The Use of Algebra in Senior High School Students' Justifications

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Students are documented as struggling in high school with using formal proofs In particular, students believe that algebra is expected by teachers but is a component of responses not easily understood by themselves This research documented the use of algebra by 35 students from a range of high schools throughout New Zealand when responding to two justification question in an externally set end-of-high-school examination Although the numbers of students are small, some differences according to gender and socioeconomic background and question format suggest that further research needs to be done

Being able to justify one's approach and response is an important part in doing mathematics Almeida (1995), for example, said that '[m]athematics is not just about identifying the truth but also about proving that this is the case' (p 171) Mathematical proofs have been described as 'perhaps the ultimate in justifications' (Sowder & Harel, 1998, p 670) In a study of 182 high-attaining 14 and 15 year olds, it was found that proofs based on algebraic arguments were considered by students as the ones that would gain the most marks (Healey and Hoyles, 2000) However, arguments that they adopted for themselves were those which they could evaluate and these rarely included algebra Koedinger and Nathan (2004) also found that when given a choice of solution methods, students were more likely to use ones not requiring algebra In Lannin's (2002) research on middle schools students using recursive and explicit reasoning, it was found that '[o]verall, the students appeared to prefer the use of verbal descriptions, perhaps due to their concern for clarity of their representations rather than conciseness' (p 11) Lowrie (1996, cited in Ellerton, Clements & Clarkson, 2000) found that when they were comfortable with the mathematics, primary students were more likely to use the types of responses that teachers used Although research has been done on primary and junior secondary school students' explanations and justification (Meaney, 2002 and Meaney & Irwin, 2005), little work has been done with senior high school students Especially at the end of high school, there is an expectation that algebra will be the medium through which students' explanations and justifications are described Sweetman, Walter and Ilaria (2002) in reviewing the literature of case studies of secondary students in regard to mathematical thinking found that students generally preferred algebraic to geometric representations

With the belief that algebra is the medium through which a mathematical argument is developed, there is a corresponding expectation that the type of evidence used to support the argument will be analytical Sowder and Harel (1998) investigated high school and college students' responses and classified them according to the evidence used in them These were: externally based proof schemes; empirical proof schemes; analytic proof schemes It would appear that the students in Healey and Hoyles's (2000) study 'were aware that empirical arguments [which they favoured] had limitations; they knew more was expected of them' (p 410) Therefore, as the use of algebra as a medium reflects the need for the mathematical argument to show analytical reasoning It would seem that using algebra in mathematical justifications is not something that students learn to do easily However, little research has been done at the final stages of school mathematical learning to see how well students have mastered the use of algebra in their justifications

There is also little research on whether different groups of students use algebra in their explanations and justification in different ways Previous research on how primary school students structured verbal explanations and justifications for mathematical tasks found differences according to age, gender and socioeconomic background (Meaney & Irwin, 2005) The appearance of certain combinations of features were also related to the accuracy of the responses and to the features of the mathematical task undertaken Differences related to gender, with girls performing better than boys, were also noted in Healey and Hoyle's (2000) research This paper documents the findings from a pilot study which investigated the use of algebra in responses by different groups of senior high school students

Methodology

35 students' responses to questions from the end-of-high school mathematics examination which students sat to gain university entrance were analysed to determine how algebra was used in justifications In 2003 in New Zealand, mathematics was divided into two separate examination papers: Mathematics with Calculus; and Mathematics with Statistics The statistics examination was done by more students and contained some questions requiring algebraic reasoning External examinations were sat in November and the papers sent to the New Zealand Qualifications Authority (NZQA), before being sent to markers in other parts in New Zealand The marks were then collated with the students' other marks, before being released in January 2004 The examination papers were then sent to students later that month After the exam had been sat, it was decided for this research to analyse student responses to two parts of Question 3 (3a and 3b3) which required justifications, both involving algebra The questions can be seen in Figure 1

QUESTION THREE (13 marks)

(a) Amy claims the solution to the following set of equations is $x = 6$, $y = -1$, $z = 4.5$.
Betsy claims the solution to the set of equations is $x = 2$, $y = -1$, $z = 1.5$.

Without solving the set of equations, establish which student is correct and justify your answer.

$$3x + 5y - 4z = -5$$
$$-3x - 2y + 4z = 2$$
$$6x + y - 4z = 17$$

(2 marks)

(b) The following set of equations has an infinite number of solutions.

$3x + 5y - 4z = 7$ .... (1)

$-3x - 2y + 4z = -1$ .... (2)

$6x + y - 8z = -4$ .... (3)

1. Eliminate x from equations (1) and (2) to form a single equation. (1 mark)
2. Eliminate x from equations (2) and (3) to form a single equation. (1 mark)
3. From **(b) 1.** and **(b) 2.** or otherwise, justify why there is an infinite number of solutions to this set of equations. (1 mark)
4. State ONE solution set if $x = 7$. (1 mark)

Figure 1 2003 Bursary statistics examination paper: Question 3

This was a small pilot study and so it was decided to seek responses from 36 students Half would be girls and half boys One third would have attended low decile schools, another third would have attended middle decile schools and the remaining third would have attended high decile schools In New Zealand it is accepted practice to equate decile level of the school with socioeconomic level (Bicknell, 1999)

Previous research with primary school children's responses suggested that even with such small numbers some patterns in responses were likely to emerge (Meaney & Irwin, 2005) Although NZQA was easily able to identify students attending different decile school, students could not be identified according to gender Thus the only way to identify six students of each gender at the different decile levels of school was to approach students attending single-sex schools With very few single-sex, low-decile schools, this process of identifying students did restrict the sampling process However, the 6 students were usually drawn from at least two schools Students who sat the Bursary Statistics exam and attended appropriate schools were sent letters late in December 2003 providing information about the project and requesting permission to photocopy and analyse their responses to Question 3 About twelve students for each group were sent letters in anticipation that a number of students would not wish to participate Once permission had been granted, the Chief Examiner for Bursary Statistics forwarded the photocopied responses to Question 3 to be analysed Enough students for each group responded except for boys from low decile-level schools Although follow up letters were sent, only 5 boys from low-decile schools gave permission for their responses to be included in the study The text structures of the responses and their use of equations is discussed in detail in the next section

Findings and Discussion

Question 3a

Question 3a required students to know that to 'establish which student is correct and justify your answer' that they had to show that one and only one set of solutions gave valid equations In order to gain the two possible marks for this question, students just had to substitute Amy's values into the three equations and show that they remained true Just over half of the students gained full marks for this question Given that students only had to substitute Amy's variables into all three equations, it is not surprising to find that only two students gained the intermediate mark of 1

Of the 34 students who attempted this question, only one provided a response which just contained algebra This boy provided an incomplete solution to the three equations, which students were specifically told not to do It may be that a narrative explanation may have also been provided if he had finished Three students also provided just narrative explanations such as 'Neither students are correct as both students values are inconsistent' (MM) and all were incorrect The majority of students provided a combination of narrative description with some algebraic equations However, as can be seen in Figures 2 and 3, the balance of equations to narrative description can be quite different between students

However, all these responses contain similar text elements to those found in primary students' mathematical explanations (Meaney & Irwin, 2005) These elements are Premise, Consequence, Conclusion and Elaborator In both examples, 'Amy is correct' are Conclusions as they explicitly respond to the actual question In the first example, 'as they don't fit the last equation' would be a Premise as it is the starting point from which the mathematical argument is developed 'Because Betsy's values for x, y, z don't give the correct solution' is a Consequence as it is a statement directly built on the Premise 'e g 6(2) + (-1) – 4 (1 5) = 12 – 1 – 6 = 5' would be considered an Elaborator If the equations are converted into clauses, this Elaborator could be considered to have Premise (6(2) + (-1) – 4 (1 5)) and Consequence elements (= 12 – 1 – 6 = 5) In the other example, only the

Conclusion is provided as a narrative description, the other parts of the justification are provided as equations which could be equated with Premises and Consequences

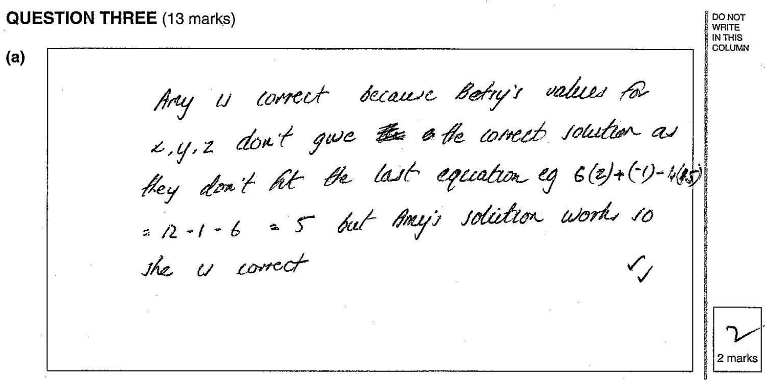

Figure 2 Student response showing the use of equations within the narrative justification (Boy from Low Decile School)

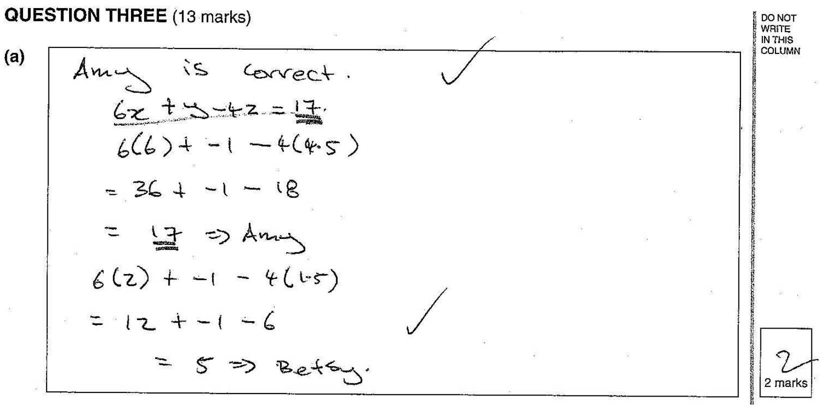

Figure 3 Student showing as a stand-alone section of the justification (Girl from Low Decile School)

These examples should not be considered as distinct categories but rather as two ends of a continuum However, when looking at student responses it was clear that they clustered at these ends with few in the middle It was therefore possible to categorise the responses based on how the equations were used in the justifications Twelve students used equations within their narrative justifications whilst 17 used them in a stand-alone section which fulfilled the role of Premise-Consequence combination Although the numbers are small, slightly more students used equations as a stand-alone section with most of these extra students being girls and students who did not attend low decile schools Equal numbers of boys and girls used equations within their justifications It may also be that students from low decile schools were more likely to use equations this way than to have them as a stand-alone section

The situation looks more interesting when the distribution is related to the correctness of students' responses which is shown in Table 1 The responses which gained 1 or 2 marks were considered correct

Table 1
Use of Equations in Responses to Question 3a

	Gender		School Decile Level			Total
Correct	Girls	Boys	Low	Medium	High	
Equations within the narrative justification	2	3	3	1	1	5
Equations as a stand-alone section of the justification	11	6	3	6	8	17
Incorrect						
Equations within the narrative justification	4	3	3	2	2	7
Equations as a stand-alone section of the justification	0	0	0	0	0	0
No equations used	1	4	2	2	1	5
Only equations used	0	1	0	1	0	1

In Table 1, responses which used equations within a narrative justification are fairly evenly spread between correct and incorrect responses However, students who used equations as a stand-alone section of their justifications were all considered to have the correct answer It would seem that if girls or students from high decile schools gave a correct response, they were more likely to use equations in a stand-alone section Bills (2002) suggested that in primary students' verbal descriptions of how they did mental calculations, certain language such as the use of 'I' or 'you' and logical connectives were more likely to be used with correct responses Thus language could provide an insight into students' thinking It may be that for older students, the way that they incorporate equations into their justifications reflects how they view not only mathematical justifications but also their conceptual understanding of the topic It was interesting to note that the model answer provided in the report on Bursary Statistics paper (NZQA, 2004) used a stand-alone section of equations More research is needed to see whether this way of incorporating equations is considered more appropriate by mathematicians at universities where these students will be continuing their mathematical education

Question 3b3

This question required students to understand that the 'set of equations had an infinite number of solutions' meant that the planes which each equation represented met along one line and so any point along this line would be a solution This understanding would enable them to check that they gained the same answer to Questions 3b1 and 3b2 These results could then be incorporated into their justification in Question 3b3 19 of the 35 students were unable to give a correct justification suggesting that this was not well understood

As was the case with the responses to Question 3a, the solutions could be separated into those which incorporated the equations into their justifications and those which needed the reader to be aware of the equation manipulations in the answers to 3b1 and 3b2 Figure 4 shows a response where the narrative justification incorporates equations into the actual response The example given in Figure 5 uses the word 'formulas' to refer back to the previous answers but does not actually provide them As was the case with some of the responses to Question 3a, the manipulation of the equations has become a stand-alone section of the justification which fulfils the role of a Premise-Consequence combination

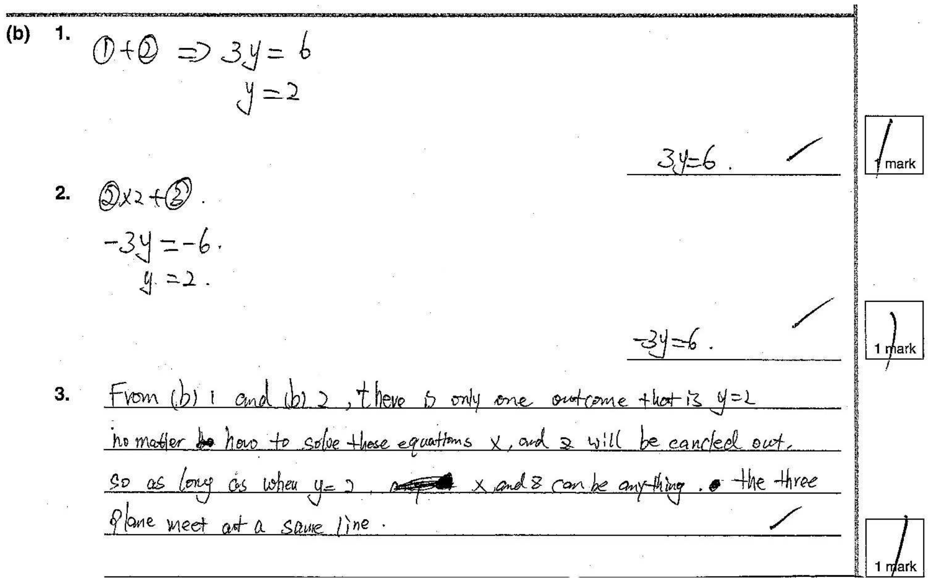

Figure 4 Justification including direct use of equations (Girl from Middle Decile School)

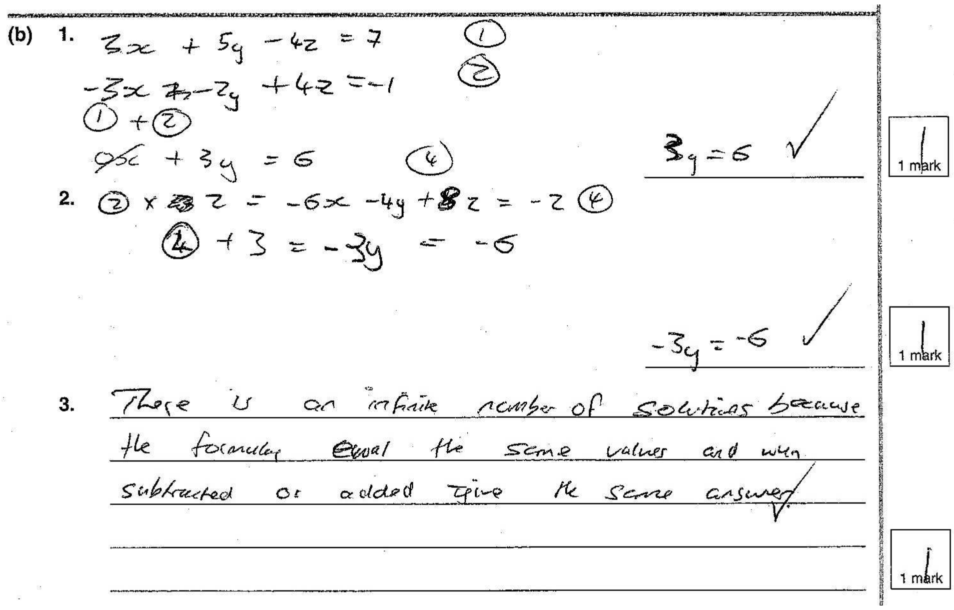

Figure 5 Justification with reference to previous parts of the question (Boy from Middle Decile School)

Fifteen students used equations within their narrative justifications whilst 18 students expected the reader to refer to the responses to 3b1 and 3b2 when reading their justifications Unlike the responses to Questions 3a, there do not appear to be any differences according to gender or decile level of school attended This lack of differences can also be seen when the type of response is related to its correctness Table 2 sets out the distribution of students who gave correct and incorrect responses and how they used or referred to the equations manipulated in the previous parts of this question

It is difficult to know why the differences which appeared in the responses to Question 3a are not apparent in these ones Part of the reason could be the larger numbers of students who did not understand the underlying concept of ‘an infinite set of solutions’ and therefore struggled to know what would constituted an appropriate justification For most students responding to Question 3a seemed to be easier If they did not get full marks, it was usually because they made calculation errors when substituting values rather than because they did not understand what constituted a solution to a set of equations However, it is interesting to see that students who did give correct answers to 3b3 were fairly evenly divided between incorporating equations into their responses and just making reference to them in their narrative justifications It may be that the instruction in Question 3b3 to incorporate the responses to 3b1 and 3b2 in their justifications meant that students who in responding to 3a had a separate equations section believed that in this response they should include the equations into their justifications The model answer provided for this question incorporated the equations into the justification (NZQA, 2004) This would suggest that the numbers of students who used equations in their justifications should have increased in both correct and incorrect solutions but only a small increase was detected It is most likely a combination of variables which contributed both to the appearances of differences in responses to Question 3a and the non-appearance in responses to Question 3b3

Table 2
Use of Equations in Responses to Question 3b

	Gender		**School Decile Level**			**Total**
Correct	Girls	Boys	Low	Medium	High	
Equations as part of the narrative justifications	3	4	1	3	3	7
Equations as a stand-alone section of the justification	3	6	2	3	4	9
Incorrect						
Equations as part of the narrative justifications	5	3	3	2	3	8
Equations as a stand-alone section of the justification	7	2	4	3	2	9
No equations used	0	2	1	1	0	2

Conclusion and Implications for Further Research

Little work has been done on how senior high school students' structure their justifications and explanations, yet these are important in understanding students' beliefs about proof We also do not know whether the typical ways that different groups express themselves affect how they construct mathematical justifications and explanations In this research, justifications for Question 3a showed differences according to gender and to decile level of school attended but these differences were not replicated in justifications for Question 3b3

This research was done with just 35 students, all of whom attended single-sex schools To determine whether there are differences in how students incorporate equations into their justifications, more students' responses need to be analysed and to a range of questions requiring justifications using algebra It would also be interesting to see whether students attending co-educational high schools exhibit any differences in their responses related to their gender This could help determine whether any differences which might be found are related more closely to question type or to student variables

There is also a need to determine whether differences in how students incorporate equations into justifications reflect differences in beliefs about the concepts underlying the questions Often there are several appropriate ways to express oneself in any situation and they do not reflect any difference in understanding the situation However, without further research, we cannot presume this to be the case in the way that senior high school students incorporate equations into their justifications

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References

Almeida, D (1995) Mathematics undergraduates' perceptions of proof *Teaching mathematics and its applications, 14*(4*)*, 171-177

Bicknell, B (1999) The writing of explanations and justifications in mathematics: Differences and dilemmas In J M Truran & K M Truran (Eds), *Making the difference* (Proceedings of the 22nd Annual Conference of the Mathematics Education Research Group of Australasia, pp 75-83) Sydney: MERGA

Bills, C (2002) Linguistic pointers in young children's descriptions of mental calculations In A Cockburn & E Nardi (Eds), *Proceedings of the 26th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol 2, pp 97-104) Norwich: PME

Ellerton, N , Clements, K , & Clarkson, P (2000) Language factors in mathematics education In K Owens & J Mousley (Eds), *Research in mathematics education in Australasia* (pp 29-98) Sydney: MERGA

Healey, L , & Hoyles, C (2000) A study of proof conceptions in algebra *Journal for Research in Mathematics Education, 31*(4), 396-428

Koedinger, K R , & Nathan, M J (2004) The real story behind story problems: Effects of representations on quantitative reasoning *Journal of the Learning Science, 13* (2),129-164

Lannin, J (2002, April) *Developing middle school students' understanding of recursive and explicit reasoning* Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, Louisiana

Lowrie, T J (1996) *Visual imagery in primary school mathematics* Unpublished doctoral thesis University of Newcastle, NSW

Meaney, T (2002) Aspects of written performance in mathematics learning In K Irwin, B Barton, M Pfannkuch & M Thomas (Eds), *Mathematics education in the South Pacific* (Proceeding of the 25th Annual Conference of the Mathematics Education Research Group of Australasia, pp 481-488) Sydney: MERGA

Meaney, T , & Irwin, K C (2005) *Language used by students in mathematics for quantitative and numerical comparisons* NEMP Probe Study Report Dunedin: EARU, University of Otago

New Zealand Qualifications Authority (NZQA) (2004) *University entrance, bursaries and scholarship examination mathematics with statistics 2003: Marking schedule and examination commentary* Wellington: Author

Sowder, L , & Harel, G (1998) Types of students' justifications *The Mathematics Teacher*, *91*(8), 670-675

Sweetman, T D , Walter, J G , & Ilaria, D R (2002, April) *Understanding, justification and representation: Secondary students and emergent strands in mathematics education case study literature* Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, Louisiana