The Mathematical Discourse of Advanced Undergraduate Mathematics

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This paper describes the first phase of a study that aims to identify the specific English language discourse features that cause problems for English as an additional language (EAL) students in advanced undergraduate mathematics. In this phase observations of lectures and an examination of course notes have been undertaken to try to identify the detailed nature of mathematical discourse features that might cause problems. Building on literature about general features of mathematical discourse, particular examples are identified and analysed, from the point of view of both English and mathematics.

Educational institutions in New Zealand and Australia are continuing to accept immigrant, refugee and minority students who have a language background that is not English, the language of instruction. There is thus on-going interest in language requirements for tertiary study, and in the provision of programmes that will assist students in their studies. At tertiary level, students with poor English language take mathematics under the impression that they will not be so disadvantaged. Many perceive it to be relatively language-free. There have been several investigations into language issues in mathematics education at secondary level and in bridging or entry-level tertiary programmes (e.g., Adler, 1998). This paper describes a study at advanced undergraduate level that aims to investigate how students who have English as an additional language (EAL) are understanding mathematics differently compared with students who have English as a first language (L1). What are the discourse features that cause problems for EAL students?

The study reported here is the third in a series of studies aimed at examining the difficulties that (EAL) students have when studying undergraduate mathematics in English. The first study, involving 80 students, used text, symbolic and diagrammatic questions to indicate the extent of textual difficulty experienced by EAL students in first year undergraduate mathematics (Barton & Neville-Barton, 2003). It indicated that, in comparison with native speakers of English, EAL students have a 10% disadvantage in achievement through lack of understanding mathematical text. This is about the same as the disadvantage experienced in Arts subjects. Other results concluded that technical mathematical discourse is a more important factor than general English and that EAL students unjustifiably rely on symbolic modes to make up for textual disadvantages.

The second study (involving nearly 400 students) found that EAL students who had recently arrived in New Zealand self-reported levels of understanding similar to those of English first-language students on common types of questions (Barton & Neville-Barton, 2004). Those EAL students who had been in New Zealand for 6 years or more self-reported much lower levels of understanding. Why are EAL students with *more* experience of English understanding mathematical discourse *less well* than new EAL students? Is it because the new EAL students have developed a mathematical discourse in their own language and can recognise it in English, whereas the older EAL students were struggling with English at the time when they developed their mathematical discourse (that is, during secondary education), and thus it is less well-developed despite their greater English experience? This suggestion parallels Cummins' (1986) ideas concerning the effect of high levels of development in L1 on understanding in another language.

This third study is part of a five-pronged project aiming to examine the detail of the discourse features causing difficulties. This study focuses on advanced (third-year) undergraduate mathematics, whereas the other parts examine senior secondary level. We are interested in more advanced mathematics because we notice that the proportion of EAL students taking mathematics drops dramatically at this level. There may be other explanations—for example cultural preferences for other major subjects—however we also suspect that there is a change in the nature of mathematical discourse and its relation to mathematics itself at this level. Proof and mathematical argumentation, formal mathematical expression, and heightened roles for definitions and axioms are examples.

Theoretical Background

Mathematics education research in bi- and multi-lingual settings has identified language as a social tool in the classroom and as a vehicle for mathematics learning as important areas of investigation (Gorgorio & Planas, 2001, Secada, 1992). Moschkovitch (2002) notes the shift in attention of researchers towards communication in the classroom, and that much of this research is on monolingual classes. This study returns to the issues for bi- and multi-lingual learners, but seeks to understand their situation in a wider context. Research in schools where students have poor ability in the language of instruction has shown the depth and complexity of the learning disadvantage, including, for example, a flow-on effect to the use of visual imagery (Gorgorio & Planas, 2001), and exclusion from significant mathematical discussion (Setati & Adler, 2001). In undergraduate learning some competence in the language of instruction can be assumed, and learning may take place in the first language where there is a large group of learners with that language. However language factors need to be researched with respect to texts, lectures and assessments.

MacGregor and Moore (1991) suggest several reasons why language is important in mathematics education, and Cummins (1986) long ago postulated threshold levels at which advantages may apply for speakers of more than one language. This study focuses attention on a group of mathematics learners who are well into the transition from one language to another. It is to be expected that threshold levels may apply for these students. What are these levels for successful study in undergraduate mathematics? What are the language characteristics that create advantage or disadvantage?

There is a considerable body of literature that examines the nature of mathematical discourse. Halliday is usually credited as focusing researchers' attention on mathematical language as special. He wrote (1978, p. 65):

> We can refer to a 'mathematics register', in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is, not mathematics itself), and that a language must express if it is used for mathematical purposes. ... It is the … styles of meaning and modes of argument, that constitute a register, rather than the words and structures as such. We should not think of a mathematical register as solely consisting of terminology, … [There] are many different ways in which a language can add new meanings and inventing words is only one of them.

Dale and Cuevas (1987) describe the mathematics register in terms of unique vocabulary and syntax (sentence structure), and discourse (whole text features). Examples of these features are commonly taken from elementary or secondary level mathematics (e.g. Esty, 1992; Houston, 1989). One of the aims of this study is to identify examples at advanced levels, and, in particular, to identify *which* features are problematic for EAL students.

However, there are also more generic features that need attention. Also mentioned in the literature above are such features of mathematical discourse as its density, logical complexity, heavy demand on reader's memory, unpredictability, and the mix of prose, symbol and diagrams. While these are postulated as potential sources of difficulty for EAL students, whether in fact they do present problems, particularly at higher levels of mathematics, is the issue being approached here.

Clarkson (1991), Moschkovitch (2002) and Setai and Adler (2001) all review research for bi- and multi-lingual learners. Moschkovitch also discusses ways in which these studies can be conceptualised, listing learning vocabulary, constructing meanings, and participating in discourses. This study focuses on the second of these, and asks how third-year EAL students make sense of changes in mathematical discourse at this level, taking into account the fact that they have been successful (and therefore, presumably, have successfully negotiated mathematical discourse) in previous courses.

This consideration raises the issue of Cummins' language hypotheses and the idea of threshold levels (Cummins, 1986; explained for mathematics in Clarkson, 1991, p. 13-14). The EAL students may not have reached the threshold level in English, but perhaps they have reached a threshold level in mathematical discourse in their own language. Does this make it easier for them to understand that discourse in English, as is suggested by the results of the previous study?

The Study

This study involves two third-year undergraduate mathematics courses in the Department of Mathematics at The University of Auckland. One of these courses is about topology and geometry (8 students of whom 3 are EAL students), the other is about complex analysis (20 students of whom 18 are EAL students). Two of the researchers are the lecturers of these two courses. The study has four phases. The first phase involves the researchers attending lectures and examining texts and course notes, and trying to identify significant mathematical discourse features at this level of mathematics. In particular differences in discourse between mathematical presentations at this level compared with first- and second-year levels are being noted. These features are then being transformed into test/questionnaire items that will be presented to students of these courses in the second phase. Also part of the second phase is an examination of student assignment and test scripts for instances of linguistic misunderstanding or misuse.

The third phase will involve interviews with up to ten students. These interviews will check the exact nature of linguistic difficulties that have been identified in the questionnaire or scripts. The fourth phase will be the construction of a more comprehensive and better targeted test instrument to be used with a larger group of advanced students so that the extent of difficulties identified can be assessed.

In addition, information on the English and mathematical ability of the participating students will be gathered. The first phase is reported in this paper. The second and third phases will be completed by June, 2004, and will be reported in the conference presentation of this paper. The fourth phase is scheduled for the second half of 2004.

Results and discussion

The focus of the lecture observations and text examinations has been to produce examples of the mathematical discourse features that appear in the literature, with particular attention to differences at this level compared with earlier courses.

The Role of the Lecture

When sitting in a lecture listening for things that might pose difficulties for EAL students, there are many occasions when just distinguishing words might be a problem. Take, for example, this statement from one of the lectures: "Let, ah, delta be less than epsilon minus ar one". The meaning is "$\delta < \varepsilon - r_1$" not "$r\delta < \varepsilon - r_1$". Also observed was the phrase: "cosine ex times aitch" where only intonation distinguishes between "cos(x).h" and "cos(x.h)". Students who had been listening to lectures for three years should cope with such vocal anomalies and the enunciation of Greek letters should not be a problem to them.

However, an important difference was noted. In lower level courses the lecturer usually closely follows a text or course notes. Furthermore, more often than not, they are performing a calculation or procedure. Thus what they say is usually being written down while they are speaking, or an example of it is available, giving visual clues to help students to listen. In the examples above, however, there was no visual clue present. The lecturer was calling on an example from previous work, it is true, but that example was not present in the notes or on the overhead being discussed. The statement was a supplementary explanation to the main point of the discussion, which itself was an example of a theorem.

In general, the role of the lecturer changes at advanced levels. The observations made in both classes were that there was less describing and explaining, and more defining, linking and illustrating. This imposes different demands on a listener. Describing and explaining usually have a known focus, there is much redundancy in the discourse, and the style is familiar. Once a student understands what is happening, then they can predict what will be said, and listening becomes a confirming activity.

Defining, linking and illustrating mean that apparently random mathematical topics may suddenly arise in the course of a lecturer's delivery. All the talk is vital, and it may jump unexpectedly. Illustrations or examples are usually different from each other in a fundamental way. In calculations or procedures the point is that there is a similarity about them, and once you can do an arbitrary one you can attempt all of them. However, once you are shown an example of a theorem it does not mean that you could necessarily find or even identify all other examples. Listening tasks that may have been within the range of EAL learners previously, may now not be so easy.

Vocabulary

The technical vocabulary has many of the same characteristics as the technical vocabulary at earlier levels, and students are likely to be familiar with it. However, in addition, technical-like uses of general English words begin to be used. For example, in the topology course, it was noted that T_1-spaces are 'weaker' than Hausdorff spaces. This use of 'weaker' is unusual for two reasons. One is that it may be intuitively opposite to the normal sense of the word. T_1-spaces are weaker because there are *more* of them. Whereas weakness is usually associated with less of something. It is also unusual because it does not refer to the *object*, it refers to the separation property that that object represents. It is weaker because of how it is used, not because of what it is or what it does.

Another example of this is the use of the word 'important' in the summing up sentence: "Compact Hausdorff are among the most important of all topological spaces".

A second vocabulary issue that arises with more frequency at this level of mathematics is the use of suffixes to form a technical term, for example 'analyticity' and 'differentiability'. Such constructions are easy for a native speaker, as the meaning is intuitively apparent. For an EAL student the meaning needs to be learned.

Density of Syntax

The density of mathematical discourse is frequently referred to in the literature. This density occurs in several ways, one of which is the use of multiple adjectives or adjectival phrases. Consider this statement that was part of a theorem: "Let u and v be two continuous real-valued functions of two variables having continuous first partial derivatives that satisfy the Cauchy-Riemann equations in some domain D". There are eight aspects to u and v that we need to keep in mind from this statement.

In this example there is also the joining together of a lot of ideas to form one idea. When discussing this research with a colleague who learned psychology in English as a Dutch speaker, we were told that even after 20 years she still has to translate back to Dutch when she meets combined words (such as "projective identification"). Mathematics at third year level uses such combinations frequently: "the subset of the union of all balls".

In general English, adjectives may be indicative or descriptive. "Look at the green trees on the hill". "Look at the tallest tree on the hill". In the first sentence the word 'green' may be more or less redundant. In the second sentence the word "tallest" indicates exactly what is being referred to. In mathematical discourse adjectives are always essential, that is to say they are always indicative. For example, consider: "the set of real numbers $\boldsymbol{R}$". The word 'numbers' is not really needed. In the section above, the word "Hausdorff" is used on its own to refer to a "Hausdorff space". Thus every adjective must be comprehended exactly, increasing the memory as well as cognitive load.

While observing this aspect of the texts in the lectures, it was noted that every part of the diagrams that appeared in the notes or were drawn by the lecturer is also important. Unlike illustrations in prose, where there are many superfluous aspects to a drawing, a diagram is dense and attention must be paid to every part.

Logical Complexity

Another way in which mathematical discourse is dense is the complexity of its logical structures. For example: "An interior point of a set is a point such that you can construct an open interval on that point which is entirely in the set." The formal logical structure of the sentence is: "X is a Y such that Y has a Z that is a W". Not only is the logical structure complex, but X, Y, Z, and W may also be complex entities. Each one might be a concept that is made up of more than one sub-concepts as illustrated in the previous section.

This example is from a definition, and thus it can be expected that considerable attention will be given to the complex meaning of the statement. However long, intricate statements occur elsewhere. An example from each course underlines this feature.

In the proof of a topological theorem is the statement: "If $x = \{x_i\}$ and $y = \{y_i\}$ are two distinct points in X, then we must have $x_{i(0)} \neq y_{i(0)}$ for at least one index $i(0)$." This has the logical structure: "If A and B are C, then D for at least one E." An equivalent general English statement is: "If some of the trees and some of the shrubs in the park need pruning, then there is several hours work for one of the gardeners."

In an explanation after stating the definition of a complex number, we have the statement: "If we denote (*x, y*) by $x + iy$, where $i^2 = -1$, then we can denote the ordered pair (0, 1) by i so that $i^2 = (-1, 0)$." This has the logical structure: "If A so that B, then C when D." An equivalent general English statement is: "If I go to the shops so that I pass the park, then I will see the flowers provided they are blooming."

Both these examples, and there were plenty more, occur as part of a longer piece of text which is intended to be explanatory—a proof in one case and the description of a consequence of a definition in the other. They are not the most important piece of information being presented at that time. Thus students are not expected to spend much effort in understanding the statement. However, when these statements are transformed into English equivalents, the linguistic complexity becomes apparent. Readers are invited to read each of the general English statements again and to identify the most important idea. In the first case, it is that there is a lot of work to be done. In the second it is that the flowers can be seen. In reaching this conclusion, the authors of this paper needed a discussion to overcome initial disagreement. In the mathematical context the part of the statement that is the main subject needs to be identified quickly and unequivocally if it is to make sense.

Statements of this type are common at third-year level, but rarely occur in first-year courses. The added complexity of the mix of text and symbols is likely to be familiar to EAL students from earlier experiences. But the step-up in frequency of logically complex statements, and their use in routine discourse is different at this level.

The Logic of Mathematics

At advanced levels of mathematics there is an increasing emphasis and sophistication about the use of the words 'show' and 'prove'. The context in which these words are used becomes much more significant. What can be assumed, what results can be used, what level of explanation is required? Consider the following two requests that were observed close to each other in a lecture:

> "Show that if $f(x) = x^3$ then $f'(x) = 3x^2$". Implied (and not stated) in this statement was that a proof from first principles was required.
>
> "Show that if $f(x) = 2\sin(x^3)$ then $f'(x) = 6x^2\cos(x^3)$". This statement implied the use of the Chain Rule.

As in some of the other examples above, students, including EAL students, who have been familiar with earlier undergraduate mathematics courses will be used to interpreting these statements. But such skills are dependent on considerable familiarity with the type of question being asked. However, also observed in these courses, were requests to demonstrate the truth of a result that had a form unlike anything that had been seen before. For example, early in the topology course, when students had only recently been introduced to the idea of closed and open sets, their text contained theorems such as: "Every compact subspace of a Hausdorff space is closed", and "In a Hausdorff space, any point and disjoint compact subspace can be separated by open sets, in the sense that they have disjoint neighbourhoods". These theorems were on the same pages as definitions of T_1-spaces and Hausdorff spaces. Furthermore, proofs of these theorems included assumptions that were not stated in these pages (for example, that an empty set is open).

In order to be able to provide a suitable proof, the student must know considerable detail about the context of the question. In advanced mathematical courses these contexts are often new, and usually are subtle. They are indicated linguistically by unusual relationships between things being discussed (for example, in the example above, the objects T_1-spaces and Hausdorff spaces were presented as "separation properties").

A separate issue about the logic of mathematics is illustrated the following paragraph:

> We can represent the complex number $z = x + iy$ by the point (x, y) in the complex plane, where the rectangular Cartesian coordinates x and y have their usual meanings. Alternatively, we may represent $z = x + iy$ by a directed line segment (vector) from the origin to the point (x, y). … It is also convenient to express a complex number z in polar coordinates $(r, \o)$, where r is the distance of z from the origin, and $\o$ is the angle which the line from the origin to z makes with the positive real axis.

There are four options here for the diagrammatic representation of a complex number: point or vector in Cartesian coordinates, or point or vector in polar coordinates. This is *arbitrary* as indicated by 'can' 'may' 'alternatively'. We can use one or the other depending on what is useful. At earlier levels, mathematics is usually presented as predetermined. At advanced levels the arbitrary and contingent nature of mathematics emerges. We can define objects as we want them to be. There are subtle changes of language indicating this shift. For example, the phrase "we can …" can mean "it is possible or not …" or it can mean "it is the case". Consider the difference between "I can go to the movies (I am permitted if I so choose)" and "I can drive a car" (this is a skill I have).

Test Items

The observations made in the first phase of this study are now the basis for constructing a test instrument to discover whether the features identified actually cause third-year EAL students some difficulty. In doing this, care is being taken that mathematical knowledge does not interfere with the items. One technique being used is the creation of 'new' mathematical objects. For example, one question asks students to identify from a series of twelve graphs those that represent a K-function, where a K-function is defined as "a non-negative, continuous, monotone decreasing function".

Other questions so far constructed include one that asks students to distinguish between definitions and theorems, and another that asks students to identify the truth or falsity of statements derived from inverting a definition.

Conclusion

The observations reported from this study into advanced undergraduate mathematical discourse reflect those features that are reported in the literature at earlier levels. However there is some evidence that there is a fundamental change in the nature of the discourse: not only do the normal features continue to get more complex, but also the use of mathematical discourse changes in several ways. Logical statements become the essence of mathematical meaning, not just a way of describing mathematical relationships. The roles of definitions, axioms and theorems in mathematical argumentation are subtly indicated in their linguistic expression. General English is used in increasingly creative ways to describe the increasingly sophisticated nuances of mathematical concepts.

Understanding mathematical discourse successfully at earlier levels may not be sufficient to understand it at advanced levels. The reported comments of students that they do not have language problems with mathematics like they do with other subjects belie some evidence that EAL students "reach a wall" in mathematics at this level (to quote an experienced mathematician who has been lecturing such students for many years.

We look forward to the future phases of this study to further illuminate the EAL mathematics students' experiences.

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