Supporting 5 Year Olds to Represent Their Mathematical Reasoning

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This paper illustrates how young students aged 5 years old were supported to develop their mathematical representations. In seven lessons, students progressed from using concrete manipulatives to representing their mathematical ideas in more sophisticated ways. The teacher utilised specific instructional actions including holding high expectations for all students to engage with collaborative discussion, paying close attention to the students' reasoning and representations and drawing on these to extend them further. The findings offer potential for how teachers of young students can support them to develop reasonable mathematical representations.

The New Zealand Mathematics Curriculum (Ministry of Education, 2024) outlines expectations that students are supported to engage in mathematical practices such as explanation, justification, generalisation and representation. New Zealand children begin school when they are 5 years old and bring with them a wealth of prior life experiences and skills. These include making informal representations to show "how many" or "which are longer or shorter", however, the process of connecting these to formal mathematical concepts requires support (Bobis & Way, 2018).

Mathematical representations can be made in a range of different ways, for example, using concrete materials, manipulatives, or drawing pictures or diagrams, or by using number sentences or formal equations. The purpose of representing mathematics in these ways is to show or explain mathematical reasoning (Bicknell et al., 2016; Bobis & Way, 2018; Pape & Tchoshanov, 2001). Accurately representing mathematical thinking is particularly challenging for young students as representing mathematical reasoning is a complex process requiring students to transfer internal thought to an external medium (Pape & Tchoshanov, 2001). For young learners, there is often an emphasis placed on using some type of materials to support mathematical learning. However, depending on how and why certain materials are used, some students may find it challenging to reasonably represent their mathematical ideas.

Teachers often use material representations when modelling or showing students how to do certain things such as counting using counters or blocks or using tens frames to count and group to ten (Bicknell et al., 2016; Obersteiner et al., 2014). In their study, Donovan and Alibali (2021) investigated how students aged 7-8 years-old used non manipulatives (toys) as "tools for doing maths" (p. 296). While the study showed success in the students manipulating these materials, this occurred through the researcher (in the teacher role) specifically instructing the students to use the toys as tools for mathematics. Donovan and Alibali concluded that it is necessary to make explicit to students *how* materials are to be used to represent mathematical reasoning. A New Zealand study by Bicknell et al. (2016) examined how the teacher made deliberate decisions about which manipulatives 5-year-old students would use in a mathematics lesson involving multiplication and division. For example, when working out the number of egg cartons required for a selected number of eggs, egg cartons and counters were provided to support the students in their representation of division. In a follow-up interview, the teacher in Bicknell et al.'s study shared that at times the students would revert to using representations

(2025). In S. M. Patahuddin, L. Gaunt, D. Harris & K. Tripet (Eds.), *Unlocking minds in mathematics education. Proceedings of the 47th annual conference of the Mathematics Education Research Group of Australasia* (pp. 357–364). Canberra: MERGA.

that they were comfortable with rather than engaging with the formal representations modelled by the teacher. To support the students the teacher would either co-construct number equations with the students or remind the students to represent using equations. Pape and Tchoshanov (2001) emphasise the important role teachers play in facilitating students to reasonably represent their reasoning. They highlight that students require regular exposure to multiple representations and the opportunities to practice applying these mathematical representations in mathematical activity. Additionally, Gravemeijer (1999) found that careful teacher scaffolding is required for students to develop their representations from early models to more formal models.

For young learners, the use of drawing as a mathematical representation is a powerful way to articulate thinking (MacDonald, 2018). A study by Roche et al. (2020) with 5-year-old students highlighted how students drew ducks to represent their emerging reasoning, however what was not evident in the drawings was the exact mathematical understanding the students held. Several researchers (e.g. Bicknell et al., 2016; Bobis & Way, 2018; Cartwright, 2023; MacDonald & Murphy, 2019) have shown that while drawings and mathematical representations can provide teachers the opportunity to understand students' reasoning, at times deeper insight is required from the students to completely understand their representation. Additionally, within a sociocultural framework, Pape and Tchoshanov (2001) highlight how representations can be used as tools for explaining mathematical reasoning and that teachers should aim for a balance between student representations and their verbal explanations. In an Australian study with students aged 6-years-old, Herbert and Williams (2023) focused on one teacher's actions to elicit mathematical reasoning. The teacher used in-the-moment-questioning to focus the students to make connections between the language and representations required.

Building on this research literature, we aim to add to the field by investigating how young students just beginning school can learn to develop mathematical representations. In this paper, we outline how one teacher facilitated young students across three mathematics lessons to develop several mathematical representations. The aim of this paper is to highlight how, over a short period of time, young learners can be supported to make and extend reasonable mathematical representations. Our research question is: How can young students be supported to develop mathematical representations?

We situate this research study in a sociocultural framework. Sociocultural theories of learning emphasise the dynamic nature of learning, with key components highlighting how students learn through interaction with others and their setting. Within mathematics classrooms, a sociocultural framing is characterised by students working collectively on mathematical activity and engaging in mathematical discourse to reason about important concepts (Boaler & Sengupta-Irving, 2016).

Research Methods

This qualitative study was conducted in one New Zealand primary school classroom with 10 students aged 5-years-old. The teacher was an experienced junior primary school teacher. The data used for this paper were drawn from these students' first three formal mathematics lessons at school and formed part of a six-week-long investigation examining teacher actions to support young learners to engage in mathematical practices. From their first lesson, the students were both expected and required to communicate their thinking with their peers. All ten students were grouped together heterogeneously. Each mathematics lesson followed a similar structure. Firstly, a mathematics task was presented to the group. The teacher read the problem to the students and then used talk moves, such as repeating, revoicing, adding on, wait time, and reasoning (Chapin & O' Connor, 2007) to engage the students in discussion. This part of the lesson (usually 5 minutes) continued until the teacher was sure all students understood the context and knew what they were expected to find out. The students then worked in pairs to solve the task together. While the groups worked, the teacher monitored their activity, intervening when she thought necessary to scaffold, prompt, or support them. After 5 minutes working together, the students formed a bigger group where ideas were shared, while the teacher recorded explanations on the board. To conclude the lesson, the teacher connected students' thinking to highlight important mathematical concepts.

Data collection involved classroom observations which were video recorded and later transcribed. These provided opportunities for the researchers to review the data multiple times. In addition, field notes and photographs of student work were collected. Thematic analysis was used to analyse the data. Thematic analysis involves seeking meaning from the data to answer the research questions (Clarke & Braun, 2017). Additionally, comparison between the field notes and video observation occurred to ensure all sets of data were interpreted accurately. For this paper, the authors undertook an initial individual examination of the video recorded observations to identify emerging themes. This included identifying the teacher actions both before and after the representations presented here. The authors then cross checked for validity of emerging themes. The subsequent analysis involved the comparison of the developing themes to ensure they addressed the research question.

Findings and Discussion

In this section, we present the teacher actions that created opportunities for students to develop mathematical representations. They also demonstrate the shifts in teacher actions over the lesson sequence that lead to the students successfully representing their thinking. These teaching moments occurred over seven lessons during an algebra patterning unit.

Initial Teacher Modelling

In the first lesson, the teacher presented a task which required the students to copy and extend an ABCABC pattern using Unifix cubes. The teacher instructed the students to finish their pattern at the 12th cube. While working in pairs, the teacher noticed one group sharing ideas, talking together saying, "yellow, pink, purple, yellow, pink, purple" and using the cubes to support their ideas. The teacher selected the pair to explain how they had constructed their pattern. The two students counted out loud together saying, "one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve" as they connected each cube to the next. While the students explained and showed their thinking to the big group, the teacher wrote the count on the board. By notating the count on the board, the teacher provided an opportunity for all students to see the count and connect what they could hear to the written form.

The teacher then drew the other students into a discussion, stating "I could hear lots of groups counting as they were making their patterns, well done." This statement explicitly emphasised to the students that what they were doing was useful for their mathematical reasoning. As the other groups had successfully copied and extended the pattern to 12 with the cubes, the teacher sought to extend their initial ideas. While drawing squares in a modelling book and colouring these in the colours of the cubes, the teacher explained what she was doing saying, "look this is what we can do. I am drawing a square to represent the cube, then I am colouring it in to show what colour each cube is". Her action of explicitly modelling for the students is similar to what Herbert and Williams (2023) found in their investigation. Additionally, the teacher modelled how to translate a pattern from concrete materials to a diagram. This supports what Gravemeijer (1999) advocates that careful teacher scaffolding is required for students to develop their representations from early models to more formal models. The teacher also notated numbers under the cubes to support the mathematical ideas that each colour represents a position in the pattern, reenforcing their counting prior.

Immediately following her modelling, the teacher then asked the pairs to think about how they could represent their own patterns as a drawing. Figure 1 shows one group's representation.

Figure 2

One Groups Representations

Figure 1 highlights how students used their concrete model to create a visual model of their pattern. As all the pairs represented their patterns in similar ways, the teacher extended the students' initial reasoning and modelled two further representations - drawing the pattern and writing the numbers while counting out loud or describing the pattern out loud. Additionally, by counting out loud together, students could further reinforce their counting. In her final discussion with the students in the first lesson, the teacher briefly showed the unit of repeat - marking these on the student's drawn representation in Figure 1. While this was not the focus of the lesson, the teacher had noticed and used the opportunity to expose the students to the mathematical term.

In the subsequent lessons, the teacher prompted the students to represent their patterns. Firstly encouraging the students to describe their patterns verbally then by making the pattern with materials. Once the pattern was made, the teacher would prompt the "now how could you represent this pattern in your books?" encouraging the students with an open prompt to represent their pattern.

Extending Mathematical Understanding through Representation

In lesson three, the students had been asked to copy a flower pattern of a lei (a Pasifika flower necklace or garland that are presented to visitors on arrival to the islands or worn at celebrations or special occasions) and repeat this pattern three times (AAAAB). Figure 2 highlights how one group used red and yellow (pre)cut out flowers to copy the pattern.

Figure 3

One Groups Representation – Materials

The group worked together to arrange their flowers into the pattern of the lei and counted "one two three four red and then one yellow – one two three four red and one yellow". This group counted the units of repeat rather than counting the individual number of flowers. When the teacher noticed the students had successfully copied the pattern, she extended their representations of using concrete manipulatives and asked them to draw their patterns. Figure 3 shows this representation.

Figure 4

Student Representation - Drawing

The visual representation highlights how the group were able to extend their representation from materials to a drawing. While all the groups had successfully made their patterns with the materials, this group was the only one to successfully extend to a drawing.

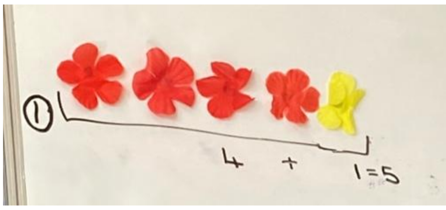
Notably while all the groups were attempting to extend their representations to a drawing, the teacher consistently validated their reasoning saying, "you are all so clever, you have represented your pattern in two different ways; you have represented it with materials and now you have represented it by drawing". Validating the students' ideas emphasised the usefulness of representations expressing mathematical reasoning. Additionally, by always naming the mathematical practice of representation, the teacher's actions support what Selling (2016) found, that naming the mathematical practice facilitates students to learn and use them.

Extending to a Mathematical Equation

Continuing in lesson three, the teacher asked the students for the total number of flowers in one unit of repeat. Chloe responded by saying, "five". The teacher validated her response by saying "wow, you are such a good mathematician". Using Chloe's response, the teacher modelled how their drawings of the unit of repeat could be extended to a number sentence. By revoicing Chloe's reasoning the teacher emphasised the unit of repeat saying "so you thought if we had four [draws a line under the four red flowers on the whiteboard] and one more [draws a line under the one yellow flower] in our unit of repeat we have five, is that right?" (Chloe nods). These actions highlight the teacher publicly validating Chloe's reasoning and providing a model for other students. The teacher then wrote numbers under the lines and flowers stating "so four plus one equals five". Figure 4 shows the teacher's model on the whiteboard.

Figure 5

Model of Mathematical Equation



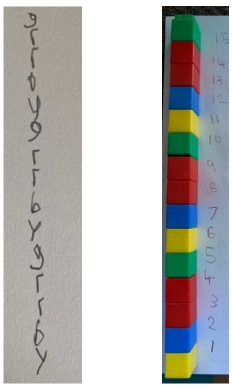
The teacher had modelled how the students' drawing of the unit of repeat could be represented as a mathematical equation. By paying close attention to how the students represented their reasoning, the teacher was able to use this as an opportunity to extend mathematical thinking. She made the students' implicit understanding of 4 + 1 = 5 explicit through the modelling of the equation. These actions are similar to what Bicknell et al.'s (2016) found where the teacher purposefully modelled representations as equations. As the lesson continued, the teacher asked the students to represent the total number of flowers in three units of repeat. Whilst two groups reverted to 1:1 counting, the other two groups attempted the use of an equation. This highlights the potential of exposing students to formal representations early in their mathematics education.

These first three lessons served as a foundation the teacher built on in the subsequent four lessons. Throughout lessons four to seven the teacher continued to be purposeful in her public praise as students represented their mathematical reasoning in various ways. For example, as students were working in their groups solving their task, the teacher could be heard saying "Wow, I love how this group have represented their thinking in two ways". In addition, the teacher continued to specifically use the term "represent", for example, she asked a group "share your ideas with us and I will represent it on the board" or "what did we do last week to represent our thinking? We showed our pattern in colours and letters". The ongoing naming of the mathematical practice of representation supported the students to know and use this practice effectively.

The progression of learning to know and use mathematical representation was highlighted in lesson seven. In this lesson, the students were working on a pattern task that involved working out further points within a pattern (ABCCD). The students were provided one unit of repeat (yellow, blue, red, red, green) and then asked to find the sixth colour and 15th colour of the pattern. During the launch of the task the teacher asked the students "can you work out the colour of the sixth and 15th block and record it – represent your thinking". This was an open-ended prompt, as the teacher did not specify how the students should solve and represent their reasoning. The students began working in groups, all making the same pattern (yellow, blue, red, red, green) with Unifix cubes. Immediately following their use of Unifix cubes, all students began drawing their patterns in their books. As they did this, the teacher praised one group saying, "I love how Lukes' group is representing, they are drawing squares to show each block". This statement prompted a response from Luke, "we don't know what colour they are." The teacher replied, "well, we only have pencils today so what could we do?" In their groups, all students immediately began discussing how they could show which block represented which colour. Rather than telling or showing the students how to resolve their dilemma, the teacher provided an open-ended prompt that initiated active engagement from all students and provided an opportunity for them to find a reasonable way to represent their thinking. Figure 5 show two group's representations.

Figure 6

Group One and Group Two's Representations



Group One Group Two

Group One's representation shows a tower of letters, each letter representing the coloured blocks y = yellow, b=blue, r=red, g=green. As this group shared their representation with the group, they counted each letter aloud, stating that fifteenth block would be green. Group Two's representation shows the tower carefully aligned to numbers that they had written. Both representations reflect the progression from the students' initial representations with concrete materials (Lesson One) to more sophisticated mathematical representations.

Conclusion and Implications

As prior research has found (Pape & Tchoshanov, 2001), supporting students to accurately represent their mathematical thinking is particularly difficult for younger students, as it requires the transference of internal thought to an external medium. However, the progression of sophistication of mathematical representation over a short period of time evident here, highlights how teachers can purposefully support young students to make sense of mathematics. To begin with, the teacher provided concrete materials to support the students to sense-make by representing a pattern. The students were then shown how to transfer from one representation model to another. These findings add to prior research (e.g., Bicknell et al., 2016; Donovan & Alibali, 2021, Obersteiner et al., 2014) focusing on scaffolding students to use concrete materials as tools for learning mathematics. To achieve progression in learning, the teacher then provided explicit models for students to replicate that built on their prior understanding and lead them to further explore ways to show their reasoning. Furthermore, adding to what Pape and Tchoshanov (2001) emphasised, the teacher in this study provided consistent exposure to multiple mathematical representations and opportunities for the students to explore these.

This paper has outlined one model of how young students entering school can progress from representing their mathematical ideas using materials, to using drawings and diagrams to represent reasoning to equations. To achieve this, the teacher used a range of pedagogical actions. From the start, the teacher held high expectations for students to engage in collaborative discussion while working on mathematics activity. Noticeably, she paid close attention to the students' reasoning, drawing on their reasoning and building on the students' ideas to extend them. The teacher carefully scaffolded new learning by explicitly modelling how to transfer concrete representations to visual representations. This meant that by the third lesson, the teacher had afforded the students an opportunity to see how concrete and visual representations of mathematics could be represented numerically.

The data presented in this study are based on a small-scale study. Therefore, more research is needed to explore how teachers can support young students to represent their mathematical thinking in their first few weeks of school across multiple settings. Nevertheless, the study provides a strong exemplar of how teachers can build upon student generated representations to develop mathematical understanding in the moment.

References

Bicknell, B., Young-Loveridge, J., & Nguyen, N. (2016). A design study to develop young children's understanding of multiplication and division. *Mathematics Education Research Journal*, *28*(4), 567–583. https://doi.org/10.1007/s13394-016-0180-4

Boaler, J. (2008). Promoting 'relational equity' and high mathematics achievement through an innovative mixed-ability approach. *British Educational Research Journal*, *34*(2), 167–194. https://doi.org/10.1080/01411920701532145

Boaler, J., & Sengupta-Irving, T. (2016). The many colors of algebra: The impact of equity focused teaching upon student learning and engagement. *Journal of Mathematical Behavior, 44, 179-190.*

Bobis, J., & Way, J. (2018). Building connections between children's representations and their conceptual development in mathematics. In V. Kinnear, M. Y. Lai, & T. Muir (Eds.), *Forging Connections in Early Mathematics Teaching and Learning* (pp. 55–72). Springer Singapore. https://doi.org/10.1007/978-981-10-7153-9_4

Chao, S.-J., Stigler, J. W., & Woodward, J. A. (2000). The effects of physical materials on kindergartners' learning of number concepts. *Cognition and Instruction*, *18*(3), 285–316. https://doi.org/10.1207/S1532690XCI1803_1

Chapin, S. H., & O'Connor, C. (2007). Academically productive talk: Supporting students' learning in mathematics. *The Learning of Mathematics*, 113–128.

Clarke, V., & Braun, V. (2017). Thematic analysis. *The Journal of Positive Psychology*, *12*(3), 297–298. https://doi.org/10.1080/17439760.2016.1262613

Donovan, A. M., & Alibali, M. W. (2021). Toys or math tools: do children's views of manipulatives affect their learning? *Journal of Cognition and Development*, *22*(2), 281–304. https://doi.org/10.1080/15248372.2021.1890602

Evans, R., & Cleghorn, A. (2022). Do student teachers see what learners see? – Avoiding instructional dissonance when designing worksheets. *South African Journal of Childhood Education*, *12*(1). https://doi.org/10.4102/sajce.v12i1.1015

Fereday, J., & Muir-Cochrane, E. (2006). Demonstrating rigor using thematic analysis: A hybrid approach of inductive and deductive coding and theme development. *International Journal of Qualitative Methods*, *5*(1), 80–92. https://doi.org/10.1177/160940690600500107

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102_4

Herbert, S., & Williams, G. (2023). Eliciting mathematical reasoning during early primary problem solving. *Mathematics Education Research Journal*, *35*(1), 77–103. https://doi.org/10.1007/s13394-021-00376-9

MacDonald, A. (2018). *Mathematics in early childhood education.* (1st ed.). Oxford University Press Australia and New Zealand.

MacDonald, A., & Murphy, S. (2019). Using the drawing-telling approach to reveal young children's mathematical knowledge. *Mathematics Education Research: Impacting Practice.*, 99–116.

McNeil, N., & Jarvin, L. (2007). When theories don't add up: Disentangling the manipulatives debate. *Theory Into Practice*, *46*(4), 309–316. https://doi.org/10.1080/00405840701593899

Ministry of Education. (2024). *Mathematics and statistics in the New Zealand curriculum years 0–8*. Draft.

Obersteiner, A., Reiss, K., Ufer, S., Luwel, K., & Verschaffel, L. (2014). Do first graders make efficient use of external number representations? The case of the twenty-frame. *Cognition and Instruction*, *32*(4), 353–373. https://doi.org/10.1080/07370008.2014.948681

Pape, S. J., & Tchoshanov, M. A. (2001). The role of representation(s) in developing mathematical understanding. *Theory Into Practice*, *40*(2), 118–127. https://doi.org/10.1207/s15430421tip4002_6

Papic, M. M. (2013). Improving numeracy outcomes for young Australian indigenous children. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing Early Mathematics Learning* (pp. 253–281). Springer Netherlands. https://doi.org/10.1007/978-94-007-6440-8_13

Punch, K., & Oancea, A. (2014). *Introduction to research methods in education*. SAGE Publications Ltd. http://digital.casalini.it/9781473909199

Roche, A., Ferguson, S., Cheeseman, J., & Downton, A. (2020). *Making equal groups: The case of 12 little ducks*.

Selling, S. K. (2016). Making mathematical practices explicit in urban middle and high school mathematics classrooms. *Journal for Research in Mathematics Education*, *47*(5), 505–551.