A Study of Grade 10 Students' Conceptions of Proof in a Singapore Secondary School

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This paper examines Grade 10 students' conceptions of proof in a secondary school in Singapore. Using a purposive survey of 9 mathematical items, proofs by 8 students for two items, one on Number and Algebra and another on Geometry and Trigonometry, were coded using Harel's proof schemes. The findings show that the students' proofs for the two questions displayed a range of proof schemes, except the authoritative scheme. With the exception, of the perceptual scheme, the predominance of other schemes was item specific. It was also apparent from the proofs written by the students that they struggled to articulate their arguments using correct mathematical language.

Mathematical proof and the aspects of proof form the very foundations of mathematics. The arguments, justifications and reasoning typically used in a proof are ubiquitous in all of mathematics. The formulae or theorems we apply to solve problems are proven true via deductive means. Even though most of these proofs that we speak of are conceptually out of reach for the average student at secondary level, the aspects mentioned are crucial in forming solid mathematical foundational knowledge. It is surprising then that the concept and process of proving have not been ascribed sufficient importance in the secondary mathematics curricula worldwide until recently (Ball et al., 2003; Healy & Hoyles, 1998). There has always been a belief that proof should not be covered in an in-depth manner in the secondary mathematics curriculum as it is beyond students' access and teachers' expertise (Philipp, 2007; Ko, 2010; Stylianides, 2013). The applications of mathematics were deemed as more critical than the proofs of mathematical theorems since only professional mathematicians actively pursue the construction of rigorous proofs.

Students' conceptual knowledge (or conceptions) related to learning proofs encompasses their understanding of the fundamental principles and logical reasoning involved in constructing valid mathematical arguments (Basturk, 2010). It involves grasping abstract concepts, recognizing the logical structure of proofs, and comprehending the significance of proof techniques such as direct proof, proof by contradiction, and proof by induction. Such conceptual knowledge also includes understanding the role of definitions, axioms, and theorems in establishing mathematical truth, as well as recognizing the connections between different mathematical concepts and how they can be applied in the context of proof construction (Basturk, 2010).

In the Singapore secondary school mathematics curriculum (SMC), there is an acceptance that proof and its aspects of it (for example, argumentation, justification, and reasoning) are essential, as evidenced in the mathematics curricular documents (Ministry of Education, 2013a, 2013b, 2018). However, despite the statements of intent stated in the curriculum documents, there is little evidence of the enactment of proof and proving throughout the mathematics curriculum. Most proofs seem to appear mainly in the topic of Geometry in both Elementary Mathematics (EM) and Additional Mathematics (AM). All students in secondary school offer EM as a school subject, while only students in the same grades who are capable of doing more mathematics offer AM.

(2025). In S. M. Patahuddin, L. Gaunt, D. Harris & K. Tripet (Eds.), *Unlocking minds in mathematics education. Proceedings of the 47th annual conference of the Mathematics Education Research Group of Australasia* (pp. 413–420). Canberra: MERGA.

The Study

The study reported in this paper is part of a larger study on proof and proving in Singapore (Thanabalasingam, 2024). As part of an investigation related to "What shapes students' learning of mathematical proof in a secondary school" students' conceptions, experiences, approaches, perceptions and challenges were investigated. However, in this paper we will limit the scope to students' conceptions related to learning proof.

Conceptual Framework

The study adopted a purposive framework, shown in Table 1, for classifying the types of justification students adopt in their proof schemes. Harel's (2007) proof scheme comprises three main categories, namely i) the *External conviction proof scheme,* ii) the *Empirical proof scheme,* and iii) the *Analytical proof scheme.* In the external conviction proof scheme category, individuals prove via their existing conceptions of proof purely through an external source such as a textbook or teacher. In this process, individuals do not try to develop new proof; instead, claims are made based on rote learning. This scheme is further elaborated into three categories, which are:

- *Authoritative* - proof based on student, teacher, book or any other external authority.
- *Ritual* - proof is made based on opinion and form not on the content.
- *Non-Referential Symbolic* - individual tries to produce mathematical arguments through symbolic representations and statements without knowing the meanings of symbols.

Table 1

Harel's Proof Scheme (Harel, 2007)

External conviction	Empirical	Analytical
○ Authoritative	○ Inductive	○ Transformational
○ Ritual	○ Perceptual	○ Axiomatic
○ Non-referential symbolic		

In the empirical proof scheme category individuals make or develop claims based on their intuitions through examples. This scheme is further elaborated into two categories, which are:

- *Inductive* - individuals try to produce arguments with examples and trials. Generalisation is made with one or several examples.
- *Perceptual* - individuals try to produce a proof by making deductions based on perceptual representations.

The analytical proof scheme category consists of arguments based on logical deductions. Proofs in this class are based on definitions with formal structures. Formal concepts are developed through theoretical definitions, not in line with existing ideas. Verification is achieved using axioms, and the difference between an axiom and a theorem is grasped. This scheme is further elaborated into two categories, which are:

- *Transformational* - mental guesses are made regarding the accuracy of the proposition and deductions are made based on these guesses. It features generalisation, operational thinking, and logical deduction.
- *Axiomatic* - proof is made by using axioms and theorems.

Method

The Conceptions Survey

The conceptions survey comprised nine questions involving Algebra and Geometry. The questions were modified from past national level EM and AM examination questions. These modified questions were more general, thus providing students with opportunities to solve them in ways they deem fit. This would allow us to decipher their proof schemes from their solutions. In this paper, we will focus on two of the questions, question 2 (Number and Algebra) and question 7 (Geometry and Trigonometry).

Subjects

A purposeful sample of eight Grade 10 students (four males and four females) from a secondary school in Singapore participated in the study. The students were representative of mathematics ability range in their respective classes. They were taking both EM and AM as school subjects. The students were keen to participate in the study and assess their readiness for their upcoming end-of-grade 10 national examination. Participation of the students in the study was governed by ethical requirements of the Nanyang Technological University (National Institute of Education) in Singapore.

Data Collection and Analysis

Students were given one week to complete their take-home survey. The survey had to be completed in one-sitting within a duration of 150 minutes. This arrangement was based on mutual trust and a belief in students' integrity. The rationale for allowing a take-home survey was due to a tight curriculum schedule at school, as they were preparing for their end-of-grade 10 national examination. The researcher collected the completed surveys from the students a week after they were given out.

The solutions provided by the students were coded using Harel's (2007) proof schemes shown in Table 1. The two coders were both experienced mathematics teachers with deep mathematical and pedagogical knowledge for teaching of mathematics at the secondary school level. The simple percentage agreement (McHugh, 2012) between the 2 coders was near perfect (greater than 85%) for all solutions provided by the students. Figures 2 and 3 show sample solutions of students for questions 2 and 7 respectively, and the proof schemes adopted by the students in their solutions together with the proof schemes identified by the coders.

Figure 1 illustrates that Student 1-1 employed ritualistic, non-referential symbolic, inductive, and transformational proof schemes in her solution. The ritualistic nature of her approach was evident in the formation of an equation rather than demonstrating that the difference between the products of opposite corners of the square is always 8, as required. Her assumption that $x = 1$, followed by the formation of an equation summing four arbitrary numbers within a 2×2 grid to 260, reflects a non-referential symbolic scheme. Additionally, the lack of distinction between x as a constant (1 in this case) and as a variable further exemplifies this scheme. The inductive proof scheme was apparent when she attempted to identify a pattern by computing the difference between vertically adjacent numbers (e.g., $9-1=8$), although this was not applicable in the given context. Her final reasoning suggested an attempt at proof by contradiction, as indicated by the conclusion that x resulted in a non-integer, which contradicts the problem constraints. The transformational scheme was evident in her indirect but logically structured approach.

Student 7-2, in contrast, demonstrated perceptual, transformational, and axiomatic proof schemes. The perceptual scheme was reflected in his introduction of the variable x without explicit definition, relying on an implicit assumption that x must be a positive integer. Similar to Student 1-1, he successfully transformed the problem and proved it by contradiction. His axiomatic proof scheme was observed in the logical deductions made in part (a) of the question.

Figure 1

The Coded Solutions of Students 1-1 and 7-2 for Question 2

Question

2. The diagram shows part of a number grid.

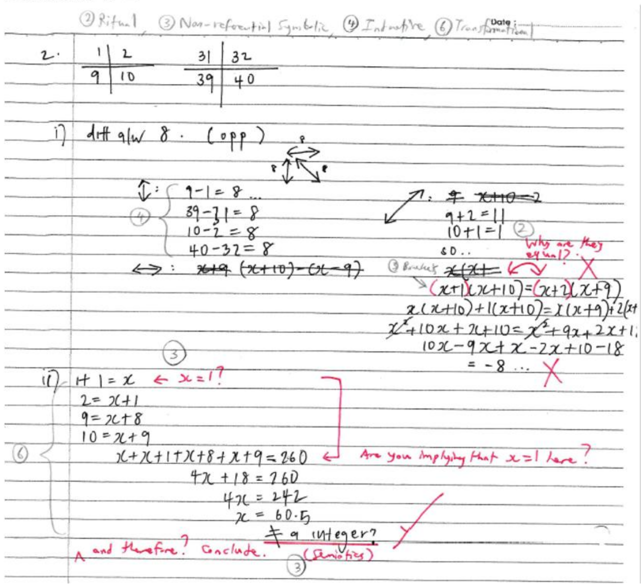
1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
…..							

A square outlining four numbers, as shown, can be placed anywhere on the grid. Above are 2 examples of such a case.

(i) Show that the difference between the products of the numbers in the opposite corners of the square is always 8.

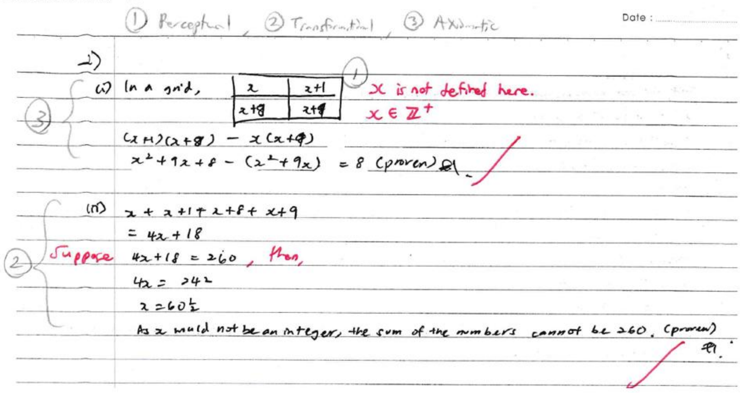
(ii) Show that the sum of the four numbers in the square cannot be 260.

Student 1-1



1. Authoritative
2. Ritual
3. Non-Referential Symbolic
4. Inductive
5. Perceptual
6. Transformational
7. Axiomatic

Student 7-2



1. Authoritative
2. Ritual
3. Non-Referential Symbolic
4. Inductive
5. Perceptual
6. Transformational
7. Axiomatic

Figure 2 illustrates that Student 2-1 employed ritualistic, non-referential symbolic, and perceptual proof schemes in his solution. The ritualistic aspect of his approach was evident in the incorrect application of the Pythagorean Theorem when the problem required the use of its Converse. The non-referential symbolic scheme emerged in his statement, *"∠ABC* = Right angled triangle*"* erroneously equating an angle to a geometric figure. Additionally, his perceptual proof scheme was reflected in his assertion that *"ACB* should form a right-angled triangle.*"* This statement suggests an internal assumption that *ACB* represents a triangle rather than an angle, as he did not explicitly define it. Furthermore, rather than proving the claim as required, he treated it as a given.

In contrast, Student 6-2 demonstrated perceptual and axiomatic proof schemes. His perceptual reasoning was evident in the sketching of a distorted yet conceptually intended circle passing through points *A*, *B*, and *C*. Despite minor errors in his argument, the overall approach was axiomatic, as demonstrated by his application of the theorem stating that if the product of the gradients of two lines equals -1, then the lines must be perpendicular.

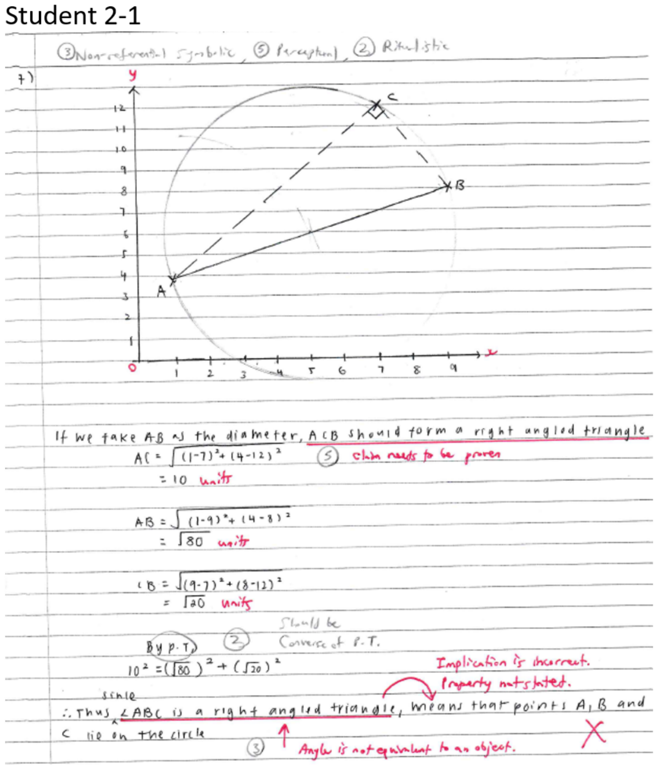
Figure 2

The Coded Solutions of Students 2-1 and 6-2 for Question 7

Question

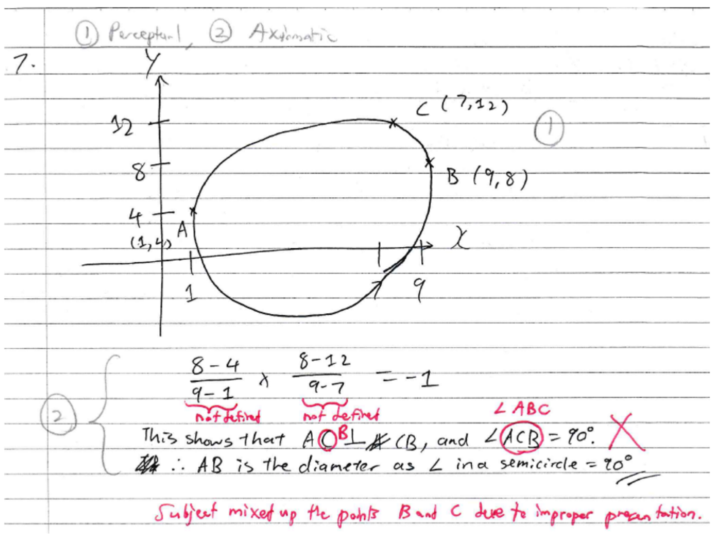
7. Three points are given by $A(1,4)$, $B(9,8)$ and $C(7,12)$. Explain why A, B and C lie on a circle with exactly two of these points forming the diameter.

Student 2-1



1. Authoritative
2. Ritual (coded)
3. Non-Referential Symbolic (coded)
4. Inductive
5. Perceptual (coded)
6. Transformational
7. Axiomatic

Student 6-2



1. Authoritative
2. Ritual
3. Non-Referential Symbolic
4. Inductive
5. Perceptual (coded)
6. Transformational
7. Axiomatic (coded)

Findings and Discussion

Question 2 (EM Number and Algebra strand) involves concepts in Number Theory. Students were required to recognise the number pattern in the number grid given and prove in general that the difference between the products of the numbers in the opposite corners of the square is always 8 (for part (i)) and that the sum of the four numbers in the square cannot be 260 (for part (ii)). The question required students to generalise and find an association between the four terms in each 2×2 grid and form the relevant equations to solve the problem. Table 2 summarises the proof schemes the eight students displayed in their solutions.

Table 2

Summary of Students' Proof Schemes for Question 2

Main	Sub	1-1	2-1	3-1	4-1	5-2	6-2	7-2	8-2
External Conviction	1. Authoritative								
	2. Ritual	√							
	3. Non-Referential Symbolic	√				√			
Empirical	4. Inductive	√		√	√				√
	5. Perceptual				√	√		√	√
Analytical	6. Transformational	√	√	√	√		√	√	
	7. Axiomatic		√	√			√	√	

Legend: √ indicates presence of respective proof scheme

It is apparent from Table 2 that the proof schemes which occurred most regularly were the Inductive (4 out of 8 times), Perceptual (4 out of 8 times), Transformational (5 out of 8 times) and Axiomatic (4 out of 8 times) proof schemes. Evidence of the inductive scheme was seen in the solutions of students trying to verify that a statement is true using particular values instead of proving the general case. Students 1-1, 3-1, 4-1 and 8-2 used arbitrary starting values and tried to explain that the statement was true in general by verifying only a few instances.

Evidence of the perceptual proof scheme was seen in the solutions of students who either misunderstood the problem or perceived the nature of the variables used in their generalisation. Student 3-1 misunderstood that the number grid is limited to the 40 numbers shown when it was supposed to be unrestricted. Students 4-1 and 7-2 did not define the variables used in their general terms. Student 5-2 took the vertical difference in each column instead of considering the difference in the product of the diagonal entries. Evidence of the transformational proof scheme was seen in all the solutions of the students except students 5-2 and 8-2. Most of them managed to transform the problem into one that required proof by contradiction and correctly proved the result. The axiomatic proof scheme was evident in the solutions of students who solved the problem using generalised terms and strung together a logically well-connected argument.

Question 7 (AM Geometry and Trigonometry) required students to explain that points *A*, *B* and *C* lie on a circle with exactly two of these points forming the diameter. They were expected to show that the triangle *ABC* is a right-angled triangle in a semi-circle with diameter *AC*. They could do this by either showing that line *AB* is perpendicular to line *BC* or stating the converse of the Pythagoras Theorem. Table 3 summarises the proof schemes the eight students displayed in their solutions.

Table 3

Summary of Students' Proof Schemes for Question 7

Main	Sub	1-1	2-1	3-1	4-1	5-2	6-2	7-2	8-2
External Conviction	1. Authoritative								
	2. Ritual	√	√	√	√				
	3. Non-Referential Symbolic		√		√				
Empirical	4. Inductive								
	5. Perceptual		√		√	√	√	√	√
Analytical	6. Transformational								
	7. Axiomatic			√			√	√	

Legend: √ indicates presence of respective proof scheme

It is apparent from Tabe 3 that the proof schemes which occurred most regularly were the Ritual (4 out of 8 times) and Perceptual (6 out of 8 times) proof schemes. Evidence of the ritual proof scheme was observed in the solutions of students who did not have a clear idea as to what was necessary to prove the problem or assumed the result that needed to be proved. Students 1-1 and 3-1 found all the three gradients, to observe which paired product will give an output of -1, when it was only necessary to prove that *AB* is perpendicular to *BC*. This was intuitively clear from the respective positions of the three points on the Cartesian plane. In student 4-1's initial attempt, he incorrectly assumed that points *A*, *B* and *C* lay on the circle and used that assumption to find the length of the circle's diameter, and hence, its equation.

Evidence of the perceptual proof scheme was seen in the solutions of students who were not rigorous enough in their proof or had misconceptions about the nature of what a proof should be. Students 2-1 and 8-2 showed that triangle *ABC* is right-angled by showing that the converse of the Pythagoras theorem holds without explicitly stating which angle is right-angled. Students 4-1, 6-2 and 7-2 used diagrams, according to their own perceptions, to explain their proof. While a diagram could have been used as an aid to their understanding, it should not have been referenced in the proofs.

It is apparent from the proofs written by the eight students for questions 2 and 7 that none of the students adopted the authoritative proof scheme in their solutions. For question 7, none of the solutions the students wrote displayed the inductive or transformational proof schemes. However, for question 2, the inductive and transformational proof schemes were present in the solutions of four or more students. It appears that the nature of the questions does lend students to varied proof schemes. For both the questions, several student solutions displayed the ritual, non-referential, perceptual and axiomatic proof schemes.

It was also apparent from the proofs written by the students that they struggled with articulating their arguments using correct mathematical language. They misused symbols, connectors, or semantics, leading to incorrect or incoherent explanations. At times when they relied on inductive reasoning, they attempted to verify a statement by using specific values or examples instead of establishing a general case. They made assumptions or drew conclusions without properly defining variables, proving statements, or considering all possible cases. For question 2, where six out of the eight students demonstrated the transformational proof scheme, they successfully transformed a problem into one that required a different approach or proof technique. Students did rely on memorized steps or formulas without fully understanding when and how to use them when proving and this was apparent when they adopted the ritual proof scheme.

As the students who participated in the study were at the end of their ten years of schooling in a secondary school in Singapore, it is apparent from their proof schemes that they lacked the knowledge and skills for the construction of robust proofs. This has implications for secondary school mathematics teachers in the school where the study was carried out.

Concluding Remarks

The findings of this study have implications for teachers enacting the school mathematics curriculum in secondary schools, specifically in Singapore. There appears to be a need for teachers to focus on the construction of rigorous proofs by students. They may do this via explicit demonstration of proof types and construction of mathematical arguments using accurate mathematical language. Proving, like other mathematical processes such as evaluating and solving must be placed at the core of mathematics instruction as proofs are the foundational building blocks of mathematics.

Acknowledgement

The study reported in this paper was granted approval (IRB-2021-02-030) by the Research Integrity and Ethics Office, Nanyang Technological University (NTU) Institutional Review Board. No funding was involved as the study was the PhD thesis of Navinesh Thanabalasingam at the National Institute of Education, Nanyang Technological University, Singapore.

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