Experienced and Novice Teachers' Choice of Examples

Chris Bills
University of Cental England, UK
chris.bills@uce.ac.uk

Liz Bills
University of East Anglia, UK
liz.bills@uea.ac.uk

The decisions that teachers make in choosing *particulars to illustrate the general* are important aspects of their pedagogic role. Examples are used by teachers to exemplify concepts and to illustrate procedures. Worked examples frequently form the basis of a teacher's explanation. This paper focuses on the examples used by experienced teachers and the justification they give for their choice to novice pre-service teachers.

The relation "is an example of" is fundamental to the discipline of mathematics. Its importance is reflected in popular understanding of both what it means to learn and to do school mathematics. Furthermore this learning is judged by our institutions largely on the basis of success in working examples under exam conditions. Examples are thus an everyday part of life for a mathematics teacher. A typical UK secondary school mathematics teacher will choose examples to introduce ideas to the class and for the class to work on independently of the teacher. Text books typically contain large numbers of worked examples and examples for students to attempt.

The language of examples used in school is potentially ambiguous. We talk, sometimes interchangeably, of examples of a concept (examples of triangles, integers divisible by 3, polynomials etc.) and also examples of the application of a procedure (finding the area of a triangle, finding if an integer is exactly divisible by 3, finding the roots of a polynomial etc).

In this paper we are interested in decisions that teachers make about choice of examples of concepts and examples of application of procedures, and in particular the way in which novice teachers are inducted into this decision making. Our data is collected from conversations between a group of student teachers and the experienced teachers who were to act as their mentors during their practicum.

Background

More than thirty years ago, Skemp (1971) wrote about the role of examples in the teaching and learning of mathematics. Skemp's basic model for the learning of mathematical concepts was abstraction from examples, which meant that the teachers' choice of which examples to present to pupils was crucial. His advice on this topic includes consideration of *noise*, that is the conspicuous attributes of the example which are not essential to the concept, and of *non-examples,* which might be used to draw attention to the distinction between essential and non-essential attributes of the concept. Once the concept is formed, later examples can be assimilated into that concept (Skemp, 1979) and a more sophisticated *concept image* formed (Tall & Vinner, 1983).

Another aspect of the role of examples centres on the notion of generic example, or prototype. A generic example:

> involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class. (Balacheff, 1988, p. 219)

Freudenthal (1983) describes examples with this potential as *paradigms*. A strand of psychological research beginning with Rosch (1975) has explored how *prototypes* (representatives of categories) are used in reasoning. Hershkowitz (1990) has drawn attention to the tendency to reason from prototypes rather than definitions in mathematics, and the errors that this kind of reasoning can produce.

A teacher's choice of examples may, then, be guided by a sense of the role of example in concept formation. Their choices are likely to be different if the example is seen as one of a set from which a concept is to be abstracted, or as a paradigmatic example which should contain the essence of the concept. It may also be influenced by the teacher's own understanding of the concept. In particular, choices which take account of using the ideas of noise and non-examples to address the boundaries of the concept are likely to reflect understanding of and familiarity with the *dimensions of variation* of the concept (Marton & Booth, 1997).

In a recent study Rowland *et al* (2003) considered the way in which student teachers give evidence of their subject knowledge in their teaching of mathematics to primary school children. One of the aspects of teaching considered was choice of examples. The authors gave instances of choices which, in their words, "obscured the role of the variable" (p. 244). In one of these a student teacher illustrated the reading of a clock face set at half past the hour by using the example of half past six. In another the first example chosen to illustrate the addition of nine by adding 10 and subtracting one was to add nine to nine itself. Here the 'special' nature of the example could mislead.

These theoretical conceptions of the role of examples in teaching and learning, and the limited empirical evidence of student teachers' use of examples form the background to our consideration of experienced and student teachers' discussion of examples.

Context

The data were collected at a day-conference in January 2005 at an Initial Teacher Training Institution for Secondary School (age range 11-18 years) Mathematics Teachers, in the UK. The purpose of the sessions was to encourage the school-based trainers (mentors) and their trainees to collaborate on developing classroom activities which provoke and promote mathematical thinking. The morning sessions had focussed on: the developing mathematical reasoning and problem solving. The afternoon activities centred on *teacher-generated examples*. This included: choosing an example with which to introduce a topic; analysing sets of examples (exercises); developing an exercise for a purpose; creative use of text book exercises. In this paper we discuss data from the first task of the afternoon, where participants were asked to consider 'good' first examples. There were 12 mentors and 14 student teachers present and they were divided into five groups. Mentors were grouped with student teachers who would be working with them in their schools for the forthcoming practicum. In each group there were at least two mentors and at least two student teachers. Discussions were audio-taped and transcribed.

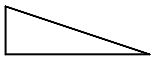
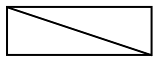
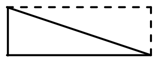
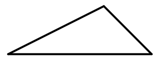
The Task

The first 'teacher-generated example' task required both the experienced and student teachers to consider the examples they would choose as the first one to use in introducing a particular topic. This was described as a 'particular to illustrate the general' or a 'generic' example. They first wrote or drew examples as individuals then discussed them as a group.

Examples for Area of a Triangle

Participants were first asked to: “Give the example you would use to introduce the calculation of ‘area of a triangle’”. The table below sets out some differences between the written responses. The distinction which drew most attention during the group discussions was that between a first example which involved a right angled triangle and one which did not. All but one of the mentors chose to start with a right-angled triangle, whereas only eight out of the fourteen student teachers did so (see Table 1).

Table 1
First Examples for Area of a Triangle

		Right-angled triangle	Rectangle halved	Rectangle completed	Apex above base
					
Mentors	12	3	5	3	1
Students	14	5	1	2	6

In group one the first mentor (1M1) explained her choice of a right-angled triangle in terms of developing from this first example to a more general argument:

1M1 I would start with a right angled triangle, ‘cause I would develop from the rectangle and I would develop how we work out the triangle by using the fact that we know the rectangle. Having got that, I would then develop that into something that’s non-right-angled.

In group two one of the student teachers (2S1) asked the three mentors (2M1, 2M2 and 2M3), who had all suggested a right angled triangle as first example, why they had made this choice. Mentor 3 has drawn on his response sheet both a right-angled triangle and, secondly, one which is not, and the group is pointing to these two drawings as they speak.

2S1 Why would you … ?
2M1 because you’d make it an easy rectangle–
2M2 –because it’s showing the idea that it’s half of a rectangle, yeah. And that’s where the formula derives from.
2M3 You can then go on to show that any triangle is half of a rectangle.
2M1 But it’s easier to see that that one is.
2M3 That they’re the same height and –
2M1 –it’s easier to see that as half a rectangle, because you have to create two extra triangles to make that (pointing to an apex-above-base example)

In group three all five participants chose a right-angled triangle to start and made little comment on this. In group four, although one of the student teachers gives a first example which is not right-angled, in her first contribution she acknowledges that she is in a minority and the conversation turns to whether the first triangle should be half a square or half a non-square rectangle, both mentor and student teacher assuming that the latter is more complicated than the former.

4S1 …I was thinking, go in for a square …you know a triangle which is half a square … then I was thinking maybe we should go in with something that’s a bit trickier.
4M1 I think we’re coming up with the different levels here now. … I want to see what people have forgotten first. I’ve gone right to the basics at first, I’m afraid I’ve done a square

In group five a mentor explains his choice as follows:

5M1 Yes it does depend where you're starting, but if you're starting off introducing the idea of areas of rectangles, you would start with a right angle triangle, … I would start, by talking about, but I would even possibly start with one where the rectangle was there in the first place, with a diagonal line across it and ask what fraction of the rectangle was the triangle, how do we find the area of the rectangle, therefore the area of the triangle?

There is a clear emphasis, on the part of the experienced teachers, on building up from a simple case to develop an argument to support a formula for the area of any triangle. The right-angled triangle (or in one case, an isosceles right-angled triangle) is seen as the simplest case from which to start.

There were two voices making a different case. One of these was the mentor who drew a non-right-angled triangle as his first example and an obtuse-angled triangle as a second. He argued:

1M2 I wouldn't start off like that. I'd show them like that. Show them a non right angle one, then I would, like … you see you want to emphasise it's the *height* of the triangle, *irrespective* of whether there's a right angle there or not

The second was a mentor who, although he gave a right-angled triangle as his example, expressed a similar argument in favour of moving on to a more general triangle:

4M2 –well I went for a rectangle, but I think the concept of base and height needs to be explained using those sorts of triangles, where you've not always got a right angled triangle, but where you have to drop the perpendicular down to see what your height is. Often, just given the numbers they will do this, if you start off with those sorts of questions, then they come across a triangle like this, they just multiply the two sides together and then divide by two, they will not realize the perpendicular height is that height there. Yeah, that needs to be explained as well to them.

Examples for Addition of Fractions

The participants were next asked to give the first example they would use in introducing addition of fractions. The conversations focused on distinctions between sums to which pupils might or might not be expected to know the answer, and between pairs of fractions where both denominators were the same or were different. Table 2 gives the frequencies with which participants chose these types of fractions for their first example. Most participants gave more than one example, and four student teachers left this section blank.

Table 2
First Examples for Addition of Fractions

		½ + ½	¼ + ¼	Other fractions, same denominator	½ +¼	Other fractions, different denominator
Mentors	12	2	3	2	3	2
Students	10	2	0	2	4	2

Two thirds of the mentors chose a starting example which involved only halves and/or quarters. These mentors in groups one and five gave typical explanations of this choice:

1M1 Again, for fractions I would start very, very simple, because they know what one quarter and one quarter is. They *know* the answer to that, they don't have to think about it, they know one quarter and one quarter is a half, it's two quarters. And by doing that, I'm illustrating that this does not change, because otherwise you get two fifths plus three fifths is five tenths, you know. They will add the bottoms as well.

5M2 As soon as you mention the word 'fraction' to a class, half of them will switch off. So you've got to keep it very simple and you've got to be very systematic in the way you – I've chosen half plus a half because most of them will know the answer already, but I want to demonstrate that you've got a common denominator, they need a common denominator and that you don't add the bottom ones. That's what I'm trying to do here.

In group two the argument shifted slightly to address the role of known facts and procedures in building understanding.

2M2 –it's quite good a half and a quarter because they mostly can work out the answer without really –
2M1 –having to know how to do it.
2M2 –having to, yeah. Which should be encouraged in a way, shouldn't it, rather than teaching them a method.
2M3 Well what is the answer? You go back to why rather than …, you get the answer first and then write it down and then you get the understanding.

Mentors in group three also recommended using a familiar example. Their justification is in terms of pupils being able to relate the new work to their existing knowledge.

3M1 I would do a half and a quarter because they know the answer. They'd immediately and instinctively know the answer.
3M2 Which has an advantage doesn't it, because anything you say after that makes some sense to them.
3M1 If you choose anything harder, they don't know the answer, they can't relate to it.

Again in this case, the main emphasis from mentors is to build up from a simple case. However because the context for this example is numerical there is the additional possibility of using 'known facts', which gives another interpretation of 'simple'.

Examples for Solution of Linear Equations

The third prompt for initial examples was 'solution of linear equations'. As this was the last of the three topics, there were some groups that ran out of time before they discussed their suggestions. Table 3 illustrates differences between responses of mentors and student teachers in their first given example. Two of the student teachers recorded no example.

Table 3
First Examples for Solution of Linear Equations

		Positive whole number solution	Unknown appears first	Single operation
Mentors	12	11	12	10
Students	12	5	8	6

The mentors were almost unanimous in choosing examples where there was a single operation on the unknown, where the unknown appeared first in the equation ($x+1=3$ rather than $1+x=3$ or $3=x+1$) and where the solution was a positive whole number (in the exception to this, the solution was zero). Student teachers' responses were more variable.

Two brief conversations in the groups confirmed the general intention to 'keep things simple' but with some questioning. The first was in group three:

3S1 I've got x add seven equals ten.
3M2 We've got to keep it simple to start off with.
3S1 Why? ten is a common number if you had seven what do you need to add to make ten–
3S3 –you're teaching them the concept, there's no point in involving difficult numbers

In group one there was again some dissent from the general agreement on the principle of simplicity:

1M2 if you start off by basic, too basic, they seem to get the idea it's going to be basic anyway. Rather than if you can like put them in the deep end straight away, yeah? I chose this equation for instance, 2 x plus 1, so you know. I gave them this the other day and they–
1M1 –well they'll do that by trial and error, won't they to begin with.
1M2 I asked them to explain how you done it, we took away one, then we divided it by 2, then we got the answer. I didn't teach them that. Do you know what I mean? I mean if I'd started off with something like this sum, x plus 1 equals 30 …

Here the same mentor who argued against using a right-angled triangle as a first example was concerned about the effect of using a single-operation equation. His concern that "they seem to get the idea it's going to be basic anyway" may be parallel to his recognition of the need for pupils to explore the concept of *height* of a triangle, and that this may not arise in the context of the right-angled triangle.

Themes Emerging from the Discussions

Our analysis of the written responses and discussion has identified a number of themes, of which we discuss two here. The first theme is the 'simple example' as a first step in the development of an argument or understanding of a procedure. This is apparent as a majority view amongst mentors in relation to both area of a triangle and addition of fractions. Although there is not enough recorded discussion to provide evidence of this kind of reasoning in the case of solution of linear equations, the mentors' choice of examples is consistent with this principle.

The second theme is the use of simple examples to avoid confusion. The notion of simplicity extended beyond using a right-angled triangle and using simple fractions. For example these participants each talked about a different kind of simplicity:

3S3 That's right, use simple numbers. No point writing nineteen times eighteen, times that by a half. They might recognize the triangle, the formula, they can actually work out the formula they might just not know how to work out nineteen times eighteen.

5M1 The biggest piece of advice is keep it simple in the first place, like in this one creating an example where there is no cancelling to be done afterwards, because that just confuses them. We can deal with that, once they can do the mechanics of it …the first example I would use where the numerators are one because it makes explaining what you're doing much easier. I'd then go on to something like that, but very carefully picking every time in the first instance, ones that don't give you an improper fraction as the answer. Because the simplifying improper fractions bit is a further complication …

In each of these cases the speaker might be seen to be aiming to allow the pupils to concentrate their attention on the new procedure by removing the need for them to focus attention on any other, possibly imperfectly mastered, procedures. At a surface level these concerns may appear to be linked to Skemp's notion of *noise.* They are concerned with the role of detail which is, in a sense, irrelevant to the concept being studied. The link, though, is superficial, since Skemp's notion of *noise* concerns attributes which define concepts, rather than elements in procedures.

A further aspect of the concern to avoid confusion is exemplified by the following:

> 3M2 What I was trying to say about the different numbers… you don't really want like seven and seven. You know, if you've got three different parts to the question, you want three different digits so that when you're referring to them, there's no confusion about exactly which, you know, when you say *the two*, you know they know what the two is.

This is an instance of the kind of awareness identified by Rowland et al (2003) of choice of examples which "obscure the role of the variable". The two instances of such choices in our data are provided by a mentor and a student teacher who each chose a right-angled triangle with two equal sides.

Discussion

"Keeping things simple" is a key message that comes out of the discussions we have reported. None of the 'simple' first examples offered by mentors would satisfy the requirement of a generic example or paradigm, or stand as a 'particular to illustrate the general' as our instruction requested. Right-angled triangles, fractions with equal denominators and equations for which pupils can "see" the solution can all be seen as examples which avoid the main concept, rather than examples which avoid unnecessary complication. This appears to be the kind of argument that mentor 1M2 has in mind when he criticizes choices of right-angled triangles and simple equations. It is also something that at least one of the students recognizes as an issue, related to earlier discussions with his university-based tutor:

> 5S1 … at some stage we've been told that maybe it is bad to keep it simple because more difficult examples can work better than the easy ones. The second point is this confusion, and it may confuse them, how valid is this as a reason for simple examples?

However there is a lot of evidence that the majority of mentors are not using the model of the generic example to think about first examples. Rather they are seeing examples as a means to build up an argument. The way in which mentors aim to develop the argument is different in the cases of the triangle and the fractions. In the case of triangles, the aim is to encourage pupils to 'see' that the triangle is half of the rectangle, and several mentors talk about ways of making this visible by use of practical equipment. This is the first step in a chain of deductive reasoning which establishes a general formula for the area of a triangle. In the case of addition of fractions the reason for choosing simple examples in the first instance is that pupils will know the answer without needing to use a procedure. For some this is seen as a way of allowing pupils to relate the new procedure to their existing knowledge. For others it leads to a means of confronting what mentors see as a common misconception, that the denominators are added when summing the fractions.

The evidence that we have analysed consists of records of what mentors and student teachers say they would do in terms of choice of examples. It is not derived from observation of classroom practice and needs to be treated with caution on that account. However we seem to have evidence that established teachers are able to draw on their experience to predict complications and choose examples to avoid them. This is seen, for instance, in the mentors' concern to avoid equations with solutions which are not positive whole numbers. On the other hand, the majority of mentors do not use models of generic example or paradigm, nor awareness of issues of prototype or concept image, in making their choices of examples. Rather they are using models which have the potential for them to build deductive arguments.

Conclusion

After the group discussions on these examples the participants were invited to feed back their responses to the question "What advice would you give to student teachers about how to choose examples to use?" Within the groups the following were offered as summaries:

1M1 Well I think you choose an example which is very easily comprehensible to begin with, that they can get their teeth into and understand exactly where it's come from and then develop it on from there. That's my theory.

2M3 –so the first example is something they that they can–
2M1 –do.

3M2 I mean the main thing we've picked on is small numbers isn't it?

4M2 I would say you start by establishing that pupils know the theory behind the topic that we're teaching and we do that through very simple examples … once you feel confident the concept is there then you move on to slightly harder questions in each particular case.

5S1 Simplification, keep it simple.
5M2 Something they can relate to perhaps as well.
5M1 Yeah.
5M2 You know if possible, but keep it simple.
5M1 Keep it simple.

Despite the sophistication of the arguments explored earlier in some of the discussions, these summaries present misleadingly straightforward advice on example selection. There is a danger that, under the time pressure typically prevailing in a school maths department, the pedagogic content knowledge of the mentor becomes compressed into advice of this nature. The message received by the student teacher may be to use 'simple' examples in order to offer pupils a straightforward procedure that can be followed in an unproblematic way. The examples the student teachers choose may then not prepare pupils to work with the range of variation they meet subsequently. The more complex purposes behind the mentors' choices are easily lost in the 'Keep it Simple' prescription.

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