Top Number or Numerator? Teachers' Choices of the Mathematics Register in Mathematics Teaching Talk

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Recent research has called for more attention in the understanding and developing of mathematics teaching talk, focusing on the specific use of language as a resource to help students learn mathematical concepts. Based on a task-based interview with eleven teachers, this paper attempts to utilise the Mathematics Register Knowledge Quartet to examine what these teachers know and may do when teaching fractions. Consequently, the analysis of the teachers' responses, in relation to the four dimensions of the quartet, provided insights on how teachers might attend to or use (or not) the mathematics register in the mathematics teaching talk of fractions and related concepts.

Language as a Resource for Teaching and Learning Mathematics

Though language as a research focus in mathematics education is a relatively young sub-field, a wide range of work has been done in this area for at least the last half of the century. This included research such as those focusing on language's nature and role in mathematics teaching and learning and those focusing on the sociopolitical dimensions of language in mathematics education (Barwell, 2021). Research perspectives on the role of language in mathematics education have also evolved from being deficit-oriented initially (e.g., Cummins, 1979, as cited in Schütte, 2018) to more resource-oriented (e.g., Adler, 2000; Planas, 2018).

The Mathematics Register (*vis-à-vis* Mathematical Language)

Particularly, the mathematics register (Halliday, 1975), often synonymously referred to as the formal or school mathematical language, has been largely researched upon, in terms of its role as an important resource in the developing and acquiring mathematical ideas or concepts (e.g., Sigley & Wilkinson, 2015). Being a complex notion, a register is often misunderstood simply as a collection of highly technical vocabulary terms in the language used to describe objects in an academic discipline (Halliday, 1975). In the same manner, the mathematics register is not just a collection of mathematics-related words or terms. Instead, the mathematics register further determines how these words or terms are used or structured within the natural language to form unique phrases or clauses that can precisely represent and communicate both explicit and implicit mathematical meanings or relationships (Wilkinson, 2018).

To sum up, the mathematics register serves the function of thinking about (and communicating in spoken or written forms) mathematical ideas and meanings (Pimm, 1987; Vygotsky, 1934/1986). Consequently, the mathematics register is a rich and useful resource for mathematics teaching (and learning), particularly in mathematics teaching talk (as defined by Planas et al., 2023a), which has the potential to "open up or close opportunities for participating in the mathematical discourse for school learners" (p. 522). However, this is only possible if teachers have actually developed an understanding of "the forms and the meanings and ways of seeing enshrined in the mathematics register" (Pimm, 1987, p. 207) or have internalised the mathematics register as a tool for thinking for themselves (Wilkinson, 2018), such that they could appropriately mediate its use in learning processes. Unfortunately, this is not necessarily true for all teachers. For instance, in a study (Lane et al., 2019) on pre-service teachers' mathematics register proficiency, it was found that these teachers generally do not have sufficient knowledge and understanding of the mathematics register. Similar findings were observed in the case study of six early career teachers' understandings of the register (Turner

(2025). In S. M. Patahuddin, L. Gaunt, D. Harris & K. Tripet (Eds.), *Unlocking minds in mathematics education. Proceedings of the 47th annual conference of the Mathematics Education Research Group of Australasia* (pp. 429–436). Canberra: MERGA.

et al., 2019), which resulted in variations in terms of their practices in using the mathematics register as a resource for teaching. While both studies presented what could be seen as deficit-oriented narratives of pre-service or beginning teachers' mathematics register proficiency, they also highlighted the importance of thinking about how the mathematics register can be better used as a resource in both mathematics and mathematics *teacher* education.

The Mathematics Register Knowledge Quartet (MRKQ)

In the same study by Lane and colleagues (2019), a framework adapted from the Knowledge Quartet framework (Rowland et al., 2005) was conceptualized with the intent of analysing pre-service teachers' ability in using the mathematics register to facilitate a peer-teaching segment of a teachers' education course. In particular, they (Lane et al., 2019) argued that, while there has been much research done in terms of pre-service teachers' mathematical knowledge for teaching (e.g., Ball et al., 2008), little attention has been given to mathematics language as being part of knowledge for teaching, though it is deemed as an essential component of mathematics teacher education (Cramer, 2004, as cited in Lane et al., 2019). Hence, they attempted to provide a glimpse into pre-service teachers' knowledge of the mathematics register by "redefining" how teachers' knowledge of the mathematics register may look like in relation to the four dimensions of the Knowledge Quartet (Rowland et al, 2005). A full illustration of the MRKQ with examples can be found in their study (see Table 1 in Lane et al., 2019, p. 793). Briefly, the four dimensions of the MRKQ are defined as follows:

- *Foundation* dimension focuses on teachers' knowledge and understanding of the register and their awareness of differences between everyday language and the register.
- *Transformation* dimension focuses on teachers' *knowledge-in-action*[5] through their planning and actual teaching, in terms of how they plan for mathematical language and use representations and analogies to elicit mathematical meaning for students.
- *Connection* dimension focuses on teachers' consistency in the use of the register within and between lessons, and across different mathematics topics, coupled with an awareness of students' difficulties with the register.
- *Contingency* dimension focuses on teachers' *knowledge-in-interaction*[5] observed through their abilities to interpret students' register in line with the mathematics register and facilitate an adherence to the mathematics register during classroom interactions.

Notably, both Knowledge Quartets by Rowland et al. (2005) and Lane et al. (2019) were primarily developed for the purpose of understanding pre-service teachers' knowledge for teaching and their teaching practices. However, I see value in how the constructs within the four dimensions can similarly be used to better understand the existing state of experienced teachers' knowledge of the mathematics register, as well as, how and why their mathematics teaching talk are shaped in a certain manner. As argued by Rowland et al. (2005), the Knowledge Quartet "provides a means of reflecting on teaching and teacher knowledge, with a view to developing both" (p. 257) – a framework to analyse and discuss teachers' mathematical content knowledge. Correspondingly, the MRKQ (Lane et al., 2019) can be used as a frame to think about mathematics teaching practices with a focus on teachers' knowledge of the mathematics register. Thus, in this paper, I hope to inform the existing state of how teachers are attending to language as a resource in their classroom – an aspect which appears to be lacking in recent research – using the MRKQ. Specifically, I looked at the following research question: How are teachers' knowledge and potential usage *(knowledge-in-action* and *knowledge-in-interaction)* of the mathematics register featured through their responses to a task on fractions?

[5] These terms *knowledge-in-action* and *knowledge-in-interaction* were defined in the original Knowledge Quartet (Rowland et al., 2005).

Method and Data

The data for this paper was part of my PhD research study, which investigated the same phenomenon in greater depth and scope. To address the research question, an overall qualitative approach was taken to collect and analyse data. The data was collected through semi-structured interviews with nine experienced mathematics teachers based in Canada in the first quarter of 2022. All the teachers had taught or were still teaching mathematics in English-medium classrooms, ranging from elementary to tertiary levels. As the data was collected during the early post-COVID19 pandemic period, I was limited in terms of my research approach, due to restrictions on face-to-face interactions with my participants. Hence, I designed a task-based interview protocol where the teachers were asked to respond to a series of eight tasks. These tasks were created and presented in the form of a fictional dialogue between a pair of student characters to illustrate classroom-based situations which might lead to language-related dilemmas and challenges when using the mathematics register relating to a range of concepts across different content foci (numbers, fractions, geometry and graphs). After reading each task, the teachers were asked to reflect upon what they noticed about the language used by the students and how they would respond if they were the teacher in that situation. Further details on the specific tasks and the methodology can be found in my doctoral thesis (Tiong, 2024).

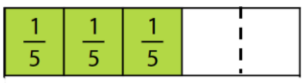
Considering the size of this paper, the remaining discussion is based on interview data for the task which focused on the concept of fractions – an important yet challenging mathematical concept for many children and even adults (e.g., Kieran, 1992; Siegler & Lortie-Forgues, 2017). In a review conducted by Siegler and Lortie-Forgues (2017), they attributed the difficulties of the fraction concept to *inherent* factors such as its confusion with whole number arithmetic. and culturally contingent factors such as teachers' knowledge of fractions and the teaching of fractions. For example, students tend not to understand or view fractions as numbers – an important aspect in the concept of fraction which is often neglected. Instead, they may consider a fraction as being a mathematical object which is made up of two separate whole numbers. This misconception is often compounded with the colloquial use of the *top number* and the *bottom number* when referring to the *numerator* and the *denominator* of a fraction respectively. The following task simulates a fictional dialogue between two elementary-level students (S1 and S2) who are working with fraction strips to show the fraction $\frac{3}{5}$ [see Figure 1], as follows:

S1: Because the bottom number is five, we need to use the green piece (which denotes the fraction $\frac{1}{5}$).

S2: And we need three of them to get the top number three.

Figure 1

Diagram of Fraction Strips Given in the Task



Here, S1 and S2 seemed to understand what each other meant by the *top number* and the *bottom number* when discussing about how they should represent the fraction $\frac{3}{5}$, using the fraction strips. In other words, they appeared to have a common student mathematics register, which is more colloquial, in relation to fractions. While there did not seem to be any apparent misconception due to the use of the student mathematics register (which differs from the formal mathematics register) in the task, their understanding of the concept of fractions, due to the use of colloquial terms, was up for interpretation. In particular, the task aimed to prompt the participants to think about whether (and how) they would mediate the use of language in this instance and if such language use was appropriate in the mathematics teaching talk of fractions.

Teachers' Choices of Mathematics Teaching Talk on Fractions

In analysing the teachers' responses, I used the four dimensions of the MRKQ as the lens to identify quotes which were indicative of the teachers' knowledge and use of the mathematics register in the mathematics teaching talk of fractions, based on how they would respond to these tasks – their articulated *knowledge-in-action* and *knowledge-in-interaction*. Notably, most quotes identified are not unique to only one dimension, but often relevant to other dimensions of the MKRQ (Rowland et al, 2005). The following is an example of a quote, which showed a participant's understanding of the mathematics register surrounding the concept of fractions within the *Foundation* dimension, as well as her ability to interpret students' register and inferred their understanding of fractions within the *Contingency* dimension.

> […] there is no reference to that relationship, part-whole. So it is like that they are separate but procedurally. But they will see that the bottom, it should be divided to that one in the top, it should be divide. […] so it feels like they didn't build a meaning for this part-whole relationship here […] they don't develop this relationship, they see them as separate numbers.

The subsequent discussion is substantiated with teachers' (identified with participant numbers T1 – T9, in order of appearance) responses where relevant.

Foundation Dimension

All participants were cognizant of the mathematics register, such as *numerator*, *denominator*, *parts* and *whole*, surrounding the concept of fractions. They also showed understanding of what *numerator* and *denominator* referred to or meant in relation to the relative positions in the fraction representation, analogous to the top and bottom notions that the students ascribed to in the task. The part–whole relationship in fractions was also referred to when they elaborated on the meaning of the two words. For example, T1 defined a fraction as "pieces of a whole, where the whole is the denominator and the number of pieces you have is the numerator". T2 stated that the denominator represents "how many pieces the whole [is] divided into". Their explanations mostly focused on the definitions of *numerator* and *denominator* in the case of proper fractions, which might not be applicable to other types of fractions such as improper fractions or compound fractions. Yet, differences were observed in how they described the concept of fraction. For instance, only T3 and T4, explicitly mentioned the idea of fractions as numbers. Specifically, T3 articulated that she would want her students to "think of one-fifth as the number, not like the five as a number and the one is a number". This concept has implications on how students may understand what the numerator and the denominator mean in relation to any given fraction due to its symbolic representation.

In terms of the awareness of the differences between the everyday and the mathematics registers, several teachers hinted at how there might be a disconnect in terms of how fractions are used within the everyday and the mathematics register. For example, T5 mentioned how "there's a gap between how we talk about fractions in real life, versus the terminology", when asked if she would be concerned with students using terms such as *top* and *bottom numbers*, instead of *numerator* and *denominator*. She further highlighted how terms specific to the concept of fractions, are rarely mentioned or heard outside of the mathematics classroom. T5's viewpoint was similarly shared by T4, who added that perhaps even *top number* and *bottom number* did not seem to be common terms that would be heard in the everyday context. However, to T4, these terms seemed more concept-related, and she wondered "where this language (*top number* and *bottom number*) comes from". She highly doubted that it would be "out of their natural home experience" and elaborated that parents would more likely be using names of specific fractions (e.g., a half or a third) if they were to teach or talk to their children about fractions. One might more likely hear a phrase such as "cut that apple in half", than a discussion on the *top* and *bottom numbers* of a fraction, in the everyday context.

Transformation Dimension

As most teachers focused on the use of *top* and *bottom numbers* as being indicative that the students likely understood the fractional representation, there was not much in-depth discussion of the approaches to how they would specifically plan for the teaching of fractions. Primarily, their articulated *knowledge-in-action* related to how they would introduce the mathematics register, specifically *numerator* and *denominator*, in relation to the task. Interestingly, though the concept of fractions is often taught with representations and analogies, most teachers did not go into that discussion. Instead, the interviews revolved around whether and how they would teach words in the register. In retrospect, I wondered if it was due to the design of the task, or due to how the connection between the learning of language and the learning of mathematics concepts might have gone unnoticed by the participants. Nonetheless, there was one teacher, T2, who elaborated on how she would use other representations to help students develop deeper understanding of the part–whole relationship in the concept of fractions. She felt that the brief dialogue between S1 and S2 was insufficient to inform her about their understanding of fractions. Other than asking students to explain what they meant, with reference to "what's the part and what's the whole", she mentioned the importance of using different (and atypical) representations to reinforce their understanding of the part–whole relationship. She further described how she would "switch up the wholes and switch up the parts", by using hexagons to represent the whole and parts of the whole, instead of the typical pie or number strip.

Connection Dimension

With the interview data to the task alone, it was insufficient to ascertain the teachers' use of the mathematics register across topics and lessons. However, some observations relating to their consistency in the mathematics register relating to the concept of fractions could still be made. Notably, the teachers seemed to be less consistent in terms of their use of the mathematics register while relating to the concepts of fractions. Many teachers mentioned (implicitly and explicitly) that they would not mind the student mathematics register presented in the task and even code-switch between the student mathematics register and mathematics register. For example, T6 would "call it the *bottom number*" as she introduced the concept of the *denominator*. Subsequently, while she would model the term *denominator* in her teaching, she would constantly remind students that it "means the *bottom number*", until they become "comfortable with the math". In contrast, only T3, who explicitly mentioned the idea that fractions should be seen as numbers, shared that she was not comfortable with students using *top* and *bottom numbers*, in the context of fractions. To her, the student register, in this case, would likely be inconsistent with their prior or future understanding of rational numbers.

In relation to why the students might have chosen to use *top* and *bottom* numbers, rather than *numerator* and *denominator*, all the teachers attributed it to at least one of the following three difficulties – the unfamiliarity with the terms *numerator* and *denominator*, the perceived lack of usefulness of *numerator* and *denominator*, and the challenge with describing *numerator* and *denominator* other than as numbers. Firstly, the unfamiliarity with the terms were pointed out by both T4 and T7. Notably, the terms *numerator* and *denominator* are rarely used in the everyday context and very specific to the concept of fractions in the mathematical context. As such, students would mostly likely think of them as analogous to unfamiliar "foreign language". Secondly, some teachers (e.g., T2 and T5) voiced their opinion regarding the perceived lack of usefulness of these terms, although they valued the precision of the mathematics register. To them, learning the terms might not necessarily be helpful in developing students' understanding. Possibly because the part–whole relationship is not obvious in the terms *numerator* and *denominator*, T5 even commented how she would not "necessarily think that the terminology itself is super-useful". Lastly, T3 shared her struggled attempt to avoid referring to *numerator* and *denominator* as two separate numbers. She commented how "the language here is tricky" and questioned how teachers could name and refer to *numerator* and *denominator* without using the word *number*. Specifically, she asked, "one-fifth is a number, so what should we call that bottom five?" Consequently, it made me reflect about how many teachers (including myself) might likewise refer to the one and five in the fractional representation for one-fifth as numbers, for lack of a better term.

***Contingency* Dimension**

Intentionally, the design of the task focused the teachers' attention to the student mathematics register (*top* and *bottom numbers*) and how they interpreted it with the formal mathematics register (*numerator* and *denominator*). In responding to what they would do as a teacher in this task, the extent to which they would adhere to the mathematics register during their interactions with students varied, pending different considerations. Generally, all the teachers were able to interpret how the students were referring to the *numerator* and the *denominator* when they used *top number* and *bottom number* respectively. Two teachers further shared their inference of the two students' understanding of fractions, based on their more colloquial mathematics register. For instance, T7 felt that by using "*bottom number* is five, *top number* is three" in the description of the fraction $\frac{3}{5}$ indicated a rather procedural understanding of fractions. The students seemed to be thinking how "the only relationship between the numbers is where they stay, top or bottom". This lack of "reference to that part–whole relationship" further reinforce the possibility that they were seeing the *numerator* and the *denominator* as separate numbers. On the same note, T4 commented, "it is clear here from this discourse that fraction is not seen as a number, it's seen as something separate".

In terms of how the teachers would facilitate an adherence (or not) to the mathematics register in mathematics teaching talk on fractions, many shared that they would model the use of the mathematics register. However, most of them would not necessarily correct the students' usage of *top* and *bottom numbers* in this instance. They would instead consider students' readiness and whether they might still be at the early stages of learning about fraction concepts. Some teachers were even doubtful about the connection between the students' usage of the mathematics register and their understanding of the concepts as they considered these terms as specialised terms which only exist in the mathematics register. For example, T7 was not confident that students who "say denominator" would demonstrate a different (or higher) level of understanding as compared with those "who would describe it as a *bottom number*" by explaining how these terms "are so mathematical, and it's not used in daily life that much".

Consequently, most teachers would likely push for a greater adherence to the mathematics register after the students developed a more conceptual understanding of fractions. They generally agreed that mathematics register could be acquired gradually when students are ready. For instance, T3 commented how "the class would eventually pick up on that" while T8 suggested that the mathematics register "would be something to develop and grow" as students become familiar with fraction concepts but "not at the elementary level" necessarily. Similarly, T6 would "step in and actually emphasise the correct word because they have the conceptual understanding" and encourage them to "practise using it", since "they're both there and they both can practise at the same time". T6's actions would probably resonate with T9 who strongly believed that to "build our knowledge base, we have to know the language as well" and explained, "if they don't use these terms, how are they going to get comfortable with them?"

Discussion and Conclusion

Collectively, all the experienced teachers demonstrated an understanding of the content-specific mathematics register related to the concept of fractions within the *Foundation* dimension of the MKRQ (Lane et al., 2019). However, this knowledge might not be translated into actions as they would still plan for the teaching of fractions without necessarily involving the mathematics register (*Transformation* dimension). Moreover, there appeared to be a consensus in how they would mediate the use the content-specific mathematics register in this case. The teachers would generally model the use of the specialised terms, *numerator* and *denominator* while code-switching with the use of *top number* and *bottom number* (*Connection* dimension). They would also not expect or insist that their students use these terms, which are seen as specialised within the mathematics register (*Contingency* dimension).

While I am cognisant that mathematics teaching talk can occur without the use of the mathematics register[6], I wonder if the lack of emphasis on the use of the register in mathematics teaching and learning processes might reduce the opportunities for students to engage in richer discussions and deeper reasoning around fractions. As proposed by Kieren (1992), a potential gap exists between students' "conceptual models of fractional numbers" (p. 326) and their experiences working with the symbols of fractions. On the same note, I wonder if a gap may also exist between students' conceptual understanding and their language or verbal experiences with fractions. Notably, studies (e.g., Siegler & Lortie-Forgues, 2017) have suggested how the mathematics register in some East Asian languages, such as Korean and Mandarin, might better develop students' understanding of fractions as the meanings of *numerator* and *denominator* are explicated more clearly through the way fractions are named in these mathematics registers.

Hence, in addressing the possible misconception where fractions are often not considered numbers by students, it may be necessary for teachers to bring in the etymology of the terms *numerator* and *denominator* in their mathematics teaching talk. Specifically, the word *numerator* has origins related to ideas of enumerating or counting, while the word *denominator* is linked to the act of naming. In other words, the *numerator* enumerates the number of *denominations* that has been named in the fraction. Moreover, this understanding can be further extended to that of rational numbers where integers are fractions that enumerate in denominations of ones while fractions such as $\frac{3}{5}$ enumerate in denominations of fifths. An emphasis on the *counting* of the respective denominations named in different fractions may in turn help students perceive fractions as numbers (as single entities) instead of viewing the "numerator and denominator as two independent parts" (González-Forte et al., 2018, p. 465). On the same note, Planas and colleagues (2023a) also argued how there might be "meaning encoded in the mathematics register" (p. 524) that requires teachers to decode for students during mathematics teaching talk to allow students "who have little to no experience with making specialised word use function in order to discuss and reason mathematically" (p. 523).

Fundamentally, such an approach may also help to strengthen the multiplicative relationship between the numerator and the denominator which students tend to struggle with – as seen from their erroneous thinking in the comparison of fractions where they either compare numerators and denominators separately or compare the gap between them. By speaking of and seeing a fraction as enumerating a certain denomination (equivalently, m times of the $\frac{1}{n}$ unit in the fraction $\frac{m}{n}$), it might lower the likelihood of students reasoning additively with the difference between the numerator and the denominator. Consequently, this approach may address the

[6] In their recent discussion piece, Planas and Pimm (2023b) argue for the potential of developing the notion of a *mathematics communication register*, which encompasses both linguistic (e.g., the mathematics register) and non-linguistic (e.g., gestures) aspects of mathematical communication.

perceived lack of usefulness or relevance of the terms *numerator* and *denominator* by decreasing the seemingly arbitrary nature of these specialised terms in the eyes of both teachers and students. Hence, a next step of my research will be to expand the dimensions of the MRKQ to include an additional focus on the integration of knowledge relating to the etymology of specialised terms in the mathematics register within topic-specific mathematics teaching talk.

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