Making Multiplication Structure Visible: Spatial Artefacts, Instrumental Genesis, and Emerging Distributive Reasoning

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This study extends instrumentation theory by examining how spatial artefacts support Year-3 students' development of multiplicative thinking through composition and decomposition. Drawing on classroom observations, we trace students' instrumental genesis—how they reconfigure artefacts (instrumentalisation) and develop utilisation schemes (instrumentation). The findings show a trajectory toward deliberate product composition and flexible decomposition, laying the groundwork for distributive reasoning. We introduce *spatially mediated instrumental genesis* to characterise this spatially grounded scheme development.

Introduction

Understanding multiplication as more than repeated addition remains a central challenge in the early years mathematics (Fosnot & Dolk, 2001). A key step is the development of *decomposition strategies* (e.g., 5×12 as 5×10 + 5×2), which support flexible computation and students' emerging appreciation of distributive reasoning (Fosnot & Dolk, 2001; Kinzer & Stanford, 2013). Spatial representations—including arrays, area-based rectangles, and bar-type models—are commonly used to foreground such composition and partitioning (Barmby et al., 2009; Harries & Barmby, 2006), yet their mathematical potential depends on how learners take them up as resources for thinking and justifying rather than as display forms alone (Fosnot & Dolk, 2001; Putrawangsa & Patahuddin, 2022). From the instrumental approach, introducing an artefact does not itself ensure mathematical productivity. An artefact becomes an instrument only through learners' developing utilisation schemes. However, how students' scheme development is shaped by *spatial structuring work*— particularly when learners move across related spatial artefacts—remains under specified (Rabardel, 2002).

This paper analyses Year-3 students' engagement with two spatial artefacts—*multiplication bars* (pre-made rectangular tiles labelled a×b, physically arranged to compose and decompose products - see Figure 1) and a *bar model* (a drawn rectangular/tape representation flexibly partitioned to express multiplicative structure – see Figure 5)—within a three-task classroom intervention. Using instrumentation theory, we examine how students' ways of acting with and reasoning through spatial artefacts support their composition and decomposition of multiplication, and how this activity contributes to emerging distributive reasoning. We argue that making spatial mediation explicit extends instrumentation theory for domains in which conceptual development is tightly linked to spatial structuring. This paper addresses the following research question: *How do spatial artefacts mediate Year-3 students' instrumental genesis for decomposition strategies and emerging distributive reasoning in multiplication? In other words,* we elaborate spatially mediated instrumental genesis by specifying how students' utilisation schemes depend on aligning, matching, and partitioning spatial measures across spatial artefacts.

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Theoretical framework

The instrumental approach distinguishes artefacts from instruments. An artefact is the material or symbolic object made available for activity, whereas an instrument is a mixed entity comprising the artefact and the utilisation schemes through which a user organises and carries out goal-directed action (Rabardel, 2002; Shvarts et al., 2021). This distinction foregrounds that learners must construct ways of using tools and representations if these are to become mathematically functional for learners—for example, to support the composition and decomposition of products and the justification of distributive properties.

Instrumental genesis refers to the process through which an artefact becomes an instrument (Artigue, 2002; Rabardel, 2002). Rabardel (2002) characterises instrumental genesis as a dual, intertwined process: instrumentation (processes oriented toward the subject, involving the emergence and evolution of schemes) and instrumentalisation (processes oriented toward the artefact, involving selection, transformation, regrouping, and attribution of functions). For analysing students' work with spatial representations, this duality foregrounds both how artefacts shape activity and how students progressively invest the artefacts with mathematical functions (e.g., by selecting multiplication-bar tiles whose side lengths match the intended factors, aligning tiles edge-to-edge to compose a target product, splitting a composed arrangement into sub-rectangles that correspond to a chosen decomposition $12 = 10 + 2$), and re-representing that spatial structure as a drawn bar model by marking partitions and labelling the associated partial products.

To trace learning in tool-mediated activity, the instrumental approach commonly draws on the notion of *scheme*, defined as a relatively stable organisation of activity for a class of situations (Shvarts et al., 2021; Vergnaud, 1998). In classroom settings, schemes are inferred from students' techniques and from recurring patterns in their interactions with artefacts. These techniques may serve both pragmatic and epistemic functions: pragmatic in producing results, and epistemic in supporting mathematical meaning-making and justification. This epistemic function, however, often depends on careful task design and classroom institutionalisation (Artigue, 2002; Shvarts et al., 2021).

In this paper, the focal artefacts are spatial—representational and manipulable resources whose mathematical potential depends centrally on spatial organisation and transformation (e.g., covering, composing, partitioning, recombining). Building on our prior work on spatialised instrumentation (Putrawangsa & Hasanah, 2020) and research examining mechanisms that support students' mathematical understanding during spatial reasoning interventions (Lowrie & Logan, 2023; Lowrie et al., 2020), we propose spatially mediated instrumental genesis as an extension of instrumentation theory that foregrounds how scheme development is mediated by spatial structuring operations. Analytically, we trace (a) instrumentation through the emergence of utilisation schemes for coordinating the two factor dimensions (e.g., keeping one factor fixed while partitioning the other to support decomposition), and (b) instrumentalisation through students' spatial actions on the artefacts (e.g., rotating, swapping, composing multiplication bars; creating and labelling partitions on the bar model).

Method

This study reports a focused qualitative analysis situated within a broader design-based research (DBR) project on spatialising multiplication learning for Year-3 students in Indonesia. DBR blends theory-driven design with systematic study of learning in authentic settings, aiming to contribute to both practice and theory (Design-Based Research Collective, 2003).

The classroom experiment involved 24 Year-3 students (aged 9–10) in an Indonesian primary classroom. The intervention comprised three sequenced tasks using spatial artefacts (i.e., multiplication bars and a bar model). In Task 1 (compare areas), students used a set of multiplication bars (from 1×1 to 5×5, see Figure 1) to compare the size of two rectangles (A and B, see Figure 2). Students first predicted which rectangle was larger and then used bar configurations to validate their prediction inferring dimensions and products.

Figure 1

A set of multiplication bars

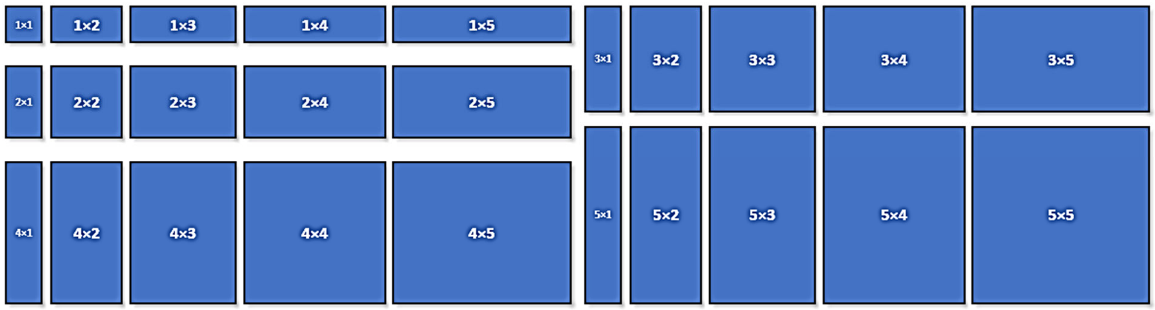
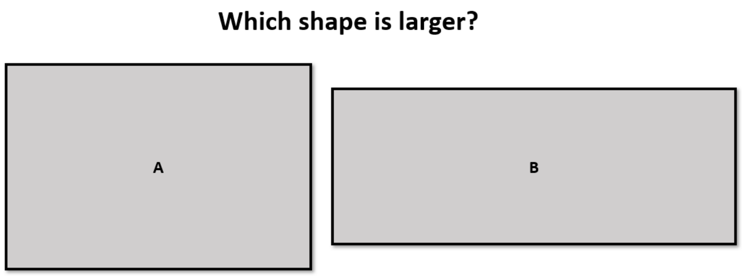


Figure 2

Two rectangles for comparison task



Following the first task, Task 2 (composing a multiplication) asked the students to generate a rectangle by combining at least two multiplication bars and to determine the dimensions and product of the resulting composite rectangle. In the final task, Task 3 (decomposing a multiplication), students solved 5×12 without multiplication bars by partitioning a bar model into two or three parts, computing partial products, and recombining them.

Data were collected through classroom observation conducted during the intervention, including video-recorded classroom interaction and detailed transcripts capturing teacher prompts and student explanations.

Data were analysed qualitatively to characterise students' instrumental genesis as they worked with multiplication bars and the bar model to develop and justify decomposition strategies. Drawing on the instrumental approach, we coded episodes for evidence of instrumentation (the emergence and evolution of utilisation schemes shaped in part by spatial constraints and affordances) and instrumentalisation (students' purposeful transformations and reconfigurations of the artefacts). This analysis focused on how students used spatial structuring to reason about decomposition and to articulate emerging distributive principles. The data points presented in this paper represent the predominant strategies employed by the students in each task.

Findings

Across the three tasks, students' activity illustrates spatially mediated instrumental genesis: that is, the spatial artefacts (multiplication bars and the bar model) progressively became instruments for coordinating dimensions, composing products, and justifying whole products as sums of partial products, thereby fostering decomposition strategies and emerging distributive reasoning in multiplication.

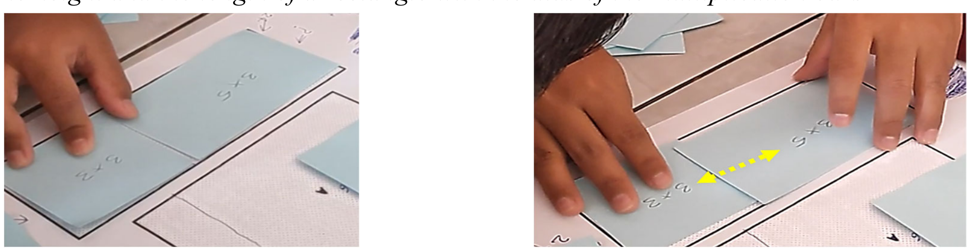
Task 1: Relating factor pairs through composed multiplications

Task 1 was designed to support understanding that different factor pairs can yield equivalent products and that such products can be interpreted as compositions of related multiplications (e.g., 4×6 as 4×5 + 4×1; 3×8 as 3×5 + 3×3). During the comparison task, initially, students justified their comparisons perceptually ("wider" vs "taller"), suggesting that factor relations were not yet their primary basis for judging equivalence. Through working with the multiplication bars, however, students began to coordinate factor relations explicitly and to use composition strategies to infer dimensions and determine the rectangles' products.

In terms of instrumentalisation, students initially undertook substantial technical work with the bar set—scanning available bars, selecting candidates, testing placements, and repeatedly rotating, swapping, and replacing pieces to construct a complete covering configuration for each rectangle (Figure 3a). This instrumentalisation was mathematically consequential because it produced configurations that instantiated a common factor (a shared dimension) and a composed factor (a sum of lengths), thereby making factor relations perceptible and available for reasoning.

Figure 3

Tracing the height and the length of a rectangle with the aids of the multiplication bars



a b

Instrumentation and an emerging utilisation scheme were evident during this task. Once a configuration was established, students developed a scheme called *common-factor coordination* consisting of epistemic techniques such as (1) identifying the common factor from the shared dimension of the bars; (2) composing the other factor by adding the length labels; (3) interpreting the whole product as the sum of partial products; and (4) restating the whole as a single multiplication. For Shape B, for instance, they used 3×5 and 3×3, composed 8 as 5 + 3 with the common factor 3, and concluded that 3×8 equals 24 (Figure 3b). The bars thus shaped students' activity toward recognising equivalence through composed multiplications.

From a learning-process perspective, students' instrumental genesis was evident. By the end of Task 1, multiplication bars functioned as an instrument for relating two multiplications through factor composition. Here, students could justify that 3×8 was equal to 3×5 + 3×3 by expressing 3×8 as a composition of two related products sharing a common factor, providing an early form of distributive reasoning tied to factor relations.

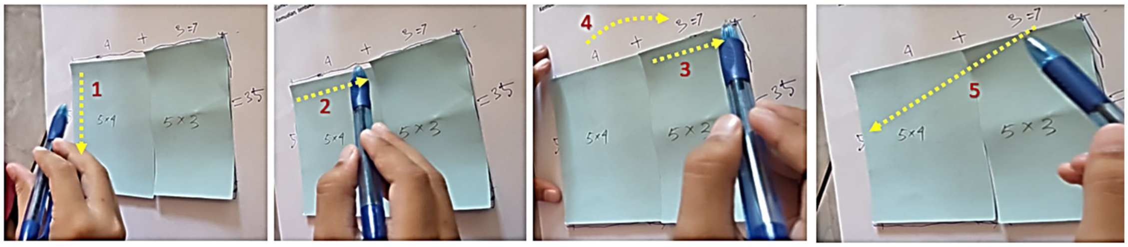
Task 2: Composing a multiplication from shared factor multiplications

Task 2 aimed to make explicit that a new multiplication can be composed from two known multiplications that share a common factor (e.g., 5×7 from 5×4 and 5×3). This shifted students from recognising composed multiplications in Task 1 to intentionally producing them.

Instrumentalisation was witnessed as the students actively built new composite artefacts by selecting bars with the same width, aligning them side-by-side, and attributing the composite rectangle a single multiplication identity (Figure 4). In doing so, they repurposed the bar set from ‘measurement’ to ‘construction’—a clear instrumentalisation move oriented to generating a new product.

Figure 4

Composing 5×7 from 5×4 and 5×3



a b c d

Instrumentation and an emergent utilisation scheme became evident as students engaged with the task. They developed a stable *compose-and-name scheme* linking factor composition to product composition. The scheme led students to (1) keep the common factor invariant (e.g., the shared width); (2) add the remaining factor values to form the composed factor; (3) state the resulting multiplication; and (4) connect it to a sum of related products. For example, one student justified that 5×7 can be generated from 5×4 + 5×3 by summing 4 and 3 to obtain 7 while holding 5 invariant as the shared factor (Figure 4). This foregrounded multiplication composition as a structural operation rather than merely an isolated arithmetic fact.

In this task, the instrumentalisation and instrumentation jointly supported students’ instrumental genesis. By the end of Task 2, multiplication bars functioned as instruments for composing products from related products: students could coordinate a common factor and justify how factor addition (b+c) corresponds to product addition ($ab + ac$), preparing the conceptual ground for distributivity.

Task 3: Decomposing a multiplication on a bar model

Task 3 aimed to promote decomposition of a multiplication by partitioning one factor while keeping the other factor invariant, and to justify the total product as a sum of partial products (e.g., 5×12 as 5×9 + 5×3 or as 5×5 + 5×4 + 5×3). Importantly, physical multiplication bars were removed to test whether students could apply the scheme using a bar model as a spatial record.

Instrumentalisation was noted as students created partitions on the bar model (two-part and three-part splits), chose accessible addends, labelled each segment with a multiplication, and organised calculations to recombine partial products. These actions show instrumentalisation of the bar model: students transformed a generic bar representation into a structured decomposition artefact tailored to their chosen factor split (Figure 5 and 6).

Instrumentation and an emergent utilisation scheme were clearly observable. The bar model shaped students’ activity toward *a partition–compute–recompose scheme* as they (1) ensured the partitioned addends sum to the original factor (e.g., 9+3=12); (2) computed each partial product with the invariant factor 5; and (3) recombined partial products to justify the whole product. Students explicitly checked the sum of partitions and confirmed that recomposition (15+45=60) matches 5×12, treating recomposition as justification (Figure 5). Three-part decompositions further indicated flexibility and generality of the scheme (Figure 6).

In relation to students’ instrumental genesis, by the end of Task 3, the bar model functioned as an instrument for multiplication decomposition: students could independently partition a factor, coordinate it with an invariant factor, and justify the product as a sum of partial products, articulating an emerging distributive principle without reliance on physical bars.

Figure 5

A decomposition strategy for 5×12, splitting the bar into two partitions, 5×9 and 5×3.

a b c

Figure 6

A decomposition strategy for 5×12, splitting the bar into three partitions, 5×5, 5×4, and 5×3.

a b c

Overall, the three-task sequence supported a progression from noticing equivalence (Task 1), to intentionally composing new products (Task 2), to independently decomposing and justifying products on a bar model (Task 3). By the end of the sequence, students could coordinate a common factor with composed or partitioned factors and justify product equivalence as a sum of related products—an emergent, spatially grounded form of instrumental genesis for distributive reasoning.

Discussion and Contribution

This paper extends instrumentation theory by foregrounding spatially mediated instrumental genesis for early multiplication. The findings demonstrate that multiplication bars and a bar model did more than support 'area reasoning'; they mediated the genesis of instruments for specific multiplication concepts: (i) relating factor pairs of two products, (ii) composing a product from related products sharing a factor, and (iii) decomposing a product into partial products to justify an emerging distributive principle.

Interpreted through Rabardel's framework, the observed progress reflects the development of instruments as artefact-plus-scheme entities (Rabardel, 2002). Across tasks, instrumentalisation was visible in students' selection, rearrangement, composition, and partitioning of spatial artefacts, while instrumentation was visible in the emergence of utilisation schemes, such as *common-factor coordination scheme* (Task 1), *compose-and-name scheme* (Task 2), and *partition–compute–recompose scheme* (Task 3) that coordinated common factors and composed or decomposed the other factor.

Artigue (2002) cautions that technical work can remain implicit, and that the epistemic value of techniques does not arise automatically. To this point, the mathematical significance of the process goes beyond the operational—and in fact may not arise automatically from practice. Rather, the instrumental genesis emerges from the intersection of how an artefact shapes the user's activity and how the user shapes the artefact. The present sequence made technical work visible and progressively mathematical by requiring students to (a) express a bar configuration as a single multiplication, (b) justify a composed multiplication using relations among known multiplications, and (c) justify decomposition through recomposition.

Purposeful task design and teacher mediation supported students' technical work in progressively mathematical ways.

In Artigue's terms, these tasks supported a shift from *pragmatic* technique, where students focus on fitting, aligning, or adjusting bars to achieve the desired configuration, to *epistemic* technique, in which the bar model becomes a genuine *instrument* for establishing and justifying multiplicative relationships. Through this instrumental genesis, techniques gained theoretical value: they became not merely ways of acting on artefacts but ways of *knowing*, *explaining*, and *justifying* mathematics (Artigue, 2002). Theoretically, spatially mediated instrumental genesis enhances the instrumental approach for settings where mathematical content is tightly connected to spatial structuring operations. The construct highlights how spatial artefacts can stabilise common-factor structure and make factor composition/decomposition actionable, thereby shaping the trajectory of scheme development for distributive reasoning (Artigue, 2002; Rabardel, 2002).

Practically, the three-task sequence offers a compact design heuristic for supporting Year 3 students' understanding of multiplication as a structure, particularly as a foundation for distributive reasoning (Fosnot & Dolk, 2001; Kinzer & Stanford, 2013). For classrooms similar to the study site, the findings suggest that distributive reasoning can be developed through carefully sequenced tasks that keep one factor invariant, invite students to compose the other factor, and connect multiplication expression to sums of related products. This has an important classroom implication: rather than introducing the distributive property only as a formal rule, teachers can foster students' structural understanding through task design, shared representations, and discussion that makes factor-product relationships explicit. The findings also indicate that classroom institutionalisation is critical. Teachers need to compare strategies, highlight invariant relationship across tasks, and establish shared mathematical language so that distributive reasoning becomes a recognised and reusable classroom technique rather than a local strategy. In this sense, the sequence supports instrumental genesis toward distributive reasoning as an endorsed mathematical technique (Artigue, 2002).

These implications should, however, be considered in light of the study's limitations. The study was conducted in a single classroom context and focused on one short instructional episode. As such, the findings are best interpreted as offering a situated account of how distributive reasoning may emerge through task sequencing, rather than as evidence of broad generalisability. Future research should examine whether the developed schemes transfer to new numbers, problem contexts, and representations, and how classroom institutionalisation practices support consolidation of distributive reasoning over time.

Conclusion

By analysing a three-task sequence through instrumentation theory, this paper showed how spatial artefacts can mediate students' instrumental genesis for understanding multiplication through composition and decomposition. Multiplication bars and a bar model supported the emergence of utilisation schemes that relate factor pairs, compose products from related products, and decompose products into partial products, grounding an emerging distributive principle. The notion of spatially mediated instrumental genesis offers a theoretically grounded way to describe and design for these developments in early multiplication learning. The findings of this study build on a focused body of research literature that explores the (potential) mechanisms for supporting students' mathematics understandings during spatial reasoning interventions (Lowrie & Logan, 2023; Lowrie et al., 2020).

Acknowledgments

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