Development of a Web-Based Learning Tool to Enhance Formal Deductive Thinking in Geometry

Madduma Bandara Ekanayake
University of Wollongong
<mbe81@uow.edu.au>

Christine Brown
University of Wollongong
<christine_brown@uow.edu.au>

Mohan Chinnappan
University of Wollongong
<mohan_chinnappan@uow.edu.au>

Preparing students for proof type geometry problem solving has become a key issue for mathematics educators. Prevailing instructional strategies have been shown to be inappropriate to address the complexities of deductive geometry. In this report, we propose a design of a learning environment to address the above issue. We argue that this model has a potential to help students to make progress up to van Hiele Level 3 and to acquire skills required to solving a range of proof type geometry problems.

Geometry has been recognised as a potential subject for solving problems in a wide range of real life situations. It can foster the knowledge base with rich and effective information that enhances spatial reasoning in making sense of the environment (National Council of Teachers of Mathematics, 2000). In addition, future trends indicate that there will be enormous opportunities in the exponentially growing visual technologies field for people who possess geometry knowledge. Students start learning formal deductive geometry including formal proof type problems at senior secondary level (SSL). If successful this promotes logical thinking (Ernest, 1991), deductive reasoning (Polya, 1945) and visual reasoning (Duval, 1995; Fischbein, 1993) more than any other subject at that level. However, students rate it as their least favourite subject (Anderson, 1995) and they demonstrate little interest and low achievement (Charalambos, 1997; Chinnappan, 1998).

Van Hiele Theory suggests that students in the same class may be at different thinking levels of geometry: "… in any given classroom, one might find the teacher, text and students function at different thinking levels" (Senk, 1989, p. 309). Teachers have problems supporting student development of formal deductive reasoning when their students are either disinterested or are scattered across several levels of understanding. No single teaching strategy can possibly meet these diverse needs. Students may require individualised support. In the study reported here, we address the instructional problem by assessing the range of needs, translating these into a set of design principles, and applying those principles to the development of a learning design prototype.

Theoretical Considerations

When information processing theory is applied to the process of solving well-structured proof type geometry problems, four key events emerge. First, the student reads the problem in the text to understand it (*Analysis*). As analysis occurs, the problem is diagrammatically represented externally on a sketchpad as well as internally in working memory (WM) (*Representation*). Representation stimulates a search of long-term memory (LTM) for required information such as strategies (*Planning*) and useful operators. These are used to transform the initial representation in WM until the goal is achieved (*Use of* *memory retrievals*). Metacognition oversees these cognitive functions (Brown, Hedberg, & Harper, 1994; Schoenfeld, 1985).

Geometry teachers have to facilitate this process through instructional strategies. Research literature reveals that geometry at SSL has intrinsic difficulties that create obstacles for the instructional process. This section poses some of them.

Sudden Shifts in Geometric Thinking Process: van Hiele Levels

According to the van Hiele theory (Clements & Battista, 1992; Senk, 1989) student geometric thought develops in five discontinuous levels called van Hiele Level (VHLs) forming four shifts to complete geometric thought. Transition (a shift) from one level to another is very difficult as two levels belong to two different paradigms. The first shift is from visual (van Hiele Level 0 - VHL 0) to analysis (VHL 1); the second is from analysis to non-formal deduction (VHL 2); the third shift is from non-formal deduction to (formal) deduction (VHL 3). This is the vital prerequisite for learning deductive geometry.

Senk (1989) found a significant correlation between van Hiele levels and proof writing achievements. Students need to develop highly abstract concepts based on definitions, postulates and axioms to learn formal deductive geometry. The new thinking process opposes their familiar inductive reasoning in which concept development takes place through generalisation of patterns and commonalities of concrete object-specific examples (Senk, 1985).

Students at SSL may be scattered through all levels of geometric thought. Of 241 American students that Senk (1989) observed 27% were at VHL 0, 51% at VHL 1, and 15 were at VHL 2 confirming only 7% were ready to start learning activities of deductive geometry. This implies that the instructional process will fail unless remedial measures are taken to support students to make the prerequisite shifts.

Complexity of Geometry Problem Solving Processes

Geometry problems differ from other mathematical problems, as geometry problems have no set procedures or algorithms. Students use logic extensively and this can lead to difficulties:

> Geometry proof problem solving is hard. … Of the 27 definitions, postulates and theorems that are introduced prior to such a problem in a traditional curriculum, 7 can be applied at the beginning of this problem. Some of these rules can be applied in more than one way yielding 45 possible inferences that can be made from this problem's givens. … The number of options continues to increase at further layers … at minimum it takes 6 such layers of inferences to reach the problem goal (Koedinger, 1993, pp. 16–17)

Most of these inferences are correct but irrelevant to the problem at hand and novices have difficulty discerning relevance. For instance the number of inferences related to a given square with its centre exceeds 50 relationships, while only one or two may be relevant to a given problem. Thus inferring is a critical skill. Backward inference, forward inference, bi-directional inference, and drawing auxiliary lines are not rules but heuristics that reduce unnecessary inference. The relevance of general strategies like heuristics and planning on geometry has been highlighted (Chinnappan & Lawson, 1996; Koedinger, 1993; Schoenfeld, 1985).

Visual Representation of Geometry Concepts and Relationships

Although proof type problems are given in words, the actual problem execution space is diagrammatic. Once the problem is precisely converted into a diagram, the student solves the problem on the diagram itself. The diagram represents the problem situation with necessary information. Effective diagrammatic representation of the problem promotes intuition for conjectures (Charalambos, 1997). The diagram also represents geometric knowledge (Koedinger, 1993) including concepts and schemata (Chinnappan, 1998). Therefore, it involves visual reasoning. Thus, visual representation and related strategies play a dominant role in proof type geometry problem solving.

In the problem solving process, diagrams are not perceived as holistic images. Their parts are considered in pairs to obtain new information or to make conjectures. Students have to separate relevant parts in complex diagrams. As there is no universal rule to separate them, students find it difficult when they lack experience. Teachers find it difficult to help students as selection depends on the situation rather than a rule. Effective visual strategies may be critical to support students learning geometry problem solving.

Role of Worked Examples in Learning Proof Type Geometry Problems

The novelty of proof type problems makes no provision for instructions to foster skills in making inferences and diagrammatic reasoning as rules or principles. They are developed as problems become familiar. The commonly used strategy for familiarisation is worked examples.

Presenting critical knowledge along with worked examples is a crucial issue. Although a teacher usually explains everything to the class, they are only likely to record the solution on the board in a format that matches the textbook. Students record this in their workbooks for future reference. They do not see any justification for making decisions on inference selection or figural selection in worked examples in the textbook or workbook. They will also miss the logical flow of generating the solution (Anderson, 1995; Koedinger, 1993). Thus the expert's thought process is not modelled in the worked examples for novice use. Crucial information can be provided through explanatory information.

Providing explanatory information has been extensively evaluated in the context of cognitive load theory (Sweller, 1999). As the execution space exists on the diagram and the solution is presented in text form, interrelated visual information is physically separated. Too much explanatory information may be redundant. When the worked example is only the solution the student might not need to think. In summary, worked examples should not cause cognitive load.

The theoretical consideration concludes that students have difficulties in constructing an extensive knowledge base required for formal proof. Van Hiele Theory suggests that students at SSL are a highly heterogeneous group. To engage in formal deductive proof type geometry problems they will need to develop planning skills and heuristics, represent and interpret complex diagrams, and are most likely to gain insight into this process through the strategy of worked examples.

Method

Although a three-phase method was used in this study, the focus of this paper is on phases two and three. Phase One details are provided, as it is contextually critical to understand the latter phases.

Participants. The participants in Phase One of this study were 166 students from four schools in Sri Lanka.

Instrument. Three tasks were designed—a Geometry Problem Solving (GeoPS) Task, a Geometry Content Knowledge (GeoCK) Task, and a General Problem Solving (GenPS) Task. Each comprised a one-hour written test.

Scores on student mathematical problem solving (MatPS) were determined from continuous assessment records.

Procedure. In *Phase One*, the participants were tested to determine the predictive indicators for success in the geometry problem solving process. Students completed the three tasks—Geometry Problem Solving (GeoPS), Geometry Content Knowledge (GeoCK) and General Problem Solving (GenPS). Their scores on mathematical problem solving (MatPS) were standardised. It was assumed that these MatPS scores would represent student's inductive and non-formal deductive reasoning skills. The statistical package SPSS was used to perform a linear multiple regression analysis of all scores.

Phase Two took the results of Phase One and coupled them with an extensive literature review of both theoretical considerations of formal deductive thinking in proof type geometry problems and reported teaching concerns. These were used to identify instructional needs as a basis for the design of a supportive learning environment.

Phase Three took a specific instructional topic—congruency of triangles—as an example around which to develop an instructional prototype of a key component of the *Phase Two* environment —the use of the worked example.

Results and Discussion

The results of *Phase One* are not reported in depth here. In summary, the regression analysis yielded the finding that GeoPS is mainly dependent on GeoCK, though the three independent variables (GeoCK, GenPS, MatPS) are interrelated.

Phase Two looked at the implications of this for a learning environment to support formal deductive reasoning in proof type geometry problems. Information related to proof type geometry problems is highly specific to the domain of geometry therefore geometry knowledge is the critical prerequisite for solving such problems. Thus, a device for providing geometry content knowledge is a major component of the learning environment.

This content knowledge corresponds to the VHL 3. However, as students are at multiple levels some of them are not in a position to perform these activities. They should be provided with remedial activities and then they should be promoted to VHL 3. Appropriate maturity at VHL 3 is a prerequisite to proof type problem solving and those who have that capacity are ready to start proof type problem solving.

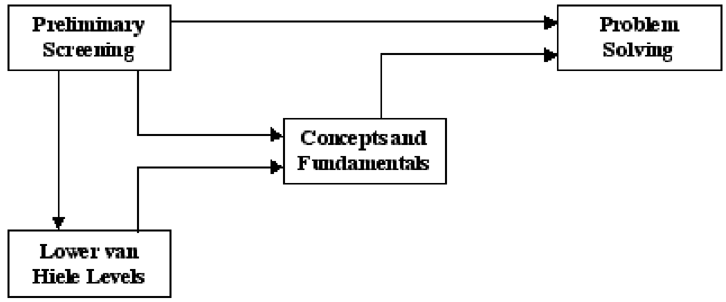

Figure 1. Screening to establish appropriate student support.

A supportive learning environment should provide the means to identify these three groups: students ready to problem solve; students still learning at VHL 3; and students below VHL 3. Figure 1 indicates the need for some preliminary screening process to allow students to access material appropriate to their level of understanding.

Students who are ready to solve problems are directed to *Problem Solving*. Students who can perform learning activities at VHL 3 are directed to *Concepts and Fundamentals*.

The *Lower van Hiele Levels* in Figure 1 represents all students who are below VHL 3, however they could not be treated as a ubiquitous group. According to the theory, each student must complete all previous VHLs, therefore students would progress through a structure such as that represented in Figure 2.

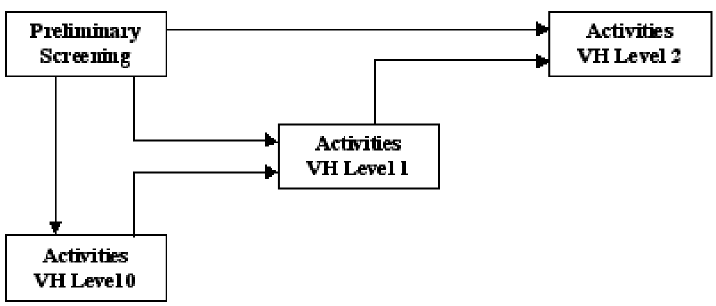

Figure 2. Suggested mechanism for completion of lower van Hiele levels.

The implications of this theory are considerable. To support the content knowledge base for formal deductive proof type problems in geometry, you would need to consider the design of an integrated system for the whole of geometry, if you were to provide support for a class where students could be at any van Hiele level.

Ideas for preliminary screening could be obtained from the diagnostic tests developed by Mayberry (1981) or Lawrie (1998). The activities within a level could be designed in accordance with van Hiele phases of instruction: inquiry, directed orientation, explicitation, free orientation and integration. Problems could be incorporated simultaneously as activities progress. Mistretta (2000), Lawrie (1998), Clements and Battista (1992), Fuys, Geddes, & Tischer. (1988) and Mayberry (1981) provide ideas for activities. Each level would end with a post-test that would serve as a passport to the next level.

Phase One results and van Hiele theory suggest the need for this large-scale multi-level design. Instructional strategies within a level might differ from those required to support the transition or shift in paradigm to the next level.

The literature was a rich source of ideas regarding other needs this learning environment should address. Each of the concerns presented in the theoretical considerations has design implications. The complexity of the geometry problem solving process and the difficulty novices experience determining the relevance of inferences could be addressed through multiple examples where thought processes are shared and modelled. The vital nature of visual representation in geometry suggests students may value modelling of transfer of problem information into diagrammatic form, and ways to interpret information on a diagram. This could be provided through a range of strategies, such as a "think-aloud" that accompanies a movie demonstration; sequential unfolding of a problem with diagrammatic representation alongside text; or embedded annotations within a movie.

Worked examples are a way to familiarise students with novel problems. The question arises regarding the cognitive load afforded by explanatory information. There is also a need to deliver metacognitive support in such a way that its cognitive load does not negate its benefits.

Information processing theory unpacks a series of steps in the problem solving process—analysis, representation, planning and use of knowledge retrievals. The use of a series of guided steps through problems might reduce the cognitive load of rich metacognitive support and provide a process framework to assist planning and self-regulation.

Phase Three took these core design ideas from the conceptual "big picture" level, and through an iterative and collaborative design process, applied them to the topic of congruency of triangles. Each design meeting varied in one or all of the following three aspects—the scope of the tool design discussed, the activity sequence for the worked example, or the nature of metacognitive support that would be provided.

Phase Two design principles were important to keep prototype ideas on track. Initial diagrammatic representation of the flow through a worked example was our starting point. These steps were related to information processing, and the terms were translated into terms students would understand. Example problems were then used to test and refine this structure in a process of rapid prototyping. The development environment required a simple web authoring tool and a graphics program for the development of diagrams.

Problems were not seen in isolation. Early discussion focused on the initial student decision to seek feedback, or gain access to a richly guided support process. If the support option was chosen, then students ought to be able to attempt a similar problem. Having attempted (successfully or with support) the first problem, a more advanced problem would allow them to test their understanding. Thus worked examples were clustered as a repeating structure of core elements illustrated in Figure 3.

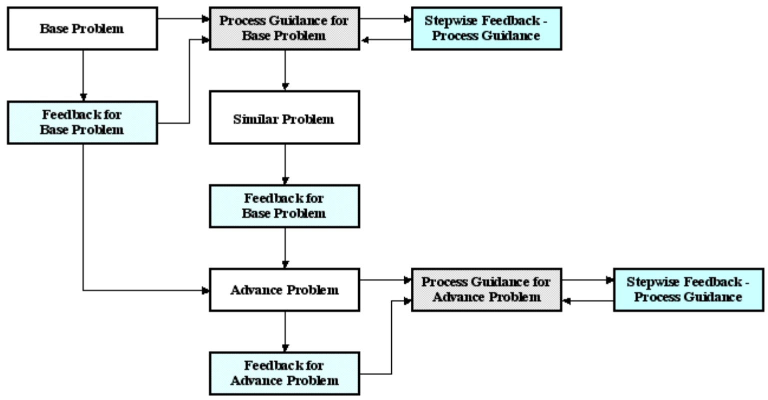

Figure 3. Design for presenting worked examples.

Worked examples are provided in sets. Each set has a base problem, a similar problem, and an advanced problem. The similar problem provides a second attempt and the advanced problem represents learning transfer. Students are expected to complete the problem off-screen in a workbook. Then they seek feedback or process guidance, both of which provide worked examples with extended information.

Process guidance helps develop problem-solving strategies. This structure is presented as a three-column table. The first column provides instructions to identify events in geometry problems in general. These steps are closely related to events in the problem solving process. For instance, reading the problem, drawing the diagram, marking givens on the diagram and writing givens and proof are related to analysis and representation. Thinking about the key idea and missing information are related to a searching or planning event. Deducing new information and deriving solutions are related to the use of memory retrievals.

The second column of the process guidance provides information to familiarise students with the sub-goaling process of the particular problem. For instance, the first goal has been decomposed into 12 sub-goals. Process guidance across a range of different problems helps students to generalise that problem planning differs from problem to problem.

The third column of the process guidance provides links to illustrate the step. Essential background knowledge is embedded in these steps for “just in time” access.

When data and theoretical considerations are balanced with rapid prototype development, instruction can be designed from both a conceptual and grounded perspective. Early student feedback lends support to the core design components of this prototype. It seems the process guidance and problem loop structure are effective ways to engage students in the development of formal deductive problem solving in geometry.

Reference

Anderson, J. R. (1995). *Learning and memory: An integrated approach*. New York: John Wiley & Sons.

Brown, C., Hedberg, J. & Harper, B. (1994). Metacognition as a basis for learning support software, *Performance Improvement Quarterly, 7*(2), 3–26.

Charalambos, L. (1997). A few remarks regarding the teaching of geometry through a theoretical analysis of the geometrical figure. *Proceedings of the 2nd World Congress of Non-linear Analysis,* 2087–2095.

Chinnappan, M., (1998). Schemas and mental models in geometry problem solving, *Educational Studies in Mathematics, 36*(3), 201–217.

Chinnappan, M., &Lawson, M., (1996). The effect of training in use of executive strategies in geometry problem solving, *Learning and Instruction, 6*(1), 1–17.

Clements D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York: Macmillan.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammanna & V. Villani (Eds.), *Perspective on the teaching of geometry for the 21st Century: An ICMI Study*. Dordrecht: Kluwer.

Ernst, P. (1993). *The philosophy of mathematics education.* Dordrecht: Kluwer.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139–162.

Fuys, D., Geddes, D., & Tischer, R. (1988). The *van Hiele model of geometry thinking in geometry among adolescents.* Reston, VA: National Council of Teachers of Mathematics**.**

Koedinger, K. R., & Anderson, J. R. (1993). Reifying implicit planning in geometry: Guidelines for Model-based intelligent tutoring systems design. In S. P. Lajoie (Ed.), *Computers as cognitive tools* (pp. 15–45). NJ: Lawrence Erlbaum.

Lawrie C., (1998). *An investigation into the assessment of students' van Hiele levels of understanding in geometry*. Doctoral dissertation, Australia, University of New England.

Mayberry, J. W. (1981). *An investigation in the van Hiele levels of geometry thought in undergraduate preservice teachers*, Doctoral dissertation, USA: University of Georgia.

Mistretta R. M., (2000). Enhancing geometric thinking. *Adolescence, 35*(138), 365–379

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*, Reston, VA: National Council of Teachers of Mathematics.

Polya G. (1945). *How to solve it*, Princeton, NJ: Princeton University Press.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Berkeley, CA: Academic Press.

Senk, S. L. (1989). van Hiele levels and achievement in writing geometry proofs. *Journal of Research in Mathematics Educati*on*, 20*(3), 309–321.

Sweller, J. (1999). *Instructional design in technical areas*. Sydney: John Sweller.