Two Pathways to Multiplicative Thinking

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This paper presents the learning pathways of two children who are part of a larger study of the development of multiplicative thinking. The two children changed their approach to the mental solution of two-digit by one-digit multiplication problems over the course of eight mathematics lessons. They began the unit of lessons using repeated addition strategies and ended it using multiplication strategies. The children differed, however, in the range of strategies they used and in the path they took towards more sophisticated understandings.

Understanding multiplication is a significant step in learning mathematics. Learners who understand how multiplication works and can solve multiplication problems readily are termed 'multiplicative thinkers' (Ministry of Education, 2004). Multiplicative thinking requires a new level of sophistication in thinking about numbers and operations. This sophistication is inherent in the nature of multiplication (Davydov, 1992; Jacob & Willis, 2001; Schwarz, 1988; Vergnaud, 1983).

Cross-sectional and longitudinal studies have revealed the stages children pass through as they become multiplicative thinkers (Anghileri, 1989; Jacob & Willis, 2001; Kouba, 1989; Mulligan & Mitchelmore, 1997; Mulligan & Wright, 2000). They begin using counting strategies, progress to strategies based on repeated addition, and lastly use features of multiplication, such as commutativity and known basic facts, to solve problems. While we can observe these stages in children's thinking about multiplication, they do not tell us how children come to think in these ways. We know that some children never reach the stage of effectively using multiplicative thinking (Jacob & Willis, 2001), so the progression is not an automatic one. How, then, do children negotiate the transition between these approaches to multiplication problems?

Data on how children are thinking about multiplication commonly comes from observing and categorising children's responses to problem situations. The strategies that children use to solve the problems are taken as an indication of their thinking about that problem. These problems can be word problems (Nesher, 1988, for example) or tasks with a multiplicative structure (Jacob & Willis, 2001 for example). Mulligan & Mitchelmore (1997) show that children's strategy choice is determined by the structure they impose on the problems they are given, rather than on the mathematics inherent in the problem. The children's strategy choice thus gives important information about their thinking.

If the strategies children use to solve problems tell us about their thinking, then observed changes in strategy use can be used to indicate transitions. Siegler proposes the microgenetic method as a way of observing this change in strategy use (Siegler, 2000; Siegler & Crowley, 1991). The microgenetic method involves making dense observations of children's strategy use around a period of anticipated change. Siegler & Crowley (1991) used this method to observe the transition from counting-on from the first number in an equation, to counting-on from the larger number. From the results of this and other similar studies, Siegler proposes an overlapping waves theory (Siegler, 2000). The overlapping waves theory says that strategies do not replace one another in a linear fashion, with less sophisticated strategies disappearing as new strategies are acquired. Instead, strategies wax and wane, emerging and co-existing in patterns particular to individuals. While overall trends may be maintained, there is variance between individuals in the ways in which the milestones of thinking are achieved.

This study addresses the question of how children negotiate the transition between additive and multiplicative thinking, as evidenced by changes in their use of strategies in solving multiplication problems.

Methodology

Participants

This study took place in a school of approximately 400 pupils in a middle-class area. The focus was the classroom of Mrs P, a teacher with more than twenty years experience, who had taught at the upper primary level at this school for six years. She was the senior teacher for a cluster of six classrooms. The teachers of these six classrooms wrote a pencil-and-paper test for the children to sit, in order to sort them into classes for a three-week unit on multiplication. The children in Mrs P's class for the multiplication unit scored 3, 4 or 5 out of 30 on this test, and were unable to go beyond two-digit by one-digit multiplications.

This paper focuses on two students from this class, Le and Lo. Le (female) was 10, and Lo (male) was 9:11. I interviewed all the children in Mrs P's class, and selected six children to follow intensively. This paper reports on Le and Lo because they both made changes to the way the solved multiplication problems during the course of the unit.

Materials

The interview was based on a grid of fifteen problems. This grid had been trialled in a previous study and found to elicit useful information about children's thinking. Five problem types were presented at three levels of difficulty – two or three times table; six, seven or eight times table and two-digit by one-digit multiplication. The children solved the problems mentally. The five problem types were drawn from analyses by Mulligan & Mitchelmore (1997), Vergnaud, (1998) and Nesher, (1988). They were equal groups, rate, multiplicative comparison, array and Cartesian product. Each problem was read to the children from a problem card which they could then refer to. Once they had solved each problem I asked them to talk about how they had worked it out. If they were struggling to solve the problem I prompted them to talk out loud about what they were trying to do.

Each week I wrote a semi-structured interview protocol based on the lessons from that week. This included the opportunity to mentally solve two-digit by one-digit multiplications under similar conditions to the grid-based interviews. These discussions were audio-recorded, and any written products collected

Method

The children were interviewed at the beginning of the study, after the lessons were completed and six months later. During the three-week unit, I videotaped each mathematics class, audio-taped the target children and met with each participant on a weekly basis. These meetings were fifteen minutes long. There were thus six opportunities for the children to explain their thinking about multiplication in a one-to-one setting: three interviews based on the problem grid and three semi-structured interviews. These interviews took place in a withdrawal room adjacent to the project classroom, on days when there were no mathematics lessons.

The lessons that Mrs P took emphasised basic facts, and then moved to a written method for solving two-digit by one-digit multiplication. This written multiplication method shares characteristics with the 'split and add' mental method. Termed 'Bill's method' in the class textbook, children are asked to multiply the ones and record the answer, then multiply the tens and record the answer. These answers are added together to give the product.

Method of Analysis

To understand how Lo and Le's thinking changed during the multiplication lessons I looked at their strategy use (Siegler, 2000), which I defined as the ways in which they reported solving multiplication problems mentally. The data reported here is from their solution of two-digit by one-digit multiplications. I sorted the strategies mentioned in the literature, and those I found when interviewing children, into a hierarchy of increasing sophistication. This hierarchy followed the developmental trend outlined in the introduction and considered the mathematics inherent in each of the strategies. I then plotted each child's use of strategies against the six times that they were interviewed, giving a picture of the strategies they chose to use at each interview.

Results

Figures 1 and 2 show the strategies that Lo and Le used on each of the six occasions that I met with them.

T1 is the initial interview. T2, T3 and T4 are the semi-structured interviews, conducted weekly. T5 is the interview immediately following the unit, and T6 is the interview six months later.

These figures show that the participants' strategy use changed over the course of the unit, and that these changes were still evident six months later. Lo abandoned his initial, inaccurate use of additive strategies and focused exclusively on a split-and-add strategy for these problems. Le used a wider range of strategies throughout the study, maintaining her additive strategies and adding new multiplicative strategies.

	T1	T2	T3	T4	T5	T6
Compensating						
doubling and halving						
split and add			●		●	●
Algorithm with carrying				●		
use tables						
add some/multiply some						
repeated addition – split		◎		●		
repeated addition – doubling	◎					
repeated addition – whole			●			
skip counting		●				
counting all						

Key: ◎ inaccurate use ● accurate use

Figure 1. Strategies used by Lo to mentally solve 2-digit x 1-digit multiplication problems.

Lo's Pathway

Lo began the unit unable to do two-digit by one-digit multiplication mentally. He could find the answer to 6x8 by using a repeated addition with doubling strategy (8+8=16, 16+16=32…), but had trouble keeping track of the number of times he had added the starting number when he attempted two-digit by one-digit multiplication.

After three lessons (T2), Lo had appropriated part of the method he was being taught. He had begun splitting the two-digit number into tens and ones. Dealing with the ones first, he added repeatedly, then multiplied the tens and added the two totals together. For example, solving 17 x 3 – '7 + 7 + 7 is 21 add on 3 times ten is 41 all up'. He was inaccurate with this method, however, either miscalculating the repeated addition or the final addition. A related method involved skip counting, and was accurate. Still dealing with the tens and ones separately, Lo solved 24 x 6 by skip counting in fours and then skip counting in twenties before adding the two together. He described what he had learned during the week as "working out normal sums the long way" and remembered doing "number stories".

The next semi-structured interview (T3) took place after Lesson 6. By this point Lo could recall the name the teacher was using for the multiplication method, saying he had learned 'Bill's method'. He was able to use Bill's method mentally to solve 18x4, doing the multiplication of the ones first and the tens second, as taught in the written procedure. Lo was able to recall that 8x4 was 32 without hesitation, which contrasts with his lengthy doubling repeated addition procedure to solve basic facts in the initial test two weeks earlier. He also used repeated addition of the whole number to solve a subsequent problem.

In the final semi-structured interview (T4), Lo used the algorithm with carrying to mentally solve 24x6. Despite not being instructed in this method at all during the unit, he repeated the algorithm as he worked out the sum: "4 times 6 is 24, put down 4, carry the 2. Two times 6 is 12 and 2 is 14 – one hundred and forty-four." He also split the number and added repeatedly – with a variation. To add six twenties he added 2 six times, and then says "put a zero on that". Previously he added or skip counted in tens to solve this part of the equation. When asked to brainstorm what he knew about multiplication he wrote three things: 'division is the opposite' (of multiplication), 'it's like doubling' and 'it's an improved way of plus'.

The final interview (T5) had the same format as the initial interview. This time Lo was able to answer all five of the two-digit by one-digit multiplications correctly. He did each of them by splitting and adding, beginning with the ones (Bill's method).

When interviewed six months later (T6) Lo recalled 'things on the mat with the boards', 'working in my maths book with the text books' and 'learning multiplication and basic facts'. He used the split and add strategy to solve the five problems with accuracy.

	T1	T2	T3	T4	T5	T6
Compensating						
doubling and halving		●	●		◎	
split and add			●	◎	●	●
algorithm with carrying		◎	●	●		
use tables						●
add some/multiply some						
repeated addition – split						
repeated addition – doubling	●		●			●
repeated addition – whole	●	●		●		
skip counting		●	●	●		
counting all						

Key: ◎ inaccurate use ● accurate use

Figure 2. Strategies used by Le to mentally solve 2-digit x 1-digit multiplication problems.

Le's Pathway

Le began the unit as a very confident and accurate user of repeated addition by doubling. To solve 13x7 she says "13, 26, 52, that will be four, add 3 more plus 13 is 65 plus 26 is 91." She was capable of monitoring the number of 'thirteens' she had used, and could work out the additions accurately.

At T2, after one week, Le used a range of strategies, some of which were additive and some of which were multiplicative. For the problem 23x4 she used the algorithm, although this had not been taught, and was not accurate – "3 times 4 is 12, 2 at the bottom, 1 by the 2. Two times 4 is 8, add the 1 is 5. Put the 1 next to the other number is 52." However, for the problem 24x6 she produced an unusual and complicated strategy based on doubling and halving. After she described it verbally I asked her to record what she did. To solve 24x6 she halved the 24 twice, to get 6x6, and multiplied 36 by 4. Le described this as 'halving it until it gets easier' and said she had learned it 'because I tried it once and got it right so I thought I'd try it again'.

Le recalled details about the lessons she was involved in at T3. She used 'Bill's method' as a term to describe the split and add strategy she now used for some problems. Her repeated addition – doubling strategy was still used, as was the algorithm, although she was accurate this time. The doubling and halving strategy had also developed. To solve 18 x 4, Le said: "Halve 18 is 9. 9 times 8 ... 8x9 …. 8x10 is 80 take away 8 is 72." In this solution path both sides of the equation are proportionally adjusted. I asked a further question to find out more about this strategy. For 134x4 she halved 134 and doubled 4 (67 x 8), using skip counting in sixties and in eights to help her work out the answer.

The double and halve strategy did not emerge at T4, although the question 24x6 was presented again. Le retained her skip counting and repeated addition strategies, used the algorithm accurately and used split-and-add inaccurately. When multiplying the tens in the split and add she forgot the zero (she forgot she was working with tens), suggesting far less understanding of multiplication than the complex strategies used the previous week. Le's brainstorm about multiplication was complex, including eighteen 'bubbles' of linked ideas. She linked the multiplication symbol to four ideas – adding, patterns, doubling and methods. Her listed methods were 'Bill's method', 'long ways' (double and halve), 'down ways' (algorithm) and 'normal ways' (repeated addition).

At T5 Le used the split-and-add strategy for examples with teen numbers. For the questions with numbers larger than twenty she tried to use her double and halve strategy. This strategy seemed to have broken down. She began by halving – to solve 42x6 she says "42 – 21—10 + 11 , so 10x6 and 11x6". Le forgot to double her answer – this happened on two of the five problems.

Six months later (T6) Le still used the split-and-add strategy for one of the five problems. For the others she used a combination of multiplication and addition based strongly on doubling. As at the first interview she was accurate with these methods despite their relative complexity.

She recalled learning a method, but could not remember the details: "I forgot what it was called but we did it in a way that…we only…..we used more lines and stuff, but I don't remember how we do it."

Features of the Pathways

Both participants began the unit with a similar approach to multiplication problems, and ended it with a similar approach to multiplication problems. From my initial interview I characterised them both as additive thinkers. If their most sophisticated strategy at the final interview is considered to represent their thinking, then they both end the unit as multiplicative thinkers. Their pathways between these two points vary however. Lo moved through strategies and seemed to leave them behind, focusing on one method for all problems. Le built up strategies, retaining her strong sense of doubling and addition and adding new multiplicative strategies. Looking at their patterns of strategy use they seem like quite different learners despite the similarities in their progress.

Both pathways feature strategies that were not taught. The algorithm with carrying was not taught during the unit, yet both participants used it to solve problems. Le invented a strategy with more sophistication than what she had been taught; one which bore little relationship to the mathematics being covered in class.

Although instruction focused on a pencil-and-paper method for solving two-digit by one-digit multiplications, the children did not become reliant on pencil and paper to solve the problems in the interviews. They both developed a mental analogue of 'Bill's method' – the split-and-add strategy.

Discussion

Lo and Le changed the strategies they used in response to two-digit by one-digit multiplications over a period of three weeks, and maintained these changes six months later. They changed in the expected direction, moving from additive strategies to multiplicative strategies. If the participants had only been tested before and after the multiplication unit, we would have seen the expected change, but we would know very little about its genesis. These results show us that while children may change in the expected ways, they do so by following a path which is unique to them. This path is determined not only by developmental trends, but by instruction, expectations, interpretations and prior experience.

Le invented a doubling-and-halving strategy which was unlike the instruction she was receiving. Her own preference for doubling, shown by her ongoing use of additive doubling as a solution strategy, fed into the development of a more sophisticated way of looking at doubling within multiplication. Her description of 'trying it out' suggests an expectation that mathematics is a place for experimentation, and her mind-map and solution paths show that she has a drive to make sense of the mathematics she is grappling with. However, alongside this personal sense-making path lies a path influenced by instruction. 'Bill's method' uses different properties of multiplication from the doubling methods. Le also uses the algorithm with carrying, a strategy which seems to emanate from the participants in the lessons without ever being referred to directly. There is not enough evidence here to explain why Le does not use her double-and-halve strategy six months after instruction. However, it was not supported by the instruction she received, and Le settled for the approved, or taught, method, backed up by her additive doubling method. This 'back up' is a strong feature of Le's thinking about multiplication.

Lo had a more linear pathway to the use of multiplicative strategies. He was not able to find an answer to a two-digit by one-digit multiplication when the unit began. Lo gained, and stuck with, the method that gave him accurate solutions. His methods strongly reflect the instruction he received and abandon the doubling ideas that he began with. The repeated addition method he used involved splitting and adding, as did the multiplication method. Like Le, he also used the algorithm with carrying at one point. Lo's strategies seem to have a narrower base than Le's, but he can still confidently use them six months after instruction, so they are clearly part of his repertoire.

These results support Siegler's (2000) overlapping waves theory, where strategies are seen to emerge, disappear and co-exist as learning progresses. This emergence and disappearance of strategies shows the nature of the bridge that learners are making between old understandings and new ones. The strategies that the learners gain and neglect give their understandings a particular character which will influence how they develop in the future. Although we can identify stages in children's thinking about multiplication (Anghileri, 1989; Kouba, 1989; Mulligan & Watson, 1998), the results of this study suggest that there are different ways of negotiating transition between these stages. Instruction, prior knowledge, preferences and expectations interact with the known trend in development to produce individual, unique and influential pathways to more sophisticated thinking.

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