An Exploration into Secondary Mathematics Teachers' Instruction of Mathematical Proof in Singapore: Understanding Their Conceptions, Beliefs, and Challenges Related to Teaching Proofs

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This study examines factors shaping teachers' instructions of mathematical proof in Singapore. Using a qualitative case study, two teachers' conceptions, beliefs, and challenges were analysed. Findings reveal reliance on empirical and axiomatic proof schemes, shaping instruction. Teachers faced challenges such as limited curriculum emphasis and lack of training. Instructional gaps include insufficient focus on proof tasks and inadequate pedagogical resources. Recommendations include enhancing teacher training, curriculum integration, and assessment reforms. Addressing these issues ensures proof is integral to mathematical reasoning in classrooms.

Mathematical proof represents a hallmark of rigorous reasoning in the discipline of mathematics. In school contexts, proof serves not only to validate mathematical truth but also as a vehicle for cultivating students' logical thinking, justification, and communication of mathematical ideas. Despite its pedagogical significance, the teaching of proof continues to present challenges in secondary school classrooms around the world (Aaron & Herbst, 2019; Ball et al., 2003; Healy & Hoyles, 1998; Mariotti et al., 2018; Sears, 2019).

How proof is taught depends largely on teachers' instructional decisions, which are shaped by their conceptions and beliefs. Teachers play a pivotal role in mediating students' engagement with proof, yet they often encounter challenges that affect how proof is introduced and emphasised in lessons (Aaron & Herbst, 2019; Ayalon & Hershkowitz, 2018; Mukuka et al., 2023). Although the Singapore secondary mathematics curriculum formally recognises the importance of proof – evident in the syllabi for Elementary Mathematics (EM) and Additional Mathematics (AM) (Ministry of Education, 2013a, 2013b, 2018) – proof remains underemphasised in enacted instruction. Furthermore, the absence of proof-related items in national assessments discourages teachers from investing instructional time in developing students' proving abilities.

The study reported in this paper is part of a larger study on proof and proving in Singapore (Thanabalasingam, 2024). As part of an investigation related to "What shapes teachers' instruction of proof in secondary school", teachers' conceptions, beliefs, and challenges were investigated there. However, in this paper we limit our attention to a proper subset of findings found therein. This paper examines how teachers' varied experiences with proof – some having had strong undergraduate exposure, others with limited formal training – affect their beliefs about teaching. It also highlights constraints such as limited curriculum materials and the lack of emphasis in assessments, both of which lead to an over-reliance on textbook demonstration and reduced opportunities for student-centred proving activities. By unpacking these factors, this study contributes to our understanding of the enacted curriculum for proof instruction and points to areas for improvement in teacher preparation and curriculum development. The

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findings aim to inform ongoing efforts to strengthen the teaching of proof in secondary schools and ensure that students acquire reasoning skills that go beyond procedural competence.

Conceptual Framework

This study adopts a dual-framework approach by integrating Harel's (2007) Proof Scheme Framework and Cabassut et al.'s (2012) Belief System Framework, offering a holistic lens to examine both the cognitive and affective dimensions of teachers' proof instruction.

Teachers' Conceptions of Proof: Harel's (2007) Proof Scheme Framework

Harel (2007) prioritised proof function as a credible argument. He obtained from this idea ways to "classify individuals' argumentative behaviours" through the type of reasoning they tend to use and referred to proof schemes which were classified into three categories:

- External Conviction – Reliance on authority (e.g., "My teacher said this is true"), procedures, or symbols without understanding. This is further subcategorised into 'Authoritative', 'Ritual' and 'Non-referential symbolic'.
- Empirical – Validation through examples or visual patterns, including inductive or perceptual reasoning. This comprises subcategories of 'Inductive' and 'Perceptual'.
- Analytical – Deductive reasoning using formal logic, subdivided into transformational (structured-based) and axiomatic (based on definitions and theorems). This consists of 'Transformational' and 'Axiomatic' as subcategories.

Analysing teachers' responses to proof tasks through this framework helps identify their underlying proof schemes and the implications for their instructional practices.

Teachers' Beliefs About Teaching Proofs: Cabassut et al.'s (2012) Belief System Framework

Cabassut et al. (2012) studied teachers' beliefs about proof and devised categories to describe these distinct beliefs. They emphasised how beliefs shape teaching decisions, outlining:

- Beliefs about the nature and role of proof in mathematics.
- Beliefs about proof's place in the school curriculum.
- Beliefs about the teacher's own mathematical ability.
- Beliefs about instructional strategies for proof.

These beliefs influence how teachers prioritise, deliver, and assess proof instruction in classrooms.

An Integrated Framework for Studying Teachers' Instruction of Proof

Although Harel's (2007) Proof Scheme Framework and Cabassut et al.'s (2012) Belief System Framework originate from different strands of mathematics education research – one focusing on cognitive conceptions of proof and the other on affective and pedagogical beliefs – there is a strong theoretical alignment between them. Both frameworks recognise that how proof is taught depends not only on the teacher's conceptual understanding of proof but also on their beliefs about its role in mathematics education. The integration of these frameworks is not arbitrary; it is based on the premise that teachers' conceptions of proof influence their beliefs about proof instruction, and these beliefs, in turn, shape their enacted teaching practices. For example, a teacher who predominantly operates within an Empirical Proof Scheme (Harel, 2007) may hold a belief that proof should be taught through example-based exploration rather than formal deduction (Cabassut et al., 2012). By linking these two perspectives, the integrated framework allows for a holistic understanding of proof instruction – one accounts for both the cognitive processes involved in doing proof and the affective, pedagogical, and epistemological considerations involved in teaching proof. Thus, it provides a structured way to analyse the enacted curriculum of proof instruction in Singapore's secondary schools and helps identify areas for improving teacher education and curriculum design.

Methodology

Research Design

This study employs a qualitative case study approach to examine the factors shaping teachers' instruction of mathematical proof in Singapore secondary schools. A multiple-case design was adopted, involving two in-service secondary school mathematics teachers. While the small sample size limits broad generalisability, case studies allow for rich, context-specific insights (Creswell, 2012). The selection of one junior teacher (10 years of experience) and one senior teacher (39 years of experience) ensures a comparative perspective, capturing a range of instructional practices. Data was collected through two survey questionnaires and semi-structured interviews, with responses analysed using both qualitative and quantitative techniques (e.g., counting the frequencies of various proof schemes). Triangulation of data sources (questionnaire responses, task performance, and interviews) was employed to enhance validity and mitigate self-report bias.

Participants

Two secondary school mathematics teachers were purposefully selected based on the following criteria: (1) Mathematics training background – Both teachers held a degree in pure mathematics and a postgraduate diploma in education; (2) Years of teaching experience – One teacher (Teacher 1) had 10 years of teaching experience, while the other (Teacher 2) had 39 years, providing perspectives from both junior and senior educators; (3) Teaching responsibilities – Both teachers were teaching Elementary Mathematics (EM) and Additional Mathematics (AM) at the upper secondary level. These selection criteria ensured that the study included teachers with a strong mathematics background and extensive classroom experience in proof instruction.

Instruments

Three instruments were used for data collection: (1) Conceptions Questionnaire – A set of nine modified Ordinary-Level (O-Level) proof-related questions from the EM and AM syllabi. The O-Level is a national examination which secondary school students sit for at the end of their secondary education in Singapore. Teachers' responses were analysed using Harel's (2007) Proof Scheme Framework to categorise their proof schemes. Inter-rater reliability was conducted to ensure that the classification process was rigorously reviewed to ensure consistency and to avoid bias. (2) Beliefs Questionnaire – a 40-item Likert-scale survey adapted from Cabassut et al. (2012) Belief System Framework (Thanabalasingam, 2024). Questions were categorised into: (i) Teachers' beliefs about the role of proof in school mathematics; (ii) Teachers' beliefs about themselves as mathematical thinkers; and (iii) Teachers' pedagogical beliefs about proof instruction. While self-reported data may be influenced by social desirability bias, the study mitigated this through data triangulation, cross-referencing questionnaire responses with task performance and interview insights. (3) Semi-structured Interviews – Follow-up interviews were conducted to probe teachers' reasoning in responding to the questionnaires and to identify additional challenges in proof instruction. The interviews were audio-recorded and transcribed for qualitative analysis, which we elaborate in the section on Data Analysis. Interview protocols were developed based on prior literature and reviewed to minimise interviewer bias.

Procedures

Teachers were briefed on the study's objectives and provided consent to participate. They completed the Conception Questionnaire as a take-home task within one sitting (150 minutes). While external influences could not be entirely controlled, interview cross-checking was used to verify the authenticity of responses. One week later, they completed the Beliefs Questionnaire. Immediately after the Beliefs Questionnaire, semi-structured interviews were conducted with each teacher (about 30 minutes per session).

Data Analysis

Teachers' conceptions of proof were analysed using Harel's (2007) Proof Scheme Framework, categorising responses into External Conviction, Empirical, and Analytical Proof Schemes. Inter-rater reliability checks were in place. Teachers' beliefs about proofs were analysed using Cabassut et al.'s (2012) Belief System Framework, examining their epistemological, didactical, and pedagogical perspectives. Interview data was transcribed and coded to identify recurring themes and challenges in proof instruction. Coding consistency was ensured through independent interviews and iterative refinement. This methodological approach acknowledges its limitations while strengthening its credibility through triangulation, cross-checking procedures, and transparency in data collection. The study provides a nuanced, in-depth examination of how teachers' conceptions and beliefs shape their enacted curriculum for proof instruction in secondary school mathematics classrooms.

Findings and Discussion

We now present the findings from the study, organised according to five themes as categorised in Thanabalasingam's (2024) study: teachers' conceptions, experiences, approaches, perceptions, and challenges related to teaching proofs. Each theme is supported by evidence from the Conceptions Questionnaire, the Beliefs Questionnaire, and semi-structured interviews.

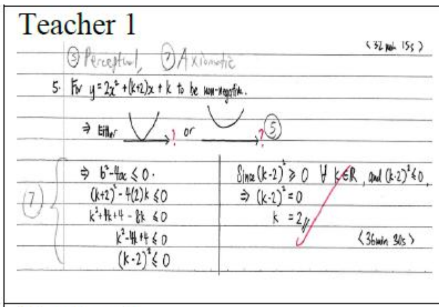
Teachers' Conceptions of Proof

Taking note of the number of times, each proof scheme surfaced for each question, the findings indicate that teachers exhibit varying proof conceptions, aligning with Harel's (2007) Proof Scheme Framework. While one teacher predominantly relied on Empirical Proof Schemes, frequently validating mathematical statements through pattern recognition rather than formal deduction, the other demonstrated a more Analytical Proof Scheme, emphasising logical transformations and axiomatic reasoning. This is exemplified in their attempts of Question 5 (Algebra Strand) which required them to prove that there exists only one value of the constant k for which $y = 2x^2 + (k + 2)x + k$ cannot be negative and to state its value (see Figure 1).

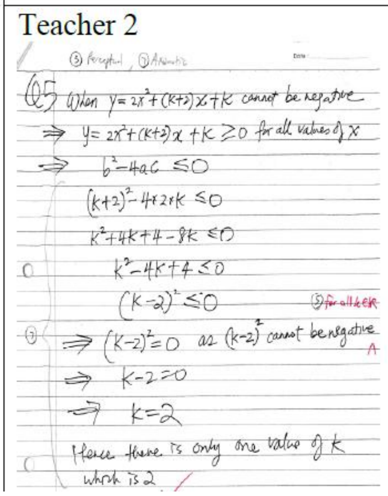
Both teachers applied axiomatic reasoning in structuring their arguments using standard algebraic theorems. However, aspects of perceptual reasoning emerged – Teacher 1, for instance, sketched quadratic graphs without explicitly defining the horizontal axis, assuming it was understood to represent x. Similarly, Teacher 2 omitted crucial definitions of a, b, and c in the discriminant expression, perceiving it as a norm to represent the discriminant in this 'standard' form. Also, she perceived that $(k - 2)^2$ cannot be negative always *without* indicating clearly that this was quantified over all real numbers – it may be negative if complex numbers are allowed. Thus, Perceptual and Axiomatic proof schemes were the most observed schemes among the teachers.

Figure 1

Responses of Teachers and Respective Proof Schemes (Question 5) and Harel's Proof Schemes



1. Authoritative
2. Ritual
3. Non-Referential Symbolic
4. Inductive
5. Perceptual
6. Transformational
7. Axiomatic



1. Authoritative
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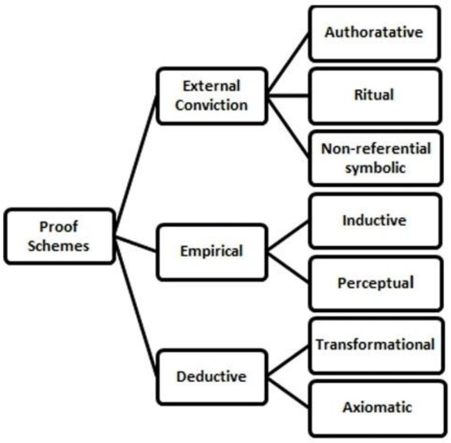

Teachers' Experiences in Teaching Proof

Both teachers had limited exposure to proof instruction during their teacher training. The junior teacher, Teacher 1 (10 years of experience), noted that proof was once emphasised more in Singapore's curriculum but has diminished over time. The senior teacher, Teacher 2 (39 years of experience), admitted that proving was largely absent from their pre-service education, leaving them to learn proof-teaching techniques through trial-and-error. Additionally, their teaching experiences related to teaching proof differed in the following ways: Teacher 1 had a more positive experience with proof instruction. He actively used diagrams and multiple examples when teaching proofs, emphasising a visual and structured approach. Teacher 2 had a mixed or neutral experience with proof instruction. She did not regularly encourage students to construct or present proofs and felt less confident in incorporating proof-based activities in class. Furthermore, findings from the teachers' beliefs system questionnaire and insights from the subsequent semi-structure interviews indicated that while both teachers regularly presented proofs, neither strongly emphasised student-centred proof construction (e.g., working in groups or presenting proofs). Notably, Teacher 2 said "… but I'm not trained to teach proofs in the in-service courses…Uh, yes. I mean lack of training courses in teaching proofs. Because there are many courses on heuristic problem-solving, but not on proof". The lack of formalised training in proof instruction suggests a gap in teacher preparation programs. Teacher 1 said "…back to the question as teaching proofs uh, I find myself needing to bring in a lot of my prior experience". So, if teachers are not systematically exposed to proof pedagogy, their instruction may be shaped more by personal experiences than by research-informed strategies.

Teachers' Approaches to Teaching Proof

The study found significant differences in how the two teachers approached proof instruction. Teacher 1 adopted a student-centred approach, encouraging peer discussions and allowing students to play a role in proof development. He emphasised logical reasoning and engagement in understanding proofs rather than passive notetaking. Teacher 2 took a teacher-centred approach, relying more on textbook examples and direct instruction with limited student involvement in proof construction. Survey results further indicated that while Teacher 1 actively encouraged student participation, he did not integrate proof construction regularly into classroom activities or homework assignments. Teacher 2, on the other hand, rarely engaged students in proof discussions and placed a strong emphasis on direct explanation. To illustrate this point, we note that in teaching congruence proofs in geometry, Teacher 1 encouraged students to verify congruence properties with physical cut-outs before moving to formal proof, fostering student engagement. In contrast, Teacher 2 followed a structured approach using textbook examples and guided practice, with minimal student involvement in constructing proofs. These findings suggest that instructional approaches to proof are strongly shaped by teaching experience. Younger teachers may be more open to student-centred discovery approaches, whereas experienced teachers may adhere to traditional didactive instruction. This highlights the need for professional development programs to encourage diverse and effective proof-teaching strategies.

Teachers' Perceptions to Teaching Proof

Teachers' perceptions of proof instruction varied. While both teachers strongly agreed that proofs are essential in mathematics, their views on student involvement differed. Teacher 1 strongly valued student engagement, encouraged classroom group discussions, and believed students should actively participate in constructing proofs. Teacher 2, though generally positive about proof, was less convinced that all students should learn to read and write proofs. She had a neutral stance on whether students should play a significant role in constructing proofs. Both teachers agreed that it is the teacher's responsibility to present proofs, rather than relying on students to construct them independently. Teacher 1 supported active student contributions, while Teacher 2 remained neutral on this point.

Teachers' Challenges to Teaching Proof

Teachers face several challenges when teaching mathematical proofs, as surfaced during the semi-structured interviews. These challenges are influenced by various factors. Curriculum materials play a role in preparation, but teachers find them lacking in explicitly demonstrating the proof process. One teacher noted that the curriculum does not sufficiently emphasise proof instruction. Training is another factor—while undergraduate studies help develop mathematical maturity, there is a gap in in-service training on how to teach proofs effectively. Personal experience also shapes teaching ability; Teacher 1 credited his struggles as a student for his understanding, more than formal training. Determining the correct approach to proofs poses difficulties, as evidenced in the conception's questionnaire, with one teacher struggling under time pressure, while another found problems easier due to prior exposure. Teaching proofs effectively is a challenge, as students often struggle with clearly expressing their reasoning and lack motivation to learn proof techniques, especially if they already find mathematics difficult. Finally, both teachers suggested curriculum and training improvements. Teacher 1 recommended access to exam markers' reports to address student misconceptions and better guidance for teachers in supporting students' proof development. Teacher 2 emphasised increasing the inclusion of proofs in the curriculum, more in-service courses on proof teaching, and greater emphasis on proof-related questions in exams. Overall, the key challenges involve curriculum adequacy, training gaps, personal experiences, proof-solving difficulties, teaching struggles, and areas for improvement in curriculum and teacher preparation.

Summary of Findings and Identified Gaps in the Enacted Curriculum

Triangulation of data identified several factors shaping teachers' instruction of proof. Teachers' proof schemes, mathematical knowledge, reasoning patterns, and misconceptions influence their conceptions of proof and instructional approaches. Their experiences are shaped using visual aids and examples. Instructional style varied between student-centred and teacher-centred methods, differing in proof development, student involvement, and assessment strategies. Teachers' beliefs about the purpose of proof in mathematics guided their teaching priorities. Challenges such as curriculum constraints, limited training, time pressures, and assessment demands also significantly affected instruction. Understanding these factors can inform improvements in professional development, curriculum design, and assessment policies.

Key gaps in the enacted curriculum hinder effective proof teaching in secondary mathematics. From this study, some of these include: (1) Guidance on Proof Development – Teachers need structured frameworks to guide students through the proof process. (2) Teacher Training – There is a lack of specialised training in proof pedagogy; in-service programs should address this. (3) Recognising Students' Thinking – Teachers need better strategies to identify and develop students' reasoning in proof. (4) Curriculum Improvements – Teachers suggested greater access to markers' reports, more proof problems, and support for developing mathematical maturity. By addressing these gaps, curriculum developers can better support teachers in delivering effective instruction and fostering a deeper understanding of proof among students.

Implications and Recommendations

The findings from this study may have some implications for the teaching of mathematical proof in secondary school mathematics in Singapore, granted the limitation of small sample size. In this section, we present key recommendations aimed at addressing the challenges identified in teachers' conceptions, experiences, approaches, perceptions, and challenges related to proof instruction.

Strengthening Teacher Training in Proof Pedagogy

Given that teachers had limited formal training in proof instruction, pre-service and in-service professional development programs should include explicit training on teaching proof. Workshops can focus on different proof schemes, reasoning patterns, and instructional strategies to enhance teachers' confidence and competence in proof instruction. Training programs should incorporate case studies and lesson demonstrations to illustrate proof-teaching strategies.

Addressing Curriculum Gaps in Proof Instruction

The Ministry of Education and curriculum developers should consider reviewing current textbooks and instructional materials to ensure they provide adequate support for teaching proof. Additional teaching guides can be developed to offer step-by-step strategies for guiding students in proof construction. Efforts should be made to align teacher training programs with curriculum reforms, ensuring that new proof-related content is well-supported by pedagogical training.

Encouraging Professional Learning Communities

Schools should establish professional learning communities (PLCs) where teachers can collaborate, share best practices, and reflect on their proof instruction experiences. Cross-school collaborations and lesson study groups can be formed to allow teachers to observe proof-teaching strategies in a supportive environment. Online forums and teacher networks should be strengthened to provide a platform for ongoing discussions and resource-sharing among mathematics educators.

Conclusion

This study set out to investigate what shapes teachers' instruction of mathematical proof in secondary school mathematics classrooms. By examining teachers' conceptions, experiences, approaches, perception, and challenges, we have uncovered key influences on proof instruction and identified gaps in the enacted curriculum. The findings underscore the importance of strengthening teacher training, improving curriculum resources, refining assessment practices, and fostering professional collaboration to support proof instruction.

Inevitably, this study contributes to the broader discourse on mathematics education and proof pedagogy, providing valuable insights for educators, curriculum developers, and policymakers. Future research can further explore how these proposed changes impact classroom practices and student learning outcomes, ensuring that proof remains an integral and meaningful component of mathematics education.

Acknowledgement

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