical reasoning process in an intact class of 33 students (aged 11) in a primary five mathematics classroom in a Singapore school. Classroom discussions during task implementation are centred on the mathematical processes (e.g., interpreting, reasoning, justifying) that students demonstrated as they developed models representing the real-world problem. Findings are presented from the lens of the regular mathematics teacher of the class who typically delivers teacher-centred prescriptive mathematics lessons. Implications from these findings serve to propose areas of focuses in teacher education as well as the possibility of more varied engagement involving the use of mathematical modelling in primary classrooms.

Theoretical Underpinning

Model development is an important part of the Models-and-Modelling Perspective (MMP) (Lesh and Lehrer, 2003) which this study adopted. MMP emphasizes 'thinking mathematically' and using mathematical interpretations on real-world situations. MMP suggests that beyond content mastery and abilities in solving traditional textbook word problems, it is important that students acquire the ability to represent real-world problems in mathematical terms and to construct models as a solution to the problems. When students are engaged in modelling, their ways of thinking do not manifest as a single, one-dimensional sequence but instead as a series of cycles, in which their mental models representing the given situation are expressed, tested, and revised. In other words, knowledge develops along multiple dimensions from comparing to contrasting, from concretizing to abstracting, from specific to general, from simple to complex or from a collection of un-coordinated, immature ideas to more co-ordinated, mature, and effective knowledge (Lesh & Zawojewski, 2007). The express-test-revise process results in students making better representations or models of the problem situation through purposeful description, explanation, or conceptualization. In a sense, students become involved in mathematizing (quantifying, dimensioning, coordinating, categorizing, algebratizing, and systematizing relevant objects, relationships, actions, patterns, and regularities) (Mousoulides, Sriraman, & Christou, 2007), which is a key aspect of the modelling process.

Research Methodology

This study adopted design research methodology. The researchers and teachers worked closely together in two iterative cycles comprising three phases over a four-month period: design, teaching experiment, and retrospective analysis (Research Advisory Committee, 1996). Each implementation cycle became an opportunity to collect data to inform subsequent design. Design researchers, upon a parallel and retrospective process of reflection of the design and its outcomes, refined their initial hypothesis and principles towards a coherent theory concerning their understanding of their design experience (Edelson, 2002).

Specifically, the first cycle of this study familiarized the teacher with features of mathematical modelling tasks and some facilitation skills to promote children's mathematisation process. In the design phase of this cycle, the teacher completed a

researcher-designed model-eliciting “Bus Route Task” (see Appendix) while working with her colleagues in small group discussions facilitated by the researchers. The task encouraged a diversity of mathematical approaches. It provided a platform for students to abstract mathematical relationships among key variables (i.e., distance, time, and costs) interpreted within the real-world experiences of the children. The teacher subsequently implemented the same task with her primary 5 students during three consecutive one-hour sessions over three days in the teaching experiment phase whilst the researchers took the role of non-participant observers in the class (Cohen, Manion, & Morrison, 2011). Following each implementation session, the teacher engaged in a 20-minute interview to reflect on her facilitation strategies, outcomes, and students’ engagement with the task. In the retrospective analysis phase of the first cycle, the teacher engaged in an overall post-task reflection with the researchers. Here, the teacher’s reflection is a critical instrument to inform subsequent design of the teaching experiment during the second cycle (Dolk, Widjaja, Zonneveld, & Fauzan, 2010). The intent is to build the teacher’s capacity to design, facilitate, and evaluate learning from her own mathematical modelling task.

The following section presents findings from teacher reflection in the first cycle during the teaching experiment and retrospective analysis phases involving the Bus Route Task. Transcripts of the video-recording of the teacher’s discourse with students working in groups of four served as supplementary data to complement the teacher’s reflection accounts. The focal point of analysis is the teacher’s views on the potentials of the Bus Route Task on children’s mathematisation processes.

Findings and Discussion

The potentials of the Bus Route Task on children’s mathematisation process as identified by the teacher in her reflection centred around three predominant areas.

(a) Identification of Variables and Formulation of Relationships between them

The excerpt below extracted from teacher interviews during the teaching experiment phase revealed the teacher having detected that all eight groups of students had successfully identified the mathematical topics from their prior learning which were relevant for mathematising the given problem situation (line 5).

Researcher	1	...So in terms of mathematical thinking and presentation, outcomes and all that,
	2	are there things that you’re particularly happy about?
Teacher	3	...generally, I was quite happy with the progress they made because initially
	4	I thought they would not be able to bring the maths out, but they were
	5	actually able to do that. They were able to identify the topics that were
	6	relevant for this task.
Researcher	7	are there particular questions that you’re very happy that you’ve asked in
	8	order to steer them towards to what you were intending to do?
Teacher	9	...they were able to actually see that these (the variables) are not just isolated
	10	data that they just going to present, that they could actually find the
	11	relationship (between the variables)...So they could tell me things like you
	12	know when it’s (the bus route is) longer, it’s more expensive and (a) shorter
	13	(bus journey) is cheaper.

In addition, the students were also able to articulate crucial relationships between key variables (i.e., distance, cost) in the problem during her facilitation (line 9 in the excerpt below). Some of the groups even went on to identify other variables affecting the choice of the most efficient bus route such as the time taken for the entire journey, the number of bus

stops, and speed of the buses (a topic they have yet to learn at the point of implementation of the task).

Teacher	1	So what is your, what is your recommendation?
Student	2	Our recommendation route is the pink route.
Teacher	3	So what are you basing this on?
Student	4	From the bus fare and the, the...the length of the...
Teacher	5	The distance?
Student	6	Ya.
Teacher	7	Okay, you are going to base on bus fare and distance. Is there any relationship
	8	you found? Between the two.
Student	9	The shorter the distance the cheaper the bus fare.
Teacher	10	The cheaper the bus fare.

The students were observed to have demonstrated the ability to draw upon their school-based mathematical knowledge in their model development. They made sense of the data in the map, selected the relevant quantities for comparisons, and defined the operations to create a meaningful representation of the real-world problem situation. Such findings were contrary to the teacher's initial assumptions about her students' ability in mathematical modelling (line 4 in the first excerpt). To this end, it appeared that her students have demonstrated that they were able to represent real-world problems mathematically in their first attempt at a modelling task.

(b) Relating School-Based Mathematics to Real-World Experiences

In mathematising the Bus Route Task, the students drew on their real-life experiences to make sense of the situation to create a model. In particular, speed was identified to be an important variable determining the efficiency of travelling at the various given bus routes. However, the students also realised that there was insufficient information about the speeds of the buses in the task. Hence they decided to make assumptions about the speeds of the buses in their mathematical approach. This was noticed by the teacher. During the retrospective phase of the second implementation session, the teacher articulated that '*there was even one group that talked about speed. They suddenly brought in speed. Then they realised that the speed actually is also relative. I mean you know it's ... they were going to assume that all the buses were going to move at the same speed.*' The following excerpt records the conversation between the teacher and a student group illustrating how a group member reasoned that the speed of the bus would have an impact on whether the bus route was efficient (lines 18-19).

John	1	We used a string and a ruler to measure how long the route is. First we used
	2	the string and place it on the map according to the highlighted route then after
	3	that we make a marking on the string. We used the ruler to measure how, how
	4	long the distance is.
Teacher	5	Okay. That is for the...shortest (route).
John	6	Shortest (route).
Teacher	7	How about fastest? You said fastest also. (Being the most) Efficient (route)
	8	means fastest (journey). How do you...how do you show that? Uh you are
	9	saying that the pink route is fastest? Okay. How do you show that?
John	10	Because uh...it has the shortest distance and you can reach uh...the destination
	11	the fastest.
Teacher	12	Okay. Fastest and shortest. Do you agree that it's actually linked?
John	13	No.
Teacher	14	Shortest means fastest. How many of you uh agree with them that shortest is
	15	the fastest? [Nods from some students in the group.] So you are in agreement
	16	with them. How about those...is there anyone who disagrees? Shortest may

	17	not be fastest. Yes Tom, why do you feel that way?
Tom	18	Because the bus may go very slow. Do not mean it's (the bus journey of a
	19	specific route will be) the fastest (most efficient).
Teacher	20	Okay, so how did you cover (investigate) that? He says that...you are talking
	21	about the speed of the bus right? Did you cover that part of it? What did you
	22	say about that?
Tom	23	Uh...it has...(we assume that) all the buses are (travel) at the same speed.

Besides speed, the students also went beyond their textbook knowledge to articulate other variables that are based on their real-world experiences. For instance, some students tried factoring in population density as part of the decision making process on route efficiency. They believed population density has an effect on the popularity of the bus stop which in turn could have affected the boarding time of each bus service (lines 12 to 15 in the excerpt below from the same student group).

Teacher	1	So if they (the other students) assumed that the buses are travelling at the same
	2	speed, then will you accept the argument that the shortest (route) is the fastest
	3	(most efficient bus route)? You do? Is there anyone has any other comments
	4	on this? John?
John	5	What if there's....one of the, one of the shortest route has a traffic jam?
Teacher	6	[Directing at Tom] Okay so did you make any assumptions then? [Repeating
	7	John's comments to Tom] He says shortest route may have more traffic jam.
	8	What do you have to say to justify what you have said against what he said?
Tom	9	(Maybe we can assume that) there are no traffic lights...
Teacher	10	Any other, any other factors you think that need to be taken into account here
	11	for fastest (the most efficient bus route)? Yes?
Harry	12	You know sometimes you go stop at bus stop, there are a lot of people queuing
	13	up. You know like...just like morning one we go to school you know.
Tom	14	Maybe and probably that the bus stop got a lot of people queuing up to go up
	15	the bus then will delay uh.
Teacher	16	Okay do you agree with Harry?
Harry	17	Ya.

A large part of the ability of the students in the sample in making connections between their real-world experiences and the demands of the task could have resulted in the teacher's focused questioning during facilitation. For example, the teacher prompted her students to look more closely at the information given in the map so as to select other useful information or '*more realistic factors*' affecting bus route efficiency. She wanted them to '*see from the real life point of view.*' As a result, some groups were able to use variables such as the number of bus-stops and traffic conditions as part of their arguments for or against bus route efficiency. In developing their model, students' knowledge of pertinent mathematics (e.g., length, time, distance, average) and their flexibility in drawing upon their real-world experiences on identifying these variables were visible in the mathematising process. They were observed to link the mathematics they learnt in the classroom to what they understand in the real world.

(c) Mathematical Justification of Models

In addition, the students also attempted to justify their models. For instance, in the excerpt below, the group tried to explain their model comprising of sets of distance measurements of each bus route provided in the problem. They not only measured the routes with a string and ruler but also took an average value of the multiple measurements of each route for accuracy. Their model led to a conclusion that the most efficient bus route would be the route with the shortest distance measured. They later proposed verifying this

conclusion by calculating the bus fare for the journey. This is interesting because bus fares in the Singapore transport system are already pro-rated according to the distance travelled.

Teacher	1	This (the distance recorded here) is based on?
Student	2	Ruler length (we used the ruler to measure the length of the string used to trace the bus route on the map)
Teacher	3	The length.
Student	4	The average also.
Teacher	5	Okay, the distance. What else are you going to do? Is that enough you think? No? How else would you want to get evidence (to determine the most efficient bus route)?
Student	6	The bus fare. Find the bus fare.

Moreover, the teacher detected that the justification process of model development task allows for in-depth communication of mathematical reasoning during group discussions as compared to her regular mathematics lessons. Indeed, the teacher reflected on how some group members "argued" for their choice of the most efficient route based on their mathematical reasoning and subsequently reached a conclusion within the group:

"When they were talking about (which was) the shortest route, then the other one (student) said no. Then they will argue away. Of course they have to prove why is the shortest (the most efficient route) and things like that...But they have reached a conclusion (later). They are quite sure that they can justify. It should be okay. They have their arguments."

It is evident that from the teacher's interview, she could see that the students refined their models that suggested their growth in refining their mathematical ideas by way of their arguments and their reasoning.

Implications and Conclusion

This paper reports findings from teacher reflection on the potentials of mathematical modelling tasks at primary five. The opportunities afforded by the task were appreciated by the teacher who was surprised at the quality and sophistication of the mathematisation processes during the task. The teacher detected three areas that the Bus Route Task provided a platform for: (a) the identification of key variables and their relationships, (b) relating school-based mathematical knowledge and skills to the real-world experience, and (c) justification of mathematical models developed. These findings echo the arguments made by the proponents of model-eliciting tasks in mathematics classrooms (e.g., English, 2010, Lesh and Lehrer, 2003). In this regard, mathematical modelling has not only opened the eyes of the teacher towards a more student-centred approach mathematics lesson but also allow for richer communication of ideas by the students. In justifying their models, students communicate their thinking more explicitly than they would in their regular mathematics class.

Although the possibility of including mathematical modelling tasks in Singapore primary mathematics classrooms may not be as remote as one may think, there has to be concerted efforts at guiding teachers in facilitating and evaluation of learning from such tasks in order for the children to reap benefits from these tasks. In addition, as Stillman (2010) emphasised, a positive climate for active, purposeful mathematical discussions would be necessary to set the stage for learning during modelling tasks.

References

- Ang, K. C. (2010). Mathematical modelling in the Singapore curriculum: Opportunities and challenges. In A. Araújo, A. Fernandes, A. Azevedo & J. F. Rodrigues (Eds.), *Educational interfaces between mathematics and industry: Proceedings of the EIMI 2010 Conference* (pp. 53-62). New York: Springer.
- Chan, C. M. E. (2011). *Conceptions of mathematical modelling*. Paper presented at the Connecting to practice: 15th International Conference on the Teaching of Mathematical Modelling and Applications.
- Cohen, L., Manion, L., & Morrison, K. (2011). *Research methods in education* (7th ed.). London: Routledge
- Dolk, M., Widjaja, W., Zonneveld, E., & Fauzan, A. (2010). Examining teacher's role in relation to their beliefs and expectations about students' thinking in design research In R. K. Sembiring, K. Hoogland & M. Dolk (Eds.), *A decade of PMRI in Indonesia* (pp. 175-187). Bandung, Utrecht: APS International.
- Edelson, D. C. (2002). Design research: what we learn when we engage in design. *The Journal of the Learning Sciences*, 11(1), 105-121.
- English, L. D. (2010). Young children's early modelling with data. *Mathematics Education Research Journal*, 22(2), 24-47.
- English, L. D., & Watters, J. J. (2005). Mathematical modelling in third-grade classrooms. *Mathematics Education Research Journal*, 16(3), 59-80.
- Lesh, R., & Lehrer, R. (2003). *Models and Modeling Perspectives on the Development of Students and Teachers*. In *Mathematical Thinking and Learning*, 5(2&3), (109-129). Mahwah, NJ: Lawrence Erlbaum Associates.
- Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 763-803). Charlotte, NC: Image Age Publishing.
- Ministry of Education [MOE]. (2007). *Ministry of Education Mathematics Syllabus – Primary*. Singapore: Curriculum Planning and Developmental Division.
- Mousoulides, N., Sriraman, B., & Christou, C. (2007). From problem solving to modelling: The emergence of models and modelling perspectives. *Nordic Studies in Mathematics Education*, 12(1), 23-47.
- Ng, K. E. D. (2011). *Facilitation and scaffolding: Symposium on Teacher Professional Development on Mathematical Modelling - Initial perspectives from Singapore*. Paper presented at the Connecting to practice - Teaching practice and the practice of applied mathematicians: The 15th International Conference on the Teaching of Mathematical Modelling and Applications, Australian Catholic University, Melbourne, Australia.
- Stillman, G. A. (2010). Implementing applications and modelling in secondary school: Issues for teaching and learning. In B. Kaur & J. Dindyal (Eds.), *Mathematical applications and modelling: Yearbook 2010 of the Association of Mathematics Educators* (pp. 300-322). Singapore: World Scientific.
- Research Advisory Committee. (1996). Justification and reform. *Journal for Research in Mathematics Education*, 27(5), 516-520.

Appendix 1 – The Bus Route Task

Determining the Most Efficient Bus Route

Ms Chang has recently moved to Block 297C Punggol Road. She is going to start teaching at Punggol Primary school next week and needs to know how to travel to the school. However, the MRT is always too crowded for her to take and it also requires her to take a feeder bus which results in inconvenience. Ms Chang realizes that there are three bus services that ply different routes to her school. Help her to find the most efficient route to travel by bus from her home to the school. The location of her home is marked in the map. Currently the three bus services that are available for Ms Chang to choose are Service 124, Service 62 and Service 89. The routes for Service 124, Service 62 and Service 89 are marked as blue, yellow, and pink lines respectively on the map. The bus stops along each bus route are marked with stickers with corresponding colours.

Your task is to give Ms Chang a proposal consisting of the following:

1. How your group determines what is meant by the “*most efficient*” bus route
2. *Assumptions* about the problem your group made in order to help Ms Chang
3. The *mathematics* used to decide which route is the most efficient
4. How your group *justifies* that the selected route is the most efficient
5. The final *recommended* route for Ms Chang

For us to better understand your work, you can attach the following to your proposal:

- (a) A map containing the chosen bus route.
- (b) The information you found useful for this task

Bus Fares

<i>Distance Range</i>	<i>Bus Fare (Cash)</i>
First 1 km	\$1.10
Up to every 0.7 km increase	30 cents increase

Note: The student-groups were given a map in which the three different routes for selection were marked using pink, blue, and yellow colours. Bus stops were indicated along the routes using coloured stickers. In Singapore, bus fares are calculated according to the distance travelled.

Empirical Evidence for Niss' *Implemented Anticipation* in Mathematising Realistic Situations

Gloria Stillman

Australian Catholic University (Ballarat)

<gloria.stillman@acu.edu.au >

Jill P. Brown

Australian Catholic University (Melbourne)

<jill.brown@acu.edu.au>

Mathematisation of realistic situations is an on-going focus of research. Classroom data from a Year 9 class participating in a program of structured modelling of real situations was analysed for evidence of Niss's theoretical construct, implemented anticipation, during mathematisation. Evidence was found for two of three proposed aspects. In addition, unsuccessful attempts at mathematisations were related in this study to inability to use relevant mathematical knowledge in the modelling context rather than lack of mathematical knowledge, an application oriented view of mathematics or persistence.

Mathematical applications and modelling of real world situations are receiving increased emphasis in several curricula internationally at the moment (e.g., Ministry of Education, 2006). The teaching and learning of applications and modelling has been the subject of on-going research for many years (Blum et al., 2007; Kaiser et al., 2011; Niss, 2001). Two of the areas receiving on-going attention have been the mathematisation (i.e., translation into mathematics) of the idealised problem formulated from the real situation and the reverse process, de-mathematisation (i.e., interpretation of mathematical outputs of modelling in terms of the real situation). Recently, Niss (2010) has added to the theoretical models informing research in these areas.

The notion of mathematising has been promoted by the mathematics frameworks for PISA 2003, 2006, 2009 and 2012 assessing the mathematical literacy of 15 year olds roughly at the end of compulsory schooling. In the earlier frameworks what seemed to be presented as mathematisation was the entire mathematical modelling cycle (see OECD, 2009, p. 105). The mathematical modelling cycle is described as a key feature of the draft PISA 2012 mathematics framework (OECD, 2010) but mathematising is less prominent. It is one of seven fundamental mathematical capabilities that underpin the framework. Mathematising is taken to mean the fundamental mathematical activities that are involved in "transforming a problem defined in the real world to a strictly mathematical form...or interpreting or evaluating a mathematical outcome or a mathematical model in relation to the original problem" (OECD, 2010, p. 18). The latter would be termed "de-mathematisation" by Niss (2010). Mathematising is thus seen in PISA as one of the cognitive capabilities that can be learnt through schooling so as to enable students to understand and engage with the world in a mathematical manner. The purpose of this paper is to demonstrate whether or not there is empirical evidence for one of the main explanatory constructs of Niss's model of mathematisation processes (Niss, 2010, p. 57).

Theoretical Framework

Researchers in the area of applications and modelling often use diagrams of the "so-called" modelling cycle to discuss what appears to be happening at a task and mental level during modelling. These are mere simplifications but they are a useful means of communicating amongst international researchers. The diagram used by Niss (2010) is reproduced in Figure 1. This shows two quite disparate domains in this particular representation of modelling: the extra-mathematical domain (i.e., the real world situation

and the modeller's idealisation of this) and the mathematical domain. Others such as Stillman (1998) argue for the importance, particularly in the context of schooling, of also including in these diagrams a representation of the blending of the real world and mathematical world. Idealisation occurs through making assumptions and identifying elements in the situation which are of interest which are then formulated, that is, specified into a problem statement which may take the form of a question. The mathematical domain includes the mathematical model that has been made of the situation, and mathematical artefacts (such as graphs and tables) that might be used in solving the mathematical model. The idealised situation is mathematised through a process of translation into mathematics and similarly mathematical outputs need to be de-mathematised, that is, interpreted in terms of the idealised situation and the real situation which was the launching point for the modelling in the first place.

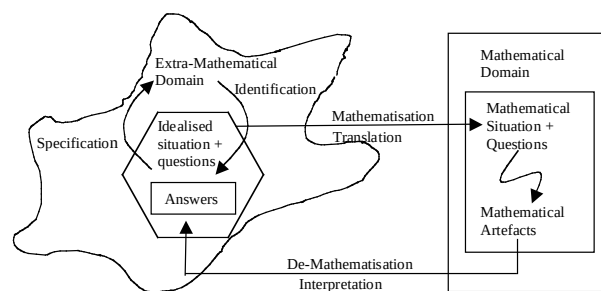


Figure 1. Modelling processes (after Niss, 2010, p. 44)

In order to produce a theoretical model of the mathematisation process, Niss (2010) introduces the construct, “implemented anticipation” (p. 54). Successful mathematisation, according to Niss, involves anticipating what will be useful mathematically in subsequent steps of the cycle and implementing that anticipation in decision making and carrying out actions. Firstly, the idealisation of the real situation from the extra-mathematical domain involves implementing decisions about what elements or features are essential as well as posing any related question or statement of the problem in light of their anticipated usefulness in mathematising. Secondly, when mathematising this formulation of the problem situation the modeller needs to do this by anticipating mathematical representations and questions that, from previous experience, have been successful when put to similar use. Thirdly, when anticipating these mathematical representations, the modeller has to be cognisant of the utility of the selected mathematisation and the resulting model in future solution processes to provide mathematical answers to the mathematical questions posed by the mathematisation. This involves anticipating mathematical procedures and strategies to be used in problem solving after mathematisation is complete. Thus successful implemented anticipation involves a three step foreshadowing process.

To be able to successfully use implemented anticipation in mathematising a real or realistic situation modellers need to: (1) possess relevant mathematical knowledge, (2) be capable of using this when modelling, (3) believe a valid use of mathematics is modelling real phenomena, and (4) have persistence and confidence in their mathematical capabilities (Niss, 2010, p. 57). Clearly, this is a challenging process. It is reasonable to expect that modellers, especially new modellers, would experience this challenge and have difficulties related to the three foreshadowing aspects of implemented anticipation. These difficulties

might be explained by these four requisites. Both successful and unsuccessful attempts at modelling or applying mathematical knowledge to real situations are opportunities for developing deeper “metaknowledge about modelling and mathematisation, in particular” (Schaap, Vos, & Goedhart, 2011, p. 145) and thus should be the foci of any study of mathematisation.

Researchers (e.g., Galbraith & Stillman, 2006; Schaap et al., 2011; Stillman, Brown, & Galbraith, 2010) have found evidence of beginning modellers in secondary schools having difficulties with mathematising because of impeding formulations of the problem statement. However, no one to date has attempted to use Niss’s model in analysing classroom data. In this paper we will attempt to use the model as the basis for our analysis of data from a year 9 class of beginning modellers who had participated in a program of quite structured modelling over one year. To operationalise the mathematising construct for research purposes we take as its starting point the formulated statement of the problem situation. This may or may not be formulated as a question. The end point of mathematising will be the mathematical model.

The Study

As part of the RITEMATHS project, a series of three modelling tasks were used in a class of 21 Year 9 students. The data used in this paper relate to the implementation of the last of these tasks, *Shot On Goal*. Names used are pseudonyms. In brief, the task uses a soccer context (see Figure 2) where the modelling problem involves optimising a position for an attacking player to attempt a shot on goal whilst running parallel to the sideline (for details, see Stillman et al., 2010). The teacher anticipated that students might have difficulty with task formulation so he used a nearby soccer field to provide an outdoor demonstration at the beginning of the lesson sequence. The teacher used a rope parallel to the sideline (i.e., perpendicular to the goal line) as a run line for several students to run down and stop when they had the best shot on goal. One student then stayed on this run line, marking the average of their estimates. The process was repeated for a run line closer to the goal. Students then discussed the effect on the average position for the best shot before returning to the classroom to begin the task. They worked in 7 groups of 2 to 4 students. The teacher allocated each group a particular distance for their run line from the near goal post.

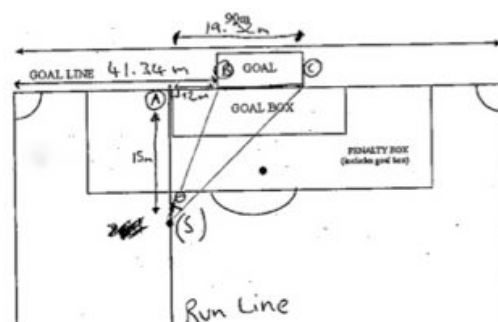


Figure 2. Student diagram of *Shot on Goal*

The main part of the task (Tasks 1 to 10) offered structured scaffolding. Two questions at the end included no details of how to mathematise or approach them mathematically (see Figure 3). Fifteen students attempted part or all of these. Tasks 11 and 12 are the focus here. The task was implemented over three lessons on consecutive days, and these final tasks were

attempted during the third lesson. Data collected which is relevant to the focus of this paper consisted of transcriptions of 3 video-recordings and a further audio-recording of groups working on the tasks, individual task scripts from 15 students and interview responses of nine of these students (see Appendix for example questions).

TASK 11 – CHANGING THE RUN LINE

Investigate whether the position of the spot for the maximum shot on goal changes as you move closer or further away from the near post. [Collect data from other students results to help you see if there are any patterns in the position of spots for the maximum angle.] What does the relationship between position of the spot for the maximum shot on goal and the distance of the run line from the near post reveal?

TASK 12 – CHANGING THE RULES

Soccer is often a low scoring game. Some have suggested that it would be a better game if the attackers had more chance of scoring, so the width of the goal mouth should be increased. Others claim it would be a more skilful game if the goal keeper was given more of a chance to stop goals by reducing the width of the goal mouth. Investigate what effect changing the width of the goal mouth would have on the position of the maximum shot on goal for the run lines and give your recommendation.

Figure 3. Tasks for mathematising at end of *Shot on Goal*

The research questions addressed in this paper are:

1. Is there evidence for the existence of Niss's *implemented anticipation* in mathematisations occurring in the classroom?
2. Do Niss's four requisites explain unsuccessful mathematisations?

To analyse the data student responses to Tasks 11 and 12 were classified as (a) mathematisations showing (i) successful or (ii) unsuccessful implemented anticipation (b) qualitative statements (i) identifying a relationship between relevant variables or (ii) identifying variables but not supported, in either case, by any mathematical objects or representations and (c) incomplete as only raw data with no translation into mathematics or interpretations recorded. When a(ii), (b) or (c) classifications were given, interview data and video and audiotape data were scrutinised in detail for explanations. These were then compared to Niss's four requisites for successful implemented anticipation.

Results and Analysis

Fifteen students from 6 different groups attempted Task 11. Group 5 was a long way behind other groups because of difficulties with formulation of the task (see Stillman, et al., 2010). Two students from Group 7 were also behind because of difficulties they were experiencing using their graphing calculator. These students did not attempt tasks 11 or 12.

Ned from Group 6 recorded his results of collecting data for Task 11 in a partially ordered table which he called a "commentary table". He correctly identified the three relevant variables "run line to goal post", "dist along RL", and "Angle" ("Max angle" used in recording the raw data). There was an error in the data which Ned and Len created themselves and recognised at the time of data collection as they decided Group 4's angle for a 14 m run line distance maximised at 18 m as the data were recorded to only one decimal place and thus the angle was the same for 16 to 19 m. They did not correct this datum as they felt the rest of the data supported their conclusion.

After tabulating the data in this fashion from Len's systematically organised two column recording of the data and scrutinising it for less than 15 seconds, Ned exclaimed:

Ned: Check it out. The distance along the run line you have to be is, with the exception of-f-f 14, ah, is 3 more than the distance, what the distance is from the run line to the goal post. [Video, Group 6]

This was correct. In recording his interpretation of the data Len added: “It also proves our theory that the closer you come to goal, the closer you have to be on your run line to achieve maximum angle.” Neither student recorded this symbolically. However, when interviewed Len, wrote “ $y = x + 3$ where maximum angle is $= y$ and distance from near post $= x$ ” in response to being asked to write his answer algebraically. The group’s work was classified as successful implemented anticipation (ai).

Ozzie and Jaz of Group 2 used correctly ordered tables with Jaz also identifying and labelling the variables in the columns correctly on his script. Ozzie concluded:

The closer to the post, the higher the maximum angle you can achieve, but you have to run in [along the run line] closer to an extent. If [the run line is] further away from post, the goal is largest from far [meaning the maximum angle occurs further out]. [Script, Ozzie, Group 2]

This was typical of the conclusions of all three group members. When asked in interview if he was able to write his answer algebraically, Ozzie responded: “I think you could but I would have to think about it.” However, he recognised the response from Group 6 as being a linear function that produced a straight line graph. In interview, Jaz claimed to have seen “there was a relationship, three metres” from the start of the task. When asked if he could have expressed this algebraically he said he “could have” but his written conclusion (as above) was trying to capture this. He elaborated:

Jaz: Yeah, yeah. Like the closer you get, the closer your run line is to your goal post the bigger angle, you get a much bigger maximum angle but you have to run towards the goal further but I think it is every 3 metres you have to run in by figuring out all of this [seems to be describing: position of maximum angle on run line $=$ run line distance from post $- 3$]. [Interview, Jaz, Group 2]

When asked to show Group 6’s model algebraically, he used a diagram and d , the distance from the near post to the run line, to show the maximum shot angle occurring on a run line at $d + 3$. He wrote $y = 2d + 3$ and labelled $d + 3$ on his diagram also as y . When asked what y equalled, he replied, “ $y = 2d + 3$ ” but his explanation described $y = d + 3$. The work from these two Group 2 members was classified as unsuccessful implemented anticipation (aii) as their ordered table did not help produce a mathematical answer although the representation was correctly anticipated as potentially allowing this. The response of Sven from Group 7 also used an ordered table but no conclusion was drawn. It was classified similarly as (aii). This student appeared to run out of time.

Molly and Christine (Group 4) collected data from five groups and systematically recorded these in two-celled rows across the page. When they attempted to translate these, together with their own results for a run line of 14, into an ordered mapping of *run line distance from near post* $\rightarrow$ *shot distance down run line* $\rightarrow$ *maximum angle of shot*, three mappings had *run line distance* $\rightarrow$ *shot distance down run line* reversed. The results in this format were interpreted as showing: “In most cases the further away the run line from the goal post the smaller the angle [meaning maximum angle] and the larger the distance down the run line.” This work was classified as unsuccessful implemented anticipation (aii) as the mapping does not support this conclusion even though the representation is correctly anticipated as a useful tool to do so.

The four members of Group 4 all produced qualitative descriptions as did both members of Group 1 and the third member of Group 2. Simon, Max and Lori (Group 4) showed only a relationship between the run line distance from the goal post and the maximum angle (bi). Rose (Group 4), Raza (Group 2), and Jim (Group 1) described two relationships between relevant variables (bi), for example:

The relationship reveals the angle of “shot on goal” gets bigger as you get closer to the goal. This is effective for both “position of the spot for the maximum shot on goal” and “distance of the run line from the near post”. [Script, Rose, Group 4]

Ahmed (Group 1) merely identified the variables, “the further from post” and “distance on the run line to a certain point”, as playing “a crucial factor in getting the best angle” (bii). The work of Rod (Group 7) was classified as incomplete (c) as it merely listed data. The video record showed he ran out of time as he was recording data at the end of the lesson.

Eleven students from 5 different groups also attempted Task 12. The others ran out of time. The mathematisation required that the students realise that the representations and models they had used earlier for finding the angle of the shot using their particular run line distance from the near post and a standard soccer goal width of 7.32 m could be used to capture the changed conditions created by varying the width of the goal. Work from four groups (1, 2, 6 and 3) was classified as showing successful implemented anticipation (ai).

Groups 1 and 2 used their original equations and then changed the widths in the formula so they could graph the three functions showing size of shot angle versus distance along run line on their graphing calculator screen (Figure 4) to find where maximums were occurring and make comparisons. They recorded examples of the functions they had used with the exception of Ahmed who merely recorded goal mouth width, run line distance for maximum shot angle and size of the maximum angle for three specific cases. In addition, Ozzie showed where the changes in the algebraic model came from in the real context.

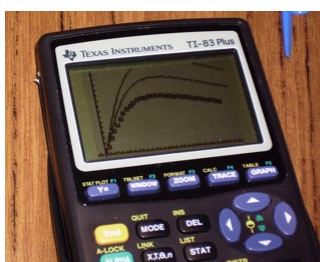


Figure 4. Group 1’s graphs for goal widths of 5, 7.32 and 10m for a run line from near goal post of 10 m.

For example, as Group 2’s distance of their run line from the near post was 12 m and they narrowed the goalmouth to 5 m, their function became:

$$y = \tan^{-1}(17/x) - \tan^{-1}(12/x); 17 \rightarrow 12 + 5 \text{ [Script, Ozzie, Group 2].}$$

Ozzie then observed that: “if the goal’s width is smaller, the total maximum of the ‘angle on goal’ is smaller. You must also go in closer. The max angle was 9.93° at 14 metres.” They then looked at the case of the goal being 10 m in width. Neither group made a definite recommendation.

Group 6 also used their previous formula varying it to generate numerical data for two cases in their calculator LISTs. They then compared with their original data set for their run line distance. Finally, they produced two tables of ordered data for goal widths of 8 m and 6 m. These showed distances of 1 to 5 metres from the goal line along the run line and corresponding angles. These were labelled “examples” but it was clear from the video that distances much further down the run line were examined which allowed them to see where the angle was becoming a maximum. Their observations and recommendation were:

Based on calculations, widening the goal increases the distance away from the goal required to achieve the maximum angle. However, the overall shot angle is increased. The reverse is also true,

Narrowing the goal means the distance from the goal is less, but the shot angle is decreased. I would recommend shortening the goal width, so players can get closer to the goal and still have a large chance of scoring. [Script, Ned, Group 6].

Group 3 again used an ordered mapping of *shot distance down run line* $\rightarrow$ *angle of shot* for several shot distances along their particular run line for a goal width of 9 m and then 6 m (although the latter was not stated by either student). The mapping was correctly recorded on both scripts. Angle data were generated by changing formulae in the calculator for a larger then smaller goalmouth. Only Christine noticed that the angle was reaching a maximum then decreasing but she mentioned this only for the smaller goalmouth case: “if you decrease the goalmouth to make it smaller, the angles grow but become smaller”. The group recommendation, recorded on Christine’s script as Molly had already filled her page, was: “Make goals larger for more scoring shots”.

Simon and Max (Group 4) made only a qualitative statement indicating “changing the width of the goal would change the angle”. Max added that “the bigger the width of the goal the bigger the angle” and the reverse of this, indicating it was a direct relationship. The former was classified as identifying variables (bii) and the latter as identifying a relationship (bi) but both were without mathematical support other than collected data.

With respect to explanations of students’ unsuccessful mathematisations, confidence and persistence were not considered to be playing a part for the 15 students who attempted these tasks. All students were persisting on the task as a whole at the end of the three lessons (i.e., 165 minutes). Students who were unable to complete because of time pressure or for whom there was no interview data of relevance were eliminated from the analysis leaving the work of nine. Evidence of students’ appreciation of modelling real world phenomena and references to reality in mathematics was gleaned from the video and audio interactions in their group and through responses to interview questions. Only Raza was negative towards the utility in the real world of the mathematics they were using saying his father was 40 years old and used algebra only once since school. Rose and Lori were more interested in mathematical exercises but conceded others such as sports people could use mathematics to inform decisions. Ned remained ambivalent towards the need for a task context to be real or not although he appreciated mathematics being in some sort of context. Thus, appreciating the modelling of real phenomena was not seen as discriminating between successful or unsuccessful mathematisation. As Task 12 did not prove to be discriminating either, it was also eliminated from the analysis. To see if students whose responses to Task 11 received classifications other than (ai), possessed the relevant mathematical knowledge of linear relationships, their responses in interview to being asked to describe and graph such a relationship revealed that all but Jaz did this correctly. Jaz’s response was, however, a linear function. Rose used the relation correctly but only in terms of mappings of coordinate points. Clearly, being able to use this mathematical knowledge in a modelling context, not possessing the knowledge itself, was the difficulty for those students who were unsuccessful in mathematising Task 11.

Discussion and Conclusions

As the tasks were already formulated as mathematical statements or questions by the task setter, all the successful mathematisations could be said to have involved students identifying relevant variables, foreshadowing representations that would be useful in identifying relationships between variables and producing mathematical answers, and realising that the representations and models used earlier could be used to capture changed

conditions. This is thus empirical evidence for the last two aspects of Niss's implemented anticipation (Niss, 2010). A more complex task requiring student input to formulation would be required to demonstrate the first aspect.

Where evidence was available, unsuccessful attempts at mathematisations in a classroom context were in this study related to inability to use relevant mathematical knowledge in the modelling context rather than lack of the relevant mathematical knowledge per se, an application oriented view of mathematics or persistence on the task. As this was only the third in a series of modelling tasks the students had attempted as their first experience of modelling, it is not surprising that this was the most discriminating of Niss's four requisites (2010) for successful implemented anticipation in mathematisation.

Acknowledgement

Data used in this paper were collected by the authors as part of the RITEMATHS project (an ARC funded linkage project – LP0453701).

References

- Blum, W., Galbraith, P. L. Henn, H.-W., Niss, M. (Eds.) (2007). *Modelling and applications in mathematics education: The 14th ICMI Study* (New ICMI Study Series, Vol. 10). New York: Springer.
- Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. *ZDM*, 38(2), 143-162.
- Kaiser, G., Blum, W., Borromeo Ferri, R., & Stillman, G. (Eds.). (2011). *Trends in the teaching and learning of mathematical modelling*. New York: Springer.
- Ministry of Education. (2006). *Mathematics syllabus: Secondary*. Singapore: Author.
- Niss, M. (2001). Issues and problems of research on the teaching and learning of applications and modelling. In J. F. Matos, W. Blum, S. K. Houston, & S. P. Carreira (Eds.), *Modelling and mathematics education* (pp. 74-88). Chichester, UK: Horwood.
- Niss, M. (2010). Modeling a crucial aspect of students' mathematical modeling. In R. Lesh, P. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical competencies* (pp. 43-59). New York: Springer.
- OECD. (2009). Mathematics framework. In *OECD PISA 2009 assessment framework: Key competencies in reading, mathematics and science* (pp. 83-123). Paris: OECD Publishing. DOI: 10.1787/19789264062658-4-en
- OECD. (2010). *Draft PISA 2012 mathematics framework*. Retrieved 29 March 2012 from <http://www.oecd.org>
- Schaap, S., Vos, P., & Goedhart, M. (2011). Students overcoming blockages while building a mathematical model: Exploring a framework. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in the teaching and learning of mathematical modelling* (pp. 137-146). New York: Springer.
- Stillman, G. (1998). The emperor's new clothes? Teaching and assessment of mathematical applications at the senior secondary level. In P. Galbraith, W. Blum, G. Booker, & I. D. Huntley (Eds.), *Mathematical modelling* (pp. 243-253). Chichester, UK: Horwood.
- Stillman, G., Galbraith, P., & Brown, J. (2010). Identifying challenges within transition phases of mathematical modelling activities at Year 9. In R. Lesh, P. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical competencies* (pp. 385-398). New York: Springer.

Appendix: Selected Interview Questions

- Q10.1 Could you have written an algebraic model for your answer to Task 11?
- Q10.2 One group [Group 6] said that the position of the spot for the maximum shot on goal was 3 metres more than the distance of the run line from the goal post. What type of mathematical relationship is this?
- Q10.2.2 Draw me a graph to show it.
- Q11.1 Do you like doing challenging tasks like this in maths? Can you elaborate on that.
- Q11.1.1 What types of maths tasks do you prefer?
- Q11.2 What is the purpose of tasks such as Cunning Running and Shot on Goal?
- Q11.3 Do you like the fact these tasks are set in a real world context?

Supporting Teachers in Choosing and Using Challenging Mathematics Tasks

Peter Sullivan

Monash University

<peter.sullivan@monash.edu>

Debbie Michaels

Monash University

<deborah.michels@monash.edu>

Doug Clarke

Australian Catholic University

<doug.clarke@acu.edu.au>

Angela Mornane

Monash University

<angela.mornane@monash.edu>

Anne Roche

Australian Catholic University

<anne.roche@acu.edu.au>

The research reported here was motivated by curiosity about ways of suggesting tasks and activities to teachers that assist them in their planning and which allow them to see the purpose of the suggested tasks. A group of junior secondary teachers involved in a larger project worked through a set of tasks that sought to address the content they were about to teach, and then completed a survey about those tasks. A comparison group also completed the survey. There was consensus across the groups of teachers about the potential of the tasks, they felt they could implement them and that such tasks were preferable to comparable textbook exercises. There are implications for both teacher educators, and for resource developers.

Making Suggestions about Mathematics Tasks to Teachers

It can be assumed that teacher educators and others who seek to support teachers in improving the mathematics learning of their students focus on developing teachers' knowledge. This knowledge includes knowledge about curriculum, about pedagogy, about students and about mathematics. Teacher educators also seek to develop positive teachers' attitudes towards mathematics and its teaching, ways for teachers to elicit positive motivation in students and to foster in them an orientation to overcoming barriers and mastering content. The following is a report of an aspect of a larger project¹ that is focusing on the latter two of these aspects. The project is encouraging teachers to present students with challenging tasks and to describe the actions that teachers might take when students are working on those challenging tasks to maximise student persistence. We use the term *persistence* to describe the category of student actions that include concentrating, applying themselves, believing that they can succeed, and making an effort to learn. We term the tasks that are likely to foster such actions *challenging*, in that they allow the possibility of sustained thinking, decision making and some risk taking by the students.

The rationale for the project is based on the widespread generic advice that teachers should pose challenging tasks as a way of engaging students (see, e.g., City, Elmore, Fiarman, & Teitel, 2009). It is also founded on a belief that learning mathematics is not about students learning a large number of routines and small steps but about them connecting concepts, linking ideas together, planning multiple steps, and thinking about their mathematical

¹ The EPMC project is funded through an ARC Discovery Project (DP110101027) and is a collaboration between the authors and their institutions. The views expressed are those of the authors. We acknowledge the generous participation of the project schools, McKinnon Secondary College and St Peter the Apostle School, Hoppers Crossing.

knowledge in new ways (Hiebert & Carpenter, 1992).

One aspect of the project is focusing on the nature of support that teachers might need for implementing challenging tasks and in particular finding ways to offer teachers ideas about tasks that have the potential to be productively challenging.

A framework that guided our thinking about tasks was proposed by Stein, Grover, and Henningsen (1996) who argued that the consideration of classroom tasks by teachers goes from the ...

- Mathematical task as presented in instructional materials
 - which, influenced by the teacher goals, their subject matter knowledge, and their knowledge of students, informs ...
- ... mathematical task as set up by the teacher in the classroom
 - which, influenced by classroom norms, task conditions, teacher instructional habits and dispositions, and students learning habits and dispositions, influences ...
- ... mathematical task as implemented by students
 - which creates the potential for ...
- ... students' learning.

In this framework, the first step for the teacher is to identify some suitable tasks, whether from instructional materials or from some other source. As Sullivan, Doug Clarke and Barbara Clarke (2009) found, one of the major challenges that teachers experience is finding the time for detailed planning. Teachers have limited time for examining texts and other teacher resources, or trawling the internet for tasks and ideas, or even scouring their own files. It is likely to be helpful for teachers if support is offered in a format that assists them in identifying tasks with potential to foster student engagement and learning efficiently, or in the form of suggestions of tasks for which they see the purpose and potential.

While teachers may need support with expanding their pedagogical repertoire, and even in developing key aspects of their mathematical knowledge for teaching (Hill, Ball, & Schilling, 2008), suggestions of suitable tasks and activities in a form in which the potential of the task is clear would seem helpful. The aspect of our project that is exploring the nature of that support is the focus of this paper.

A Model of Influences on Teacher Actions

The model of influences (Figure 1) that guides the design of the overall project was informed by Clark and Peterson (1986) and used successfully by Sullivan et al. (2009). It includes four sets of background variables: teacher knowledge; teacher attitudes, beliefs, and self-goals; situational and other constraints; and teachers' intentions. The first three of these influence each other and are presumed to inform decisions teachers make when selecting and using tasks.

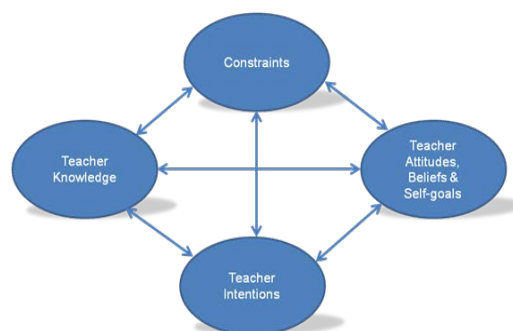


Figure 1. Influences on teacher actions

This model also informed the data collection for this aspect of the project. In the model, it is assumed that the teacher's knowledge influences the choice of task. This knowledge might be about the mathematics that is needed to appreciate the potential of the task (see, e.g., Ball & Bass, 2000; Chick, 2007) or about the pedagogical actions needed to implement the task (see Hill et al., 2008). This knowledge informs the articulation of clear mathematical goals for the students and the selection of tasks at the appropriate level of challenge.

The model further recognizes that a fundamental assumption is that teachers' beliefs, attitudes, and self-goals influence the extent to which they maintain the challenge in tasks that they pose (e.g., Middleton, 1999). Sullivan et al. (2009) reported that their project teachers seemed reluctant to pose challenging tasks for fear of negative student reactions. Stein et al. (1996) and Tzur (2008) also described similar responses from teachers when implementing tasks. Related factors that have been identified as influential are teachers' beliefs about mathematics and learning mathematics (Thompson, 1992) and their attitudes towards mathematics generally (Hannula, 2004). Another relevant factor is self-goals theory (Dweck, 2000) which refers to the extent to which teachers see their own and students' ability as incremental and amenable to improvement through effort, for example. These variables manifest themselves in the tasks posed and the forms of affirmation that teachers use.

The third aspect refers to constraints experienced by teachers when seeking to challenge students. We infer that there are powerful constraints since it appears that, in many Australian classes, the tasks used are often routine and unlikely to be challenging (see Hollingsworth, Locan, & McCrae, 2003). Some of the constraints on teachers include the diversity in students' cultural backgrounds, language fluency and readiness to learn (see Delpit, 1988), the ways in which the structure of classrooms and common responses of students to schooling restrict teacher choices (see Doyle, 1986), and that the skill levels of the students inhibit their participation in challenging tasks (see Stein & Lane, 1996). In addition, we anticipate that access to resources, available planning time and fear of negative student reactions also act as constraints on the teachers.

The above sets of variables inform teachers' intentions, meaning what they plan to do in their lessons. The research questions informing the data collection of this aspect of the project are as follows. With respect to the tasks suggested by us, do teachers:

- i. see the mathematical and pedagogical potential?
- ii. anticipate constraints in using these tasks?
- iii. consider the tasks to have more potential than texts or other resources to which the teachers have access?
- iv. need to be familiar with the implied pedagogies? and
- v. have completed the task itself?

The intention is that the results can inform future support for, and advice to, mathematics teachers.

The Context and Methods of the Data Collection

In the overall project, as well as in this aspect, we are adopting a design research approach which "attempts to support arguments constructed around the results of active innovation and intervention in classrooms" (Kelly, 2003, p. 3). In particular, we are intervening to prompt the use of tasks that are more challenging than those commonly used by many teachers. We ensure that we are suggesting tasks that align with the teachers' current topics to avoid the constraint that teachers often experience due to what they see are

the imperatives for content coverage. We encourage our project teachers to work through tasks prior to using them in their classrooms, and we do this in teacher learning sessions.

We present the tasks to the teachers in sequences, generally of increasing difficulty. This is partly because it is hard to anticipate what tasks will be challenging for a particular class, so the sequences allow teachers to make that decision. It is also partly because, for the tasks to be accessible yet challenging, some prior experiences are often needed. The earlier tasks in the sequences are not necessarily intended to be challenging. While we do make various pedagogical suggestions, the planning of the lessons is left to the teachers.

The teachers at our secondary level project school had suggested that they were about to teach the topic of surface area and volume to their year 8 classes. We proposed that the goals of the tasks that we suggested were to introduce students to: the nature of volume of prisms and cylinders (area of the end $\times$ length); surface area of prisms (as the total area of the faces); efficient methods of calculating volume and surface area of rectangular prisms; and ways in which surface area and volume are different. The particular sequence of tasks that was the basis of the data collection reported below was as follows:

Task 1: This is a rectangular prism made from cubes. What is the volume of this prism? What is the surface area? (a photo of a prism made from cubes was presented with the task)

Task 2: A set of 36 cubes is arranged to form a rectangular prism. What might be the dimensions of such prisms? What is the surface area of each of your prisms?

Task 3: The volume of a rectangular prism is 600 cm^3 . What might be the surface area of the prism?

Task 4: A rectangular prism is made from cubes. It has a surface area of 22 square units. Describe what the rectangular prism might look like?

Task 5: The surface area of a closed rectangular prism is 94 sq cm. What might be the dimensions of the prism?

Task 6: (The students were presented with a photo of a “child” made with cubes) What is the volume and surface area of the child? Imagine an adult made with twice as many cubes. What is the volume and surface area of the adult? Imagine a giant made with three times as many cubes. What is the volume and surface area of the giant? What can you say about the ratio of surface area to volume in each case?

Task 7: Design a 3L milk container that will fit into the refrigerator

Task 8: What is the volume and surface area of a Toblerone box? I want to wrap up some gifts made up of Toblerones. What is the surface area and volume of a gift made out of 3 Toblerones? ... out of 4 Toblerones? What other designs could you make? Find their surface area and volume.

Task 9: We can make open top boxes out of a sheet of paper by cutting squares from the corners and then folding up the sides. If a sheet of paper is 200 mm by 300 mm, what might be the volume of some boxes that we can create? (a relevant diagram was presented with the task)

Tasks 1 to 5 were intended to be a sequence using open-ended approaches to allow students choice in approach and level of complexity and to contrast surface area and volume. Tasks 6 to 9 were intended to be contextualised tasks that connect the study of surface area and volume (capacity) to students’ experience. The overall mathematical focus should be clear to the project teachers since this was the topic they were about to teach, the tasks are simply and succinctly phrased, and it was intended that the sequence of the first five tasks would be clear to teachers, as would the practical potential of the latter tasks.

The project is intensive and so we are working with a small group of teachers. For the project teachers, we worked through each of these tasks in a teacher professional learning session, and then invited them to complete the relevant aspect of a survey after each task. To expand the number of responses, another group of junior secondary teachers who were not part of our project were also presented with the tasks (on a powerpoint) and the same survey

questions. This comparison group did not work through the tasks before completing the survey but did have an opportunity to discuss them. Both groups were secondary mathematics teachers and seemed to have little difficulty with the required content.

Recognising that the project teachers may have been oriented to giving us the responses they anticipated we wanted to see, the other group of teachers were much less so. The tasks in the survey were not directly connected to the content of the professional learning sessions, and their anonymity would certainly have been protected in the larger group. The responses of this comparison can be taken as an accurate reflection of their opinions (as far as that is possible). The project teachers' responses can also be taken as an indication of their actual opinions in that the teachers subsequently choose some tasks for lessons that were observed by the project team indicating that they were comfortable enough with the tasks to use them in that public setting (see Sullivan, Clarke, Cheeseman, Michels, Middleton, Mornane, & Roche, 2011, for a report on those observations).

Note that the collection of responses such as those reported below is time consuming for teachers in that it required consideration of at least nine specific tasks, including working through them in the case of the project teachers, and for the teachers to respond to around 10 items for each of these tasks. Even though the total number of responses is small, it still represents a substantial and intensive data collection exercise.

Results

The results reported below the responses to the survey about the nine tasks presented in Figure 1. Table 1 presents the profile of responses to task 1 above of all teachers combined (31 teachers altogether), with the number of responses per category from the six project teachers in brackets. This table is presented to indicate the range of responses from both groups of teachers. The teachers were invited to respond to the prompts in the left hand column using the code 1 for strongly disagree, 2 for disagree, 3 for agree, and 4 for strongly agree.

Table 1

Responses of Teachers to Survey Prompts Related to Task 1 (above)

	SD	D	A	SA	Mean
a. I would use the task and I would let the students work out how to do it for themselves	0	3	12 (2)	10 (4)	3.3
b. The task would be organisationally difficult with my class	9 (3)	13 (2)	2 (1)	0	1.7
c. The task would provide experiences to develop important knowledge about these concepts	0	0	12	13 (6)	3.5
d. The task needs more information for the students before it could be used	7 (4)	13 (1)	4 (1)	1	2.0
e. The task would result in better learning about these concepts than the ones in the text book	0	1 (1)	17 (4)	6 (1)	3.2
f. I know a bunch of tasks on this topic that are more interesting than this one	2	22 (5)	1 (1)	0	2.0

Note that prompts a and c refer to research question *i*, prompts b and d seek data on research question *ii*, and prompts e and f address question *iii*. The overwhelming majority of the teachers felt that they would use this task as we intend, they felt able to use it, they recognized the potential of the task, did not feel more information was needed, that the task would be better than text exercises, and is more interesting than other similar tasks that they know.

Even though there are clear trends in the responses, there was also a spread indicating that teachers respond in different ways to suggestions of tasks and there is no such thing as a perfect task. In terms of providing suggestions, it is therefore important that teachers be offered options and choices.

In comparing the responses of the project teachers with the other teachers, there are some interesting results. Recognising that the number of project teachers is small, they were more likely to strongly agree that the task would provide important experiences for the students than the other teachers (the main figure in the table is the response of all teachers). The project teachers also were more likely to strongly disagree with the proposition that the task needed more information for the students (two thirds of the project teachers vs one fifth of the others). It seems that having worked through the task in a professional learning session helps to convince teachers that there is enough information for the task to be completed and so resist the related student demands for hints that are not needed. The project teachers also seemed more likely to strongly agree that they would let the students work out how to do the problem by themselves (two thirds vs one third). This suggests that it is helpful for teachers to work through tasks before using them. It seems that having worked through the tasks and having the opportunity to discuss and share possible solutions with colleagues helped the teachers to see the potential in the tasks. It seems that working through challenging tasks with planning teams before using them in their classrooms will assist teachers in anticipating the potential of tasks.

Recognising that there were some slight differences in the strength of the responses of the two groups of teachers both to the prompts listed in Table 1 and in responses to the other tasks, the directions of the responses were the same. For simplicity and space reasons, responses of both groups to the other tasks are combined and means presented to indicate the overall direction of the responses.

Table 2 presents the mean of the responses to the same six prompts for the other tasks. Note that a mean of 2 indicates overall disagreement with the proposition, and less than 2 is stronger disagreement. A mean of 3 indicates overall agreement with the prompt and a score of more than 3 is stronger agreement. Between 2 and 3 can be considered balanced or neutral.

As with Table 1, it seems that for each of the tasks, the teachers would be willing to allow students to work through the tasks by themselves, they do not consider the organisational challenges to be significant, they recognise the potential of the tasks for developing conceptual knowledge, they think the information in the task is enough, and that the tasks offer better learning opportunities than those in texts or the ones they know already.

Table 2

Mean Responses of Both Groups of Teachers to Tasks 2 to 9 (from Figure 1)

	Task 2	Task 3	Task 4	Task 5	Task 6	Task 7	Task 8	Task 9
I would use the task and I would let the students work out how to do it for themselves	2.9	2.8	2.8	2.9	2.6	3.1	2.8	3.0
The task would be organisationally difficult with my class	1.9	2.0	2.2	2.0	2.0	2.0	1.9	1.9
The task would provide experiences to develop important knowledge about these concepts	3.2	3.4	3.3	3.5	3.4	3.3	3.3	3.4
The task needs more information for the students before it could be used	2.2	2.3	2.4	2.1	2.8	2.5	2.4	2.2
The task would result in better learning about these concepts than the ones in the text book	3.2	3.2	3.2	3.2	3.2	3.3	3.1	3.4
I know a bunch of tasks on this topic that are more interesting than this one	2.1	2.0	1.8	2.0	1.8	2.0	1.8	1.8

There is surprising consistency in the responses across the tasks. To take two examples, task 2 and task 6 are very different in what is required of teachers and students. While the former, although open-ended with a variety of possible responses, provides quite clear indication to students of what they should do, the latter requires substantial judgment and inference by the students, and presumably would be more difficult organisationally, if not mathematically. Noting that the more teachers felt task 6 needed more information, otherwise the teachers' responses are similar to the tasks. It is hard to know whether the teachers consider the former more difficult pedagogically than we do, or whether they consider the latter to be less difficult for the students than we think it would be.

Conclusion

As part of a larger project that is suggesting examples of challenging tasks to teachers so that we can study the impact of those tasks on student learning, we explored the ways that teachers interpreted the tasks that we suggested. After working through a particular set of tasks as part of a professional learning session, our junior secondary project teachers completed a survey on aspects of their responses to those tasks. A comparative group of similar teachers also completed the survey but did not work through the tasks nor did they participate in professional learning on this content.

In terms of the research questions presented above, it seems that both groups of teachers saw the mathematical and pedagogical potential of the tasks, they did not feel constrained by the ideas, and they felt the tasks had more potential than those that they find in texts. While it is not critical that teachers have worked through tasks for them to see the potential in the tasks, this may help teachers to avoid providing too much information and may also allow deeper consideration of the related pedagogies.

We feel that succinct presentation of tasks fits with the time pressures that teachers are experiencing, and also that task suggestions need to align with what the teachers are planning to teach. There are clear advantages in offering tasks to teachers in sequences in that teachers can start the sequence with a task that is at the appropriate level of difficulty

for their class. Perhaps making a decision on the starting point for the their sequence can encourage teachers to maintain the challenge in the task.

It also seems that while there were strong trends, there was also variability in the responses of the teachers, indicating that teachers have different responses to different tasks. It would be helpful for teacher educators and those who propose resources for teachers to be aware of the need for variety in tasks suggested for teacher use.

References

- Ball, D., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Westport, CT: Ablex.
- Chick, H. (2007). Teaching and learning by example. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 3-21). Adelaide: MERGA.
- City, E. A., Elmore, R. F., Fiarman, S. E., & Teitel, L. (2009). *Instructional rounds in education*. Mass.: Harvard Educational Press.
- Clark, C. M., & Peterson, P. L. (1986). Teachers' thought processes. In M. C. Wittrock (Ed.) *Handbook of research on teaching* (pp. 255-296). New York: Macmillan.
- Delpit, L. (1988). The silenced dialogue: Power and pedagogy in educating other people's children. *Harvard Educational Review*, 58(3), 280-298.
- Doyle, W. (1986). Classroom organisation and management. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 392-431). New York: Macmillan.
- Dweck, C. S. (2000). *Self theories: Their role in motivation, personality, and development*. Philadelphia: Psychology Press.
- Hannula, M. (2004). *Affect in mathematical thinking and learning*. Turku: Turun Yliopisto.
- Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 65-97). New York: Macmillan.
- Hill, H., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372-400.
- Hollingsworth, H., Lokan, J., & McCrae, B. (2003). *Teaching mathematics in Australia: Results from the TIMSS video study* (TIMSS Australia Monograph No. 5). Camberwell, Victoria: Australian Council for Educational Research.
- Kelly, A. (2003). Research as design. *Educational Researcher*, 32(1), 3-4.
- Middleton, J. A. (1999). Curricular influences on the motivational beliefs and practice of two middle school mathematics teachers: A follow-up study. *Journal for Research in Mathematics Education*, 30(3), 349-358.
- Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, 33(2), 455-488.
- Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason and analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, 2(1), 50-80.
- Sullivan, P., Clarke, D., & Clarke, B. (2009). Converting mathematics tasks to learning opportunities: An important aspect of knowledge for mathematics teaching, *Mathematics Education Research Journal*, 21(1), 85-105.
- Sullivan, P., Cheeseman, J., Michels, D., Mornane, A., Clarke, D., Roche, A., & Middleton, J. (2011). Challenging mathematics tasks: What they are and how to use them. In L. Bragg (Ed.), *Maths is multi-dimensional* (pp. 33-46). Melbourne: Mathematical Association of Victoria.
- Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.
- Tzur, R. (2008). A researcher perplexity: Why do mathematical tasks undergo metamorphosis in teacher hands? In O. Figuras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proceedings of the 32nd Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol.1, pp. 139 - 147). Morelia: PME.

Insights into Ways that teachers plan their Mathematics Teaching

Peter Sullivan

Monash University

<peter.sullivan@monash.edu>

David Clarke

University of Melbourne

<d.clarke@unimelb.edu.au>

Doug Clarke

Australian Catholic University

<doug.clarke@acu.edu.au>

Peter Gould

NSW DECS

<peter.gould@det.nsw.edu.au>

David Leigh-Lancaster

Victorian Curriculum and Assessment Authority

<leigh-lancaster.david.d@edumail.vic.gov.au>

Gerard Lewis

Catholic Education Office, Melbourne

<GLewis@ceomelb.catholic.edu.au>

Insights into aspects of teachers' planning reported here were gathered as part of a broader project examining aspects of the implementation of the Australian curriculum in mathematics. In particular, responses of teachers to a survey of various aspects of decisions that inform their planning are discussed. While there is diversity in processes teachers use for planning, a consistent theme was that teachers make active decisions at all stages in the planning process. There were slight differences in the ways that primary and secondary teachers plan. There are important implications from our findings for those who support teachers in the transition to the Australian Curriculum: Mathematics.

Introduction

The Melbourne Declaration (MCEECDYA, 2008) articulates an agreed commitment of Australian states and territories and the Commonwealth on the educational entitlement of all young Australians. This Declaration underpins the development of an Australian curriculum, and includes mathematics as one of eight learning areas where “English and mathematics are of fundamental importance in all years of schooling and are the primary focus of learning in the early years”. In enacting this, the Australian Curriculum Assessment and Reporting Authority (ACARA) (2012) outlines the rationale and guiding principles for the development of the Australian Curriculum: Mathematics (AC:M). That document explains the decision to develop discipline specific curricula, and also describes structures and processes to build connections between those disciplines. The overall Australian Curriculum, in addition to the discipline specific documents, describes cross curriculum priorities (e.g., Aboriginal and Torres Strait Islander histories and cultures), and general capabilities (e.g., numeracy, creative and critical thinking). There is an accompanying *Mathematics Shape Paper* (ACARA, 2009) which recommends reforms in curriculum documentation. For example, changes recommended are that:

- the content be succinctly and clearly described to support teacher decision making;
- the process strands, termed proficiencies (understanding, fluency, problem solving and reasoning), be integrated with the content strands;
- the curriculum be less “crowded” allowing for greater depth of study; and
- there be greater attention to statistics and probability at all levels of schooling.

Subsequent drafts of documents constituting the AC:M interpret these principles, outlining content descriptions and elaborations, achievement standards, and annotated student work samples.

There is an underlying assumption that curriculum documentation significantly informs the way teachers plan, teach and assess, and that improved planning can result in improved

teaching and learning. The following is a report of one aspect of a research project¹ that is seeking to elaborate the relationship between curriculum documentation and mathematics teaching. The focus of this report is on ways that teachers report that they plan.

There is a direct connection between planning and teaching. Teachers are best able to support students when they have a clear sense of what they intend students learn. Hattie and Timperley (2007), for example, reviewed a range of studies on characteristics of effective classrooms and found that feedback was one of the main influences on student achievement. They listed the key elements as being that students receive information on “where am I going?”, “how am I going?”, and “where am I going to next?” Presumably part of the planning process is anticipating how such questions might be answered for students, a key aspect of which is choosing the important aspects of the topic to be taught. Charles (2005) argued that focus on important ideas “had not become part of mainstream conversations about mathematics standards, curriculum, teaching, learning and assessment” (p. 9). One aspect of the overall project is exploring the extent to which the AC:M assists teachers in identifying and articulating the important ideas informing their teaching.

The overall project includes three data collection phases: a baseline survey; interventions; and development of exemplars. The following presents results from the baseline survey and in particular we describe teachers’ responses to a prompt related to their planning.

Curriculum Knowledge and Curriculum Planning

A second underlying assumption is that teachers’ planning decisions are informed by their knowledge of curriculum generally and the curriculum they are intending to teach in particular. There are two major categories of knowledge needed for the teaching of mathematics: subject matter knowledge; and pedagogical content knowledge. Hill, Ball, and Schilling (2008) described diagrammatically components of these two categories of knowledge. When converted to text, the sub categories of *Subject Matter Knowledge* were: common content knowledge; specialised content knowledge; knowledge at the mathematical horizon. The subcategories of *Pedagogical Content Knowledge* were knowledge of content and teaching; knowledge of content and students; knowledge of curriculum. Hill and colleagues did not elaborate what they meant by the term knowledge of curriculum. In discussing the issue of the knowledge of curriculum that teachers need, Shulman (1986) earlier said:

The curriculum is represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances. (p. 10)

Note that this statement does not mention knowledge of the type of curriculum documentation that has been produced by the various Australian jurisdictions previously, the documentation associated with the AC:M, or indeed, more general discourse internationally on such matters, such as the core standards in the United States.

¹ This is a report of the Peopling Education Policy project that is funded by the Australian Research Council (LP110100062) with additional funding provided by the NSW Department of Education and Training, Victorian Curriculum and Assessment Authority, Catholic Education Office Melbourne and the Australian Curriculum Assessment and Reporting Authority. The project is collaboration between Monash University, Australian Catholic University, University of Melbourne, University of Newcastle, University of Sydney, and University of Technology Sydney. The content is the responsibility of the authors and the views expressed do not necessarily represent the views of the universities or the partners.

The assumption mentioned above is that knowledge of curriculum informs planning. Yet it seems that the ways teachers plan is very much dependent on the overall schooling context and what they take the term curriculum to mean. Kilpatrick, Swafford, and Findell (2001), for example, argued that planning is critical, takes much effort and is the core activity of experienced teachers. Yet they claimed that the planning of US teachers consists of choosing activities and anticipating ways in which they will organise the activities. Similarly the significant research on teaching reported by Stein, Grover, and Henningsen (1996) described the initial phase in planning as the teacher taking the mathematical task as presented in instructional materials which, influenced by the teacher's goals, subject matter knowledge, and knowledge of students, informs the way they teach. In other words, these US teachers use instructional materials as the starting point for planning.

Our experience is that the planning of teachers in Australia is quite different. In all jurisdictions, schools develop yearly plans of the topics that form the skeleton of the curriculum. Teachers then, individually or in teams, plan units of work (or sequences of lessons) using the constructs, structures and detail of the relevant curriculum. It is assumed that previously the various state curriculum documents informed both types of planning in some way and that, in the future, teachers will consider the AC:M (or at least some variation of it) as the basis of both types of planning. It is, however, not clear when and how curriculum documents are consulted, and how that information informs planning. That is the focus of the data reported below.

The Context and Methods of the Data Collection

The overall focus of our project is on describing influences on teachers' knowledge and practice, and especially the ways teachers interpret and use curriculum documents. The project is using an adaptation of the schematic of Clark and Peterson (1986) that proposes that teachers' intentions (that is their planning) is informed, interactively, by their knowledge of curriculum and pedagogy, their dispositions (their beliefs, attitudes and values), and the opportunities and constraints arising from the context of teaching. We hoped to gain insights into ways these factors interact through seeking insights into teachers' planning.

To minimise preconceptions about the types of documents that teachers consult, the authority they attribute to those documents, the role of their experiences, the resources to which they may have access, and so on, we first conducted eight focus groups of primary teachers and four focus groups of secondary teachers. The prompts for those focus groups were intended to promote discussion and even argumentation among the teachers. For example, we presented participating teachers with extracts from the AC:M and asked questions such as: What does this extract mean to you? To what extent does this suggest changes to your current practices? What would you need to know to implement this aspect of the curriculum in your classroom? How might you assess whether students learn the content suggested by the curriculum content descriptions? How might this influence your planning?

The focus group interviews were recorded and transcribed, and the transcripts read to determine themes in the teachers' responses. These themes were used as the basis of a survey on which we sought responses from a broader range of teachers. The survey was piloted with one primary and one secondary group of teachers to check for coherence, ease of completion, and the clarity of the prompts.

There were six sections to the survey: the range and authority of documents that teachers consult; the ways that teachers describe important ideas in topics they were planning to teach, the processes teachers use in their planning; ways teachers match classroom tasks

with curriculum statements; the extent to which teachers incorporate aspects of mathematical reasoning into their planning and teaching (a key aspect of the AC:M); and what teachers identify as their professional learning needs associated with the interpretation of curriculum. Teachers were invited to complete an on-line version of the survey via a link on the web site of the *Australian Association of Mathematics Teachers*, and some groups of teachers to which we had occasional access completed a paper version of either the whole survey or particular whole sections of the survey.

The particular research questions that focus on teachers' planning were:

What is the comparative emphasis that teachers place on activities or curriculum documents in making planning decisions?

In what ways is the planning of primary and secondary teachers different?

Teachers' Planning Processes

At the start of the survey, teachers were invited to indicate the frequency with which they consulted various documents and made decisions. David Clarke and colleagues (in press) reported that teachers' decisions are informed just as much by their experience, their colleagues in the case of primary teachers, and their assessments of students as they are by formal curriculum documents.

Next, teachers were asked to respond to the following prompts:

What is the next mathematics topic you are planning to teach?

What is the most important idea that you will focus on for that topic?

Nearly all teachers responded to the first prompt by listing a topic that was easily recognisable as part of the mathematics curriculum such as Algebra or Addition and subtraction. Doug Clarke and colleagues (in press) reported that responses to the second prompt indicated that the important ideas listed by many teachers would not provide the sort of focus that Charles (2005) was proposing, with around 32% of primary teachers simply listing a sub-topic of the larger mathematical topic (e.g., measurement—perimeter and area, and 13% making a statement which could apply to most other topics and not specifically an important idea tied to the particular topic (e.g., real life applications). The equivalent percentages for secondary were 56% and 17%, respectively.

The items that form the basis of the data below follow on directly from these prompts. This particular item was:

Rank the following statements in order from 1 to 6 in terms of "closeness" to what you will do when you are planning and teaching that mathematics topic (use 1 for most like you, 2 for the next most like you ... 6 for least like you)

The responses from secondary and primary teachers to this prompt and a subsequent one are presented separately.

Secondary Teacher Responses

Table 1 presents responses of the secondary teachers. In Table 1 (and Table 2 below), even though the numbers represent ranks, the mean is presented to act as a general guide to the responses overall (with a low score representing a higher rank), along with the percentage ranked by teachers in the top two categories, providing a sense of the priority teachers give to the particular planning strategy.

Table 1

Ranking of Statements on Specific Planning Process in Order (secondary) (n = 130)

	Mean	% of teachers ranking this in the top two
Based on what I know about the students already, I will collect the relevant classroom activities to which I have access, choose the ones that fit the topic, and then decide the order in which I will use them.	2.5	56
I will read the official curriculum documents to identify the specific learning goals and then plan my teaching based on that.	3.2	43
I will look through a textbook or other teacher resource and plan teaching which relates to the given content.	3.3	36
I will look at the school-based curriculum documents then identify specific learning goals, and select activities from there that I will use with my students.	3.4	32
I will meet with other teachers, we will share our experience and ideas, then plan the activities and resources we will use.	3.6	27
I will read the official curriculum documents to identify the specific learning goals and then use the teaching ideas and resources that they suggest.	4.5	18

There were no significant differences between the responses of teachers to these items, based on their age, experience, system, gender, or whether they completed the survey online. In the case of this table, of some interest is whether there were differences by state. For these teachers the ranking and proportions in the top two categories were similar across the three states that had enough responses to allow comparisons.

The first observation from the table is the diversity of responses suggesting that teachers have different priorities and processes for their planning. Some base their planning on judgments and activities, others refer to official curriculum documents, others to texts (although fewer than we had anticipated for these secondary teachers) and a smaller number rely on school developed documents. Indeed, even the most popular option, that of collecting relevant classroom activities, was only selected in the top two factors by just over half of the teachers. While some secondary teachers report using a text, only around one third ranked this in the top two categories, suggesting that the common impression that secondary teachers rely on largely on texts for planning may be incorrect for this sample.

A second inference from the responses is that teachers make judgments about what their students need and plan accordingly. Rather than the common conception that teachers are passive users of curriculum, it seems that these teachers at least make active decisions about the content of their teaching.

It also seems that planning is an individual rather than collaborative enterprise for these secondary teachers, with only 27% listing meeting with other teachers in the first two ranks.

Because we were not sure that the options offered in Table 1 represented the planning processes of all teachers, the teachers were also invited to respond to the following prompt:

If none of the statements in the question above are at all close to a description of how you would plan this topic, please write a description of how you would plan your next mathematics topic:

There were only 16 written responses indicating that most teachers felt that their specific planning processes were represented in the list of options. On other similar items, there were many more teachers who gave written responses. Of the teachers who did respond, the

responses were sophisticated and detailed, and the most common category of response was those that considered the options presented (shown in Table 1) did not capture the complexity and subtlety of their planning. For example, the following are some of those written responses:

Identify the learning from Australian curriculum document. Look to see how new texts have interpreted the content. Reflect on what they have said, consider what I know already about the topic and the students' existing knowledge. Consider where the topic goes and what preparation is needed for future topics. Gather all relevant and interesting resources. Select for engagement.

I have developed notes specific to that topic by following the syllabus then finding the appropriate resources from text books, web sites or other resources that would suit the students in that particular class adjust or change my teaching to suit their needs.

I would obtain the topic to be taught from the schools' program. Using the official text book I will prepare a lesson plan, develop warm-up exercise, lesson presentation (may involve, overheads, Powerpoints, physical models), classwork (taken from textbooks, or material I have available (or available from school), homework, end of topic test.

I check the school based documents to check our timing, confirm the learning goals in the official curriculum documents and then will select from many sources the resources that meet the needs specific to my class and the students in it.

These examples are representative of the responses of other teachers. The responses indicate that these teachers have thoughtful and multi-step processes for their planning, they plan idiosyncratically, and make active decisions. This is discussed further below.

Primary Teacher Responses

The primary teachers also were invited to rank the same items after they had nominated the next topic they were to teach. Their responses are presented in Table 2.

Table 2

Ranking of statements on the planning process in order (primary) (n = 97)

	Mean	% Teachers in top two
Based on what I know about the students already, I will collect the relevant classroom activities to which I have access, choose the ones that fit the topic, and then decide the order in which I will use them.	2.5	55
I will read the official curriculum documents to identify the specific learning goals and then plan my teaching based on that.	2.9	49
I will meet with other teachers, we will share our experience and ideas, then plan the activities and resources we will use.	3.3	46
I will look at the school-based curriculum documents then identify specific learning goals, and select activities from there that I will use with my students.	3.8	15
I will read the official curriculum documents to identify the specific learning goals and then use the teaching ideas and resources that they suggest.	3.8	23
I will look through a textbook or other teacher resource and plan teaching which relates to the given content.	4.7	10

The top two options for these primary teachers were also the most highly ranked by secondary teachers, with the main difference being the higher ranking of planning with other teachers. As with the secondary teachers, there is a diversity of ways that these teachers plan, and it can be inferred that they make active decisions in their planning. The

primary teachers generally gave a lower ranking to referring to a text as part of the planning process, but a higher ranking to meeting with other teachers to plan.

As with the secondary teachers, the primary teachers were also invited to describe their planning if none of the above descriptions matched the way they plan. There were also just 12 replies, as with the secondaries, indicating that the above were meaningful descriptions of their planning. Some of the additional descriptions focused on the role of the team:

We meet as a team and share out the term's maths planning. Each teacher volunteers to prepare two or three weeks' learning for the whole stage. When it is your turn, you write up the next week's maths on a proforma and put a copy of the plan and any relevant worksheets and activities in each person's pigeonhole. A copy is also placed on (xxx) and referred to in the following year. If appropriate, a Notebook file is developed on the topic and saved on (xxx) for use with interactive whiteboards.

Other comments emphasised the role of pedagogy such as:

Although I rated the above 1-6 in reality I pre-test the children on the concept then through discussion we share their results and knowledge then I pose questions relating to the concept (to engage their lateral thinking skills) we then go back and discuss then I show examples then I ask them to write examples and come to front of their class and reflect. The children then reflect on that child's knowledge etc. This is a process and a cycle and over an hour lesson we can reflect on many concepts and take the understanding of each concept to a deeper level. I use texts, curriculum doc to guide me. I rarely give out a work sheet.

One of the responses indicated an emphasis on integration across topic boundaries.

I will consider my students' current understanding of the topic (sampling and distribution) based upon the level of understanding they have demonstrated already. I will then devise an inquiry topic which will necessitate deep exploration of sampling (beyond their current understandings) in order for the students to resolve the inquiry question, regardless of what line of approach they take. I will ensure that the inquiry cannot possibly have a 'correct' answer which will necessitate the students gathering and presenting evidence to argue and justify their solutions in class. I will then look at the curriculum documents for my state, recognise that my students are working well above where they are 'supposed' to be, shrug it off and advise them of that because it amuses them that EQ thinks they are incapable of such deep understandings at their age.

Again these free format responses illustrate the complexity and sophistication of teachers' planning routines, the active role that they play, and their willingness to consult with planning team members. These issues are discussed further below.

Discussion and Conclusions

The results reported are from a survey on aspects of teachers' planning processes. There are important implications for those who educate and support mathematics teachers.

It seems that, from these data, there is a wide diversity of emphases and sequences of actions in the planning processes reported above. This suggests that any professional learning on planning should start from the processes that teachers use, and that teacher educators and others, rather than imposing particular routines on teachers, should help them to find ways to make their own processes better.

It can also be inferred that teachers do not follow recipes and instructions but make decisions based on their own judgments, and those of colleagues (in the case of primary teachers), on ways to increase the chances of their students learning. Curriculum developers, administrators, policy makers, school leaders, teacher educators and others should find ways to support teachers in their decision making rather than trying to curtail it. In particular, attempts to be overly prescriptive or to provide "teacher-proof" lessons will be counterproductive.

A surprising finding is that there were few differences in the spread of responses between the categories of teachers. One difference was that the primary teachers reported more emphasis on collaborative planning and secondary teachers claimed to plan by themselves. Another difference was the secondary teachers used texts more than primary teachers, although interestingly still less than we had anticipated that they would. It seems that it would assist secondary teachers if routines in schools were established to facilitate collaborative planning.

It also seems that the planning of many teachers is substantially informed by curriculum documentation. Our project will continue to explore ways that teachers use such documentation and which aspects they find helpful. It is surprising, given the resource intensive nature of the preparation of such curriculum documents, that there is limited recent Australian research on how they are used. Certainly, an Australian mathematics curriculum, with common documentation and supporting resources provides an ideal context for such research to receive renewed attention.

References

- ACARA (2012). The shape of the Australian Curriculum. Accessed Jan 2012 from [http://www.acara.edu.au/verve/resources/The Shape of the Australian Curriculum V3.pdf](http://www.acara.edu.au/verve/resources/The%20Shape%20of%20the%20Australian%20Curriculum%20V3.pdf)
- ACARA (2009). Shape of the Australian Curriculum: Mathematics. Downloaded in Feb 2012 from [http://www.acara.edu.au/verve/resources/Australian Curriculum - Maths.pdf](http://www.acara.edu.au/verve/resources/Australian%20Curriculum%20-%20Maths.pdf)
- Charles, R. (2005). Big ideas and understandings as the foundation for early and middle school mathematics. *NCSM Journal of Educational Leadership*, 8(1), 9-24.
- Clark, C. M., & Peterson, P. L. (1986). Teachers' thought processes. In M. C. Wittrock (Ed.) *Handbook of research on teaching* (pp. 255–296). New York: Macmillan.
- Clarke, D. M., Sullivan, P., & Clarke, D. J. (in review). Important ideas in mathematics: What are they and where do you get them? *Australian Primary Mathematics Classroom*.
- Clarke, D. J., Clarke, D. M., & Sullivan, P. (in review). Resources that support teachers' planning. *Australian Primary Mathematics Classroom*.
- Hattie, J., & Timperley, H. (2007) The power of feedback. *Review of Educational Research*, 77(1), 81-112.
- Hill, H., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39, 372-400.
- Ministerial Council for Education, Early Childhood Development and Youth Affairs (MCEECDYA, 2008). Melbourne Declaration on educational goals for young Australians. Downloaded on Feb 2012 from http://www.mceetya.edu.au/mceecdya/melbourne_declaration,25979.html
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, 33(2), 455–488.

Students' Ways of Knowing and Learning Mathematics and Their Ways of Interacting with Advanced Calculators

Hazel Tan

Monash University

<hazel.tan@monash.edu>

In this paper, the relationships between students' beliefs about knowing and learning mathematics, and how they engage with calculators, are investigated. An online survey was conducted for 964 Singaporean and 176 Victorian senior secondary students. Students' *connected knowing–deep approach* conception of mathematics was found to be associated with their use of *calculators as collaborator*, and their *separate knowing–surface approach* conception of mathematics was associated with use of *calculators as master*. Gender differences in students' beliefs were also found.

From past research, students' conceptions about mathematics are found to be related to their learning approaches and their academic achievement, and influenced by the learning contexts they experience (Muis, 2004). Since the inclusion of technology in mathematics education, particularly the use of graphic calculators (GC) and calculators with computer algebra systems (CAS), there have been few studies in which the relationships between students' conceptions of mathematics, approaches to learning mathematics and how they interact with such technologies have been investigated. The aim of this study was to address this gap in the literature.

Background Literature

There are many studies on students' epistemological beliefs (beliefs about knowledge and ways of knowing) both in general (e.g., Hofer & Pintrich, 1997), and about mathematics (e.g., Muis, 2004; Roesken, Pepin, & Toerner, 2011). There are many theories in both fields, but no unified framework exists; mathematics education researchers have generally investigated beliefs about mathematics as a construct under the affective domain (Muis, 2004). The interest in affect and beliefs in mathematics started in the 1980s from investigations of teachers' beliefs which influenced their instructional approaches and it has since blossomed into a field encompassing a multitude of theories and views about the various aspects of beliefs and belief systems (Roesken, Pepin, & Toerner, 2011). In a review of research on epistemological beliefs, Muis (2004) found that students generally believed in an innate ability to do mathematics and considered knowledge to be unchanging, composed of unrelated components, and handed down by an authority figure.

Students' beliefs have also been found to be related to their learning strategies; for example a belief in mathematics as a list of isolated facts and fixed procedures might encourage students to concentrate on memorising lists and procedures as a study strategy (Muis, 2004; Schommer-Aikins & Easter, 2006). Although there were significant associations between students' epistemic beliefs about mathematics and the types of behaviours students engaged in when learning mathematics, no strong evidence of causality was identified (Muis, 2004). Muis (2004) suggested that there could be other types of relationships between beliefs and learning strategies, such as a reciprocal relationship.

As technology becomes more integrated into the mathematics curricula in different countries, there are many studies conducted which investigate how students interact with the advanced calculators and their learning outcomes (e.g. Burrill et al., 2002). Students' use of

calculators generally resulted in positive attitudes and improvement of learning, and was found to be influenced by factors such as the role of calculators in assessment and instruction, and students' familiarity with and attitude towards the tool. Students tended to use calculators for quick and accurate graphing, and for problem solving and investigations; however evidence suggests that calculators might be underutilised (Burrill et al., 2002). There were mixed results regarding gender differences in students' use of calculators, with some studies showing no evidence of gender difference, and others with a gender difference favouring females (Burrill et al., 2002) or males (e.g. Tan & Forgasz, 2011).

Even though students' beliefs and attitudes are generally acknowledged as influenced by socio-cultural contexts, there were few cross-country comparisons (Roesken, Pepin, & Toerner, 2011). There does not appear to be research studies in which the relationships between students' gender, beliefs, learning strategies and their engagement with technologies have been investigated. It is thus of interest in the current study to investigate students' mathematical beliefs and learning approaches in technology-rich environments where advanced calculators are used. Two research questions are the focus of this report which formed part of a larger study:

1. Are there any differences in students' ways of knowing and approaches to learning mathematics (a) by region, (b) by gender within each region?
2. Are there any relationships between students' ways of knowing, approaches to learning mathematics, and their ways of interacting with the advanced calculators, (a) by region, (b) by gender within each region?

The Study

Theories Underpinning the Study

Students' ways of knowing. Becker (1995) applied the *Women's ways of knowing* model by Belenky, Clinchy, Goldberger and Tarule (1986) to mathematics. She described two types of knowing, Separate and Connected. *Separate Knowing* (SK) deals with "logic, rigour, abstraction, rationality, axiomatics, certainty, deduction, completeness, absolute truth, power and control, algorithmic approach, structure and formality" (Becker, 1995, p. 167). *Connected Knowing* (CK) deals with "intuition, creativity, hypothesizing, conjecture, experience, relativism, induction, incompleteness, personal process tied to cultural environment, contextual" (Becker, 1995, p. 167). These ways of knowing have been found to be gender-related, with males tending to score higher in separate knowing than females (e.g., Schommer-Aikins & Easter, 2006).

Students' approaches to learning. Building on the early work of Marton and Säljö (1976), Biggs (1987) described students' learning approaches to be composed of two components: motive – the intention to learn, and strategy – the way to go about learning. A *Deep Approach* (DA) is intrinsic and meaningful, where students "study to actualize interest and competence" (Biggs, 1987, p. 11) through seeking understanding and relating to ideas. A *Surface Approach* (SA) is instrumental and reproductive, where the "main purpose is to meet requirements minimally... (and to) limit target to bare essentials and reproduce (content) through rote learning" (Biggs, 1987, p. 11). It was found that students who thought of mathematics as a fragmented body of knowledge tended to learn it using surface approaches, whereas students who viewed mathematics as an interconnected complex system tended to use deep approaches (Crawford, Gordon, Nicholas, & Prosser, 1994). Findings on gender

differences are ambivalent. Gordon (1997), for example, found that females score higher than males on surface approaches, while Kılıç and Sağlam (2010) found that males scored higher on rote learning (surface approach) and females scored higher on meaningful learning (deep approach).

Students' ways of interacting with technology. In order to describe the ways teachers and students interact with technology for teaching and learning mathematics, Goos, Galbraith, Renshaw and Geiger (2000) proposed four metaphors for the role of technology that comprise the Master, Servant, Partner, and Extension of self (MSPE) framework:

- Master (M), where one is subservient to technology (e.g., blindly follow calculator steps in the textbook without understanding);
- Servant (S), where one uses technology for menial or tedious tasks (e.g., using calculators to replace mental or pen-paper calculations);
- Partner (P), where one treats technology as an equal that provides resources and information (e.g., using calculators as a platform for collaborative inquiry); and
- Extension of Self (E), where one engages with the technology intimately and seamlessly (e.g., integrating calculators appropriately in mathematical explorations as part of one's intellectual repertoire of mathematical tools and skills).

Tan (2009) developed an instrument based on the MSPE framework and administered it to Singaporean and Victorian senior secondary mathematics students. Factor analysis revealed three, rather than four, factors. The technology as Partner and Extension of Self items loaded as one factor that was named technology as Collaborator (Tan & Forgasz, 2011). In this paper, further findings from the same study are reported. The focus is on the relationships between students' ways of knowing and approaches to learning mathematics, and their ways of interacting with advanced calculators.

Instruments, Sample, and Data Analysis

The main study was conducted in Singapore (on GC) and Victoria (on CAS calculators) using an online survey. Invitations were sent to students through schools in both regions to participate in the study. Additionally an advertisement was put up in Facebook to recruit student participants from Victoria (Tan & Forgasz, 2011).

Instruments. The relevance, applicability, and length of pre-existing instruments associated with the various theoretical frameworks described above were closely considered in the development of the instruments used in the present study. Some modifications were needed.

1. *Measuring students' ways of interacting with technology.* Students' ways of interacting with technology (GC and CAS calculators) were measured using a researcher developed instrument adapted from the MSPE framework. Following a pilot of the instrument with multiple items, the final instrument was comprised of 12 items measuring Calculator as Master (Cal_Ma), as Servant (Cal_Se), and as Collaborator (Cal_Co) - see Tan (2009) for more details.
2. *Measuring ways of knowing and learning mathematics.* Students' separate (SK) and connected knowing (CK) were measured using a researcher developed instrument adapted from Ocean's (1998) survey items. Students' approaches to learning (surface [SA] and deep [DA]) were measured by adapting the 22 items from Biggs' Learning Process Questionnaire (Kember, Biggs, & Leung, 2004) to mathematics. A 5-point Likert type format (1=strongly disagree, 5=strongly agree) was used for consistency.

Analysis of data from a pilot study of 189 Singaporean students showed that there were moderate to strong positive correlations between the CK and DA scores ($r=0.606$) and between SK and SA scores ($r=0.401$). Since the close relationship between students' beliefs and their study behaviours found in this data analysis is also found in the literature, it was decided to pool the items from the two instruments to increase the reliability of the subscales and shorten the length of the instrument. An iterative process of analysing reliability tests and principal component analyses (PCA) was employed to eliminate poorly performing items. The PCA confirmed the existence of two factors, with seven items from connected knowing and deep approach forming one factor, and seven items from separate knowing and surface approach forming another. The combined instrument, *Ways of Knowing and Learning Mathematics*, measured *Connected Knowing–Deep Approach* (CK-DA) and *Separate Knowing–Surface Approach* (SK-SA) (see Table 1) and had Cronbach- α reliability measures within acceptable limits ($\alpha_{\text{CK-DA}}=0.792$; $\alpha_{\text{SK-SA}}=0.683$). This new scale was used in the main study.

Table 1

Items in 'Ways of Knowing and Learning Mathematics' Instrument by Construct

Connected Knowing and Deep Approach (CK-DA)	Separate Knowing and Surface Approach (SK-SA)
CK	SK
Maths makes you think creatively	In maths, something is either right or it is wrong.
Good maths teachers show students several different ways to look at the same question.	To solve maths problems you have to be taught the right procedure or you can't do anything.
In maths you can be creative and discover things for yourself.	When I solve maths problems, I'm often stuck if I can't remember the next step.
DA	SA
I try to relate what I have learned in maths to what I learn in other subjects.	When I score poorly on a maths test, I worry a lot about how I will do on the next one.
I work hard at my studies because I find mathematics interesting.	I see no point in learning material which is not likely to be in the examination.
As I am reading, I try to relate new concepts and ideas to what I already know about that topic.	Even when I have studied hard for a maths test, I worry that I may not be able to do well in it.
I frequently think about how to solve maths problems even while on the bus or lying on my bed.	I learn maths formulas by heart even if I don't understand them.

Sample. There were 964 Singaporean students (37.1% males, 62.9% females), and 176 Victorian students (31.3% males, 68.8% females).

Data analysis. Data were collected and analysed using Statistical Package for Social Sciences 18.0 (SPSS) software. T-tests and Mann-Whitney U tests were used to compare regional and gender differences. Pearson product-moment correlation coefficients were used to analyse the relationships between the variables.

Results and Discussion

Factor analysis of the 14 items in the Ways of Knowing and Learning Mathematics revealed similar results to the pilot study: the seven CK and DA items formed one component, and the seven SK and SA items the other (see Table 1). Cronbach- α values for the two subscales by region were: CK-DA ($\alpha_{S'pore}=0.774$; $\alpha_{Vic}=0.798$), and SK-SA ($\alpha_{S'pore}=0.693$; $\alpha_{Vic}=0.625$); all were reasonable (Pallant, 2007).

Regional and Gender Differences in Ways of Knowing and Learning Mathematics

Mean subscale scores were calculated and reduced to the range 1 to 5 for ease of interpretation. The results of comparisons of the mean subscale scores by region, and by gender within region, using t-tests and Mann-Whitney U tests are shown in Table 2.

Table 2

Regional and Gender Comparisons: N, Mean Scores, Test Statistics, and Effect Sizes

Factor	Gender	Singapore				Victoria			
		N	Mean	Test statistic	Effect size	N	Mean	Test statistic ¹	Effect size
CK-DA	Overall	928	3.28			125	3.34		
	Female	586	3.21	-3.67, $p<0.001$	0.15	100	3.27	2.04, $p<0.05$	0.18
	Male	342	3.39			25	3.61		
SK-SA	Overall	933	3.47			124	3.50		
	Female	587	3.51	2.46, $p<0.05$	0.08	100	3.55	-2.29, $p<0.05$	0.21
	Male	346	3.40			24	3.28		

¹ Mann-Whitney U test was used for Victorian data due to small sample size.

Interestingly, there were no significant regional differences in the CK-DA and SK-SA scores of Singaporean and Victorian students. Students in the two regions have comparable results on connected knowing-deep approach and separate knowing-separate approach. Also, the mean scores for SK-SA (S'pore=3.47; Vic=3.50) were higher than that for CK-DA in both regions (S'pore=3.28; Vic=3.34), with the difference being significant only for the Singaporeans: $t(921) = -5.362$, $p<0.001$. This implies that students agree more strongly with having a separate knowing-surface approach than to having a connected knowing-deep approach to the learning of mathematics. Since both the Singaporean and the Victorian senior secondary mathematics curricula culminate in high stake examinations, this finding of high SK-SA scores is consistent with the conclusion from other studies that argue that an extreme emphasis on academic performance and examination success is associated with a tendency to adopt a surface learning approach (e.g., Kılıç & Sağlam, 2010).

As can be seen from Table 2, there are clear gender differences in students' ways of knowing and learning mathematics which are common to both regions. On average, males scored higher than females in CK-DA, whereas females had higher SK-DA scores than males; the Victorian data had higher effect sizes than the Singaporean data. Overall, these findings suggest that males were more likely to hold connected knowing conceptions about mathematics and use deep approaches to learn mathematics, whereas females were more likely to hold separate knowing conceptions and use surface approaches. This finding is

consistent with Gordon's (1997) result that females were more likely to employ a surface approach to mathematics learning. However, the finding that males were more likely than females to employ connected knowing is different from what the gender literature suggests (Becker, 1995).

Relationships between Ways of Knowing and Learning Mathematics and Ways of Interacting with Technology

The correlation coefficients between students' CK-DA, SK-SA and their ways of interacting with calculators are found in Table 3.

Table 3

Correlations between CK-DA, SK-SA, and Calculator as Master, Servant and Collaborator

			Cal _ Ma	Cal _ Se	Cal _ Co
CK-DA	S'pore	Overall	−0.069*	−0.094**	0.412**
		Female	−0.028	−0.029	0.384**
		Male	−0.101	−0.202**	0.426**
	Vic	Overall	−0.230*	−0.108	0.405**
		Female	−0.199	−0.053	0.366**
		Male	−0.118	−0.222	0.520*
SK-SA	S'pore	Overall	0.362**	0.213**	−0.095**
		Female	0.351**	0.240**	−0.117**
		Male	0.368**	0.183**	−0.043
	Vic	Overall	0.519**	0.342**	0.007
		Female	0.478**	0.335**	−0.028
		Male	0.614**	0.148	0.220

* $p < 0.05$. ** $p < 0.01$. *** $p < 0.001$.

It can be seen in Table 3 that common associations are evident in both regions:

- Using calculator as Master (Cal_Ma), i.e., being subservient to technology, is positively associated with having a Separate Knowing and Surface Approach towards mathematics learning ($r_{S'pore}=0.362$; $r_{Vic}=0.519$). Students who learnt mathematics by memorising the steps without real understanding also tended to have a technological or mathematical dependence on the calculators.
- Using calculator as Collaborator (Cal_Co), i.e., engaging with the calculator as a Partner and as an Extension of Self, is positively associated with having a Connected Knowing and Deep Approach towards mathematics learning ($r_{S'pore}=0.412$; $r_{Vic}=0.405$). Students who learnt mathematics through intrinsic interest and deep understanding also tended to use the calculators at a more sophisticated level for problem solving and mathematical exploration.
- Using calculator as Servant (Cal_Se), i.e., using calculators for tedious, repetitive computation and to replace pen-paper calculations, has a weak to moderate positive association with Separate Knowing and Surface Approach ($r_{S'pore}=0.213$; $r_{Vic}=0.342$). Students who held a rigid conception of mathematics and use a rote learning

approach to mathematics learning are also likely to use the calculators for menial tasks.

Using a Fisher transformation to compare the correlation coefficients between the two regions, only one significant regional difference was found: between SK-SA and Cal_Ma ($z = -1.98$, $p < 0.05$). The correlation between SK-SA and Cal_Ma was stronger for the Victorian ($r = 0.519$) than the Singaporean sample ($r = 0.362$). Interestingly, scatterplots of the SK-SA and Cal_Ma scores for both regions revealed that there were higher proportions of Singaporean than Victorian students who had low SK-SA scores and high Cal_Ma scores. This group of Singaporean students did not hold Separate Knowing conceptions or use Surface Approaches, yet used their calculators as Masters. This finding was consistent with the finding reported in an earlier paper that Singaporeans were less fluent in using GCs than Victorians were in using CAS calculators (Tan & Forgasz, 2011).

It can be seen in Table 3 that the coefficients were generally comparable across gender. Analysis using a Fisher transformation resulted in only one significant gender difference: the correlation between CK-DA and Cal_Se in the Singaporean sample. For Singaporean females there was no association between CK-DA and Cal_Se, whereas for males there was a weak negative correlation. Since these correlations were absent or weak, the difference was not considered to be educationally significant.

Conclusion

In summary, students' ways of interacting with advanced calculators are found to be associated with their beliefs about mathematics and approaches to mathematics learning. The use of calculators as Master (being subservient to technology) is associated with high Separate Knowing-Surface Approach, and the use of calculators as Collaborator (high level of sophistication) is associated with high Connected Knowing-Deep Approach. These associations were common across the two regions in this study and gender. Although causality cannot be confirmed, an implication for teaching and learning is that promoting connected knowing and deep learning approaches might simultaneously help students develop more sophisticated ways of engaging with the calculators for mathematical inquiry and exploration. Conversely, teaching that promotes separate knowing and surface approaches (particularly when there is an overly strong focus on examination success) might encourage students to use calculators at a lower level, by blindly following instructions and procedures, and/or replacing pen-paper computations.

Additionally, there is a gender difference between students' ways of knowing and learning mathematics, with males higher on Connected Knowing-Deep Approach than females, and females higher on Separate Knowing-Surface Approach. Given that these beliefs and approaches are found to be associated with students' ways of engaging with the calculator and could potentially influence students' mathematics achievement, further study is needed to investigate and address the gender difference within the two regional contexts.

Acknowledgments

I am immensely grateful to my supervisor, Associate Professor Helen Forgasz, for her advice and guidance in the conduct of the research and the writing of this paper.

References

- Becker, J. R. (1995). Women's ways of knowing in mathematics. In P. Rogers, & G. Kaiser (Eds.) *Equity in mathematics education: Influences of feminism and culture* (pp. 163-174). London: Falmer Press.
- Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986). *Women's ways of knowing: The development of self, voice, and mind*. NY: Basic Books.
- Biggs, J. B. (1987). *Student approaches to learning and studying*. Melbourne: Australian Council for Educational Research.
- Burrill, G., Allison, J., Breaux, G., Kastberg, S., Leatham, K., & Sanchez, W. (2002). *Handheld graphing technology in secondary mathematics: Research findings and implications for classroom practice*. Dallas, TX: Texas Instruments.
- Crawford, K., Gordon, S., Nicholas, J., & Prosser, M. (1994). Conceptions of mathematics and how it is learned: The perspectives of students entering university. *Learning and Instruction*, 4(4), 331-345.
- Goos, M., Galbraith, P., Renshaw, P., & Geiger, V. (2000). Reshaping teacher and student roles in technology-enriched classrooms. *Mathematics Education Research Journal*, 12(3), 303-320.
- Gordon, S. (1997). Students' orientations to learning statistics – Profiles of experience. In F. Biddulph & K. Carr (Eds.), *People in mathematics education* (Proceedings of the 20th annual conference of the Mathematics Education Research Group of Australasia, pp. 192-199). Rotorua, NZ: MERGA.
- Hofer, B. K., & Pintrich, P. R. (1997). The development of epistemological theories: Beliefs about knowledge and knowing and their relation to learning. *Review of Educational Research*, 67(1), 88-140.
- Kember, D., Biggs, J., & Leung, D. Y. P. (2004). Examining the multidimensionality of approaches to learning through the development of a revised version of the Learning Process Questionnaire. *British Journal of Educational Psychology*, 74(2), 261-280.
- Kılıç D., & Sağlam, N. (2010). Investigating the effects of gender and school type on students' learning orientations. *Procedia - Social and Behavioral Sciences*, 2(2), 3378-3382.
- Marton, F., & Säljö, R. (1976). On qualitative differences in learning-I: Outcome and process. *British Journal of Educational Psychology*, 46, 4-11.
- Muis, K. R. (2004). Personal epistemology and mathematics: A critical review and synthesis of research. *Review of Educational Research*, 74(3), 317-377.
- Ocean, J. (1998). Identifying separate and connected knowing in mathematics education. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times* (Proceedings of the 21st annual conference of the Mathematics Education Research Group of Australasia, Gold Coast, pp. 428-435). Sydney: MERGA.
- Pallant, J. (2007). *SPSS survival manual: A step by step guide to data analysis using SPSS*. Sydney: Allen & Unwin.
- Roesken, B., Pepin, B., & Toerner, G. (2011). Beliefs and beyond: Affect and the teaching and learning of mathematics. *ZDM: The International Journal of Mathematics Education*, 43(4), 451-455.
- Schommer-Aikins, M., & Easter, M. (2006). Ways of knowing and epistemological beliefs: Combined effect on academic performance. *Educational Psychology*, 26(3), 411-423.
- Tan, H. (2009). Development of an instrument for ways of using graphic calculators. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides*. (Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia, pp. 531-538). Palmerston North, NZ: MERGA.
- Tan, H., & Forgasz, H. (2011). Students' ways of using handheld calculators in Singapore and Australia: Technology as master, servant, partner and extension of self. In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices* (Proceedings of the 34th annual conference of the Mathematics Education Research Group of Australasia and the Australian Association of Mathematics Teachers, Alice Springs, pp. 728-735). Adelaide: AAMT and MERGA.

Pedagogical Content Knowledge in Mathematical Modelling Instruction

Liang Soon Tan

*National Institute of Education, Nanyang
Technological University*
<liangsoon.tan@nie.edu.sg>

Keng Cheng Ang

*National Institute of Education, Nanyang
Technological University*
<kengcheng.ang@nie.edu.sg>

This paper posits that teachers' pedagogical content knowledge in mathematical modelling instruction can be demonstrated in the crafting of action plans and expected teaching and learning moves via their lesson images (Schoenfeld, 1998). It can also be developed when teachers shape appropriate teaching moves in response to students' learning actions. Such adaptive development of teachers' pedagogical content knowledge may in turn be supported by their knowledge of the mathematical modelling process and Ang's (to appear) proposed framework for planning mathematical modelling instruction.

Introduction

The enactment of mathematical modelling instruction within the context of our Singapore mathematics curriculum structure requires a major paradigm shift in teachers, who must adopt new approaches and new roles to such mathematical modelling instruction, as well as the acquisition of new knowledge. To implement such innovations in mathematical modelling instruction successfully will therefore require teachers to have the requisite knowledge in mathematical modelling instruction that provides a basis for classroom practices. To achieve such changes, teachers will need to learn for teaching mathematical modelling (Sherin, 2002; Shulman, 1986b). Such "learning for teaching" approach requires a more adaptive style of instruction in which teachers understand the connections of mathematical ideas in these tasks, and adjust their pedagogical strategies to augment the implementation of these tasks (Chazan & Ball, 1999; Lampert, 2001, Sherin, 2002).

Even though secondary mathematics teachers have access to materials and resources in the form of booklets containing collections of modelling and application activities (for example, Ang, 2009; Galbraith and Carr, 1987), the lack of knowledge in mathematical modelling instruction may inhibit novice teachers of mathematical modelling from structuring and engaging students with meaningful and effective mathematical modelling learning experiences (Shulman, 1986a; Ball, 2000).

Recently, a new framework for mathematical modelling instruction (Ang, to appear) had been proposed to guide and facilitate novice teachers of mathematical modelling in translating their modelling ideas into a sequence of modelling lessons. It seems likely that Ang's framework for mathematical modelling instruction was derived based on his pedagogical content knowledge (Shulman, 1986a) or craft knowledge in teaching mathematical modelling. Such craft knowledge comprises the accumulated and integrated set of knowledge and beliefs developed over a period of time while teaching and practising mathematical modelling.

The discussion to follow is based on the premise that the challenges faced by novice teachers of mathematical modelling in our secondary school classrooms may be due to a lack of teacher knowledge in structuring and implementing mathematical modelling learning experiences rather than to a lack of mathematical modelling instructional resources or ideas. Although much has been written about aspects of mathematics teachers' content and pedagogical content knowledge in specific mathematics topics, there has been a scarcity

of studies on the kinds of teacher knowledge required for mathematical modelling instruction.

In the subsequent sections, we present some background considerations which are necessary as a basis for what is to be discussed. Specifically, we will attempt to discuss how Ang's (to appear) proposed framework and knowledge of mathematical modelling process can be used to support the adaptive development of teachers' pedagogical content knowledge in the continuum of mathematical modelling instructional contexts.

Mathematical Modelling Process

We begin by clarifying the mathematical modelling process and how its components relate with one another. We posit that such considerations of the modelling process can influence the cognitive and metacognitive aspects of structuring mathematical modelling instructions using the proposed framework.

Essentially, the mathematical modelling process is characterized by the iterative negotiation of learning between the real and mathematical world. A typical mathematical modelling learning task traces the following trajectory.

The mathematical modelling process begins with a mathematical modelling problem motivated by practical concerns oriented in the real world. The real world problem is then formulated into a mathematical problem. The process of formulating the real world problem into a mathematical problem entails four more sub-processes which are not claimed to be very original but can be useful for understanding the model formulation process:

1. The modelling purpose is first clarified in terms of identifying the predictive, explanatory or prescriptive functions of the model. Its corresponding modelling approach would then be employed. Some of the common modelling approaches include empirical modelling, simulation modelling and deterministic modelling (Ang, 2009).
2. Other subject discipline knowledge may then be used to identify the relevant elements, relations and structures that characterize the real world situation to be modelled.
3. The necessary assumptions and conditions are then made to consider an idealized real world situation so that its mathematical representation is made more tractable.
4. The mathematical problem is then formulated by being mindful of past mathematical associations or any theoretical connections from other subject discipline that can be used to translate the relevant elements, relations and structures in the idealized real world situation into its corresponding mathematical objects, relationships and structures.

The mathematical problem is analyzed and solved using known mathematical methods. The mathematical solution is then being interpreted into some plausible real world solution. Validation of the mathematical model necessitates checking the accuracy of the mathematical solution, or relating it with observed data, or comparing it with other models and established theory for a measure of the degree of representation of the idealized situation. The reasonableness in the mathematical model will then determine if the process needs to be iterated by re-examining how the mathematical problem has been formulated.

Forms of Teachers' Knowledge

Teachers' knowledge for mathematical modelling instruction can determine to a large extent how teachers perceive and respond to curriculum innovation efforts to infuse mathematical modelling learning experiences in the secondary mathematics syllabus. To

think about knowledge of mathematical modelling instruction requires going beyond knowing the mathematical modelling process. It may require understanding the complex interplay among aspects of other forms of teacher knowledge in the mathematical modelling teaching and learning environment.

Shulman (1986b) argued that behavioural research on teaching actions can lead to overly simple prescriptions of teacher effectiveness. Shulman's (1986b) discussion of the categories and forms of teacher knowledge has provided a more comprehensive consideration of the factors influencing teaching performance. Of particular note is his introduction of the broader domain of pedagogical content knowledge that highlighted that teacher knowledge depends on more than subject matter knowledge. Shulman (1986a) represented pedagogical content knowledge as the blend of content and pedagogy for the understanding of how particular aspects of the subject matter are organized, adapted and represented to teach for understanding by considering students' conceptions, abilities and interests.

Following the work of Shulman (1986b), other researchers had examined the nature of pedagogical content knowledge in more detail (Chazan & Ball, 1999; Ball, 2000; Lampert, 2001, Sherin, 2002). Ball (2000) had deconstructed mathematical content knowledge and mathematical pedagogical content knowledge into its key components. Like Shulman (1986b), Ball (2000) argued that both content and pedagogical content knowledge are critical for effective teaching. However, such models of teachers' knowledge generally do not explain how this knowledge is used in teaching and learning situations.

Schoenfeld (1998) had developed a model of teaching in context. The model described the mechanisms in which teachers' beliefs, goals, and knowledge interact in specific teaching situations. This serves as a framework that fosters meaningful discussion about factors that may influence teachers' decisions and actions during the teaching process. The concept of lesson image can be particularly useful here. Lesson image is organized in chunks of action plans that may take the form of routine, script, mini-lecture and simple talk. Teachers' content and pedagogical content knowledge are in turn organized and accessed to frame these action plans.

Schoenfeld (1998) noted that the model worked well as an analytic tool to understand the act of teaching as it unfolded in particular contexts. These include lesson contexts where teachers had clear and un-conflicting learning goals and their lesson images and agendas were not co-constructed with students during the lesson. In this regard, we argue that the mathematical modelling lessons structured by Ang's (to appear) proposed framework may be of the types described by Schoenfeld. This point will be elaborated later when we consider the proposed framework in more details. Therefore, it is possible that Schoenfeld's model of teaching in context can contribute in part towards theoretically grounded understandings of the teaching process and development of teachers' knowledge in mathematical modelling lessons structured by Ang's (to appear) proposed framework.

Ang's proposed framework for mathematical modelling instruction is basically a set of decision procedures aimed at scaffolding novice teachers of mathematical modelling to translate their ideas into a series of modelling learning tasks (see Table 1).

Some researchers have pointed out that cognitively demanding modelling tasks need to be appropriate with respect to the kinds of students' prior knowledge in order for them to complete the tasks successfully (Galbraith, 2006; Stillman et al., 2007). Three levels of learning experiences in mathematical modelling were defined to guide teachers in pitching their modelling tasks at a level where students have the necessary skills and competencies to complete them successfully (Ang, to appear).

Specifically, level 1 tasks focus on students acquiring mathematical modelling skills that they may relate to in future modelling tasks. Level 2 tasks are essentially about guiding students in applying models known to the students to new modelling situation. This is done with the purpose of developing in students the necessary modelling competencies for meaningful engagement in the modelling process. Level 3 tasks focus on facilitating students in undertaking a mathematical modelling task where they should be ready to build models deem new to them or to modify known models.

Table 1

Framework for Planning/Designing Mathematical Modelling Learning Experience (from Ang, to appear, pg. 4)

Framework Component	Explanation
1. WHICH Level of Learning Experience?	Decide which level (Level 1, 2 or 3) of mathematical modelling learning experience that we wish to focus on.
2. WHAT is the Skill/Competency?	List all the specific skills and competencies (mathematical or modelling) that we target in this learning experience; State the problem to be solved, if applicable.
3. WHERE is the Mathematics?	Write down the mathematical concepts or formulae or equations that will be needed in this learning experience.
4. HOW to Solve the problem/model?	Prepare and provide plausible solutions to the problem identified in this learning experience.
5. WHY is this experience a success?	List factors or outcomes that can explain why this experience is considered successful and look out for them during the activity.

We note that the first two questions in the proposed framework can guide teachers in specifying appropriate and clear learning goals for the modelling tasks. Doyle (1988) argued that it is important for teachers to be cognizant of the extent to which students are explicitly expected to demonstrate understanding of the mathematics underlying the activities in which they are engaged. The third question in the proposed framework makes provision for that by asking teachers to draw explicit connections between the mathematical ideas and the modelling tasks being planned.

The fourth question encourages teachers to work out and be familiar with the solution space of the modelling task. This can help teachers in facilitating students' learning during the task implementation (Blum & Borromeo Ferri, 2009). Importantly, this may help teachers determine if the modelling task is really a good fit for the learning goals and hence the need to iterate the planning process back to the first two questions. The fifth question encourages teachers to monitor the progress of the modelling task. Such self-monitoring can increase teachers' sense of competence and control and in turn, their motivation to carry out such modelling tasks.

Structuring Mathematical Modelling Lessons

Arising from the use of this framework and integrating knowledge of mathematical modelling process and mathematical content knowledge, one could construct lesson images to structure mathematical modelling lessons. In this section, we show how two lesson

images can be constructed to structure mathematical modelling lessons based on two level 2 modelling tasks.

Task A To explain how it is possible for a detective to estimate a perpetrator's height based on the size of the footprint in a crime scene investigation situation. Teacher A1 intends to structure a level 2 task for students to apply known direct proportional relationship model to explain the novel situation, thereby developing in students the mathematical modelling competencies of model formulation and validation.

Lesson Image A1

Since the purpose of the modelling task is explanatory in nature, empirical modelling approach will be employed. After discussing Task A situation, Teacher A1 will lead students to realize that they will have to investigate if a relationship exists between a person's height and his foot length. Teacher A1 will question students on what they need to know before they can estimate a person's height based on his foot length.

Teacher A1 then question students on the types of data that needs to be collected in order to investigate the existence of the relationship between height and foot length. Teacher A1 will discuss with students factors that may affect one's height, such as genes, race, gender, nutrition, maturation etc. The discussion of such factors can then lead students to see the need for making necessary assumptions with regards to the type of data to be collected. For example, students can make the assumption that they are only examining the data from a particular type of population. At this point, teacher A1 will then provide students with the required data set from internet sources.

After plotting the data points on their graphing tools, teacher A1 will prompt students to reason their choice of linear function that will fit the observed trend of data points. Where there are students who do not get the expected direct proportional relationship for height and foot length due to some solution error, teacher A1 will ask students to check their model with some Science theory that states a taller person will need a longer foot span to maintain balance. Students will then be asked to use their model to explain how they can verify their own height based on their measured foot length. Students will also be asked to explain any deviations based on the implications of the assumptions made earlier.

Task B To determine the "best" design for the cover of a link way. Since the purpose of the modelling task is prescriptive in nature, deterministic modelling approach will be employed. Teacher B1 intends to structure a level 2 task for students to apply known quadratic function models and trigonometric ratios of acute angles to solve the design problem.

Lesson Image B1

Teacher B1 will lead students to see the need to quantify functionality of the cover in terms of its "cover effectiveness" from rain. Teacher B1 will get students to take pictures of their school's link way when it is raining. Teacher B1 will then get students to note and abstract the important features of the sheltered link way in the raining situation from their pictures and generate a similar diagram as in Figure 1. These important features will translate into relevant factors that affect the "cover effectiveness" in terms of the amount of pavement that gets wet. These factors include width of pavement (indicated as "P"), span of cover (indicated as "s"), height of support (indicated as "h") and angle of strike of rain from the

horizontal (indicated as “ θ ”). Teacher B1 will also lead students to see the need to assume these factors to be constant in order to compare the extent of cover provided by the two different designs.

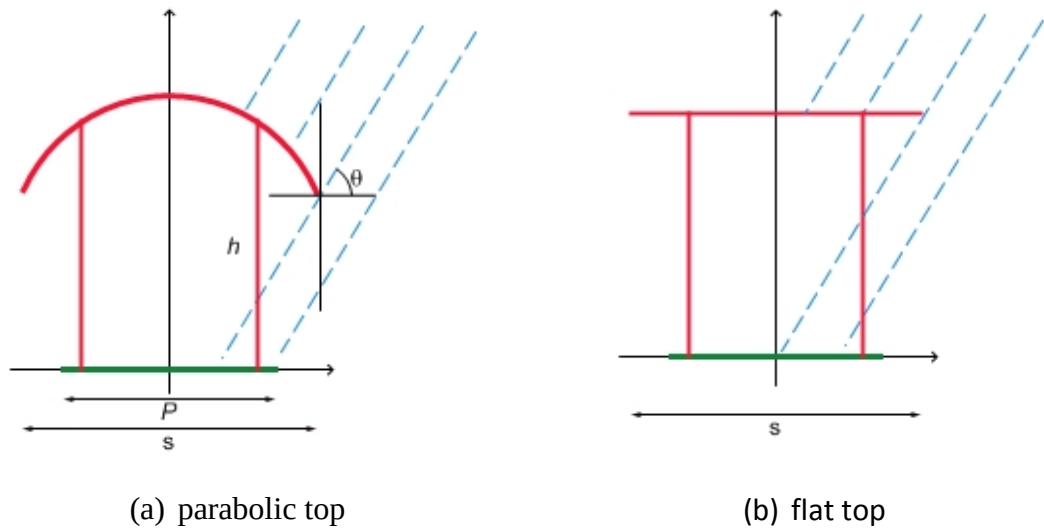


Figure 1: Diagram that shows two possible cross-sectional shapes of the cover for the walkway.

In order to facilitate students to determine the dimensions of the parabolic cross-sectional shape of the cover that will provide maximum cover from the rain, teacher B1 also worked out a plausible mathematical solution as follow:

Referring to Figure 1(a),

Let cross-sectional curve of shelter be $f(x) = ax^2 + b$

Assuming some span of cover, s , width of pavement p such that $s > p$, height of support h , and angle of strike of rain, θ , then,

Cross-sectional length of pavement and some constant length of $\frac{s}{2} - \frac{p}{2}$ that will get wet from rain =

$$\begin{aligned}
 & f\left(\frac{s}{2}\right) \tan\left(\frac{\pi}{2} - \theta\right) \\
 &= \left(\frac{as^2}{4} + b\right) \tan\left(\frac{\pi}{2} - \theta\right), \text{ where } a\left(\frac{p^2}{4}\right) + b = h \\
 &= \left(\frac{as^2}{4} + h - \frac{ap^2}{4}\right) \tan\left(\frac{\pi}{2} - \theta\right)
 \end{aligned}$$

Therefore, maximum cover occurs when

$$\left(\frac{as^2}{4} + h - \frac{ap^2}{4}\right) = 0.$$

That is, $a = \frac{4h}{s^2 - p^2}$ and $b = \frac{-hs^2}{p^2 - s^2}$.

Use of Lesson Images to Develop Teachers' Pedagogical Content Knowledge

We note that teacher A1 has integrated his knowledge of mathematical modelling process to a large extent in using the proposed framework to structure and craft his lesson image. This lesson image therefore holds the potential for students to meaningfully experience the mathematical modelling process. This adaptive style of teaching may also develop new pedagogical content knowledge in teachers implementing the lessons as they shape appropriate teaching moves in response to their interpretations of novel student learning actions (Chazan & Ball, 1999; Sherin, 2002).

On the other hand, the lack of deep understanding of the mathematical modelling process may limit teachers in using the proposed framework to structure the lesson in a way that provides students only with procedural understanding of the mathematical modelling process. Teachers are also likely to evoke associated familiar or fixed teaching and learning moves in response to unexpected students' learning actions, hence hindering the development of their pedagogical content knowledge.

In addition to integrating his knowledge of mathematical modelling process in structuring the lesson, teacher B1 has also rigorously applied the framework to structure his lesson. By not working out plausible mathematical solution to the design problem as teacher B1 has done, teachers may encounter conflicting learning goals and hence a coherent sequence of modelling process may not be experienced by the students. For example, the modelling problem may be defined as designing a more functional sheltered walkway, but the latter part of the lesson may have been planned for students to be engaged in subjective ranking of various images of shelters based on some criteria which may not be quantifiable.

The proposed framework is in fact currently being tested in a trial with several secondary schools in Singapore. Teacher participants have been introduced to the proposed framework and given some training in using it to develop modelling lessons. Data, in the form of artifacts such as teachers' notes and documentation, students' sample work and videos of lesson implementations, are currently being collected. Further analysis of the match between teachers' lesson images and the actual lesson implementations will be carried out to examine these teachers' development of pedagogical content knowledge in mathematical modelling instruction. Results of this trial are not available as yet but will be presented in future reports.

Nonetheless, through this discussion, we have elucidated the importance of structuring mathematical modelling lessons via teachers' lesson images, augmented by Ang's proposed framework and knowledge of the mathematical modelling process. Such lesson images can form an important starting point to the adaptive development of teachers' pedagogical content knowledge in the continuum of mathematical modelling instructional contexts.

References

- Ang, K.C. (2009). *Mathematical modelling in the secondary and Junior College classroom*. Singapore: Prentice Hall.
- Ang, K.C. (in press). Mathematical Modelling in Singapore schools: A framework for Instruction. In N. H. Lee, & K. E. D. Ng (Eds.), *Mathematical Modelling – From Theory to Practice*. Singapore: World Scientific.
- Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, 51(3), 241-247.
- Blum, W. and Borromeo Ferri, R. (2009). Mathematical Modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Applications*, 1(1), 45-58.
- Chazan, D. & Ball, D. L. (1999). Beyond being told not to tell. *For the learning of Mathematics*, 19(2), 2-10.

- Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction, *Educational Psychologist*, 23, 167-180.
- Galbraith, P. (2006). *Real World Problems: Developing Principles of Design*. Paper presented at the 29th annual conference of the Mathematics Education Research Group of Australasia, (RP242006).
- Galbraith, P. & Carr, A. (1987). *Practical Applications of Mathematics – Proposal for an Australian Curriculum Project*, Churchill, Victoria.
- Lampert, M. (2001). *Teaching problems and problems of teaching*. New Haven, CT: Yale University Press.
- Sherin, M. G. (2002). When teaching becomes learning, *Cognition and Instruction*, 20(2), 119-150.
- Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context, *Issues in Education*, 4, 1-94.
- Shulman, L. S. (1986a). Paradigms and research programs in the study of teaching: a contemporary perspective. In M. C. Witrock (Eds.), *Third Handbook of Research on Teaching* (pp. 3-36). New York, Macmillan.
- Shulman, L. S. (1986b). Those who understand: knowledge growth in teaching, *Educational Researcher*, 15(2), 4-14.
- Stillman, G., Galbraith, P., Brown, J. and Edwards, I. (2007). *A Framework for Success in Implementing Mathematical Modelling in the secondary Classroom*. Paper presented at the 30th annual conference of the Mathematics Education Research Group of Australasia, (Proceedings Vol. 2, 688-697).

Developing Mathematical Knowledge through Social Justice Pedagogy with Young Adult Arab Women

Mohammed Goma Tanko
Higher Colleges of Technology, UAE
<gtanko@hct.ac.ae>

Bill Atweh
Curtin University - Australia
<batweh@curtin.edu.au>

The study involved a group of Middle Eastern Muslim women learning mathematics through social justice pedagogy. The findings suggest that the involvement of students in social justice issues has not lead into a decline in their opportunity to learn mathematical content as set in the course. However, this approach has helped them to develop significant mathematical knowledge beyond the basic numeracy skills expected of them. In addition, this study demonstrated how a focus on mathematical content can be maintained while students are involved in social issues in mathematics learning.

In many school curricula around the world, mathematics is privileged for its perceived importance both for society and the individual student. The social importance of quality mathematical knowledge is often justified due to its role in technology and science which is seen as essential for the economic development of a society and its standard of living. At the personal level of the student, mathematics is seen as opening the doors to many careers and courses of further study. However, as Atweh and Brady (2009) cautioned, both reasons should not be taken uncritically. Borrowing the terminology from Down, Ditchburn and Lee (2007), Atweh and Brady added that it is possible to conceive the importance of mathematics for the student can be understood at three levels. Mathematics education can contribute to the students' *conforming*, *reforming* or *transforming* society. This understanding of the role of mathematics education is consistent with the critical mathematics movement (Skovsmose, 1994). Using Freire's terminology, Gutstein (2006) argues that developing mathematical knowledge helps students to not only "read the world", i.e., understand it, but it should lay the foundation for their capacity to "write the world", i.e., change it.

This paper discusses how mathematics learning of a group of Arab women was facilitated by using a social justice pedagogy based on Gutstein's model. Gutstein's framework of teaching mathematics for social justice identified two sets of goals: Mathematics goals and social justice goals. The mathematics goals are: To read the mathematical world; to succeed academically in the traditional sense; and to change his or her orientation to mathematics. The social justice goals are: To read the world with mathematics; to write the world with mathematics; and, to develop positive cultural and social identities. This paper addresses only 2 elements of the mathematical goals, namely, "to succeed academically in the traditional sense" and "change ones orientation towards mathematic".

Theoretical Background

The social justice approach adopted here is based on the work of Gutstein and his colleagues (Gutstein, 2001, 2006; Gutstein & Peterson, 2005) in a Chicago school in the USA which argues that engaging students in mathematics within a social justice context increases students' interest in mathematics and also helps them learn important mathematics. A number of researchers investigating issues similar to the one presented in this paper have documented how students' mathematical knowledge and understanding could be enhanced through social justice teaching. For example, Gutstein (2001)

documented how middle school students developed their understanding of the concept of “area” by examining the distortion of the world map projection used in their school (typically, Mercator’s) by enlarging Europe and Greenland and shrinking the African continent. Turner (2003) described how urban middle school students developed their understanding of measurement and ratios by investigating the level of crowdedness of their school in comparison to others in the district. More recently, Tracey (2007) documented how students in a Maths Club used mathematics as a tool to fight against a plan to close their school by considering the costs of bussing them to the new school and the subsequent overcrowding and adverse learning conditions that may result at the receiving school.

Researchers have also documented the challenges associated with teaching mathematics for social justice. For example, Taysum and Gunter (2008) said “it is arguably dangerous to attempt to work for social justice in the school when the writers [Leaders] of policy tests are people far removed from the messiness of the day to day realities of the people whose identities they are shaping” (p. 187). Gutstein and Peterson (2005) argued that social justice teaching cannot be easily done when using a rote, procedure-oriented mathematics curriculum. A test-driven, teacher centred approach does not foster the kind of teaching environment necessary for teaching for social justice.

Jacobsen and Mistele (2010), in their investigation of the challenges preservice teacher face with teaching for social justice concluded that the major challenge was what they called the “problem of balancing” (p. 8). In other words, the problem of how to get a balance between the mathematical concepts and the social justice issues addressed in given tasks. They provided 4 possible manifestations of the “problem of balance” (1) the use of mathematics without mathematics instruction by the teacher (2) use of traditional methods of teaching and/or nonchallenging mathematics (3) trivializing of social issues, and (4) disconnect or artificial connections between social issues and the mathematics.

More recently, Atweh (2012) in a response to a chapter by Dowling and Burke (2012) argues that a reading of the world necessarily is subjected to the simplification of the phenomenon being read and the simplification of the mathematics used to read it (based on the level of knowledge and resources available to the student). Hence, a prerequisite for a productive reading of the world through mathematics is an awareness of the limitations and assumptions one makes about the mathematics, the world, and the viewer. Atweh concludes by saying “Under the right conditions, I would like still to hope that one can be to be a good mathematics teacher and political activist at the same time” (p. 110).

The Context

This research was carried out in one of 17 HCT colleges around the United Arab Emirates (UAE). At the time of this study, at entry, students with reasonable Common Educational Proficiency Assessment (CEPA) scores in English and Mathematics are admitted into the Higher Diploma Foundation Program (HDF) and the remainder into the Diploma Foundation Program (DF). Upon passing the DF, students can commence their HDF courses. In other words, DF students are perceived, by the system, as mathematically weak, and the course aims to develop rather basic numeracy skills in order to prepare students for higher studies.

Further, contextual information about the College that is relevant to the content of the projects developed by the students in this study, is that more than 50% of the students came from outside the city of Abu Dhabi. The college provides bus services, at a subsidised rate, to students who require the service. Some students are driven, or drive themselves to

college. Those who drive themselves are mostly part-time working students. The college has student parking area and staff parking area within its premises.

The mathematics course in which the participants were involved in consists of 8 modules: Whole numbers; time; fractions and decimals; measurement; fractions, decimals and percentage; ratio and proportion; time calculations; and tables, graphs and formulas. The research presented in this paper focused on 3 content areas, namely: Percentage, time calculation and graphs. Experience from previous years shows that these topics remain a particular challenge for many students undergoing a traditional teaching approach. They are also useful topics to deal with many demands of the daily life.

Methodology

This study involved a group of 20 women participants, from the age of 16-36 years in the DF Mathematics Course. The sample was purposefully chosen (Creswell 2005) because they were taught by the first author and, relative to other classes, the majority had a reasonable command of both written and spoken English. Triangulation (Stake, 2003) was achieved through multiple sources of data including (1) student presentations (2) participants' test results from the end of the module (3) participants' reflective questionnaires, and (4) focus groups. Grounded theory approach was used to analyse the data. In particular, the approach discussed by Strauss and Corbin (1998) was utilised to organise and interpret the raw data in this research.

Procedures

At the start of the project, students engaged in a brainstorming session to identify issues related to social justice of concern to them to investigate using mathematics resulting in three projects: Time of Travel, Career Aspirations and Car Parking.

The Time of Travel group was concerned with the issue whether the means of transport provided by the college could be improved to meet the needs of a number of students who lived outside the city. Each student in the class recorded their travel times to and from college for a week. This data formed the basis of learning mathematics as well as investigating the challenges some students faced in their travels to and from College. The Car Parking was concerned with the allocation of parking at the College to see if it was fairly distributed between staff and students. To obtain their data, they measured the dimensions of all car parks and worked out how many cars could park at any given time. The Career Aspirations group used mathematics to investigate needed action to increase the level of information available to students with regards to available career opportunities. They constructed a questionnaire which was completed by 7 classes out of the 9 classes in DF Semester 2 at the college for their data collection.

Five hours every week for the duration of 6 weeks were devoted to working and reporting on the projects. During the early stages of the projects, it was noted that the group discussion tended to focusing less on the mathematics involved and more on the investigated social issues. This is the "problem of balance" identified by Jacobsen and Misteale (2010) above. In seeking to meet this challenge, the first author introduced the Mathematical Enriched Worksheets to focus the students on the mathematics. A separate sheet was developed to consider the data from each of the 3 projects. All students in the class attempted all 3 worksheets; first, individually in class and at home; then, they discussed their results within their project groups. Finally, the project group working on the topic presented their answers to the relevant Mathematical Enriched Worksheet to the whole class.

Results

Overall Performance of the Students

One common concern by many teachers about engaging students in project work is that the time “wasted” on project may prohibit “covering the curriculum” and hence disadvantage students’ performance on set tests. Hence, it is useful to compare the performance of these students with other groups also taught by the first author following the traditional pedagogies in the course. At the end of each Module, all the DF students would take the same pen and paper end of module examination. The examination consisted of questions contributed by the teachers who taught the course. Table 1 shows the result of the participating class (Class A) compared to other classes. Sections A, B and C were taught by the first author but only Section A students were exposed to teaching for social justice.

Table 1
Summary of Test Results (Class A followed the Social Justice Pedagogy)

Section	Percentage				Time Calculations				Graphs			
	A	B	C	D	A	B	C	D	A	B	C	D
Student Number	20	19	20	18	20	19	20	18	20	19	20	18
Mean	72	76	68	81	70	74	64	74	78	69	62	71
STDEV	8	15	18	12	7	11	13	9	10	12	18	12

Two observations stand out from the Table 1. First, the performance on class A was the highest of all classed on the Graphs test, about average on the Time Calculations test and slightly lower than average on the Percentage test. However, over all, not a single student in Class A has failed the tests compared to 3, 6 and 1 failures from Classes B, C and D respectively. It is also interesting to note that these tests recorded the highest failure rate of all modules, indicating that students in general found this content to be ones of the hardest to master. Also, arguably, these modules required students to demonstrate understanding of the content as against recall of procedures and formulas which constitute the content of the other modules. Hence, overall this approach to teaching mathematics did not disadvantage students on their performance on the set tests. We can only conjecture that had the context of the questions on these tests been related to the context of the projects (thus creating a synergy between pedagogy and assessment) the performance of the students in Class A might even have been higher.

However, the second interesting observation from the above data is that on all tests, Class A demonstrated the lowest standard deviation. This lower variation means that the gaps in knowledge between the top and bottom students are less pronounced in this class. Therefore, rather than learning in this class being individualistic, it was more communal.

However, other observations from the data show that there are significant benefits in students learning as a result of these activities. These will be outlined below.

Deep Mathematical Knowledge

As discussed above, the course consisted of foundational level skills expected of students to enter the Diploma courses. With this in mind, these projects provided students with activities that were beyond the level of difficulty represented in the workbooks adopted in

the course. For example, the students involved in the Car Parking project designed their own sketch of the car parks at the College and successfully calculated their areas (see Figure(1)). Not only have students not studied this content in their class, but it is also much more complex than the simple calculations expected of students. Of additional interest is that the students obtained the necessary formulas on their own. One student asserted that “this project has made us to search for information about area of shapes and drawing that we didn’t know before”.

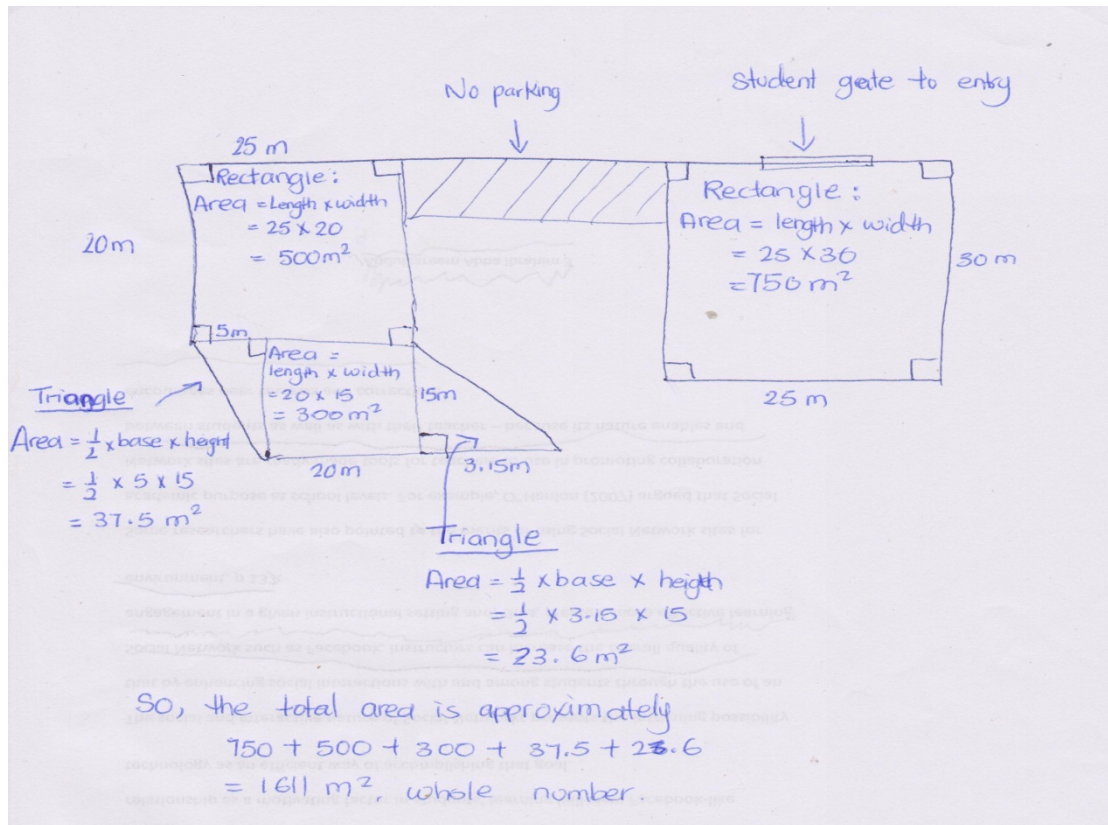


Figure 1. A sketch of the students’ car parking area

The Time of Travel Mathematical Enriched Worksheet contained the following question.

A journey on a bus from city A to city B will normally last for two hours, and every passenger pays Dh15 for this journey. Use this information to calculate how much it would have cost you to travel to college using public bus. From your answer, do you think those of you who travels by bus to college are getting a good deal? Explain your answer fully.

To arrive at the answer, they had to multiply the cost (15) by their total hours per week, and then divide the answer by 120. They then multiplied the answer by the total number of weeks in a semester. The question involved multiple skills and steps; hence, it was more complex than the direct questions that the students are used to in their modular workbooks.

Shaikha spends approximately 2 hours on the bus every week. Below is her response to the question.

$$120n = 15 \times 533$$

$$n = (15 \times 533) \div 120 = \text{Dh } 66.63 \text{ per week}$$

One semester is about 20 weeks, so I multiplied the answer by 20,
 $66.63 \times 20 = \text{Dh } 1332.60$, one semester.

It is cheap[er] for me to travel by public transport, only Dh1332.60 in one semester. But, I pay Dh1900 to travel in the [school] bus.

However, significantly, Shaikha adds “I still take the college bus because it is better than public because I travel with my friends and is safer and I don’t worry when I am late to college because we are many”.

One very interesting aspect of Shaikha’s work is that it reminds us of one of the limitations of traditional mathematical problems in which the mathematics produces the “correct” answer to real world problems. This is far from authentic problems where mathematics can assist in decision making, but the adopted solution may not be mathematical at all.

However, not all students demonstrated the ability to make sense of the problem in terms of the real world situation that they were investigating. One of the questions on the Time of Travel Worksheet was:

Find out how many students come from outside the city in your class.

For this academic year (2008-2009), there are approximately 2500 students registered at the College. Based on your answer above, calculate the percentage of students you would expect to come from the city.

For many students this is a “trick” question. While the first part involves number of students living outside the city, the second part involved calculations of the number of students from the city itself. Sumaya seems to have fallen into this trap; her solution was:

Number of students from outside Abu Dhabi = 11

Number of students from Abu Dhabi at ADWC = $(11 \div 20) \times 2500 = 1375$ students

So, percentage = $(1375 \div 2500) \times 100 = 55\%$

Arguably, she did not read the question with much care even though her knowledge and skill in percentages was appropriate. Furthermore, this reasoning was not isolated; a few of the students gave the same answer. Of interest here is to speculate as to why this misreading has occurred. While the worksheets used in this study were provided to assure a focus on the mathematical knowledge and skills, for some of the students who were not involved in the Time to Travel group, the exercises in them were not necessarily as meaningful. They did not generate the questions themselves. Hence these questions may not have been different to the usual problems that appear in textbooks – where shallow reading is not uncommon.

Finally, interviews and reflective questionnaires with the students at the conclusion of the study show that the majority of students have gained confidence in their mathematical understanding and skills. One of the questions on the reflective questionnaire was “*what do you like or not like about the mathematics projects you have completed this semester?*” One of the participants Bedour wrote:

I now see math as useful in my life. I didn’t like math before but now I go to class early no late again. I can calculate percentage in fun way also I can add and subtract time. Before is all hard to understand.

Another participant Afra, wrote “I understand math better and I want to do math all the time”.

Discussion and Conclusions

The findings discussed above suggest that the involvement of students in the social justice pedagogy implemented as projects chosen by the students, assured there was no decline in their opportunity to learn mathematical content as set in the course. It has helped them develop significant mathematical knowledge beyond the basic numeracy skills

expected of them. However, the findings also show that by developing mathematics within real world contexts that are meaningful to them, mathematics has become meaningful to them, and they have developed confidence in its learning. We conclude this paper by a reflection on its design and findings using the four challenges to a social justice approach to teaching mathematics identified by Jacobsen and Mistele (2010) and discussed above.

Mathematics without Mathematics Instruction

The use of student selected projects necessarily raises a dilemma for teachers. On one hand, they need to avoid taking the responsibility away from the students for decisions as to what mathematics is needed to investigate the selected social phenomenon; and on the other hand abdicating their accountability as a teacher responsible for meeting the set outcomes of the subject. In this project the students decided what kind of data was needed to investigate the issues they had selected. They may not have been consciously aware of the particular content of mathematics required in the module. However, this was important part of the teachers concern. The lengthy discussion within the groups and in consultation with the teacher and with the use of the mathematics enriched worksheets, it was possible to achieve both challenges above. While the students were completing the worksheets, the teacher partly assumed the traditional role of a teacher in checking students work and scaffolding their mathematical understanding. Hence, scaffolding the students is possible without taking away their sense of agency in their design and implementation of the project. Our suggestion is in line with Gutstein's (2006) dictum that in the absence of concrete direction and guidance in the mathematics content, one does not easily become a "reader" of the world using mathematics.

Non-challenging Mathematics

A related dilemma which presents itself for teachers using social justice projects in their teaching is the balance between the mathematics needed to investigate the phenomenon (which may be considered as basic level as far as the curriculum is concerned) and the level of mathematics expected in the formal curriculum. Once again, the teacher in this particular class opted to supplement the mathematics actually used by the students to investigate their phenomena with questions that highlighted the mathematics in the curriculum. However, as we argued above, these worksheets were still based on the actual data that the students have generated, hence, may have contributed to keeping the mathematics meaningful for the students.

Trivializing of Social Issues

Arguably, some of the social issues investigated in these projects, e.g., parking and cost of travel, may not be seen as significant social justice issues for the students and their society. However, they were selected by the students themselves. They are meaningful to them in their particular context.

Disconnect or Artificial Connections between the Mathematics and the Social Issues

The last challenge identified by Jacobsen and Mistele is the tendency of teachers raising questions that have marginal connection to social issues in real life - that is using questions that may have some mathematical value but no one really needs to answer in real life. In a real classroom context, perhaps this is inevitable. Particularly when different groups of students are working on social justice issues that they select, a challenge for the teacher is to

assure that all students would have a chance to experience all the mathematics content covered in the curriculum. Hence, it may be unavoidable at times to raise some mathematical questions that may be artificial in that particular context of the social issue. In this particular study, it was decided that all students attempt all worksheets including those based on the data of the other groups.

Finally, we note that this study has made a contribution to the limited, yet expanding empirical literature on a social justice approach to mathematics education by investigating its use in mathematics bridging course in higher education as well as in the context of traditional cultures such as the Middle East.

References

- Atweh, B. (2012). Commentary on the chapter by Paul Dowling and Jeremy Burke, "Shall we do politics or learn some maths today? Representing and interrogating social inequality". In H. Forgasz & F. Rivera (Eds.), *Towards equity in mathematics education: Gender, culture and diversity* (pp. 105-110). Dordrecht: The Netherlands, Springer.
- Atweh, B., & Brady, K. (2009). Socially response-able mathematics education: Implications of an ethical approach. *Eurasia Journal of Mathematics, Science and Technology Education*, 5(3), 267-276
- Charmaz, K. (2008). Grounded theory in the 21st century. In N. K. Denzin & Y.S. Lincoln (Eds.), *Strategies of qualitative inquiry* (3rd ed.). Thousand Oaks, CA: Sage Publications.
- Creswell, J.W. (2005). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research* (2nd ed.). Upper Saddle River, NJ: Pearson Education.
- Dowling, P. & Burke, J. (2012). "Shall we do politics or learn some maths today? Representing and interrogating social inequality. In H. Forgasz & F. Rivera (Eds.), *Towards equity in mathematics education: Gender, culture and diversity* (pp. 87-103). Dordrecht: The Netherlands, Springer.
- Down, B., Ditchburn, G., & Lee, L. (2008) *Teachers' ideological discourses and the enactment of citizenship education*. *Curriculum Perspectives*, 28 (2). pp. 1-27
- Gutstein, E. (2001). Math, maps, and misrepresentation. *Rethinking Schools: An Urban Educational Journal*, 15(3), 6-7
- Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*. New York, NY: Routledge.
- Gutstein, E., & Peterson, B. (2005). *Rethinking mathematics: Teaching social justice by the numbers*. Milwaukee, WI: Rethinking Schools.
- Jacobsen, L. J., & Misteale, M. J. (2010). Please don't do "connect the dots": Mathematics lessons with social issues. *Science Education and Civic Engagement*, 2(2), Summer 2010.
- Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Dordrecht, The Netherlands: Kluwer.
- Strauss, A., & Corbin, J. (1998). *Basics of qualitative research* (2nd ed.). Thousand Oaks: Sage.
- Taysum, A., & Gunter, H. (2008). A critical approach to researching social justice and school leadership in England. *Education, Citizenship and Social Justice*, 3, 183.
- Tracey, K. (2007). *The impact of human resources capital on ethnic minority middle school students, engagement in a mathematics community of practice and their mathematics identity*. PhD Thesis, University of North Carolina at Greensboro, USA
- Turner, E. E. (2003). *Critical mathematical agency: Urban middle school students engage in significant mathematics to understand, critique, and act upon their world*. Unpublished Doctoral Dissertation.
- Stake, R. E. (2003). Case studied. In N. K. Denzin & Y. S. Lincoln (Eds.), *Strategies of qualitative inquiry* (2nd Ed., pp. 133-164). Thousand Oaks, CA: Sage.

Developing Mathematical Resilience among Aboriginal Students

Steve Thornton

Charles Darwin University

<stephen.thornton@cdu.edu.au>

Joanne Statton

Alberton Primary School

<joanne.statton@albertonps.sa.edu.au>

Sophie Mountzouris

Alberton Primary School

<sophie.mountzouris@albertonps.sa.edu.au>

The processes of mathematisation, the use of mathematical models and representations of real world contexts, and contextualisation, the embedding of mathematical ideas into a meaningful context, are key aspects of students' mathematical learning. We present a conceptual framework for thinking about mathematising and contextualising developed as part of the *Make it Count*, a national project that seeks to develop an evidence base of practices that improve Indigenous students' learning in mathematics. We suggest that an intentional focus on mathematisation and contextualisation helps to make mathematics meaningful, particularly for Indigenous students. In particular we suggest that such a focus has the potential to enhance the mathematical resilience of Aboriginal students.

Much of the present teaching of mathematics, particularly in the primary years, has Aboriginal students doing mathematics that is not related to their world and their everyday experiences. As a result, by the time many Aboriginal students have reached the latter years of primary school they have been alienated from mathematics. (Matthews, Howard & Perry, 2003, pp. 12, 13)

Make it Count is a national project, conducted by the Australian Association of Mathematics Teachers (AAMT) Inc., that seeks to develop an evidence base of practices to improve the learning outcomes of Indigenous students in mathematics. It is part of the Australian Government's 'Closing the gap – expansion of intensive literacy and numeracy programs initiative'. This initiative seeks to halve the gap in reading, writing and numeracy achievements for children within a decade (Commonwealth of Australia, 2009). The project has established eight clusters of schools across Australia, supported by AAMT, critical friends and consultants, to develop culturally responsive mathematics pedagogy that will both engage Indigenous students and contribute to improved outcomes.

Specifically *Make it Count* aims to:

- document and share effective models of teacher professional development, whole school change and community engagement in relation to mathematics and numeracy;
- develop whole school approaches to mathematics and numeracy that result in markedly improved achievement by Indigenous students;
- build and participate in networks and professional learning communities; and
- act as catalyst and support for action by others.

In the Alberton cluster of schools in SA teachers have attempted to develop culturally responsive mathematics pedagogy by focusing on the value of context in the learning of mathematics. In this paper we discuss the theoretical framework that has been developed by teachers in the schools, and particularly on the role of mathematisation and contextualisation in planning effective mathematics learning for Indigenous students. We report on the project as it has unfolded at the schools in the cluster. Rather than presenting this as a traditional report structured as literature, research question, methodology and results, we allow the narrative to unfold. This reflects the emergent and ongoing nature of the project. We use Sophie's reflections as a stimulus to show the emergence of a research question that focuses

on the potential of mathematisation and contextualisation for enhancing the mathematical resilience of Indigenous students.

The Alberton Cluster of Schools

The Alberton cluster of schools consists of three schools, Alberton Primary School, Northfield Primary School and Ocean View Birth to 12 College, each of which has a significant proportion of Aboriginal¹ students. The cluster is located to the North and West of Adelaide in suburbs that were typically populated by blue-collar workers with a high proportion of migrants and Indigenous people. Like many dockyard and near-city suburbs they are undergoing rejuvenation, with a growing sense of community pride and identity. This sense of pride is reflected in the schools, where students and teachers are generally engaged and enthusiastic.

It is important to note that the majority of the Aboriginal students at these schools speak English as a first language and live in an urban environment. While these students belong to families that are deeply embedded in the everyday social and economic fabric of second order Australia (Watson, 1988), they nonetheless face significant challenges. They are caught between two cultures, on the one hand maintaining the strong sense of group and family identity and associated practices such as communal sharing and collaborative success, and on the other hand being immersed in a culture that values individual achievement through material success (Malin, 1994). As Watson (1988) says “the particular experience of biculturalism that this group knows is the most important characteristic of the form of life of this category of Aboriginal people to consider, when we come to mathematics education” (p.260). Yet there is little research related to the mathematics education of urban Aboriginal children, and almost none that deals with non-cognitive aspects of mathematics learning.

A Purposeful Approach to Mathematisation and Contextualisation

There is a considerable body of literature that suggests that the use of meaningful and relevant contexts may be important for Aboriginal students learning mathematics. Frigo (1999) suggests that “(c)ontextualising mathematics ... means finding ways of providing experiences and strategies in which students can gain meaning and develop the appropriate language that enables them to extend their skills in Western mathematics” (p.13). Although recognising that contextualising mathematics for Indigenous students is not straightforward and involves the overturning of power relations and tacitly held beliefs about Indigenous students, Matthews, Watego, Cooper and Baturo (2005) suggest that it “has the potential to change the educational environment so that Indigenous cultures and their way of knowing are valued rather than devalued and that Indigenous students have pride in their culture and believe that they can perform well in the education system” (p.519).

However Brown (2008) suggests that context alone may not be enough.

Further research is required that specifically targets three things: Indigenous students’ perceived value of mathematics, their conceptualisations of the usefulness of their school mathematics beyond the classroom, and the ways in which their underachievement can be improved. Insufficient research has been performed to reveal new or alternative mathematics teaching strategies that can contextualise mathematics for Indigenous students. Contextualising the mathematics curriculum

¹ We use the term Aboriginal when discussing the students attending these schools, as they identify as Aboriginal rather than more generically as Indigenous.

would be an effective means to significantly improve both Indigenous and non-Indigenous students' understandings of mathematics. (Brown, 2008, p.93)

This caution is echoed by Sullivan, Zevenbergen and Mousley (2003) who suggest that contexts, both mathematical and pedagogical, may exclude some children. They argue that the mathematical context is often artificial, alienating, or excluding and always requires recontextualisation to draw out the mathematics, and that the pedagogical context must be made explicit to enable children to understand the purpose and make sense of the mathematics. It seems that elements such as social justice, empowerment, engagement, reconciliation, self-determination, connectedness and relevance (Matthews, Howard & Perry, 2003) need to work together to ensure that learning mathematics is meaningful for Indigenous students.

The Alberton cluster approach seeks to link mathematics and context through the deliberate acts of mathematisation and contextualisation. We developed a model of teaching and learning (Figure 1) that would be more nuanced and powerful than merely suggesting that teachers should use real-life examples to teach mathematics. In making use of this model we described contextualisation as the shift from mathematics to context, a process in which mathematical ideas are deliberately embedded in everyday contexts. These contexts then provide a springboard for the learning of mathematical ideas. Contextualisation thus provides a strong sense of purpose, in that it has a meaningful learning outcome for students in terms of the mathematical understanding developed (Ainley, Pratt & Hansen, 2006). Mathematisation is then the reverse process in which everyday contexts are expressed mathematically, and mathematics becomes the vehicle for solving real problems. This process shows the utility of mathematics, in that it is used to make sense of and solve a meaningful task (Ainley, Pratt & Hansen, 2006).

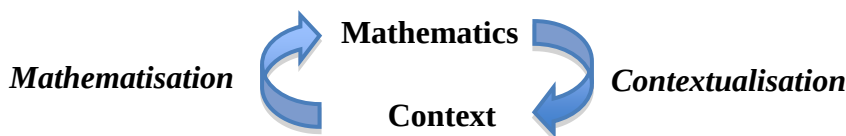


Figure 1. The Alberton cluster model of mathematisation and contextualisation

The deliberate acts of mathematisation and contextualisation described above also address the oft-criticised use of pseudo-contexts in mathematics (Olive, Makar, Hoyos, Kor, Kosheleva, & Sträßer, 2010). Rather than presenting students with artificial and contrived problems typically found in school textbooks, the deliberate acts of contextualisation and mathematisation enable teachers to ensure that students solve rich mathematical problems with an authentic purpose. In this way they closely align with the principles espoused by the Dutch Realistic Mathematics Education program, in which students are guided through the process of constructing mathematical tools and principles in meaningful contexts (Van den Heuvel-Panhuizen, 2003).

The model illustrated in Figure 1 evolved after almost two years of discussion and reflection on teaching practices among the teachers at the Cluster schools. The project was stimulated by the question asked by one Aboriginal student of her teacher, Sophie, "Why can't we draw all day?" Teachers were then challenged to think about how they might make the teaching and learning of literacy and numeracy more meaningful by embedding it in contexts such as art, design, technology, sport or enterprise. Students in years 3, 4 and 5 were then offered the choice of learning contexts, providing, at a school level, empowerment, engagement, self-determination, connectedness and relevance (Matthews, Howard & Perry, 2003). The *Make it Count* project then provided the means and stimulus to

develop a model of teaching that connects mathematics and context in a purposeful and meaningful way. For Sophie, who is also studying architecture outside the school environment, this led to the establishment of the “Deadly Design” studio.

Sophie and Debra²: An Indicative Case Study³

The Deadly Design studio has been a place of incredible change in the last three years. It is a shared space, in which I team-teach with Danielle. Powerful changes in attitude and learning have been experienced both by students and by me. To illustrate this I would like to share my educational journey with an Aboriginal student called Debra from whom I learnt a great deal about my own teaching practices.

Debra is a smart and precocious year four student who came to the Deadly Design studio learning environment with a range of behaviours and attitudes that were productive and some that were clearly counterproductive to acquiring the mathematical knowledge skills needed to perform well in a range of mathematical situations. Debra has had significant support in both English and mathematics, however has continued to underperform, especially in mathematics. To Debra’s credit with all of the difficulties she has experienced she has continued to attend and participate in the curriculum on offer.

At the start of the year, prior to the commencement of the Deadly Design studio, she seemed disconnected from learning. She reacted negatively to the shock of a new set of circumstances, such as a new space or a relief teacher, often challenging staff or failing to follow organisational structures. She became loud and often created conflict with other students. She would respond to learning tasks by stating “I’m not doing this!” and would sit in her seat distracting the students around her. When asked to complete her task she would often call out “I can’t do it!” and chat about unrelated matters. Even when she tried to start a task Debra would quickly say “I need help!” Debra rarely took responsibility for her learning. Aside from calling out for help Debra rarely put her hand up during group discussions. She spent a great deal of the time either sitting quietly or distracting others.

On one occasion Debra was working with a relief teacher in Danielle’s space. She had an argument with another student and spent a great deal of the lesson crying. Mid way through the lesson she gave up being responsible and she was about to be sent to buddy class. I decided to intervene and asked her to come and work on my side of the design space. She obliged quite happily as I have a strong connection with her, having taught three generations of aunties and uncles and brothers and sisters within her family. All of their “Sophie” stories are positive. I believe that coupled with the contextual learning this relational connection had a profound impact on Debra, and created a place where she had the space and confidence to learn. We were connected on a personal level, and also connected to our environment through the use of context as an entry point to learning. This connection is in contrast to the usual separation of teacher and student and the imposition of performance based criteria which would have only fuelled Debra’s resistance by reinforcing what she did not know.

In the ensuing lesson on measurement I asked her if like Aunty Alicia she had used a tape measure. She replied that she had and proceeded to share an occasion when this had occurred. I also knew that Alicia was studying to gain a horticultural certificate, giving me a direct hook into the ANZAC garden project we were doing. I asked Debra if she would like to investigate how our garden could include Aboriginal plants and which ones were the

² Names of students are pseudonyms.

³ This section is written by Sophie in the first person as her reflections of the program.

most appropriate. She got very excited by this and was extremely keen to get home and start the conversation with her family. She returned with a list of names and is in the process of inquiring about prices and quantities needed to plant out the garden.

I believe that the conscious mathematisation and contextualisation used in planning has led me to deliberately break with the content-based approach I used previously. A linear, hierarchical curriculum tends to impose a teaching methodology that always starts with the mathematics and seldom makes connections with children's lived realities. On the other hand the Alberton cluster *Make it Count* model encourages both relational and contextual connections. Along with the other teachers in the project I have developed an increasingly complex planning approach that sustains mathematical skills for the long term and allows for multiple entry points into the mathematics curriculum, particularly for Debra and the other Aboriginal students in my group. Figure 2 shows how planning the ANZAC garden provided the context for learning about concepts of measurement and place value.

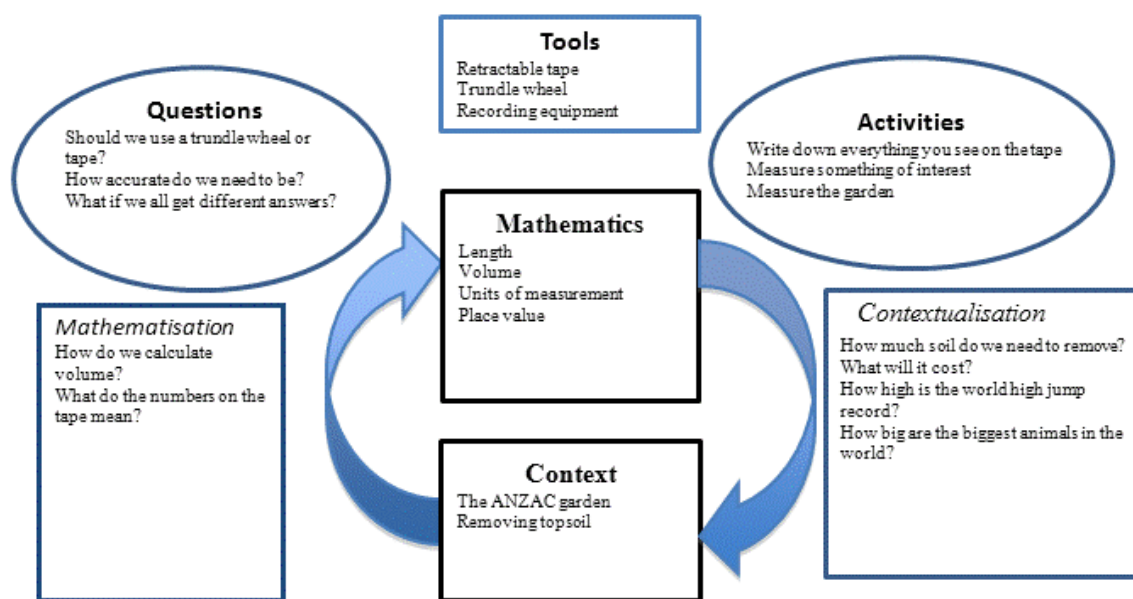


Figure 2. Mathematising, contextualising and the ANZAC garden.

Six weeks down the track my observations and reflections are showing that Debra has made great shifts in her approach to mathematics. As illustrated below in the graphic of the lesson plan, we used the tape measure as a way into looking at hundreds, tens and units. This mathematisation was an entry into a lesson on place value, during which Debra was paired up with a student with whom she had never worked before. At the start of the year this would have upset Debra significantly as she sought only friends or family to work with. Instead, Debra quickly asked her learning partner where she wanted to work and together they collected popsticks and other materials. She independently set out her book with the required tally table, asked her partner who should go first and started the game. She trusted her partner to help her and even asked other students for assistance. While playing the game she made mistakes but she did not get shamed and try to hide her lack of understanding by becoming withdrawn or angry. Instead she tried to solve the problem of how many bundles were allowed in the tens column by checking the procedure as she understood it, then with peers and finally with an adult.

At the end of the game the class reflected on the learning they had done. I asked the group why the pop stick game might be important to their mathematical learning. Immediately Debra's hand was up to participate in the conversation.

Sophie: Why do you think the pop stick game might be important to our maths learning?

Because we need to use the tape measure correctly (A).

Because we need to learn our units, tens, hundreds and thousands (D).

For carrying numbers (A).

Helps maths and you learn more about thousands (N).

Learning more about different numbers (N).

Teamwork (A).

Sophie: How did teamwork help you?

It's better having a go with someone (A).

Taking a risk (N).

Encouraged other (D).

Researched with my partner (D).

Solved the problem together (N).

Collaborated to work out the problem (D).

D = Debra's response

A = Another Aboriginal student's response

N = non-Aboriginal student's response

Articulating the Research Focus and Methodology

Like Sophie, the project teachers have maintained reflective journals that focus on the responses of Aboriginal students to approach to learning described above. We shared and discussed these reflections to draw out common themes. Almost universally the teachers highlighted students' increased engagement, and their enhanced understanding of the nature of mathematics. One teacher related how an Aboriginal student who had previously been negative about mathematics had become much more confident, actively looking forward to learning mathematics. On entering the classroom he now asked questions about what he might be learning, rather than expressing his dislike for learning. Another reported that an Aboriginal student was able to articulate what she saw as the distinction between mathematics and numeracy in terms of using mathematical ideas to solve everyday problems. All teachers reported on the sense of pride students took in their work and on their capacity to articulate the concepts they had learnt. They considered this increased engagement particularly important for Aboriginal students, who are often quick to opt out of learning when they see it as lacking in meaning or purpose (Purdie and Buckley, 2010).

However the teachers' reflections encompassed more than just engagement. They suggested that the Aboriginal students' orientation to learning may have been positively impacted by the approach. Like Debra, rather than being performance-driven the students were more learning-driven, which in turn led to a greater sense of purpose and a willingness to ask questions or embrace mistakes (Dweck, 1999; Sullivan, Tobias & McDonough, 2006). The project team adopted the concept of mathematical resilience (Johnstone-Wilder & Lee, 2010a) to describe this orientation to learning, and considered it crucial to improved outcomes in mathematics.

All learning requires resilience; however we contend that the resilience required for learning mathematics ('mathematical resilience') is a particular construct as a consequence of various factors including: the type of teaching often used ..., the nature of mathematics itself ... and pervasive beliefs about mathematical ability being 'fixed'. Helping learners to develop mathematical resilience enables them to adapt positively to the difficulties presented by mathematics and to be in a position to consider continuing to develop their mathematics beyond compulsory age. (Johnstone-Wilder and Lee, 2010b, p. 4, in-text references omitted)

Like Sophie, the other teachers in the team felt that the approach to teaching and learning in the *Make it Count* project was, in fact, changing the type of teaching and addressing students' perceptions of the nature of mathematics as a fixed and disconnected body of knowledge. Hence it had the potential to engage disinclined learners, particularly Aboriginal students, and to encourage them to maintain positive attitudes to mathematics beyond primary school. We identified five key aspects of mathematical resilience: having a growth mindset shown through behaviours such as learning from mistakes; meta-cognition shown through a willingness to reflect on answers and problem solving processes; adaptability shown through a willingness to try new strategies or start again; inter-personal aspects of learning such as seeing asking questions as clever rather than an admission of lack of knowledge; and a sense of purpose shown by a student's desire to seek meaning in his or her learning.

Sophie's reflections on Debra's learning powerfully illustrate each of these aspects of mathematical resilience. She was prepared to make mistakes; she took responsibility for her learning; she checked procedures and started again when she was unsure; she willingly worked with and asked questions of other students; and she gave insightful responses to Sophie's questions about the purpose of the game. Throughout the project the team has gathered data relating to the mathematical resilience of their Aboriginal students through structured observations of indicative behaviours and fortnightly reflections. As Johnstone-Wilder and Lee (2010a) state "if [mathematical resilience] is important, why are we not measuring it?" (p.41)⁴.

Conclusions

The project is ongoing. We are conscious that, at this point, we are not in a position to report on the impact of the Alberton model on students' cognitive outcomes. However, as Sophie's story vividly shows, we are confident that teachers have developed an approach to teaching that is both complex and purposeful. It requires deep thinking about mathematics, context and the relationships between them. We have developed a conceptual framework that values the inherent unpredictability of learning and the emergence of new knowledge, and that explicitly builds connections between mathematics and the real world of Aboriginal learners. Debra's response to Sophie's challenge to find native plants for the ANZAC garden shows the power of such a framework in creating authentic and meaningful learning.

We are confident that this approach is impacting positively on those aspects of students' learning behaviour that we have described as mathematical resilience. Debra's story is but one example of how Aboriginal students in particular have benefitted from the approach. It has been repeated across classes and across the three schools involved in the project. We believe that paying attention to the development of productive dispositions to learning such as mathematical resilience is a key to better learning outcomes for Aboriginal students.

⁴ The project team also identified the capacity to transfer knowledge as an important cognitive outcome of the project. This will be the subject of future publications.

References

- Ainley, J., Pratt, D. & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal*, 32(1), 23–38.
- Brown, K. (2008). Employing mathematical modelling to respond to Indigenous students' needs for contextualised mathematics experiences. In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, pp. 93-99). Brisbane: MERGA Inc.
- Commonwealth of Australia (2009). *Closing the gap: targets and building blocks*. Accessed 30 March, 2012 from <http://www.fahcsia.gov.au/sa/indigenous/progserv/ctg/Pages/targets.aspx>
- Dweck, C. (1999). *Self theories: Their role in motivation, personality, and development*. Philadelphia: Psychology Press
- Frigo, T. (1999). *Resources and Teaching Strategies to Support Aboriginal Children's Numeracy Learning: A Review of the Literature*. Sydney: NSW Board of Studies
- Johnston-Wilder, S. & Lee, C. (2010a). Mathematical Resilience. *Mathematics Teaching*, 218, 38-41
- Johnston-Wilder, S. & Lee, C., (2010b). *Developing mathematical resilience*. Paper presented at the BERA Annual Conference, 1-4 Sep 2010, University of Warwick.
- Malin, M. (1994). Why is life so hard for Aboriginal students in urban classrooms? *Aboriginal Child at School* 22(2), 141-154
- Matthews, S., Howard, P. & Perry, B. (2003). Working together to enhance Australian Aboriginal students' mathematics learning. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Mathematics education research: Innovation, networking opportunity. Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia* (pp. 9-28). Sydney: MERGA Inc.
- Matthews, C., Watego, L., Cooper, T. & Baturo, A. (2005). Does mathematics education in Australia devalue Indigenous culture? Indigenous perspectives and non-Indigenous reflections. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.) *Building connections: Theory, research and practice. Proceedings of the 28th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 513-520). Melbourne: MERGA Inc.
- Olive, J., Makar, K., Hoyos, V., Kor, L. K., Kosheleva, O. & Sträßer, R. (2010). Mathematical Knowledge and Practices Resulting from Access to Digital Technologies. In C. Hoyles & J-B. Lagrange (Eds) *Mathematics Education and Technology-Rethinking the Terrain. New ICMI Study Series Vol 13 Part 2* (pp. 133-177). US: Springer
- Purdie, N. & Buckley, S. (2010). *School attendance and retention of Indigenous Australian Students. Issues paper no. 1 produced for the Closing the Gap Clearinghouse*. Retrieved 30 March 2012 from http://www.aihw.gov.au/closingthegap/documents/issues_papers/ihw/33/12176.pdf
- Sullivan, P., Zevenbergen, R. & Mousley, J. (2003). The contexts of mathematics tasks and the context of the classroom: Are we including all students? *Mathematics Education Research Journal*, 15 (2), 107-121
- Sullivan, P., Tobias, S. & McDonough, A. (2006) Perhaps the decision of some students not to engage in learning mathematics in school is deliberate. *Educational Studies in Mathematics*, 62, 81–99
- Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in Realistic Mathematics Education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, 54, 9–35.
- Watson, H. (1988). Language and mathematics education for Aboriginal-Australian children. *Language and Education*, 2(4), 255-273

Use of Practical Worksheet in Teacher Education at the Undergraduate and Postgraduate Levels

Pee Choon Toh

Nanyang Technological University
<peechoon.toh@nie.edu.sg>

Foo Him Ho

Nanyang Technological University
<foohim.ho@nie.edu.sg>

Tin Lam Toh

Nanyang Technological University
<tinlam.toh@nie.edu.sg>

Khiok Seng Quek

Nanyang Technological University
<khiokseng.quek@nie.edu.sg>

We have applied the ‘practical paradigm’ in teaching problem solving to secondary school students. The key feature of the practical paradigm is the use of a practical worksheet to guide the students’ processes in problem solving. In this paper, we report the diffusion of the practical paradigm to university level courses for prospective and practising teachers. The higher level of mathematics content would demand higher order thinking skills. Learners without a model of problem solving would often revert to solving by referring to many examples of the same ‘type’ of problem. Polya-type problem solving skills framed by the practical worksheet was used as an attempt to elicit more effective problem solving behaviour from them. Preliminary findings show that they were able to use the practical worksheet to model their solution of problems in the courses.

Since the 1980s, problem solving has been at the heart of the Singapore mathematics curriculum. Some recent studies in Singapore schools (e.g., Foong, 2009, Teong et al., 2009), suggest that problem solving, especially that of non-routine problems, is done mostly as enrichment activities and not part of the core mathematics curriculum.

To address the issue of problem solving being regarded as peripheral, rather than central to the curriculum, the Mathematical Problem Solving for Everyone (MProSE) project was formed. The theoretical underpinnings for the MProSE design experiment can be found in Quek, Dindyal, Toh, Leong, & Tay (2011). Based on these design parameters, a package of problems, lessons and materials suitable for implementation into the mathematics curriculum was developed (Toh, Quek, Leong, Dindyal, & Tay, 2011). An account of the implementation of this problem solving module into a local independent Secondary school in 2009 is described in Dindyal, Tay, Toh, Leong & Quek (2012). Since then, the problem solving module has been incorporated in another three Secondary schools (Leong et al, 2012).

One distinct feature of the MProSE project is the practical paradigm through the use of the Practical Worksheet for the problem solver to work out his/her solutions. This worksheet contains sections that guides the problem solver through the four stages of Polya’s model (Polya, 1945), and also incorporates Schoenfeld’s framework (Schoenfeld, 1985), highlighting the cognitive resources, use of heuristics, control, and belief systems of the problem solver. A condensed form of the worksheet together with a suggested assessment rubric can be found in Toh, Quek, Leong, Dindyal, & Tay (2009).

We believe that a critical success factor for bringing problem solving back to the centre of the curriculum lies in having well-prepared teachers. In the MProSE project, teacher preparation is an important component (Leong, Dindyal, Toh, Quek, Tay, & Lou, 2011). That teachers spend a substantial amount of time in actual problem solving is a critical part of the teacher preparation process.

A prior effort to use the practical worksheet in a postgraduate course on “Discrete Mathematics and Problem Solving” for 21 practising teachers was reported in Tay, Quek, Dindyal, Leong & Toh (2011). In that paper, the reflections of the teachers as they embarked on their learning journey were reported and analysed, and it appeared that the practical worksheet was useful in bringing about the awareness of the problem solving process. In this paper, we report on how problem solving using the practical worksheet was incorporated into an undergraduate mathematics content course for prospective teachers, a postgraduate mathematics content course for practising teachers, as well as a postgraduate prospective teacher education course.

Undergraduate Mathematics Content Course

One of the authors—henceforth known as the lecturer— taught a 36-hour course introducing number theory to 59 undergraduate students. The content of the course was typical of similar courses taught elsewhere and includes divisibility, congruences, Diophantine equations, Euler’s generalization of Fermat’s little theorem etc. Most of the students were in the first year of their B.A. (Ed.) or B.Sc. (Ed.) programme, and had so far been mainly learning content mathematics. About 85% of them had not undergone the “teaching of mathematics” component that would have introduced to them Polya’s problem solving framework.

The lecturer having previously taught calculus to the same group of students recognised that many of them would face difficulties in number theory because it was atypical of the mathematics that they were used to in their pre-university education. When faced with a problem like “prove that if m is a composite number, then $2^m - 1$ is also composite,” most of them would simply freeze and not know how to proceed. Unless one has seen the solution before, a possible approach would be to use heuristics like “substitute numbers (for m)” and “search for patterns”. Probing further, one could try to “solve a simpler problem”. Simple examples of composite numbers are even numbers (greater than 2) and, with a little experimenting, students would generally be able to see that $2^{2^k} - 1$ seems to be always divisible by 3. If they could then prove this conjecture, they would have partially solved the problem¹.

To help students overcome these difficulties, the lecturer tried to teach *through* problem solving (Shroeder & Lester, 1989) using Polya’s model. He began with the first three stages of (1) understanding the problem, (2) devising a plan and (3) carrying out the plan, without explicit mention of Polya as he wanted to convey the idea that these are natural processes that mathematicians use to solve mathematics problems. During the lectures, the lecturer would demonstrate how he understood the problem, what kind of heuristics he would use, as well as possible plans for solving the problem, before finally carrying out the plan. This departs from the usual theorem–proof, theorem–proof type of exposition that is commonly used in teaching advanced mathematics. Gradually, the job of solving the problem was passed on to the students and the lecture notes would only have the names of the Polya stages, followed by spaces for students to work on. About a third of the way through the course, Polya’s model, including Stage 4 which we renamed as “Check and Expand”, was introduced to the students. They were also given the practical worksheet to be used for their problem solving assignments.

¹ The problem solving attempt that was described actually took place during a consultation session with three students.

Data from one of the problem solving assignments is presented in Table 1. The problem was

Problem: *Find a million consecutive composite numbers.*

This was “non-routine” for the students as there were no similar examples in the lecture notes and the prescribed textbook. Among the 57 assignments received, 51 students managed to solve the problem, 50 of them displayed evidence of using the Polya stages in their solution, and slightly less than half (26 students) used the heuristic of working with a smaller number of consecutive composites first before going on to solve the problem for a million. The lecturer was pleasantly surprised that so many of the students were able to successfully use Polya’s framework to solve the problem. It was also encouraging to see that 37 of the students went on to Stage 4 and attempted to generalize the problem. Some students also demonstrated higher order thinking skills. For example in *Figure 1*, Student X’s working indicated that he started by searching for consecutive composites in the hope of finding a pattern. His metacognition process was clearly spelt out in the control column of the practical worksheet. Due to the lack of space, we do not reproduce how he successfully solved the problem in his second attempt.

Table 1

Data from Undergraduate Mathematics Content Course

Description	Number of Students
Practical Worksheets Received ¹	57
Solved the problem	51
Displayed Evidence of Using the Polya Stages	50
Use of at least one Heuristic	26
Attempted to Check and Expand	37

¹ 2 students did not submit their assignments.

III Carry out the plan (You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.) (i) Write down in the Control column, the key points where you make a decision or observation, for e.g., go back to check, try something else, look for resources, or totally abandon the plan. (ii) Write out each implementation in detail under the Detailed Mathematical Steps column.	
Detailed Mathematical Steps	Control
<u>Attempt 1</u> 1st 3 consecutive composite numbers 8, 9, 10 Let n be the number of consecutive composite numbers 1-1 $n=3$ $8=3+5$ $9=3+6$ $10=3+7$ } is this a pattern? 1-2 $n=3$ Find another 3 consecutive composite numbers 26, 27, 28 (check: all 3 are composite and are consecutive) $26=3!+20=4!+2$ $27=3!+21=4!+3$ $28=3!+22=4!+4$ } Are both of these patterns? ✓ confused, identified lots of patterns, but has a general idea that to find n consecutive composite numbers, we have to find a number which can be divided by numbers from 1 to n. <u>Attempt 2</u>	- Decide to find whether there is a pattern behind the question - Try out with different sets of 3 consecutive numbers composite - Identified many patterns, but no idea whether there is any link. - Stuck, try again from the start.

Figure 1. Student X's working and his control statements.

Postgraduate Mathematics Content Course

The practical paradigm of mathematical problem solving was also introduced to practicing teachers attending a Masters Degree course on Theory and Applications of Differential Equations. The students (8 in total) were introduced to the practical paradigm during the first introductory lesson of the module when the tutor (one of the authors) introduced the qualitative analysis, and finding the solution of an ordinary differential equation analytically and numerically. The tutor introduced the use of the practical worksheet alongside Polya's problem solving stages in the context of differential equations. During the first four lessons, practical worksheets were used by the participants. In the second lesson, the tutor modeled the use of the practical worksheet in solving a non-routine differential equation with the participants.

It was observed during the third lesson that generally the students were reluctant to use the practical worksheets, especially for Polya Stage 1 (Understand the Problem) and Stage 2 (Devising a Plan). Most of them attempted to solve the problem directly. It was also evident that they were not comfortable to proceed to Stage 4 (Check and Expand) of the worksheet.

A test on the content material of the first four lessons was conducted in the fifth lesson. The students were informed in the first lesson of this test, of which a significant portion would consist of a mathematics practical test. The test problem was

Problem: Draw the direction field for $\frac{dx}{dt} = x^2 t$.

This is non-routine problem for the students, as most of the examples considered in qualitative analysis in this course were autonomous equations of the form $\frac{dx}{dt} = f(x)$. Moreover, lecture examples used x and y to represent the independent and the dependent variables respectively. Furthermore, the shape of the direction field for this differential equation is not as easily obtained as for those in the lecture examples.

A pleasant surprise turned out in the analysis of the students' scripts. All the eight students filled up all the four pages corresponding to the four stages, even though in the first four lessons the students had expressed their reservations in completing Stages 1 and 2 of the worksheet. In communicating their feelings about this problem (in Stage 1), some of the students' responses were:

- Rather daunting as RHS of DE involves both the independent and dependent variables.
- Problems seem challenging since there are two variables on the RHS of the expression.
- Looks ok. However, RHS involves x and t .
- The problem scares me.
- This problem can be rather challenging as we are used to $\frac{dy}{dx}$ and not $\frac{dx}{dt}$.
- Seems manageable. Need to identify the variables involved.
- Need to construct a slope field & observe pattern.
- (One student began with substituting values into the right hand side of the equation to obtain values of the slopes.)

Further, it was heartening to observe that all the students attempted Stage 4 (Check and Expand) of the practical worksheet. We use student Y's script (Figures 2 and 3) as illustration. At Stage 3 (Carry out the plan), the student had only plotted values for various values of (t, x) as, according to him, "the direction field looks odd" (Figure 2). He progressed to Stage 4 and solved the differential equation analytically to obtain the general solution (Figure 3). This gave him a clearer picture of the slope field, and he reverted to Stage 3 to complete the sketch of the slope field.

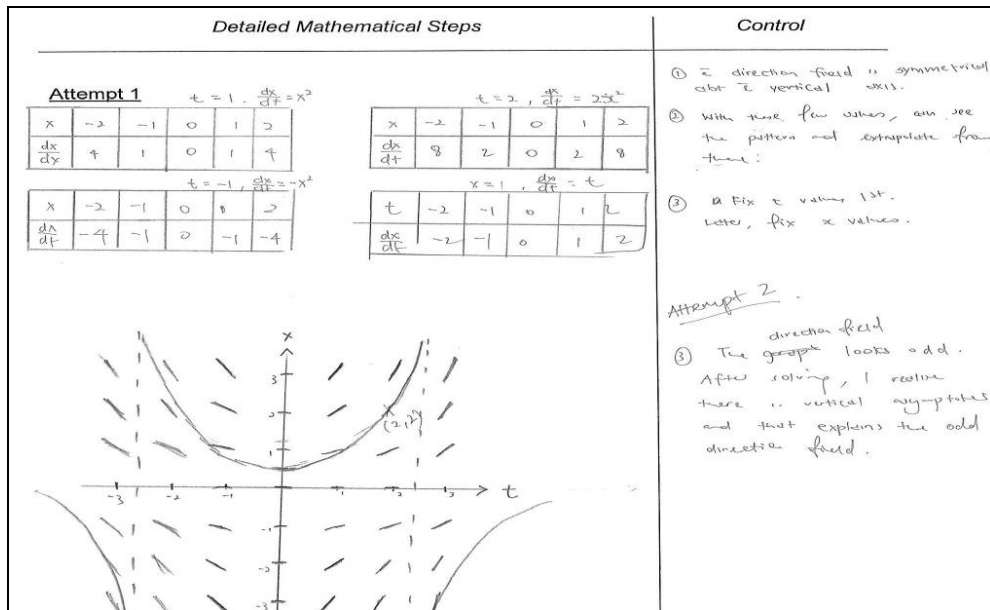


Figure 2. Student Y's completed Stage 3 of the worksheet.

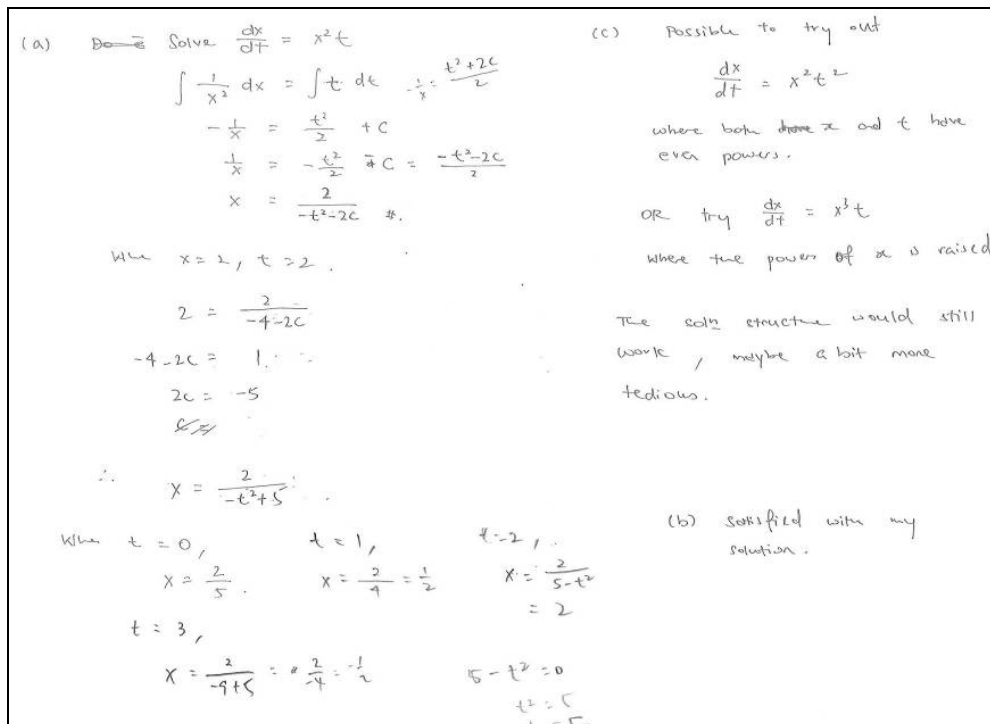


Figure 3. Student Y's completed Stage 4 of the worksheet.

Teacher Education Course for Prospective Teachers

One of the authors—henceforth known as the tutor—taught the mathematics methods course in the Postgraduate Diploma in Education (PGDE) programme which aims to prepare university graduates in Mathematics or in a Mathematics-related discipline such as Engineering to become Secondary school mathematics teachers. About six hours of the instructional time of the module was devoted to equipping prospective teachers with the necessary skills and pedagogy for teaching problem-solving and teaching mathematics through problem solving. In the past, the tutor conducted these problem-solving lessons without using the practical worksheet approach. To help them organise their problem

solving processes, the tutor used the practical paradigm for a class of 24 prospective teachers in the 2011 cohort. The rest of this section is a brief description of the middle portion of this module on problem solving.

The tutor started by explaining what a mathematical problem is, emphasizing that it is different from a routine exercise and that it requires time and effort to solve. Both Polya's model and Schoenfeld's problem solving framework were highlighted so that prospective teachers understood that problem solving requires not only subject matter knowledge, but also heuristics, helpful beliefs, and control. The tutor demonstrated and discussed how different heuristics were used to solve some problems before he focused on how the practical worksheet should be used by going through a specific problem.

Problem : *ABC is an equilateral triangle. P is a point inside the triangle such that the distances from its three sides are 4, 5 and 6 cm. Find the length of one side of the triangle.*

The tutor noticed that in their first attempt to use the practical worksheet, many students did not follow the instructions on the worksheet and proceeded to solve the problem directly. In particular, there were hardly any comments on the thinking process in the "control column" of the practical worksheet. This may be due to the fact that most of them could solve the problem in about 15 minutes. Thus, they may think that control was irrelevant and unnecessary for them. During the class discussion, the tutor deliberately emphasised the importance of Polya's Stage 4 (Check and Expand section in the practical worksheet) for developing a deeper understanding of the problem. Indeed, the class suggested the following extensions:

- If a point P is inside a given equilateral triangle, the sum of the distances from each of the three lengths to the point is always a constant and equal to $\frac{\sqrt{3}}{2}$ times of the side of the equilateral triangle.
- If the point P is inside a given square (or rectangle), it is obviously true that the sum of the distances from each of the four sides to the point is a constant—equal to the perimeter of the rectangle.
- The same observation is true when the point P is inside a given pentagon with equal sides but not true if the sides of a pentagon are not all equal (the tutor used the *Geometers' Sketchpad* to verify these two observations in class).

At the end of the segment, the class came up with the following conjecture: "In a regular n -sided convex polygon, the sum of the distances from a point P inside the polygon to each of the n sides of the polygon is always a constant²."

Despite their initial reluctance at filling up the earlier sections of the practical worksheet, one encouraging sign was that the fourth page of the worksheet points to the need to push forward to expand the solution instead of stopping at the solution. One prospective teacher in particular commented: "I didn't see the usefulness of the template [referring to the practical worksheet] until the last bit. It is very rewarding when we extend the problem. It was really fun and exciting".

Upon reflection, the tutor thinks that the practical worksheet approach to problem-solving creates a platform for students to focus on Check and Expand. As the "looking-back" stage is part of the worksheet, they will likely sense that their attempts are not

² One of them subsequently managed to prove that the constant is equal to $\frac{n}{2 \tan \frac{180^\circ}{n}}$ times the length of the sides of the n -gon.

complete when they stop at Stage 3. There is thus a greater impetus to re-examine the solution process and look beyond the problem—the process of expanding would be useful in helping students to make generalisations and achieve a deeper understanding of the problem.

Summary

We reported on the use of the practical worksheet to help prospective and practising teachers experience problem solving through the Polya's model and Schoenfeld's framework. The lecturer/tutors are encouraged to see some signs of success. The emphasis on the problem solving model as embodied in the practical worksheet has shown that the learners generally exhibited behaviour which we think (from our experience teaching these courses in the past) they would usually not show (e.g. Check and Expand). We believe that this is one of many *starting* steps that need to be taken in bringing problem solving back to the heart of the curriculum.

References

- Dindyal, J., Tay, E. G., Toh, T. L., Leong, Y. H. & Quek, K. S (2012). Mathematical Problem Solving for Everyone: A New Beginning. *The Mathematics Educator*, 13(2), 1-20.
- Foong, P. Y. (2009). Review of research on mathematical problem solving. In K. Y. Wong, P. Y. Lee, B. Kaur, P. Y. Foong, & S. F. Ng (Eds.), *Mathematics education: The Singapore journey* (pp. 263-300). Singapore: World Scientific.
- Leong, Y. H., Dindyal, J., Toh, T. L., Quek, K. S., Tay, E. G. & Lou, S. T. (2011). Teacher education for a problem-solving curriculum in Singapore. *ZDM: The International Journal on Mathematics Education*, 43(6-7), 819-831.
- Leong, Y. H., Toh, P. C., Dindyal, J., Quek, K. S., Ho, F. H., & Tay, E. G. (2012). *Diffusion of the Mathematics Practical to Mainstream Secondary Schools in Singapore*. Paper to be presented at the 35th annual conference of the Mathematics Education Research Group of Australasia.
- Polya, G. (1945). *How to solve it*. Princeton: Princeton University Press.
- Schoenfeld, A. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
- Shroeder, T., & Lester, F. (1989). Developing understanding in mathematics via problem solving. In P. Traffon & A. Shulte (Eds.) *New directions for elementary school mathematics: 1989 Yearbook* (pp. 31-42). Reston, VA: NCTM.
- Tay, E. G., Quek, K. S., Dindyal, J., Leong, Y. H. & Toh, T. L. (2011). Teachers Solving Mathematics Problems: Lessons from their Learning Journeys. *Journal of the Korean Society of Mathematical Education Series D: Research in Mathematical Education*, 15(2), 159-179.
- Teong, S. K., Hedberg, J. G., Ho, K. F., Lioe, L. T., Tiong, J., Wong, K. Y. & Fang, Y. P. (2009). *Developing the repertoire of heuristics for mathematical problem solving: Project 1*. Final technical report for project CRP 1/04 JH.
- Toh, T. L., Quek, K. S., Leong, Y. H., Dindyal, J. & Tay, E. G. (2009). Assessment in a Problem Solving Curriculum. In Hunter, R., Bicknell, B., Burgess, T. (Ed.) *MERGA 32 Conference Proceedings* (pp. 686-690). Wellington, New Zealand: Massey University.
- Toh, T. L., Quek, K. S., Leong, Y. H., Dindyal, J. & Tay, E. G. (2011). *Making Mathematics Practical: An Approach to Problem Solving*. Singapore: World Scientific.
- Quek, K. S., Dindyal, J., Toh, T. L., Leong, Y. H. & Tay, E. G. (2011). Problem Solving for Everyone: A design experiment. *Journal of the Korean Society of Mathematical Education Series D: Research in Mathematical Education*, 15(1), 31-44.

The Effect of Small-Group Game Play Activities on Number Sense Performance

Rashidah Vapumarican

Convent of the Holy Infant Jesus (Kellock)

<arif_hong_chu_sen@moe.edu.sg>

Manu Kapur

National Institute of Education

<manu.kapur@nie.edu.sg>

This study examines the effect of small-group game play before and after a Number Sense Test (NST). Fifty-seven 7-year old children were assigned into two conditions: (a) NST-Game Play, in which students sat for the NST before game play; or (b) Game Play-NST, in which students experienced game play before sitting for the NST. Findings suggested that Game Play-NST students outperformed the NST-Game Play students in number sense performance. Having Game Play-NST experience appeared to be more effective in enhancing students' performance in number sense.

Background

Number sense is one's general understanding of number and operations which includes the ability and inclination to flexibly use this understanding to make mathematical judgements and develop efficient strategies when handling numerical situations (McIntosh, Reys & Reys, 1992). The importance of the development of number sense is evident mathematics curricular documents such as the National Council of Teachers of Mathematics of America (NCTM, 2000), the Australian Council of Education (AEC, 1991) and the Singapore Ministry of Education (MOE, 2006).

Gersten and Chard (1999) has emphasized that the construct of number sense is essential in conceptual understanding. These students have a good sense of magnitude of numbers, observe number patterns and are able to recognise benchmark numbers (Case, 1998). On the other hand, children who do not have a good number sense tend to be inhibited in their learning of mathematics (Ekenstam, 1977) and will have persistent problems in many areas of mathematics especially in relation to conceptual understanding and application of procedural knowledge (Woodward & Baxter, 1997). In addition, a study by Griffin, Case and Siegler (1994) underscored the relationship between instruction in number sense activities and its effect in reducing failure in early mathematics.

Research has also brought to light the role of informal learning of number sense. Gersten and Chard (1999) assert that most children develop number sense through informal interactions with parents and siblings before entering kindergarten. Griffin et al. (1994) echoes similar thought, suggesting that number sense is informally acquired prior to formal school and that it is necessary for learning formal arithmetic in elementary grades. Howden (1989) has also cited on the intuitive nature of number sense and the way it gradually develops in learners by exploring numbers, visualising them in different context and relating them beyond traditional algorithms. Research has also indicated that prior knowledge is an essential variable in learning (Alexander, Kulikowich, & Schulze, 1992; Dochy, 1994). Glaser and De Corte (1992) describe prior knowledge as a "springboard for future learning" (p. 1). If students' intuitive, informal and prior knowledge are important resources, how do we activate such knowledge and leverage them for both teaching and assessment of number sense into the formal setting of mathematics in school?

The learning of mathematics as a product of social activity is widely accepted (Cobb, Gravenmeijer, Yackel, McLain & Whitenack, 1997; Sfard, 2001). Through social activity, students would be put into situations where they need to communicate and reason

mathematics ideas with their peers. In lending support to social learning, the significance of context of play in the study of the learning process of mathematics in early years and primary schools has been highlighted by Edo, Planas and Badillo (2009) and Peters, (1998). They contend that classroom play situations tend to promote construction of mathematical knowledge, development of mathematical problem-solving strategies and enhancement of individual mathematical thinking. Such interactions in a game play setting make it possible for learners to communicate and appreciate the different viewpoints too.

“Number sense exhibits itself in various ways as the learner engages in mathematical thinking” (McIntosh et al., 1992, p. 3). Over the past decade or so, many researchers have assessed number sense through pencil-and-paper tests, interviews or individually administered tests, combination of both pencil-and-paper test and interview and even computerised test (Aunio, Ee, Lim, Hautomaki & Van Luit, 2004; Dunphy, 2006; Markovits & Sowder, 1994; Markovits & Pang, 2007; McIntosh, Reys, Reys, Bana & Farrell, 1997; Yang, Li & Li, 2008). The assessments of number sense cited may not have drawn upon the characterization of number sense that learners possess. The setting for the computerised test, though different from the pencil-and-paper test, is still one of a question-and-answer type where students read the questions and then choose or give their answers. Number sense being an intuitive feeling of numbers and its informal acquisition may not be uncovered or manifested in such assessments.

While there has been substantive research on leveraging children’s intuitive ideas and knowledge for designing better teaching and learning (e.g., Kapur & Bielaczyc, 2012; Kapur, 2012), work on how assessment of learning could also be influenced by activation of such resources remains under-researched by comparison. Therefore, we designed this study to examine how assessment of number sense that students have already learnt over the school year can be influenced by game play activities that activate students’ formal as well as intuitive priors about number sense. More specifically, we examine the effect of having students take number sense test (NST), followed by a Game Play activity (NST-Game Play condition) or engage in Game Play followed by NST (Game Play-NST condition) on their performance on the NST.

Methodology

Participants and Design

The sample of 57 primary 1 students (about 7 years old) consisted of the main ethnic groups in Singapore. They come mostly from the neighbouring residential and were from the middle to low Socio-Economic Status (SES) groups. Students were assigned to two groups to experience either the NST-Game Play condition (N=33) or the Game Play-NST condition, (N=24). Students’ year-end mathematics scores were taken as a measure of mathematics ability. The year-end mathematics scores comprised an aggregate of several standardised pencil-and-paper tests and performance task assessments. Table 1 shows the mean and standard deviations of NST-Game Play students and Game Play-NST students. A univariate analysis of variance, ANOVA, showed no significant effect of the experimental condition (NST-Game Play vs. Game Play-NST) on mathematics ability, $F(1,55) = 0.539$, $p = .466$, partial $\eta^2 = .010$.

Table 1

Descriptive Statistics of Math Scores by Experimental Condition.

Conditions	M	SD
NST-Game Play	86.08	9.92
Game Play-NST	88.13	11.10

NST-Game Play students sat for the NST test first followed by game play on the following day and the reverse sequence for Game Play-NST students. In both conditions, students sat for the pencil-and-paper NST. In the game play students were put into small groups of fours. In the game play, students recorded their responses after each round on the game sheets provided. The study was conducted towards the end of the school year term. Hence the students had completed the primary 1 mathematics syllabus.

Instrumentation

A Number Sense Test (NST), adapted from the Number Sense Item Bank (NSIB) by McIntosh et al. (1997), was used for this study. Twenty-one items were set based on five strands, namely; number concepts, computing and counting, equivalent expressions, multiple representations and effect of operations. The content and conceptual knowledge in the test items were consistent with the Singapore primary mathematics syllabus for primary 1. The items were either multiple-choice variety or blanks for students to fill in their own answers.

The game play comprised of two card games; Game 1-“Whose the Closest?” and Game 2-“Flip 4 and Subtract!”. Both games, adapted from Overholt, Holtz & Dickson (1999), were designed to match five of the NST items. Game 1 matches similar outcome as NST items 8 and 9 while Game 2 matches similar outcome as NST items 19, 20 and 21. The game play provides an informal setting that stands in sharp contrast to the NST. It provide opportunities for students to activate their intuitive, informal and formal number sense, hence affording them to a) explain, reason and justify their strategies, b) observe what other players do, c) listen and question others’ viewpoints, d) discover and generalise strategies, e) build on one another’s contribution and f) error correct themselves and that of the peers. A pilot study was conducted with a class of 28 primary 1 students with similar background as the study groups. The purpose of the pilot study was to test the difficulty and structure of the test items as well as the game play.

Results

The scores of the students for all the twenty-one items were calculated. Each correctly answered item was awarded one mark and zero mark for any partially correct answer or wrong answer. No half mark was awarded. The NST had scale reliability (Cronbach’s alpha) of .78.

NST Items Matching with Game Play

Analysis of the experimental condition performances on the matching NST items to the two games revealed that students in the Game Play-NST condition outperformed the NST-Game Play students. Figure 1 indicates that Game Play-NST students performed more than 1.5 times better than NST-Game Play students in NST items 8 and 9 (Game 1 NST items) and almost 2.5 times better than NST-Game Play students in items 19, 20 and 21 (Game 2 NST items). Having played the game first before sitting for the NST seemed to enable Game Play-NST students to translate the cognitive gains from the game play to the pencil-and-paper NST.

A multivariate analysis of covariance, MANCOVA, indicated statistically significant multivariate effects of: (i) the experimental condition (NST-Game Play vs. Game Play-NST) on NST scores by items that matched with the game play, $F(2,53) = 11.272$, $p < .001$, partial $\eta^2 = .298$; and (ii) mathematics ability on NST items that matched with the game play, $F(2,53) = 9.035$, $p < .001$, partial $\eta^2 = .254$. We can infer that performance on NST

items that matched game play was significantly dependent on the sequence of the NST and game play. The ANCOVAs revealed statistically significant effect of the experimental condition (NST-Game Play vs. Game Play-NST) on NST items matching with Game 1 and NST items matching Game 2; (i) for NST items 8 and 9 that matched with Game 1 (Game 1 NST items), $F(2,54) = 10.582$, $P = .002$, partial $\eta^2 = .164$; and (ii) for NST items 19, 20 and 21 that matched Game 2 (Game 2 NST items), $F(2,54) = 8.442$, $P = .005$, partial $\eta^2 = .135$.

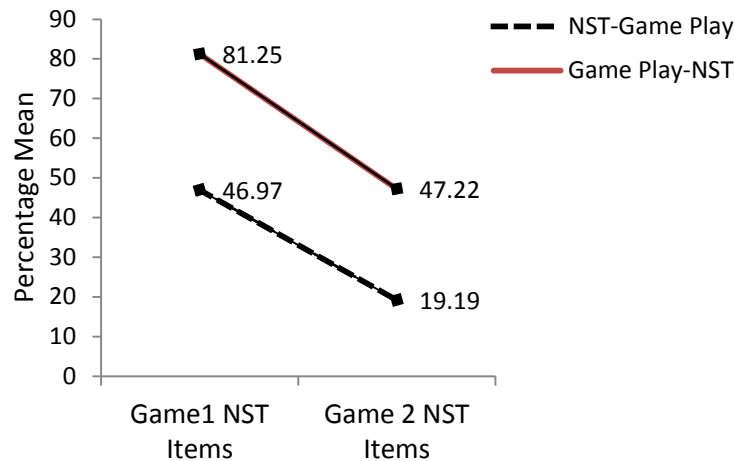


Figure 1. Mean percentage of experimental condition performance on NST items matching with game play

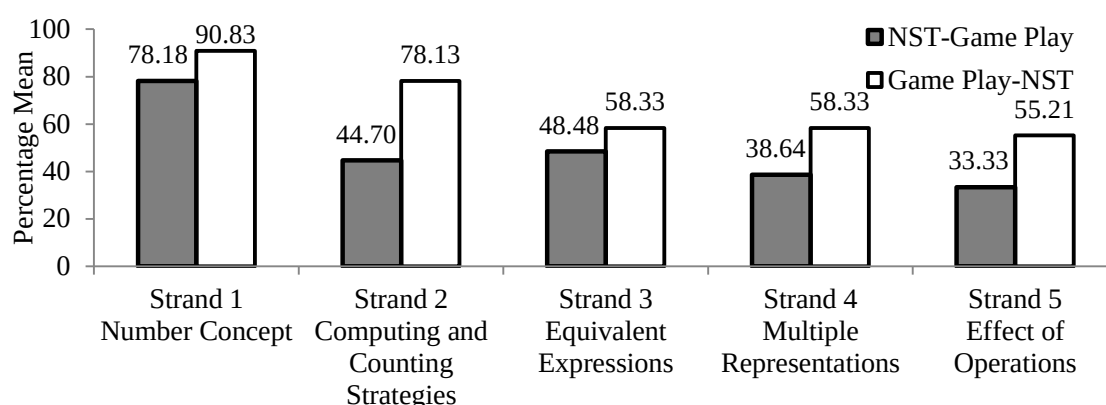
NST Items by Strands

The NST items were categorised into five strands. NST items 8 and 9 matched Game 1 were categorised in Strand 2 while NST items 19, 20 and 21 which matched to Game 2 were found in Strand 5. As seen in Figure 2, Game Play-NST Students did better than NST-Game Play Students, but greatest differences of 33.43% and 21.88% were observed in Strand 2 and Strand 5 respectively. The results suggest that having game play prior to NST had a positive effect on students' performance in NST.

A MANCOVA showed statistically significant multivariate effects of: (i) the experimental condition (NST-Game Play vs. Game Play-NST) on NST scores by strands, $F(5,50) = 6.553$, $P < .001$, partial $\eta^2 = .396$; and (ii) mathematics ability on NST scores by strands, $F(5,50) = 12.778$, $p < .001$, partial $\eta^2 = .561$. Therefore, we can infer that performance on the various strands in NST was significantly dependent on the sequence of the NST and game play.

The ANCOVAs revealed statistically significant effect of the experimental condition (NST-Game Play vs. Game Play-NST) on NST scores by strands: (i) for Strand 2 in which the two of the NST items matched with Game 1; $F(2,54) = 25.529$, $p < .001$ and partial $\eta^2 = .321$ and for (ii) for Strand 5 in which three of the NST items matched with Game 2; $F(2,54) = 6.513$, $p = .014$ and partial $\eta^2 = .108$.

Similarly, the ANCOVAs showed statistically significant effect of the students' mathematics ability on NST scores by strands: (i) for Strand 2, $F(2,54) = 8.915$, $p = .004$, partial $\eta^2 = .142$; and (ii) for Strand 5, $F(2,54) = 24.122$, $p < .001$, partial $\eta^2 = .309$.



NST Scores by Strands

Figure 2. Mean percentage of experimental condition performance on NST scores by strands

Performance on Game Play

In the game play, one point was given for a completely correct response in the games and zero for a partially correct response or a wrong response. These scores were then converted to percentage and analysed. Table 2 shows the percentage score for NST-Game Play and Game Play-NST conditions correct responses in Game 1 and Game 2.

Table 2

Descriptive Statistics for Correct Response in Game 1& 2 by Experimental Conditions

	Conditions	M	SD
Percentage of Correct Response in Game 1 ^a	NST-Game Play	76.77	20.28
	Game Play - NST	73.82	19.98
Percentage of Correct Response in Game 2 ^a	NST-Game Play	76.28	30.25
	Game Play-NST	68.75	25.80

^aFor response to be considered correct in Game 1 and Game 2, students must at least score 2 out of 3 rounds correct or 3 out of 4 rounds correct or 4 out of 5 rounds correct in both games.

A MANCOVA revealed a significant multivariate effect of mathematics ability on composite game scores, $F(2,53) = 11.305$, $p < .001$, partial $\eta^2 = .299$. Composite game scores refer to the total scores for Game 1 and Game 2. However, there was no significant effect of the experimental condition (NST-Game Play vs. Game Play-NST) on composite game score, $F(2,53) = 1.484$, $p = .236$, partial $\eta^2 = .053$. From these results, we can infer that the performance on the game play was not significantly dependent on the sequence in which the students in NST-Game Play condition and Game Play-NST condition played the games. In other words, when the students had the game play had no significant effect on their performance on the two games. Though we observe a slight difference in the mean for NST-Game Play condition as compared to Game Play-NST condition, the difference was proven to be insignificant. For NST-Game Play students, having the game after taking NST test did not afford them to have a significant difference in their performance during the game play when compared Game Play-NST students who had the game play first before sitting for NST.

The ANCOVAs indicated statistically significant effects of mathematics ability on the experimental condition (NST-Game Play vs. Game Play-NST) students' scores in game play; for Game 1, $F(2,54) = 6.973$, $p = .011$, partial $\eta^2 = .114$; and for Game 2, $F(2,54) = 14.775$, $p < .001$, partial $\eta^2 = .215$. However, the ANCOVAs revealed no statistically significant effects of the experimental condition (NST-Game Play vs. Game Play-NST) on their scores in the game play; for Game 1, $F(2,54) = .692$, $p = .409$, partial $\eta^2 = .013$; and for Game 2, $F(2,54) = 2.175$, $p = .146$, partial $\eta^2 = .039$. We can say that there was no main effect of the sequence of the game play which the students in the NST-Game Play and Game Play-NST conditions experienced on their performances in the two games; Game 1 and Game 2.

Discussion

This study was designed to find out how the different experiences students went through; NST-Game Play condition and Game Play-NST condition, effected their performance in number sense. We also aim to explore the possibility of how engaging students in game play could support the activation of prior and informal knowledge of number sense and hence bridge it to the formal assessment of number sense and vice versa.

The performance of the experimental condition on the NST test scores by strands presented significant differences in their mean scores. This suggests strong evidences that Game Play-NST students performed significantly better than their counterparts in the NST-Game Play condition. This was despite their matching mathematical ability. These results are aligned to a previous study conducted by Saxe & Guberman (1998) on mathematics learning in collective play with third and fourth graders. The post-test which they conducted after the Treasure Hunt game play on arithmetic problem with base-10 blocks, revealed that students who had game play produced more adequate solutions than students who were non-players. In addition, third graders who played with fourth graders demonstrated acquisition of strategic knowledge as they had many opportunities to construct such knowledge in their interactions with the older students. Similar situation could be inferred in the present study in which the small group game play could have facilitated and supported such learning and performance in the formal assessment – NST.

Further investigation on the NST by items that corresponded to the game play presented significant difference in the mean scores between the two conditions. What differentiated the two conditions was the sequence of the game play and NST. Hence it is apparent that the game play had contributed to the difference. Having played the game first before sitting for the NST seemed to enable Game Play-NST students to translate the cognitive gains from the interactions in the game play to the pencil-and-paper NST.

Another interesting result was that the differences in mean percentages for the correct responses for Game 1 and Game 2 in both conditions were found to be insignificant. This implies that the sequence of the game play did not contribute to the correct responses of the students' number sense in the games. In other words, having sat for the NST, which contained items with matching outcomes as those in Game 1 and Game 2, did not enable NST-Game Play students to translate it into the game play. Why did the students in the Game Play-NST performed better than students in the NST-Game Play in the NST? Why students in the NST-Game Play did not perform better in the game play after having sat for the NST test earlier? What were the kinds of processes that Game Play-NST students were engaged in that contributed to their better performance in the NST?

One explanation could be that the game play activity may have afforded a mechanism for activating students' intuitive and informal knowledge of number sense. The diversity of the students' cognitive level, prior knowledge and experience may have

contributed largely to the group's knowledge construction, giving opportunities for students to be engaged in reasoning, questioning, discovering and generalising (Kumpulainen & Wray, 2002). In addition, while sharing their views and perspectives, students can also build on each other's contribution to re-construct new interpretations and views.

Another explanation could be error correction. During the game play, students were observed giving suggestions and correcting their peers on their choices of numbers on the cards. In situations where a child makes an error and a peer tries to help in clarifying his or her thinking, learning opportunity arises (Yackel, Cobb & Wood, 1991). Such learning of mathematics through errors has been pointed out by Labinowicz (1987) as a potential educational role which children's errors can play. The child who is assisting his peer could have leverage on his prior knowledge of number sense. Forman & Cazden (1985) have highlighted in their research on the benefits of the acts of verbalisations or giving instructions to peers on the learner who is giving the instructions. In a game play, learning can also occur when children attempt to reach consensus. In such situation, the child needs to explain and justify his point of view or solution to others before the group is able to accept and agree to it

On the contrary, NST-Game Play students did not do better than Game Play-NST students in both games which are parallel to the NST items. This result demonstrates that pencil-and-paper setting does not necessarily afford the learner to explicitly capitalise on their informal or prior knowledge or intuition in number sense. Yet, after experiencing the formal assessment setting, NST-Game Play students were of the same footing as Game Play-NST students in the game play. They could not leverage the experience they had in the formal setting to help them perform better in the game setting. Number sense goes beyond the formal setting and it is insufficient to rely on formal setting in assessing students' number sense.

Limitations

One of the limitations of this research study is the small sample size of 57 students from one school, and therefore not representative of the larger population. Nevertheless, the data and results in this study may be relevant to students, classes and schools similar to those participating in the study. Also, students in this study are traditionally exposed to learning mathematics through "drill and practice" and through written computation. Although most of the primary 1 teachers find the items in the number sense test interesting, they have also expressed their views that the items are not the typical mathematics the students do in the classroom or in school tests. Therefore, the students who participated in the study were not familiar with the types of items they encountered in the number sense test. Finally, the study lacks an analysis of the small-group interaction during the game play. Though a quantitative and qualitative approach is time consuming and requires extensive data collection and analysis, nevertheless analysis of the small-group interaction would provide a better understanding of the research study. The analyses of the interactions that occur among students during the game play would provide valuable data of in-depth information on the interactions and the difficulties teachers have in implementing such interactions in their mathematics classrooms.

Implications and Conclusion

Students bring along with them a myriad of priors and informal knowledge into the classroom and teachers could tap on these students' priors and informal knowledge before even embarking on formal teaching. As evident in the results of the study, the informal

setting of game play empowers students to use their prior knowledge, explicitly displaying what they know and what they do not. Through game play, the participatory structure of the setting allows for students to be engaged in an interaction to construct the knowledge scaffold by the teacher. Educators need to rethink their pedagogical practices to better understand how meanings and knowledge are could be constructed. The significant effect interaction has on students' learning and performance needs to be brought to the forefront in education especially to educators who miss on the broader goals of education and are results driven.

From this study, we could also draw implications on assessment. Activation of resource such as priors, intuitive and informal knowledge helped students to connect with the formal knowledge and ultimately pushes their performance higher. Formal assessment such as pencil-and-paper test can still play a role in teachers' daily practice in assessing students and may be a good platform to assess learners' competency and learning, but clearly it has its own limitations (Fan, 2011). The experience the students in NST-Game Play condition had in the formal assessment setting in NST in which they could not leverage that experience to help them perform better in the number sense in another setting (game play) further emphasised on limitation of such assessment. As evident in this study, small-group game play setting could afford learners to manifest what they know about number sense. Number sense goes beyond the formal setting and it is insufficient to depend on formal setting in assessing students' number sense. To rely solely on a pen-and-paper test to screen pupils into a programme or stream or to determine a child's level of competency may not be accurate. Other informal platforms such as clinical interviews should be explored.

References

- Alexander, P. A., Kulikowich, J. A. & Schulze, S.K. (1992, July). *How subject-matter knowledge affects recall and interest*. Paper presented at the XXV International Congress of Psychology, Brussels.
- Aunio, P., Ee, J., Lim, A. S. E., Hautomaki, J., & Van Luit, J. E. H. (2004). Young children's number sense in Finland, Hong Kong and Singapore. *International Journal of Early Years Education*, 12(3), 195-216.
- Australian Education Council. (1991). *A national statement on mathematics for Australian schools*. Melbourne: Curriculum Corporation.
- Case, R. (1998, April). *A psychological model of number sense and its development*. Paper presented at the annual meeting of the American Educational Research Association, San Diego, CA.
- Cobb, P., Gravemeijer, K., Yackel, E., McLain, K., & Whitenack, J. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one first grade classroom. In D. Kirshner & J. Whitson (Eds.), *Situated Cognition* (pp. 151-233). Lawrence Erlbaum Associates.
- Dochy, F. J. R .C. (1994). Prior knowledge and learning. In T. Husen & T.N. Postlethwaite (2nd Eds.), *International Encyclopedia of Education*, (p .4698-4702). Oxford New York: Pergamon Press.
- Dunphy, E. (2006). The development of young children's number sense through participation in sociocultural activity: Profiles of two children. *European Early Childhood Education Research Journal*, 14(1), 57-76.
- Edo, M., Planas, N., Badillo, E., (2009). Mathematical learning in a context of play. *European Early Childhood Education Research Journal*, 17(3), 325 – 341.
- Ekenstam, A. (1977). On children's quantitative understanding of numbers. *Educational Studies in Mathematics*, 8, 317 – 332.
- Fan, L. H. (2011). *Performance assessment in mathematics: concepts, methods and examples from research and practice in Singapore classrooms*. Pearson-Prentice Hall.
- Forman, E. A., & Cazden, C. B. (1985). Exploring Vygotskian perspectives in education: The cognitive value of peer interaction. In J. W. Wertsch (Ed.), *Culture, communication, and cognition* (pp. 323-347). Cambridge: Cambridge University Press.
- Gersten, R., & Chard, D. (1999). Number sense: Rethinking arithmetic instruction for students with mathematical disabilities. *Journal of Special Education*, 33, 18–28.
- Glaser, R., & De Corte, E. (1992). Preface to the assessment of prior knowledge as a

- determinant for future learning. In F.J.R.C. Dochy (Ed.), *Assessment of prior knowledge as a determinant for future learning* (pp. 1-2). Utrecht/London: Lemma B.V./Jessica Kingsley Publishers (<http://ericae.net/books/dochyl/>).
- Griffin, S. A., Case, R., & Siegler, R. S. (1994). Rightstart. Providing the central conceptual prerequisites for first formal learning of arithmetic to students at risk for school failure. In K. McGilly (Ed.), *Classroom lessons: Integrating cognitive theory and classroom practice* (p. 25 – 49). Cambridge, MA: MIT Press.
- Howden, H. (1989). Teaching number sense. *Arithmetic Teacher*, 36(6), 6 - 11.
- Kapur, M., & Bielaczyc, K. (2012). Designing for productive failure. *The Journal of the Learning Sciences*, 21(1), 45–83.
- Kapur, M. (2012). Productive failure in learning the concept of variance. *Instructional Science - An International Journal of the Learning Sciences*. doi: 10.1007/s11251-012-9209-6
- Kumpulainen, K., & Wray, D. (2002). *Classroom interaction and social learning: From theory to practice*. Routledge Falmer, London
- Labinowicz, E. (1987). Children's right to be wrong. *Arithmetic Teacher*, 35(4), 2.
- Markovits, Z. & Sowder, J.T. (1994). Developing number sense: An intervention study in grade 7. *Journal for Research in Mathematics Education*, 25, 4–29.
- Markovits, Z. & Pang, J (2007). The ability of sixth grade students in Korea and Israel to cope with number sense tasks. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, p 241-248. Seoul: PME.
- McIntosh, A. J., Reys, B., & Reys, R. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics*, 12(3), 2 – 8.
- McIntosh, A., Reys, B. J., Reys, R. E., Bana, J., & Farrell, B. (1997). *Number sense in school mathematics: Student performance in four countries*. Perth: MASTE.
- Ministry of Education (2006). *Curriculum planning & development division ministry of education: Mathematics syllabus primary 2007*. Singapore.
- National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Overholt, J. L., Holtz, J. W., Dickson, S. S. (1999). *Big Math Activities for Young Children for Preschool, Kindergarten and Primary Children*. Delmar Publishers, New York.
- Peters, S. (1998). Playing games and learning mathematics: The results of two intervention studies. *International Journal of Early Years Education*, 6(1), 49 – 58.
- Saxe, G. B. & Guberman, S. R. (1998). Studying mathematics learning in collective activity. *Learning and Instruction*, 8(6), 489 – 501.
- Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, 46, 13-57.
- Woodward, J., & Baxter, J. (1997). *Rules and reasons: Decimal instruction for academically low achieving students*. Paper presented at the annual meeting of the American Educational Research Association, Chicago.
- Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education*, 22(5), 390 – 408.
- Yang, D. C., Li, M. F., Li W. J. (2008). Development of a computerized number sense scale for 3rd graders: reliability and validity analysis. *International Electronic Journal of Mathematics Education*, 3(2), 110 – 124.

Average Revisited in Context

Jane Watson
University of Tasmania
<Jane.Watson@utas.edu.au>

Helen Chick
University of Tasmania
<Helen.Chick@utas.edu.au>

This paper analyses the responses of 247 middle school students to items requiring the concept of average in three different contexts: a city's weather reported in maximum daily temperature, the number of children in a family, and the price of houses. The mixed but overall disappointing performance on the six items in the three contexts indicates the need for concerted efforts to link numeracy across the curriculum as required in the new *Australian Curriculum*.

The concept of average, reflected mainly in relation to the arithmetic mean, was one of the first areas of mathematics education research that touched on statistics. Starting in the 1980s, researchers considered tertiary students' understanding of the weighted mean, with some success reported for interventions to improve performance (e.g., Mevarech, 1983). Somewhat later, other researchers considered school students' appreciation of properties of the arithmetic mean (e.g., Strauss & Bichler, 1988), culminating in the seminal work of Mokros and Russell (1995) in considering average more generally in terms of ideas related to mean, median, and mode but categorised as representative or non-representative. At the same time, procedural issues related to the algorithm for calculating the mean and the concept of the mean as a balance point were being considered by others (e.g., Cai, 1998). The first longitudinal studies of school students' understanding of average employing a cognitive development model were reported by Watson and Moritz (1999, 2000). After 2000 the interest moved to teachers, with Jacobbe and Carvalho (2011) providing an overview of teachers' and pre-service teachers' understanding.

The word "average," *per se*, is not used in the *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2011) in any descriptors or elaborations associated with Data Representation. The word "mean" appears at Years 7, 8, 9, and 10A; "median" appears at Years 7, 8, and 9; and "mode" appears at Year 7. The most meaningful descriptors and elaborations are presented for Year 7 (see Table 1). Centre, spread, and outliers are important statistical words in the elaborations, as are the references to some sort of context, in this case "compar[ing] land use in the local municipality" and "connecting them to real life." In Year 8 there is reference to "using mean height for a class," whereas in Year 9 the elaboration "comparing the rainfall in various parts of Australia, Pakistan, New Guinea and Malaysia" could employ the mean and median, mentioned elsewhere in Year 9.

The occasional insertion of context within Mathematics relates to the seven General Capabilities required across the entire Australian curriculum. One of the seven is Numeracy, and each other curriculum area—for example, History or Science—includes a statement on how the Numeracy Capability is developed within its discipline. Across all disciplines students

need to recognise the mathematical basis of authentic problems and engage constructively in their solution. The identification of mathematical demands in learning areas enables students to:

- transfer their mathematical knowledge and skills to problem solving in those learning areas
- recognise the interconnected nature of mathematical knowledge, learning areas and the wider world
- become confident and willing users of mathematics in their lives.

Table 1
Year 7 Mathematics Curriculum extracts (ACARA, 2011)

Data representation and interpretation	Elaborations
Calculate mean, median, mode and range for sets of data. Interpret these statistics in the context of data (ACMSP171)	• Understanding that summarising data by calculating measures of centre and spread can help make sense of the data • Calculating mean areas set aside for parkland, manufacturing, retail and residential dwellings to compare land use in the local municipality
Describe and interpret data displays and the relationship between the median and mean (ACMSP172)	• Using mean and median to compare data sets and explaining how outliers may affect the comparison • Locating mean, median and range on graphs and connecting them to real life

With context playing an explicit role in the *Australian Curriculum* and in light of previous research the following research question is considered in this report.

What levels of understanding do middle years students display in relation to linking their definitional knowledge of average to the context within which it is placed?

Methodology

Instruments

The items about “average” used in this study were adapted from earlier research. They are presented in Figure 1 in the order in which they appeared on a larger survey. The first item, Q5, about the average family having 2.3 children, was originally used as part of an interview protocol by Watson and Moritz (2000). In the original study this question followed a question about what it means for a family to have 2.3 children, which was helpful in setting the context for the item as used there. The second and third survey items, Q9 and Q10, were part of a protocol used by Watson and Kelly (2005) exploring students’ understanding of variation in the context of a city’s daily maximum temperature over a year. As discussed earlier, spread, outliers, and range provide context for considering the mean and its contribution to an overall description of a distribution. The last three items, Q21 to Q23, were first used by Watson and Moritz (1999) in a survey format with 1654 students in grades 6, 8, 9, and 11. The context of median house prices is one of the most common applications of average in social settings, especially as a contrast to mean. Due to time constraints not all study participants completed the final three items.

Sample

The 247 students who completed the survey were part of the StatSmart project (Callingham & Watson, 2008; Callingham, 2010) and had completed one similar survey previously during the project, including Q21 to Q23 but not Q5, Q9, or Q10. The students came from three Australian states (Tasmania, 136; Victoria, 64; South Australia, 47). The number of students at each year level is given in Table 2. For the purpose of analysis, the year groups were combined in pairs, 6/7, 8/9, and 10/11. In one state Year 7 was a primary grade and there were relatively few students in Year 6. Years 8 and 9 are, according to the

curriculum, appropriate for consolidating the concept of average. There were very few Year 11 students in the study. Overall there were 47% females and 53% males.

Item 5: The average number of children in 10 families in the neighbourhood is 2.3. One family with 5 children leaves the neighbourhood. What is the average number of children per family now? Show your work here.

Item 9: Some students watched the news every night for a year, and recorded the daily maximum temperature in Hobart. They found that the average maximum temperature in Hobart was 17° C

Item 10: Ben drew the following graph to show the maximum temperatures in Hobart throughout the year. Explain what the graph shows.

Maximum temperature in Hobart for 365 days.

Max. temp. (°C)	No. of days
9	1
10	1
11	1
12	1
13	2
14	3
15	4
16	6
17	10
18	8
19	6
20	4
21	3
22	2
23	1
24	1
25	1

This article appeared in a newspaper.

Hobart defies homes trend

AGAINST a national trend, Hobart's median house price rose to \$88,200 in the March quarter - but, Australia-wide, the average wage-earner finally can afford to buy the average home after almost two years of mortgage pain.

Item 21: What does "average" mean in this article?

Item 22: What does "median" mean in this article?

Item 23: Why would the median have been used?

Figure 1. Items used in survey.

Table 2
Students in Each Year

Year	6	7	8	9	10	11	All
N	14	56	36	59	78	4	247

Analysis

The rubrics used to score responses to the six items are given in Table 3. The codes reflect the influence of the SOLO Model in recognising greater structural complexity at higher levels but also the required correctness or appropriateness of responses. The rubrics for Q5, Q9, and Q10 are based on those used in earlier interview settings by Watson and Moritz (2000) and Watson and Kelly (2005). Those for Q21 to Q23 are found in Watson and Callingham (2003).

Results

A summary of the percent of responses by item and level is given in Table 4. For Q21 to Q23 the percentages refer to the number of students reaching these items. NA records the number of missing students. Performance across the codes for the items are similar, but with the mean performance of Year 8/9 students better than Year 6/7 students on all items and marginally better than the Year 10/11 students on all but Q9. Because of the difference in pattern of performance across the years for Q9 and Q10 and the closer relationship of Q5 and Q21 to Q23 to the definitions of mean and median in the *Australian Curriculum* (ACARA, 2011) these four items are presented first, followed by Q9 and Q10.

Table 3
Rubrics for the Six Survey Items

Code	Criteria
Question 5, 2.3 children	
0	No response; incorrect answer with no explanation or unintelligible reasoning
1	Partial attempt – can recognize some aspect of the problem
2	Correct answer with no explanation or plan without explicit answer
3	Correct answer with appropriate explanation
Question 9, 17 degrees	
0	No response; imaginative or idiosyncratic comments
1	A comment about an aspect of temperature OR a statement describing 17 on a continuum
2	A comparison of temperature with other places
3	A comment about the temperature with an acknowledgment of variation focusing around 17; or an explicit reference to variation away from 17
Question 10, 17 degrees and graph	
0	No response; idiosyncratic responses or misreading details of the graph
1	A comment about the <u>shape of graph</u> ; however, no understanding of graph purpose evident
2	Statement about <u>frequency</u> and understands graph <u>purpose</u> , but no focus on importance of 17
3	Focus on 17 only: more frequent temperature
4	Statement about graph's frequency and purpose, and reference to the <u>importance of 17</u>
Question 21, "average" in article	
0	No response; No idea of central tendency, often tautological
1	Single idea not related to context
2	Describes central tendency for a data set or method of obtaining average from a data set (maybe related to context) – (needs to state both central tendency and the associated data set)
Question 22, "median" in article	
0	No response; No idea of central tendency, often tautological
1	Single idea not related to context
2	Describes the central tendency for a data set or the method of obtaining the median from a data set (sometimes related to context)
Question 23, Why "median"	
0	No response; Response that does not refer to question (e.g., language/price)
1	Usefulness or fairness (without explicit mention of outliers)
2	Mention of outliers or extreme values

Table 4
Responses at Each Code Level and Mean for Items and Year.

Year	6/7 n = 70	8/9 n = 95	10/11 n = 82
Item and code levels			
Q5, 2.3 children (0, 1, 2, 3)	(73, 17, 4, 6) mean = 0.40	(34, 17, 6, 43) mean = 1.57	(45, 20, 4, 32) mean = 1.20
Q9, 17 degrees (0, 1, 2, 3)	(59, 23, 6, 13) mean = 0.73	(43, 37, 4, 16) mean = 0.93	(56, 17, 4, 23) mean = 0.94
Q10, 17 degrees and graph (0, 1, 2, 3, 4)	(30, 11, 19, 26, 14) mean = 1.81	(15, 13, 33, 17, 23) mean = 2.21	(26, 9, 21, 20, 26) mean = 2.11
Q21 "average" in article (0, 1, 2, [NA*])	(55, 34, 11, [25]) mean = 0.57	(16, 61, 23, [21]) mean = 1.07	(31, 51, 18, [21]) mean = 0.87
Q22 "median" in article (0, 1, 2, [NA*])	(66, 16, 18, [26]) mean = 0.52	(31, 34, 35, [21]) mean = 1.04	(28, 47, 25, [25]) mean = 0.96
Q23 why "median" (0, 1, 2, [NA*])	(86, 14, 0, [26]) mean = 0.17	(62, 24, 14, [21]) mean = 0.51	(73, 18, 9, [26]) mean = 0.36

($x_0, x_1, x_2, \dots, x_n$) indicates the percent attempting the item who performed at levels (0, 1, 2, ..., n)

* Number who did not reach the item.

Item Q5, 2.3 Children

Of the half of the 247 responses that were coded 0, most (89) put question marks, said they did not know, or left the space blank. Of the 37 responses coded 0 that wrote something, 13 put down a number different from 2 with no explanation (e.g., 1.3, 1.6, 2.1, 5, 1.15). Two responses were drawings of boxes with tallies (likely to represent families with children) with no conclusion. Two responses questioned the existence of 0.3 of a child. Ten responses provided calculations that could not be interpreted (e.g., $2.3 \div 9 = 4.23$, $10 \div 4 = 2.3$). Three responses provided a written answer indicating that the average did not change because “the maximum number of children would be 5 to start with most likely,” because “the family had more than the average amount of children,” or “2.3 cause it doesn’t say that the family was the only one that had five kids.”

Of the 47 responses coded 1, 24 worked with the number 23 (10×2.3), subtracted 5, but divided by 10 rather than 9, obtaining an answer of 1.8. Other answers had errors in calculations using numbers that indicated the students knew or selected numbers relevant to the solution, perhaps not completing the process. Six of these answers were larger than 2.3, indicating that students did not check their answers, as the question implied a reduction in the average due to more than the average being removed. Four responses showed some understanding of the meaning of average but claimed that there was not enough information provided to work out the answer (e.g., “It would depend on how many children the other 9 houses had”).

Of the responses coded 2, three responses showed more complete intuition but did not actually calculate the new average (e.g., “Less than 2.3,” “Decreases, because a high number was taken out which was keeping the average high”). Nine others gave an answer of 2.0 but with no explanation or unclear justification. Of the 71 students who showed appropriate calculations to reach the answer of 2.0 (coded 3), there was a variation in the amount of detail provided. Eight of these responses created 10 families with appropriate numbers of children to equal 23, with one of 5; that family was removed and the average shown for the 9 remaining. It would appear that the context supported these students in solving the problem. Others followed the procedure suggested above dividing by 9.

Items 21 to 23, Average and Median

As shown in Table 4 these items were more difficult for students than the others. Across Q21 to Q23, of the responses coded 0, an increasing number wrote that they did not know or could not remember the concept involved or why the median was relevant (20 for Q21, 48 for Q22, and 66 for Q23). A further 12 responses to Q21 were tautological, as was 1 response to Q22. Other answers to Q21 that were coded 0 focussed on irrelevant aspects: “average means decent,” “average means higher number,” “a house that could be any sort of house.” Q22, about the meaning of median, had responses such as the following: “the median means the price of houses rose,” “most popular,” “maybe the main price.” The modal idea occurred on several occasions. Many of the responses to Q23 did not focus on dealing with outliers when explaining why the median was used: “because people are most likely to see a house that costs 88 700,” “to determine if the average worker can afford the house,” “because the average is already used,” “to show an approximate house price in Hobart,” “to add in statistical stuff and make it interesting.”

For Q21, on the meaning of average, responses did not have to focus on a particular meaning of average. Responses coded 1 reflected a single appropriate aspect of any type of average: “the normal wage earner,” “the most common wage earner,” “in the middle.” A

code of 2 was given for responses that added to this the context and/or discussion of the data set from which the average is obtained: “The average means the workers whom are not high nor low, but middle,” “It’s all the data added and divided by the number of data,” “The average house means not an expensive house but not a cheap one either. In other words the common home.”

Similarly, for Q22 on the meaning of median, code 1 responses reflected a single aspect of the concept: “I think it means the same as average,” “standard house price.” Code 2 responses were more specific about the properties of median, although perhaps more colloquial than a text book definition: “Median is the amount of money used for the houses that sold in between the highest and lowest,” “Median means the middle house price of the March quarter,” “The complete middle of ranked data.”

For Q23, code 1 responses reflected the usefulness of the median without referring to outliers: “Because the median is the middle range amount that people spend,” “So that readers don’t think the house prices are too high or too low,” “Average would be inaccurate.” Code 2 responses acknowledged the need to avoid outliers: “To exclude any extremely high or low prices,” “Because if the mean was used, the price of a really expensive house would put up the mean. The median is more accurate.”

Items Q9 and Q10, Weather

Items Q9 and Q10 were more open-ended than the other items, inviting students to reflect on the contextual meaning of an average rather than its mathematical definition. Fewer students than for Q5 left these items blank or said they did not know (30 for Q9 and 18 for Q10), suggesting that the context and question encouraged engagement. For many of the items within year groups the mean performance compared to the total possible score was better for Q10.

For Q9—which asked about average temperature—there were 13 tautological responses. Other responses coded 0 for Q9 presented views about the weather not related directly to the average maximum temperature or involving a misinterpretation of the average. These included: “This tells us that the temperature rarely rises over 17°C,” “That the temperature is usually 17°C maximum,” “We might have had a drought.”

Code 1 responses to Q9 contained a single comment about an aspect of temperature or about 17° on a continuum: “It tells us that most days the temperature in Hobart is 17°C or close to it,” “That it is 17°C. We have had some cold days, we’ve had some hot days.” Code 2 recognised responses that compared Hobart temperatures with other places. As seen in Table 4, there were few of these: “Hobart’s weather for maximum temperature is around 17°C, not going up much like the mainland,” “Well when you think of WA their average would most likely be in the 20 range. So that tells us that a lot of the time it is unpredictable but I would say Hobart is a cold place.”

In contrast, the highest level of response (Code 3) acknowledged variation close to or away from 17°C: “the average maximum temperature is not very warm. It’s probably because Hobart has cold winter days and not-so-hot summer days,” “It may be hot in summer but it’s colder for the rest of the year which brings the whole average down,” “It is quite cool. Probably ranging from about 0° → 28° mostly in the 17° area,” “Even though there would be plenty of hot ones there were also cold ones to keep the average down.”

Item Q10, which referred to a graph of the year’s daily maximum temperatures, was the most open-ended of the six items analysed here. Its purpose was to assess the link between the knowledge that the average maximum for the year was 17°C and a graph depicting the

frequency of temperatures throughout the year. The graph was created as a “student response” and not based on actual data. Student responses coded 0 were idiosyncratic or misread details of the graph: “It’s hot around 16 17 18,” “17°C was the highest,” “The graph shows that the maximum degrees in Hobart is 17°C and it shows that it is a vertical graph,” “That it’s ‘averaged’ by mode.”

Responses coded 1 noted the shape of the graph but misinterpreted its purpose: “The temperature rises then drops,” “It shows the mean, mode and median of Hobart’s weather throughout the year,” “The most frequent temperature,” “In the middle of the year the temp has gotten hotter and in the beginning and end it dropped,” “It goes in a pattern, starts low, goes up, then low again.” Code 2 responses made statements about frequency and the purpose of the graph but not the importance of 17°C: “How high the temp got to in °C and how many days it got to that temp,” “That the temperature was hardly ever extremely high or extremely low, it stayed in the middle,” “It shows the range of temperature maximums: 9-25. It also shows how many days were a certain temp relative to other temps,” “That we are mainly average temperature but we do have a few hot days and few cold days.”

Code 3 responses focussed on 17°C only as the more frequent temperature, sometimes with additional erroneous interpretation: “That it was 17° a lot of days throughout the year,” “It is most likely to be 17° in Hobart,” “This graph shows that 17°C was a very popular temperature in Hobart. That the temperature goes up to 17°C then down,” “I think it shows that the average maximum temperature was 17°C that year.” Code 4 responses, in contrast, attended to multiple aspects of the graph, and made statements about frequency and the purpose of the graph including acknowledgment of the importance of 17°C. Examples of these responses include: “It rarely gets to 9 and 25°C but it does a lot get to 17°C,” “It shows us the maximum temperature did not exceed 25° or come below 9°. It was 17 almost twice as often as it was 15, etc, though it doesn’t actually display the no. of days,” “It show that most of the time the temperature is around 15-19°C, with 17°C being the most. This is most likely why the average temperature is 17°C as that is the range where the temperatures mostly were,” “That 17 was the most common temp but it varied alot,” “That a majority of the days were 17° and many were close to 17°,” “That most days were seventeen. We have less days in each temperature as we move away from 17°C.”

Discussion

The outcomes of the analysis of these three contexts point to some intriguing conclusions and messages for the curriculum, assessment, and teachers. When students are asked specific procedural or definitional questions in context (Q5, Q21-Q23) they do not perform as well as when they are asked more general questions placing central tendency in a context that encourages the engagement with variability in the overall context (Q9, Q10). Given the description and elaboration in Table 1 from the curriculum, students at least from Year 8 would be expected to perform better on all items, and certainly on Q5, Q21-Q23 in particular, than seen in this study. These four items reflect an emphasis—on real world properties of mean and median—that has been in the curriculum for many years. Q9 and Q10 reflect a more recent focus on the contribution of variation. The fact that the students in this study have been more engaged and successful with Q10 would appear to support Watson’s (2005) view that although expectation and variation are the foundations of the Chance and Data curriculum, variation is more intuitive, developing earlier.

At least two implications arise for teaching. On no items is there a noticeable continued improvement in performance from Years 8/9 to 10/11. Further explicit engagement with the

concepts appears to be required. If teachers can appreciate the diversity in the levels of comprehension found here, and the aspects of understanding that are more difficult for students to reach, then they should be able to devise remedial strategies to assist students to come to a stronger understanding. The limited nature of some responses suggests that greater care and attention should be given to encouraging in-class discussion of how statistical information complements and enhances understanding of real world contexts, and to sharing different appropriate strategies for making sense of data in context.

The writers of assessment items, for example for NAPLAN, may need to rethink what questions are asked and how, if they want to encourage students to have a wider view of the application of the concept of average than just the definitions and procedures. It is easy to write an item like Q5 but not as easy to devise good multiple choice distracters for items like Q9 and Q10. In the future the demands of the *Australian Curriculum* will require both procedural and conceptual understanding, and due consideration of context.

Teachers who are aware of the General Capabilities in the *Australian Curriculum* also need to be aware of the need to satisfy not only the requirements of Mathematics but also the more general Numeracy needs of other areas of the curriculum and the goal to “become confident and willing users of mathematics in their lives” (ACARA, 2011). Tasks related to Q9 and Q10 are more likely to be meaningful for these aims than items like Q5.

Acknowledgements

This research was partially funded by ARC Linkage Grant No. LP0669106.

References

- Australian Curriculum, Assessment and Reporting Authority (ACARA). (2011). *The Australian Curriculum: Mathematics, Version 1.2, 8 March 2011*. Sydney, NSW: Author.
- Cai, J. (1998). Exploring students' conceptual understanding of the averaging algorithm. *School Science and Mathematics*, 98, 93-98.
- Callingham, R., & Watson J. M. (2008). Overcoming research design issues using Rasch measurement: The StatSmart project. In P. Jeffery (Ed.), *Proceedings of the AARE annual conference, Fremantle, December, 2007*. Available at <http://www.aare.edu.au/07pap/cal07042.pdf>
- Callingham, R. (2010). Trajectories of learning in middle years' students' statistical development. Refereed paper in C. Reading (Ed.), *Data and context in statistics education: Towards an evidence-based society*. (Proceedings of the 8th International Conference on the Teaching of Statistics, Ljubljana, Slovenia, July). [CD-ROM] Voorburg, The Netherlands: International Statistical Institute.
- Jacobbe, T., & Fernandes de Carvalho, C. (2011). Teachers understanding of average. In C. Batanero, G. Burrill, C. Reading, & A. Rossman (Eds.), *Teaching Statistics in School Mathematics – Challenges for Teaching and Teacher Education* (pp. 199-209). New York: Springer.
- Mevarech, Z. (1983). A deep structure model of students' statistical misconceptions. *Educational Studies in Mathematics*, 14, 415-429.
- Mokros, J., & Russell, S. J. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education*, 26, 20-39.
- Strauss, S., & Bichler, E. (1988). The development of children's concept of the arithmetic average. *Journal for Research in Mathematics Education*, 19, 64-80.
- Watson, J.M. (2005). Variation and expectation as foundations for the chance and data curriculum. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, pp. 35-42). Sydney: MERGA.
- Watson, J. M., & Callingham, R. A. (2003). Statistical literacy: A complex hierarchical construct. *Statistics Education Research Journal*, 2(2), 3-46.
- Watson, J. M., & Kelly, B. A. (2005). The winds are variable: Student intuitions about variation. *School Science and Mathematics*, 105, 252-269.
- Watson, J. M., & Moritz, J. B. (1999). The development of concepts of average. *Focus on Learning Problems in Mathematics*, 21(4), 15-39.
- Watson, J. M., & Moritz, J. B. (2000). The longitudinal development of understanding of average. *Mathematical Thinking and Learning*, 2(1&2), 11-50.

Teaching for Abstraction: Collaborative Teacher Learning

Paul White
Australian Catholic University
<paul.white@acu.edu.au>

Sue Wilson
Australian Catholic University
<sue.wilson@acu.edu.au>

Michael Mitchelmore
Macquarie University
<mike.mitchelmore@mq.edu.au>

Working collaboratively with the researchers, a small team of teachers developed and taught a lesson based on the Teaching for Abstraction model (White & Mitchelmore, 2010). This paper reports how one teacher learned about the model and implemented it in her lesson. It was found that she had assimilated several key features of the model, such as starting with several embodiments of the target concept and guiding students to look for similarities between them. However, she had some difficulty focusing students' attention clearly on the underlying mathematical similarity. It was concluded that teachers need to experience more examples of Teaching for Abstraction before they can to reify and apply the model faithfully.

In 1963, Zoltan Dienes advanced a theory that all elementary mathematical concepts are the result of abstraction and generalisation from common experiences. Later, Richard Skemp summarised some key terms in this theory:

Abstracting is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. ... To distinguish between abstracting as an activity and abstraction as its end-product, we shall ... call the latter a *concept*. (1986, p. 21, italics in original)

Over the past decade, the first and last author have been developing a teaching model, called Teaching for Abstraction, which is based on Dienes's theory. This model (described in more detail in White & Mitchelmore, 2010) consists of four phases:

- *Familiarity*. Students explore a variety of contexts where a concept arises, in order to form generalisations about individual contexts and thus become familiar with the underlying structure of each context.
- *Similarity*. Teaching then focuses on helping students recognise the similarities and differences between the underlying structures of these contexts.
- *Reification*. The general principles underpinning the identified similarities are drawn out, and students are supported to abstract the desired concept into a mental object that can be operated on in its own right.
- *Application*. Students are then directed to new situations where they can use the concept.

In our investigations to date, also described in White and Mitchelmore (2010), we have designed Teaching for Abstraction units on angles, decimals, percentages and ratio and prepared corresponding teaching materials. Our studies of their implementation across Years 3-8 indicate that the model has promise in terms of student learning of key concepts and generalisations but that further exploration and development is still needed.

In particular, it was repeatedly found that some teachers had difficulty understanding and therefore implementing the Teaching for Abstraction model. Some teachers followed the materials provided to the letter, not being in a position to adapt them to their particular

classroom situation; others reversed the suggested teaching order and thus subverted the essence of the model. A possible reason for this result is that the teachers were not involved in planning the unit or developing the teaching materials. As Pegg and Panizzon (2008) warn, teachers need to be actively involved if innovative ideas are to be accepted.

We therefore decided to investigate a different method of implementing the model, in which we would work with teachers to develop Teaching for Abstraction lessons that would better fit their classroom situations. We hypothesised that such a procedure would lead to improved teacher learning about the model and hence a more faithful implementation.

The Present Study

As a theoretical framework for assessing the impact of this new method, we use the Interconnected Model of Professional Growth (Clarke & Hollingsworth, 2002). This model contains four domains and various interactions between them, as shown in Figure 1. Note that most interactions are bidirectional. For example, teachers enact a new idea, belief or practice in their classroom and then reflection provides (positive or negative) feedback on the critical features of that innovation and its value to them.

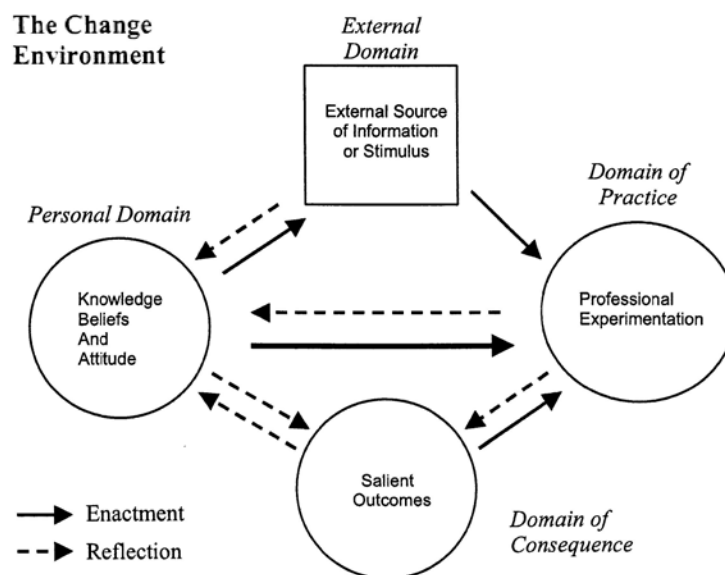


Figure 1. The Interconnected Model of Professional Growth (from Clarke & Hollingsworth, 2002).

As an initial exploratory study, we worked with a small group of teachers in a small regional town to prepare a number of unconnected lessons. The teachers then taught these lessons in a kind of lesson study mode (Hart, Alston, & Murata, 2011). In terms of Figure 1, the External Domain was the collaborative lesson preparation and study, the Personal Domain was teachers' knowledge and acceptance of Teaching for Abstraction, the Domain of Practice was the lessons taught, and the Salient Outcomes consisted of student outcomes from the lessons taught. Our basic research question was:

- Can collaborative lesson development lead to a faithful implementation of the Teaching for Abstraction model?

To provide answers to this question, data were collected on all four components of the professional growth model in Figure 1. We report a case study of one teacher involved in the collaborative planning and delivery of one Teaching for Abstraction lesson.

Method

Participants

The study took place in Grade 6 at St Joseph's,¹ a small primary school in regional New South Wales. Initially, three teachers volunteered to participate: David, Bridget and Uarda. However, David was unwell during our visits and Bridget had several other calls on her time. Only Uarda participated fully, and she is therefore the focus of this paper. Uarda had been teaching for 5 years, and was enrolled in a master's degree at the time.

Procedures

The authors paid two 2-day visits to St Joseph's. Both visits followed the same pattern: The research team (the authors, Uarda, and whichever of the other teachers were available) met for an initial discussion on the afternoon of Day 1 to decide on the aims for the following day. Early on Day 2, the team developed a lesson and prepared the necessary materials. Later in Day 2, Uarda taught the experimental lesson while the others observed, after which the team discussed the lesson and modified it as they felt appropriate. Bridget then taught the revised lesson while the others observed. Because of space limitations, only the first visit is reported here.

Data Collection and Analysis

The research team's discussion sessions were audio recorded and transcribed, but no recordings were made of the lessons. Instead, one member of the research team acted as a lesson recorder, taking detailed notes of the lessons that included time markers for the major transitions. The other members subsequently added their individual observations.

To assess short-term student learning, the recorder noted students' comments during the face-to-face teaching and all members of the team circulated and observed students during the small group work phase, occasionally interacting with them to clarify what they were attempting to do. In addition, the teachers administered a short quiz at the end of the first visit and a short questionnaire at the end of the second visit.

The analysis of how Uarda interpreted and applied the Teaching for Abstraction model, as well as its resulting effectiveness and potential, focussed on the four components shown in Figure 1. Firstly, each author formed an interpretation of each of these components on the basis of their own observations, notes and informal discussions during the site visits. The three authors then cross-validated and synthesised their separate interpretations during extensive discussions, frequently re-examining transcripts and field notes to reach consensus.

Results

Initial Discussions

At the first meeting of the research team, the authors outlined the Teaching for Abstraction model and gave some examples of their previous research. In particular, they provided the teachers with copies of the instructional materials developed for a previous

¹ We use pseudonyms for the school and the teacher participants.

investigation of teaching percentages in Year 6 (White, Mitchelmore, Wilson, & Faragher, 2008). They also explained the purpose of the study, and asked teachers for their reaction to the model and its potential in their situation.

The teachers expressed interest in experimenting with the model. Uarda indicated that she regularly trialled novel approaches that she believed might be beneficial to her students. Teaching for Abstraction had a definite resonance for her because she was particularly keen on the use of realistic scenarios and always tried to embed mathematics in contexts that she felt would be familiar and interesting to students.

The teachers then outlined a number of topics where they felt their students were having most difficulty, and it was agreed that the next day's lesson would focus on place value in decimals. It was also agreed that Uarda and Bridget would both teach the same lesson with their own classes, with time for discussion and revision between them.

Lesson Planning

The teachers reported that many students were still unable to tell whether, for example, 0.65 was bigger than 0.8. It was decided to focus the lesson on this topic, restricting the content to 1- and 2-place decimals. This topic was considered to be sufficiently narrow for a single lesson that could nevertheless be of significant value to students. The teachers could extend students' understanding to other decimals later.

The team then sought to identify a small number of familiar contexts involving one- or two-place decimals, which could then be compared to identify and abstract the underlying similarity. The teachers initially had some difficulty with this task but, after some suggestions from the authors, the team agreed on four contexts: money (\$0.65 vs \$0.8), length (0.65 cm vs 0.8 cm), fractions of a box of 100 lollies (0.65 vs 0.8) and fractions of a 10×10 square of chocolate (0.65 vs 0.8). It was decided to write four scenarios and to break each class into four groups that would (after an initial introduction) circulate around the four tasks, spending no more than 5 minutes on each. This would leave time for a final discussion in which generalisations about decimal place value could be abstracted.

The teachers then left to collect or create materials for the lesson (counters for money and lollies, rulers and butcher paper for measurement, and Dienes blocks for the chocolate). After they returned, discussion continued on how to phrase the tasks to include a natural comparison. The entire planning session took just under an hour.

The Lesson

Uarda started the lesson by writing the four scenarios on the whiteboard. She explained the rotation procedure for the lesson (which was already familiar practice in her classroom), divided the students into four approximately equally sized groups, and instructed them to start working on their first scenario. No other introduction was given.

During the small group work, the group working on the money task finished first at each rotation. The lollies group took a long time to count out two lots of 100 counters, and the measurement group was slow to draw their two lines. So although the first rotation took only 3 minutes, as planned, the group work took up 26 minutes in all.

Uarda then commenced the final section of the lesson. Asked to identify similarities between the four scenarios, students initially remarked only on superficial aspects: All the scenarios involved Tom and Judy and the numbers 0.8 and 0.65, and Tom always won. Asked how we know that 0.8 is bigger than 0.65, some students demonstrated considerable insight with the responses "0.8 is 0.80", "0.65 is 6 and a half, whereas 0.8 is 8" and "0.8 is different from 0.08". Uarda then gave an explanation using Dienes blocks, taking the flat to represent 1 unit and having the students recognise that a long then represents 0.1 and a cube

represents 0.01. Many students appeared to realise that 0.8 is represented by 8 longs and that 0.65 corresponds to 6 longs and 5 cubes, but one student suggested that a cube represented 0.1 and several students called 0.65 “point sixty-five”.

Lesson Discussion

After the lesson, Uarda remarked on how well the students enjoyed the hands-on aspect of the lesson. She was particularly struck by the fact that some students who are normally silent had participated actively: “Monica put up her hand at the back a few times. She’s somebody that generally has no idea. And she was hands-up confidence.”

Students’ tendency to notice irrelevant similarities was noted, and it was decided for the second trial lesson to vary the children’s names and to ask alternately for the larger or smaller number. It was also decided to compare 0.4 and 0.25 in order to make the lollies and measurement tasks more manageable. Bridget left to write the revised scenarios on the whiteboard in her classroom in preparation for her lesson.

Discussion then turned to the question of how general students’ understanding of place value was. It was felt that the approach taken had tended to reinforce whole number thinking (e.g., treating 0.8 as 80 out of 100 rather than 80% of the flat). It was decided to put more emphasis on decimals as fractions of a whole rather than numbers of parts of the whole, even by including questions such as “What is 0.4 of a student’s pony tail?” Ideas were canvassed on how to question students in order to take them beyond the specific insights they had shown, by focussing more strongly on the similarities between the four scenarios. The plan for the repeat lesson (not reported here) was modified accordingly.

As a measure of how much students had learned from the lesson, Uarda suggested a short quiz in which students would be asked to identify the smaller of two decimals; the team then constructed this quiz jointly. Figure 2 shows the resulting quiz.

Circle the smaller number:					
(1) 0.8	(2) 0.52	(3) 0.6	(4) 0.8	(5) 1.6534	(6) 2
0.65	0.7	0.298	0.09	1.72	2.2

Figure 2. Short quiz questions.

Assessment

Shortly after the lesson discussion, Uarda administered the quiz to a total of 24 students in her class. About 70% answered Items 1, 2 and 6 correctly, while Items 3, 4 and 5 were only answered correctly by just over 50% of the students. However, five students marked all the *larger* numbers correctly. It could be argued that they understood how to find the larger or smaller of two decimals but had not read the instructions carefully and had proceeded as in the scenarios they had just experienced. On this assumption, the percentages correct would have been about 20% greater than those just quoted.

If this assumption is correct, the result for Item 2 would suggest that the majority of Uarda’s students had generalised their knowledge of 2-place decimals to numbers other than the ones in the given scenarios. However, the zero in Item 4 seems to have introduced difficulties for several students. Most students appeared to be able to cope with numbers with no decimal places, as in Item 6, but it was clear from Items 3 and 4 that more work needed to be done on decimals with more than 2 places.

The assessment results thus confirmed the post-lesson conclusion that the discussion of similarities and differences needed to put greater emphasis on the underlying structure of the decimal notation system.

Discussion

We discuss the results in terms of the four components of the Integrated Model of Teacher Professional Development (Figure 1). Because Application was deliberately avoided in the experimental study, we consider only the first three phases of the Teaching for Abstraction model.

The Domain of Practice: Implementation of the Teaching for Abstraction Model

The lesson certainly incorporated major elements of the Teaching for Abstraction model. Most obvious was the emphasis on exploring a variety of contexts where a concept arises (Phase 1 of the model). This characteristic was probably the result of the team's careful planning, firstly to clearly identify a focus concept or generalisation and secondly to select scenarios that embodied this abstraction and would be familiar to the students.

Similarity recognition (Phase 2) was also present. However, students showed a strong tendency to focus on superficial similarities, and it was difficult for Uarda to bring students' attention to mathematical similarities.

Neither lesson proceeded very far with reification (Phase 3), where the underlying structure could have been laid bare, explained, and formalised. Dienes blocks did not seem to relate well to the similarities that students had observed and were almost a distraction.

The Domain of Consequence: Student Outcomes

The assessment of student outcomes suggested that most students had achieved the narrow objective of ordering one- and two-place decimals. However, understanding was not deep enough to allow many students to extend this knowledge to decimals containing zeros after the decimal point or with more than two places. We attribute this finding to shortcomings in the Similarity and Reification phases of the lesson.

The lesson had some other positive outcomes: The realistic nature of the activities appeared to engage some students more than usual, and there was clear evidence that students were responding well to the chance to think for themselves.

The Personal Domain: Teacher Learning

Uarda emphasised right from the first meeting her belief in the importance of embedding mathematics in realistic, familiar contexts. However, the idea of using more than one context for each concept was clearly new to her, and in the initial discussion sessions the authors had to repeatedly stress the importance of similarity recognition in the Teaching for Abstraction model. Consequently, she put a lot of thought and effort into the process of identifying and selecting appropriate contexts for the trial lesson. Our observations showed that she was skilled in managing a classroom so that students could work on several contexts in one lesson, leaving sufficient time to discuss the students' findings. We conclude that Uarda had successively learned to implement the Familiarity phase of the Teaching for Abstraction model.

Uarda also attempted to implement the Similarity phase in that she challenged students to identify similarities in, and make generalisations from the exploratory activities they had just carried out. However, she did not clearly distinguish superficial and mathematical similarities and did not probe for explanations or forge links to students' existing knowledge. As a result, Uarda did not effectively address the Reification phase. It would seem that she had only just begun to form a concept of the content and purpose of these two phases.

Uarda's use of Dienes blocks had not been discussed in the lesson planning sessions and represented a return to a more didactic pedagogy. It was an honest attempt to address a known student difficulty, but it only served to disrupt the structure of the Teaching for Abstraction lesson. In this respect, Uarda's modification was similar to the way teachers in earlier studies had subverted the model by returning to a more familiar lesson structure (White & Mitchelmore, 2010).

The External Domain: Collaborative Lesson Planning

The collaborative lesson planning went according to plan, although it would certainly have been more effective had all the teachers been able to participate to the extent that Uarda did. She certainly had ownership of the Teaching for Abstraction part of her lesson.

However, it is now clear that the professional development we provided for Uarda and her colleagues had several shortcomings. Firstly, we had ignored Dienes's (1963) dictum that "all variables need to be varied if the full generality of the concept is to be achieved" (p. 158) and had not stressed the importance of varying irrelevant factors when choosing the initial scenarios. However, this principle soon became obvious and was readily corrected in subsequent lessons.

More importantly, we did not stress strongly enough the significance of the crucial steps of similarity recognition and reification. It is clear in hindsight that the teachers needed to spend more time reflecting on mathematical similarities—how to distinguish them from superficial similarities, how to draw them out through careful questioning, how to derive adequate explanations, how to link students' learning to their previous knowledge, and how to reify similarities to allow for the abstract manipulation of mathematical concepts.

Conclusions and Implications

To answer our research question, it appears that collaborative lesson development of the type we provided did not in this case lead to faithful implementation of the Teaching for Abstraction model. In particular, there were serious difficulties in implementing the Similarity and Reification phases of the model. As a consequence, although students gained from the use of familiar contexts and the challenge of reaching their own conclusions, it seems that they did not reach the desired depth of understanding.

Because most modern curriculum innovations stress the importance of linking abstract concepts to familiar situations, it appears to be relatively easy for teachers to learn the Familiarity phase of the Teaching for Abstraction model. However, the Similarity and Reification phases are unique to the model, and are therefore likely to be completely novel to teachers. Moreover, the whole approach is the reverse of the traditional ABC (Abstract Before Concrete) method of teaching mathematics (Mitchelmore & White, 2000). As Bridget put it, "It is the opposite of what we're doing in school now. It is starting with the blurred and being revealed. Not doing specific instructions first, more a thinking thing." As Sullivan, Clark, and Clark (2009) found in a similar context, teachers may need a lot of help in developing such new skills when they diverge so widely from their normal practice.

We conclude once again that the type of one-off intervention that was the subject of the present study is unlikely to achieve the desired effects. More time is needed for teachers to experiment with the new lesson structure and to reflect on what happens in the classroom and how it affects student learning (see Figure 1). Even if researchers do work collaboratively with teachers to develop potentially more effective Teaching for Abstraction lessons, professional development clearly needs to allow teachers to experience several examples of the application of the model. Only then could we expect them to recognise the

underlying structure of the model, reify it into a set of principles that become part of their pedagogical knowledge and beliefs, and apply their understanding to the design of new lessons. In other words, we probably need to teach teachers Teaching for Abstraction by abstraction.

Acknowledgements

The study reported in this paper was supported by a grant from the National Centre of Science, Information and Communication Technology, and Mathematics Education for Rural and Regional Australia (SiMERR).

References

- Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, 18, 947-967.
- Dienes, Z. P. (1963). *An experimental study of mathematics-learning*. London, England: Hutchinson.
- Hart, L. C., Alston, A. S., & Murata, A. (Eds.). (2011). *Lesson study research and practice in mathematics education: Learning together*. New York: Springer.
- Mitchelmore, M. C., & White, P. (2000). Teaching for abstraction: Reconstructing constructivism. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, pp. 432-439). Perth: MERGA.
- Pegg, J., & Panizzon, D. (2008). Addressing changing assessment agendas: Impact of professional development on secondary mathematics teachers in NSW. *Mathematics Teacher Education and Development*, 9, 66-80.
- Skemp, R. (1986). *The psychology of learning mathematics* (2nd ed.). Harmondsworth, England: Penguin.
- Sullivan, P., Clarke, D., & Clarke, B. (2009). Converting mathematics tasks to learning opportunities: An important aspect of knowledge for mathematics teaching. *Mathematics Education Research Journal*, 21(3), 85-105.
- White, P., & Mitchelmore, M. C. (2010). Teaching for abstraction: A model. *Mathematical Thinking and Learning*, 12, 205-226.
- White, P., Mitchelmore, M. C., Wilson, S., & Faragher, R. (2008). Teaching percentages: Professional learning in three regional Catholic schools. *Journal of Catholic School Studies*, 80(2), 55-62.

Australian Pre-Service Teachers Overseas Tour: Implications for Mathematics Teaching and Learning

Allan Leslie White
University of Western Sydney
<al.white@uws.edu.au>

Australian pre-service teachers have to acquire layers of knowledge as school classrooms are multicultural in composition. To what extent does the experience of an overseas professional experience tour contribute to the development of pre-service teachers in meeting recognised professional teaching standards? This paper describes the perceptions of Australian pre-service mathematics teachers who participated in educational and cultural activities during planned tours to Malaysia. The data set was collected through the use of questionnaires, interviews and focus group discussions.

The University of Western Sydney (UWS) is unique in the extent of cultural diversity among the students and staff, with students originating from around 170 different international backgrounds (Ingleson, 2009). Nearly half of the Australian students are from non-English speaking backgrounds. Teaching is becoming increasingly globalised in terms of teachers and their careers (Ministerial Council on Employment, Education, Training & Youth Affairs (MCEETYA), 2002), and teacher education cohorts in Australia are more diverse with increasing numbers of international students (Lee, 2011). UWS reflects the composition of Australian school classrooms which are multicultural and thus broad cultural understanding and knowledge are important to pre-service teachers' knowledge base (Clarke, Shimizu, Ulep, Gallos, Adler, & Vithal, 2006). One of the goals of UWS policy is to "maximize the number of UWS pre-service teachers who have an international experience during their program of study" (Ingleson, 2009, p. 3). In response to the goals of UWS, the School of Education has a history of conducting Overseas Professional Experience Program (OPEP) for pre-service teachers in such countries as Malaysia, Fiji, Cook Islands and China. Currently OPEP is a fundamental platform of the UWS School of Education's strategy to 'internationalise' its teacher education programs, but there is considerable outside pressure to discontinue this program.

The pressure on overseas professional experience comes from a variety of sources such as the formation of national and state institutes of teachers which have influenced pre-service education programs to focus more strongly upon preparing teachers for local schools. Thus universities offering overseas professional experience programs have dropped significantly because they are seen as a tourist activity rather than a preparation for pre-service teachers who will need to meet local accreditation requirements. This study challenges this view by analysing the learning of pre-service teachers who have completed an OPEP tour in terms of Australian teaching standards and draws on data from two evaluation studies (Ng & White, 2011; White & Ng, in press). The question addressed in this paper is: To what extent does the experience of the OPEP tour contribute to the professional development of pre-service teachers in meeting the Australian Association of Mathematics Teachers (AAMT) professional teaching standards?

Review of Literature

Pre-service teachers require particular types of knowledge (Shulman, 1986, 1987; Hill, Ball & Schilling, 2008). Ponte and Chapman (2008) in their review of pre-service mathematics teachers' knowledge and development, gave special attention to: "(1) the

development of the mathematical knowledge of prospective teachers; (2) the development of their knowledge of mathematics teaching; and (3) the development of their professional identity" (p. 254). The acquisition of this knowledge occurs in a socio-cultural environment and involves other teachers as mentors and university teacher educators. There have been a number of other proposed lists or frameworks. One arising out of the USA context by Villegas and Lucas (2002) recommended that pre-service teacher programs systematically interweave the following six salient characteristics throughout the coursework, learning experiences, and fieldwork of pre-service teachers to better prepare them to work successfully in culturally and linguistically diverse classrooms. These include: (i) socio-cultural consciousness, that requires pre-service teachers to critically examine their own socio-cultural identities and the inequalities between schools and society that support institutionalized discrimination; (ii) an affirming attitude toward students from culturally diverse backgrounds through respecting cultural differences and by adding activities related to the culture of the students; (iii) commitment and skills to act as agents of change to confront barriers to change, and skills for collaboration and to assist schools in becoming more equitable over time; (iv) constructivist views of learning that assumes all students are capable of learning, and that teaching promotes critical thinking, problem solving, collaboration, and the recognition of multiple perspectives; (v) learning about students' past experiences, home and community culture, and world both in and outside of school to build relationships and use of these experiences in the context of teaching and learning, and; (vi) culturally responsive teaching strategies to assist students construct knowledge, build on their personal and cultural strengths, and experience the curriculum from multiple perspectives.

Pre-service teachers sometimes report experiencing resistance by students to attempts to introduce newer pedagogies while on professional teaching experience placements in schools. The research by Brousseau (1984) on 'didactical contracts' helps to explain the pre-service teacher's frustration. The didactical contract encompasses the conscious and subconscious beliefs, behaviours and relationships that guide and control what teachers and students do within mathematics lessons. His research shows that once teachers and students become accustomed to an approach then they resist any change. Teachers and students develop sets of ingrained actions that arise from, yet simultaneously determine, didactical contracts. The teachers and their students have reciprocal expectancies, and their actions tend to become economical in the sense that they are guided by expectations of what can and cannot be done in lessons. Although these expectations generate common classroom practices, it is usually the case that neither the teacher nor the students subject the expectations to reflection or scrutiny. A teacher attempting a different pedagogy may be subjected to cries of 'is this in the test?' Similar findings have been reported in Brunei (Lim, 2000), and New Zealand (NZ) schools. Barton (2003) maintained that in NZ classroom settings, didactical contracts influence teachers' aims, methods, behaviours, content covered, and choice of procedures for assessing learning.

The AAMT (2002) published the *Standards for Excellence in Teaching Mathematics in Australian Schools* document as representing a consensus view, by the profession for the profession, describing the knowledge, skills and attributes required for good teaching of mathematics. These *Standards* are structured using the three domains of professional knowledge, professional attributes and professional practice.

Professional knowledge comprises: knowledge of students including their social and cultural contexts, the mathematics they know and use, their preferred ways of learning, and how confident they feel learning mathematics; knowledge of mathematics; and, knowledge of students' learning of mathematics. Professional attributes comprises: personal attributes;

personal professional development; and, community responsibilities. Professional practice comprises: the learning environment; planning for learning; teaching in action; and, assessment. These three domains will be elaborated in more detail as needed during the discussion section.

Studies involving international professional experience programs make claims of greater professional autonomy and resourcefulness (Hill, Thomas & Cote, 1997), empowerment (Yarrow & Miller, 2006), and greater resilience, personal growth and confidence to innovate (Fitzsimmons & McKenzie, 2006).

It can be argued that the OPEP tours fall into the area known as service learning, that is pre-service teachers learn while serving the students in culturally and educationally different schools. The basic components of service learning are experience, reflection, and knowledge which are regarded as mutually interdependent and reciprocal. The structure of the tours creates opportunities for all the participants to interact with these basic components. The long term benefits, as the literature suggests, are that the pre-service teachers participating in service learning are highly likely to: improve academically (Sax & Astin 1997; Fredericksen 2000); use a more engaged form of learning in their classrooms (Eyler & Giles 1999); and use the skills of critical thinking and analysis in other arenas (Roschelle, Turpin & Elias, 2000; O'Hara 2001).

The AAMT framework was chosen because it is applicable to the participants in this study. This study seeks to test these claims by examining if the experience of the tours contributes to the professional development of pre-service teachers in meeting the AAMT professional teaching standards.

Methodology

This paper reports a selection of data from two larger evaluation studies of the 2010 and 2011 OPEP tours (Ng & White, 2011; White & Ng, in press). The studies used a mixed model research design with a combination of quantitative and qualitative aspects while the theoretical basis for the evaluation was guided by the principles articulated by Guskey (2000).

Sample

The tour groups include pre-service teachers from secondary (across a range of subjects), primary and early childhood learning contexts. Australian pre-service teachers teach mathematics as either a specialist in the discipline (such as in secondary schools) or along with other subjects (such as in primary schools). In Malaysia, primary and secondary teachers are specialists in the discipline.

In 2010 the pre-service mathematics teaching sample of the tour participants included one primary teacher and three secondary mathematics teachers, and in 2011 there were four primary teachers and one secondary mathematics teacher.

The pre-service teachers spend two weeks teaching in Penang Island schools. In 2010 six Malaysian schools were involved (1 primary, 5 secondary schools) and in 2011 there were four schools (2 primary and 2 secondary). The schools are allocated by the Penang State Education Department.

Instruments

Data were collected at different times using a variety of instruments and groups of participants for the purposes of triangulation. The data came from UWS pre-service teachers and staff, Malaysian students, school administrators and teachers. Data collection was

mainly done via questionnaires administered to school Malaysian administrators or teachers and Australian pre-service teachers, interviews with pre-service teachers, and focus group interviews conducted with groups of Malaysian primary and secondary students.

The pre-service teachers' questionnaire was a combination of a short answer demographic information section, a Likert scale section based on the first four levels of Guskey's (2000) scale, and three open-ended questions probing responses to the Likert scale. The questionnaire was adapted from the instruments administered during similar programs in the Malaysian context from previous years. The pre-service interviews consisted of a number of open-ended questions and were analysed according to emerging themes. The focus group interviews of Malaysian students in both primary and secondary schools consisted of a number of open-ended questions and students were encouraged to volunteer their opinions. Focus group discussions were recorded, transcribed and key themes identified.

Data were extracted for the mathematics teaching pre-service teachers from the larger evaluation data set.

Results and Discussion

The limitations of the data arising from the small sample size and because it was collected for other purposes demand caution when discussing the significance and implications of the findings. However, within these limitations, the data reveals some interesting observations that could provide direction for future studies. The tour provided a unique context for pre-service teacher learning. During the school weeks the teachers worked together every night with the two UWS staff in an intensive process of preparing, discussing and reflecting upon their teaching. This section is divided according to the dimensions listed in the AAMT Standards (2002) and will be supported by the use of the participants' voice through direct quotes. However, the content of the quotes often ranges across more than one dimension.

Professional Knowledge

This section in the Standards is divided into knowledge of students, of mathematics, and of students' learning of mathematics.

An issue reported by all pre-service teachers was that of dealing with differences in context and culture from a position of being an outsider. Although most of the pre-service teachers had taught in Australian classrooms with mixed abilities, a range of learning styles and communication issues, they found that in the Malaysian classes their awareness of these issues was enhanced. "By being in the minority instead of the majority, I believe I have developed a greater empathy with students in the same position" (pre-service teacher Epsilon).

The pre-service teachers' responses to the open questions reflected a greater awareness of the needs of students which motivated the pre-service teachers to reassess their teaching. The following responses are typical of the sentiments expressed by the pre-service teachers: "I will be more tolerant and patient with [students of] different cultural background or [who] are struggling with English" (pre-service teacher Alpha). And,

I have developed a better understanding of the Muslim religion, Malay, Chinese and Indian cultures, and that students no matter what [the] cultural background have different learning styles and [it] confirmed my beliefs that a mixture of strategies need to be used to facilitate their learning. (student Delta).

From the analysis on participants' reactions towards the tour, all reported they had benefited from the program in terms of the involvement with the teaching and learning process, the positive interaction with the school students, resulting in the pre-service teachers developing an affirming attitude toward culturally diverse backgrounds and in respecting cultural differences. The tour contributed to participants' socio-cultural consciousness, and allowed the pre-service teachers to critically examine their own socio-cultural identities and the inequalities between schools and societies (Villegas & Lucas, 2002).

Knowing what I know now about their culture will completely change the way I run my classroom. You may think that a student with poor English is a little bit different, but it is more than that. You really have distinct differences. I need to listen more and allow the kids to teach me before I start to teach them. This tour has opened my eyes (pre-service teacher Gamma).

When asked about the mathematics that they had to teach, the pre-service teachers reported that it was generally similar to what they were expected to teach in Australia. However the primary pre-service teachers commented upon the Malaysian primary schools being examination focused, having specialist subject teachers and the teaching of subjects rather than integrated learning units. Both primary and secondary pre-service teachers observed that students were reticent in answering questions unless they were certain they were correct. The students were also more focused on the answer and not on the process for obtaining the answer.

The Malaysian people are perfectionists in education. Students are taught to stand and greet the teacher on entering the room. Some students will thank the teacher at the end of the lesson. Students are unwilling to volunteer information unless they are 100% sure they are correct. They don't care how the answer was worked out, as long as it is correct (pre-service teacher Zeta).

The students were contributors to the pre-service teachers' new knowledge of students' learning of mathematics. There was common agreement across all student focus groups that they enjoyed and were excited at having a pre-service teacher as their teacher. When asked why, they used words such as: 'newness', 'different', 'younger', 'interesting', 'explain clearer' and 'funnier'. They all mentioned enjoying talking to the pre-service teachers as an opportunity to use and improve their English language. They also liked telling the pre-service teachers about their lives and culture. The Malaysian teachers commented positively on the more relaxed relationship between the students and the Australian pre-service teachers.

While it is unrealistic to think that a two week interaction will greatly change the students' proficiency with English, what came through strongly from all groups, was that they were motivated to communicate in English. The students felt comfortable with making errors in their engagement with the pre-service teachers. Whether the excitement and motivation will remain after the teachers leave was beyond the scope of this study.

Professional Attributes

This dimension includes personal attributes, personal professional development, and community responsibilities. The OPEP tour is voluntary and fully funded by the pre-service teachers. UWS only covers staff costs. The demographic information was very diverse. The pre-service teachers listed enthusiasm for mathematics as an essential attribute for engaging the class.

When I got passionate about what I was teaching the children became excited. They also responded to any jokes I told no matter how lame they were. Four of the girls told me that my lessons were the

first mathematics ones they had enjoyed. I felt so good and I was inspired to do better (pre-service teacher Kappa).

The majority of pre-service teachers responded positively regarding the impact of the collaboration and cohesive group relationships that developed because of the nightly lesson planning process. They felt the process accelerated their personal professional development as teachers and used words such enhanced self-efficacy, fostered critical reflection, reassessed ways of thinking, new ideas and problem-solving.

I learnt so much from the others and made such good friends that I will find it difficult when I am on my own. I will probably keep in touch using Facebook so I won't be fully on my own. For the first time I really feel that I can be a teacher (pre-service teacher Iota).

Professional Practice

This dimension includes the learning environment, planning for learning, teacher in action, and assessment.

The schools altered their organization to cater for the pre-service teachers through flexible schedules for teaching, observation and an alterable timetable to enable involvement in other co-curricular activities and enrichment exposure, e.g. sports, trips, staff development, in-service training or meeting.

All pre-service teachers commented on the development of a sense of confidence and of autonomy. They compared overseas with local professional teaching experiences and reported having made greater advances as a consequence of their teaching overseas. A typical response was:

Thank-you for giving me the opportunity to be part of this amazing experience. I have been stretched 'out of my comfort zone' and this has helped me to hone my skills and give me confidence in my ability to facilitate students learning (pre-service teacher Beta).

A key area was in the understanding of cultural and language issues and how to support second language learners and how to organise for inclusive education in a mathematics classroom. The pre-service teachers discussed the strategies that they developed such as constantly assessing comprehension, allowing more think time, speaking slower and repeating key phrases and breaking down tasks in their planning. Most reported modeling, and explaining mathematics in spoken, written, non-verbal and visual forms. They taught students how and why to do group work in mathematics, to speak to peers and in front of the class and how to listen to English instructions. They reported a greater awareness of the literacy issues around the teaching of mathematics.

I developed pedagogical skills for teaching English to second language learners. Children are becoming more globalised, so similar interests between students can be used to engage students. [I learnt] not to underestimate some abilities and how to modify lessons not only for ability but cultural understanding (pre-service teacher Epsilon).

The primary pre-service teachers made some observation regarding the workings of the primary classrooms. The teaching of mathematics, while taught by a mathematics specialist, was largely focused on the external examination and the pedagogy concentrated upon the procedures for getting a correct answer. When the pre-service teachers tried getting students to develop an understanding of the underlying concepts before mastering the procedure they met resistance from the students. They found that they adjusted their pedagogy to follow the Malaysian teacher and to look for and use teachable moments to strengthen students' understanding of the concepts.

When they couldn't do the algorithm, then they would listen to me explain why and how the algorithm worked. I was just waiting for them to fail so that I could help them understand what they were trying to do (pre-service teacher Iota)

The research by Brousseau (1984) helps explain this observation. He highlighted the important role of mutual expectations between teachers and pupils in the construction of mathematics learning environments. Thus an established didactical contract in a classroom may need to be renegotiated. The Malaysian students were working with a one constructed with their regular classroom teacher. This contract appears to value the end product over the process. The students resisted the pre-service teacher's attempts to make a change.

Conclusion

The paper has briefly argued that teaching is becoming increasingly globalised in terms of teachers and their careers (MCEETYA, 2002). Teacher education cohorts in Australia are more diverse with increasing numbers of international pre-service teachers (Ingleson, 2009; Lee 2011) and there is a need for pre-service teachers developing skills to work in a range of Australian teaching contexts that are multicultural.

This paper examined the OPEP program's contribution to the professional development of pre-service teachers in meeting the AAMT (2002) professional teaching standards. The data presented in this paper, while limited, suggest that for this small sample there is evidence that the experiences of the tour made a positive contribution to the pre-service teachers professional knowledge, professional attributes and professional practice. The tour contributed to the development of pre-service teachers' cross-cultural understandings, empathy and skills as a result of experiencing different traditions. Whether this can be generalized for all tours and for larger cohorts must be left to further research.

Thus an overseas professional experience program is much more than a mere tourist activity and exhibits the potential for making a positive contribution to the preparation of pre-service teachers.

References

- Australian Association of Mathematics Teachers (AAMT)(2002). *Standards for excellence in teaching mathematics in Australian Schools*. Adelaide: Author.
- Barton, B. (2003). The Mathematics Enhancement Project: Using the concepts of cultural conflict, critical mathematics education, and didactic contract. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 137–143). Geelong, Australia: Mathematics Education Research Group of Australasia.
- Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H. G. Steiner (Ed.), *Theory of mathematics education* (pp. 110–119). Bielefeld, Germany: Universität Bielefeld.
- Clarke, D. J., Shimizu, Y., Ulep, S. A., Gallos, G., Adler, J., & Vithal, R. (2006). Cultural diversity and the learner's perspective: Attending to voice and context. In F. K. S. Leung, K-D Graf, F. J. Lopez-Real (Eds.), *Mathematics education in different traditions – A comparative study of East Asia and the West* (pp. 353-380). New York, NY: Springer.
- Eyler, J., & Giles, D. E. (1999). *Where's the learning in service-learning?* San Francisco: Jossey-Bass.
- Fitzsimmons, P., & Mackenzie, B. (2006). Super or superfluous in the South Pacific? Unearthing the benefits of an overseas teaching practicum. *Journal of Research in International Education*, 5, 177- 89.
- Fredericksen, P. (2000). Does service learning make a difference in student performance? *The Journal of Experiential Education*, 23, 64-74.
- Guskey, T. R. (2000). *Evaluating professional development*. Thousand Oaks, California: Corwin Press.
- Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39, 372–400.

- Hill, B., Thomas, N. & J. Cote (Eds.) *Into Asia: Australian Teaching Practicums in Asia*. Carlton, Victoria. National Project of the Partnerships and Professional Development Program, Asia Education Foundation.
- Ingleson, J. (2009). *International framework 2009 - 2013*. Penrith, Sydney: University of Western Sydney.
- Lee, J. K. F. (2011). International Field Experience – What do student teachers learn? *Australian Journal of Teacher Education*, 36, 10, 1-22.
- Lim, T. H. (2000). *The teaching and learning of algebraic equations and factorisation in O-level Mathematics: A case study*. Unpublished M.Ed dissertation, Universiti Brunei Darussalam.
- Ministerial Council on Employment, Education, Training & Youth Affairs, (MCEETYA) (2002). *Demand and supply of primary and secondary school teachers in Australia*. Retrieved October, 10, 2011 <http://www.mceetya.edu.au/public/demand.htm#intro>
- Ng, K. T., & White, A. L. (2011). *Programme Evaluation Report: Australian Pre-Service Teachers Overseas Educational and Cultural Tour 2010*. Penang, Malaysia: RECSAM
- O'Hara, L. S. (2001). Service-Learning: Students' transformative journey from communication student to civic-minded professional. *The Southern Communication Journal*. 66, 251-266
- Ponte, J. P., & Chapman, O. (2008). Preservice mathematics teachers' knowledge and development. In L. D. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 223-261). New York: Routledge.
- Roschelle, A. R., Turpin, J., & Elias R. (2000). Who learns from service learning? *The American Behavioral Scientist*. 43, 839-847.
- Sax, L. J., & Astin, A. W. (1997). The benefits of service: Evidence from undergraduates. *Educational Record*, 78, 25-32.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1-22.
- Villegas, A. M., & Lucas, T. (2002). *Educating culturally responsive teachers*. Albany, NY: State University of New York Press.
- White, A. L., & Ng, K.T. (in press). *Similarities and differences: A Malaysian and Australian research study*. Penang Malaysia: SEAMEO RECSAM.
- Yarrow, A., & Millwater, J. (1992). *The mathematical practicum in teacher education: A research report of the work of one institution*. Brisbane, Queensland Institute of Technology.

Investigating Pre-service Teachers' Mathematics Anxiety Using the Revised Mathematics Anxiety Scale (RMARS)

Sue Wilson
Australian Catholic University
<sue.wilson@acu.edu.au>

Pre-service primary (elementary) teachers' mathematics anxiety affects their engagement with and future teaching of mathematics. The study measured the range of mathematics anxiety in 219 pre-service teachers starting a teacher education course in an Australian university. They responded to the Revised Mathematics Anxiety Scale (RMARS) and a set of demographic questions. Age differences in anxiety were found to be significant, and relationships were found between the RMARS scores and students' self-perceptions of their current mathematics anxiety levels.

Mathematics is essential for everyday life. Failure in mathematics can have a powerful emotional impact that may extend far beyond the mathematics classroom (Boaler, 1997), culminating for many in an ongoing state of mathematics anxiety.

Mathematics anxiety is an important research topic in the mathematics education community. Many students suffer from mathematics anxiety and this seems to be independent of whether they are good at mathematics or not (Furner, 1996).

Pre-service primary (elementary) teachers' mathematics anxiety has important impacts on their studies and on their future students. Previous research has investigated the use of bibliotherapy to address pre-service teacher mathematics anxiety (Wilson & Thornton, 2008; Wilson, 2007). This paper, part of a larger project to build on that research, examines the range of first year primary pre-service teachers' (PST) mathematics anxiety.

Theoretical Framework and Literature Review

This research is located at the intersection of the literature on the impacts of mathematics anxiety on primary teacher mathematics education, and bibliotherapy.

Mathematics Anxiety

Mathematics anxiety is more than a dislike for mathematics (Vinson, 2001). Among the early researchers of mathematics anxiety, Dreger and Atkin (1957, p. 344), identified "emotional reactions to arithmetic and mathematics". Richardson and Suinn, (1972, p. 551) elaborated "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations". Smith (1997) characterises mathematics anxiety in a number of ways ranging from uneasiness when asked to perform mathematical task, to avoidance, feeling of physical illness, dread and panic, thus theoretical models of mathematics anxiety have multidimensional forms that incorporate attitudinal (dislike), cognitive (worry) and emotional (fear) aspects, (Hart, 1989; Wigfield & Meece, 1988).

Gender differences in mathematics anxiety have been extensively studied. The results are inconsistent, with a number of studies reporting that females have higher levels of mathematics anxiety than males (for example, Alexander & Martray, 1989) and others not confirming significant differences. Baloglu and Kocak (2006), controlling for mathematics experience, found that gender effects of mathematics anxiety varied with the context. Age is another factor where contradictory findings are reported in the literature. Hembree, (1990),

did not find any age-related differences, but Baloglu and Kocak (2006) found that older students exhibited more total mathematics anxiety than younger ones, particularly in mathematics testing and course situations.

Impact of Teaching

Mathematics anxiety has been associated with inappropriate teaching practices (Hasbee, Sam, Nur, & Tan, 2009, Uusimaki & Nason, 2004). The teacher's attitude has been identified as a major factor (Vinson, 2001). Teachers' beliefs about their own ability are significant in their approach to teaching mathematics and can produce unwillingness to teach upper primary classes (Wilson, 2009).

High mathematics anxiety impacts on performance and achievement in mathematics (Betz, 1978; Ma, 1999; Sheffield & Hunt, 2006). Mathematics anxiety is related to success in higher education (Stubblefield, 2006). Students with high mathematics anxiety avoid mathematics in their courses and careers (Scarpello, 2005). Researchers have found that high levels of teacher mathematics anxiety can be perpetuated in classrooms (for example, Martinez, 1987, Furner & Berman, 2005). Therefore, "preventing math anxiety begins by helping teachers confront and control their own fears of math" (Martinez, 1987, p. 117).

Many students come to tertiary teacher education with limited mathematics understandings, and a pattern of avoidance and anxiety. Researchers of primary PST report high levels of mathematics anxiety, low confidence levels to teach mathematics and low mathematics teacher efficacy. For a more detailed discussion of these issues see Wilson (2009). This affects not only their current study but also their future teaching of mathematics and hence the attitudes of their future students. Researchers have identified teacher preparation programs as sources of mathematics anxiety (Uusimaki & Nason, 2004, p. 370). Dunkle (2010) warns of the consequences of this:

A model of anxiety with regard to mathematical concepts can be passed on to the next generation of students by their teachers. Thus, there is an urgent need to overcome this anxiety in preservice teachers so that they may more appropriately model their skills to their students, and thus break the cycle of math anxiety that appears to be becoming generational in nature. (p. 14)

The way individuals perceive themselves as learners of mathematics is integral to their subsequent identity as teachers. In previous research (Wilson & Thornton, 2008; Wilson, 2007) many PST described an interaction during their schooling that led to them identifying themselves as persons who couldn't learn mathematics, and said that this still impacted on their self-images as future teachers of mathematics. Some of the consequences of these experiences can be seen in their levels of mathematics anxiety and self-perceptions as future teachers of mathematics. Identities remain open to revision during the time of their course. Hence, Walshaw (2004, p. 557) argues that "teacher education must engage the identities of pre-service students".

In summary, PST with mathematics anxiety are less likely to engage with mathematics, and have low confidence and low self-efficacy, impacting on their identity as teachers of mathematics. It is for these reasons that teacher education has become a crucial site for further research.

Mathematics Anxiety Scales

Researchers have sought way to assess levels of mathematics anxiety. Anxiety is a construct and not able to be measured directly. We have international units and standards for some measurements, but to attempt to quantify a construct such as anxiety is much more problematic. Observed or reported activities lead to inference of anxiety in anxiety scales.

Dreger and Atkin, (1957), developed the first standardised mathematics anxiety instrument, the Number Anxiety Scale. In 1972, the Mathematics Anxiety Rating Scale, (MARS), a major scale used in research and clinical studies, was developed by Richardson and Suinn. It has been widely used with high reliability and validity reported.

Alexander and Martray developed the Revised MARS in 1989, by reducing the original 98-item MARS (Richardson & Suinn, 1972), to 25 items. The original MARS was found to be time-consuming to administer because of its length. It also interpreted mathematics anxiety as one factor. The RMARS measures three factors - mathematics test anxiety, numerical task anxiety and mathematics course anxiety. The RMARS has been widely used in academic research, rigorously tested, and found to be psychometrically sound (Revak, 1996; Bowd & Brady, 2002; Haynes, 2003, Baloglu & Kocak, 2006). Baloglu and Zelhart (2007) reported that the RMARS “was found to be a valid and reliable measure of college students’ mathematics anxiety levels” (p, 608). They reported that, based on reliability investigations, the total scale and subscales were highly reliable, and they confirmed the three-factor structure. Recently, Dunkle (2010) used the RMARS to provide valid and reliable results for measuring PST math anxiety over time. A range of other instruments has been developed but their analysis is beyond the scope of this paper.

Bibliotherapy

Bibliotherapy is a technique that was developed in psychology (Shrodes, 1950), and library science. It involves guided reading of written materials used in gaining understanding or solving problems, followed by individual or group discussion in a non-threatening environment. Previous research used bibliotherapy during mathematics units for PST to examine their attitudes towards themselves as learners and teachers of mathematics (Wilson & Thornton, 2008; Wilson, 2007). Changes in response to the bibliotherapy process in previous research strongly suggest the importance of insight as a major factor in bringing about a positive projective identity. Identity brings together affective qualities and cognitive dimensions. Ricoeur (1994) suggests that people make sense of their own personal identities in a similar way to their understanding of the identity of characters in stories.

The Study

This paper reports the first stage of ongoing research investigating maths anxiety in PST in education courses in an Australian university. The project aims to investigate the range of anxiety that first year PST feel towards mathematics, and ways that they might change the way they approach their mathematics studies. The study will explore the use of bibliotherapy as a tool for reflecting on and changing affective responses to mathematics and enhancing primary PST engagement with mathematics in their university mathematics units.

Research Questions

In this mixed methods study, bibliotherapy will be used as a tool for addressing affective responses to mathematics. The goal is to understand what influences bibliotherapy has on PST in their engagement with mathematics units and their beliefs, and its impact on their developing professional identity. The following research questions will be investigated:

1. With what range and extent of mathematics anxiety do PST present?
2. What influenced the way PST think about themselves as a learner and potential teacher of mathematics?
3. What do PST identify as causes of their mathematics anxiety?

4. How do PST describe their mathematics anxiety experiences?
5. How do PST with high levels of mathematics anxiety respond to bibliotherapy readings?
6. How do PST describe their experiences of bibliotherapy that lead to reduced anxiety?
7. Does bibliotherapy lessen the level of PST maths anxiety as measured by the short MARS-R?

Four instruments will be used:

1. Participants will be surveyed at the start and the end of the year with an instrument to collect data on student mathematics anxiety. They will be asked to rank the extent to which they feel anxiety about activities involving maths, and to score their anxiety about mathematics.
2. PST who volunteer to take part in the extended project will complete a critical incident report, via email, on a past experience that contributed to their self-perceptions as a learner and teacher of mathematics.
3. PST on-line reflections and discussions will be stimulated by selected readings on mathematics anxiety, and will be posted in an on-line forum with access limited to the volunteer group.
4. Semi-structured interviews after the first mathematics unit, will focus on the insights that the PST developed as part of the bibliotherapy process. The reflections and interviews will be related to the outcomes of previous research using the bibliotherapy framework.

This paper reports the results of the first part of this project, which aimed to investigate the range of anxiety that first year PST feel towards mathematics, using the RMARS survey.

Methods

The Sample

Two cohorts of PST were the samples for the study. They comprised first year primary PST enrolled in the Bachelor of Education (Primary) degree course, from two campuses of an Australian university. Sample 1 came from a regional campus and Sample 2 from a metropolitan campus. Samples of 57 (98% response rate) and 162 (94% response rate) completed the survey instrument. They were surveyed in first few weeks of the first semester of their university course.

Instrument

A range of mathematics anxiety surveys was considered and RMARS (Alexander & Martray, 1989) was chosen because of its length, fit with the research question, appropriateness for group and strong psychometric information. The RMARS is a 25-item, five point (1 = not at all, to 5 = very much) Likert-type instrument. It has three sub-scales Mathematics Test Anxiety (items 1-15); Numerical Task anxiety (items 16-20); and Mathematics course anxiety (items 21-25). The original RMARS was used with minor modifications for the Australian context. The RMARS rating forms and a set of demographic questions were used in the study. These asked for information such as age and gender, and mathematics courses studied in high school, and the number of years/months since their last mathematics course.

Procedure

Ethics approval was obtained from the university ethics committee, and agreement to use the RMARS survey was received from the author. In addition to the survey the PST were asked to rate their general mathematics anxiety level, confidence and current mathematics anxiety level after completing the questions on a scale between 0 and 100, where higher scores related to higher levels. As current literature suggests that mathematics anxiety is a transitory-state construct (Baloglu & Zelhart, 2007), this question sought to obtain a measure of the mathematics anxiety that PST were currently experiencing when completing the survey.

Data were coded onto an excel spreadsheet and arranged so they could be analysed with the software program Statistical Package for Social Sciences (SPSS) 20.0.

Results and Discussion

Means and standard deviations for the total scale scores on the RMARS were computed. The PST exhibited a broad range of anxiety levels. Scores ranged from 31 - 116, with a mean score of 66.48 and a standard deviation of 16.74.

Comparing PST from the two campuses, Sample 1 had a mean score of 66.02 and a standard deviation of 19.19, and sample 2 had a mean score of 62.78 and a standard deviation of 17.86. No significant differences were found between the cohorts from the two campuses. Sample 1 and sample 2 were statistically equivalent on the total RMARS scores, ($p < 0.25$), as well as the three subscales (Mathematics Test Anxiety (MTA), $p < 0.33$; Numerical Task Anxiety (NTA), $p < 0.09$; and Mathematics Course Anxiety (MCA), $p < 0.73$). These results indicate that there is a wide range of mathematics anxiety within the cohorts, but that the cohorts as a whole are not significantly different.

Gender differences were examined for the total scale scores on the RMARS as well as the three subscales. No significant differences were found between females and males on the total RMARS scores, $p < 0.61$, nor on the three subscales (MTA, $p < 0.32$; NTA, $p < 0.71$; and MCA, $p < 0.30$). The results for gender differences reported by researchers vary and this underlines the complexity of this issue.

Significant differences were identified between age cohorts. For the purpose of this study, and in line with published research, mature-age PST were defined as those 25 years and over. Age differences were examined for the total scale scores on the RMARS as well as for the three subscales. The older group demonstrated higher levels of mathematics anxiety than the younger group. Figure 1 shows the percentages of the younger (SL) and mature-age (Mature) cohort's anxiety scores at each level of anxiety.

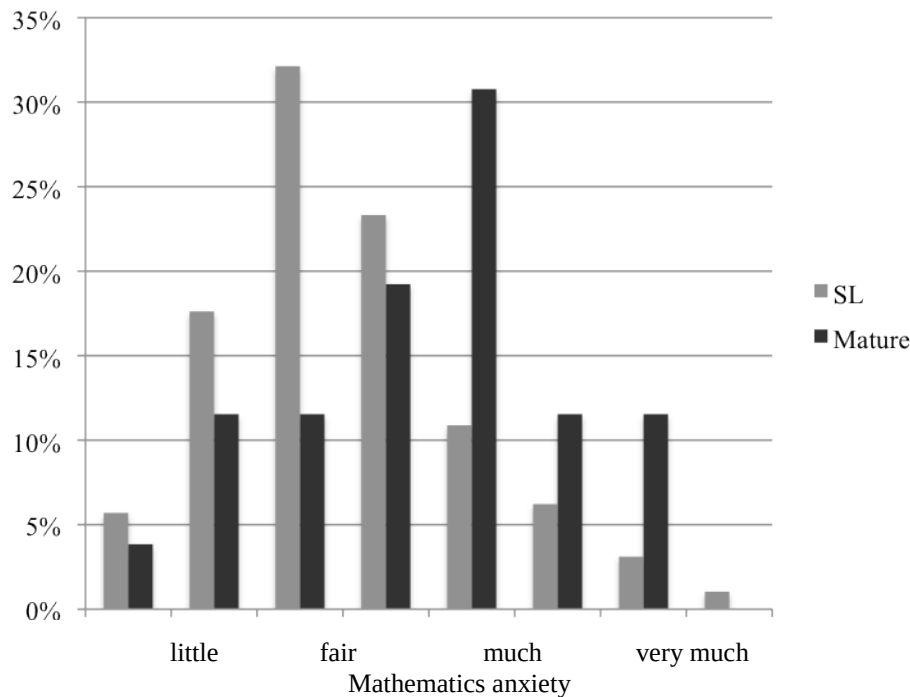


Figure 1. Percentages of mathematics anxiety scores of younger (SL) or mature age (Mature) PST.

Significant differences at the 95% level were found between the scores of the younger (mean 62.44, standard deviation 17.73) and mature-age (mean 73.58 standard deviation 19.75) on the total RMARS scores, ($p < 0.003$); and on the three subscales (MTA, $p < 0.035$; NTA, $p < 0.001$; and MCA, $p < 0.016$), with MTA and MCA significant at the 95% level and NTA significant at the 99.9% level. This supports the findings of Baloglu and Kocak (2006) that older college students show higher levels off mathematics anxiety than younger ones, however they reported that MTA and MCA contributed more to the anxiety of their older cohort than NTA.

Concurrent validity of the instrument was investigated by examining the relationships between the RMARS scores and students' self-perceptions of their general and current mathematics anxiety levels, using paired t-tests. The analysis showed that there was a correlation between the PST' estimates of their general levels of mathematics anxiety and their current level of mathematics anxiety. The RMARS scores were significantly different to the general mathematics anxiety level rating ($p < 0.009$), but not to their current level of mathematics anxiety. This indicates that the RMARS scores were a better indication of their perception of their current level of mathematics anxiety, than their general level.

Conclusions

This research reports on preliminary data on the range of mathematics anxiety experienced by PST in their first few weeks of their teacher education course at university, and shows that in some cases these anxieties may present differently when taking a mathematics test, doing mathematical computations, or undertaking a mathematics course. Teacher educators should be aware of the extent of range of anxiety that PST experience at the beginning of their teacher education course, and especially that mature-age PST experience anxiety differently in mathematics courses compared to younger PST, and hence

the needs of mature-age students coming in to teacher education may be different to those of younger students.

The next part of the study will involve the bibliotherapy approach, which is relatively unexplored in mathematics teacher education. The findings from this study will shed light on the extent to which bibliotherapy can foster a more positive orientation towards mathematics in pre-service teachers. As Wolodko, Willson and Johnson (2003) write “our challenge is to help pre-service teachers confront their past experiences and anxieties about teaching and learning of mathematics. If these are openly dealt with during their university education, fewer teachers may be content to teach just as they have been taught” (p. 224).

Of particular interest to teacher educators will be the impact of the participation in writing about experiences and reflections on the development of PST evaluations of themselves as future teachers of mathematics, and sharing these with a supportive on-line community. The reflections produced during the project will be used, with permission, as tools for developing teacher educators and teachers’ personal and professional knowledge. This has the potential to transform learning and teaching beyond that of the PST to their future teaching of mathematics and hence the attitudes of their future students. It is also anticipated that examples related to reflective practice and developed during this project may be useful for a wider audience of teacher educators and PST, extending the strategies available in teacher education courses to address mathematics anxiety.

References

- Alexander, L., & Martray, C. (1989). The development of an abbreviated version of the mathematics anxiety rating scale. *Measurement and Evaluation in Counseling and Development*, 22, 143–150.
- Baloglu, M. & Kocak, R. (2006). A multivariate investigation of the differences in mathematics anxiety. *Personality and Individual Differences*, 40, 1332-1335.
- Baloglu, M. & Zelhart, P (2007). Psychometric properties of the revised mathematics anxiety rating scale. *The Psychological Record*, 57, 593–611.
- Betz, N. (1978). Prevalence, distribution, and correlates of math anxiety in college students. *Journal of Counseling Psychology*. 25(5)441-448.
- Boaler, J. (1997). *Experiencing school mathematics: Teaching styles, sex and setting*. Buckingham: Open University Press.
- Bowd, A. & Brady, P. (2002). Factorial structure of the Revised Mathematics Anxiety Rating Scale for undergraduate education majors. *Psychological Reports*, 91, 199–200.
- Dunkle, S. (2010). *Remediation of math anxiety in preservice elementary school teachers*. Unpublished doctoral dissertation, Faculty of D'Youville College Division of Academic Affairs.
- Furner, J. (1996). *Mathematics teachers’ beliefs on using the National Council of Teachers of Mathematics Standards and the relationship these beliefs have on students’ anxiety toward mathematics*. Paper presented at the Mid-South Educational Research Association’s 1996 annual meeting, November 6, The University of Alabama, Tuscaloosa. (ERIC Document Reproduction Service No. TM 026 376).
- Furner, J. & Berman, B. (2005). Confidence in their ability to do mathematics: The need to eradicate math anxiety so our future students can successfully compete in a high-tech globally competitive world. *Dimensions in Mathematics*, 18(1), 28–31.
- Hart, L. (1989). Describing the affective domain: Saying what we mean. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 37–45). New York: Springer.
- Hasbee, U., Sam, H., Nur, A., & Tan, K. (2009). Factors causing mathematics anxiety among undergraduate students. *Proceedings of CoSMEd 2009: Third International Conference on Science and Mathematics Education*. Penang, Malaysia.
<http://www.recsam.edu.my/cosmed/cosmed09/AbstractsFullPapers2009/Abstract/Mathematics%20Parallel%20PDF/Full%20Paper/M11.pdf>
- Haynes, S. (2003). *The Effect of Background Music on the Mathematics Test Anxiety of College Algebra Students*. Unpublished doctoral dissertation, West Virginia University.
- Ma, X. (1999). A meta-analysis of the relationship between anxiety toward mathematics and achievement in mathematics. *Journal for Research in Mathematics Education*, 30(5), 520–540
- Martinez, J. (1987). Preventing math anxiety: A prescription. *Academic Therapy*, 23, 117–125.

- Revak, M. (1996). *The Effect of Distributed Practice Homework on Precalculus Achievement at a Military Academy*. Unpublished doctoral dissertation, Graduate School of Florida Institute of Technology
- Richardson, F. C., & Suinn, R. M. (1972). The Mathematics Anxiety Rating Scale: Psychometric data. *Journal of Counseling Psychology*, 19, 551–554.
- Ricouer, P. (1994). *Oneself as other*. (Trans. K. Blamey). Chicago, IL: University of Chicago Press.
- Scarpello, G. (2005). *The effect of mathematics anxiety on the course and career choice of high school vocational-technical education students*. Unpublished Ph.D. Dissertation, Drexel University.
- Sheffield, D., & Hunt, T. (2006). How does anxiety influence math performance and what can we do about it? *MSOR Connections*, 6(4), 19–23.
- Shrodes, C. (1950). *Bibliotherapy: A theoretical and clinical- experimental study*. Unpublished doctoral dissertation, University of California at Berkeley.
- Smith, S. (1997). *Early childhood mathematics*. Boston: Allyn & Bacon.
- Stubblefield, L. (2006). Mathematics anxiety among GED recipients in four-year institutions. *Journal of Mathematical Sciences and Mathematics Education*, 1(2), 19–23.
- Uusimäki, L. & Nason, R. (2004). Causes underlying pre-service teachers' negative beliefs and anxieties about mathematics. *Proceedings of the 28th conference of the international group for the psychology of mathematics education*, 4, 369–376.
- Vinson, B. M. (2001). A comparison of preservice teachers' mathematics anxiety before and after a methods class emphasizing manipulatives. *Early Childhood Education Journal*, 29(2), 89–94.
- Walshaw, M. (2004). Becoming knowledgeable in practice: The constitution of secondary teaching identity. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010. Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia* (pp. 557–563). Townsville, Qld: MERGA.
- Wigfield, A., & Meece, J. L. (1988). Math anxiety in elementary and secondary school students. *Journal of Educational Psychology*, 80, 210–216.
- Wilson, S. (2007). My struggle with maths may not have been a lonely one: Bibliotherapy in a teacher education number theory unit. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice. Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (pp. 815–823) Hobart, Tas: MERGA.
- Wilson, S. (2009). "Better you than me": Mathematics anxiety and bibliotherapy in primary teacher professional learning. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing Divides: Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 603–610). Palmerston North, NZ: MERGA.
- Wilson, S. & Thornton, S. (2008). "The factor that makes us more effective teachers": Two pre-service primary teachers' experience of bibliotherapy. *Mathematics Teacher Education and Development*, 9, 22–35.

A Revolving Model of Pre-service Teacher Development in Mathematics

Susanna Wilson
University of Canterbury
<sue.wilson@canterbury.ac.nz>

This paper presents a theoretical model for the development of knowledge for teaching, based on a study of primary pre-service teachers (PSTs) in their final year of an Initial Teacher Education Programme. This model arose from findings about PSTs' perceptions of knowledge for mathematics teaching that were related to knowledge of the curriculum, mathematical content knowledge and knowledge of school contexts. The theoretical model describes how PSTs developed this knowledge for teaching mathematics. There are three phases in this model; *recognising* crucial aspects of teaching, *reconceptualising* these aspects, and *realizing* these aspects in a mathematics classroom. In conclusion I discuss the possible use of this model for my work as a teacher educator.

Knowledge needed for teaching is an important consideration for teacher educators because they are charged with the responsibility of preparing teachers to teach in the classroom. The seminal work of Shulman (1986) highlights the complexity of this knowledge and lists different categories of knowledge for teaching. Grossman (1990) represents these categories of knowledge for teaching to include, "general pedagogical knowledge, subject matter knowledge, pedagogical content knowledge and knowledge of context" (p. 6). These four categories are significant as they influence and inform the content of Initial Teacher Education (ITE) programmes, whose goal is to prepare PSTs for teaching. In the field of mathematics education several authors agree that the task of learning to teach mathematics is complex (Ball, 1993; Hill, Sleep, Lewis & Ball, 2007). PSTs need to not only *know* what to teach that is subject or mathematical content knowledge; they also need to know how to teach this content. Sowder (2007) describes the knowledge needed for teaching mathematics as "a blend of subject knowledge and an understanding of how to deliver this" (p.173). Mathematics educators within the ITE setting then need to decide how to support PST to develop this knowledge. Ball (2000) describes this as needing to identify content knowledge that matters for teaching, understanding how such knowledge needs to be held, and understanding what it takes to learn to use such knowledge in practice.

Within the ITE setting PSTs develop knowledge for mathematics teaching from different sources such as course work and Professional Practice experiences. In addition they bring with them their own knowledge of mathematics for teaching which Shulman (2004) describes as their "intellectual biography." PSTs need to construct knowledge for teaching mathematics from all of these sources and transform this knowledge for use in a mathematics classroom (Fennema & Franke, 1992; Winsløw et al., 2009). This paper presents a model for the development of knowledge for teaching mathematics in an ITE setting. This model evolved from a research project that aimed to identify the needs and concerns of third and final year PSTs as they prepared to teach mathematics in a primary school (Wilson, 2010). In this paper I firstly describe this model which is entitled, "The Revolving Model of Pre-service Teacher Development"; secondly I present some data relating to the model and finally I discuss the use of the model for my work as a mathematics teacher educator.

The Emergence of the Model.

One of the mathematics courses I teach within the Bachelor of Teaching and Learning degree is a third and final year course which is held over five weeks. This optional course was the context for the research project. Each year I endeavour to include course curriculum that meets the needs of the students, within a tight timeframe. Ensor (2001), citing the work of Bernstein (1996), refers to course curriculum as “privileged repertoire” which is defined as, “the set of symbolic and material resources that teacher educators (and teachers) select and configure in order to shape their classroom practice” (p. 299). As I prepare this curriculum I am faced with the dilemma of *what* to select and *how* best to present this. My challenge is to select and teach a “privileged repertoire” to prepare the PSTs for the realities of teaching mathematics in their own classrooms. I also need to take into account the knowledge about mathematics teaching they bring with them from their previous two and a half years of teacher education. Twelve course members volunteered to participate in the study and data was collected over the five weeks, using two questionnaires and two interviews of focus groups. The data were analysed for themes and resulted in two significant findings. The first finding related to *what* knowledge the PSTs needed for mathematics teaching; curriculum knowledge, content knowledge and knowledge of school contexts, as explained by the work of Shulman (1986) and Grossman (2000). The second finding revealed a process related to *how* they constructed this knowledge for mathematics teaching. I identified three active phases of knowledge construction which are; *recognising* aspects of knowledge for mathematics teaching, *reconceptualising* this knowledge for teaching, then *realising* it for enactment in a mathematics classroom. This evolved into a model for teaching, entitled “The Revolving Model of Pre-service Teacher Development”. Each of these phases is described below:

Recognising: this refers to the aspects of knowledge for mathematics teaching the PSTs *recognised* when they engaged with the different sources of knowledge in their ITE course. “Recognising” means to notice, and to identify elements of effective mathematics teaching (Walshaw & Anthony, 2007). This phase is closely aligned with the work of Mason (2002) and Sherin and van Es (2002) who write about the importance of noticing as a process for effective mathematics teaching. Sherin and van Es (2002) describe a key aspect of noticing as being able to identify what is important or noteworthy when teaching mathematics.

Reconceptualising: refers to how PSTs formed and reformed their ideas about mathematics teaching, after the process of recognising. This includes thoughts, feelings and actions they engaged with to develop their own understanding of effective mathematics practices. This reconceptualising phase is also informed by two other important aspects of noticing described by Sherin and van Es (2002), as being to make connections between classroom interactions and broader principles of teaching, and to learn and use what one knows about the context to reason about classroom events. This phase acknowledges a constructivist view of learning where new learning is connected to existing knowledge (Borko & Putnam, 2000).

Realising: refers to the processes they engaged with to realise, or “make real,” their new knowledge about mathematics teaching for use in the classroom. This includes both intended implementation of knowledge in practice such as what they might plan to do in their classrooms, as well as the actual implementation in a mathematics classroom. For PSTs this can occur within course work by engaging with tasks which mimic “real” tasks teachers would do, for example when planning for mathematics teaching, and when teaching mathematics during Professional Practices. This phase aligns with the work of

Ensor (2001) who emphasises the importance of enacting plans as a crucial part an important part of learning to teach.

Each of the three phases can be considered as a discrete process, however when put together presents a connected and revolving process for knowledge construction. To best represent this connected process I developed a visual model which is illustrated in Figure 1 below:

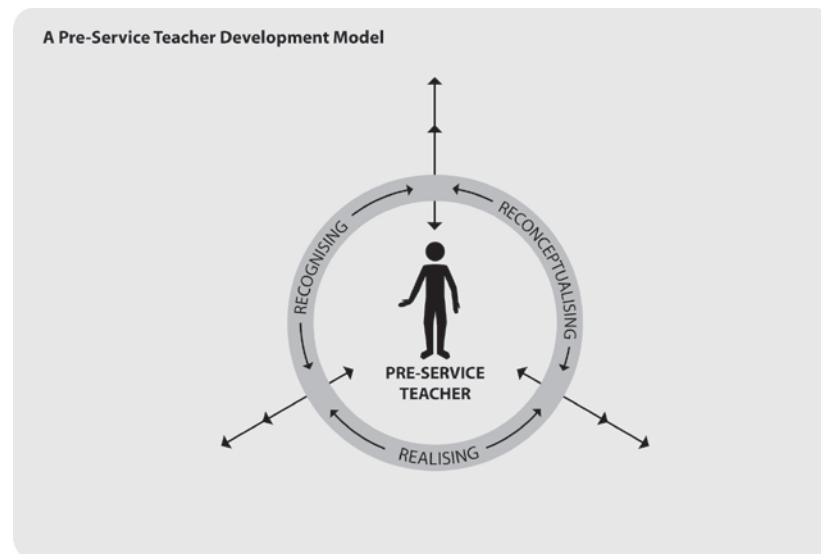


Figure 1. A revolving model of pre-service teacher development.

The PST is placed at the centre of the model because their data informed the construction of the model. The three phases of recognising, reconceptualising and realising are presented in a circular form to illustrate the cyclic revolving motion, both forward and backwards, of knowledge construction for each individual PST. “Revolving” relates to ideas about moving and turning, as well as thinking, considering and focusing, all of which are applicable to a dynamic learning process. The model engages the three domains of curriculum knowledge, mathematical content knowledge and knowledge of school contexts and occurs over different time spans and different contexts for each PST. When combined, the model presents both *what* the PSTs wanted to know and *how* they came to know it within their ITE course. This is presented in Figure 2 below:

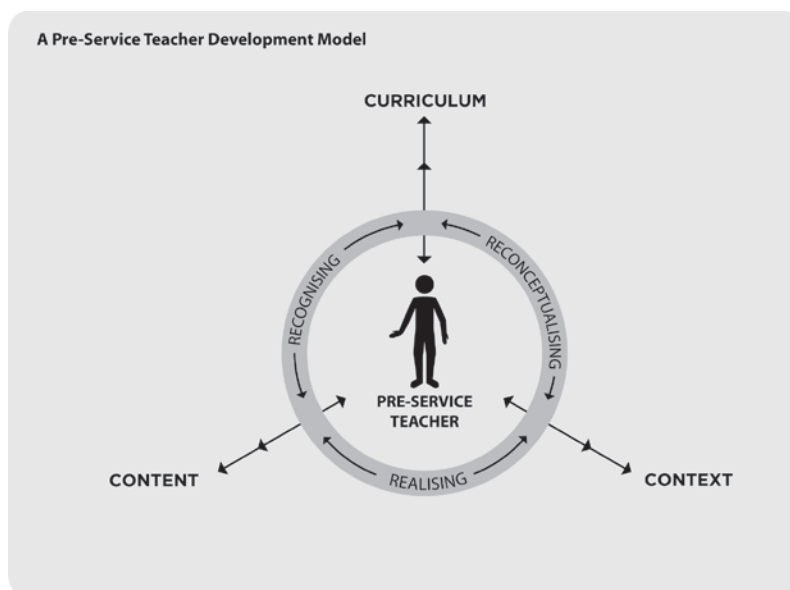


Figure 2. A revolving model of pre-service teacher development.

The following examples of the model in action are drawn from data from a group of PSTs during the development and completion of a course assessment task. This task required the completion of a long-term plan in mathematics, based on the Mathematics and Statistics learning area of *The New Zealand Curriculum*, (2007). For the PSTs in the study, this was the first time they had worked with the new curriculum. They had to engage with the curriculum, select and combine achievement objectives to form units of work and write key mathematics ideas to summarise the mathematics content for each unit.

The Revolving Model in Action

Recognising: PSTs recognised the long term plan as an important document they had seen on Professional Practice that contained curriculum content and provided direction for teacher's planning. Despite this they were unsure about didn't the content of the plans, the purpose of the plans or how they had been put together. On receipt of a plan on placement Marie described it as "just words on a page...blah,blah,blah", and Daniel admitted it was a document he thought he had to collect to meet a requirement, "when I get a long term plan on Professional Practice I don't pay much attention to it- it's there to fill up my folder". When working with the Mathematics and Statistics learning area of the curriculum to complete their own plans they started to notice different aspects. They recognised the strand headings i.e. number and algebra, geometry and measurement and statistics as being a reconfiguration of the old curriculum document headings. They noticed the inclusion of content from The Number Framework (Ministry of Education, 2000). While they were familiar with this content they had questions about how The Number Framework aligned with the new achievement objectives. An important part of noticing is recognising what they don't know, which occurred for several PST when they engaged with the curriculum achievement objectives. Within the objectives they identified problematic and unknown mathematical topics and vocabulary, for example, fractions, decimals, algebra, knowledge and strategy. Tui described some of the vocabulary in the objectives as "lingo", stating "it's just a bunch of words". She explained, "Algebra is just a word. I don't really know, I couldn't tell you what it is". In relation to The Number Framework content she said, "I didn't know the difference between knowledge and strategy...I've heard the terms before,

but didn't know what they meant". The PSTs recognised they needed an understanding of the mathematical content included in the objectives and actively sought resources to develop this knowledge.

Reconceptualising: PSTs scrutinised and analysed the curriculum achievement objectives looking for clues for information about the mathematics they had to teach and how they could teach it. The glossary from the old curriculum and a variety of dictionaries were used to define unknown terms for example, decimals, fractions, and algebra. Some relied on the verbs in the objectives to explain how to teach the content; "like when I looked at the achievement objectives I found them really useful for examples, if you're saying describing and investigating then that's what you have to do. Yip I found that really useful" (Daniel). Cross checking the old achievement objectives with the new, helped to construct content understanding. Teaching resources were used extensively to clarify the meaning of mathematical terms and for guidance about how to teach the concepts. Ministry of Education resources were also used extensively, particularly web based resources. Teaching resources that originate from the Ministry of Education are trusted as reliable sources and resources of knowledge to supplement curriculum and content understanding (Wilson & McChesney, 2010). The PSTs engaged in a sense making process by continually reasoning about curriculum and mathematical content, and moving between the curriculum and other resources. By deconstructing then reconstructing curriculum content they were able to develop both curriculum and mathematical content knowledge.

Realising: within the context of writing the long term plans, the PSTs were able to realise curriculum and mathematical content knowledge. While their mathematics plans were not able to be implemented in a classroom the task allowed them to engage in a realising process in the context of this course. They rated the task as a highlight of the course, because it simulated both a product and process they would engage with as "real" teachers. Armed with this knowledge they were excited by the prospect of being able to, as beginning teachers, work alongside teachers and contribute to collegial curriculum planning. Kate explains this, "You can now participate in unit planning and long term planning because you have done it. You can feel a part of this going into a school, rather than sitting there and going, I have no idea how to this". The planning task not only helped them to transform curriculum and content knowledge for teaching, it also helped their transition into the role of a beginning teacher. The intended realisation of the plan completed in the course setting, helped prepare them for the actual realisation of a plan in their own classrooms. This was a realising of both future teaching and a connection with future classroom responsibility.

Use of the Model for ITE.

The experience of each phase of The Revolving model highlights the complexities of the processes that PSTs engage with to develop knowledge for teaching while they move between the different components of their ITE programme. For PSTs there is a danger that different course components can become separate and disconnected. In order for learning to occur across and between course components, PSTs and teachers educators need to "bridge the gap" between these experiences (Ball, 2000). The three phases of the teaching model; recognising, reconceptualising and realising, show how this could occur. The model represents how PSTs recognise familiar and unfamiliar aspects of privileged repertoire for mathematics teaching, make sense of these, and then realise them in practice. In addition the model provides a sense making process when course components present contrasting and/or conflicting messages about effective mathematics teaching (Goos et al., 2009). It places the

PST at the centre of a decision making process and encourages them to construct knowledge for mathematics teaching from a diverse range of course experiences.

In the model the PST is positioned in the centre of the diagram with an open posture and an outstretched hand. I decided to do this deliberately to acknowledge a quote by Shulman (2004) who describes the complexity of pedagogical knowledge for teaching as being “both a handful and mind-full” (p. 513). The “mindfull” represents the vast amount of knowledge available for mathematics teaching, for example, the mathematics, the curriculum, resources, activities, and assessment practices, while the “handful” represents the portions of this knowledge PSTs can select for implementation in a mathematics classroom. Placing them at the centre of the diagram surrounded by a circular process emphasises their ongoing and active process to construct knowledge about the curriculum, mathematical content and school contexts. It also reinforces the PST as the learner, who as a novice teacher is beginning to make their ‘own’ decisions about a privileged repertoire for mathematics teaching. While experienced teachers have their own repertoire to draw from for enactment in the classroom, PSTs need to develop this. “Experienced teachers may possess rich repertoires of metaphors, experiments, activities or explanations that are particularly effective for teaching a particular topics, while beginning teachers are still in the process of developing a repertoire of instructional strategies and representations” (Grossman, 1990, p.9). The revolving motion of the circle also highlights that knowledge construction involves more than imitating the privileged repertoire presented by teachers while on Professional Practice, or in course work, rather PSTs need to generate their “own” repertoire. This occurs over time and at different rates depending on PSTs experiences. The outstretched hand also indicates the collegial nature of the knowledge construction process because the PSTs developed knowledge by working both individually and collectively. Collegial processes within the ITE setting align with collegial processes in school settings and help prepare PSTs for transitioning between the two (Winsløw et al, 2009). Within the course the PSTs participated in a workshop which simulated a teacher meeting to develop their long-term plans. The ‘hand’ in the diagram acknowledges the input of others in the knowledge construction process.

The circular diagram also challenges the notion of “learning to teach” as being a linear process. This study has highlighted the many cycles of learning for the PSTs both within between and course experiences. For each PST these cycles accumulate along a continuum of learning, commencing within an ITE programme and continuing when they move into school settings. The model has potential as a professional learning model in their own classrooms, where they will continue to develop curriculum, mathematical content knowledge and knowledge of their contexts by recognising, reconceptualising and realising aspects of effective mathematics teaching from their individual school settings.

The identification of the revolving model has helped my work as a teacher educator because it presents categories of knowledge for teaching mathematics as well as a process for developing this knowledge. The identification of knowledge about curriculum, mathematical content and contexts helps to inform decisions I make about the privileged repertoire of mathematics teaching included in the course. Rather than making assumptions about this course curriculum I can feel assured that by addressing these three areas I will be meeting the needs of the PSTs. The identification of curriculum content highlights the need to include in-depth work on the Mathematics and Statistics learning area of NZC (2007) and the Number Framework document. The identification of mathematical content highlights the need for the inclusion of this content within an ITE programme and raises questions about how best to teach this within the constraints of short courses and within already crowded

ITE programmes. As a teacher educator, prioritising what content to include always comes at a cost of leaving content out. This continues to be a dilemma for me.

The identification of the long-term plan as a valued task by the PSTs stresses the importance of selecting and providing tasks which simulate the reality of the classroom. This is particularly important for PSTs who experience separate and short Professional Practice experiences and who do not have their own classrooms to situate their learning (Borko & Putnam, 2000). I did not set out to evaluate the long-term plan as an assignment, but PSTs unanimously agreed it was one of the most useful assignments they have done in their ITE programme because it provided a “real” task that real teachers did. It was also valued because the task was a vehicle for developing both mathematics curriculum and content knowledge. This information serves to support the decisions I make about the benefits and appropriateness of mathematics assessment tasks selected for inclusion in the mathematics course, and the role they play in preparing PSTs for transitioning to various classroom settings as Beginning Teachers.

The model is an important contribution to the field of mathematics education, because it articulates a process for learning to teach mathematics both within the ITE setting and beyond. Much is known about *what* PSTs need to know in order to teach mathematics, but not a lot is known about *how* they transform this knowledge for teaching (Fennema & Franke, 1992). Ball (2000) summarises this by stating, “It’s not just what mathematics teachers know, but how they know it and what they mobilize mathematically in the course of teaching” (p. 243). The revolving model sheds lights on this process; *recognising* emphasises the need for PSTs to recognise both familiar and unfamiliar, congruent and incongruent aspects of effective mathematics teaching, *reconceptualising* acknowledges the need for time, spaces and resources to deconstruct and reconstruct meaning from the recognising process, and *realising* highlights the importance of grounding these understandings in the intended or actual reality of the mathematics classroom. In a recent article, Spalding, Klecka, Lin, Wang and Odell (2011) ask how teacher education programmes can support and challenge PSTs to develop “self” as a teacher, this paper presents one model which could help address this.

References

- Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, 93(4), 373-397.
- Ball, D. (2000). Bridging practices: intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, 51(3), 241-247.
- Borko, H., & Putnam, R. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher*, 29(1), 4-15.
- Ensor, P. (2001). From pre-service mathematics teacher education to beginning teacher: A study in recontextualising. *Journal for Research in Mathematics Education*, 32(3), 296- 320.
- Fennema, E., & Franke, M. L. (1992). *Teachers’ knowledge and its impact*. In D. A. Grouws (Ed.), *The handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.
- Goos, M., Arvola, B., Bednarz, N., DeBlois, L., Maheux, F., Morselli, F., & Proulx, J. (2009). School experiences during pre-service teachers education from the students perspective. In R. Even & D. Ball (Eds.), *The professional education and development of teachers in mathematics: The 15th ICMI study* (pp. 83-92). New York: Springer.
- Grossman, P. (1990). *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.
- Hill, H., Sleep, L., Lewis, J., & Ball, D. (2007). Assessing teachers’ mathematical knowledge: What knowledge matters and what evidence counts? In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 111-155). Charlotte, NC: Information Age Publishing.
- Mason, J. (2002). *Researching your own practice. The discipline of noticing*. London: Routledge Falmer.
- Ministry of Education, (2000). *Book 1 - The Number Framework*. Wellington: Learning Media.

- Ministry of Education, (2007). *The New Zealand Curriculum*. Wellington: Learning Media.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Shulman, L. (2004). *The wisdom of practice. Essays on teaching, learning and learning to teach*. San Francisco: Jossey-Bass.
- Sowder, J. (2007). The mathematical education and development of teachers. In F. K. Lester Jr (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 157-223). Charlotte, NC: Information Age Publishing.
- Spalding, E., Klecka, C., Lin, E., Wang, J. & Odell, S.(2011). Learning to teach: It's complicated but it's not magic. *Journal of Teacher Education*.62(1),3-7.
- van Es, E. & Sherin, M. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Technology Education*, 10, 571-596.
- Walshaw, M. & Antony, G.(2007). *Effective pedagogy in mathematics/pangarau. Best evidence synthesis iteration (BES)*. Wellington: Ministry of Education.
- Wilson, S. (2010). *Knowledge for teaching mathematics in a primary school: Perspective of pre-service teachers*. Unpublished master's thesis, University of Canterbury, Canterbury, New Zealand.
- Wilson, S. & McChesney, J.(2010). "I always feel more confident when I know where things are going"; How do pre-service teachers engage with mathematics curriculum documentation? In B. Kissane, L. Sparrow, & C. Hirst (Eds.), *Shaping the future of mathematics education*. (Proceedings from the 33rd annual conference of the Mathematics Education Research Group of Australasia, Fremantle, pp. 649-656). Fremantle: MERGA.
- Winsløw, C., Bergsten, C., Butlen, D., David, M., Gomez, P., Goos, M., ... Wood, T. (2009). First years of teaching. In R. Even & D. Ball (Eds.), *The professional education and development of teachers in mathematics: The 15th ICMI study* (pp. 93-102). New York: Springer.

Nature of an *Attitudes toward Learning Mathematics* Questionnaire

Khoon Yoong Wong

National Institute of Education, Nanyang
Technological University

<khoonyoong.wong@nie.edu.sg >

Qian Chen

National Institute of Education, Nanyang
Technological University

<qian.chen@nie.edu.sg >

Students' attitudes toward mathematics and its learning have been subject to numerous studies in the past six decades. These studies treat such attitudes as both desirable learning outcomes and correlates of mathematics achievement. Many Likert-type attitude scales have been devised to measure significant constructs underlying mathematics-related attitudes, such as confidence, anxiety, and utility of mathematics. The psychometric properties of these attitude scales may be culture and age dependent. As part of a research project called Singapore Mathematics Assessment and Pedagogy Project (SMAPP), an effort was made to devise and validate an attitude toward learning mathematics scale that can be used with lower secondary school students in Singapore. This paper explains the use of exploratory and confirmatory factor analyses to reduce an initial 57-item questionnaire to one with 24 items that cover these six dimensions: *Checking solutions*, *Confidence*, *Enjoyment*, *Use of IT in mathematics learning*, *Multiple solutions*, and *Usefulness of mathematics*. The data comprise responses from about 890 Secondary 1 (Grade 7) students in 2010, who took the 57-item questionnaire, and another 850 students who took the 24-item questionnaire in 2011. The nature of the final questionnaire is discussed. This effort contributes to the continual effort to devise validated attitude scales that are suitable for different cultures and student groups.

Introduction

Positive attitudes are desirable learning outcomes for most school subjects, including Mathematics. They are also important and positive correlates of mathematics achievement. Numerous studies, including large international comparative studies, such as the Trends in Mathematics and Science Study (TIMSS), have included student attitude as a significant variable of interest (House, 2006; Mullis et al., 2000; Schreiber, 2002; Van den Broeck, Opdenakker, & Van Damme, 2005), because, according to the National Mathematics Advisory Panel (2008) of the United States, "Children's goals and beliefs about learning are related to their mathematics performance" (p. xx). These studies have used Likert-type attitude scales to measure significant constructs underlying mathematics-related attitudes, such as confidence, anxiety, and utility of mathematics. Well-known scales developed by researchers such as the Aiken, Fennema, Sandman, and Sherman, have been modified and used extensively in many studies (e.g., Fennema & Sherman, 1976; Fogarty, Cretchley, Harman, Ellerton, & Konki, 2001; Tapia & Marsh, 2004), and comprehensive reviews of this research area have been provided by researchers over the past thirty years, such as Kulm (1980), Leder (1992), Ma and Kishor (1997), Maasz and Schlöglmann (2009), McLeod (1992), Sriraman (2008), Zan, Brown, Evans, and Hannula (2006).

The psychometric properties of attitude scales toward learning may be culture dependent because the practices of teaching and learning are strongly influenced by cultural milieu, which Schmidt et al. (1996) referred to the pedagogical flow of different countries. Furthermore, the responses gathered from students using such scales also depend on their age. As part of a research project called Singapore Mathematics Assessment and Pedagogy Project (SMAPP), an effort was made to devise and validate an attitude toward learning mathematics scale that can be used specially with lower secondary school students (Grade 7, age 13+) in Singapore. This part of the project attempts to address the culture and age factors of using a Likert-type attitude scale with school students. This questionnaire was

used to measure the attitudes of participating students towards mathematics and its learning before and after the SMAPP intervention. Hence, the research question of this paper is: What were the psychometric properties of the SMAPP *Attitudes toward Learning Mathematics* (ALM) questionnaire? The following sections will describe the design and validation process that was undertaken to answer this research question.

Procedure and Results

Many researchers, including those mentioned above, have pointed out that *attitude* is an ambiguous construct with many shades of meanings overlapping the affective and cognitive domains. However, even a cursory examination of published attitude scales will have identified items used to describe emotions, beliefs, anxiety, confidence, and so forth. Thus, for this project, we have taken a working definition of *attitudes* to encompass items commonly found in these published scales that we feel are within the learning experiences of lower secondary students.

Step 1: Build on a Previous Project

The initial questionnaire was built from one used in a previously funded project called the *Mathematics Assessment Project* (MAP). This earlier project investigated the use of four types of alternative assessment strategies in mathematics learning at primary and secondary levels and included a questionnaire about students' attitudes toward mathematics and its learning (Fan & Zhu, 2008). The 2010 version of ALM consisted of 57 items, 12 of them were from the MAP project. These 57 items were classified under three general categories: (1) Attitudes towards Mathematics; (2) Mathematical Thinking and Reasoning; and (3) Others. The questionnaire was a 9-point scale: 1 (Disagree totally); 2 (Disagree a lot); 3 (Disagree); 4 (Disagree a little); 5 (Neither Disagree nor agree); 6 (Agree a little); 7 (Agree); 8 (Agree a lot); 9 (Agree totally). This 9-point scale allows for greater variation in opinions, hence may be closer to an interval scale, than the more common 5-point scale, but it has the disadvantage that some students may not be able to make such fine distinct judgments about their beliefs. The students took about 15 minutes to answer the questionnaire on-line.

Step 2: Constructs Underlying the 57 Items

The next step was to identify the underlying constructs of the 57 items beyond the three broad categories mentioned above. Two datasets were collected in 2010: (a) a "pre" dataset obtained from 893 Secondary 1 Express¹ students from 5 schools in February; (b) a "post" dataset from the same students with reduced size ($n = 751$; 84%) due to absentees in October.

The Cronbach's alpha for the whole questionnaire, after reverse scoring of negative item, was 0.967 for the pre-data and 0.963 for post-data. Although these values were acceptable, our aim was to design a shorter questionnaire that still reflected the underlying constructs of ALM. This shorter version should have clear psychometric properties and it will save administration time for the students and teachers who may use it in their own research in the future. To achieve this, exploratory factor analysis (EFA) was used.

Both datasets were subject to Principal Component Analysis (PCA) with Varimax rotation. Items with factor loadings less than 0.4 in magnitude or free loadings or cross loadings were mostly ignored. With this procedure, both the pre-data (57.0% of total

¹ Secondary school students in Singapore attend one of three courses: *Normal (Technical)*, *Normal (Academic)*, or *Express*. Students in the *Express* course have average to high ability performance.

variance) and post-data (56.5%) seemed to cover six similar factors: *Checking solutions*, *Confidence*, *Enjoyment*, *Use of IT in mathematics learning*, *Multiple solutions*, and *Usefulness of mathematics*. Some factors (e.g., *Multiple solutions*) were more well-defined than others (e.g., *Confidence*), and some items were loaded on different factors in the two solutions. Given the theoretical links between some of the factors such as confidence and enjoyment, this result is not unexpected for a long questionnaire.

Step 3: Construction of the Shorter Version

The construction of the shorter version of ALM was guided by the above EFA results and more importantly theoretical meanings to be attached to the retained constructs. Briefly, the theoretical ideas are as follows:

1. The *Checking solutions* scale was based on the looking back feature of the Polya's problem solving framework, which is used extensively in Singapore mathematics teaching.
2. The *Confidence* scale covered students' self-concept of their ability to do the mathematics.
3. The *Enjoyment* scale examined the degree to which students enjoyed studying mathematics.
4. The *Use of IT* scale dealt with to what extent students believed that IT could support their learning of mathematics, and this scale was the weakest one in the original questionnaire.
5. The *Multiple solutions* scale dealt with students' propensity to look for multiple solutions, a trait about flexibility in problem solving that is emphasized in the mathematics education literature but may not be widely practiced in Singapore classes.
6. The *Usefulness of mathematics* scale was a ubiquitous one covering students' beliefs about the usefulness and relevance of mathematics to their daily life; this aspect was the main motivation for SMAPP.

Four items were to be used to measure each scale, giving a total of 24 items. Fourteen original items were retained based on the factor solutions and ten new ones had to be created where the original items were deemed to be inappropriate. The Cronbach's alphas of the six scales using the retained items were re-computed using both datasets, with results shown in Table 1. These were within commonly reported values about attitude constructs.

Table 1
Cronbach's Alphas of 24 items (2010 data)

Scales	No. of Retained Items	Pre α	Post α	No. of New Items
Checking solutions	3	0.777	0.816	1
Confidence	3	0.799	0.728	1
Enjoyment	3	0.887	0.830	1
IT and mathematics learning	1	NA	NA	3
Multiple solutions	2	0.782	0.810	2
Usefulness of mathematics	2	0.752	0.786	2

The 24 items are shown in Table 2. The related items for each scale were placed at intervals of six; for example, the *Check* items appeared as items 1, 7, 13, and 19. For convenience, the scales to which the items belong are shown in Table 2 but they did not appear in the actual questionnaire. Both positive and negative items were used. The same 9-point scale was retained to allow for comparison of responses in the 2010 and 2011 data.

Step 4: Analysis of Data of February 2011 using the Short Version

This short version was administered to 849 and 854 Secondary 1 Express students from eight schools in February and October. Only results of the February 2011 data are reported here.

As shown in Table 3, the *Enjoyment* scale was the most reliable one, while the *Multiple* scale was the least reliable. The reason could be that the items in the *Enjoyment* scale were readily understood by the students, but they may not have adequate learning experience about finding multiple solutions in problem solving to give consistent responses to the items in the *Multiple* scale. The alphas of the other four scales were also not as high as those reported in Table 1. This suggests that the selection of items for these scales based on earlier factor solutions and the design of new items at Step 3 needs to be further fine-tuned.

Table 3 also shows that *Confidence* and *Enjoyment* had the strongest correlation, and this reinforces the notion that both constructs are highly related. Beliefs about use of IT to learn mathematics were weakly related to all the other scales, and its highest correlation with *Multiple* (.279) suggests that IT could be an environment to encourage students to seek for multiple solutions. Further research is needed to verify these observations.

Step 5: Confirmatory Factor Analysis (CFA)

Confirmatory Factor Analysis (CFA) was conducted using AMOS 19 to check whether the hypothesized six-factor structure of ALM has empirical support. The tested model initially consisted of six factors with four indicators per factor. After several rounds of modification, it was found that acceptable model fit could be achieved only with item 24 deleted from the analysis. This item was newly added to form this short version, and it seems to differ from the other items by being very specific about certain aspects of the usefulness of mathematics. Whether or not this item format is the cause of its misfit in the model requires further investigation. Hence, the final model works for 23 items, with only three items for the *Usefulness* factor.

Table 2
Items in the Attitudes toward Mathematics Learning (ALM): Short Version

Item No.	Scales	Items
1.	Check	When I know I have made a mistake in solving a problem, I will try to find out why.
2.	Confidence	I am good at using mathematics to solve real-life problems.
3.	Enjoy	I enjoy doing mathematics.
4*.	IT	I do not like to use the computer to learn mathematics.
5*.	Multiple	I do not like to think of other ways to solve the same problem.
6.	Usefulness	Mathematics is important.
7.	Check	After I have solved a problem, I will go through the solution again and check if I have made any mistakes.
8.	Confidence	I am confident in solving mathematics problems.
9*.	Enjoy	I find mathematics boring.
10.	IT	I can learn mathematics from playing computer games.
11.	Multiple	I often figure out different ways to solve mathematics problems.
12.	Usefulness	I think mathematics is useful in solving real world problems.
13*.	Check	Once I have worked out an answer to a problem, I do not check my answer.
14.	Confidence	I find mathematics easy.
15.	Enjoy	Overall, I have good feelings about mathematics.
16.	IT	IT (Information Technology) has been helpful to my mathematics learning.
17.	Multiple	I try to understand the different solutions given by my classmates.
18*.	Usefulness	I think mathematics is useful only for tests.
19.	Check	After I have solved a problem, I will ask myself if the answer makes sense to the given problem.
20*.	Confidence	I am not good at giving reasons in mathematics.
21.	Enjoy	Solving mathematics problems is fun to me.
22.	IT	Mathematics software (e.g., graphing) helps me to learn mathematics.
23.	Multiple	After I have solved a problem, I will look for other methods to solve it.
24.	Usefulness	Mathematics helps me to understand reports and advertisements about prices, sale, percentages etc.

Note: * Negative items.

Table 3
Cronbach's Alphas and Correlations of ALM Scales: Short Version (n = 849)

	Check	Confidence	Enjoyment	IT	Multiple	Usefulness
Check	1	.458**	.517**	.176**	.583**	.524**
Confidence		1	.709**	.177**	.491**	.526**
Enjoyment			1	.114**	.512**	.590**
IT				1	.279**	.209**
Multiple					1	.509**
Usefulness						1
Cronbach's α	0.714	0.695	0.898	0.693	0.633	0.711
Mean	6.66	5.21	6.15	6.03	5.93	6.91
SD	1.38	1.43	1.92	1.60	1.32	1.36

Note. **Correlation is significant at the 0.01 level (2-tailed).

Following the suggestions by Meyers, Gamst, and Guarino (2006) and others, we examined four model fit measures. The chi-square had a value of 465.972 ($df = 184$, $N =$

849), $p < 0.01$, and this indicates a mismatch between the proposed model and the observed data. However, the sole use of this index in judging the overall fit of the model is not appropriate because of the sensitivity of the chi-square to large sample, which is the case here. The three other indices indicate good model fit: NFI = .953 (acceptable values are .95 or above), CFI = .971 (similar to NFI), and RMSEA = .04 (acceptable values are .08 or below). In short, the CFA showed that the six-factor model was a well-fitting model, with the deletion of item 24.

The standardized regression weights for this solution are summarized in Table 4. These weights are useful in deciding to what extent the items are related to the underlying purported latent factors (Brown, 2006, p. 131). Although the six-factor structure of ALM was basically confirmed, items 5, 13, and 20 had coefficients much lower than those of the other items in the same factors. They were all negative items, and some lower secondary students may have difficulty agreeing or disagreeing with negative items that demand complex thinking processes. This observation need to be further examined.

Table 4
Standardized Regression Weights

Estimate				Estimate			
1	<---	Check	.624	*4	<---	IT	.548
7	<---	Check	.696	10	<---	IT	.552
*13	<---	Check	.392	16	<---	IT	.804
19	<---	Check	.664	22	<---	IT	.671
2	<---	Confidence	.575	*5	<---	Multiple	.205
8	<---	Confidence	.882	11	<---	Multiple	.842
14	<---	Confidence	.781	17	<---	Multiple	.606
*20	<---	Confidence	.238	23	<---	Multiple	.555
3	<---	Enjoyment	.894	6	<---	Usefulness	.794
*9	<---	Enjoyment	.664	12	<---	Usefulness	.772
15	<---	Enjoyment	.887	*18	<---	Usefulness	.524
21	<---	Enjoyment	.842				

Note: * Negative items (recoded).

Discussion and Concluding Remarks

In the above sections, we have described a rather elaborate process of developing a short version of an “attitudes towards learning mathematics” questionnaire that our project SMAPP had used to measure changes of the attitudes of lower secondary students during the implementation of the project. The practical value of the above validation is that the ALM questionnaire is a relatively short one with some known psychometric properties, and it will be disseminated as a research tool to other secondary schools in Singapore, so that it can be used by school teachers in their own action research, currently widely promoted in Singapore, or by other researchers in Singapore or elsewhere. Its relatively short length suggests that ALM can be an efficient research tool for the busy teachers.

Given that there are many mathematics-related attitude items in the literature, writing and compiling items for a Likert-type attitude questionnaire should at first glance be relatively straightforward. However, the above efforts show that validating the psychometric properties of a questionnaire against the putative theoretical structure is far from an easy

task; instead, it requires considerable statistical work and substantive interpretations. Even if a satisfactory solution is attained for the moment, new data will have to be collected from the intended target population and subject to further analysis so that better solutions can be found. The possibility of replication is a hallmark for scientific advancement, and this is especially important in the social sciences where replication can be difficult to arrange and is often not attempted. The next phase of our work is to replicate the above factorial structure in at least three ways.

The first way is to test the model with new data. The original data were not drawn randomly from the target population because participation in the SMAPP project was voluntary, even though the sample was large and came from different schools. Thus, findings about the attitudinal structure of ALM may not be considered as normative for other student groups and across countries. This is related to the issues mentioned at the beginning about the influence of age, culture, and ability on measurement of attitudes.

Secondly, it should be beneficial to examine to what extent the less acceptable items are due to item format (as noted above), language usage, or the ability of students to make meaningful judgment about their own thinking, namely, their metacognitive awareness. Interviews could be conducted with students to gain insights into how they might interpret the items. Examples discussed by Blunch (2008) show that respondents may have paid attention to key words in statements different from those the researchers have in mind, thus leading to unexpected CFA findings. Further conceptual considerations may lead to deletion or re-writing of some of these items.

The third approach is prompted by the pattern of correlations given in Table 3. The *Confidence* and *Enjoyment* scales had very strong correlation and they may be combined into a single factor. The *IT* scale with its low correlations with the other scales seems to stand alone. The other three scales (*Check*, *Multiple*, and *Usefulness*) form a third factor. A general attitude factor can be hypothesized as a second-order factor with no indicators of its own such that it serves as a common cause of the three first-order factors in a hierarchical CFA model, parallel to the g-factor in the structure of cognitive ability (Kline, 2011). This is an interesting hypothesis to be tested, and it might contribute to the scientific investigation of the nature of attitude about mathematics or other subjects.

To conclude, similar efforts to create efficient and validated questionnaires to measure mathematics-related attitudes among students have practical values and research implications. Obviously such efforts are never-ending and unexpected new insights are truly the intrinsic rewards for conducting these studies.

References

- Blunch, N. J. (2008). *Introduction to structural equation modelling using SPSS and AMOS*. Thousand Oaks, CA: Sage Publications.
- Brown, T. A. (2006). *Confirmatory factor analysis for applied research*. New York, NY: Guilford Press.
- Fan, L. & Zhu, Y. (2008). Using performance assessment in secondary school mathematics: An empirical study in a Singapore Classroom. *Journal of Mathematics Education*, 1(1), 132-152.
- Fennema, E., & Sherman, J. (1976). Fennema-Sherman Mathematics Attitudes Scales: Instruments designed to measure attitudes toward the learning of mathematics. *Journal for Research in Mathematics Education*, 7(5), 324-326.
- Fogarty, G. J., Cretchley, P., Harman, C., Ellerton, N., & Konki, N. (2001). Validation of a questionnaire to measure mathematics confidence, computer confidence, and attitudes to the use of technology for learning mathematics. *Mathematics Education Research Journal*, 13(2), 154-160.
- House, J.D. (2006). Mathematics beliefs and achievement of elementary school students in Japan and the United States: Results from the third international mathematics and science study. *The Journal of Genetic Psychology*, 167(1), 31-45.

- Kline, R. B. (2011). *Principles and practice of structural equation modeling* (3rd ed.). New York, New York: Guilford Press.
- Kulm, G. (1980). Research on mathematics attitude. In R. J. Shumway (Ed.), *Research in mathematics education* (pp. 356–387). Reston, VA: National Council of Teachers of Mathematics.
- Leder, G. (1992). Attitudes to mathematics. *Mathematics Education Research Journal*, 4(3), 1–7.
- Ma, X. & Kishor, N. (1997). Assessing the relationship between attitude toward mathematics and achievement in mathematics: A meta-analysis. *Journal for Research in Mathematics Education*, 28, 26–47.
- Maasz, J., & Schlöglmann, W. (Eds.). (2009). *Beliefs and attitudes in mathematics education: New research results*. Rotterdam, the Netherlands: SensePublishers.
- McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–595). New York: Macmillan.
- Meyers, L. S., Gamst, G., & Guarino, A. J. (2006). *Applied multivariate research: Design and interpretation*. Thousand Oaks, CA: Sage Publications.
- Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., Gregory, K. D., Garden, R. A., & O'Connor, K. M. (2000). *TIMSS 1999 international mathematics report: Findings from IEA's repeat of the third international mathematics and science study at eighth grade*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.
- National Mathematics Advisory Panel (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: Author. Available from <http://www.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf>
- Schmidt, W. H., Jorde, D., Cogan, L. S., Barrier, E., Gonzalo, I., Moser, U., Shimizu, K., Sawada, T., Valverde, G. A., McKnight, C., Prawat, R. S., Wiley, D. E., Raizen, S. A., Britton, E. D., & Wolfe, R. G. (1996). *Characterizing pedagogical flow: An investigation of mathematics and science teaching in six countries*. Dordrecht: Kluwer Academic Publishers.
- Schreiber, J. B. (2002). Institutional and student factors and their influences on advanced mathematics achievement. *The Journal of Educational Research*, 95(5), 274–286.
- Sriraman, B. (Ed.). (2008). *Beliefs and mathematics: Festschrift in honor of Guenter Toerner's 60th birthday*. Missoula, MT: Information Age Publishing.
- Tapia, M., & Marsh II, G. E. (2004). An instrument to measure mathematics attitudes. *Academic Exchange Quarterly*, 8(2), 16–21.
- Van den Broeck, A., Opdenakker, M.-C., & Van Damme, J. (2005). The effects of student characteristics on mathematics achievement in Flemish TIMSS 1999 data. *Educational Research and Evaluation*, 11(2), 107–121.
- Zan, R., Brown, L., Evans, J., & Hannula, M. S. (2006). Affect in mathematics education: An introduction. *Educational Studies in Mathematics*, 63(2), 113–121.

Profiling Students' Capacities to Link Number and Algebra in Years 5, 6 and 7 in Nanjing, China

Wenbin Xu
Nanjing Normal University
<xuwenbin@njnu.edu.cn>

Max Stephens
The University of Melbourne
<m.stephens@unimelb.edu.au>

Qinqiong Zhang
Wenzhou University China
<zhang922@yahoo.com>

This study investigates how 217 students in Years 5, 6 and 7 from three schools in Nanjing, China, link number and algebra (called relational thinking in this study). It categorizes their performances in terms of five levels, and uses these levels to create profiles of algebraic thinking across Years 5, 6, and 7. The study examines the interrelationships between the different types of questions used, and highlights the importance of connecting students' arithmetic learning and the development of their algebraic understanding.

Rationale for the Research and Relevance to the Curriculum

In China, "Number and Algebra" is one out of four content strands in the National Mathematics Curriculum Standards (Ministry of Education, 2001; 2011). China's official curriculum endorses the building of closer relationships between the study of number in the primary school and the development of algebraic thinking. This is very different to what in past curriculum documents was an almost exclusive focus on arithmetic methods in the primary school years (Stephens and Zhang, 2011).

Some Chinese researchers, such as Xu (2003), suggest that curriculum reform requires a closer alignment between the study of number and number relationships in the primary school and the study of algebra in the secondary school. Jacobs, Franke, Carpenter, Levi and Battey (2007) also argue that a more coherent treatment of number and algebra requires changed emphases in and treatment of the way number and number relations are taught and understood by students. While admitting that there is still room for debate and whether *relational thinking* in arithmetic represents a way of thinking about arithmetic that provides a foundation for learning algebra, *or is itself a form of algebraic reasoning*, Jacobs et al. (2007) emphasise that "one fundamental goal of integrating relational thinking into the elementary curriculum is to facilitate students' transition to the formal study of algebra in the later grades so that no distinct boundary exists between arithmetic and algebra" (p. 261).

Five key ideas underpin our theoretical perspective on relational thinking and these constitute a bridge between number and number operations and early algebra thinking. These ideas are all now prominent in the research literature on early algebra:

- Structure of number sentences (Cai, Ng and Moyer, 2011; Jacobs, Franke, Carpenter, Levi and Battey, 2007);
- Equivalence (Kaput, Carraher and Blanton, 2008; Lins and Kaput, 2004);
- Compensation using equivalence according to specific operations (Britt and Irwin, 2011; Stephens and Wang, 2009; Irwin and Britt, 2005);
- Numbers that can vary (Cooper and Warren, 2011; Fujii and Stephens, 2001); and
- Generalization (Cooper and Warren, 2011; Mason, Stephens and Watson, 2009).

All the above ideas rest on deep understanding of number sentences and are often left implicit in text book treatment of algebra in junior secondary school, where it is introduced as the generalization of arithmetic and formal use of letters in equations. As Cooper and Warren (2011) argue, “quasi-generalisation in an elementary (school) context appears to be a necessary precursor to expressing the generalisation in natural language and algebraic notation” (p. 193). Moreover, assessment frequently emphasises procedural fluency assuming that procedural success carries with it conceptual understanding. Unless students experience these five key ideas in the context of number sentences and number operations in the primary school, they will usually have a difficult transition to learning algebra in the junior secondary school.

Methodological Position

The Sample

Three schools in Nanjing were involved in this study: two of them were primary schools (respectively named as Pn and Pj in this study), and one was a junior secondary school (named as Sh). All three participating schools are considered to be among the middle group of achievement. In Pn, the same questionnaire was given to 83 students, 45 from a Year 5 class and 38 students from a Year 6 class. In Pj, it was given to 69 students, 35 from a Year 5 class and 34 from a Year 6 class. In the junior secondary school Sh, there were 65 students from two Year 7 classes; one class had 32 students and another had 33. In summary, 217 students from six classes across three Year levels in three Nanjing schools completed the same questionnaire.

Design of Questionnaire

The questionnaire followed that used by Stephens and Wang (2009), and contained three types of mathematical sentences for each of the four operations. In Type 1 number sentences (single unknown), students were given four questions based on one of the four operations and asked to find the value of a missing number and to explain their thinking. The following addition sentences are provided as examples showing how the placement of the unknown number was deliberately varied.

For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.

$$23 + 15 = 26 + \square$$

$$43 + \square = 48 + 76$$

$$73 + 49 = \square + 47$$

$$\square + 17 = 15 + 24$$

Following each set of four Type 1 number sentences, the subsequent page for each operation consisted of related Type 2 and Type 3 sentences. Taking addition as an example, the questions for Type 2 sentences all followed a similar format:

Can you think about the following mathematical sentence?

$$18 + \square = 20 + \square$$

Box A
Box B

a) *In each of three sentences below, can you put numbers in Box A and Box B to make each sentence correct?*

$$18 + \square = 20 + \square$$

Box A
Box B [2 more templates given]

b) *When you make a correct sentence, what is the relationship between the numbers in*

Box A and Box B?

- c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?
- d) If any number was put in Box A, could the number sentence still be true? Please give your explanation.

Part e then followed presenting students with a Type 3 sentence, using literal symbols c and d in place of Box A and Box B (see Table 1). Following Hart (1981), students were asked: What can you say about c and d in this mathematical sentence? As can be seen, the Type 3 sentence for addition, $c + 2 = d + 10$, is structurally similar to its corresponding Type 2 Number Sentence. Students needed to describe the relationship between c and d that permits this mathematical sentence to be true. The total length of the questionnaire was eight pages.

Table 1

Type 2 and Type 3 Sentences for the Other Three Operations

Operation	Type 2 sentences		Type 3 sentences
Subtraction	$72 - \square = 75 - \square$		
	Box A	Box B	$c - 7 = d - 10$
Multiplication	$5 \times \square = 20 \times \square$		
	Box A	Box B	$c \times 2 = d \times 14$
Division	$3 \div \square = 15 \div \square$		
	Box A	Box B	$c \div 8 = d \div 24$

Qualitative Categorization of Students' Responses

Students' Responses to Type 1 Sentences

Students' responses to Type 1 sentences were classified as Computational, Computational/Relational Mixed or Relational. Computational thinking was clearly shown where students had calculated the result of the two given numbers. In other responses, students showed relational thinking clearly using words or by means of arrows or diagram (see figure 1). Incorrect relational thinking was sometimes evident when some students solved a subtraction sentence, such as $104 - 45 = \square - 46$, giving 103 as the missing number, by applying an incorrect direction of compensation; that is, using the same direction of compensation as in addition. When correct answers were given with no explanation or other evidence of a student's thinking, responses were classified as Computational.

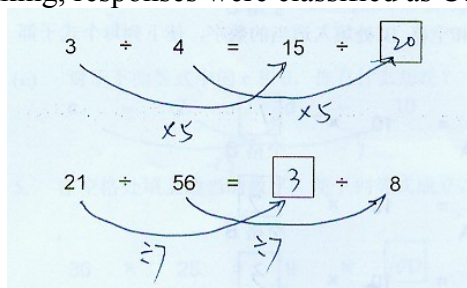


Figure 1: Arrows used by students to solve Type 1 sentences.

Because some students may have been more confident or comfortable with computation, Type 1 number sentences may not fully disclose the capacity for relational thinking. For that reason, the questionnaire needed to include additional Type 2 and Type 3 questions in order to distinguish between students who preferred to solve Type 1 questions by computation from those students whose thinking was restricted to computation.

Students' Responses to Type 2 and Type 3 Sentences

To discuss students' responses to Type 2 sentences, three questions were considered: (1) Were correct numerical exemplifications given in Part a? (2) How did students describe the relationship between the numbers in Box A and Box B in Parts b and c? (3) Could they generalise the relationship between numbers in Box A and Box B and could they utilise that generalisation in answering Part d? Following Stephens and Wang (2009), the following categorisations were used to classify responses to Type 2 and Type 3 sentences. Table 2 illustrates how some students described the relationship between the numbers in Box A and Box B and between c and d .

Table 2

Categories of Students' Responses to Type 2 and Type 3 Questions

Response type	Examples for Type 2 and Type 3 questions
Incorrect Relation	Students use 'difference' to describe multiplication and division the difference (between c and d) is always 16, in $c \div 8 = d \div 24$
Non-directed Relation	they would always be 5 apart, as in $3 \div (\text{Box A}) = 15 \div (\text{Box B})$ there is always a 3 difference, as in $72 - (\text{Box A}) = 75 - (\text{Box B})$ the numbers have a distance of 2, as in $18 + (\text{Box A}) = 20 + (\text{Box B})$
Directed Relation: Non-referenced* or No magnitude	so long as the number in Box B is larger, as in $72 - (\text{Box A}) = 75 - (\text{Box B})$; or as in $3 \div (\text{Box A}) = 15 \div (\text{Box B})$ d will be more than c , as in $c - 7 = d - 10$, or in $c \div 8 = d \div 24$ one number is always 2 higher than the other number, as in $18 + (\text{Box A}) = 20 + (\text{Box B})$ one is 4 times more than the other, as in $5 \times (\text{Box A}) = 20 \times (\text{Box B})$
(Fully)Referenced and Directed Relation	one is 3 more than the other, Box B is bigger (subtraction); or $B - 3 = A$ c is 8 more than d , as in $c + 2 = d + 10$; or $c = d + 8$ A is 5 times less than B , as in $3 \div (\text{Box A}) = 15 \div (\text{Box B})$; or a difference of 2 with A larger, as in $18 + (\text{Box A}) = 20 + (\text{Box B})$; or $A - 2 = B$

*"Non-referenced" means not to point out the relational objects, such as Box A and Box B, or c and d

Categorising Students' Relational Thinking on Type 2 and Type 3 Sentences

According to their responses on Type 2 and 3 questions, five categories of students' relational thinking were created: Established; Highly Consolidating; Consolidating; Emerging; and Non-Relational. Two categories, Highly Consolidating and Non-relational were not included in the Stephens and Wang (2009) study.

Established Relational thinkers gave responses to Type 2 and Type 3 sentences across all four operations that showed fully referenced and directed relations. These students were almost always able to: (1) specify the relationship between the numbers in Box A and the numbers in Box B with clear references to the magnitude and direction of the difference between them; (2) employ the form of words used to describe this relationship as a part of the condition that describes how *any* number can be used in Box A and still make a true sentence; (3) explain clearly how c and d are related for the Type 3 sentence to be true, treating c and d as general numbers.

Highly Consolidating Relational thinkers' responses to Type 2 and Type 3 sentences across all four operations demonstrated clear relational thinking in Parts a, b and c. They *occasionally* had difficulty with one of Part d and/or e using some form of incomplete relational thinking. In some cases their responses to three of the four operations were fully relational but the questionnaire was not entirely completed.

The third category, Consolidating Relational, included students who, in their responses to Type 2 and Type 3 sentences across the four operations demonstrated clear relational thinking in Parts a, b and c, but *almost always* gave incomplete responses to one or both of Parts d and e. These students: (1) were *almost always* able to specify the relationship between the numbers in Box A and the numbers in Box B with clear references to the magnitude and direction of the difference between them; (2) *rarely* gave a complete explanation as to how any number might be used in Box A and still have a true sentence; and (3) *usually* referred to some feature of the relationship between *c* and *d*, typically giving only a specific pair of values for *c* and *d*.

Emerging Relational thinkers were *unable* to give a complete relational description in Parts b and c to the relationship between the numbers in Box A and in Box B. Their descriptions were *typically* non-directed or non-referenced or written without specifying the magnitude of the relation. *Invariably*, their incomplete description of the relationship between the numbers used in Box A and Box B determined their explanation in Part d of how any number can be used in Box A and still have a true sentence. In Part e, they either attempted to give a correct pair of values for *c* and *d*, or they omitted this question altogether.

Non-Relational responses typically gave correct numerical results for Part a, but could not relate or see connections between the numbers involved in Parts b, c, d and e. This is illustrated by a Year 5 student who answered Part b by saying: "*Meets the relation that the sum is the same*"; and Part d by saying: "*Cannot, because if any number is filled in, it's very easy to have incorrect (numbers) filled.*"

Coding Procedures for Student Questionnaires

Two groups of graduate students independently coded questionnaire responses from each grade using the above classifications and their indicators. The coders then compared results to identify any differences. For Type 1 number sentences, complete agreement was almost always achieved. However, where students had not completed all questions or where they had given some incorrect answers, an on-balance judgement was necessary. Students' responses to Type 2 and Type 3 questions across all four operations were used to decide which category best described their performance. Differences in coding occurred in approximately 10% of questionnaire responses, where an overall response might be classified one level higher or one level lower among the five categories. For these discrepancies, researchers and graduate scorers were almost always able to reach an agreed classification.

Quantitative Analysis of Students' Responses

Summary Profiles of Years 5, 6 and 7

Table 3 presents the summary statistics for each Year level. It first shows how students from each Year level responded to the Type 1 number sentences using the three classifications: Computational, Computational/Relational Mixed, and Relational. It then shows how these different responses to Type 1 number sentences were spread across the

five categories of responses to Type 2 and Type 3 sentences: Established Relational (EsR), Highly Consolidating Relational (HCR), Consolidating Relational (ConR), Emerging Relational (EmR) and Non-Relational (NonR).

Table 3
Summary Statistics for Type 1, Type 2 and Type 3 Responses

Year level	Type 1 Responses		Type 2 and Type 3 responses				
			NonR	EmR	ConR	HCR	EsR
Year 5 (77 valid)	Computational	43	11	16	15	1	0
	Comp/Rel Mixed	19	0	10	7	2	0
	Relational	15	0	5	4	5	1
Year 6 (72)	Computational	22	7	9	4	2	0
	Comp/Rel Mixed	24	2	6	7	5	4
	Relational	26	3	7	8	4	4
Year 7 (65)	Computational	16	0	4	5	6	1
	Comp/Rel Mixed	20	0	2	6	8	4
	Relational	29	0	1	5	17	6

In Year 5, of the 77 valid responses out of 80 from the two classes in schools Pn and Pj, more than half of the students (43 out of 77) were still using computational thinking for Type 1 number sentences. All 11 Non-Relational responses on Type 2 and Type 3 sentences and 16 out of 31 Emerging Relational responses on Type 2 and Type 3 sentences – 27 out of 43 students – came from the Computational group on Type 1 sentences. Of the 34 students who used either Relational or a mix of Computational and Relational thinking on Type 1 number sentences, most (26) were Emerging or Consolidating Relational thinkers on Type 2 and Type 3 sentences. However, nine Year 5 students were either Established or Highly Consolidating, with only one using Computational thinking on Type 1 sentences. Reliance on computation to solve Type 1 sentences appears to be linked to limited capacity for Relational thinking on Type 2 and Type 3 sentences in the Year 5 group.

Year 6 included 72 samples from two classes in schools Pn and Pj. In Year 6, nearly one third of the students (22 out of 72) were still using computational thinking for Type 1 number sentences, showing a trend towards relational thinking on Type 1 sentences. Seven of the 12 Non-Relational students on Type 2 and Type 3 sentences and 9 of the 22 Emerging Relational thinkers on Type 2 and Type 3 sentences – 16 out of 34 students – used Computational thinking on Type 1 sentences, suggesting that reliance on computation on Type 1 sentences is still associated with limited capacity to deal with Type 2 and Type 3 sentences. Of the 50 students who used either Relational or a mix of Computational and Relational thinking on Type 1 number sentences, 5 were Non-Relational, 13 were Emerging, 15 were Consolidating, 9 were Highly Consolidating. All 8 students who were Established Relational thinkers on Type 2 and Type 3 sentences used either Relational or a mix of Computational and Relational thinking on Type 1 sentences.

All 65 samples from two Year 7 classes came from one junior secondary school Sh. No Year 7 students were Non-Relational on Type 2 and Type 3 sentences. The 16 students out of 65 who used computational thinking for Type 1 number sentences were spread across

Emerging (4), Consolidating (5), Highly Consolidating (6) and Established (1) on Type 2 and Type 3 sentences. Reliance on Computational thinking to solve Type 1 sentences here appears to be more of a choice. Of the 49 students who used either Relational or a mix of Computational and Relational thinking on Type 1 number sentences, 3 were Emerging, 11 were Consolidating, 25 were Highly Consolidating and 10 were Established. Ten of the 11 students who were Established Relational thinkers on Type 2 and Type 3 sentences used either Relational or a mix of Computational and Relational thinking on Type 1. Forty two of the 65 Year 7 students were classified as Highly Consolidating or Established Relational thinkers. While this may be explained, in part, by exposure to literal symbolic algebra in Year 7, it seems to have had no major benefit for the other 23 students.

Quantifying Students' Responses

We used the following weighting system to give a numerical profile of relational thinking for each class and grade level: 0 for Non-Relational, 1 for Emerging, 2 for Consolidating, 3 for Highly Consolidating, and 4 for Established. We applied these numbers to responses at each Year level and divided by the number of students to obtain a mean score for each Year level and for each class.

Table 4
Mean Measures of Students' Relational Thinking

Year	5		6		7	
Class	Pj-5	Pn-5	Pj-6	Pn-6	Sh-7a	Sh-7b
Class Mean	1.424	1.455	1.735	1.737	2.875	2.545
Year Mean	1.442		1.736		2.708	

From Table 4, it can be seen that the Year means of students' relational thinking – 1.442, 1.736 and 2.708 respectively – show an increase from Year 5 to Year 6, and an even greater increase from Year 6 to Year 7. There was no statistically significant difference between the two participating classes at each Year level.

Research Findings and Implications for Teaching

In each class, the profile of students' performances was spread across 4 to 5 levels. While there was evident growth between Year 6 and Year 7, there remained for each Year level a wide range of ability to express mathematical thinking. Many Year 7 students were still thinking mathematically well below the capacity of some students in Year 5 and 6. This is a challenge for teaching at every Year level.

The results of this study confirm the various affordances and impediments that, as discussed by Stephens and Wang (2009), connect different responses to Parts b and c for Type 2 sentences and students' likely success in answering the associated Part d question, and its related Part e question. If students used Non-directed, or Directed (no magnitude), or Directed (non-referenced) relational thinking in their responses to Parts b and c, they were almost always unable to give a complete mathematical generalisation in Part d of how any number can be used in Box A and still have a true sentence, *and* they were also unable to describe the mathematical relationships between *c* and *d* in Type 3 sentences. Students who successfully completed both Part d and Part e appeared to recognise the structural similarity between the two questions. Some students who successfully completed Part e were unable

to give a successful generalisation needed for Part d. Teachers need to help their students articulate referenced and directed relational descriptions by highlighting to students the disadvantages and advantages that different descriptions offer.

All three Types of sentences appear to have a high teaching potential for developing and refining mathematical thinking with respect to: equivalence and compensation; attention to structure and operations; attending to the range of possible variation (which numbers vary, which numbers stay the same); and generalisation. Relying on Type 1 number sentences alone to achieve these goals is probably insufficient. Strategies, such as teaching with variation, can help students to see that, for both Type 2 and 3 sentences, the permissible range of variation can include rational numbers and negative numbers. This can help to build a stronger foundation for the subsequent idea of variable in high school mathematics.

References

- Britt, M., & Irwin, K. (2011). Algebraic thinking with and without algebraic representation: A pathway for learning. In J. Cai & E. Knuth (Eds.), *Early Algebraization: A global dialogue from multiple perspectives*, pp. 137-159. Heidelberg: Springer.
- Cai, J., Ng, S., & Moyer, J. (2011). Developing students' algebraic thinking in earlier grades: Lessons from China and Singapore. In J. Cai & E. Knuth (Eds.), *Early Algebraization: A global dialogue from multiple perspectives* (pp. 25-41). Heidelberg: Springer.
- Cooper, T., and Warren, E. (2011). Year 2 to Year 6 students' ability to generalize: Models, representations and theory for teaching and learning. In J. Cai & E. Knuth (eds.), *Early Algebraization: A global dialogue from multiple perspectives*, (pp. 187-214). Heidelberg: Springer.
- Fujii, T., & Stephens, M. (2001). Fostering understanding of algebraic generalisation through numerical expressions: The role of quasi-variables. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *The Future of the teaching and learning of algebra. Proceedings of the 12th ICMI study Conference* (Vol. 1, pp. 258-264). Melbourne, Australia.
- Hart, K. M. (1981). *Children's understanding of Mathematics*. London: John Murray.
- Irwin, K., and Britt, M. (2005). The algebraic nature of students' numerical manipulation in the New Zealand Numeracy Project. *Educational Studies in Mathematics*, 58, 169-188.
- Jacobs, V., Franke, M., Carpenter, T., Levi, L. and Battey, D. (2007). Developing children's algebraic reasoning. *Journal for Research in Mathematics Education*, 38(3), 258-288.
- Kaput, J., Carraher, D., & Blanton, M. (Eds.), *Algebra in the early years*. New York: Lawrence Erlbaum Associates and National Council of Teachers of Mathematics.
- Lins, R., and Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of algebra: the 12th ICMI Study* (pp.47-70). Boston: Kluwer Academic Publishers.
- Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal*, 21(2), 10-32.
- Ministry of Education of PRC (2001). *Mathematics curriculum standards for compulsory education* (experimental) (in Chinese). Beijing: Beijing Normal University Press.
- Ministry of Education of PRC (2011). *Mathematics curriculum standards for compulsory education* (revised) (in Chinese). Beijing: author.
- Stephens, M., & Wang, X. (2009). Some key junctures in relational thinking: A study of Year 6 and Year 7 students from Australia and China. In R. Hunter, B. Bricknell, & T. Burgess (Eds.), *Crossing Divides: Proceeding of the 32nd annual conference of the Mathematics Education Group of Australasia* (pp. 499-506). Wellington, New Zealand: MERGA.
- Stephens, M., & Zhang, Q. (2011). Teacher capacity as a key element of national curriculum reform in mathematics: a comparative study between Australia and China. In Clark, J., Kissane, B., Mousley, J., Spencer, T. & Thornton, S (Eds.), *Mathematics: traditions and (new) practices. Proceedings of the 23rd biennial conference of the Australian Association of Mathematics Teachers & 34th annual conference of the Mathematics Education Group of Australasia* (pp. 702 - 710). Alice Springs: AAMT & MERGA.
- Xu, W. (2003). Algebraic thinking in arithmetic: Quasi-variable expressions (in Chinese). *Xueke Jiaoyu (Journal of Subject Education)* 11, 6-10, 24.

Problem Posing in Mathematical Investigation

Joseph B. W. Yeo
Nanyang Technological University
<josephbw.yeo@nie.edu.sg>

This paper reports on the types of problems that high-achieving students posed when given investigative tasks that were constructed by opening up some mathematical problems. A teaching experiment was conducted to develop the students' thinking processes during mathematical investigation, and each student was videotaped thinking aloud during a pretest and a posttest. The findings show that some students were unable to pose the original intended problems and what Krutetskii (1976) called problems that 'naturally follow' from the task, including extending the task to generalise. The implications of the difficulty encountered by these students for teaching and research will also be discussed.

Background

Mathematical investigation is emphasised in many school curricula (e.g., Australian Curriculum, Assessment and Reporting Authority, 2012; Ministry of Education of New Zealand, 2007) because it is believed that investigation helps students to think and learn mathematically (Jaworski, 1994), and that investigation reflects the practices of mathematicians (Civil, 2002). Although there is plenty of research on problem solving (Cai, Mamona-Downs, & Weber, 2005), there are not many empirical studies on investigation despite a thorough search of current literature. Moreover, such research on investigation is usually restricted to examining the general benefits that investigation has offered to the students (e.g., Bailey, 2007; Tanner, 1989), but empirical studies on the types of processes that students actually engage in when doing investigation are rare. The study described in this paper addresses this 'gap' by investigating how students think during investigation. With a clearer understanding of the nature and development of these thinking processes, it might help teachers to cultivate these processes in their students.

Unlike mathematical problems, e.g., Task 1, that typically contain the question in the task statement, open investigative tasks, e.g., Task 2, usually do not contain any question but students are expected to pose their own problems to solve or investigate (Orton & Frobisher, 1996). Frobisher (1994) suggested that it is "nearly always possible to restate [a problem] in order to make it into an investigation" (p. 158). Using Task 1 as an illustration, Frobisher's idea would be to remove the *intended problem* from the task and replace it with the word 'investigate' (see Task 3), which he believed students would ultimately pose the intended problem and generalise by finding the shortest time to toast n slices of bread.

Task 1: Mathematical Problem (Toast)

Three slices of bread are to be toasted under a grill. The grill can hold exactly two slices. Only one side of each slice is toasted at a time. It takes 30 seconds to toast one side of a slice of bread, 5 seconds to put a slice in or to take a slice out, and 3 seconds to turn a slice over. What is the shortest time needed to toast the three slices of bread?

Task 2: Investigative Task (Square Each Digit and Add)

Choose any number. Square each digit of the number and add to obtain a new number. Repeat this process for the new number and so forth. Investigate.

Task 3: Investigative Task (Toast)

Three slices of bread are to be toasted under a grill. The grill can hold exactly two slices. Only one side of each slice is toasted at a time. It takes 30 seconds to toast one side of a slice of bread, 5 seconds to put a slice in or to take a slice out, and 3 seconds to turn a slice over. Investigate.

The purpose of opening up a problem to an investigative task is to allow students to pose their own problems to solve. But the concern is whether the students know how to pose the intended problems. There also seems to be two different types of investigative tasks that tend to elicit somewhat different kinds of processes: Type A (e.g., Task 2) involves searching for any pattern by trying examples so as to generalise, while Type B (e.g., Task 3) requires students to *first* pose a problem like finding the time taken to toast the three slices and then solve it by using other heuristics such as deductive reasoning. It is beyond the scope of the paper to examine all the thinking processes that students engage in when performing both types of investigative tasks, so the paper will focus only on the problem-posing processes for Type B tasks. Hence, the research question for the paper is:

- What kinds of problems did the high-achieving students in this study pose when given Type B investigative tasks?

Literature Review

Thinking Processes

Clement (2000) observed that one of the most important needs in research on students' thinking processes is the need for insightful explanatory models of these processes, and that such research is most fruitfully undertaken using the thinking-aloud method (Duncker, 1926) and teaching experiments (Lesh & Kelly, 2000). Mason, Burton and Stacey (1985) described four main processes in mathematical thinking: specialising (or trying examples), conjecturing, justifying and generalising. Some researchers (e.g., Height, 1989; Cifarelli & Cai, 2004) have also designed a model to depict the thinking processes in investigation. Based on these models but with some modifications, an investigation model (see Fig. 1) was developed for this study to describe the interaction of these processes. An important difference between a mathematical investigation model and a problem-solving model is the additional phase of problem posing after understanding the task in investigation.

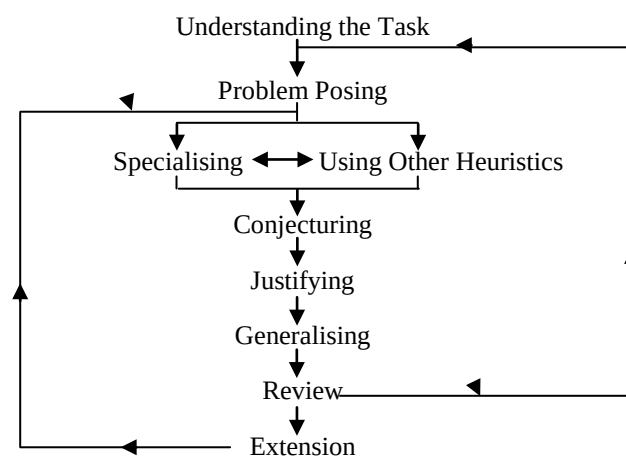


Figure 1. Investigation model of thinking processes

Problem Posing

Silver (1994) observed that “problem posing can occur before, during, or after the solution of a problem” (p. 19). For investigation, problem posing occurs even before the students start to think of a solution because they need to pose their own problems to solve. Brown and Walter (2005) discussed two categories of problem posing. The first category is to accept the given in the task statement. There are at least two types of problems in this category. The first type is to search for patterns, e.g., “Is there a pattern? If yes, what is the pattern?” The second type is to generalise if possible, e.g., “Is there a formula? If yes, what is the formula?” To illustrate the first category of problem posing using Task 2, students can try examples to look for patterns and then generalise without changing the given.

The second category is to extend the task by changing the given using the ‘What-If-Not’ strategy. Kilpatrick (1987) discussed taking something that is a constant in the task statement and let it vary, while Orton and Frobisher (1996) used the “What if” question to extend a problem by changing the given. Using Task 3 as an illustration, students can ask, “*What if* there are four slices to be toasted? *What if* the number of slices the grill can hold is *not* two but three?” This kind of extension can also lead to the finding of a general formula for the time taken to toast the bread. The difference between the process of generalising in the first and second categories of problem posing is that the former does not involve changing the given while the latter requires changing the given to extend the task.

Krutetskii (1976) argued that there were some problems that ‘naturally follow’ from the given task, and he found that high-achieving students were able to pose them directly but low-achieving students were unable to do so. Using Task 3 as an illustration, students may ask, “What is the time taken to toast the three slices?” or even pose the intended problem: “What is the shortest time needed to toast the three slices?” Students are also expected to generalise by asking, “Is there a formula to toast n slices?” In fact, they can even go to the next level of generalisation by asking whether there is a formula to toast n slices if the grill can hold exactly m slices.

Since these problems may be ‘natural’ from the perspective of the person who designs the task but they may not be ‘natural’ for students to pose, there is a need to distinguish the difference. In this paper, this kind of problems will be called ‘logical problems’ that follow from the task because these problems are posed based on a logical sequence. For example, before you can find the least time taken to toast the three slices, you need to find the time taken to toast the three slices using a few methods first. In line with the research question for this paper, an area of interest is whether students find it ‘natural’ to pose such ‘logical problems’ that follow from the given task, including the intended problem.

Methodology

A teaching experiment was conducted with a class of 19 high-achieving students, who had no prior experience in investigation, to develop their processes when they attempted both types of investigative tasks. They were chosen from a bigger pool of high-achieving students from the same school by giving each one of them an investigative task and asking them questions such as whether they had seen this kind of tasks before, and whether they knew how and what to investigate. Only students who answered ‘no’ for all these questions were selected. As it was beyond the scope of the present research to study a large group of students with different academic attainments, high-achieving students were chosen because

an initial exploratory study by the author (Yeo, 2008) has suggested that they were more likely to exhibit a wider range of investigation processes to inform the investigation model (shown in Fig. 1) than average or low-achieving students.

However, the same exploratory study has also revealed that most of the high-achieving students with no prior experience in investigation did not really know what and how to investigate because they did not understand the task requirements. Thus the students in the current study were given a two-hour familiarisation lesson before the pretest to teach them *what* to investigate. After the pretest, the students were given five two-hour developing lessons to develop their processes further, i.e., to teach them *how* to investigate. The posttest was conducted within one week after the last lesson. The pretest and the posttest each contained two tasks, one of which was a Type B task. The students were given 30 minutes to do each task. For the pretest, the Type B task was Task 3 described earlier in this paper, while the Type B task for the posttest was as followed:

Task 4: Investigative Task (Sausages)

I need to cut 12 identical sausages so that I can share them equally among 18 people. Investigate.

Unlike traditional textbook exercises where parallel items mean changing the numbers while retaining the same method of solution, parallel items for investigative tasks could not be designed in this way, or else the posttest would just be an exercise for the students with nothing new to investigate. Thus the posttest tasks must have different patterns but still retain the same structure and elicit similar processes as the pretest tasks. For example, Task 3 (Pretest) and Task 4 (Posttest) are of the same Type B, and both involve minimising (find least time and least number of cuts) and generalising using two variables (toast n slices in a grill that can hold exactly m slices, and share n sausages equally among m people).

In addition, each student was videotaped thinking aloud during the tests to capture his or her thinking processes, so as to identify the types of processes in order to study their interaction among one another, and their development. To prepare the students to think aloud during the tests, they practised thinking aloud during the familiarisation lesson and the last developing lesson. The verbal protocols of the students during the tests were then transcribed and coded to identify the types of processes. Four representative transcripts with a total of 1258 codes were given to two experienced coders to code. The inter-coder agreement was 93%, suggesting that the coding scheme developed by the author was reliable enough. The coded transcripts were then manually analysed to study the nature and development of thinking processes in mathematical investigation.

To answer the research question in this study on the kinds of problems posed by the students, there was a need to look beyond the students' test answer scripts to their thinking-aloud protocols, because sometimes the students posed a problem verbally without writing it down in their answer scripts, and it was not easy to guess from their working what their problem was. Occasionally, the students posed a problem verbally but they then changed their minds, so they did not write the problem down in their answer scripts. Therefore, the students' thinking-aloud protocols complemented their answer scripts in providing a fuller picture of the kinds of problem the students posed, so as to answer the current research question more reliably.

Results and Discussion

The logical problems to pose for Task 3 (Pretest), including the intended problem, in sequential order of reasoning as explained earlier in this paper, are:

- Logical Problem 1 (LP1): Find the time taken to toast the three slices of bread.
- Logical Problem 2 (LP2): Find the shortest time to toast the three slices of bread (Intended Problem).
- Logical Extension 1 (LE1): Find the shortest time to toast n slices of bread.
- Logical Extension 2 (LE2): Find the shortest time to toast n slices of bread if the grill can hold m slices.

The logical problems to pose for Task 4 (Posttest) are:

- LP1: Find the number of cuts to share the 12 sausages among the 18 people.
- LP2: Find the least number of cuts to share the 12 sausages among the 18 people (Intended Problem).
- LE1: Find the least number of cuts to share n sausages among m people.

Unlike Task 3 which has two extensions because it is more logical to first change the number of slices to be toasted than the number of slices the grill can hold, Task 4 has only one extension because it cannot be determined which of the two variables is more logical to change first. In fact, data analysis has shown that most of the students in this study changed both the number of sausages and the number of people at the same time. Another interesting finding for Task 4 was that most students (79%) found it even more natural to pose a problem that was not LP1 or LP2:

- LP0: Find how to cut the 12 sausages to share among 18 people.

This problem is called LP0 since, in sequential order of reasoning, one has to find how to cut the sausages before finding the number of cuts (LP1). Therefore, a similar LP0 was also created for Task 3 as shown in Table 1, which contains a summary of the findings for the present study. Most students posed more than one problem for each task.

Table 1

Summary of Findings (total number of students in this study = 19)

Task 3 (Pretest)	No.	Task 4 (Posttest)	No.
Logical Problem 0 (LP0): Find how to toast the 3 slices	0	LP0: Find how to cut 12 sausages to share among 18 people	15
Logical Problem 1 (LP1): Find time taken to toast the 3 slices	11	LP1: Find number of cuts to share 12 sausages among 18 people	3
Logical Problem 2 (LP2): Find shortest time to toast 3 slices (Intended Problem)	11	LP2: Find least number of cuts to share 12 sausages among 18 people (Intended Problem)	6
Logical Extension 1 (LE1): Find (shortest) time taken to toast n slices	7		
Logical Extension 2 (LE2): Find (shortest) time taken to toast n slices if grill can hold m slices	0	LE1: Find (least) number of cuts to share n sausages among m people	5
Attempted any kind of extensions	12	Attempted any kind of extensions	19

Problem Posing

For Task 3 (Pretest), data analysis shows that 11 of the 19 students (58%) posed LP1, while 6 students posed LP2, almost immediately after understanding the task. Only two students did not know what to pose. Of the students who posed LP1, 5 of them also posed LP2 almost immediately after solving LP1. Thus a total of $6 + 5 = 11$ students (58%) found it natural to pose LP2, which was also the intended problem. For Task 4 (Posttest), data analysis shows that 3 students posed LP0 explicitly, but 12 students just went ahead to find how to cut the sausages without posing LP0 at all. It seems that Task 4 was phrased in such a way that it was very natural for the students to proceed to solve LP0 without even posing it. Thus we might say that a total of $3 + 12 = 15$ students (79%) found LP0 natural enough to pose although the majority did not pose it directly. This was unlike Task 3 where no one posed LP0 as the students found it more natural to pose LP1 and LP2 instead.

However, only 3 students (16%) attempted to find the number of cuts (LP1) for Task 4, as compared to Task 3 where it was natural for the majority (58%) to find the total time taken to toast the three slices (LP1), probably because the time for each process, such as putting in or taking out a slice, was given in the task statement. For Task 4, a total of 6 students (32%) posed LP2 naturally: three of them after solving LP1; one of them after solving LP0 without posing LP1; and the last two students after reading the task only once, without even posing LP0 or LP1. For example, one of the last two students said:

“I need to cut 12 identical sausages so that I can share them equally among 18 people. Investigate. Cut 12, um, maybe, like, the minimum number of cuts.” [S17; Posttest]

From the above analysis, the context and phrasing of a task seem to affect whether the students found it natural to pose the logical or intended problems, e.g., fewer students were able to pose LP1 and LP2 for Task 4 as compared to Task 3, despite the fact that Task 4 was a posttest task after 10 hours of lessons to develop students’ investigation processes. Moreover, it was not necessary for the students to pose the logical problems in order, e.g., they could pose LP2 directly, without even posing LP0 or LP1.

Extension

Since some students did not find the shortest time (LP2) for Task 3, all students who used any toasting method to find a general formula for the toasting time, not necessarily the shortest time, will be included in the analysis. The findings show that it was only natural for 7 students (37%) to find the time taken to toast n slices (LE1). No student attempted LE2, but this might be due to the time constraint of the test: 6 of the 7 students ran out of time when trying LE1 while the last one just managed to find a general formula. Nevertheless, the last student did recognise that his formula is only true for the case where the grill can hold exactly two slices and the timings are as those given in the task statement:

“And this is the formula for this question. But very sadly, it’s only for this one. So, only when you get a question which has [pause] grill it for 30 seconds, take it down will be 5 seconds, and turn it over is 3 seconds, and this must be, and must have, can toast two bread at a time. So then you can get this number and use this formula.” [S04; Pretest]

Another 2 students attempted other kinds of generalisation: one of them tried to find a formula to toast n slices if it takes a seconds to put a slice in or to take a slice out, b seconds to toast a slice, and c seconds to turn a slice over, for the case where $0 < n \leq 2$ (the latter condition was puzzling); while the other one tried to find the time taken to toast a slice of

bread that has x number of sides (again it was puzzling how a slice of bread could have, e.g., 5 sides). Nevertheless, a total of $7 + 2 = 9$ students (47%) extended Task 3 by generalising. A possible reason why so few students generalised was that most of them still did not fully understand they could change the given in the task statement, even after a two-hour familiarisation lesson. For example, a student asked the invigilator during the pretest:

“Teacher, the investigation must be, um be, related to the question? Like if they say 3, I investigate finish, can I use 21 slices of bread?” [S12; Pretest]

However, 3 other students extended the task without generalising, e.g., toasting the three slices together in a bigger grill, not a grill that can hold exactly m slices. Thus, a total of $9 + 3 = 12$ students (63%) extended Task 3 in one way or another.

For Task 4, only 5 students (26%) tried to generalise the least number of cuts (LE1). A possible reason why so few students attempted LE1 was because most of them did not even find the number of cuts for the original task (LP1 or LP2). Nevertheless, another 7 students did try to generalise for other reasons, e.g., to find a general formula for the amount of sausage a person will get, or a general method to cut $6n$ sausages to share equally among $6(n+1)$ people. Therefore, a total of $5 + 7 = 12$ students (63%) attempted some kind of generalisation for Task 4, as compared to only 47% for Task 3. However, all the students (100%) extended Task 4 in various ways, as compared to only 63% who extended Task 3. The following show some of the extensions attempted by the students for Task 4:

“How to share usage of a computer with 9 brothers over a week?” [S02; Posttest]

“If a sausage is a triangle ... if it is a square ...” [S06; Posttest]

“We can use 12 different sizes of sausages.” [S18; Posttest]

From the above analysis, the 10 hours of developing lessons had helped the students to be more proficient in extending Task 4 (Posttest) as compared to Task 3 (Pretest), although their attempt to generalise the number of cuts for Task 4 (LE1) was hampered by their inability to pose the logical problem of finding the number of cuts for the original task (LP1 or LP2). Moreover, the quality of the extension for Task 4 was also higher: the students not only changed the numbers in the original task statement of the posttest, but they also changed other variables such as the size and shape of the sausages, and the context to sharing the usage of a computer.

Conclusion

The opening up of mathematical problems to become investigative tasks has provided students the opportunity to pose their own problems to solve. But unlike what Krutetskii (1976) has discovered, some high-achieving students in this study did not find it natural to pose problems that logically follow from the task. The analysis has also revealed that the context of the task and the phrasing of its task statement might affect whether students are able to pose the logical or intended problems, suggesting that teachers could not just design an investigative task by opening up a mathematical problem, as proposed by Frobisher (1994), without considering the possibility of ‘losing’ the logical or intended problems that they might want their students to solve. Therefore, there is a need for teachers to mull over how to guide their students to pose the intended or logical problems that follow from such investigative tasks, and yet not close up the tasks by restricting the students’ freedom and creativity to pose other types of problems to solve. On the other hand, the students had learnt

from the teaching experiment how to extend the tasks in different ways, although there was still room for improvement in their ability to generalise. This paper has contributed to the current research through documenting the types of problems that students posed during investigation, and suggesting that these problem-posing processes could be developed in a mathematically enriched environment, although more research needs to be done to study how students cultivate these processes over a longer period of time.

References

- Australian Curriculum, Assessment and Reporting Authority (2012). *The Australian curriculum*. Retrieved Feb 28, 2012, from <http://www.australiancurriculum.edu.au/Download>
- Bailey, J. (2007). Mathematical investigations: A primary teacher educator's narrative journey of professional awareness. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia (MERGA): Mathematics: Essential research, essential practice* (Vol. 1, pp. 103-112). Adelaide, Australia: MERGA.
- Brown, S. I., & Walter, M. I. (2005). *The art of problem posing* (3rd ed.). Mahwah, NJ: Erlbaum.
- Cai, J., Mamona-Downs, J., & Weber, K. (2005). Mathematical problem solving: What we know and where we are going. *Mathematical Behavior*, 24, 217-220.
- Cifarelli, V., & Cai, J. (2004, July). *A framework for examining the mathematical exploration of problem solvers*. Research paper presented to the Special Interest Group on Problem Solving of the International Congress on Mathematics Education, Copenhagen, Denmark.
- Civil, M. (2002). Everyday mathematics, mathematicians' mathematics, and school mathematics: Can we bring them together? In E. Yackel (Series Ed.) & M. E. Brenner & J. N. Moschkovich (Monograph Eds.), *Everyday and academic mathematics in the classroom* (pp. 40-62). Reston, VA: National Council of Teachers of Mathematics.
- Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Erlbaum.
- Dunker, K. (1926). A qualitative (experimental and theoretical) study of productive thinking (solving of comprehensive problems). *Pedagogical Seminary*, 33, 642-708.
- Frobisher, L. (1994). Problems, investigations and an investigative approach. In A. Orton & G. Wain (Eds.), *Issues in teaching mathematics* (pp. 150-173). London: Cassell.
- Height, T. P. (1989). *Mathematical investigations in the classroom*. Australia: Longman Cheshire Pty.
- Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry*. London: Falmer Press.
- Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123-147). Hillsdale, NJ: Erlbaum.
- Krutetskii, V. A. (1976). The psychology of mathematical abilities in school children. In J. Kilpatrick & I. Wirszup (Series Eds.), *Soviet studies in the psychology of learning and teaching mathematics*. Chicago: University of Chicago Press.
- Lesh, R. A., & Kelly, A. E. (2000). Multitiered teaching experiments. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 197-230). Mahwah, NJ: Erlbaum.
- Mason, J., Burton, L., & Stacey, K. (1985). *Thinking mathematically* (Rev. ed.). Wokingham, England: Addison-Wesley.
- Ministry of Education of New Zealand. (2007). *The New Zealand curriculum*. Retrieved Feb 28, 2012, from <http://nzcurriculum.tki.org.nz>
- Orton, A., & Frobisher, L. (1996). *Insights into teaching mathematics*. London: Cassell.
- Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics*, 14(1), 19-28.
- Tanner, H. (1989). Introducing investigations. *Mathematics Teaching*, 127, 20-23.
- Yeo, J. B. W. (2008). Secondary school students investigating mathematics. In M. Goos, R. Brown, & K. Malar (Eds.), *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia (MERGA): Navigating currents and charting directions* (Vol. 2, pp. 613-619). Brisbane, Australia: MERGA.

Predictive Validity of Numeracy Entry Requirements for University: Pre-service Teachers' Mathematics Knowledge and Attitudes

Jenny Young-Loveridge
The University of Waikato
<jenny.yl@waikato.ac.nz>

Brenda Bicknell
The University of Waikato
<bicknell@waikato.ac.nz>

Judith Mills
The University of Waikato
<judith@waikato.ac.nz>

This paper presents the findings of a study that assessed the numeracy competency of more than 200 students enrolled in pre-service primary teacher education. The Mathematics Thinking and Reasoning assessment consisted of 10 tasks that included 2-digit computations and proportional reasoning. Students rated their attitudes towards mathematics at primary and secondary school, and currently. Analysis of the data showed that university admission status was not related to students' scores on the assessment tasks. The correlation between meeting the University Entrance Numeracy Requirements and the total correct was very modest. The implications of these findings for ITE providers are presented.

Over the past few years, there has been increasing recognition among mathematics education researchers of the need for primary teachers to have sufficient mathematics subject matter knowledge (SMK), an essential component of pedagogical content knowledge (PCK) in mathematics (Ball, Hill & Bass, 2005; Ma, 1999; Moch, 2004; Rowland & Ruthven, 2011). Ball, Thames, and Phelps (2008) have built on Shulman's (1986) seminal work on teacher knowledge and provided a practice-based theory for "professionally oriented subject matter knowledge in mathematics" (p. 389). Their model further subdivides SMK into three parts: Common Content Knowledge, Knowledge at the Mathematical Horizon, and Specialised Content Knowledge. Similarly PCK is broken down into Knowledge of Content and Students, Knowledge of Content and Teaching, and Knowledge of Curriculum.

Several researchers have investigated the connections between SMK and PCK (e.g., Askew, 2008; Ward & Thomas, 2008). Evidence indicates that there is no clear linear relationship between these two categories. Having tertiary level mathematics is not necessarily an advantage, although having limited understanding of mathematics may be a problem. Ward and Thomas (2008) found that teachers with low levels of SMK also had low levels of PCK. However, those with high levels of SMK had a range of scores on the measure of PCK. In other words, there were some teachers with high levels of SMK that had low levels of PCK. This evidence supports the claims of several writers (e.g., Askew 2008; Moch, 2004) that there is a certain threshold level of SMK that is necessary for good teaching, but being able to meet this requirement is not sufficient on its own.

In most pre-service teacher education programmes it is a requirement on entry that students provide evidence of having achieved a specified level of mathematics knowledge. Recent studies continue to reveal concerning gaps in prospective teachers' mathematical understandings (Livy & Vale, 2011; Zazkis, Leikin, & Jolfaee, 2011). Currently, New Zealand University Entrance (that includes 14 credits in mathematics at NCEA Level 1, normally completed in Year 11) is being taken as providing sufficient evidence of numeracy competency. From 2012, students over 20 years of age (with Special Admission to university) will be required by the Initial Teacher Education (ITE) provider "to meet comparable numeracy requirements as those entering with University Entrance" (NZTC, 2010). It should be noted that in New Zealand primary teachers are qualified to teach up to

Year 8. In many countries, students at this level are taught by specialist mathematics teachers in the secondary school system.

Literature on the attitudes towards mathematics shows that it is important to consider how teachers and students feel about mathematics as well as looking at their mathematics achievement (McGinnis, Kramer, Shama, Graeber, Parker, & Watanabe, 2002; Southwell, White, Way, & Perry, 2005). It is thought that teachers' attitudes influence their classroom practices, and this has an impact on students' attitudes and learning. In a factor analysis of attitude questions, Southwell et al. (2005) identified two independent factors of insecurity and confidence contributing to attitudes towards mathematics and teaching mathematics. Recognition of the link between attitudes and beliefs about SMK and reform-based mathematics pedagogy in pre-service teacher education programs have led to initiatives to improve the attitudes and beliefs of prospective teachers and these have shown encouraging results (e.g., McGinnis et al., 2002).

This paper presents the findings of a study that assessed the numeracy competency of 248 students enrolled in a three-year pre-service primary teacher education program.

Method

The participants in the study included 248 undergraduate students enrolled in the first year of a three-year Bachelor of Teaching degree (see Table 1). The majority of students had been awarded University Entrance (61%), one fifth of the cohort had been given Special Admission (23%), just over one eighth had come from Other Tertiary institutions (14%), and the remainder had Discretionary Entry (3%). Numeracy Credits attained as part of other qualifications were noted on the files of those without University Entrance. This information showed that almost three-quarters of the cohort had the equivalent of 14 credits in Numeracy at Level 1 NCEA (73%), with just over one quarter having no evidence of Numeracy Credits (27%).

Table 1

Number and Percentage of Participants According to University Entry Status

Category of Entry to University	No. of students	%
University Entrance	150	60.5
Special Admission *	56	22.6
Other Tertiary Institutions *	34	13.7
Discretionary Entry *	8	3.2
<i>Total</i>	<i>248</i>	<i>100.0</i>

**NB: Of the students without UE, 30 were evaluated as having equivalent Numeracy Credits*

The Mathematics Thinking and Reasoning assessment consisted of 10 tasks that included 2-digit computations and proportional reasoning. Two of the tasks were at Level 3 of *The New Zealand Curriculum* (Ministry of Education, 2007) and seven tasks were at Level 4, corresponding to expected achievement levels for primary school students at the end of Year 8 (see Table 2). Students were asked to solve the tasks and show their thinking using words, numbers, and/or pictures/diagrams. Answers were marked as correct or incorrect, and information about the thinking used to solve the problem was also coded. Overall, total scores ranged from 1 to 10. Aggregated scores were calculated for the two Level 3 tasks and the seven Level 4 tasks. The Statistical Package for Social Sciences (SPSS) was used to analyse the quantitative data. The frequencies of correct responses and the most common incorrect responses were noted.

After completing the mathematics tasks, students were given a four-point Likert type rating scale ranging from “Really Like/d” to “Really Dislike/d” mathematics, “at primary school”, “at secondary school”, and “now”. They were also invited to make comments in a box provided to give further insights to their experiences and attitudes towards mathematics.

Results

Students’ performance was analysed as a function of University Entrance status and also whether or not they had the 14 Numeracy Credits at Level 1 NCEA (or the equivalent) required as part of University Entrance. The percentages of students who successfully completed each task according to their Entry and Numeracy status is presented in Table 2. Overall, students were more successful on tasks involving whole numbers and addition of decimal fractions. The most challenging tasks were those related to fractional quantities and proportional reasoning.

Table 2

Percentages of Students According to Entry and Numeracy Status Who Could Do Each Task

#	Question	Overall n=248	UE n=150	no UE n=98	NC n=180	No NC n=68
1	Tama has 64 stickers. He uses 27 on the first day of school. How many does he have left?	95	95	96	94	99
2	John needs \$403 to buy a stereo. He has saved \$297. How much money does he still need?	86	86	85	84	88
3	Sue used 8.3 metres of red material and 2.57 metres of blue material to make costumes for the play. How much material did she use altogether?	88	91	84	91	82
4	Ana bought 4.3 metres of rope to make skipping ropes, but only used 2.89 metres. How much rope was left over?	62	66	55	65	54
5	If 18 packets each hold 24 felt pens, how many pens is that altogether?	64	65	62	62	68
6	If 56 plums are shared among 14 people, how many plums will each person get?	92	92	91	92	90
7	Tama and Karen buy two pizzas. Tama eats $\frac{3}{4}$ of one pizza while Karen eats $\frac{7}{8}$ of the other one. How much pizza do they eat altogether?	32	37	22	33	27
8	If Ben got 72 out of a possible total of 90 marks, what percentage was that?	28	31	24	29	27
9	Jo spent \$60 on stationery. She got one-third off the original price, because she was a teacher. What was the original price?	62	65	56	63	57

Students with University Entrance did slightly better than those without University Entrance on most tasks. On average, students with University Entrance attained an average total score half a mark higher than those without ($M = 6.75$ and $M = 6.24$ for those with and without UE, respectively). The biggest differences were evident on the task involving subtraction of decimals with regrouping (66% vs. 55%), addition of common fractions (37% vs. 22%), and converting fractions to percentages (31% vs. 24%).

Those with Numeracy Credits outperformed those without Numeracy Credits for six tasks, while the reverse pattern was found for three tasks. The average score for those with Numeracy Credits was slightly higher than for those without (6.62 vs. 6.37). The difference in mean total score was only one quarter of a mark, on average. The largest differences were

evident on the task involving two-digit multiplication (65% vs. 54%) and subtraction of decimals with regrouping (91% vs. 82%).

Statistical analysis was completed to calculate the correlation between the total number of correct answers and University Entry status (using dummy variables, with 1 assigned to students with UE or with Numeracy Credits, and 0 assigned to those without). The correlation coefficient (Spearman's rho) for the relationship between UE status and number of correct answers was 0.129, while the corresponding value for the relationship between Numeracy Credits and number of correct answers was 0.066, extremely modest values at best. University Entry Status and Numeracy Credits were strongly correlated ($r = 0.760$).

Misconceptions

An analysis of common incorrect responses was completed for the cohort on tasks where at least two percent of students had given the same incorrect response (see Table 3). All of the common mistakes involved the Level 4 tasks, including adding and subtracting fractions and decimals, converting a fraction to a percentage, working out the whole from knowing a part, and multi-digit multiplication. For example some students when adding 8.3 and 2.57, treated the decimal part as whole numbers ($3 + 57 = 60$), later putting the decimal point back in the number (10.6 or 10.60). Subtraction of decimals was likewise challenging for some students who subtracted the smaller decimal part away from the larger part, confusing the subtrahend with the minuend ($4.3 - 2.89 = 1.59$). A common mistake (6% of those with UE vs. 4% of those without UE) when multiplying 18×24 was to multiply the tens ($10 \times 20 = 200$) and multiply the ones ($8 \times 4 = 32$), and then add these two partial products ($200 + 32 = 232$), disregarding the cross-products formed by multiplying each of the tens with each of the ones ($10 \times 4 = 40$, $20 \times 8 = 160$) and adding these two partial products to the others to give an answer (total product) of 432.

Two-thirds of the students had problems adding $\frac{3}{4}$ and $\frac{7}{8}$. The most common mistake was to convert $\frac{3}{4}$ to $\frac{6}{8}$ then add both the numerators and denominators to get an answer of $\frac{13}{16}$. Almost as many of those with UE made that mistake as those without UE (15% vs. 19%). Some other students did not find an equivalent fraction for $\frac{3}{4}$, instead immediately adding both numerators and denominators to get an answer of $\frac{10}{12}$ (19% of those with UE; 26% of those without UE). These mistakes clearly show that students were simply executing a mis-learned procedure and were not paying attention to the meaning of the problem. Had the students used 'number sense', they would have realised that $\frac{3}{4}$ and $\frac{7}{8}$ are both close to one whole, so the answer had to be greater than one. Just over one-quarter of the students (28% overall; 31% of those with UE and 24% of those without UE) could convert 72 out of 90 to a percentage by noticing that 72 and 90 are both multiples of 9, so the fraction could easily be simplified to $\frac{8}{10}$ and from there converted to 80%. Alternatively they might have noticed that every nine marks was worth 10% and calculated how many groups of 9 are in 72, or used benchmarks such as one half ($45 = 50\%$), one quarter ($22.5 = 25\%$), and one fifth of one quarter ($4.5 = 5\%$), then added the parts together to get 72 marks and 80% in total.

Another task that students found challenging was Question 9, asking them to find the original price given that \$60 was two thirds of that price. A common mistake was to find one third of \$60 and add it on, to give an answer of \$80 (7% of those with UE; 13% of those without UE), instead of halving the \$60 to find one third of the original price, then multiplying the amount by 3 to get a total of \$90. A similar pattern was found for the comparison of those with and without Numeracy Credits.

Table 3
Percentages of Students Who Made Common Errors on Tasks

#	Question	Overall n=248	UE n=150	no UE n=98	NC n=180	No NC n=68
3	Sue used 8.3 metres of red material and 2.57 metres of blue material to make costumes for the play. How much material did she use altogether? (Correct Ans: 10.87)	88				
	Ans: 10.6	2	1	4	1	4
	Ans: 10.60	4	3	6	3	7
	Ans: 11	2	3	1	2	2
4	Ana bought 4.3 metres of rope to make skipping ropes, but only used 2.89 metres. How much rope was left over? (Correct Ans: 1.41m)	62				
	Ans: 1.14	3	2	5	3	4
	Ans: 1.59	10	8	12	10	9
	Ans: 2.59	4	6	2	6	0
5	If 18 packets each hold 24 felt pens, how many pens is that altogether? (Correct Ans: 432)	64				
	Ans: 108	4	4	4	4	3
	Ans: 216	2	2	2	2	2
	Ans: 232	5	6	4	7	2
7	Tama and Karen buy two pizzas. Tama eats $\frac{3}{4}$ of one pizza while Karen eats $\frac{7}{8}$ of the other one. How much pizza do they eat altogether? (Correct: $\frac{13}{8}$ or 1 and $\frac{5}{8}$)	32				
	Ans: 10/12	21	19	26	21	22
	Ans: 13/16	17	15	19	16	18
8	If Ben got 72 out of a possible total of 90 marks, what percentage was that? (Correct Ans: 80%)	28				
	Ans: 0.648	3	3	3	3	3
	Ans: 0.81	3	3	3	4	0
	Ans: 0.82	12	11	14	12	13
	Ans: 72/90	10	12	7	10	10
9	Jo spent \$60 on stationery. She got one-third off the original price, because she was a teacher. What was the original price? (Correct Ans: \$90)	62				
	Ans: 180	3	3	4	3	3
	Ans: 60	2	1	3	1	4
	Ans: 80	10	7	13	8	13

Threshold Level of Mathematics Knowledge (SMK)

Students were ranked according to their successful performance on Level 4 tasks. Only one student was unable to get any of the Level 4 tasks correct. That student had met the University Entrance requirements, including the 14 Numeracy Credits. Nine students were able to get only *one* response correct for NZC Level 4 tasks. Four of these students had University Entrance, three had been given Special Admission to university, and two had come from another tertiary institution. For three of the nine students, the one Level 4 task on which they experienced success was adding 8.3 and 2.57. One student was successful in

subtracting 2.89 from 4.3. The remaining five students were able to work out that 56 divided by 14 was 4. The most common way of working this out was by drawing tallies to create physical representations, then counting groups of 14 to find the number of groups. None of these nine students could add $\frac{3}{4}$ and $\frac{7}{8}$, calculate 72/90 as a percentage, or work out the original price when 2/3 of it was \$60.

Twenty-five students were able to get *two* responses correct on NZC Level 4 tasks. Nine of these students had University Entrance, eight had been given Special Admission to university, and eight had come from another tertiary institution. Sixteen were able to add the decimal quantities. Not one of the 25 students could subtract 2.89 from 4.3, and only three students could multiply 18 x 24. Two of the 25 students could add $\frac{3}{4}$ and $\frac{7}{8}$. One student was successful in working out 72/90 as a percentage, and four worked out the original price when 2/3 of it was \$60.

Forty-two students were able to get *three* responses correct on NZC Level 4 tasks. Just over half of these (n=24) had University Entrance, 13 had been given Special Admission, four came from another tertiary institution, one had been given Discretionary Entrance. Twelve students subtracted 2.89 from 4.3 correctly and 22 were able to multiply 18 x 24. Only five of the 47 students could add $\frac{3}{4}$ and $\frac{7}{8}$, and only one student could work out 72/90 as a percentage. Altogether almost one third (31%) of the students got fewer than half of the seven NZC Level 4 tasks correct. One quarter (25%) got *four* responses correct on NZC Level 4 tasks. It was interesting to note that only ten of the 62 students successfully solved the addition of fractions task.

Affective Responses

The percentages of students who disliked (or really disliked) mathematics at primary school, at secondary school, and currently are shown in Table 4. Students were most positive about mathematics at primary school (83% liked or really liked it). Less than two-thirds (64%) of the cohort was positive at the beginning of their pre-service primary teacher education course. Those students with UE were slightly more negative towards mathematics currently than those without (38% vs. 34% disliked or really disliked maths currently). On the other hand, fewer of those with UE were as negative about the mathematics they did at secondary school than those without UE (41% vs. 50% disliked or really disliked maths at secondary school).

Students with and without Numeracy Credits were very similar in their feelings about mathematics at primary school (17% vs. 18%), and currently (37% vs. 35%). However those without Numeracy Credits were substantially more negative about mathematics at secondary school than were those who had achieved the credits (53% vs. 41%).

Table 4

Percentages of Students Who Disliked (or really disliked) Mathematics

	Overall n=248	UE n=150	no UE n=98	NC n=180	No NC n=68
At primary school	17	17	17	17	18
At secondary school	44	41	50	41	53
Currently	36	38	34	37	35

Students' comments revealed information about the reasons for their ratings. It was evident that students believed that their teachers had influenced their attitude towards and success in mathematics.

I had a primary teacher when I was ten that really knocked my confidence at maths. Since then I've found it extremely difficult and I often get confused – I use my fingers to count. (T030)

At primary [school] my ability in maths was very dependent upon the teacher. With some I 'got it' and others not. (S050).

My maths throughout my life started off great, but when it came to high school, teachers in the lower classes were too busy to teach or not that great a teacher. How can you or why would you want to try when you get that? I think with the right teacher and attitude I can really learn to be the best I can be. I would love help thanks. (T031)

Some students commented on the importance of links between mathematics and real life.

I have never been a lover of maths. When I was introduced to maths at primary I could do the basics but not well. As I progressed through school I felt embarrassed I was not on the same level as my peers. When I started working, paying bills etc, I found maths was part of everyday life. Now I have a desire to learn and I want to improve my attitude towards maths which will hopefully improve my capability and ultimately my ability to teach maths as a subject. (T004).

I found the way I use maths in day-to-day life is different to the way I was taught maths at secondary but probably more like the maths I was taught at primary school. Over the years I have developed my own way of working things out in my head. I find maths a challenge and enjoy the way it gets my mind thinking to find the solutions. (S025)

Discussion

Overall, the results were of considerable interest and concern. Although these undergraduates had three years in which to strengthen their mathematics, the current regulations only require them to do 72 hours of compulsory mathematics education papers. This part of the course is completed half-way through their programme, leaving eighteen months in which they may do no further mathematics before they become Provisionally Registered Teachers. Given the areas of weakness demonstrated here, it is questionable whether it will be possible to bring their performance up to an acceptable level within the 72-hour programme. Currently there is no assessment before students exit the programme to ensure they meet numeracy competency requirements. However, students have an opportunity to strengthen their understanding by enrolling for an optional mathematics education paper in the third year of their programme.

Analysis of the data showed that university admission status was not related to students' score on the assessment tasks. The correlation between meeting the University Entrance numeracy requirements and the total correct was very modest. Many of the lowest scorers (getting a total of 1 or 2 tasks correct) had been awarded University Entrance. By the same token, some of the students who got all ten tasks correct were those with Special Admission who had no evidence of meeting the University Entrance numeracy requirements. When the average score out of 10 was calculated for each category of admission status, those with the highest average score were those with Discretionary Entrance. Those with University Entrance were only marginally higher than those with Special Admission.

Most students used traditional algorithms to calculate their answers. This reflects a strong orientation towards procedures and rules, consistent with an instrumental approach to mathematics (Skemp, 2006). This is despite several decades of mathematics education reform that has called for greater emphasis on conceptual understanding and 'making sense' of the mathematics - Skemp calls this relational understanding. This is consistent with other findings showing that it is extremely difficult to change the ways that mathematics is taught and learned (Anthony & Hunter, 2005; Lamon, 2007). It should be noted that the Numeracy Credits for University Entrance are part of a standards-based assessment system that allows multiple opportunities for re-assessment. If students complete these requirements in Year 11,

they may discontinue studying mathematics for their remaining two years of secondary education.

The findings of this study have important implications for ITE providers. If a certain threshold of discipline knowledge in mathematics is necessary for good teaching, then it is vital that institutions assess prospective teachers to ascertain the extent of that knowledge and identify particular areas that may need to be further strengthened. The findings suggest that the use of numeracy assessment tasks to reveal important misconceptions could be helpful in determining the extent to which students are likely to meet a threshold level of proficiency in mathematics.

References

- Anthony, G., & Hunter, R. (2005). A window into mathematics classrooms: Traditional to reform. *New Zealand Journal of Educational Studies*, 40, 25–43.
- Askew, M. (2008). Mathematical discipline knowledge requirements for prospective primary teachers, and the structure and teaching approaches of programs designed to develop that knowledge. In P. Sullivan & T. Woods (Eds.), *The international handbook of mathematics teacher education: Volume 1: Knowledge and beliefs in mathematics teaching and teaching development* (pp. 13-35). Rotterdam, The Netherlands: Sense Publishers.
- Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, Fall, 14-22, 43-46. http://www.aft.org/pubs-reports/american_educator/issues/fall2005/BallF05.pdf
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407.
- Lamon, S. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Charlotte, NC: Information Age Publishing.
- Livy, S., & Vale, C. (2011). First year pre-service teachers' mathematical content knowledge: Methods of solution for a ratio question. *Mathematics Teacher Education and Development*, 13(2), 22-43.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.
- McGinnis, J. R., Kramer, S., Shama, G., Graeber, A. O., Parker, C. A., & Watanabe, T. (2002). Undergraduates' attitudes and beliefs about subject matter and pedagogy measured periodically in a reform-based mathematics and science teacher preparation program. *Journal of Research in Science Teaching*, 39(8), 713-737.
- Ministry of Education (2007). *The New Zealand curriculum*. Wellington: Author.
- Moch, P. L. (2004). Demonstrating knowledge of mathematics: Necessary but *not* sufficient. *Curriculum and Teaching Dialogue*, 6 (2), 125-130.
- New Zealand Teachers Council (NZTC) (2010). Approval, review and monitoring processes and requirements for initial teacher education programmes. Wellington, NZ: Author. Retrieved on 16 November 2011 from: <http://www.teacherscouncil.govt.nz/te/iteproviders.stm>
- Rowland, T. & Ruthven, K. (2011). *Mathematical knowledge in teaching*. Dordrecht, The Netherlands: Springer.
- Schulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15, 4-14.
- Skemp, R. (1986). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School*, 12, 88-95.
- Southwell, B., White, A., Way, J., & Perry, B. (2005). *Attitudes versus achievement in pre-service mathematics teacher education*. Retrieved on 23 January 2012 from: <http://www.aare.edu.au/05pap/sou05705.pdf>
- Ward, J. & Thomas, G. (2008). Does teacher knowledge make a difference? *Findings from the New Zealand Numeracy Development Projects 2007* (pp. 50-60). Wellington: Ministry of Education.
- Zazkis, R., Leikin, R., & Jolfaee, S. C. (2011). Contributions of "mathematics for elementary teachers" courses to teaching: Prospective teachers' views and examples. *Mathematics Teacher Education and Development*, 13(2), 3-21.

SYMPOSIA

Working Through the Practice Architectures of First Year University Mathematics Teaching

Jo Balatti

James Cook University
<Josephine.Balatti@jcu.edu.au>

Shaun Belward

James Cook University
<Shaun.Belward@jcu.edu.au>

This paper describes how a group of university lecturers are adopting an action research approach to improve the learning experience of students in first year mathematics. Using the three categories of saying/thinking, doing, and relating (Kemmis, 2009) to explore practice, it describes the new practices of the action research team, the established practices of mathematics teaching at university, and the team's trials at changing elements of that teaching practice.

There are compelling reasons for improving the mathematics learning experience of university students. Since the 1990s, in Australia and elsewhere, there has been a decline in students undertaking university mathematics (Nardi, 2008). In response, mathematicians are developing more interest in the way they teach. The first year experience is particularly important because it constitutes the transition from school to university mathematics. In this paper, we reflect on the first 18 months of our grass roots attempt as members of an action research team to improve first year mathematics at our university.

The action research approach we use involves undertaking repeated cycles of planning, acting, observing, and reflecting (McTaggart, 1997). In acting and interacting together for a common purpose over an extended period of time, we have formed a community of practice (Wenger, 1998) which we call the Mathematics Action Research Team (MathsART). However, following the action research model does not necessarily guarantee success. Neither MathsART nor its members is context free; both are embedded in larger communities of practice. In our case, MathsART is located in the Discipline of Mathematics within the School of Engineering and Physical Sciences at James Cook University. These larger entities have what Kemmis (2009, p.466) calls "practice architectures" that are the "mediating preconditions" which "shape and give content" to how we talk/think, act, and relate. The challenge for MathsART is to influence the established "practice architectures" to produce better teaching and learning.

In this paper we describe three sites of practice: the emerging practices of MathsART; the established practice architectures of the Discipline; and the changes that MathsART has trialled in the two first year mathematics subjects. For our analysis, we have revisited the written summaries of our meetings and our recorded conversations. Through a process of consensus we arrived at key elements in the "saying, doing and relating" (Kemmis, 2009, p. 466) of each set of practices.

The Practices of MathsART

In July 2010, the Head of Mathematics (Shaun) organised a workshop for the six mathematics teaching staff and the secondary mathematics education lecturer (Jo) to discuss the teaching and learning of mathematics particularly in first year. In the previous semester, the pattern of high failure rate and low tutorial attendance had continued. Furthermore, in an informal survey of the first semester cohort, only 17% had indicated that they had spent, on average, at least the recommended minimum of six-seven hours of home study each week. Jo's concern centred on the higher than average failure rate amongst the small number of education students. These students constitute fewer than 5% of the total cohort of

approximately 200 students who are mostly engineering and science undergraduates. Shortly after the workshop, the action research team was formed.

The four core members of MathsART are three mathematics lecturers (Patrick, Shaun and Wayne) and one mathematics education lecturer (Jo). The three mathematics lecturers have extensive experience teaching mathematics (over 10, 20 and 30 years respectively) including in the first and second years. Jo is the facilitator in the action research team. Her expertise is in adult learning and in using action research to generate change in professional practice. She teaches secondary mathematics education in the School of Education.

The common purpose of the team members is to increase the quality of the student learning in first year so that more students succeed in the two subjects and, as importantly, carry a deeper understanding of mathematics into their second year. The project began with attempting to increase student engagement.

Action research aims to change “practitioners’ practices, their understandings of their practices and the conditions in which they practise” (Kemmis, 2009, p. 463). Accordingly, we are developing norms concerning the talk we engage in, the work we do, and the way we relate to one another. Table 1 below summarises the main practices we are developing.

Table 1

Ways of Talking, Doing and Relating in MathsART

Developing Practice	Adopting different ways of thinking/acting/interacting
Saying/thinking	Fostering the belief that pedagogy can make a difference Deprivatising practice Decision-making based on evidence
Doing	Implementing the action research cycle Weekly professional conversations (audio-recorded and minuted) Reading literature on pedagogy and sharing reflections Posting reflections about teaching on the community website Professional development (PD) with school teacher on ICTs PD with school teacher on school curriculum and assessment Doing research on our practice for wider professional community
Relating	Peer observations of teaching Incidental conversations about teaching with peers

The Practice Architectures of the Discipline

In Table 2 we list what we believe are some of the key aspects of the entrenched practice architectures defining how mathematics has been traditionally taught at this university. The table is purely descriptive and provides no explanation for why the practice architectures have formed in the way they have.

Actions Taken to Improve Mathematics Teaching and Learning

Despite being immersed within the practice architectures described in Table 2, MathsART has implemented a range of different practices in their teaching. Some of the more significant actions trialled are listed in Table 3.

Table 2

Practice Architectures of the Mathematics Community of Practice as at July 2010

Practice architecture	Examples
Saying/thinking	
About the learner	Student performance has declined since the 1990s. Student results are primarily a function of students' ability, background knowledge and application. The school system is the main source of the problem (Belward, Mullamphy, Read & Sneddon, 2007).
About the teaching	Content and pedagogy in first year mathematics are not the cause of the problem but need to change because of the problem. Content is more important than pedagogy Staff see themselves as mathematicians first, teachers second. Pedagogy is not generally understood and is not a priority.
Doing	
Lecturing	Transmission approach
Tutorials	Traditional – students arrive with problems and tutor answers them
Assessment	Online quizzes; on-course and final exams; no calculators
Resources	Lecture notes, tutorial sheets and solutions - the same year to year
Tutors	Generally postgraduate students with no training in teaching
Relating	
Student/student	No peer collaborative work done
Student/lecturer	Little subject specific feedback sought
Lecturer/lecturer	Lecturer works autonomously; conversations on teaching are rare

Table 3

Actions Taken by MathsART over the 18-Month Period of Operation

Elements of practice	Actions taken or implemented
Saying/thinking	As for Table 1
Doing	
Lecturing approach	Interactive approach trialled
Tutorials	Structure encourages interaction; attendance monitored ¹
Assessment	Now offers students a "second chance"
Resources	Tutorial exercises revised; online quizzes evaluated ² ;
Staffing	Lecturers tutor; School teacher tutors preservice teachers
Students	Feedback sought regularly; Hons research projects on first year
Relating	
Student/student	Collaborative learning arrangements outside of class ³
Student/lecturer	Tutorial representative meetings with staff
Lecturer/lecturer	Working collaboratively on subject design

¹Note: Higgins's paper reports on tutorial attendance rates. ²Note: Read's paper reports on the evaluation of the online quizzes. ³Note: Belward's paper analyses lecturers' response to feedback on this initiative.

In Conclusion

The analytical descriptions provided have deliberately limited themselves to identifying ways that contrast the practices initiated by the action research team with the existing practices of the Discipline of Mathematics. The descriptions do not include the large number of elements that remain the same in the action research community of practice and in the Discipline. For example, the structural element of three or four contact hours weekly is one element of practice that is the same. Another is the time allocation to the lecture mode of delivery and to the tutorial. The latter may change at a later date but for now, it is another constant. In the 18 months that MathsART has been working together, the mathematics lecturers in the team taught one first year subject twice and the other once. It would be naïve to suggest that MathsART has changed the practice architectures in the Discipline, practices that have accreted over many decades. Nevertheless, there are promising signs that change might occur. Other academics within the School have expressed an interest in the action research project and some even expressed a desire to join. Also promising is that new “meta-practices”—practices that Kemmis (2008, p.21) alerts us about—at the university level appear to be supporting our attempts at transformation. For example, the peer review process that we undertook voluntarily has become, in 2012, university policy. Most promising of all, the four members of the team wish to continue with the project. We give three main reasons for doing so. Firstly, we are confident that the action research process is a systematic and robust way to guide change in practice; secondly, we are confident from the evidence we have that the changes that are in train are leading to better pedagogy and better student results; and finally, we know that the job is not yet done.

Acknowledgement

We thank Patrick Higgins and Wayne Read for their analysis and the critical review of this paper.

References

- Belward, S. R., Mullamphy, D. F. T., Read, W. W. & Sneddon, G. E. (2007). Preparation of students for tertiary studies requiring mathematics. *ANZIAM J* 47, (EMAC 2005) C840-857.
- Kemmis, S. (2009). Action research as a practice-based practice. *Educational Action Research*, 17(3), 463-474.
- Kemmis, S. (2008). Praxis and practice Architectures in mathematics education. In M. Goos, R. Brown, & K. Makar (Eds.), *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 17-28). Brisbane: MERGA
- McTaggart, R. (1997). Guiding principles for participatory action research. In R. McTaggart (Ed.), *Participatory action research: International contexts and consequences* (pp. 25-44). Albany: State of University of New York Press.
- Nardi, E. (2008). *Amongst mathematicians: Teaching and learning mathematics at university level*. New York: Springer.
- Wenger E. (1998). *Communities of practice: Learning, meaning, and identity*. New York: Cambridge University Press.

Monitoring and Analysing Attendance in First Year University Mathematics Tutorials

Patrick Higgins

James Cook University
<patrick.higgins@jcu.edu.au>

Wayne Read

James Cook University
<wayne.read@jcu.edu.au>

As part of an innovative tutorial structure introduced to a first year university mathematics subject, an attendance monitoring system was implemented. The system collected data that was used to investigate the relationship between student attendance and assessment performance which is reported here. The implementation of this system also assisted in the increase of student participation and engagement.

Over the last twenty to forty years mathematics tutorials have had a characteristic structure across the Australian university sector: A large group of students working individually on tutorial problems with one-on-one instruction from a tutor when requested. These tutorials exhibit problems of low attendance and poor student engagement.

At James Cook University (JCU), the tutorials of the mainstream first year mathematics subject MA1000, suffered from these problems with little to no student collaboration and tutor interaction that was usually transmissive. Furthermore, no systematic collection of tutorial attendance data took place in order to assist in the analysis of these issues. Therefore any suggested solution was temporary. Even though it was not monitored, anecdotal data suggested high tutorial attendance during the first third of semester followed by a decline with approximately 30% of students attending tutorials for the remaining two thirds of semester. This gave an overall average of approximately 50% of students attending MA1000 tutorials. It was believed that poor attendance and a lack of participation led to poor results in assessment, a trend that had continued over several years. With the increasing pressure on staff to maintain high pass rates, the student's lack of knowledge was carried through to higher years and other subjects.

More recently, a range of teaching strategies have been introduced into tutorials to address problems with attendance and participation (Webb, 1983). The standard tutorial no longer exists. The members of the Mathematics Action Research Team (MathsART) at JCU recently restructured the MA1000 tutorials. One feature of the new structure is the introduction of an attendance monitoring system. This system was first introduced into a first year Engineering subject at JCU by Schneider (2010). We have adapted this system to have two components: (1) an electronic barcode attendance monitor and (2) a mathematical task used to encourage participation and engagement.

In this paper, we discuss the attendance monitoring system that forms part of the new tutorial structure that we implemented. We report on attendance data collected by this system and investigate the correlation between student attendance and assessment results. This relationship has had much attention in the past with studies focussing on data collection, reasons for non-attendance and the impact on student performance. See Newman-Ford et al. (2008) for a review. We also discuss how the introduction of this system has increased participation and engagement of students when attending tutorials. This marked the first time that data were collected and analysed on this scale by the mathematics discipline at JCU.

Methods

The MA1000 student cohort comprises first year students from the disciplines of Engineering (70 - 75%), Science (20 – 25%), and Education (~5%). Contact hours for each student consists of three 50-minute lectures per week for 13 weeks and one 50-minute tutorial per week for 12 weeks. The data presented in this paper were collected over 10 weeks of tutorials with no data collected in weeks disrupted by public holidays. Tutorial classes consisted of ~25 students. The new tutorial structure was introduced into the first semester offering of MA1000, 2011. Of the 201 students that remained enrolled at the end of semester, 150 (~75%) gave consent to participate in our study. Of these, one student did not complete any assessment or attend any tutorials and has therefore been removed leaving 149 students as the cohort for this study.

The new tutorial structure incorporated into MA1000 had the following general format:

1. An informal, but mathematical warm-up exercise (5 minutes).
2. Tutorial task used to encourage participation and monitor attendance (10 minutes).
3. Task marking and discussions (5 minutes).
4. Problems arising from small group meetings from outside tutorials (15 minutes).
5. Individual one-on-one questions (15 minutes).

Monitoring Attendance

At the start of semester, students were issued with a page of barcode stickers. The barcodes contained the student's identification number and could be read by a USB barcode scanner. During each tutorial, students were given a mathematical task related to the material covered in lectures (step 2 above). This task sheet had space for the barcode sticker and room for the student to answer the task. All task sheets were collected and barcodes were scanned into the student database recording attendance. With an accurate account of student attendance, we analysed the relationship between attendance (or non-attendance) and assessment results.

Increasing Participation

The tutorial tasks were designed to increase participation and engagement in tutorials. Although we cannot collect systematic data on engagement, we can encourage participation through the marking and discussion of the task as part of step 3 in the tutorial structure above. This facilitated an interaction between students, and between the students and tutor. Also, instructing students that blank task sheets will not be collected, and therefore attendance will not be marked encourages at least some participation. With a formal tutorial structure and relevant tasks designed to support lectures and encourage interactions, the students were compelled to participate.

Results

The introduction of the new tutorial structure which incorporates the attendance monitoring system has no-doubt had an impact on the learning experience of the MA1000 students. During the semester, approximately 1200 completed tutorial tasks were collected from all tutorial classes. As an initial measure of participation, we observed that no blank tutorial tasks were submitted. This is in contrast to traditional tutorials that would contain an element of the student cohort that did not participate in any way.

We have tracked student attendance throughout the semester and calculated the total number of tutorials attended by each student. These results are shown in Figures 1 and 2.

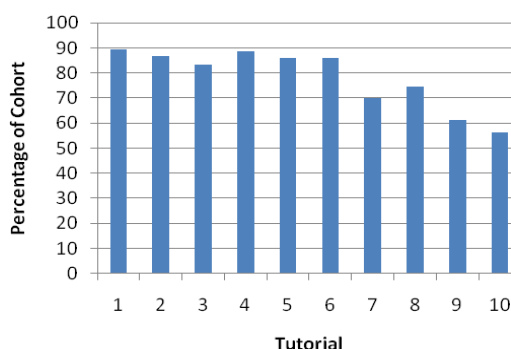


Figure 1. Percentage of cohort that attended each tutorial.

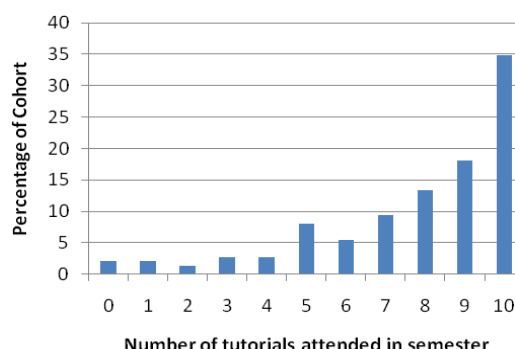


Figure 2. Percentage of cohort versus total number of tutorials attended

Figure 1 displays the percentage of the cohort that attended each tutorial during the semester. The overall average percentage attendance for all tutorials is 78% of the student cohort. This is in contrast to previous years with an anecdotal average of approximately 50%. The overall average for the last two thirds of the semester (tutorials 4 – 10) is 74% compared to 30% reported from previous years. We also note that there is a significant drop in attendance as the semester progresses. Although the decline is less dramatic, this trend is not in contrast to previous years.

Students were grouped with respect to the total number of tutorials attended during the semester. Figure 2 displays the percentage of the class in each of these groups. We note that the largest group contained those students that attended all tutorials with 35% of the cohort. We also note that 76% of the class attended seven or more tutorials.

We also compared the attendance data to assessment results. We investigated the link between final grades and overall assessment percentages with tutorial attendance. These results are shown in Figures 3 and 4. Final grade definitions are: HD – high distinction (85 – 100%), D – distinction (75 – 85%), C – credit (65 – 75%), P – pass and supplementary pass (45 – 65%), N – fail and supplementary fail (0 – 45%), and X – did not complete on-course test or final exam. Figure 3 displays the average number of tutorials that were attended by students that were allocated each final grade for the subject. This figure suggests a relationship between attendance at tutorials and performance.

We have also calculated the average overall assessment mark (as a percentage) for the groups defined for the results in Figure 2. These values are connected by the solid line in Figure 4. The error bars display the 95% confidence interval for the average mark of each group. Again, we observed that the general trend suggests that the students who attended more tutorials, performed better on assessment. We noted that students attending 0 or 1 tutorial performed better than expected. This was explained by the fact that these groups are small (three students in each) and contained two high-achieving students in each group. This is indicated by the large spread in the 95% confidence interval. These students were considered as outliers and were removed from any further calculations. The calculated correlation between tutorial attendance and overall assessment mark was $R^2 = 0.30$, which implies that 30% of the variability of the overall marks obtained can be explained by the

number of tutorials attended. The data has a significance value of $p < 0.0005$ which is highly significant and extremely unlikely to be the consequence of chance.

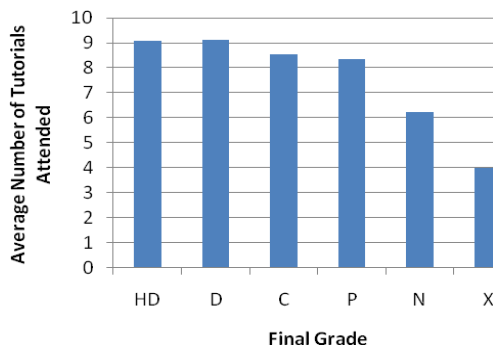


Figure 3. Average number of tutorials attended versus final grades.

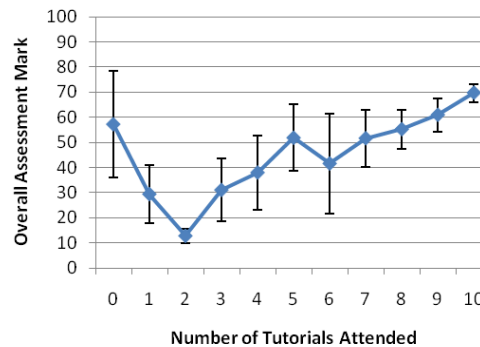


Figure 4. Average overall mark versus number of tutorials attended. Error bars indicate the 95% confidence intervals.

Conclusions and Discussions

With the implementation of an innovative tutorial structure into the mainstream first year university mathematics subject at JCU, an attendance monitoring system yielded a rich data set. This marks the first systematic collection of detailed student data for the mathematics discipline. Analysing the data shows that there is a strong relationship between tutorial attendance and assessment performance. These results imply a strong positive impact of the new tutorial structure on student learning. This system had the additional benefit of increasing student participation and engagement. As the MathsART team are dedicated to the action research methodology, this is only a first step to our ultimate goal of optimising the learning experience of mathematics students at JCU.

Acknowledgement

The authors thank Shaun Belward and Jo Balatti for their critical review of this paper and Yvette Everingham for her assistance with the statistical analysis.

References

- Newman-Ford, L., Fitzgibbon, K., Lloyd, S., & Thomas, S. (2008). A large-scale investigation into the relationship between attendance and attainment: a study using an innovative, electronic attendance monitoring system. *Studies in Higher Education*, 33(6), 699-717.
- Schneider, P. (2010). *Engaging Students by Efficiently Monitoring Attendance and Participation*. Paper presented at the 13th Pacific Rim First Year in Higher Education (FYHE) Conference 2010. Retrieved from http://www.fyhe.com.au/past_papers/papers10/index.html
- Webb, G. (1983). The Tutorial Method, Learning Strategies and Student Participation in Tutorials: Some Problems and Suggested Solutions. *Innovations in Education & Training International*, 20(2), 117-121.

Are Online Quizzes an Effective Tool for Mastering Basic Algebra?

Wayne Read

James Cook University
<wayne.read@jcu.edu.au>

Patrick Higgins

James Cook University
<patrick.higgins@jcu.edu.au >

On-line quizzes are used to help first year University Mathematics students identify weaknesses in their basic skills and improve them. Quizzes developed as a formative tool have been utilised at JCU for eight years. However, before this research no-one has questioned the effectiveness of quizzes for this task. We present a description of the quizzes used in a core first year Mathematics subject at JCU and provide a statistical overview of their usage and efficacy for the intended task.

Quizzes are a ubiquitous part of the assessment regime in tertiary education centres throughout the world. The advent of the web and Information Learning Technology (ILT) has transformed the traditional paper based quiz into a means of electronically assessing students and providing formative feedback. Several studies have shown positive correlations between student motivation, engagement and attitude and the online quiz environment, both in numerate (Angus and Watson, 2009) and literate (Johnson and Kiviniemi, 2009) disciplines. However, in the Mathematical disciplines there appears to have been little research that studies the direct relationship between the quiz questions and the student's ability to reproduce the specific techniques that the quiz questions are based on at a later date. There also appears to be little available data on the way that students use this type of quiz.

The online quizzes at James Cook University (JCU) were set up to provide a formative learning environment where first year Mathematics students could practise basic skills in their own time and pace, and allow students to develop mastery of basic building blocks in a specific subject. Online quizzes were provided in two traditional core Algebra and Calculus courses, as well as a Discrete Mathematics subject for IT students. These subjects have run for eight years, and student surveys reinforce the positive feedback reported in the literature. In particular, students believe that on-line quizzes increase their basic skill set and are a motivating factor in their Mathematical studies. However, the MathsART have raised several questions about the quiz environment. For example, 'Do students help each other?', 'Do students guess multiple choice questions?', 'Are we encouraging rote learning rather than understanding?', 'Are students mastering the basic skills required?'.

The MathsART tried to answer some of these questions, and our initial investigations revealed how little we actually knew about the learning environment that students used with the online quizzes, so we decided to take a step back and collect basic data on the quiz environment at JCU. In this paper, we provide a statistical summary of the student usage of the on-line quizzes in one of the core Mathematics subjects. We present a basic statistical analysis of student's performance on specific questions, presented both in the online quiz environment and in an invigilated environment. Students were asked to sign consent forms at the start of the subject, and 150 out of a cohort of 201 consented to the use of their data. One student did not complete any assessment leaving 149 students in the study. All the analyses presented in this paper are based on the results obtained for the consenting students that completed assessment.

The rest of this paper is organized as follows. In the next Section, we describe the structure of the subject and the nature of the quiz environment. Next, we provide a basic

statistical analysis of the results for the quizzes and finally, discussion of these results and recommendations for future study.

Quizzes and MA1000 at JCU

The online quizzes that are the focus of this paper form part of the learning and assessment environment for the first year, core Mathematics subject MA1000: Mathematical Foundations. MA1000 is a one semester subject on Algebra, Analytic Geometry, Trigonometry and Introductory Calculus, and is worth 12.5% of the points required for the first year of a Science or Engineering degree. The assessment consists of nine on-line quizzes, three on-course tests and an end of semester exam. A student can pass the subject and avoid the end of semester exam, if they pass the on-course tests. Consequently, most students take the on-course assessment very seriously. Although the quizzes form part of the summative assessment for the subject, they are only worth 10%, which is given for encouragement and does not have any significant effect on the marks for the subject.

Students are expected to access the online quizzes via LearnJCU, the Blackboard interface that JCU uses for online learning and reporting. Quizzes can be attempted from the JCU campus or remotely, and are not invigilated. The ten questions that comprise each of the nine quizzes are drawn randomly from a database of quiz questions. Students are encouraged to attempt a quiz as many times as they wish, and the highest score (out of ten) is recorded.

A quiz is generated automatically when a student accesses the system, by randomly drawing ten questions from the database. The questions are of two basic types: multiple choice and numeric answer. For the multiple choice questions, a student is given a choice of four answers. For example, a multiple choice quiz question is

Simplify: $\frac{2^{n+2}4^{n-1}}{8^{n-1}}$

Which of the following expresses the answer with positive indices?

- (a) 2^3 (b) 2^2 (c) 2^{n+2} (d) $n - 4$

The numeric questions were generated using “template” questions; a new quiz question is obtained by simply changing the numeric constants or the variable names in the template question. For example, a numeric quiz question is

Determine the value of x that satisfies

$$\frac{2^{x+1}}{4^{4-3x}} = 1$$

Each of these questions had sixteen variations, so each ten question quiz was randomly drawn from 160 possible choices. The questions on the quizzes are drawn from two categories, “simple drill” (SD) and “complex drill” (CD). Simple drill questions correspond to material that the student should be able to answer correctly 95% of the time, once they have completed that component of the subject, and these types of questions will be the main focus of this paper.

Figure 1 is a plot of the average number of attempts that students make on each of the quizzes, assuming the student attempted the quiz. We note that fifty percent of the class attempted the quiz at least twice, with the exception of quiz six, which was attempted at least once. The maximum number of attempts was thirty, for quiz eight. Fifty percent of the class attempted quiz four at least three times.

Students are expected to receive close to full marks for the quiz component of the course, and are expected to use the quizzes to master the material covered. Figure 2 is a plot of the average marks obtained for the class for each quiz, with 95% confidence intervals

marked. We note that the average mark is between 85% and 95%, except for quiz four, which was 80%. Quiz four was on the Binomial Theorem and the Binomial Series, which students found difficult.

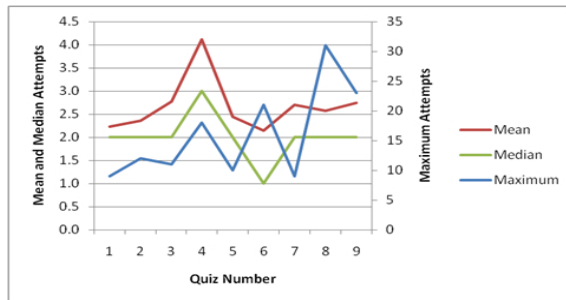


Figure 1. Number of attempts for each quiz.

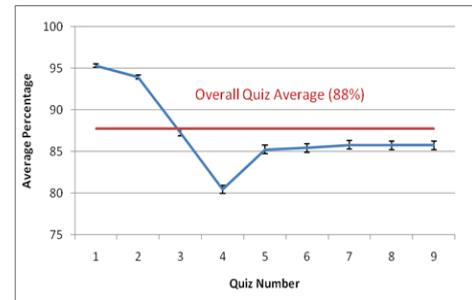


Figure 2. Average % obtained for each quiz.

The Quiz Data and Analysis

The data collected for the analyses presented in this paper is summarised as follows. Nine SD questions were selected and then examined in the oncourse tests. Four of these questions were also included in the end of semester examination. Each quiz question was worth one mark in the online quizzes. However, in the tests and examination the number of marks varied between two and three, so the averages presented are given in two ways: first, with the test and examination marks normalised so that they are out of one and secondly, with their original weighting. As noted in the previous Section, most students get close to full marks for each quiz, so we have calculated the average score for each question, where the average is taken over all of the student's attempts.

Table 1 presents the overall average marks (i.e., including SD questions) obtained in quizzes and the invigilated assessment for the subject. The subject is divided into three sections, each concluding with an on-course test. The quizzes for each section are completed before the test. The examination is at the end of the semester, for those students who do not elect to accept their grade on the oncourse assessment. Of the 149 students in our study, 37 did not sit the final exam.

Table 1

Overall Average Percentages for the Questions in the Quizzes and the Tests/Examination

Quiz Average	Normalised Test/Exam Average	Actual Test/Exam Average
71.6%	71.4%	67.6%

We note here that the quiz average and the normalised Test/Exam average are almost identical, with the actual Test/Exam average slightly smaller. Note that these results are significantly less than the target value of 95% and the average results for each quiz.

Next, we examine the average marks obtained for each of the nine quiz questions used in this study. Figure 3 is a plot of the normalised Test/Exam average and the quiz average for each question. We haven't included the actual Test/Exam averages, as they follow the same trend noted for the overall results: that is, approximately 5% less than the normalised results. Confidence intervals were not included, as the 95% intervals were almost too small to be seen on the plot.

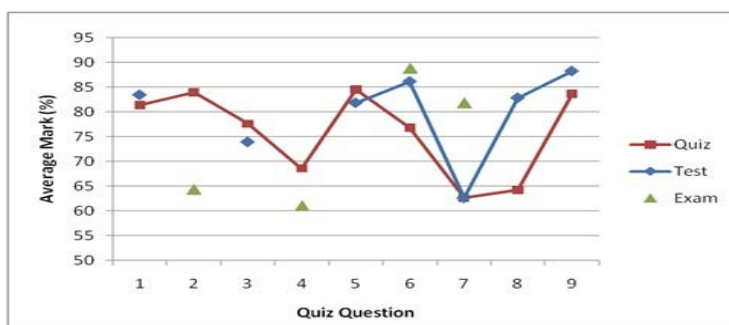


Figure 3. Average marks for the quiz questions on the quizzes, tests and exams

Close examination of Figure 3 reveals a different picture to the aggregated results presented in Table 1. First, the average scores for the quizzes are significantly higher than the overall averages for the tests and exams for most of the quiz questions. The lower averages for the overall results are probably due to the inclusion of the results for CD questions. The exceptions are the exam questions that were taken from quizzes completed earlier in the semester, and quizzes four, seven and eight. The average score for the quiz questions are a good predictor of the results for the tests for almost all the quiz questions. The only real exception is question eight, which significantly underestimates the test result.

Discussion

The analyses presented in this paper suggest that the overall quiz statistics (such as the quiz average) can be misleading. It would be easy to conclude from Figure 2 that the goal of 95% correct for the SD questions was within reach, whereas the individual quiz question data reveals that it is not. Also, the performance on the examination for quiz questions that were taught early in the semester (questions two and four in Figure 3) suggest that students are forgetting material that they have dealt with successfully earlier in the subject. The results for individual questions also suggest that the distinction between SD and CD needs to be revisited, as the average marks for questions four, seven and eight imply that these are not SD questions. These results are only preliminary, and more research needs to be carried out on the relationship between the average score per question and performance on test and exam results.

Acknowledgement

We thank Jo Balatti and Shaun Belward for their help and critical review of this paper.

References

- Angus, S.D., & Watson, J.(2009). Does regular online testing enhance student learning in the numerical sciences? Robust evidence from a large data set. *British Journal of Educational Technology*, 40, 255-272.
- Johnson, B.C., & Kiviniemi, M.T. (2009). The effect of online chapter quizzes on exam performance in an undergraduate social psychology course. *Teaching of Psychology*, 36, 33-37.

Doing it Differently: The Ups and Downs of Peer Group Learning

Shaun Belward

James Cook University
<shaun.bewlard@jcu.edu.au>

Jo Balatti

James Cook University
<josephine.balatti@jcu.edu.au>

Peer group learning is the name we have given to a particular type of collaborative learning that has been implemented as part of an action research project designed to improve teaching and learning of first year university mathematics at James Cook University. Using an innovation-decision process model we analysed the response of academics to the implementation of the peer group learning initiative. The analysis reveals that the action research methodology allows generally positive attitudes regarding the initiative to surface.

Collaborative learning takes a range of forms. It can be broadly described as “a *situation* in which *two or more* people *learn* or attempt to learn something *together*,” (Dillenbourg, 1999, p. 1). In any given circumstance the intended purpose of the collaborative learning will influence the choices made with respect to the number of participants (ranging from a pair to hundreds or thousands), what is to be learned (e.g., problem solving or communication skills), and the form of the interaction (e.g., weekly, monthly, in the same room, on-line, with peers or with a teacher). Potential benefits include the fostering of learning communities, the provision of cooperative rather than competitive environments and opportunities for active learning (Laal & Ghodsi, 2012).

The Mathematics Action Research Team (MathsART) at James Cook University (JCU) introduced collaborative learning to the first year mathematics subject MA1000 for the first time in Semester One, 2011. We describe the style of collaborative learning as peer group learning – (see next section for details). The MathsART anticipated the following benefits:

- Students would work together and need to communicate mathematics – requiring them to process ideas differently from when they study the material on their own.
- The peer groups would legitimise the practice of discussing mathematics with fellow students, thereby diminishing the common perception that it is only the lecturer or tutor who can be the teacher of mathematics. This, in turn, would facilitate a move to independent learning.
- The peer groups would help build a sense of community amongst the students, providing a social network for moral support. This is particularly useful during the transition from secondary to tertiary study, and for those students who commence tertiary study without a support network to rely upon.

The introduction of peer group learning represented a significant shift in strategy in teaching first year mathematics at JCU and caused considerable discussion and debate.

The MathsART project facilitates change in practice. One aspect of practice that changes through an action research project is that of the conditions in which the practice occurs. Kemmis (2009, p. 463) describes “transforming the conditions of practice” as “transforming the ways we *relate* to others and to things and circumstances around us.” This paper focuses on the academics involved in the implementation of peer group learning and discusses their response to this initiative after having implemented it. In particular, the MathsART response to the concept of peer group learning is explored at the time of the first formal feedback from the students regarding the teaching and learning in MA1000 in Semester One, 2011. The feedback was in the form of a tutorial representative meeting five weeks after the

commencement of the semester. This moment may be regarded as a critical incident, a time at which the transformations Kemmis (2009) describes should be apparent. It was always the intention of the MathsART to use the first formal feedback from the students as an opportunity to review the format of the peer group learning initiative and modify it as necessary. This provided motivation for the team to reflect intently on the feedback from the students and on their experiences in teaching the subject.

Changing Practice: Analysis of the Response of the MathsART to the Peer Group Learning Initiative

Description of the Peer Group Learning Initiative

The peer group learning initiative was implemented in MA1000 during the second week of semester, after students had been introduced to the concept at the lectures in the previous week. In this subject, tutorial classes comprise at most 25 students. During its first meeting, each tutorial class was informed that one of the purposes of the tutorial was: “to strengthen and develop the techniques and concepts presented in lectures through **independent** and **collaborative** learning experiences in a safe and supportive learning environment” (tutorial notes). They were also informed that the collaborative learning was to involve students working in small groups.¹ The expectations were presented as follows:

1. Each group will meet at least once per week outside of the lectures and tutorials.
2. The group should discuss the exercises that have been covered in lectures.
3. All members should participate: asking questions and helping each other.
4. One member from each group will email the tutor regarding the issues the group could not work through in the meeting.
5. Group emails will be sent to the tutor before 5pm Friday afternoon.
6. These issues will be addressed at the tutorial in the following week.

The initiative was low risk, for both academic staff and students, because groups were self-selecting and no assessment was directly linked to group tasks.

The Data Analysed and the Analytical Framework

All tutors of MA1000 in 2011 were MathsART members. They wrote weekly reflections during semester and contributed to MathsART meetings which were usually held weekly. Ethics approval was granted for collection of data from students. Consent was obtained from students to record meetings, which they participated in for the purpose of providing feedback to MathsART. Given the focus of this study, the data analysed consist of the recording of the MathsART meeting of 5th April 2011, which followed four days after the first tutorial representative meeting. A content analysis of this MathsART meeting was conducted to interpret the response of MathsART to the peer group learning initiative.

To analyse the response of the MathsART to the introduction of collaborative learning we use a model for the *innovation-decision process* presented by Rogers (2003, pp. 168-218). He describes a process through which a decision-making unit² (such as MathsART) makes a decision to adopt or reject an *innovation*. In this paper the innovation is the

¹ Recommended group size was 4 or 5, however this was not enforced.

² A decision-making unit may comprise one individual or a group of people.

introduction of peer group learning. At the coarsest level, the various stages of the model are: *Knowledge, Persuasion, Decision, Implementation, and Confirmation*.

It is the actions that a group takes at the various stages of the model that provide insight into the beliefs and attitudes they hold concerning an innovation. Since the peer group learning initiative was already implemented, the study presented here focused on the last two stages of the model, *Implementation* and *Confirmation*. Table 1 gives brief descriptions of these stages and some of the possible actions a decision-making unit might take in each that are consistent with a changing perception of the innovation. For example if the MathsART made a decision to discontinue peer group learning in MA1000 it would indicate the team's attitude toward the approach was negative in some way.

Table1

Last Two Stages in Rogers (2003) Innovation-decision Process and Possible Actions

Stage and description	Possible actions altering the form of the innovation
<i>Implementation</i> (p. 179) – occurs when a decision-making unit puts an innovation to use.	<i>Reinvention</i> (p. 180) – occurs when an innovation is changed or modified by the user in the process of its adoption and implementation.
<i>Confirmation</i> (p.189) – occurs when the decision-making unit seeks reinforcement for the innovation-decision already made.	<i>Dissonance</i> (p. 189) – may lead to (a) the search for an alternate innovation, (b) a new decision regarding, or implementation of, an alternate innovation or (c) reversal of the original innovation decision.

Note. Page numbers indicate the relevant pages in Rogers (2003).

Analysis of the Challenges and Resolutions

Eight of the 11 tutorial representatives and the three of the tutors in MA1000 attended the 70 minute tutorial representative meeting. The peer group learning aspect of the subject was discussed for approximately 20% of the time. Four days later 25% of the two hour MathsART meeting was dedicated to the peer group learning initiative. What follows is summary of challenges and associated resolutions concerning the peer group learning initiative that were discussed at the MathsART meeting.

Negative sentiments. While there was agreement that the student representative meeting presented a generally favourable view of the peer group learning experiences in MA1000 there was some concern regarding the nature of the sample of data provided. One participant of MathsART noted:

Although after the tutorial rep meeting we said, “it seems as though the groups are working well,” there was minimal feedback and I think those people that did come to the rep meeting ... were the positive ones. I think there will be more negative attitudes to group work than positive.

This comment reflects the opinion that self-selected tutorial representatives may have a bias toward positive attitudes regarding new initiatives. In response, the MathsART developed an online survey instrument to obtain feedback from a greater percentage of the class, thereby reducing the possibility of bias.

Disruptive groups. The MathsART observed some peer groups that were disruptive, either disturbing some of their own members or other classmates in the tutorial. The resolution implemented in 2011 was to assist students in finding new peer groups.

Collapsed groups. The tutorial representative meeting provided data indicating some peer groups were no longer meeting. A number of suggestions were made to overcome this challenge. These included engineering the groups according to mathematical ability, so that there was some certainty that each group had at least one member who had relatively strong mathematics skills. This suggestion was not implemented in 2011, although it may be implemented in the future. Instead the resolution to the previous challenge was adopted.

Tutor not meeting obligations to group. One tutorial representative indicated some peer groups were no longer meeting because the tutor failed to respond to all the issues that arose in the email he received listing areas of concern. While time constraints precluded attending to all concerns raised via email, this omission reduced the value of the tutorial for these peer groups. The resolutions proposed include looking at the provision of solutions that provide differing levels of scaffolding and are designed to be used by groups. This would reduce the email traffic to the tutor. This ambitious project is being implemented for 2012. For the 2011 cohort, fully worked solutions were provided immediately.

Discussion and Conclusion

The analysis presented in the previous section indicates the MathsART responded to most of the feedback from the tutorial representative meeting by resolving to alter aspects of the peer group learning initiative. With respect to Rogers (2003) model, it is possible to interpret the peer group learning initiative as belonging to either the Implementation or Confirmation stages at the time of the data collection. The resolutions may therefore be classed as Reinvention, or Dissonance. Rogers (2003, p. 189) explains that when dissonance regarding an existing innovation occurs and is reduced by a decision to implement a new initiative, the behaviour of the decision-making unit is consistent with a change of attitude. In this study the attitudes are with respect to the peer group learning initiative. Given the positive nature of the resolutions proposed, we have observed the MathsART experience positive feelings towards the peer group learning initiatives introduced. Undoubtedly, this is due to a combination of the belief that teaching and learning can be done differently and the understanding that the action research methodology offers an approach to ensuring innovations can be iteratively improved to achieve desired outcomes, or abandoned if there evidence to suggest they will not work.

Acknowledgement

We thank Patrick Higgins and Wayne Read for their critical review of this paper.

References

- Dillenbourg, P. (1999). What do you mean by collaborative learning? In P. Dillenbourg (Ed.), *Collaborative-learning: Cognitive and Computational Approaches*. (pp.1-19). Oxford: Elsevier.
- Kemmis, S. (2009). Action research as a practice-based practice. *Educational Action Research*, 17(3), 463-474.
- Laal, M., & Ghodsi, S.M. (2012). Benefits of collaborative learning. *Procedia - Social and Behavioral Sciences*, 31, 486-490.
- Rogers, E. M. (2003). *Diffusion of innovations*. New York: Free Press.

ROUND TABLE DISCUSSIONS

Students' Conceptions of Equality

Megan Anakin
University of Otago
<megan.anakin@otago.ac.nz>

Chris Linsell
University of Otago
<chris.linsell@otago.ac.nz>

Jeffrey Smith
University of Otago
<jeffrey.smith@otago.ac.nz>

Don't we know enough about students' conceptions of equality already? In this round table discussion, we will present preliminary findings from three lines of inquiry that suggests that students' conceptions of equality are more complicated than previous theoretical frameworks indicate. This research stems from recent studies in New Zealand that suggest primary and intermediate students have difficulty solving missing number problems that involve the concept of equality. Rather than viewing students' erroneous responses as problematic, we are probing into how students express their conceptions of equality. We are using diverse methods to analyse students' written, verbal, and non-verbal responses to arithmetic missing number problems collected from different assessment contexts. Two lines of inquiry involve extant data analysis from samples of over 400 Year 4 (8 and 9 year old) and Year 8 (12 and 13 year old) students that participated in the National Educational Monitoring Project in 2009. In these inquiries, one involves quantitative analyses of students' written responses and the other involves quantitative and qualitative analyses of students' oral responses that were video recorded in one-on-one interviews with an adult assessor. The third line of inquiry is a prospective classroom-based study that involves microanalysis of video recorded interactions between pairs of Year 6 (10 and 11 year old) students as they work together to solve a series of missing number problems. Because we are proposing a more complicated theoretical framework of students' conceptions of equality, we seek feedback, as well as critique, about the strengths and limitations of our diverse methods of inquiry.

Teaching Mathematics Respectfully: Preparing Culturally Responsive Mathematics Teachers

Robin Averill
Victoria University of Wellington
Te Whare Wānanga o te Ūpoko o te Ika a Māui
<robin.averill@vuw.ac.nz>

Megan Clark
Victoria University of Wellington
Te Whare Wānanga o te Ūpoko o te Ika a Māui
<megan.clark@vuw.ac.nz>

Culturally responsive teaching is widely seen to promote equity of access to achievement. The literature indicates that teacher respect is essential for culturally responsive teaching. Teachers showing respect to students has been a recurring theme in recent New Zealand studies; one student stating that the most important thing to teach prospective mathematics teachers was "to respect the kids". In this round table we present findings from a study into

the nature of respect in teachers' interactions in senior secondary school mathematics classrooms. Our study drew from mathematics students' and teachers' perspectives of teacher practice gathered using teacher and student questionnaires and semi-structured interviews, and videos of twelve mathematics lessons. Participants included six Year 12 and 13 mathematics classes and their teachers across three multi-ethnic city schools. Findings included that mathematics teachers show respect to students through their pedagogical practices, professionalism, disposition, and knowing their students well. Using students' names and providing timely and perceptive individual assistance with mathematical tasks are examples of respectful practices. Some differences between teachers' and students' views emerged with students placing greater emphasis than their teachers on professionalism and the constructive treatment of student errors. The perceived challenges to teachers demonstrating respect included factors relating to student behavior, curriculum content, and the diversity of students' mathematical abilities and ethnicities. This round table forum will begin with a short presentation of findings to initiate discussion regarding how our mathematics teacher education practice can promote respectful teacher behaviours and how teacher respect can contribute to culturally responsive mathematics teaching.

Developing Self-Regulated Teacher Learners to Improve Student Outcomes in Mathematics

Janette Bobis
University of Sydney
<janette.bobis@sydney.edu.au >

Jenni Way
University of Sydney
<jennifer.way@sydney.edu.au>

Judy Anderson
University of Sydney
<judy.anderson@sydney.edu.au>

Student disengagement in mathematics in the middle years and the related issues of underachievement and lower participation rates in higher levels of mathematics have been linked to teachers' understanding of mathematical content and their pedagogy (Ryan & Williams, 2007). Research has identified the on-going learning of teachers as key not only to improving their own knowledge, but also valued student outcomes (Timperley, Wilson, Barrar & Fung, 2007).

The round table will begin by outlining the theoretical underpinnings of a professional learning program, *Empowering Teachers of Mathematics (ETM)* and its two complementary foci: its impact on teachers (knowledge, beliefs, practices and ability to self-direct their learning); and the measurement of student outcomes (engagement and achievement). The session will provide an opportunity for international colleagues to discuss issues affecting the (a) sustainable growth of teacher mathematical content and pedagogical content knowledge; (b) the engagement of middle-years students in mathematics; and (c) how self-directed learning in teachers can be supported in a program of professional learning.

References

- Ryan, J., & Williams, (2007). *Children's Mathematics 4-15*. Maidenhead, Berkshire: Oxford University Press.
- Timperley, H., Wilson, A., Barrar, H., & Fung, I. (2007). *Teacher Professional Learning and Development: Best Evidence Synthesis Iteration [BES]*. Wellington, New Zealand: Ministry of Education. Available at www.educationcounts.govt.nz/goto/BES

Professional Conversations among Mathematics Educators

Rosemary Callingham
University of Tasmania
<Rosemary.Callingham@utas.edu.au>

Julie Clark
Flinders University
<julie.clark@flinders.edu.au>

Pep Serow
University of New England
<pserow2@une.edu.au>

Kim Beswick
University of Tasmania
<Kim.Beswick@utas.edu.au>

Merrilyn Goos
University of Queensland
<m.goos@uq.edu.au>

Steve Thornton
Charles Darwin University
<Stephen.Thornton@cdu.edu.au>

Helen Chick
University of Tasmania
<Helen.Chick@utas.edu.au>

Barry Kissane
Murdoch University
<kissane@murdoch.edu.au>

The development of survey questions to address aspects of Pedagogical Content Knowledge (PCK) in mathematics for an Australian Teaching and Learning Council project led to intense dialogue and challenging discussion among the project team. Using the questions developed as a prompt, similar opportunities were provided to mathematics educators within the project team's participating institutions. These sessions had very positive feedback. MERGA members are invited to participate in a similar experience in this round table, and to reflect upon this approach for useful professional learning.

Teachers' Mathematical Knowledge and Practice

Mohan Chinnappan
University of South Australia
<mohan.chinnappan@unisa.edu.au>

The day-to-day work of teachers of mathematics is governed by a range of factors not least their knowledge of mathematics that is relevant to their professional work. An emerging field of research is theorising about understandings of mathematics that teachers need to develop in order to support practices that will optimise mathematical learning (Ball, 2000; Ball *et al.*, 2001; Ma, 1999). The outcome of this stream of research has significant implications for the interpretation of teachers' classroom practices, teachers' professional standards, teacher education programs, quality professional development programmes, and, ultimately for arguments about professional strength of the community teachers of mathematics. This Round Table Discussion session will provide a forum for researchers to share their views and findings on the following issues. Colleagues are also invited to suggest new lines of inquiry that has the potential to add to the current debates about teachers' mathematical knowledge and the teaching of mathematics.

- Teacher's understanding of subject matter knowledge from a pedagogical perspective;
- Mathematical Knowledge for Teaching (Ball *et al.*, 2008);
- Relations between courses in mathematics and practice;
- Teachers' subject matter knowledge and representational fluency

— Teachers' subject matter knowledge and students learning outcomes

References

- Ball, D. (2000). Bridging practices. Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, 51(3), pp. 241 - 247.
- Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (pp. 433 - 456). Washington, DC: AERA.
- Butterfield, B., & Chinnappan, M. (2011). Teacher Knowledge Activated in the Context of Designing Problems. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics Traditions and [New] Practices AAMT/MERGA Conference 2011* (pp. 142-150). Alice Springs N.T.: AAMT & MERGA.
- Hill, H. C., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372-400.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4 - 14.

Models and Modelling for the Future

Lyn D. English

Queensland University of Technology

<l.english@qut.edu.au>

Richard Lesh

Indiana University

<ralesh@indiana.edu>

Kit Ee Dawn Ng

National Institute of Education

Nanyang Technological University

<dawn.ng@nie.edu.sg>

Gloria Stillman

Australian Catholic University

<Gloria.Stillman@acu.edu.au>

The terms, models and modelling, have been used variously in the literature, including in reference to solving word problems, conducting mathematical simulations, creating representations of problem situations (e.g., constructing explanations of natural phenomena), and creating internal, psychological representations while solving a particular problem.

This proposed roundtable session will focus on the models and modelling perspective first initiated by Richard Lesh. From this perspective, models may be viewed as conceptual systems or tools comprising operations, rules and relationships that can describe, explain, construct, or modify an experience or a complex series of experiences. Modelling involves the crossing of disciplinary boundaries, with an emphasis on the structure of ideas, connected forms of knowledge, and the adaptation of complex ideas to new contexts (English, in press; Hamilton, Lesh, Lester, & Brilleslyper, 2008).

The roundtable will begin with a review of the models and modelling perspective and will then be open to discussion on issues including (but not confined to):

1. Models and modelling in different nations, including sharing and collaboration;
2. The role of task context (including interdisciplinary themes) and task design;
3. The nature of the mathematical ideas and processes embedded in modelling problems;

4. Transfer of learning (e.g., from model-eliciting activities to model exploration and model adaptation activities);
5. Models and modelling with young children;
6. Across-grade sharing of modelling products (e.g., grades 2 and 7 students sharing their solutions to a given modelling problem);
7. Towards the future: advancing models and modelling.

References

- English, L. D. (In press). Learning through modeling in the primary years. In D. K. E. Ng & N. H. Lee (Eds.). *Mathematical Modelling—From theory to practice*. World Scientific Publishing Company.
- Hamilton, E., Lesh, R., Lester, F., & Brilleslyper, M. (2008). Model-eliciting activities (MEAs) as a bridge between engineering education research and mathematics education research. *Advances in Engineering Education*, 1(2), 1-25.

SHORT COMMUNICATIONS

Early Childhood Mathematics: The Case of *More* or *Less*

Mohan Chinnappan

University of South Australia

<mohan.chinnappan@unisa.edu.au>

Amy Chan

University of Wollongong

<amychan@uow.edu.au>

Young children possess informal knowledge of mathematics that is broad, complex and sophisticated. They engage in significant mathematical thinking and reasoning in many contexts. This recognition has catalysed recent reforms on the need to bring a higher level of focus on early childhood learning, in general, and early childhood mathematics, in particular (Malaysian Government, 2012; Australian Government, 2011). Early quantitative reasoning begins to develop as early as the first 2 years of life. These reasoning demonstrate robust sensitivity to numerical information in the environments including counting, numerosity and systems for representing and discriminating small and large sets.

Previous studies (Geary, 1994; Clements & Sarama, 2007) have shown that the development of numeracy skills begin before children commence their formal schooling experiences. Two concepts that are foundational to their ability to perform arithmetic operations are children's ability to discriminate between the concepts of *more* and *less*. In this exploratory study we consider pre-schoolers' (3- to 4-Year-Olds) understanding of these twin concepts. Results show that children tend to have more difficulties with the concept of *more* than *less*. Implications of the results are discussed for further investigations of studies of numeracy in early childhood mathematical thinking and numeracy.

References

- Australian Government (2011). *National Quality Framework for Early Childhood Education & Care*. Retrieved from <http://www.deewr.gov.au/EarlyChildhood/Pages/default.aspx>
- Clements, D. H., & Sarama, J. (2007). Early childhood mathematic learning. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 461-556). Reston, VA: National Council of Teachers of Mathematics.
- Geary, D.C. (1994). *Children's mathematical development: Research and practical applications*. Washington, DC: American Psychological Association.
- Malaysian Government (2012). *Pre-school education*. Retrieved from <http://www.malaysia.gov.my/EN/Relevant%20Topics/Education%20and%20Learning/Citizen/PreSchoolMain/Pages/PreSchoolMain.aspx>

Critical Reflection as an Important Pathway to Pre-service Teachers' Development (A Snapshot)

Kwee Gek Chua

Nanyang Technological University

<kweegeek.chua@nie.edu.sg>

Singapore has placed great emphasis on teacher preparation and pre-service teachers' development. Such preparation and development can be greatly enhanced through the mediation of both pre-service training at the National Institute of Education and extended classroom learning experiences in schools. Insights gleaned from the pre-service teachers

cognitive and affective contents of the weekly reflection logs and perspectives of their experiential learning during practicum and a retrospective analysis of the pre-service teachers' reflective thoughts on their cooperating teachers' pedagogical practices, setting and charting their own goals and development will be discussed.

The Use of Open-Ended Tasks as an Instructional and Assessment Tool

Kum Fong Foo
Academy of Singapore Teachers
<foo_kum_fong@moe.gov.sg>

Leng Low
Tampines Sec Sch
<low_leng@moe.edu.sg>

Yen Ping Pang
Gan Eng Seng Secondary School
<pang_yen_ping@moe.edu.sg>

Wai Leng Lye
National Junior College
<lye_wai_leng@moe.edu.sg>

Cherng Ginn Kenneth Lui
Xinmin Secondary School
<lui_cherng_ginn_kenneth@moe.edu.sg>

Open-ended mathematical tasks have the potential to not only engage students in constructive thinking but also helps them make connections to mathematical concepts they have learnt. A high quality open-ended task does not necessarily imply that the students will engage cognitively in higher order thinking to solve the task. Teachers play a crucial role in mediating the learning process in the implementation of such tasks. A guiding framework to characterise the varied degree of "openness" of the mathematical tasks is developed to ease teachers into the facilitator's role. Rubrics are designed to provide feedback both to both the teachers and students for effective task implementation.

Investigating the Interrelationships Between Teachers' Pedagogical Content Knowledge and Student Achievement Within Vocational High School Context

Vesife Hatisaru
Middle East Technical University
<vhatisaru@hotmail.com>

Students who attend vocational education are less motivated, have reluctance towards learning (Sahin & Findık, 2008), do not like academic subjects (Lewis, 2000), encounter problems with mathematics (Green, 1998; Lewis, 2000; Scarpello, 2005), and compared to other secondary students, they are usually less successful in mathematics (Bottom & Korcheck, 1989; Berberoğlu & Kalender, 2005). The knowledge of teachers who teach in vocational high schools, therefore, matters in raising student achievement (Bottoms &

Presson, 2000). The present study deals with teachers' pedagogical content knowledge and its impact on student achievement.

References

- Berberoglu, G. & Kalender, İ. (2005). Öğrenci başarısının yıllara, okul türlerine, bölgelere göre incelenmesi: ÖSS ve PISA analizi. *Eğitim Bilimleri ve Uygulama*, 4, (7), 21–35.
- Bottoms, G. & Korcheck, S. A. (1989). *Improving the Communications, Mathematics, and Science Competencies of Students Enrolled in Vocational Courses*. Southern Regional Education Board. Atlanta.
- Bottoms, G., & Presson, A. (2000). *Finishing the job: Improving the achievement of vocational students*. Southern Regional Education Board, Publication Orders Department, 592 10th St. N.W., Atlanta, GA 30318-5790. (ERIC No. ED462559).
- Green, A. (1998). Core skills, key skills and general culture: In search of the common foundation in vocational education. *Evaluation and Research in Education*. 12 (1), 23-43.
- Lewis, M. V. (2000). Vocational education and the dilemma of education. *Journal of Vocational Education Research*, 25(4), 575-584.
- Scarpello, G. V. (2005). *The effect of mathematics anxiety on the course and career choice of high school vocational-technical education students*. Unpublished Doctorial Dissertation. Drexel University.
- Sahin, İ. & Fındık, T. (2008). Türkiye’de mesleki ve teknik eğitim: Mevcut durum, sorunlar ve çözüm önerileri. *TSA/Yıl:12, S:3, Aralık 2008*, s.s.65–86.

Seeing is Believing: Building Mathematical Knowledge for Teaching Through Demonstration Lessons

Louise Hodgson

Monash University

<lmhod1@student.monash.edu>

The focus of this presentation is to outline an approach to research that aims to explore the potential of demonstration lessons as a model for teacher learning. The intent of the lessons is to allow teachers to observe the process of planning, teaching, and assessment within the context of real time classrooms. The sessions for teachers involve the preparation of lesson plans, demonstration teaching, and subsequent interrogation. The data collection will include interviews, observations and surveys and will seek to explore and describe the relationship between demonstration lessons and teacher learning.

Harnessing the Power of Cloud Computing for Mathematical Modelling in Two Different Schools

Bock Boon Lim
Ngee Ann Secondary School
<lim_bock_boon@ngeeannsec.edu.sg>

Mei Chuen Chen
Ngee Ann Secondary School
<chen_mei_chuen@ngeeannsec.edu.sg>

Trevor Redmond
A.B. Paterson College
<trd@abpat.qld.edu.au>

Joanne Sheehy
A.B. Paterson College
<jsy@abpat.qld.edu.au>

This paper describes how two schools: one from Singapore and one from Australia, harnessed the power of the emerging Cloud Computing technology to establish a cohesive social network amongst teachers between the two countries as they co-developed Mathematical Modelling lesson units. Both schools adopted the research-based Teaching for Understanding (TfU) framework as a common language and teaching philosophy for the teachers to develop and build the units. This project has demonstrated how the schools have synergistically combined new technology (Cloud Computing), innovative pedagogy (TfU) and content knowledge (Math Modelling) to enrich students' learning experiences and enhance their understanding of the mathematical ideas and concepts. This model of collaboration exemplifies how schools in different parts of the world can use new technologies to teach the 21st Century skills to students.

iMPaCT-Math: Programming as a Means to Motivate Exploration of Foundational Algebraic Concepts

Kien H. Lim
University of Texas at El Paso
<kienlim@utep.edu>

Eric Freudenthal
University of Texas at El Paso
<efreudenthal@utep.edu>

Art Duval
University of Texas at El Paso
<aduval@utep.edu>

Amy Wagler
University of Texas at El Paso
<awagler2@utep.edu>

iMPaCT-Math is a project involving the development and implementation of a set of learning modules for high-school algebra students to make connections across multiple representations: (a) statements in a program, (b) computational process; (c) graphical output, and (d) underlying mathematical concepts. These programming-related activities provide an experiential-visual context for students to engage in mathematical thinking and reinforce foundational concepts like Cartesian coordinates and slopes. Results of pilot-testing of activities in the first three modules (coordinate system, variables, and linear equations) will be shared and professional development for algebra teachers on classroom implementation will be discussed during our presentation.

Translation of Word Problems by Year 6 Lower Ability Students: An Action Research Project

Min Chern Lim

Zhenghua Primary School, Singapore
<lim_min_chern@moe.edu.sg>

Kai Kow Joseph Yeo

National Institute of Education, Singapore
<kaikow.yeo@nie.edu.sg>

Although there is general endorsement among mathematics educators and researchers on the significance of translations in mathematical comprehension, there is substantial evidence that students struggle to accurately translate verbal, tabular, graphical and algebraic representations of mathematical relations (Gagatsis & Shiakalli, 2004; Galbraith & Haines, 2000; Porzio, 1999; Wollman, 1983). This action research set out to investigate the effects of using translation on raising the understanding of word problems involving fractions among Year 6 lower ability students. The “Fast Food Approach” (FFA) was designed by integrating elements from Polya’s four stages of problem solving (1957) and The Problem Wheel (Lee, 2008) with the intent of providing a structure to enhance students’ ability to translate in order to better “understand the (word) problem”. The study employed a single-group pre-test and post-test design involving an intact Year 6 class of lower ability students from a primary school in Singapore. The findings indicated that FFA may be used to enhance the process of problem solving, as well as improve students’ attitude towards and self-confidence in mathematics and problem-solving abilities.

References

- Gagatsis, A., & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology*, 24(5), 645-657.
- Galbraith, P., & Haines, C. (2000). Conceptual mis(understandings) of beginning undergraduates. *International Journal of Mathematical Education in Science and Technology*, 31(5), 651-678.
- Lee, N. H. (2008). *Enhancing mathematical learning and achievement of Secondary One Normal (Academic) studies using metacognitive strategies*. Unpublished doctoral dissertation, National Institute of Education, Nanyang Technological University, Singapore.
- Polya, G. (1957). *How to solve it*. Garden City, NJ: Doubleday.
- Porzio, D. T. (1999). Effects of differing emphasis on the use of multiple representations and technology on students’ understanding of calculus concepts. *Focus on Learning Problems in Mathematics*, 21(3), 1-29.
- Wollman, W. (1983). Determining the sources of error in a translation from sentence to equation. *Journal for Research in Mathematics Education*, 14(3), 169-181.

Developing an Intervention Program for Students at Risk Drawing on the Strengths of Successful Existing Programs

Bernadette Long

Monash University
<bernie@stmonicasmp.catholic.edu.au>

This presentation will outline a planned study of a mathematics intervention program that draws on the successful features of existing and previously reported programs. Currently successful existing programs are being reviewed and the successful characteristics identified. Based on this, a new program will be designed. This communication will outline

the proposed research approach that aims not only to evaluate the components of the approach but also to describe the data collection methods including cognitive and affective outcomes of the intervention.

Classroom Goal Structure, Achievement Goals, and Achievement: A Multilevel Mediation Analysis of Longitudinal Data

Wenshu Luo

National Institute of Education, Singapore
<wenshu.luo@nie.edu.sg>

David Hogan

National Institute of Education, Singapore
<david.hogan@nie.edu.sg>

This longitudinal study examines the mediational role of achievement goals between previous achievement, gender, and classroom goal structure, on the one hand, and subsequent achievement, on the other. Secondary students from 115 math classes in Singapore participated in this study. Multilevel path analyses showed that at the class level, mastery classroom goal structure predicted subsequent performance through mastery approach goals, and classes with higher previous achievement and with more girls had higher subsequent performance through lower performance avoidance goals. At the student level, previous achievement predicted mastery approach goals, which in turn predicted subsequent achievement.

Transforming Children's Mathematical and Scientific Development: A Longitudinal Study

Joanne Mulligan

Macquarie University
<joanne.mulligan@mq.edu.au>

Lyn English

Queensland University of Technology
<l.english@qut.edu.au>

Michael Mitchelmore

Macquarie University
<mike.mitchelmore@mq.edu.au>

Nathan Crevensten

Queensland University of Technology
<nathan.crevensten@qut.edu.au>

A 3-year longitudinal study integrates a pedagogical approach focused on patterns and structural relationships in mathematics to science learning through novel experiences in data modelling and problem solving. Students are engaged in an innovative program, usually withdrawn in small groups and taught by the research team in collaboration with the teacher on a weekly basis for a 2-year period. The study tracks three cohorts of students, initially involved in a related study¹ when in Kindergarten, through to Grades 2, 3 and 4. In addition, two new cohorts of mathematically able students are being tracked from Kindergarten to Grade 2.

¹ Reconceptualising early mathematics learning: The fundamental role of pattern and structure

Examining Opportunities and Constraints in the Use of Context Based Experiences for Engaging Indigenous Australian Students in the Learning of Numeracy

Kate Naughtin

Monash University

<naughtin.katharine.k@edumail.vic.gov.au>

This communication will describe proposed research that examines a teaching approach aimed at improving numeracy outcomes for Indigenous Australian students. This approach uses aspects of local Indigenous Australian culture as a context for learning numeracy. The research will explore the effectiveness of teaching culture in conjunction with numeracy while examining the relationship of culture to the dimensions within numeracy. This study will also explore the engagement of students in such context based learning experiences and quantify the performance of students participating in this style of learning.

Developing Statistical Literacy: Student Learning and Teacher Education²

Hélia Oliveira

University of Lisbon, Portugal

<hmoliveira@ie.ul.pt>

Rosa Tomás Ferreira

University of Porto & CMUP, Portugal

<rferreir@fc.up.pt>

Ana Henriques

University of Lisbon, Portugal

<achenriques@ie.ulp.pt>

João Pedro da Ponte

University of Lisbon, Portugal

<jpponte@ie.ul.pt>

Carolina Carvalho

Univeristy of lisbon, Portugal

<cfc Carvalho@ie.ul.pt>

Ana Paula Canavarro

University of Évora, Portugal

<apc@uevora.pt>

Susana Colaço

College of Education of Santarém, Portugal

<susana.colaco@ese.ipsantarem.pt>

Developing students' statistical literacy is a challenging task for practicing teachers, requiring the development of new perspectives and professional knowledge in statistics. This project aims to study the development of students' statistical literacy from elementary to secondary education and is centred on two main issues: the characterization of key aspects of students' statistical literacy, particularly the ability to conduct statistical investigations, and the understanding of the development of statistical and didactical knowledge to teach this subject. The project includes teaching and teacher education experiments, using a mixed-method approach and stressing collaborative work amongst teachers and teacher educators.

² This work is supported by national funds through FCT – Fundação para a Ciência e Tecnologia, under the projects PTDC/CPE-CED/117933/2010 and PEst-C/MAT/UI0144/2011, and by FEDER funds through COMPETE.

Alleviating Maths Anxiety through Mentoring in an Emotional Intelligence Framework

Timothy Perkins
Macquarie University
<tim.perkins@nd.edu.au>

Mathematics anxiety in pre-service primary teachers can be described as a debilitating lack of confidence in one's ability both to use mathematics in a functional way and to teach mathematical content (Uusimaki & Nason, 2004). This problem is a widespread and much researched phenomenon (Haylock, 2001; Trujillo & Hadfield, 1999; Wilson, 2011). The use of mentoring to ameliorate maths anxiety has also been explored, although to a lesser extent (Beswick, Callingham, Ashman & McBain, 2011; Hudson & Hudson, 2007; Hudson & Peard, 2006). The proposed study will incorporate elements of Emotional Intelligence (Goleman, 1995; Salovey & Mayer, 1990) and Communities of Practice (Lave & Wenger, 1991) to create a mentoring system for pre-service teachers suffering from maths anxiety. Excellent mathematics teachers who rate highly on an emotional intelligence scale will mentor pre-service teachers in the hope of developing the pre-service teachers' confidence to become enthusiastic and effective mathematics educators.

Teachers' Perceptions of Student Engagement and Disengagement in Mathematics

Karen Skilling
University of Sydney
<karen.skilling@sydney.edu.au>

Janette Bobis
University of Sydney
<janette.bobis@sydney.edu.au >

Andrew Martin
University of Sydney
<andrew.martin@sydney.edu.au>

Judy Anderson
University of Sydney
<judy.anderson@sydney.edu.au>

Jenni Way
University of Sydney
<jennifer.way@sydney.edu.au>

Teachers' perceptions of their students', including whether they perceive them to be engaged or not, influences the teaching strategies they adopt, their responses to students and the efforts they make in the classroom (Hadrè, Davis & Sullivan, 2008). It is therefore important to explore the nature of those perceptions. This study explores teachers' perceptions of Year 7 students' engagement and disengagement in mathematics. Thirty-one Year 7 mathematics teachers from ten high schools located in the Sydney metropolitan region were interviewed as part of a larger project investigating student motivation, engagement and achievement in mathematics. Interviews reveal the sources from which teachers' ('accurate' and 'inaccurate') perceptions of student engagement are based and the usefulness of conceptualising such perceptions as a 'spectrum' of engagement to disengagement.

Using Students' Algebraic Thinking to Support Teacher Learning

Shikha Takker

Homi Bhabha Centre for Science Education, TIFR

<shikha@hbcse.tifr.res.in>

This paper reports a preliminary study aimed to utilise the potential of students' thinking and learning to support teacher learning. The paper begins with a review of research on algebra teaching and learning with the objective of identifying and analysing tasks that relate research on students' algebraic thinking with the practice of teaching. It then discusses a task based on students' algebraic thinking developed and tried with a cohort of middle school teachers from Mumbai. Insights from the study and its implications for continuous teacher professional development are discussed.

Developing Computational Fluency in Multi-Digit Multiplication: A Learning Trajectory Approach

Kristen Tripet

University of Sydney

<ktripet@aisnsw.edu.au>

Janette Bobis

University of Sydney

<janette.bobis@sydney.edu.au>

Jenni Way

University of Sydney

<jennifer.way@sydney.edu.au>

Learning trajectories illustrate a pathway of learning in mathematical domains. Knowing how students develop particular mathematical knowledge informs the construction of learning trajectories, providing a link between conceptual understanding and task selection (Simon & Tzur, 2004). Much of the research in multi-digit multiplication has focused on strategies used by children to solve problems, with limited research outlining successful pathways to developing understanding and how these apply to the classroom setting (Bobis, 2007; Fuson, 2003). This presentation focuses on articulating a learning trajectory for multi-digit multiplication through an examination of relevant research. This trajectory will be the basis for further study and investigation through a teaching experiment with Year 5 students. It is anticipated that the study will demonstrate how tasks can be used to effectively promote the learning process.

POSTERS

Do not Call on Me: Mathematics Anxiety among Students with Learning Differences

Michelle L.W. Bower
Landmark College
<michellebower@landmark.edu>

Suriza van der Sandt
The College of New Jersey
<vanders@tcnj.edu>

Anxiety can impact the teaching and learning of mathematics at all levels of instruction; and anxiety can affect many types of learners. Student with diagnosed learning differences are perceived to be more anxious about mathematics than other students. This survey was an investigation into some of these beliefs and feelings towards mathematics by students with diagnosed learning differences. Among the selected students the overall levels of anxiety were lower than expected. Student anxiety levels were reduced and statistically significantly after taking one mathematics course; however, these students still prefer not to engage in further mathematics applications.

Problem@Web: A Project around an On-line Problem Solving Mathematical Competition in Portugal¹

Susana Carreira
University of Algarve, Portugal
<spcarrei@ualg.pt>
Jaime Silva
University of Coimbra, Portugal
<jaimecs@mat.uc.pt>
Nélia Amado
Univeristy of Algarve, Portugal
<namado@ualg.pt>
Rosa Tomás Ferreira
University of Porto & CMUP, Portugal
<rferreir@fc.up.pt>
Isa Martins
EB2,3 Dr. Neves Júnior, Faro, Portugal
<ipati@sapo.pt>

Hélia Jacinto
Research Unit of IEUL, Portugal
<helia_jacinto@hotmail.com>
Juan Rodriguez
University of Algarve, Portugal
<jsanchez@ualg.pt>
Nuno Amaral
EB2,3 Naus, Lagoa, Portugal
<nualroam@gmail.com>
Sandra Nobre
Research Unit of IEUL, Portugal
<sandraggnobre@gmail.com>
Sílvia Reis
ES/3 Mirandela, Portugal
<silviapreis@hotmail.com>

The Problem@Web Project intends to study mathematical problem solving in a context that goes beyond the classroom. The research field involves inclusive web-based mathematical problem solving competitions aimed at middle graders (children aged 10 to 14 years-old). In this poster we address two main strands of the project: students' creativity in mathematical problem solving, and the impact of digital technologies in students' approaches when solving problems and communicating results. We provide a brief

¹ This work is supported by national funds through FCT – Fundação para a Ciência e Tecnologia, under the projects PTDC/CPE-CED/101635/2008 and PEst-C/MAT/UI0144/2011, and by FEDER funds through COMPETE.

theoretical support and a short analysis of some empirical data. We also give a summary of the research developments and data gathering for the several questions addressed.

A Longitudinal Study Preparing Preservice Teachers to Learn and Teach Mathematics with Technology

Leah A. Nillas
Illinois Wesleyan University
<lnillas@iwu.edu>

Current approaches in technology integration narrowly focus on technology alone (Harris, Mishra, & Koehler, 2009). Mishra and Koehler (2006) propose integration of multiple aspects of technology, pedagogy, and content knowledge (TPACK). This poster presents preliminary analysis of data from a four-course sequence (technology, curriculum, student teaching, and research) on the preparation of preservice teachers in integrating technology. Data from journals, curriculum, questionnaires, and sample work (projects, lesson plans, portfolios) were content analysed (Neuendorf, 2002). The goal is to examine the process of becoming technology integrators and the role of a technology course, perceptions, teaching experiences, and TPACK in that process.

References

- Harris, J., Mishra, P., & Koehler, M. (2009). Teachers' technological pedagogical content knowledge and learning activity types: Curriculum-based technology integration reframed. *Journal of Research on Technology in Education*, 41(4), 393-416.
- Mishra, P. & Koehler, M. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record*. 108(6), 1017-1054.
- Neuendorf, K. A. (2002). *The content analysis guidebook*. Thousand Oaks, CA: Sage Publications.

An Analysis of Students' Strategies for Area Measurement and its Curricular Implications

Jeenath Rahaman
Homi Bhabha Centre for Science Education
<jeenatr@hbcse.tifr.res.in>

The concept of area-measurement is particularly challenging for students as it encompasses the two critical domains – geometry and numbers. Most students face difficulty in connecting the visual, spatial and numerical aspects related to area-measurement. In line with the objectives of most curriculum, the present study tries to explore the connection between students' formal learning of area-measurement and its applications in other contexts through task based interviews. Analysis of results shows that students' greater reliance on using formal procedures limits their use of own strategies. There is a need for including contextual and visual experiences in the curriculum.