Exploring Mathematical Justifications in Collaborative Problem Solving of a Heterogeneous Group

Namirah Fatmanissa
State University of Surabaya
namirah.21027@mhs.unesa.ac.id

Mathematical justification, the act of giving arguments to defend a claim, strongly supports students' Collaborative Problem Solving (CPS). Yet studies exploring justifications in a heterogeneous-achievement group are still limited. This study describes mathematical justifications of such a group in CPS by identifying the levels of justification and the role of each student. The findings showed that the group used perceptual demonstration to generalisable arguments level of justifications. Higher-achievement students were found initiating justifications, while lower-achievement students triggered or merely accepted them.

Mathematical justification, an activity of providing arguments that support or refute a mathematical claim, is at the core of reasoning activities in problem-solving and improves reasoning skills (Brodie, 2010). In collaborative contexts, students who engage in justification activities with peers learn more as they defend their positions and resolve cognitive conflict (Fuchs et al., 1998). In a classroom problem-solving environment, teachers often group students based on their mathematics achievement to promote such activity. A heterogeneous setting was deliberately created to scaffold high-achieving students to their lower-achieving peers (Wyman & Watson, 2020). In the case of justification, this action might be precedented by the findings from the context of individual problem solving, i.e., students with high mathematics achievement tended to carry out more theoretical mathematical justifications, while students with low abilities justified perceptually more often (relying on what they saw or experienced) or did not justify at all (Chua, 2016). This is expected to lead students with lower achievement to question their claims and justify them in a more theoretical manner than their high-achieving peers. However, in the context of collaborative problem-solving, whether heterogeneous grouping mediates the quality of justification and problem-solving work remains debatable. For instance, it was found that a high-low achieving pair interacted less, as shown by offering fewer conceptual explanations, richer arguments, and more cognitive conflict resolutions than the homogeneous pairs (Fuchs et al., 1998).

We believe that investigating the quality of justification within heterogeneous grouping will inform teachers about the dynamics of heterogeneous grouping in the context of collaborative problem-solving and thus allow them to design the necessary support for the expected learning experience (e.g., arguing for claims, resolving cognitive conflict) for such a group. In this paper, we explore mathematical justifications within the problem-solving process of a group of heterogeneous achievers. Specifically, we want to answer the following questions: (1) How are the levels of the mathematical justifications in such a group? and (2) what is the role of each student in the mathematical justifications?

Theoretical Considerations

We referred to mathematical justification as an activity in which students compile arguments to support an existing mathematical claim, so that the claim is accepted by the group in which the argument is presented (Staples et al., 2012). In a collective setting, mathematical justification involves "acceptance" from the individuals involved in it (Bieda et al., 2022). This acceptance is based on the individual's agreement that a particular claim or mathematical answer makes sense. Thus, mathematical justification also considers the social context in which mathematical activities are carried out.

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Justification within heterogenous groups

Research indicates that grouping students based on mathematics achievement can expose them to activities such as justifying ideas and negotiating with peers. This exposure yielded various findings regarding the dynamics of achievement-heterogeneous grouping in producing and justifying arguments during discussions. Some studies seemed to agree that the high-achieving members of the groups showed dominance in exploring ideas and using claims and data to create arguments, while their low-achieving partners relied on these peers (Ekawati et al., 2025; Fuchs et al., 1998). These studies also found that, in heterogeneous groups, high-achieving students accommodated their lower-achieving peers by providing clearer explanations tailored to them. Yet, despite the dominance and contributions of the higher-achieving members, there was evidence that the group members, including the high-achievers, did not fully understand the quality of some justifications for the claim or how they made sense for the claim (Lin & Mintzes, 2010). It shows that there are more issues to be investigated besides the contributions of each group member and the way they respond to their peers. Two of such issues are the quality of the justification they made and how their interaction led to or otherwise hampered the quality of justifications of the constructed claims.

When analysing students' interactions during mathematical justification, it is essential to focus on those specifically directed toward claims and their justifications (Cobb et al., 2000). In that way, the analysis will focus on students' utterances toward each other that are directed to claims or mathematical justification supporting those claims. To allow for such analysis, Cobb et al. (1992) introduced the notion of a situation for justification, a situation where a mathematical justification is demanded. This situation is initiated by questions demanding reasons, clarification, or explanation of the truth of claims. The person who questions or asks for reasons acts as a trigger for the justification. For example, Fredriksdotter et al. (2022) studied justifications during the group's interactions while solving a problem: "There is a queue at the bus stop. In how many ways can 3 persons form a queue?" In their study, a discussion began when Jason claimed the answer was six ways. Milton then simply asked, "Are you sure?", which triggered other members of the group to justify their position. In response to triggers, students might develop mathematical justifications to convince others.

Mathematical justifications levels

Several studies identified levels of mathematical justification carried out by students. Simon and Blume (1996) divided students' mathematical justification into five levels, i.e., level 0 (without justification), level 1 (appeal to authority), level 2 (empirical demonstration), level 3 (deductive justification expressed in terms of example), and level 4 (deductive justification independent of example). Sowder and Harel (1998) also identified several types of mathematical justifications. They started at the externally based justification level, where students justified using an outside source. Students with better justification might give an empirical justification that was made solely based on examples. Based on Sowder and Harel (1998), the highest level of justification was analytical justification. In this level, students might try to defend a claim by reasoning toward generalising a conjecture, which was a transformational analytic justification. They also might use definitions, theorems, or logical mathematical rules, which are called axiomatic justification. Carpenter et al. (2005) identified that students' arguments to justify a claim progressed through three forms, i.e., appeal to authority, using examples, and generalisable arguments. In appeal to authority level, students used others with more authority to justify a claim. In the level of using examples, students did not think in general terms. They defended that a certain claim was true based on some specific examples. Carpenter et al. identified this level as using examples, but we considered this level as a combination of levels 2 and 3 of what other experts described. In the highest example, students gave a generalisable argument to justify a claim. This justification went beyond examples as students started to think in general terms. They used mathematical principles independent of examples. Summarising from these works (Carpenter et al., 2005; Simon & Blume, 1996; Sowder & Harel, 1998), we used the levels in Table 1 to help us describe the quality of mathematical justifications.

Table 1

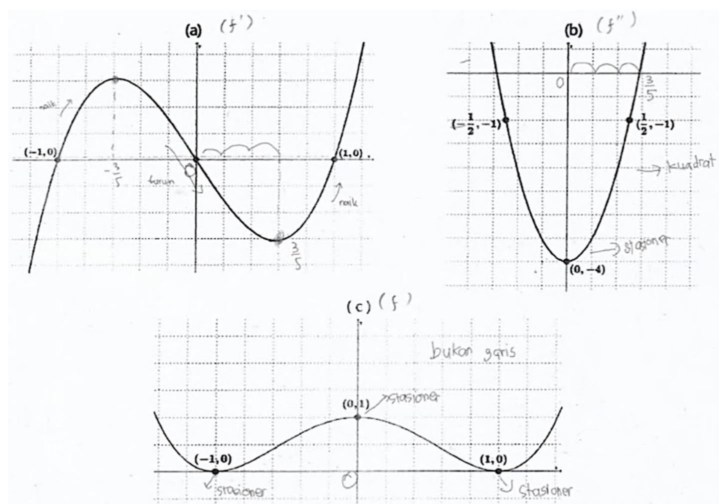
Components of Mathematical Justifications

Component	Description	Code
Trigger	Trigger of a situation for justification	T
Claim	The claim in the problem-solving process	Ci*
Levels of mathematical justifications		
No justification	Not justifying or accepting a claim without explaining why a claim is reasonable	L0
Appeal to authority	Stating the truth of claims using the authority of other people or other sources such as teachers, books, or friends	L1
Empirical or perceptual demonstration	Stating the truth of claims based on the demonstration of perceptual examples, e.g., objects, pictures, or gestures	L2
Symbolic example without generalization	Stating the truth of a claim by using some symbolic examples without generalizing those examples	L3
Generalisable argument	Stating the truth of a claim by using definitions, theorems, or rules of mathematical logic; or generalizations that do not depend on a particular example	L4

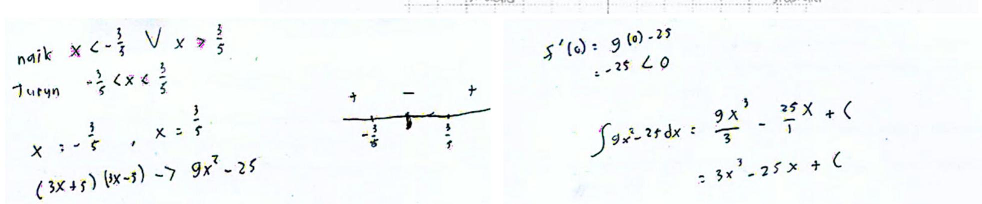
Figure 1

The task and the group work

Andi is given three graphs as follows. Among the three graphs, he is asked to determine which one is the graph for function f, f', and f''. Andi stated that a is the graph of f, b is the graph of f', and c is the graph of f''. Is Andi's statement true? Justify your answer.



Translation:
Increasing
decreasing



Methods

This paper was part of a larger study exploring justifications of four achievement-based triads, i.e., homogenous-high, homogenous-medium, homogenous-low, and achievement-heterogenous, in an eleventh-grade class of a public school in Malang, Indonesia. The class has just finished an 8-week lesson on the derivative topic without explicit instructions on justifications. The participants' selection began with administering an expert-validated mathematics test to all students and categorising them as high, medium, or low achievers based on their test scores. This paper reported the work of one male achievement-heterogeneous triad, consisting of one high-, one medium-, and one low-achievement student (pseudonyms: Hans, Mike, and Leo, respectively), on the task in Figure 1. The group worked on the task outside of the classroom schedule, and a task-based interview was done afterwards. Conversations during the group work and interviews were video-recorded, and paperwork was collected. The data analysis was carried out on group work data collected while completing the mathematical justification task, as well as on transcripts of group discussions and group interviews. Initial data condensation was conducted by coding the mathematical justification activities performed by the groups (see Table 1). Coding related to mathematical justification included the mathematical claim that appears (Ci), the trigger for the emergence of mathematical justification (T), and the level of mathematical justification (L0 to L4). The coding process began by identifying the emergence of justification situations throughout the problem-solving process, then coding the mathematical claims (Ci) that arose in these situations. The level of mathematical justification that emerged was then analysed around the claims being identified.

Results

To find the solution to the problem, the group's general strategy was to determine the degree of the polynomial function that each curve represents and their relationship (see Figure 1). First, they claimed that function a is a degree-3 polynomial by identifying its stationary points. After observing the shape of graph b and c, they also claimed that graph b showed a quadratic function and that c is not the first derivative of b. Combining these claims, they agreed that b is the first derivative of a and that Andi's statement in the task is incorrect. Due to the limited space in this paper, we presented examples of the group's mathematical justifications at different justification levels and discussed the student's role in justifying claims during the problem-solving process.

Mathematical justification levels

The use of empirical or perceptual demonstrations (L2)

We observed the group's use of level 2 justification in justifying the claim that b is a quadratic function. The group's discussion focused on the shape of the graph being concave up and that it has one stationary point as shown below.

Hans: What about this (pointing at graph b)? Andi said that b is the derivative of a. It is true, right? [T]

Mike: is a three degree? The derivative is two degree. Two degree (function) has one (stationary point), just like this (pointing at graph b) quadratic function [L2]. It is correct.

Hans: I am tired (laughing)

Mike: Two degree, right.

Leo: How is it? [T]

Hans: Yes, it is two degree (gesturing the curve shape), it is the same (agreeing).

Another example was when they observed graph c after agreeing that b is a quadratic function. The group claimed that c is not the first derivative of b since the shape of c is not a line as written in the transcript below.

Leo: Are there three stationary points (for curve c)? [T] Is its derivative b?

Mike: No [C].

Hans: If yes, c is supposed to be a line. [L2]

Mike: Why is that so?

Hans: Because it is only x. But this (curve c) is not a line. [L2]

Leo: So, it is not the derivative of b.

The mathematical justification that led to the claim that the function represented by c was not a derivative of b was put forward by Hans. He stated that if it was a linear function, it should be in the shape of a straight line. This mathematical justification was level 2 because it used an empirical or perceptual demonstration in the form of a graph shape. Mike questioned this mathematical justification and then explained again by Hans that a linear function is a polynomial function of degree one. The group agreed that the function represented by c was not the derivative of b.

The use of symbolic examples without generalisation (L3)

We observed the group's use of level 3 justification for the claim that a is a three-degree polynomial function. The group constructed an equation of $f'(x) = 9x^2 - 25$ as the first derivative of a. Hans started this discussion by marking the stationary points on graph a and he stated that the x coordinate of the point is $\frac{3}{5}$ (see Figure 1, graph a). The group then explored on this data to infer that the function was stationary at $x = \frac{3}{5}$ and $x = -\frac{3}{5}$, increasing on $x < -\frac{3}{5}$ or $x > \frac{3}{5}$, and decreasing on $-\frac{3}{5} < x < \frac{3}{5}$. This idea was confirmed by Mike and Leo by checking that the points were located three scales from O (note: five scales represented 1 unit on the graph). The group escalated the idea into constructing an equation representing the derivative of function a and finding its integral (see Figure 1, bottom part). We considered that there was also a part of perceptual reliance (L2) in this case as they relied on their identification of the exact locations of the stationary points on the graph. Nevertheless, to support the claim, the group went beyond this and moved into symbolic examples (L3) by formulating the equation of the function derivative. They formulate the equation by multiplying the factors $(3x - 5)$ and $(3x + 5)$, resulting on $9x^2 - 25$. Further, the group connected the fact that if the derivative function is two-degree, then the function is of three-degree.

Hans: It means that this $(9x^2 - 25)$ is the derivative of this (pointing at function a). [C]

Mike: Yes.

Hans: Wait a second.

Mike: So, the function is three degree? [T]

Leo: This $(9x^2 - 25)$ is the derivative? [T]

Hans: Wait (thinking).

Mike: Yes, right. It means that this one (pointing at a) is $3x^3$. [C]

Hans: Oh, I see. What did we do? I forget the name of this method.

Mike: Let's try finding the derivative (deriving $3x^3$). Yes, right. It is $9x^2$.

Hans: Do you think we find the integral? (writing $\int (9x^2 - 25)dx = \frac{9}{3}x^3 - \frac{25}{1}x + c = 3x^3 - 25x + c$).

The claim that the function represented a polynomial function of degree three was put forward by Hans. This claim was then triggered for mathematical justification by Mike and Leo. Then, Mike realized that the function was a polynomial function of degree three (i.e. with the first term of $3x^3$) by determining the integral of its derivative function. Mike reconfirmed this claim by redefining the derivative of $3x^3$ that yielded $9x^2$. This claim was confirmed by Hans, which also determined the integral of $f'(x) = 9x^2 - 25$.

The use of generalisable argument (L4)

The group's use of generalisable argument was shown during their discussion about the claim that b is the first derivative of a. In the previous discussions, they had agreed that a is a three-degree polynomial function and b is a quadratic function. After exemplifying that the derivative of a is $f'(x) = 9x^2 - 25$ (L3), they claimed that b is the first derivative of a, which directed their attention to graph b. We observed a further effort using generalisable argument when they identified that the x-intercepts on graph b are the same with the stationary points of graph a, which they argued to be at $x = \frac{3}{5}$ and $x = -\frac{3}{5}$.

Hans: It means that the stationary points are here (marking the stationary points in curve a) (see Figure 1, graph a)

…

Mike: Oh, yes (agreeing that the x-coordinate of the stationary points are $\frac{3}{5}$ and $-\frac{3}{5}$).

Hans: It is the same here, though (on graph b, scaling from O to the x-intercept and wrote $\frac{3}{5}$) (see Figure 1, graph b)

Mike, Leo: (looking at graph b)

Hans: The stationary here (pointing at graph a), intercept of x here (pointing at graph b). [L4]

Leo: Does it matter? [T]

Hans: I think it matches, right?

Mike: How? [T]

Hans: Like here is the function (pointing at graph a) and 0 here in the derivative (pointing at graph b). [L4]

Mike: But it does not change anything, right?

Hans: Yeah.

Hans associated the coordinates of stationary points of a function with the x-intercepts of its derivative function. We considered that somehow Hans understood that if f was stationary at x, then $f'(x) = 0$, which he tried to offer to his peers. Yet we felt that this justification, initiated by Hans, was accepted but not shared by his peers as they went back to the function equation for the justification later.

Student's role within group

By tracing who offered initial justifications and who triggered them, we see that each student played a different role while interacting during the problem-solving process (Table 2). We saw that the initial justifications for claims 1, 3, and 5 were offered by Hans, while Mike offered them for claims 2 and 4, and Leo repeated Hans's justification for claim 1. By questioning claims or confirming their validity, we observed that Mike and Leo often trigger others' justifications for each claim. During the problem-solving process, Leo only interacted by triggering or continuing the mathematical justifications made by others. Hans and Mike interacted by both offering and triggering mathematical justifications.

Discussions and Conclusions

The group managed to agree on levels 2 to 4 of justification during the problem-solving process. They tried to connect the data to the intended claim by considering a plausible explanation linking the data and the claim. Tracing back to the case, Hans was mostly found to initiate discussions leading to level 3 or 4 justifications (e.g., for claims 1 and 4). This finding reiterated other studies that found students of high achievement tended to offer higher-level mathematical justifications (Chua, 2016). We also found that students with low achievement offered mathematical justifications at lower levels, repeated others' justifications, or did not offer mathematical justifications at all, as reported in other studies (Whitacre et al., 2017). We also observed that mathematical justifications were mostly initiated by Hans. These mathematical justifications were then met with triggers from Mike or Leo. The triggers were then followed by either an acceptance of the initial justifications or the refinement of them by either Hans or Mike. However, for some cases, the justification offered by high achiever which was failed to be understood would be merely accepted as long as it did not change the agreement on the claim. For instance, when Hans offered the justification that if $a(x)$ was stationary at x, then $a'(x) = 0$ to support claim 3, his peers were more inclined towards the equation example related to claim 1. As triggers arose from doubts or confusion, we observed efforts to accommodate achievement differences within the group (An & Zhang, 2024) and acceptance of other students' explanations by lower-achievement students (Wu, 2024). We also noted a possibility that the lower-achievement students put their high-achievement peer as a source of authority (Bleiler-Baxter et al., 2023), which led us to reconsider our coding for level 1 for future studies, especially in the context of CPS. The group dynamics suggest that the teachers should provide scaffolding that encourages each student to construct and refine justifications, rather than relying on the high achiever in the group. Instructions should integrate prompts or structures that distribute opportunities for justification more equitably, regardless of achievement level.

Table 2

Examples of student's justifications and triggers

Claim	Sample utterances of justifications (L0-L4) or triggers (T) by Hans	Mike	Leo
Claim 1: Function a is a polynomial function of degree three	(L3) using an example that a derivative function of a is $9x^2 - 25$, which is of two-degree	(T) "There is no function (written) on the task", "the derivative is $18x$".	
	(L3) using $x = 0$ as an example that $f'(x) < 0$	(T) "is the function three degree?"	(L3) using $x = 0$ as an example that $f'(x) < 0$ (T) "is $9x^2 - 25$ the derivative?"
	(L4) determining the integral of $9x^2 - 25$		
Claim 2: The derivative of function a is a quadratic function	(T) "Is this true?"	(L4) using derivative rule to determine function a	
Claim 3: Function b is the first derivative of function a and b is quadratic	(L2) using the shape of graph b and b has a stationary point		
	(L4) stated that if $a(x)$ was stationary at x, then $a'(x) = 0$	(T) How?	(T) Does it matter?
Claim 4: The derivative of function b is a linear function		(L4) using derivative rule to determine function b	
		(T) "For x^2, the shape is like this (of the graph, pointing at curve b), what about only x? "	(T) "c has three stationary points, is c the derivative of b?
Claim 5: Function c is not the first derivative of function b	(L2) the graph of function c is not a line	(T) "Why is that so?"	
Final claim: Andi's statement in the task is incorrect	(L4) stated that Andi's statement, "graph b is f', graph c is f'' " is not supported by the data		

While this study has described mathematical justifications of students in heterogeneous groups in terms of the level of mathematical justifications and students' interactions within each group during CPS, we acknowledge some limitations. First, the notably small participant in one group limited the generalizability of the findings. Second, despite no explicit instruction to promote justification being observed prior to this study, no thorough observation was made on whether the justification norm was implicitly instilled or existed. Third, the study was set outside the classroom, which might have constrained the interpretation of the findings toward a natural classroom setting.

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