Representations of multiplicative word problems

Lynne Outhred

School of Education Macquarie University

The responses of 115 children in Years 1 to 4 to three types of multiplicative word problems were analysed. The representations that children drew to solve or to explain solutions were categorised in terms of structural characteristics and examined to assess if the drawn representation was related to success in solving the problems. There appeared to be a relationship only for the Cartesian product problem. The wording of the problem was found to influence the structure of the drawn representation.

This paper presents an analysis of children's responses to three multiplicative word problems that formed part of a sequence of tasks concerning rectangular area measurement and its relationship to multiplication concepts. The word problems included one equivalent group, one array and one Cartesian product problem; they were included in the study to vary the sequence of tasks and to explore if methods children used to solve them were related to their knowledge of array structure.

While arrays are commonly used models for illustrating multiplicative situations, there seems little research specifically dealing with children's understanding and use of such structures to represent multiplicative situations. English (1982) interviewed twenty children in Year 2 and determined the models they used when solving simple number sentences. The majority of the children used the equivalent set, rather than the array model but the reason for this preference was not discussed. Beattys and Maher (1989) presented children from Grades 4 to 6 with six pictorial models for multiplication (area, array, Cartesian product, grouping, number line, and stacks) using ten related pictures. The children were asked which pictures could be used to explain multiplication to second grade students, and how selected pictures could be used. The researchers commented that some children who were familiar with a discrete array (lines of people) did not seem to associate the area model (a rectangular array of six rows, each of four squares) with multiplication.

The aim of this paper is to investigate strategies that children of different ages use to solve different multiplicative word problems and the relation between their solution methods and their drawn representations of multiplicative problem structures.

Methodology

The tasks were presented individually to 115 children in Years 1 to 4 at four Sydney schools; approximately equal numbers of boys and girls were interviewed from each school and grade. During the interviews the methods children used to solve the word problems were monitored, that is, whether the children drew a picture, counted on their fingers, or counted "in their heads" without making use of any observable counting aids. Children who solved the word problems mentally were asked to explain their solutions and it was suggested that they draw a picture for at least one of the three problems. Thus, not all children drew representations for each problem. The observed strategies were later categorised, as were drawings that children made to help them solve the problems. The problems were as follows:

Equivalent group structure (EG)

* In the classroom there are four groups with six children in each group. How many children are there altogether?

Array structure (AR)

* Your teacher asks you to put out three lines of chairs and to put seven chairs

in each line. How many chairs do you have to put out?

Cartesian product (CP)

* A shop sells ice cream cones. There are four flavours of ice-cream (chocolate, vanilla, strawberry and banana). The cones come in three sizes, small, medium and big. How many different ice-cream cones can you choose?

The equivalent group problem was presented first, followed by the array problem, then the Cartesian product problem. The children were given a sheet of paper before each word problem was presented and were told that the paper was to use if they wished to make a drawing to work out the answer. Each problem was read to the children and they were also shown a card with the problem printed on it. The question was repeated if necessary and children were encouraged to draw a solution. If children began to draw elaborate pictures of objects (eg, of children or chairs), the interviewer would suggest that they might like to draw circles, or sticks, for the chairs (to prompt them to use abstract representations). The reason for this suggestion was that some children connected terms such as "picture" or "drawing", with an artistic representation, rather than a structural one. Occasionally children became so absorbed in drawing details of the objects that they forgot the problem itself.

Results

The difficulty levels of the equivalent group problem and the array problem were almost identical (54% and 53% of children correctly solved them respectively) whereas the Cartesian product problem was far harder (21% of the children successfully solved it). The percentage of the children in each year who solved each type of problem correctly is shown in Table 1.

Success on these word problems is clearly dependent on children's year level at school. Children found the Cartesian product problem far more difficult than either the equivalent group or array structure at each year level. These results are consistent with the findings of Mulligan (1992) who interviewed 70 children on four occasions, at the beginning and end of Year 2 and Year 3 on structurally similar problems. The Year 2 results in Table 1 can be approximately compared with her results for early in Year 2 (27%, 39%, 1% of her sample correctly solved the respective problems) and the Year 3 results with her findings for the end of Year 2 (54%, 76%, and 3%).

As can be seen in Table 1, there was a consistent increase in the percentage of children who correctly solved the equivalent group and array problems from Year 1 to Year 4. The Year 2 percentage for the equivalent group problem is higher than might be expected compared with Mulligan's results but this might have been due to sampling fluctuations. At the end of Year 3, 18% of Mulligan's sample correctly answered the Cartesian product problem she set (three sizes and two flavours of chips). Thus, while the results for the equivalent group and array problems are quite similar to Mulligan's results at the end of Year 3, early in Year 4 children in the present study seemed to do better on the Cartesian product problem, perhaps because the accompanying array tasks could be connected to the structure of this problem. The most common solution strategies were as follows:

Table 1 The percentage of children who obtained correct answers for each problem type.

	Year 1 n=30	Year 2 n=31	Year 3 n=26	Year 4 n=28
EG (4x6)	20	52	58	89
AR (3x7)	13	42	73	93
CP (3x4)	3	13	15	57

Strategies that resulted in incorrect solutions

The percentages of children using each of the following strategies for the three types of problems are shown in Table 2. These results indicate that the strategies used to represent the equivalent group and array word problems were similar, however, the strategies applied to these two problems were quite different to those applied to the Cartesian Product problem.

Specific solutions

Children made the problem specific to their own experience by, for example, drawing their own classroom groups or saying "I would like a big ice cream."

Repetition of one variable

Children sometimes appeared to have no concept of grouping and repeated one of the numbers in the problem, for example, four children or three chairs might be drawn..

Additive solutions

Additive solutions were given for all three problems, for example, ten children or chairs. Similarly, the number of ice-cream flavours and cone sizes were often added to give an answer of seven (or sometimes, eight!).

Multiplicative solutions

Some children used an incorrect representation (e.g., they counted or drew one group too many), while others drew a correct representation but counted it inaccurately.

The most prevalent incorrect solutions for the equivalent group and array problems were to add the two quantities, to draw an incorrect multiplicative representation or to count it inaccurately. When children answered the Cartesian product problem they commonly focused on specific aspects of the ice creams that were not related to the problem solution or they repeated one of the given quantities (the number of flavours or the number of cones sizes).

Many children solved the equivalent group and the array problem mentally or by counting their fingers (69% and 53% respectively). Children who initially obtained incorrect solutions mentally or using their fingers were asked to draw a picture to show how they had solved the problem (31 children for EG and 20 for AR). The initial solution method and the method indicated by the representation were then compared. The drawn representation replicated the incorrect solutions for 35% of the responses for EG and 25% of those for AR. Children used two different incorrect methods in approximately 15% of the solutions for both problems, for example, a child might answer "ten children", then draw a picture of two groups of six children. These children did not seem to see that their solutions were inconsistent. The remaining children represented the problem situation correctly in their subsequent drawing. Thus, about half the children who initially gave an incorrect solution could draw a correct representation of the problem when prompted.

Table 2 The percentage occurrence of each type of incorrect representational strategy

Incorrect strategy	EG	AR	CP
Unable to represent the problem or a specific solution given	8	4	25
Repetition of one quantity	3	6	42
Addition of the quantities	17	19	10
Incorrect multiplicative representation	9	8	1
Correct multiplicative representation (incorrect answer)	9	10	1
Total incorrect solutions	46	47	79

However, the challenge that children faced in the Cartesian product word problem was to model the problem structure because Cartesian Product problems involve the notion of unlimited categories rather than individual elements that can be directly represented. As Vergnaud (1988) and Schwartz (1988) have pointed out, in such problems children have to create a new quantity (the number of ice-cream cones) from two separate quantities (the numbers of flavours and sizes). Children in this study had great difficulty comprehending the problem; they focused on a specific aspect (for example, which flavours they would like or the number of flavours) or they added the quantities. Approximately a third of the sample (37%) thought the answer to the Cartesian product problem was either three (the number of cone sizes) or four (the number of flavours).

Such specific solutions may be a consequence of children's difficulties with either the wording or the structure of Cartesian product problems. These solutions may have been less prevalent if the children had been given a number of "hands-on" Cartesian product problems, such as those used by English (1993), to investigate children's problem solving strategies. In her study children dressed toy bears in different combinations of outfits. However, she did find that children were reluctant to select an item more than once, even when multiple items were available. She attributed children's reluctance to interpretation of the problem wording; her tentative explanation was that children interpret "different" to mean "different in all ways", rather than "all the different outfits". Comments made by children in this study would support her conclusion, eg "One flavour wouldn't have a cone.", "But there's only three sizes."

Strategies that resulted in correct solutions

Correct solutions to the equivalent group and array problems were obtained by a variety of methods: drawing each object and counting them all, counting on fingers, repeated addition and multiplication. The majority of children in Years 1 and 2 (85%) solved these problems by counting on their fingers or by individually counting objects that they drew or imagined. These children also more frequently used drawing or finger counting rather than mental strategies but in Years 3 and 4 mental strategies were the most common. The use of individual counting strategies also dropped to 36% for Year 3 and 4 children. Older children usually solved the problems mentally by grouping numbers, for example, "7+7=14, 14+7=21", or by multiplying. Children who correctly answered the Cartesian product problem usually multiplied but there were exceptions. Three of the four younger children who solved it drew ice cream cones combining each size and flavour and counted the ice-creams individually as did three of the older children.

Children's drawings

The drawings children made, either to solve the problems or to explain their solutions, were analysed. There were 108 drawings altogether but the drawings were not from the same group of children for each problem. The categories that were used to summarise the children's solution methods are illustrated in Figure 1 and are describedas follows:

The spatial structure of any drawing that the child made.

1 The drawing had no obvious structure.

2 The drawing had a linear structure,eg, a child might draw a line of circles to represent all the children in the class.

3 Non-linear groups were drawn in which the elements were not aligned.

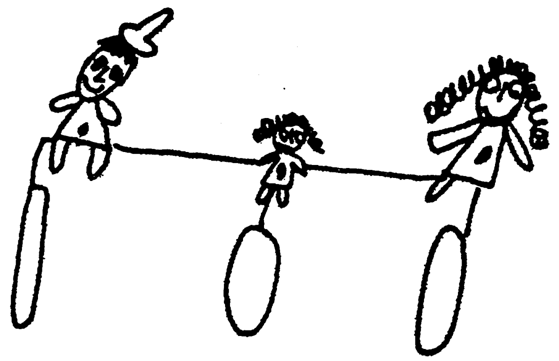
4 The child drew linear groups in which the elements were aligned in one dimension.

5 An array was drawn in which the elements of the groups were aligned in two dimensions. All the arrays drawn to represent word problems were discrete.

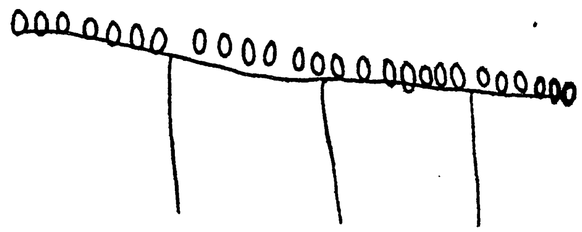
Categories 4 and 5 emphasise the structural dimensions of the array eg linear groups of 4 or 6 for the equivalent group problem but in category 4 the form of an array was not represented. The effect of key words in the problem (eg "group" or "line") influenced the type of representation that children drew. More children drew non-linear groups for the equivalent group problem (48%) than for the array problem (6%); the opposite was true for linear group and array representations (26% for the equivalent group problem comared with 64% for the array).

Figure 1 Illustrations of the spatial categories for the equivalent group word problem solutions.

No obvious structure

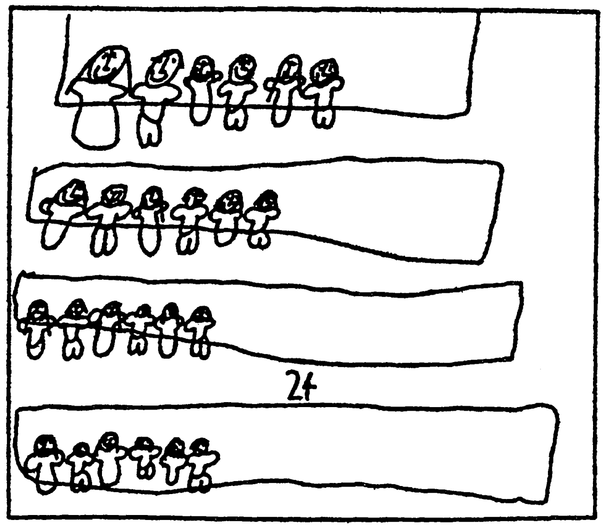


Linear structure

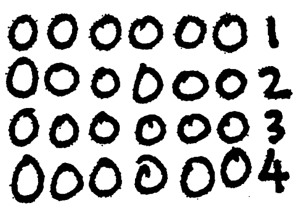


Non-linear groups

Linear groups



An array



The drawings of the children who successfully answered the Cartesian product problem (n=25) were examined to see if these children drew structured representations; 19 of these children had made a drawing to obtain or explain their solutions. All but one of these children drew either linear groups or an array representation; the children who did not draw a solution represented it numerically eg 3+3+3+3 or 3x4 to explain their reasoning. Some drawings were interesting in that they represented the process the child used to obtain the solution rather than being an enumeration of each element. However, drawing a structured solution to explain the Cartesian product problem may simply be an indicator that a child has a good understanding of multiplication. For the equivalent group problem there did not seem to be a relation between success on the problem and the structure of the representation. Similar numbers of children in Years 2 to 4 drew non-linear groups as opposed to linear groups and the success rate was similar for children using each approach. Children in Year 1 drew solutions of the first three types only.

Conclusions

This paper has presented some preliminary findings on the strategies children used to solve three different types of word problems and how success on these problems was related to age. The different types of representations children drew were strongly influenced by the problem wording and context. It was difficult to assess if some forms of representation were more effective in helping children solve the word problems than others because these findings were equivocal and the study was primarily exploratory. For the equivalent group problem it did not seem to make a difference if children drew less structured representations (non-linear groups). Too few children drew non-linear representations for the array problem for any conclusions to be drawn, however, for the Cartesian product problem, children who were successful drew structured representations.

The importance of structured representations are that they lead children towards array and area models. There are major implications for teaching if children associate equivalent set but not area models (arrays of adjoining squares) with multiplication as area models can be used to demonstrate commutativity, generalise more readily to multiplication of large numbers and they are an integral part of learning about fraction, area and volume concepts. The equivalent set model, on the other hand, becomes increasingly cumbersome for multiplication of large numbers, does not illustrate commutativity and does not play an important part in the development of higher-order concepts.

References

Beattys, C., & Maher, C. (1989). Exploring children's perceptions of multiplication through pictorial representations. In G. Vergnaud, J. Rogalski, & M. Artique (Ed.), Thirteenth International Conference for the Psychology of Mathematics Education, 1 (pp. 109-116). Paris:

English, L. (1982). An investigation of Year two children's interpretation of multiplication when presented as both a set and an array model. In Mathematics Education Research Group of Australasia, (pp. 31-47). Melbourne:

English, L. (1993). Children's strategies for solving two- and three-dimensional combinatorial problems. Journal for Research in Mathematics Education, 24(3), 255-273.

Mulligan, J. (1992) An analysis of children's solutions to multiplication and division word problems. PhD, Macquarie University.

Schwartz, J. (1988). Intensive quantity and referent-transforming operations. In J. Hiebert & M. Behr (Eds.), Number Concepts and Operations in the Middle Grades (pp. 41-52). Reston: National Council of Teachers of Mathematics.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), Number Concepts and Operations in the Middle Grades (pp. 141-161). Reston, Virginia: National Council of Teachers of Mathematics.