Fostering Students' Mathematical Understanding through Problem-Posing-Based Learning (P-PBL)

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This paper examines the nature of mathematical understanding and explores how Problem Posing-Based Learning (P-PBL) can foster it using a framework proposed by Cai et al. (2026) that conceptualizes mathematical understanding through five interconnected aspects: factual, procedural, conceptual, metacognitive, and affective. It also presents five lenses for examining and fostering understanding: mathematical, cognitive, social, diagnostic, and instructional. Future directions for research are proposed, including the use of the proposed framework to synthesize existing studies using AI and further investigate how P-PBL can support mathematical understanding.

What is mathematical understanding? How should we foster students' mathematical understanding? These are the two most debated questions in mathematics education (Cai et al., 2026), and they are important because fostering students' mathematical understanding is a widely accepted goal in the teaching and learning of mathematics (Hiebert & Capenter, 1992). In this paper, I will first discuss a framework to understand mathematical understanding and then discuss how Problem Posing-Based Learning (P-PBL) can foster students' understanding.

What Is Mathematical Understanding?

Processes versus Products

Mathematical understanding is a critical topic that has been extensively studied at least since Brownell (1935). There are at least two major debates surrounding mathematical understanding. The first is the product-process debate. Greeno (1978) stated that understanding is a process leading to "a structure that represents the relations among components of the idea that is understood" (p. 267). Similarly, Pirie and Kieren (1992, 1994) viewed mathematical understanding as a dynamic and continual process of organizing one's knowledge structure. Other researchers have used a product view to examine understanding. For example, Davis (1992) perceived understanding as "when a new idea can be fitted into a larger framework of previously assembled ideas" (p. 228). Hiebert and Carpenter (1992) defined mathematical understanding as an internal network of representations of mathematical ideas, procedures, and facts in which there are two kinds of connections: one based on the similarities and differences between different external representations or within the same form of representation and the other referring to hierarchical relationships, such as the ones between specific and general cases. The more and stronger the connections one constructs, the deeper the mathematical understanding they achieve. The word "understanding" has dual meanings, being both the present participle of the verb "to understand" (process) and a gerund acting as a noun (product).

Procedural versus Conceptual

The second major debate surrounding mathematical understanding is the procedural-conceptual debate. Hiebert and Lefevre (1986) defined conceptual understanding in terms of the richness of relationships and procedural understanding as one's familiarity with individual symbols or procedures/rules for solving mathematical problems. In the National Assessment of Educational Progress (NAEP), procedural and conceptual understandings refer to two fundamental but distinct types of mathematical knowledge tested to foster a student's mathematical understanding. Procedural understanding refers to the ability to select, apply, and

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execute mathematical rules, algorithms, and procedures correctly and efficiently. It includes executing computations, following step-by-step formulas, and using standard notations. In contrast, conceptual understanding refers to deep comprehension of underlying mathematical principles, operations, and relationships. Rather than just memorizing steps, it focuses on understanding why a formula works and how to apply those ideas to new or real-world problem settings (National Assessment Governing Board, 2002). These definitions were challenged by Star (2005), who argued that the existing literature had emphasized and valued conceptual understanding while overlooking procedural understanding. His view is consistent with the 2001 National Research Council report *Adding It Up*, which uses procedural fluency, defined as skill in carrying out mathematical procedures flexibly, accurately, efficiently, and appropriately. In this view, procedural understanding is not rote memorization of algorithms but rather a dynamic understanding of *when* and *how* to use mathematical algorithms and rules.

Features of Mathematical Understanding

Based on a literature review, Cai and Ding (2017) pointed toward the field's agreement on the following: (1) Understanding is conceptualized as both a process ("understanding" as a verb) and a result of that process ("understanding" as a noun); (2) understanding is both the making of connections and a result of connection making in a social environment; (3) understanding is a dynamic and continual process; (4) understanding may have different levels and kinds; and (5) the goal is to reach a deep level of mathematical understanding.

How to Foster Students' Mathematical Understanding?

Two Perspectives of Learning

There are two perspectives of learning which can explain ways of fostering students' mathematical understanding (Cobb, 1994). For example, in solving mathematical problems, students can use any approach they can think of, draw on any piece of knowledge they have learned, and justify their ideas in ways they feel are convincing. Learning takes place during the process of students constructing their own problem-solving solutions. In the meantime, students can share their own solutions in the classroom with their peers. The learning environment provides a natural setting for students to present various solutions to their group or class and learn mathematics through social interactions, meaning negotiation, and reaching shared understanding. Such activities help students clarify their ideas and acquire different perspectives of the concept or idea they are learning (Cai, 2003).

The same can be exhibited in P-PBL (Cai, 2022). When students pose their own mathematical problems based on a situation, they must make sense of the constraints and conditions from the given information to build connections between their existing understanding and a new understanding of related mathematical ideas (Cai & Rott, 2024). Although problem-posing activities are cognitively demanding tasks, they are adaptable to students' abilities and thus can increase students' access such that students with different levels of understanding can still participate and pose potentially productive problems based on their own sensemaking. Once students share their posed problems in the class, they not only clarify the problems they posed but also discuss possible ways of solving those posed problems.

Teachers' Learning Perspectives

In the past, mathematical understanding has mainly represented researchers' perspectives (Davis, 1992; Hiebert & Carpenter, 1992; Pirie & Kieren 1992, 1994; Richland et al. 2012; Simon, 2006). Given the important role of teachers in supporting learning, Richland et al. (2012) proposed building pedagogical frameworks that address how teachers process knowledge. They suggested the development of classroom routines that would facilitate students' mathematical understanding, including worked examples and connection making.

Cai and Ding (2017) reported that Chinese mathematics teachers emphasize the view that understanding is a web of connections, which results from continuous connection making. In contrast to the popular view which separates understanding into conceptual and procedural, Chinese teachers perceive understanding in terms of concepts and procedures. To achieve mathematical understanding, Chinese teachers take a more practical approach by making connections. Such perspectives of classroom practitioners not only inform the long-debated issue of the meaning of mathematical understanding but also provide ways to achieve it.

A Framework on Mathematical Understanding

It is quite useful to examine teachers' perspectives about how to foster students' mathematical understanding because of the important role teachers play in fostering that understanding in the classroom. A more comprehensive view of mathematical understanding is needed to provide a tool for organizing the kinds of mathematical understanding past studies have investigated. To that end, Cai et al. (2026) proposed a theoretical framework (see Table 1) outlining five aspects of mathematical understanding—factual, procedural, conceptual, metacognitive, and affective—and discussed five lenses employed by mathematics educators—mathematical, cognitive, social, diagnostic, and instructional—when considering how to foster students' mathematical understanding. The five aspects of mathematical understanding are related to the kinds of understanding reached. Researchers approach the issue of how to foster students' mathematical understanding in multiple ways, employing mathematical, cognitive, social, diagnostic, and instructional lenses that each focus on a different set of concerns.

Table 1

Fostering Students' Mathematical Understanding: A Framework

Lenses	Aspects of Mathematical Understanding				
	Factual	Procedural	Conceptual	Metacognitive	Affective
Mathematical					
Cognitive					
Social	*Fostering Students' Mathematical Understanding*				
Diagnostic					
Instructional					

Five Aspects of Mathematical Understanding

The five aspects of mathematical understanding, including factual, procedural, conceptual, metacognitive, and affective (Cai et al., 2026, pp. 3-4), are described as follows.

The *factual* aspect of mathematical understanding is knowledge that is basic to mathematics. It includes essential facts, terminology, details, or elements students must know or be familiar with to understand mathematics or solve a mathematical problem.

The *procedural* aspect of mathematical understanding refers to very specific skills, algorithms, techniques, and particular methods in mathematics. The U.S. NAEP mathematics assessment refers to procedural knowledge as students' ability to follow algorithms, perform calculations, and execute steps in a specific order to reach a solution.

The *conceptual* aspect of mathematical understanding refers to "interrelationships among the basic elements within a larger structure," which includes knowledge of classifications, principles, generalizations, theories, models, or structures pertinent to mathematics. The NAEP refers to conceptual understanding as students' understanding of the underlying mathematical concepts and relationships rather than just the ability to perform calculations. Students draw on conceptual understanding to explain the "why" behind mathematical operations and procedures.

The *metacognitive* aspect of mathematical understanding is awareness of one's own thinking and particular cognitive processes. It is strategic or reflective knowledge related to monitoring whether solution strategies are appropriate and evaluating if answers are reasonable.

The *affective* aspect of mathematical understanding falls into a noncognitive dimension including but not limited to students' attitudes, beliefs, motivation, identity, and emotion.

Five Lenses for Fostering Mathematical Understanding

Because the process of developing students' mathematical understanding is inherently complex and multidimensional, when exploring questions of how to foster that understanding, researchers usually adopt a lens suitable for that exploration. In fact, any research in mathematics education is both situated and oriented by a choice of theoretical stance and structure as well as ways of interpreting phenomena that match the purpose and nature of the research questions (Hiebert et al., 2023). The five lenses Cai et al. (2026) discussed are interrelated and often complementary for examining how to foster mathematical understanding.

The mathematical lens refers to the analysis of the mathematics itself and its practices through unpacking specific mathematical ideas and teasing out what it means to understand that idea mathematically. For example, mathematically, understanding the arithmetic average involves (1) a procedural understanding of the idea as an algorithm, (2) a conceptual understanding of the idea as an algorithm, and (3) a conceptual understanding of the concept as a statistic (the mean) to describe and make sense of and compare data sets (Cai, 2000; Mokros & Russell, 1995; Watson & Moritz, 2000).

The cognitive lens refers to applying cognitive theories to students' learning of specific mathematical topics. For example, Wearne and Hiebert (1988) examined middle school students' learning of decimal numbers, focusing on the semantic phase of the cognitive process, specifically connecting and developing. With this perspective, researchers may investigate students' difficulties understanding mathematical ideas.

The social lens acknowledges that mathematics is a social activity, one in which students can learn together through interacting with each other and teachers (Cobb, 1994). The use of the social lens in mathematics education has gained increasing attention in recent decades, reflecting the "social turn" identified by Lerman (2000) and later echoed by Inglis and Foster (2018). Research on fostering mathematical understanding that employs a social lens frequently involves language, such as mathematical discourse (e.g., Herbel-Eisenmann & Wagner, 2010) and classroom discussions (Pirie & Schwarzenberger, 1988).

The diagnostic lens recognizes the important role that assessment plays in supporting mathematics teaching and student learning. In this lens, teachers' teaching with understanding is based on the assumption that the more teachers know about students' thinking, the better they can foster students' understanding and learning. In the technology era, an example of active assessment is computerized formative assessment, which monitors students' progress and provides automatic feedback (Chang, 2025).

The instructional lens often considers learners' characteristics when exploring instructional approaches that may better facilitate students' understanding of a mathematical topic. The instructional lens is built on mathematical, cognitive, social, or diagnostic lenses. For example, Hiebert (1992) coordinated mathematical, cognitive, and instructional lenses in his analysis of the understanding of decimal fractions and how it may be fostered.

Fostering Students' Understanding through P-PBL

Problem posing in mathematics education refers to several related types of activity that entail or support teachers and students formulating (or reformulating) and expressing a problem or task based on a particular context (referred to as the *problem context* or *problem situation*; (Cai, 2022; Silver, 1994). Stanic and Kilpatrick (1989) proposed three points of view in problem-solving research: problem solving as a cognitive activity, problem solving as a learning goal unto itself, and problem solving as an instructional approach. Researchers have also conceptualized problem posing as a cognitive activity, as an instructional goal, and as an instructional approach (Cai, 2022). Fostering students' mathematical understanding through P-PBL involves viewing problem posing as an instructional approach to teach mathematics. That is, students will understand and learn mathematics through their engagement in problem posing.

Both theoretically and empirically, problem posing supports students' understanding of mathematics. In a recent meta-analysis, Ran et al. (2025) found a very positive effect of engaging students in problem-posing interventions, indicating a statistically significant positive relationship between problem-posing interventions and learners' cognitive mathematical learning outcomes (Hedges' $\bar{g} = 0.53$) and affective learning outcomes (Hedges' $\bar{g} = 0.67$).

What is Problem Posing?

In the classroom, students engage in problem-posing tasks to develop a deep understanding of mathematical concepts. Problem-posing tasks, then, are those instructional tasks that position students as generators or shapers of new problems based on real-life or mathematical situations. A problem-posing task usually includes two parts: a situation and a prompt (Cai & Hwang, 2023). The problem situation is what provides the context and data that the students may draw from (in addition to their own life experiences and knowledge) to generate problems. The prompt lets students know what they are expected to do. For example, the following is a problem-posing task used in Silver and Cai (1996): *Jerome, Elliott, and Arturo took turns driving home from a trip. Arturo drove 80 miles more than Elliott. Elliott drove twice as many miles as Jerome. Jerome drove 50 miles. Pose three different mathematical problems that can be solved based on this information.* In this problem, "Jerome, Elliott, and Arturo took turns driving home from a trip. Arturo drove 80 miles more than Elliott. Elliott drove twice as many miles as Jerome. Jerome drove 50 miles" is the situation; "Pose three different mathematical problems that can be solved based on this information" is the prompt. For the same situation, a different prompt could be used, such as "Pose one easy problem, one moderately difficult problem, and one difficult problem that can be solved based on this information."

A Case of the Gym Membership Task

One case is a lesson on linear functions described in Hwang et al. (2024). The middle school teacher in this case found a problem-solving-based lesson in her curriculum unsatisfactory and subsequently redeveloped it to better target her students' learning. Specifically, the teacher modified the lesson to have students pose problems based on the two equations $y = 1200 + 3x$ and $y = 1200 - 3x$. This activity was designed to foster students' understanding of the concept of slope, including both positive and negative slope. She also replaced the original problem-solving task with a problem-posing gym membership task: *I want to go to a gym to exercise, and I am told that there are two ways to pay for using the facility: (a) Apply for an annual card and add $1,000 to it. $10 will be deducted from the card every time I use the gym; (b) No annual card, but a charge of $30 every time I use the gym. What kind of mathematical problems do you think you can pose based on this situation to test your peers? The more problems you pose, the better. Write them on this worksheet and solve them.*

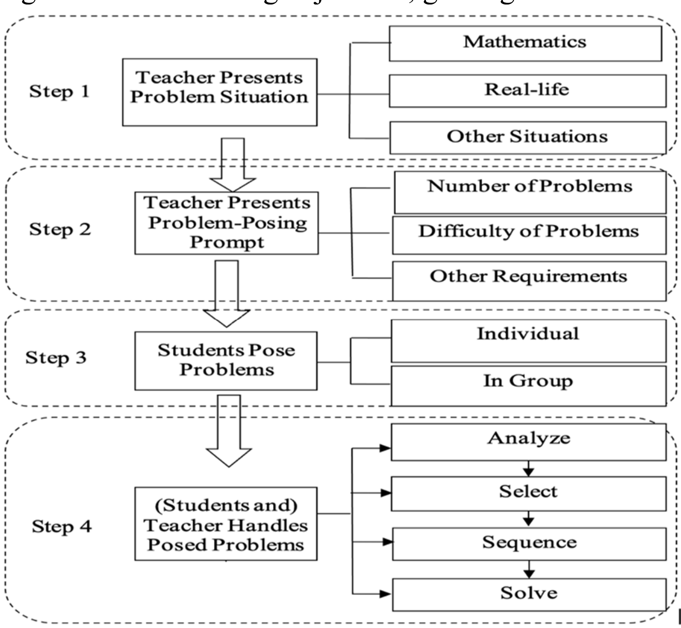
The teacher gave the students time to think about and pose problems for the gym situation. The students posed a variety of problems, such as "After *n* visits, how much money would she pay under each plan?" Through this activity, the teacher was able to elicit ideas of the *y*-intercept, slope, and so forth, connecting them to the symbolic representation.

The five lenses for exploring how to foster mathematical understanding can be applied to analyze what is being attended to or centered in this case. First, the mathematical lens is exhibited in students' understanding of linear functions. Their cognitive understanding can be seen in their process of posing various problems. Socially, the students engaged in this task in the classroom, with their activity necessitated by interactions with each other. Moreover, the task could be used as a diagnostic tool for the teacher to gauge students' understanding of linear functions. Finally, the task is an instructional tool used to further the learning goals of the lesson, such as increasing students' engagement with linear functions.

The five aspects of mathematical understanding can also be fostered through the P-PBL approach, which has increasingly been the target of research exploring students' mathematical understanding. P-PBL clearly provides a promising avenue through which to foster students' mathematical understanding along all five aspects of mathematical understanding, and research on P-PBL has made use of all five lenses.

P-PBL Instructional Model

Having shown the value of P-PBL to foster students' understanding through the above case, I now turn to presenting the P-PBL instructional model, which outlines the four steps involved in teaching through problem posing: (a) the teacher introduces a problem-posing situation; (b) the teacher provides a problem-posing prompt alongside the situation; (c) students pose problems individually or in groups; and (d) teachers address the posed problems in alignment with learning objectives, guiding students in solving selected problems (Cai, 2022).



The first two steps, the teacher introducing the problem-posing situation and the teacher providing a problem-posing prompt with the situation, are typically presented together; however, they are intentionally separated here to highlight the significance of considering both the situation and the prompt when designing problem-posing instruction. Step 3 involves students actively engaging in problem posing, either independently or collaboratively, and Step 4 focuses on how teachers handle posed problems in the classroom. This model shows an instructional flow when implementing problem posing in the classroom. It also serves as a guide for empirical studies. In what followings, I use this model to show the progress and potential of research related to the teaching and learning of mathematics.

One of the biggest challenges in P-PBL is handling students' posed problems. Although there has been wide recognition of the promise of P-PBL, additional attention needs to be paid to the specifics of handling students' posed problems (Cai et al., 2024). The four practices (analyse, select, sequence, and solve) seem to work well. In this section, I'd like to summarize the previous discussion into three main points.

The first is the importance of analyzing students' posed problems. It is very clear that students can pose very different problems. Some students even pose unexpected responses such as statements or irrelevant problems. Thus, the analysis step is critical. Analysis can be based on classification as well as clarification of the problems. Such practices can not only help students structure their problems based on different classifications but also examine the appropriateness of each of their posed problems. One recent study showed that even unexpected responses like nonmathematical problems or unclear responses can be very useful in the classroom (Ran et al., 2025). Thus, the analysis stage can help students see how such unexpected responses could be modified to become more desirable mathematical problems.

There is no way that teachers can discuss or solve all the problems students pose. Current research suggests that, based on the learning goals, teachers need to select the kinds of problems to be solved. For example, even though a problem might be very interesting and mathematically sound, if the problem is irrelevant to the lesson learning goals, teachers might decide to assign that problem as homework instead of discussing it in the lesson. Although it sounds natural to sequence the problems from simple to more difficult, it may occasionally make more sense to sequence them from difficult to simpler. In any case, it is very important to select problems to solve. Solving problems not only helps students learn further with respect to related mathematical ideas but also reinforces the problem-posing aspects of the lesson.

Looking into the Future

In this paper, I have presented a framework about mathematical understanding and fostering mathematical understanding. I have briefly discussed the potential ability of P-PBL to foster students' mathematical understanding. To close, I discuss two areas of possible future research.

The first is for the mathematics education field to use this framework to examine existing literature, including cross-cultural studies, and identify the successes and challenges we have documented regarding promoting students' mathematical understanding and ways to better support them. The framework can serve as a guide not only for individual research (as well as practice) but also for the field as a whole as we work collectively to study and deepen students' mathematical understanding. Mathematics educators will benefit from carefully evaluating how extant approaches to fostering mathematical understanding have (or have not yet) addressed each of the aspects of mathematical understanding. Moreover, mathematics education as a field has benefitted from the multiplicity of lenses that researchers and practitioners have employed, as each lens provides a complementary perspective on mathematical understanding. Especially given the rapidly evolving technological landscape, including the rise of generative artificial intelligence (AI), such synesis will be possible and feasible to identify specific and effective practices for fostering students' understanding of specific mathematical ideas. In addition, it will be critical to use these lenses to focus future research on how such technologies can support the development of each aspect of mathematical understanding in practice.

The second is continued effort on the use of P-PBL to foster students' mathematical understanding. Current research on mathematical understanding tends to focus on factual, procedural, and conceptual aspects of mathematical understanding with less research on the affective aspect even though students' deep engagement with problem posing is well-documented. Future research should explore how problem posing can be effectively integrated into mathematics instruction to foster not just students' cognitive but also affective and metacognitive development, including investigating how teachers can explicitly address students' enjoyment, curiosity, and motivation when engaging in problem-posing activities. By integrating cognitive and affective dimensions in both instructional practice and research methodologies, mathematics education can develop more effective strategies for nurturing students' problem-posing skills and intrinsic motivation.

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