Halving and Doubling in Multi-Digit Multiplication: How Students' Spatial Structuring of Arrays Supports Structural Understanding

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This paper examines how students' spatial structuring of arrays supported their reasoning about halving and doubling in multi-digit multiplication. Drawing on a teaching experiment in two Year 5 classrooms, the analysis focused on how students organised and transformed arrays when reasoning about multiplicative relationships. The findings suggest that arrays supported students not simply by representing multiplication visually, but by enabling them to test different transformations of the array. In particular, students recognised that halving and doubling required transformations that preserved equal groups and the coordinated row-and-column structure of the array.

The capacity to reason using a deep conceptual understanding of multiplicative structure is crucial to students' mathematical development (Clark & Kamii, 1996). Mathematical structure involves recognising general properties that are instantiated in particular situations as relationships between elements (Mason et al., 2009). In multiplicative contexts, this includes recognising properties such as distribution, associativity, and commutativity as generalisable relationships. It is important that students develop understanding of the mathematical properties that underlie multiplication as this knowledge forms the basis for algebraic thinking (Shifter et al., 2008).

Compared to single-digit multiplication, multi-digit multiplication requires a substantial shift in students' reasoning, as they must coordinate the multiplicative relationships inherent in place value with the distributive structure of multiplication (Izsák, 2004). Although substantial research has examined single-digit multiplication, comparatively few studies have focused on multi-digit multiplication (Hickendorff et al., 2019), and much of this work has emphasised the distributive property. Less attention has been given to how students develop understanding of the associative property in multi-digit multiplication, particularly in relation to halving and doubling, or to the role that spatial reasoning may play in supporting this understanding (Antonides et al., 2025; Battista, 2004).

This study responds to these calls by examining the role of students' spatial structuring of arrays in supporting their understanding of halving and doubling as an application of the associative property in multi-digit multiplication. Halving and doubling is treated here as an application of associative reasoning because it involves decomposing one factor and regrouping the multiplicative structure while preserving the product. The study was guided by the following research question: *How does students' spatial structuring of arrays support their reasoning about halving and doubling in multi-digit multiplication?*

Literature Review

The array is a powerful representation for multiplication because it makes the underlying spatial structure of multiplication visible (Barmby et al., 2009; Battista et al., 1998). As a two-dimensional representation, the array draws attention to equal groupings and to the coordination of composite units, enabling learners to perceive how multiplicative relationships are organised to produce a whole (Steffe, 1994). For this reason, several studies have examined students' spatial structuring when working with arrays in early multiplication and area contexts.

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Battista et al. (1998, p. 503) describe spatial structuring as mentally organising objects by identifying their parts, combining these parts into larger units, and recognising the relationships between them. In array contexts, spatial structuring is essential because it enables students to organise and enumerate spatial units and composite units, coordinate rows and columns as composite units, and recognise the multiplicative relationships represented in the array (Battista, 2004; Battista et al., 1998). Battista et al. (1998) identified three levels of sophistication through which students' progress when spatially structuring arrays. At the first two levels, students tend to focus on individual units or attend to rows and columns separately. Only at the most sophisticated level do students coordinate rows and columns simultaneously, enabling them to perceive the array as a structured multiplicative whole rather than as a collection of individual units. Consistent with this perspective, Outhred and Mitchelmore (2000) found that students often begin by covering rectangles with individual units and only gradually organise these into rows and columns that reflect the two-dimensional structure of the array.

Several studies have shown that arrays afford students access to important understandings in multi-digit multiplication. Larsson et al. (2017) argue that arrays help students move beyond additive interpretations of multiplication by making multiplicative relationships between factors visible. In particular, arrays can make the distributive property visible (Barmby et al., 2009; Izsák, 2004). Barmby et al. (2009) reported that when working with arrays, students demonstrated an intuitive sense of the distributive property.

Importantly, existing studies of students' understanding of the associative property have not examined students' reasoning in relation to arrays or considered how a representation such as the array might support students to make sense of halving and doubling. Downton et al. (2022), for example, examined students' understanding of the associative property across three applications: noticing, halving and doubling, and place value. Their findings showed that although many students could identify halving and doubling as an efficient strategy, few could explain why halving one factor and doubling another preserves the product, and students also demonstrated limited understanding of associative reasoning in place value contexts. These findings align with other research indicating that students' understanding of the associative property is often underdeveloped (Kanbir et al., 2018). Barnett and Ding (2019) similarly found that classroom teaching often left the associative property implicit and provided limited support for students to understand it as a structural property rather than as a computational rule.

Taken together, this literature suggests that while arrays support reasoning about multiplicative structure, their role in supporting understanding of halving and doubling remains underexplored. In particular, little is known about how students' spatial structuring of arrays may support understanding of halving and doubling as an application of the associative property in multi-digit multiplication. This paper addresses that gap by examining how students' spatial structuring of arrays supports their reasoning about halving and doubling in multi-digit multiplication.

Theoretical Framework

This study was informed by Realistic Mathematics Education (RME), which shaped both the design of the instructional sequence and the interpretation of students' learning. RME is grounded in the view that mathematics should be learned through students' own activity, with meaningful contexts and representations supporting the development of increasingly formal mathematical understanding (Freudenthal, 1973; Gravemeijer, 1994). From this perspective, mathematical learning involves students organising and reorganising situations mathematically, with models emerging first as representations of situated activity and later becoming tools for more general reasoning. In the present study, the array functioned in this way. Initially, it operated as a model of students' reasoning about cupcake arrangements and area, but through classroom discussion and repeated use it also became a model for reasoning about multiplicative relationships, including the associative property.

To examine how students reasoned with the array, the study drew on Battista et al.'s (1998) notion of spatial structuring. In array contexts, spatial structuring enables students to coordinate rows and columns as composite units and to perceive the multiplicative relationships represented in the array (Battista, 2004; Battista et al., 1998). In this study, spatial structuring provided the primary analytic lens for examining how students used arrays to reason about halving and doubling. Particular attention was given to the actions students performed on arrays, such as partitioning, translating, decomposing, joining, and rotating, and to whether these actions preserved equal groups, conserved the total number of units, and maintained the coordinated row-and-column structure of the array. This framework supported analysis of how students used the array to develop more generalised reasoning about multiplicative structure.

Methods

The research reported in this paper was situated within a larger project focused on articulating and refining a learning trajectory for multi-digit multiplication (Tripet, 2019). Design research methods, as articulated by Cobb and Gravemeijer (2008), were employed to enable first-hand observation of students' mathematical activity as instruction unfolded.

Context

The teaching experiment phase of the research was conducted in two Year 5 classrooms (students aged 9–11 years) in Sydney, with 23 students in Class 1 and 22 students in Class 2 (45 students in total). The classes were selected on the basis that they had not yet received formal classroom instruction in solving two-digit by two-digit multiplication problems, although some students had encountered the standard algorithm through tutoring or at home.

During the teaching experiment phase, I adopted the role of teacher, which allowed me to control the learning environment and the teaching episodes across both classes while experiencing first-hand the events of the classroom. The same instructional sequence was implemented in both classrooms over a two-week period. The sequence comprised four teaching episodes, each spanning two or three one-hour lessons. Each episode began with a problem for students to solve, followed by a whole-class discussion and time for students to explore and build on strategies shared during the discussion. This paper reports on the tasks that were the focus of the second and third teaching episodes from the teaching experiment.

Data Collection and Analysis

Three forms of data were collected: student work samples produced during the teaching episodes, transcribed video recordings of whole-class and small-group activity, and field notes recorded during and immediately after each lesson.

Consistent with teaching experiment methodology, analysis occurred in two phases. During the teaching experiment, ongoing cycles of design and analysis were used to interpret students' emerging reasoning and to inform subsequent instructional decisions. The analysis reported in this paper draws primarily on a retrospective analysis of the second and third teaching episodes.

The retrospective analysis was conducted through a spatial structuring lens. Video transcripts and student work samples were examined to identify how students organised and reorganised units and composite units within arrays when reasoning about halving and doubling. Analysis focused particularly on the spatial actions students used, drawing on the action categories identified by Antonides et al. (2025). The actions most relevant to this study are presented in Table 1. These actions were analysed in relation to whether they preserved equal groups, conserved the total number of units, and maintained the coordinated row-and-column structure of the array.

Strategies were also coded for the multiplicative reasoning they revealed, including correct and incorrect applications of halving and doubling and related alternative conceptions. The dataset was organised both by individual student and by solution method, enabling patterns to be identified within and across students. The examples presented in the results were selected as illustrative cases of broader patterns observed across the two classes.

Table 1

Coding for students spatial structuring actions (Antonides et al., 2025)

Spatial structuring actions	Descriptions
Decomposing	Dividing a spatial region into separate, disjointed, and potentially incongruent parts
Joining	Placing spatial objects adjacent to each other so that external parts of those objects coincide
Measurement-based comparison	Using numbers as measurements of spatial attributes to directly or indirectly compare two spatial objects
Numerical structuring	Constructing a coherent structure or form for a set of computations, including those involving numerical or algebraic expressions
Partitioning/Subdividing	Breaking a spatial object into separate, equal-size parts
Rotating/Orienting	Turning an object about a point, possibly in relation to a frame of reference
Spatial-numerical linked structuring	Spatial structuring that is conceptually linked to a numerical structuring
Translating	Mental act of moving an object in space while preserving its size, shape, and orientation

Results

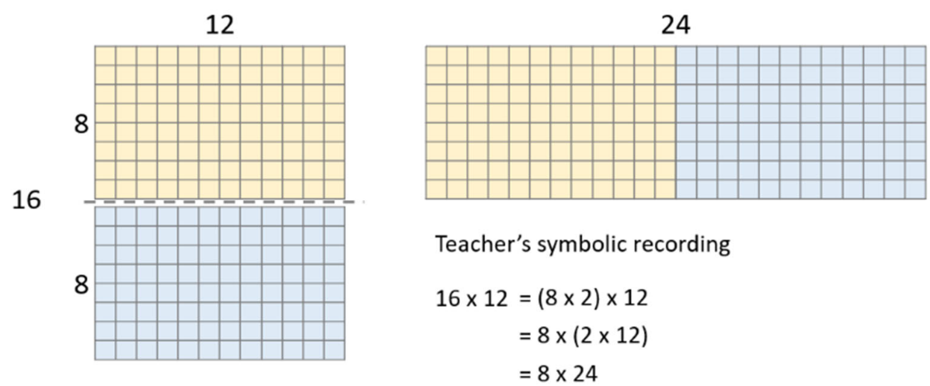
The instructional sequence followed the narrative of a bakery, with different forms of the array used to model a range of contexts. In the first teaching episode, students calculated the total number of cakes in a 24 x 8 array. Most students used the array to model their strategies, indicating confidence with the representation in its various forms. The results reported here focus on the second and third teaching episodes, where students' reasoning with arrays became the focus of analysis. The selected examples illustrate patterns observed across both classes. In each case, students' spatial structuring involved multiple actions that became visible through their interactions with the array.

In the second teaching episode, students were presented with an image of 16 filled cupcake boxes arranged in a 4 x 4 array, with each box holding 12 cupcakes. Although the individual cupcakes were not visible, students calculated the total number of cupcakes and noted that 16 x 12 was equivalent to a calculation from the first teaching episode, 24 x 8. Explaining this equivalence became the focus of investigation.

Ryan and Dylan (Class 1) constructed an array to represent the problem. Rather than representing the 16 boxes alone, they used one square unit to represent each individual cake within the boxes. In doing so, they recognised that 16 could be halved to make 8 and that 12 could be doubled to make 24. They then cut their grid array in half (partitioning) and rearranged it (translating) to create 24 x 8 array (Figure 1). Through these actions, they preserved both the total number of units and the coordinated row-and-column structure of the array (spatial-numerical linked structuring). When Ryan and Dylan later shared their work in the whole-class discussion, the associative property was introduced and explicitly linked to their halving and doubling strategy. Students were then given a series of multiplication problems to explore when halving and doubling might be useful. Through this work, they concluded that the strategy was particularly effective when it produced a multiple of ten or one hundred.

Figure 1

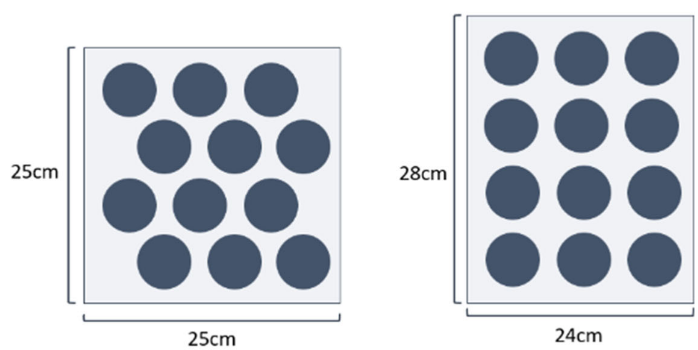
An illustration of Ryan and Dylan's representation with the teacher's symbolic recording



In the third teaching episode, students were shown an image of two cupcake trays inside cupcake boxes, one measuring 28 x 24cm and the other 25 x 25cm (Figure 2). The class discussed why one arrangement appeared skewed while the other did not, and students were asked to predict whether a 28 x 24 tray would have a larger area than a 25 x 25 tray. In both classes, some students proposed that taking 2 from 28 and adding it to 24 would produce 26 x 26, suggesting an incorrect extension of the earlier halving and doubling strategy. These responses shifted the focus of the investigation, and students were asked to test whether this reasoning was correct.

Figure 2

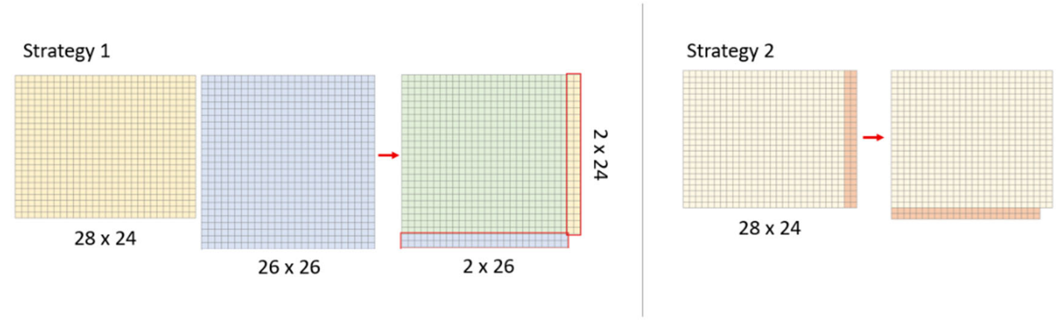
The third teaching episode and the trays that were the focus of investigation



Two significant strategies were observed across the classes (Figure 3). In the first, Mitchell and Tom (Class 1) constructed 28 x 24 and 26 x 26 arrays on grid paper and overlaid them to compare their areas directly (measurement-based comparison). Because the overhanging regions differed in size, they concluded that the two products were not equal, and that 28 x 24 was greater than 26 x 26 (spatial-numerical linked structuring). In the second, Amelie (Class 2) constructed a 28 x 24 array, cut two columns from the 28 (decomposing), and repositioned them as two additional rows beneath the existing 24 rows (joining and rotating). This revealed a 2 x 2 gap in the bottom corner, leading her to conclude that 28 x 24 must be 4 less than 26 x 26 (spatial-numerical linked structuring). In explaining her strategy, Amelie noted that the rows "don't match up", recognising that her transformation disrupted the coordinated row-and-column structure of the array.

Figure 3

An illustration of Mitchell and Tom's measurement-based comparison (Strategy 1) and Amelie's use of decomposition and rejoining (Strategy 2)



These strategies were later shared in whole-class discussion, after which students explored them further using other multiplication problems. Through area-based comparison and through decomposing and repositioning parts of arrays, students' spatial structuring supported their understanding of the distinction between additive and multiplicative compensation. This distinction was made explicit in Samar's explanation:

Samar: Last time we halved and doubled. When you halved, the two parts were the same and so you could move the parts of the array. But when you take just two off, the parts are not the same and so they won't match. So, I think that you can divide but not subtract.

Teacher: What do you mean that you can 'divide'?

Samar: You can divide by 2 and then double or divide by 3 or divide by 4…and you can keep going. But you can't just subtract 2 or 3 or 4. You need the parts [of the array] to be the same.

Over the following three one-hour lessons, the use of the associative property, particularly the use of factors to decompose numbers, became taken-as-shared in the class. This was illustrated by one student's use of factors to solve 32 x 25:

Louisa: 25 is a friendly number because you just multiply it by 4 to get 100, so you divide the 32 by 4 to get 8, so it is just the same as 8 x 100.

Discussion and Implications

These findings suggest that students' spatial structuring of arrays provided a useful basis for reasoning about halving and doubling as an application of the associative property in multi-digit multiplication. Downton et al. (2022) found that although many students could identify halving and doubling as an efficient strategy, few could explain why halving one factor and doubling another preserves the product. The present study points to a possible way of addressing this difficulty. Working with arrays in which individual units were explicitly represented appeared to provide students with a more concrete basis for reasoning about why halving and doubling preserves equivalence.

Across the examples, students chose to represent each individual unit, suggesting that explicit unit representation was important for making relationships between the factors visible. However, the findings indicate that constructing an array alone was not sufficient. What mattered was how students transformed the array and whether those transformations maintained the coordinated row-and-column structure of the array and preserved the product. Amelie's actions of (decomposing), (rotating), and (joining) disrupted this structure and did not preserve the product. This was also reflected in Mitchell and Tom's measurement-based comparison, which showed that 28 x 24 was not equivalent to 26 x 26 as the overhanging regions were different in size. By contrast, Ryan and Dylan's (partitioning) and (translating) preserved equal groups and maintained the total product. Samar picked up on this idea when she explained that halving (partitioning) produced equal-sized parts, whereas subtraction in this case did not.

Together, these examples suggest that students were better positioned to reason about halving and doubling when they could see that the product remained unchanged only when the array was partitioned using equal groups and so maintaining the overall multiplicative structure.

The findings also highlight the importance of making these structural relationships explicit (Shifter et al., 2008). Barnett and Ding (2019) found that classroom instruction often overemphasised computational strategies for obtaining an answer without making explicit links to the underlying properties that explain why those strategies work. They noted that teaching often left the associative property implicit and provided limited support for students to understand it as a structural property rather than as a computational rule. They also noted that teaching did not always adequately connect the abstract notion of the associative property to concrete forms of justification. The present study offers one example of how this might be addressed in the classroom.

In contrast, the present study explicitly connected students' strategies to the associative property and encouraged them to test halving and doubling across multiple cases. Students were afforded time to investigate their conjectures and to see why the product was preserved by making visible their spatial structuring actions, and specifically if these actions preserved the coordinated row-and-column structure of the array. Importantly, students also explored examples and non-examples of halving and doubling, which drew attention to the importance of partitioning into equal-sized groups. This focus on what remained invariant across cases may have supported students to recognise the strategy as an instance of a general property of multiplication (Mason et al., 2009), as reflected in Louisa's final solution to 32 × 25.

Through this teaching experiment, the array provided a concrete means through which students could test and justify halving and doubling as a viable strategy. More importantly, this suggests that teaching for understanding of the associative property should involve more than naming the property or demonstrating an efficient method. Students need opportunities to work with representations that make multiplicative structure visible and to examine which transformations preserve that structure and which do not.

Conclusion

Overall, this study suggests that the value of the array in teaching halving and doubling lies not simply in representing multiplication, but in enabling students to test whether transformations preserve multiplicative structure. Students' reasoning appeared to become more structurally grounded when they recognised that preserving the product depended on maintaining equal groups and the coordinated row-and-column structure of the array. In this way, the findings offer a more precise account of how arrays may support understanding of the associative property in multi-digit multiplication.

Acknowledgements

Ethics approval 14726 was granted by The University of Sydney with informed consent given by participants and parents/caregivers.

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