The implication of multiplicative structure for students' understanding of decimal-number numeration

Annette R Baturo
The Centre for Mathematics and Science Education
Queensland University of Technology, Brisbane, Australia

This paper reports on a study that examined the importance of multiplicative structure and whether, after several years of formal instruction in the decimal number system, Year 6 students had acquired an understanding of this structure. To this end, 173 Year 6 students were tested with a pencil-and-paper instrument developed to assess the decimal-number numeration processes that are normally taught in primary school and from which 45 students (32 high performing, 13 low performing) were selected for interviewing on tasks related to multiplicative structure. The interviews revealed that only the most proficient students (≥ 90% for tenths and hundredths in the test) had acquired a structural schema of multiplicativity that enabled access to application tasks

Baturo (in preparation) developed the model shown in Figure 1 to represent the "levels" of knowledge required for an understanding of place value in the decimal number system. Level 1 knowledge comprises the baseline knowledge of *position*, *base* and *order*, without which students cannot hope to function with any understanding in numeration tasks. Level 2 knowledge comprises *unitisation*, namely, the assignment of a numerical value to part/whole relationships (Behr, Harel, Post, & Lesh, 1994; Lamon, 1996) and *equivalence*, both of which require an understanding of base in processing decimal numbers. Level 2 knowledge is seen as the knowledge that connects and provides meaning for Level 1 knowledge. Level 3 knowledge comprises the structural knowledge (*additive structure, multiplicative structure* and *reunitisation*) that provides the superstructure for integrating all levels of knowledge.

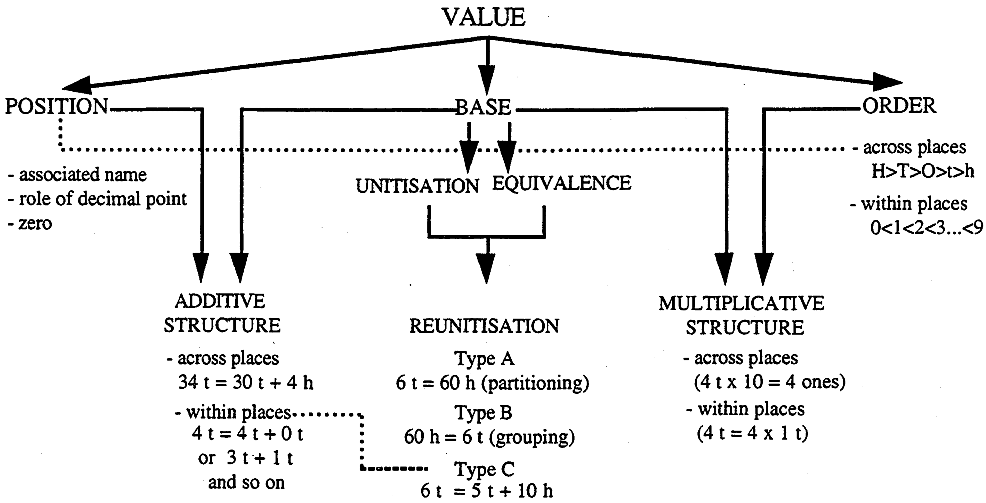

Figure 1. Cognitions and their connections embedded in the decimal number system

This paper reports on a study which explored Year 6 students' knowledge of the continuous, bi-directional and exponential properties of multiplicative structure (Smith & Confrey, 1994). Having an understanding of the multiplicative structure embedded in the decimal number system (see Figure 2) is crucial and, as argued by Baturo (1995), if not explicated for whole numbers, denies students one of the major conceptual underpinnings of decimal numbers.

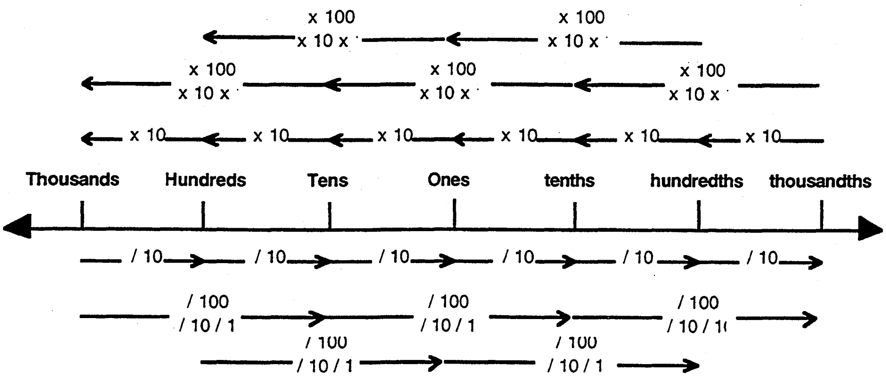

Figure 2. Multiplicative relationships embedded in the decimal number system.

The study is part of a larger project (Baturo, in preparation) which explored students' acquisition and access of the cognitions required to function competently with decimal numbers. To this end, 173 Year 6 students from two schools (different socioeconomic backgrounds) were tested with a pen-and-paper instrument that included items designed to assess number identification, place value, counting, renaming and regrouping, comparing/ordering, and approximating and estimating for: (a) tenths, and (b) hundredths.

The results showed that some students performed equally well on the tenths and hundredths components of the test whilst some performed much better on tenths than hundredths. To distinguish between the regular and irregular distributions, the students were classified as *proficient* and *semiproficient* respectively. The final interview selection comprised 12 high proficient (HP) students (test mean ≥90% for tenths and hundredths), 12 high semiproficient (HSP) students (≥85% tenths; <75% hundredths), 8 medium proficient (MP) students (80-90% tenths and hundredths), 8 medium semiproficient (MSP) students (75-85% tenths; < 65% hundredths) and 5 low proficient (LP) students (40-60% tenths and hundredths).

The 45 students were interviewed individually. All tasks were undertaken in the same order but the interviewer was free to probe responses when required. Each interview took place at the student's school during school hours and took approximately 30 minutes. The interviews were all conducted during the first quarter of the school year and were video-taped for further analysis.

Multiplicative tasks

To investigate the students' understanding of multiplicative structure, Baturo (in preparation) developed the following tasks.

Task 1. Two sets of place name cards (see Figure 3) were used in this task, one set for the interviewer and one set for the students.

thousan	thousar	hundre	tens	ones	tenths	hundrec

Figure 3. Place name cards used in Task 1.

To assess binary relationships (×/÷ 10), the interviewer put out *hundreds, tens* and asked the students if they could find another pair of places that were related in the same way and, if so, to explain their choice. (This procedure was duplicated with *hundreds, ones* to assess the ternary relationship of 100.) This first task was made as abstract as possible so that the students would be focused on the relationship between the places and not have the potential distraction of the syntactic features of the symbol in order to ascertain what components of multiplicative structure the students had acquired after several years of formal instruction.

Task 2. This task consisted of the 12 items (6 multiplication, 6 division) shown in Figure 4. In this task, the students were required to explain how they predicted a multiplicative shift (i.e., determined the finishing number when given the starting number and a multiplicative operation).

a	b	c	d	e	f
0.3 × 10 = __	0.04 × 100 = __	6.23 × 10 = __	0.7 × 100 = __	2.16 × 100 = __	0.2 × 10 × 10 =
g	h	I	j	k	k
4 ÷ 10 = __	72.5 ÷ 10 = __	0.9 ÷ 10 = __	37 ÷ 100 = __	8 ÷ 100 = __	14 ÷ 10 ÷ 10 = _

Figure 4. Task 2 items.

Task 3. The third task required the students to use a calculator to select the operation in examples similar to the following: (a) *change 7 tenths to 7 ones*; and (b) *change 8 ones to 8 hundredths*. Thus the students were provided with the starting and finishing numbers and were required to use their knowledge of the bidirectional nature of multiplicative structure to provide the operation that would make the change. For those students who were unable to do this, teaching intervention was undertaken using a place value chart and a set of digit cards to show the direction and size of the change.

Task 1 was seen as a means of determining the students' available knowledge of multiplicativity whilst Tasks 2 and 3 were seen as a means of determining whether students could access their available knowledge in application tasks.

Students' responses

Task 1

With respect to the binary relationships, 28 students (9 HP, 9 HSP, 8 MP, 2 MSP) selected an appropriate pair of place names. The most common inappropriate responses were to make no response (7 students – 1 HP, 2 MSP, 4 LP) or to select *hundredths, tenths* (7 students – 2 HP, 2 HSP, 3 MSP). Each of these latter students explained that their selection had been made on the basis of the names (a syntactic feature of the task). However, when asked to consider the order of the names, 6 (2 HP, 2 HSP, 2 MSP) of the 7 students changed their selection to *tenths, hundredths*.

When asked to explain the reasoning for their selection, 56.6% of the students (11 HP, 7 HSP, 5 MP, 1 MSP, 1 LP) gave responses which indicated that they had the appropriate knowledge of bi-directionality (×, ÷) available as well as exponentiality (the relationship of 10). Their explanations included comments such as "if you times tens by 10, you'll get a hundred; divide it by 10 to make it 10 times smaller". A further 6.7% of the students (1 HP, 1 HSP, 1 MP) indicated that they had uni-directional (×, not ÷) knowledge of the relationship available. All of these students associated the relationship from left to right (larger to smaller) with subtraction. One student (MP) had bi-directionality (×, ÷) but did not know the relationship of 10. She had selected *thousands, hundreds* and explained that *hundreds* are multiplied by hundreds to produce *thousands* and that *thousands* are divided by thousands to produce *hundreds*. Eight students (4 (HSP, 1 MP, 3 MSP – 17.8%) applied additivity (+, –) to the relationship (e.g., "add 9 tenths to make ones and take away ones to make tenths"). The remaining 8 students (4 MSP, 4 LP – 17.8%) were unable to explain their selection. (See Table 1 for the results of this task in terms of the performance categories.)

With respect to the ternary relationship (100), 71.1% (12 HP, 11 HSP, 7 MP, 2 MSP) of the students selected an appropriate pair. Their spontaneous explanations revealed that they had used the syntactic feature of position to help them select a matching pair [e.g., "the middle one (place) was missing" or "it jumps one (place)"]. However, probing their explanations revealed that 48.9% (11 HP, 6 HSP, 4 MP, 1 MSP) had full multiplicative structure (bi-directional and exponential), 6.7% (1 HP, 1 HSP, 1 MP) had uni-directional (×) and exponential (100) knowledge, 4.4% (1 HSP, 1 MP) had bi-directional but not exponential knowledge whilst 15.6% (4 HSP, 2 MP, 1 MSP) applied additivity. (See Table 1.)

Task 2

The students performed better for multiplication than for division as shown by the following results: HP (97.2%, 83.3%); HSP (79.2%, 44.4%); MP (43.8%, 10.4%); LP (25.9%, 06.7%). For multiplication, the successful students tended to use two main strategies: (a) *shifting* either the digit/s or the decimal point, and (b) *renaming* (e.g., "3 tenths times 10 is 30 tenths; 30 tenths equals 3 ones"). The unsuccessful students used one or more of the following strategies: (a) inserting a zero (the *whole-number* rule – "10 has one zero so add a zero") either at the end of the given number (e.g., 0.3 × 10 = 0.30) or at the end of the whole-number part (6.23 × 10 = 60.23); (b) *guessing* (which produced erratic responses across the tasks); and (c) omitting the task. The *shifting* strategy was successfully transferred to the division items but the latter strategy and would therefore be successful for both multiplication and division.; which would be more difficult to apply to division.

For division, the successful students used the *shift* strategy. (No other successful strategy was revealed.) The unsuccessful students used one or more of the following strategies: (a) *reversed* the operation (e.g., 10 ÷ 4 instead of 4 ÷ 10); (b) used the *wrong operation* (multiplied instead of divided); (c) inserted *zeros* inappropriately to make the number smaller (e.g., 0.0037); (d) *guessed*; or (e) *omitted* the task.

The interesting feature of this task was the way that students changed appropriate strategies for inappropriate ones. For example, 37 students (12 HP, 12 HSP, 5 MP, 5 MSP, 3 LP) used the shift strategy for at least one multiplication item but only 25 students (12 HP, 9 HSP, 2 MP, 1 MSP, 1 LP) maintained the strategy across the multiplication tasks. Furthermore, very few students transferred the shift strategy to division. One suggestion for these behaviours (failure to maintain and transfer a successful strategy) is that students whose knowledge is not semantic are swayed by the syntactic features of tasks, a phenomenon which suggests that, for many students, their knowledge of multiplicative structure is rule-based and does not include an understanding of the continuous, bi-directional and exponential properties of multiplicativity.

The students' explanations revealed that the shift strategy can be semantically-based (i.e., emanating from an understanding of the multiplicative structure of the decimal number system) or rule-based (i.e., × 10 has one zero so shift all digits one place to the left). Students who maintained the shift strategy across all the items revealed that they understood the principles underlying the strategy.

Task 3

With respect to effecting the change from 7 tenths to 7 ones, 60% only (12 HP, 8 HSP, 6 MP, 1 MSP) selected the appropriate operation (× 10), 6.7% (2 MSP, 1 LP) selected the correct direction (×) but did not know the relationship (10). With respect to effecting the change from 8 ones to 8 hundredths, all but one of the students (HSP) who was successful with the binary relationship was successful with the ternary relationships. Thus 57.8% only (12 HP, 7 HSP, 6 MP, 1 MSP) revealed an understanding of the continuous, bi-directional and exponential properties of multiplicative structure. Of the remaining 19 students, 2 (1 HSP, 1 MP) revealed that they had uni-directional knowledge (×, not ÷), 4 (3 HSP, 1 MP) revealed that they could only access full additivity (+, –) whilst the remaining 13 students were unable to do either task. (See Table 1.)

Summary

Table 1
Responses across the tasks in terms of the performance categories.

	Performance categories				
	HP (*n* = 12)	HSP (*n* = 12)	MP (*n* = 8)	MSP (*n* = 8)	LP (*n* = 5)
Task 1					
(binary)	11 (91.7%)	7 (58.3%)	5 (62.5%)	1 (12.5%)	1 (20.0%)
Full MS (×, ÷)	1 (08.3%)	1 (08.3%)	2 (25.0%)	0 (00.0%)	0 (00.0%)
Partial MS	1	1	1	—	—
– ×, –	—	—	1	—	—
– Op.$_3$, base °	0 (00.0%)	4 (33.3%)	1 (12.5%)	3 (33.3%)	0 (00.0%)
Full AS (+, –)					
Task 1					
(ternary)	10 (83.3%)	6 (50.0%)	4 (50.0%)	1 (12.5%)	0 (00.0%)
Full MS (×, ÷)	2 (16.7%)	2 (16.7%)	2 (25.0%)	0 (00.0%)	0 (00.0%)
Partial MS	1	1	1	—	—
– ×, –	1	1	1	—	—
– Op.$_3$, base °	0 (00.0%)	4 (33.3%)	2 (25.0%)	1 (12.5%)	0 (00.0%)
Full AS (+, –)					
Task M2					
Full MS	10 (83.3%)	5 (41.7%)	0 (00.0%)	0 (00.0%)	0 (00.0%)
Partial MS	2 (16.7%)	4 (33.3%)	1 (12.5%)	1 (12.5%)	0 (00.0%)
Task 3 (× 10)					
MS	12 (100%)	8 (66.7%)	6 (75.0%)	1 (12.5%)	0 (00.0%)
Partial MS	0 (00.0%)	0 (00.0%)	0 (00.0%)	2 (25.0%)	1 (20.0%)
– op.$_3$, base °	—	—	—	2	1
– op.^{8}, base ¸	—	—	—	—	—
Task 3 (÷ 100)					
MS	12 (100%)	7 (58.3%)	6 (75.0%)	1 (12.5%)	0 (00.0%)
Partial MS	0 (00.0%)	2 (16.7%)	0 (00.0%)	1 (12.5%)	0 (00.0%)
– op.$_3$, base °	—	—	—	1	—
– op.^{8}, base ¸	—	2	—	—	—

Note. MS = multiplicative structure; AS = additive structure; op. = operation.

Table 1 shows that, across the tasks: (a) the HP students generally displayed expert knowledge of multiplicative structure; (b) the HSP students had some knowledge of multiplicative structure; (d) the MP students displayed knowledge of multiplicative structure in Tasks 1 and 3 but displayed no knowledge of multiplicative structure in Task 2; and (e) the low-performing students (MSP, LP) revealed that they had no knowledge of multiplicative structure. These results suggest quite strongly that multiplicative structure is a determining factor in differentiating high-performing (HP, HSP, MP) and low-performing (MSP, LP) students and in differentiating HP students from the other high-performing groups.

Knowledge availability and access

In Task 1, the students' syntactic explanations were probed to determine whether they also had the appropriate semantic knowledge. Thus Task 1 was viewed as a means of establishing the students' available knowledge of the continuous, bi-directional and exponential properties of multiplicative structure. Tasks 2 and 3 were designed to reveal whether those students who had the appropriate available knowledge had structured their knowledge in a way that promoted access in application tasks. Table 2 provides the findings of these tasks in terms of availability and access. Note that *continuity* is not included in the table because it was not a factor in prohibiting availability or access.

Table 2
Students' availability and access of multiplicative structure in terms of the performance categories.

	PERFORMANCE CATEGORIES				
	HP (n = 12)	HSP (n = 12)	MP (n = 8)	MSP (n = 8)	LP (n = 5)
Availability (Task 1)					
– Bi-directional (×, ÷)	11	7	5	1	1
– Uni-directional (×)	1	1	1	1	0
– Exponential (B, T)	11	6	4	1	0
– Exponential (B)	1	3	1	1	1
Access (Tasks 2 and 3)					
– Bi-directional (×, ÷)	10	2	0	0	0
– Uni-directional (×)	2	6	0	0	0
– Exponential (B, T)	10	2	0	0	0
– Exponential (B only)	0	2	0	0	0

Note. B = binary; T = ternary.

The results show that the poor performance of the low-performing students (MSP, LP) was due to unavailability of knowledge rather than lack of access. The poor performance of the MP students was shown to be due to access, suggesting that their available knowledge was not connected to form an integrated schema of multiplicative structure. The HSP students generally had available and accessible knowledge of binary relationships only. The HP students alone revealed that they generally had the appropriate knowledge available for both binary and ternary relationships and could access this knowledge, suggesting that they had constructed a schema of multiplicative structure.

Conclusions

The interview tasks revealed that the HP students alone had full multiplicativity of binary relationships, could transfer this knowledge to ternary relationships and could access this knowledge. Thus they were assumed to have constructed *a complete structural schema* of multiplicativity that incorporated the three components of continuity, bi-directionality and exponentiality, and connections between the components.

Figure 4 illustrates the available knowledge of the components and the assumed degree of connections held by each performance category. The black shading indicates full knowledge of that particular component (e.g., continuity across and within domains, bi-directionality, and exponentiality for binary and ternary relationships). The grey shading indicates limited knowledge of that component (e.g., continuity within but not across domains, uni-directionality, and exponentiality for binary but not ternary relationships). No shading represents no available knowledge of the component. The arrows indicate that the knowledge components are connected and therefore the knowledge is accessible. No arrows indicates no connections and therefore no access.

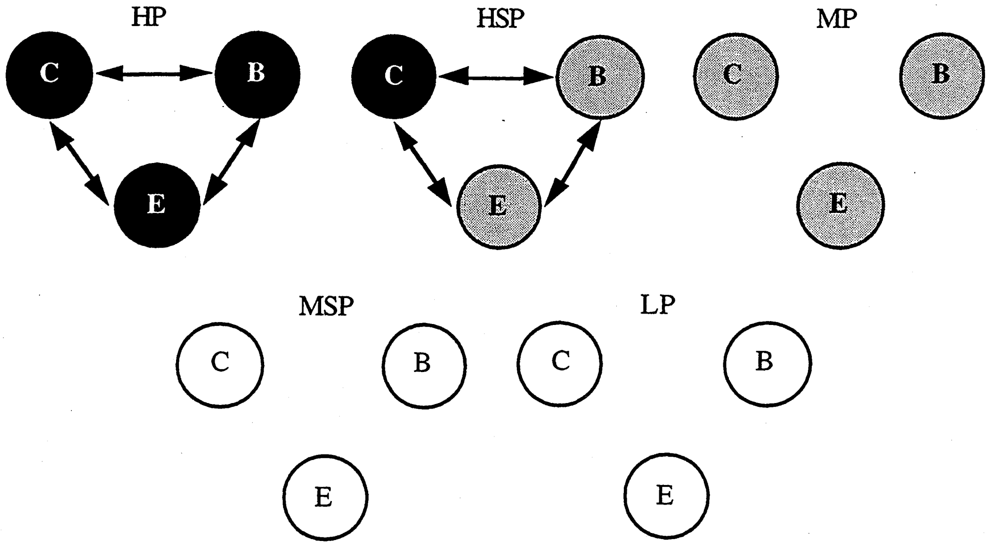

Figure 4. Structural schema of multiplicativity exhibited by the performance categories.

Two points of interest emerge from the models of multiplicative structure. One relates to the complete lack of knowledge held by the low-performing students. In the earlier interview tasks related to position and order, the students in these groups had revealed no knowledge of the decimal-fraction place names or of the value order (i.e., tenths are larger in value than hundredths) of the decimal-fraction places. This lack of knowledge was thought to be due to the absence of the exponential mental model shown in Figure 1, a supposition that is supported by their performance on the multiplicative tasks.

The HP students had scored ≥90% for both the tenths and hundredths items in the test so it seems that a complete structural schema (knowledge and connections) is essential for processing tenths and hundredths with understanding. In the test, the HSP students had scored ≥85% for tenths and <75% for hundredths so it seems that having a structural schema for tenths but not hundredths is not enough to enable processing of both tenths and hundredths with understanding. The MP students had scored 80-90% for tenths and for hundredths so having a schema of multiplicativity that is limited to uni-directional and binary relationships that are unconnected prohibits enables limited processing of tenths and hundredths. The MSP and LP students have no available knowledge so their processing of tenths and hundredths will be ad hoc and limited to prototypic tasks.

The HP students' structural schema of multiplicativity suggests that will be able to extend this knowledge with minimal difficulty to accommodate thousandths and ten-thousandths, etc. However, the HSP and MP students' structural schema indicates that they will be unable to accommodate new-decimal fraction places with any degree of understanding. The lower-performing students' lack of structural schema prohibit them from understanding tenths and hundredths and it is predicted that their problem will be exacerbated if introduced to thousandths without intervention to remediate their present knowledge of multiplicativity.

References

Baturo, A. R. (1995). *Year 5 students' cognitive functioning in the domain of hundredths*. PhD confirmation report, QUT, Centre for Mathematics and Science Education, Brisbane.

Baturo, A. R. (in preparation). *Students' cognitive functioning in the domain of hundredths*. PhD thesis, QUT, Centre for Mathematics and Science Education, Brisbane.

Behr, M., Harel, G., Post, T., & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structures. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics.* Albany, NY: State University of New York Press.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170-193.

Smith, E., & Confrey, J. (1994). Multiplicative structures and the development of logarithms: What was lost by the invention of function. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics.* Albany, NY: State University of New York Press.