Algebraic Thinking in Geometry at High School Level: Students' Use of Variables and Unknowns

Jaguthsing Dindyal
National Institute of Education, Singapore
<jdindyal@nie.edu.sg>

The research was carried out over a three-month period in two high schools in the United States. The six focus students who were selected to participate in the study were asked to solve some problems in geometry requiring the use of variables and unknowns. It was found that some of the difficulties that the students had were generic ones that students usually have in algebra but that others were mainly due to a poor understanding of the underlying geometrical or algebraic concepts.

The reform of the 1960s in mathematics education brought major changes in the U.S. school geometry content. New approaches to geometry such as coordinate, transformational, and vector approaches were emphasized in the school curriculum. Although, the reform movement met with several obstacles, it was nevertheless significant in establishing a prominent place for algebraic approaches to the teaching and learning of geometry in school curricula. There is a greater emphasis now in the geometry curriculum on writing algebraic expressions, substitution into an expression, setting up and solving equations; all of which require an understanding of the notion of variable and unknown. The terms variable and unknown will be used in the sense that Schoenfeld and Arcavi (1988) have used them. Variable means something that varies or has multiple values whereas an unknown is something that has a fixed value but that is not yet known.

Connections Between Algebra and Geometry at High School Level

Algebra and geometry have strong historical links. The use of literal symbols in the form of variables, constants, labels, parameters and so on abounds in algebra. Symbols abound in school geometry as well. Students work with variables and unknowns when generalizing results or solving problems such as finding side or angle measures. Variables are used for making general statements, characterizing general procedures, investigating the generality of mathematical issues, and handling finitely or infinitely many cases at once (Schoenfeld & Arcavi, 1988). The idea of a variable is also used in geometry using a variable point as in problems involving loci. Other simple uses of algebra in geometry as far as symbols are concerned involve labelling points or vertices, sides, and angles of figures. Some other connections between algebra and geometry in the high school curriculum arise in problem solving and modelling, and in the various modes of representations – graphical, algebraic, and numeric.

Many of the concepts in geometry have their counterpart in algebra. For example, a point in geometry corresponds to an ordered pair (x, y) of numbers in algebra, a line corresponds to a set of ordered pairs satisfying an equation of the form $ax + by = c$ ($a, b, c \in R$), the intersection of two lines to the set of ordered pairs that satisfy the corresponding equations, and a transformation corresponds to a function in algebra (National Council of Teachers of Mathematics [NCTM], 1989). Algebraic results can be achieved geometrically, and geometrical results can be demonstrated using algebra. For example, Pythagoras' theorem can be represented algebraically using the formula $a^2 + b^2 = c^2$.

The use of variables and unknowns is related to the broader idea of algebraic thinking. Research in algebraic thinking has not specifically focused on the connections between algebra and geometry. However, some studies have examined the interface between algebra and geometry (Lee & Wheeler, 1989) and the relationship between algebra and geometry (Nichols, 1986; Poehl, 1997). Algebraic thinking is generally accepted as having three related components: the use of symbols and algebraic relations, the use of different forms of representations, and the use of patterns and generalizations (NCTM, 1992, 2001; Herbert & Brown 1999; Wagner & Kieran, 1999). These three forms of algebraic thinking were investigated, but only the data pertaining to the use of variables and unknowns are reported in this paper. The concomitant research question was: In what ways do secondary students use algebraic thinking in geometry? In particular, how do secondary students use variables and unknowns in geometry and what are the associated conceptual difficulties?

Variables and Unknowns as Symbolic Representations

The use of variables and unknowns is replete in mathematics. Many students have difficulties working with variables and unknowns which they come across in mathematical problems. Students have not only to identify key components of the problems but also the underlying relationships. The representation of a mathematical problem situation is a depiction of the relationships and operations in the situation (Swafford & Langrall, 2000). The symbolic representations pose problems for the students. Duval (2002) has claimed that there is no direct access to mathematical objects other than through their representations, and thus we can only work on and from semiotic representations, because they provide a means of processing. In geometry, this implies working in different registers (natural language, symbolic, and figurative) and moving in between registers. Algebra offers geometry a powerful form of symbolic representation.

Janvier (1987) claimed that a representation could be considered as a combination of three components: written symbols, real objects and mental images. Further, representations can be external, as observable entities or internal, occurring in the minds of students. On the other hand, Kaput (1989) has mentioned that actions on a representation occur in two broad classes. First, a syntactic action involves manipulating symbols of the representation, guided only by the syntax of the symbol scheme rather than by a reference field for those symbols. For example, in the equation $x + 3 = 12$, a student may simply subtract 3 on both sides of the equations algorithmically. The focus will be on the manipulation of the symbols. Second, a semantic action is guided by the referents of the symbols rather than by the syntax. For example, a student may say: “What number added to 3 gives 12?” Kaput added that most symbol-use acts involve a mixture of the two with the syntactic/semantic distinction being polar extremes. In order to understand a statement before representing it algebraically a combination of several processes is involved, including the application of syntactic and semantic rules (MacGregor & Stacey, 1993).

A symbol is a sound or something visible, mentally connected to an idea (Skemp, 1987). The idea to which the symbol is connected is called the referent. The symbol is also called the signifier and what is symbolized is called the signified. We symbolize when we want something that is absent or missing in some way, and then we work on with the symbol as a substitute (Pimm, 1995). In algebra literal symbols are used as variables, unknowns, constants, or parameters, whereas in geometry visual symbols such as diagrams are also used. Skemp added that it is largely through the use of symbols that we achieve voluntary control over our thoughts. Literal symbols are easy to use but hard to understand (Wagner, 1983). Hence, proficiency in the use of symbols seems to be a must for students learning geometry. Most quantitative relationships are expressed algebraically but algebraic symbols do not speak for themselves (Sfard & Linchevski, 1994). For example, an algebraic expression can be conceptualised as a computational process, or as an object.

Methodology

This qualitative study took place over a three-month period during the first semester of the academic year in two large Midwestern rural high schools in the United States. One geometry class (post-Algebra I) was selected from each high school (school X and school Y). Class A, from school X, had 21 students and class B, from school Y, had 18 students. Two tests were administered to the students from these two classes: an algebra test (constructed by the researcher) and a van Hiele test developed by Usiskin at the University of Chicago, (Usiskin 1982). Based on their performances on the tests, three students were selected from each of the two classes: Anton, Beth, and Mary from class A in school X and Kelly, Phil and Ashley from class B in school Y. Anton was assigned van Hiele level 4, Kelly was next with a van Hiele level 3 whereas the other four focus students were assigned van Hiele level 1(scale 0-4). Kelly had the highest algebra test score of 27, followed by Phil with a score of 24 whereas the scores of Anton, Ashley, Beth, and Mary were respectively 23, 17, 13, and 5 (out of 30). The algebra test and the van Hiele test were used only to get some background on the students for selection and not for any extensive quantitative analysis.

The six focus students were interviewed four times for about 40 minutes each time. During these interviews the focus students were asked to solve sets of problems which involved the use of algebra in geometry. The problems were finalized based on the schools' mathematics programs with the help of three experts in the field. These problems included the use of variables and unknowns, the writing and solution of simple linear equations, the writing and solution of linear simultaneous equations in two unknowns, and the substitution of values in expressions. Besides, the two classrooms were observed for about three months and 12 lessons from each class were videotaped. Artefacts, such as tests, quizzes, and homework of the focus students were also collected. The two teachers from these two classrooms were interviewed twice for about 30 minutes each time.

Findings and Discussion

The six focus students - Anton, Beth, and Mary (from school X), and Kelly, Phil, and Ashley (from school Y) had to solve 10 problems (Problem 1 to 10), all of which required the use of variables or unknowns. Due to space constraints, all of the problems are not given, but they form part of the global discussion. The emphasis has been on the focus students' general performance, but individual solutions to a few problems have been highlighted where possible.

Problems with Known Symbols for Variables

P1: In triangle ABC, the lengths of the sides are AB = $2x + 1$, BC = $3x - 2$, and CA = $x + 4$. What is the perimeter of the triangle? What happens to the perimeter as x increases?

In Problem 1, most of the focus students understood that x had to be positive but they did not give any additional restriction on the value of x. Otherwise, they did not have many difficulties, except Mary and Beth who had some difficulties understanding how the perimeter changed when the value of x was increased. The values a variable takes are not always numbers; in geometry, variables often represent points (Usiskin, 1988). For example in Problem 7, *P* was a variable point representing the locus of a point in a plane. Problem 7 also did not seem to pose major difficulties to the focus students, except for Beth and Mary, who were the weaker students in the group.

> P 2: A plane figure with *n* sides has $n(n\text{-}3)/2$ diagonals. Using this formula, find out how many diagonals a quadrilateral, a pentagon, and a hexagon has. What is the smallest number of diagonals a plane figure can have?

In Problem 2, except for Phil and Kelly, the remaining focus students had some difficulties understanding the type of values that the variable *n* could take and what was the minimum value the expression as a whole could take. This problem required the focus students to substitute values for *n*, the number of sides of a polygon, in the expression for the number of diagonals in the polygon $n(n - 3)/2$. A formula comes up quite often in the study of algebra and also in certain parts of geometry. Substitution, which requires the passage to a single unknown (Bednarz & Janvier, 1996) into a formula was not too demanding for the students. However, the subsequent interpretation of the range of values for *n* and $n(n - 3)/2$ was problematic. Some of the students could not accept the fact that zero was the minimum value of the expression for diagonals in Problem 2. Schoenfeld and Arcavi (1988) claimed that the modern notion of a variable depends on the notion of domain and so it would be difficult to understand one without understanding the other. Thus to have a better understanding of a variable and its use, students need to pay careful attention to the domain on which the variables are defined. In this problem it was important to know what the minimum value of *n* was.

Problems with no Given Symbols for Variables or Unknowns

> P 4: The supplement of an angle is four times its complement. What is the angle?

Unlike the problems described above, in Problem 4, the focus students had to come up with a symbol for the variable, write expressions for the complement and the supplement of an angle, and set up an equation to solve. The following episode demonstrates how Kelly approached Problem 4 in her interview with the researcher (R).

Kelly: We do remember we did one in class…. How does it go? Complement equals to 90° and supplement is equal to 180°, is it?
R: hmm…
Kelly: I don't remember how we did this one in class?
R: Suppose, you had an angle of 30°, what is the complement of 30°?
Kelly: Of 30? …60.
R: And the supplement?
Kelly: 150.
R: So take any angle, suppose the angle is something.
Kelly: 40°.
R: So, what is the complement of 40°?
Kelly: 140…? The complement…50°.
R: And the supplement is 140, is that ok?
Kelly: Hmm
R: So if you don't know the angle how do you start the problem?
Kelly: With *x*?
R: Ok…let's try so, if *x* is the angle….first, what would be its complement?
Kelly: Is it 90-*x*…I don't…

R: Hmm… like for 40° it was 90° - 40° isn't it? For *x* it will be…[*She writes* 90-*x*] Ok. And what will be the supplement?

Kelly: 180-*x*?

R: Ok. So one of them is four times the other. Which one is four times the other?

Kelly: Four times its complement is the supplement.

R: Ok, alright. Can you write an equation from there? [*She writes* 180-*x* = 4(90-*x*)] Can you solve it? [*She solves the equation after a few slips in the algebra*]

During the solution process, a major difficulty for Kelly was understanding what was meant by the terms complement and the supplement of an angle. A related difficulty was producing a literal symbol for a variable for writing the complement and the supplement of a given angle. Once she was beyond these two hurdles, the rest seemed to fall into place for her. She was able to set up the correct equation and then solve it. She had a few algebraic slips, but on the whole it can be noted that her participation in this discourse helped her to solve the problem.

The focus students generally did not do well on Problem 4. Three types of difficulty were noticed: first, understanding the meaning of the words supplement and complement; second, producing a variable for writing down the expressions; and third setting up an appropriate equation to solve for an unknown. Once the equation was set up, these students found the solution relatively easy. Phil, Kelly, Beth, and Mary seemed to have the first and second types of difficulty. They did not quite understand the meaning of the words complement and supplement and could not come up with a variable on their own whereas Anton's problem was of the third type.

Two of the focus students made the same reversal error in Problem 4. Using *x* for the unknown, they both wrote 4(180-*x*) = 90-*x* for the equation. This is not an isolated incident. Many studies have reported this error, particularly in the students-and-professors problem (Clement, Lockhead, & Monk, 1981; Kuchemann, 1981; Clement, 1982; Booth, 1984; Philipp, 1992; MacGregor, 1991; MacGregor & Stacey, 1993). However, it is interesting to note that the focus students experienced this difficulty in this geometrical context. Several reasons have been put forward as possible causes of this reversal error. Herscovics (1989) referred to this error as syntactic translation, in which students formulate equations from natural language expressions. Mestre (1988) described it as a sequential left-to-right method of translation. On the other hand, Kaput (1987) stated that the major cause of the reversal error is the powerful and automatic use of natural-language rules of syntax and reference whereby algebraic letters are used as natural language nouns, and numbers are used as adjectives. Syntactic translation was noted in the responses of Beth and Mary in Problem 5.

P 5: The sum of two angles is 120° and their difference is 20°. What are the angles?

For example, Mary wrote $a + a = 120$, $a - a = 20$ for the two equations which were to be solved simultaneously, with a as the identifying symbol for angle and could not proceed any further. On the other hand, Anton used $m\angle$A and $m\angle$B as unknowns in Problem 5 and solved the problem successfully, whereas the students from school Y used *x* and *y* quite confidently as unknowns for writing their equations and get the solution. Thus, the major difficulties for the students included the use of unknowns for the two angles and the subsequent solutions of the simultaneous equations. At a more general level, the difficulties the focus students had in finding a symbol for the variable or unknown was probably due to difficulties in algebraic modelling, which White and Mitchelmore (1996) described as referring to the definitions of new variables and the symbolic expressions of relations between variables. The difficulty that the students had in coming up with a variable or unknown seemed to be related to their mental representations of the geometrical concept that they were trying to represent.

There were situations when a lack of familiarity with the algebraic concept led to an added difficulty in solving a problem. For example, in Problem 10, given below, an understanding of the term ratio was crucial to the solution of the problem.

P 10: The three angles in a triangle are in the ratio of 2: 3: 4. Find the angles.

Difficulty with the term ratio prevented some of the focus students to get the right solution although they knew that the sum of the interior angles of a triangle added up to 180°. For example Mary could not get the solution even with several prompts from the researcher (R).

Mary: I have no clue.
R: Ok, what do you know about angles in a triangle, tell me something?
Mary: They can be different.
R: Something more specific? What do you know about all the angles in a triangle?
Mary: There is only three…
R: Do you know anything about their sum?
Mary: [*no response*]
R: If I tell you that the sum is 180°, will you be able to do that now?
Mary: No…

Mary had considerable difficulty with the term ratio and how to proceed in such a problem. Phil initially had a similar problem but was eventually able to get the solution. However, Kelly, who had the highest algebra score in the group, did not understand the term ratio as well and used a trial and error strategy to get to the solution.

A variable or unknown is essentially a form of representation, showing a relationship between two or more configurations (Goldin, 2002). For Kaput (1987), the two configurations are the represented world and the representing world. In each of the above problems, the represented world refers to a geometrical concept for which the representing world is an algebraic symbol. When students were provided with the symbol for the variable or unknown, the problems were easier for the students to solve. It seemed that the students were relieved of the mental strain to think about a representation on their own and thus it became easier for them to work on the problem. However, when the students had to come up with the algebraic representations on their own they had difficulties. Hence, the relationship between the represented world (the geometrical concept) and the representing world (the variable or unknown) seemed to be problematic. Hiebert and Carpenter (1992) favoured the belief that an individual's mental representation is influenced and constrained by the external situation being represented. Accordingly, the difficulty that the students had in producing a variable or unknown seemed to be related to their mental representations of the particular geometrical concept.

Conclusion

Some of the difficulties that the students experienced are just generic difficulties that students studying algebra experience, like the reversal error which has been documented in many studies (Clement, Lockhead, & Monk, 1981; Clement, 1982; Kaput & Sims-Knight, 1983; Philipp, 1992). However, there are several that appear to be related to the geometrical concepts used in the problems. In other cases it was the algebraic concept that was problematic for the students, like in Problem 10. The study highlights the importance of algebra in geometry. It is suggested that students have a better preparation in algebra prior to joining the geometry class. A possibility is to consider an Algebra I-Algebra II-Geometry sequence instead of the popular Algebra I-Geometry-Algebra II sequence, as suggested by Nichols (1986) in her research.

The teaching of geometry at high school level should carefully focus on ascertaining that various aspects of algebraic thinking are present. The various uses of symbols as constants, unknowns, and variables should be discussed in class. In particular, the relationship between a symbol and its referent should be made very clear for the students. Students frequently treat variables as symbols to be manipulated rather than quantities to be related (White & Mitchelmore, 1996). The solution of simple equations did not seem to be a problem for students, but the solution of simultaneous equations was problematic for them. Accordingly, instruction should ascertain that geometry students adequately develop their knowledge and skills for solving such equations.

The interview tasks used in this study had to match the level of algebraic sophistication of the students. Another limitation was that this study focused on how the different interview tasks were carried out by the students rather than how students with specific characteristics approached the tasks. The focus was more on the tasks rather than on individual students. An avenue to explore would be a hierarchical framework, such as the van Hiele's, which could account for algebraic thinking as well. The use of tasks which demand more varied algebraic thinking in geometrical contexts can also be considered. Future research may also relate the problem solving abilities of students in geometry to their background in algebra.

References

Bednarz, N., & Janvier, B. (1996). Emergence and development of algebra as a problem-solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 115-136). Boston: Kluwer Academic Publishers.

Booth, L. R. (1984). *Algebra: Children's strategies and errors. A report of the Strategies and Errors in Secondary Mathematics Project*. Windsor, Berks: NFER-Nelson.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education*, *13*(1), 16-20.

Clement, J., Lockhead, J., & Monk, G. S. (1981). Translation difficulties in learning mathematics. *American Mathematical Monthly*, *88*, 286-289.

Duval, R. (2002). Representation, vision, and visualization: Cognitive functions in mathematical thinking. Basic Issues for Learning. In F. Hitt (Ed.), *Representations and mathematics visualization* (pp. 311-336). Mexico: PME-NA-Cinvestav-IPN.

Goldin, G. A. (2002). Representation in mathematical problem solving and learning. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 197-218). Mahwah, NJ: Lawrence Erlbaum Associates.

Herbert, K., & Brown, R. (1999). Patterns as tools for algebraic thinking. In B. Moses (Ed.), *Algebraic thinking: Grades K - 12* (pp. 123-128). Reston, VA: National Council of Teachers of Mathematics.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 60-86). Reston, VA: National Council of Teachers of Mathematics.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). NewYork: Macmillan.

Janvier, C. (1987). Representation and understanding: The notion of function as an example. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 67-71). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. J. (1987). Representation systems and mathematics. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 19-26). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 167-194), Reston, VA: National Council of Teachers of Mathematics.

Kaput, J., & Sims-Knight, J. E. (1983). Errors in translations to algebraic equations: Roots and implications. *Focus on Learning Problems in Mathematics*, *5*(3), 63-78.

Kuchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102-119). London: Murray.

Lee, L., & Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics*, *20*, 41-54.

MacGregor, M. E. (1991). *Making sense of algebra: Cognitive processes influencing comprehension*. Geelong, Vic, Australia: Deakin University Press.

MacGregor, M., & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education*, *24*(3), 217-232.

Mestre, J. P. (1988). The role of language comprehension in mathematics and problem solving. In R. Cocking & J. Mestre (Eds.), *Linguistic and cultural influences on learning mathematics* (pp. 201-220). Hillsdale, NJ: Erlbaum.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1992). Algebra for the twenty-first century. *Proceedings of the August 1992 Conference*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2001). *Navigating through algebra in grades 6-8*. Reston, VA: Author.

Nichols, B. W. (1986). *The effect of different sequences of geometry and algebra II on mathematics education*. Unpublished doctoral dissertation, University of Missouri, Kansas City.

Philipp, R. A. (1992). A study of algebraic variables: Beyond the student-professor problem. *Journal of Mathematical Behavior*, *11*, 161-176.

Pimm, D. (1995). *Symbols and meanings in school mathematics*. New York: Routledge.

Poehl, T. T. (1997). *Using the van Hiele model of thinking: Assessing geometry knowledge of high ability and gifted high school students in Algebra II, Trigonometry, and AP calculus*. Unpublished doctoral dissertation, University of New Orleans, New Orleans.

Schoenfeld, A. H. & Arcavi, A. (1988). On the meaning of variable. *Mathematics Teacher*, *81*, 420-427.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Sfard, A. & Linchevski, L. (1994). The gains and pitfalls of the reification – The case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Skemp, R. R. (1987). *The psychology of learning mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Swafford, J. O., & Langrall, C. W. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education*, *31*(1), 89-112.

Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry* (Final report of the Cognitive Development and Achievement in Secondary School Project). Chicago, IL; University of Chicago, Department of Education. (ERIC Document Reproduction Service No. ED 220 288).

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford & A. P. Shuttle (Eds.), *The ideas of algebra, K-12, 1988 Yearbook of the National Council of Teachers of Mathematics* (pp. 8-19). Reston, VA: National Council of Teachers of Mathematics.

Wagner, S. (1983). What are these things called variables? *Mathematics Teacher*, *76*, 474-479.

Wagner, S., & Kieran, C. (1999). An agenda for research on the learning and teaching of algebra. In B. Moses (Ed.), *Algebraic thinking: Grades K - 12* (pp. 362-372). Reston, VA: National Council of Teachers of Mathematics.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, *27*, 79-95.