Students' Attempts to Solve Two Elementary Quadratic Equations: A Study in Three Nations

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465 students, comprising students in Year 9 classes in Thailand, Year 10 students in Brunei Darussalam, and second-year university students in the United States, attempted to solve the same quadratic equations. Most of the school students and many of the university students were confused about the concept of a variable and the meaning of "solution to a quadratic equation". Most of the students in the three subsamples acquired neither instrumental nor relational understanding of elementary quadratic equations.

Stacey, Chick and Kendall's (2004) edited volume on *The Future of the Teaching and Learning of* Algebra includes 13 chapters on algebra education written by scholars from around the world. It provides a comprehensive statement on the past, present and future of algebra in the school curriculum. However, although there are some references to quadratic equations, there is no careful statement of cognitive challenges faced by students when they are trying to solve quadratic equations. Indeed, as far as the present writers are aware, such a statement is not to be found, anywhere, despite there being an abundance of evidence to show that many students find quadratic equations extremely difficult to solve.

The origins of this study were in the first author's (Vaiyavutjamai's) investigation of the teaching and learning of quadratic equations with 231 students in six Year 9 classes in two secondary schools in Thailand. The students were in two top-stream, two middle-stream, and two low-stream classes. Analyses of pencil-and-paper performance and interview data revealed that, despite having just participated in 11 lessons on quadratic equations, hardly any students had moved beyond an instrumental understanding of the mathematics associated with quadratic equations (Vaiyavutjamai, 2004a, 2004b).

In the Brunei Darussalam component of the study, 205 Year 10 students attending a secondary school in Brunei Darussalam participated in 10 lessons on quadratic equations. The third author (Clements) then asked the students to solve the same quadratic equations used in the Thai study. These Bruneian students were in eight classes – comprising two top-stream, four middle-stream and two low-stream classes.

The second author (Ellerton), having become aware of the findings in Thailand and in Brunei Darussalam, wondered whether similar patterns with respect to the learning of quadratic equations would be found in the United States of America. She investigated the extent of understanding of quadratic equations of 29 second-year students attending a mid-Western university in the United States. All 29 intended to become specialist middle-school mathematics teachers, and all had studied quadratic equations some years earlier in middle- and high-school Algebra 1 and Algebra 2 classes.

Aims of the Investigation

This paper is concerned with the extent to which students in the three subsamples (in Thailand, Brunei Darussalam, and the United States) correctly solved equations that were in the form $x^2 = K$ ($K > 0$) and $(x - a)(x - b) = 0$ (where a and b can be any real numbers).

With respect to the Thai and Bruneian components of the study, one might have expected that after the students had participated in a set of about 10 or 11 lessons on quadratic equations almost all of them would have been able to solve equations in the form $x^2 = K$ and $(x - a)(x - b) = 0$. In the Thai component of the study, however, Vaiyavutjamai's (2004a, 2004b) analyses pointed to the following four conclusions:

1. After the lessons on quadratic equations, many of the students did not realise that quadratic equations often had two solutions.
2. Some students who solved a quadratic equation correctly did not know how to check whether their solutions to the equation were correct. These students tended not to know what their solutions represented in relation to the original equation.
3. Many students (perhaps a majority of them) did not realise that if a variable (say x) appeared twice in an equation (e.g., with $x^2 - 8x + 15 = 0$, or $(x - 3)(x - 5) = 0$), then it had the same value in the different "places" in which it appeared.
4. When attempting to solve $(x - 3)(x - 5) = 0$ some Thai students "expanded" the two parentheses to obtain $x^2 - 8x + 15 = 0$ (or something similar), refactorised, and then equated each factor to zero. Lim (2000) reported the same tendency among 94 Form 4 O-level students in Brunei Darussalam.

We were particularly interested in comparing the extent to which the same four kinds of tendencies were evident within the Thai, Bruneian and US subsamples.

Review of Related Literature

It appears to be the case that difficulties that students experience in learning to solve equations in the form $x^2 = K$ ($K > 0$), and $(x - a)(x - b) = 0$ are not part of the pedagogical content knowledge of secondary mathematics teachers or, for that matter, of authors of textbooks or articles on the teaching and learning of algebra. Vaiyavutjamai (2004a) reviewed sections on quadratic equations in mathematics textbooks and teachers' guides widely used in Thailand, and found no reference to such difficulties. Likewise, in the United Kingdom, French (2002), in his book on *Teaching and Learning Algebra*, made no reference to the difficulties that students experience with quadratic equations. Authors of summary articles on algebra education in various research publications of the National Council of Teachers of Mathematics (NCTM) (e.g., Kieran, 1992; Kieran & Chalouh, 1993; Wagner & Parker, 1993) have been silent so far as the teaching and learning of quadratic equations are concerned. There was no reference to the difficulties in two chapters on the teaching and learning of algebra in *A Research Companion to Principles and Standards for School Mathematics* (NCTM, 2003). The chapter on algebra in the *International Handbook of Mathematics Education*, authored by a Mexican scholar and a European scholar (Filloy & Sutherland, 1996), did not refer to quadratic equations.

Yet, students do experience difficulties with quadratic equations. In Thailand, for example, Chaysuwan (1996) reported that immediately after 661 Grade 9 students in Bangkok had participated in lessons on quadratic equations 70% of their responses to standard quadratic equations tasks were incorrect. Sutherland, 1996).

The best discussion of the teaching and learning of quadratic equations that the writers were able to locate was the U.K. report on the teaching of algebra in schools. Ironically, the report was originally written between 1929 and 1933 by a "Boys' School Committee" of the Mathematical Association in the United Kingdom (Mathematical Association, 1962). The report has three chapters on equations, with one being devoted to pedagogical issues associated with quadratic equations. In that chapter emphasis was placed on making students aware of "fundamental principles" surrounding standard methods for solving quadratic equations. For example, in regard to the null factor law (i.e., if $P \times Q \times R \times ... = 0$, then either $P = 0$, or $Q = 0$ or $R = 0$ or ...), the report stated:

> Here we have a general principle of the highest importance, and it should be presented as such, not merely as a special dodge for solving quadratics. In its general form the principle is just as simple and easy to grasp as in the special form where there are only two factors—perhaps easier because more impressive ... (p. 29)

Even the way the word "term" should be used was the subject of comment:

> Great stress should be laid on the use, and the accurate use, of the word "term." The expression $x^2 - y^2 + 2y - 1$ has four terms, but by grouping as $x^2 - (y - 1)^2$ we reduce it to two terms and have a form to be treated precisely like $a^2 - b^2$. When an expression is factorised it has been reduced to a single term. The final process indicated by the form is multiplication, whereas in the intermediate steps leading to factorisation the process indicated is addition or subtraction. (Mathematical Association, 1962, p. 33)

The Committee consisted of 16 teachers, mostly from élite schools like Eton College, Winchester College, Harrow School, and Rugby School. The report was full of what might have been called the "wisdom of practice," but whether the advice offered is appropriate to teachers in the twenty-first century, especially to teachers in schools with students from a wide range of backgrounds, is a matter for debate. Furthermore, the report did not pretend to be a research document. It does not, for example, offer research support for its recommendation that the null factor law be presented in its most general form. Nor does it discuss issues such as the proportion of middle-secondary school students who, when asked to solve an equation like $(x - 3)(x - 5) = 0$, might think that the x in the first bracket is a different variable from the x in the second bracket.

Considering the importance of quadratic equations in the history of mathematics, and in secondary school mathematics curricula around the world, it is surprising that research into the teaching and learning of quadratic equations has been so sparse. In the chapters on algebra in the last two four-yearly research summary publications of the Mathematics Education Research Group of Australasia (see Warren, 2000; Warren & Pierce, 2004), the word "quadratic" was used just twice. That was when Warren and Pierce (2004) stated:

> In a small study, Gray and Thomas (2001) investigated the use of a graphics calculator and multiple representations to explore quadratic equations. Their study involved a sample of 25 students aged 14–15 years. The results indicated that in such environments students did not improve their ability to solve quadratic equations. (p. 300)

As Warren and Pierce (2004) noted, the concept of a variable is central to algebra. Nevertheless, the paucity of research into the learning of quadratic equations has meant that peculiarities associated with variables in quadratic equations, and in particular with the effects of these on student learning, have remained hidden. Thomas and Tall (2001) distinguished, among other things, between "algebra as generalised arithmetic" and "manipulation algebra", and commented that research indicated that students who completed secondary education had usually acquired reasonably accurate proficiency so far as substituting values for variables in expressions and equations, and were able to solve and interpret variables in symbolic and graphical contexts. However, student thinking in such contexts often appeared to be dominated by a perceived need to achieve procedural mastery, and usually there was no guarantee that relational understanding was achieved.

Secondary and middle-school teachers tend to believe that algebraic manipulations are the most important aspect of school algebra (Lim, 2000). Yet, students often acquire procedural skills without comprehending what they are doing (Vaiyavutjamai, 2004a). They learn "rules without reason" or, using Skemp's (1976) terminology, they merely acquire an “instrumental understanding”. Lim’s (2000) analysis suggested that many upper-secondary mathematics students did not even acquire instrumental competency with respect to quadratic equations.

Methodology

Although the students in the Thai, Bruneian and US subsamples solved a range of equations, in this paper the emphasis is on comparing performances of, and methods used by, the three subsamples when attempting to solve two representative equations, specifically $x^2 = 9$ and $(x - 3)(x - 5) = 0$. The Thai and Brunei students attempted these two equations, as well as many other quadratic equations, immediately before and after participation in a series of lessons on quadratic equations. The 29 students in the US subsample were also currently taking a course on algebra (as part of their preparation to become specialist middle-school mathematics teachers), and in this course the concept of a variable had been specifically dealt with. As part of this course they were asked to solve $x^2 = 9$ and $(x - 3)(x - 5) = 0$, on a pencil-and-paper test, and also to indicate how they would check any solutions they obtained for $(x - 3)(x - 5) = 0$.

Selected students from the Thai and Brunei subsamples also participated in post-teaching interviews in which they were specifically asked to explain how they solved the equation $(x - 3)(x - 5) = 0$. After setting out their solutions to $x^2 = 9$ and $(x - 3)(x - 5) = 0$, the US students were asked to respond to a series of true/false questions seeking information on how they approached the $(x - 3)(x - 5) = 0$ task.

Results and Discussion

Table 1 shows percentages of the students in the three subsamples who gave correct solutions to $x^2 = 9$ and $(x - 3)(x - 5) = 0$. The Thai Year 9 students had the highest proportion of correct answers to the $(x - 3)(x - 5) = 0$ task. Results of further analyses of the students’ pencil-and-paper responses to the $x^2 = 9$ task, for each of the three subsamples, are shown in Table 2. Entries in Table 1 indicate that most Bruneian Year 10 students failed to obtain the correct solutions to the two equations. Although most of the US students did obtain correct solutions, the proportions that did not could be regarded as educationally significant, given that all of the US subsample has been strong mathematics students when in all, all intended to be middle-school specialist mathematics teachers, and all were currently taking a course in which careful attention had been given to the concept of a variable in algebra.

Table 1
Percentages of Students in Three Subsamples Solving $x^2 = 9$ and $(x - 3)(x - 5) = 0$ Correctly

	Percentage of Subsample Responses Correct in ...		
	Thailand	Brunei Darussalam	USA
Equation	(231 Year 9)	(205 Year 10)	(29 2nd-year university)
$x^2 = 9$	37%	14%	59%
$(x - 3)(x - 5) = 0$	78%	31%	66%

Entries in Table 2 reveal that a majority of the students in the Thai and Bruneian subsamples did not know that equations in the form $x^2 = K$ $(K > 0)$ had two real number solutions. The same was true of 12 of the 29 students (41.4%) in the US subsample. In fact, all the Thai and Bruneian students had been taught to solve an equation like $x^2 = 9$ in two ways: by writing $x^2 = 9$ as equivalent to $x^2 - 9 = 0$ and then factorising the left-hand side by the difference of two squares (before applying the null factor law), and by simply asserting that $x^2 = 9$ is equivalent to $x = \pm\sqrt{9}$, or ±3, because $+3^2 = 9$ and $(-3)^2 = 9$. Twelve of the 29 US students did not obtain two correct solutions to $x^2 = 9$. Of the 17 US students who did obtain the two correct solutions, 13 reasoned that $x^2 = 9$ implies that x equals $\sqrt{9}$ which, they asserted, was equal to ±3. These students seemed not to be aware of the international convention that the "√" symbol means "the positive square root of," and therefore the correct reasoning is: $x^2 = 9$ implies that x equals $\pm\sqrt{9}$, which is ±3.

Table 2
Various Categories of Responses of Students to the Equation $x^2 = 9$

Response to the Equation $x^2 = 9$	% of Subsample Giving that Response in ... Thailand (231 Year 9 Students)	Brunei Darussalam (205 Year 10 Students)	USA (29 2nd Year University Students)
Two Correct Solutions, and Correct Working	35.9%	7.4%	13.8%
Two Correct Solutions (But After Asserting that $\sqrt{9} = \pm 3$)	1.3%	6.9%	44.8%
Only One Correct Solution ($x = 3$)	26.8%	66.5%	41.4%
Two Incorrect Solutions	10.4%	8.2%	0.0%
One Incorrect Solution	25.6%	11.0%	0.0%

Vaiyavutjamai (2004a) interviewed each of the four Year 9 mathematics teachers who participated in the Thai component of the study after they had taught the lessons on quadratic equations (but before the teachers knew how their students had responded to a set of quadratic equations that included $x^2 = 9$). In the interviews all four teachers stated that they believed that at least 75% of their students would have no trouble solving $x^2 = 9$ correctly. In fact, one of the teachers stated that he thought that that type of task was so easy he did not teach his students much about it. He said he just "asked them to do the exercises by themselves" (Vaiyavutjamai, 2004a, p. 292). Lim (2000) reported that all four of the Year 10 mathematics teachers who participated in his study seriously overestimated the number of students in their O-level classes who would be able to solve $x^2 = 9$. One of the teachers, who taught a high-stream class, thought that "almost all of his students would be able to solve the equation correctly, but in fact less than 10 percent of them actually did. Another teacher predicted about half of the students in his Year 10 O-level class would solve the equation correctly, but in fact none of them did.

Results of analyses of the students' pencil-and-paper responses to the $(x - 3)(x - 5) = 0$ task, for each of the three subsamples, are summarised in Table 3. Entries in Table 3 suggest that, on the whole, students in the Thai Year 9 subsample learned to solve equations of the type $(x - 3)(x - 5) = 0$ much better than students in the Year 10 Bruneian subsample. In fact, in Brunei Darussalam the number of students who immediately "expanded the brackets" to get $x^2 - 8x + 15 = 0$ was greater than the number who immediately equated $(x - 3)$ and $(x - 5)$ to zero. This same tendency was also found in a minority of the Thai subsample. Interview data indicated that the Thai and Bruneian students who “expanded brackets” did that because they believed that they had been taught to reduce all quadratic equations to the form $ax^2 + bx + c = 0$, and then either to factorise the left-hand side (and use the null factor law) or to use the quadratic formula. Interestingly, 11 of the 29 university students in the US subsample did the same thing.

Table 3

Various Categories of Responses of Students to the Equation (x – 3)(x – 5) = 0

	% of Subsample Giving that Response in ...		
Response to the Equation $(x-3)(x-5)=0$	Thailand (231 Year 9)	Brunei Darussalam (205 Year 10)	USA (29 2nd Year University)
Two correct solutions after immediately equating $(x-3)$ and $(x-5)$ to zero, OR after substituting $x = 3$ and $x = 5$ in the equation.	74.9%	12.8%	58.6%
Two correct solutions [after writing $(x-3)(x-5)=0$ as $x^2 - 8x + 15 = 0$].	2.6%	18.2%	6.9%
No correct solution [but wrote $(x-3)(x-5)=0$ as $x^2 - 8x + 15 = 0$].	3.5%	35.8%	31.0%
No correct solution [and in none of the above categories].	19.0%	33.2%	3.5%

Interview and Questionnaire Data in Relation to the (x – 3)(x – 5) = 0 Task

Interviews with 18 Thai students (a high-performer, a medium-performer, and a low-performer from each of the six participating classes) generated data that pointed to the conclusion that the thinking of many of the 231 students was guided by a serious misconception (even though that misconception did not prevent them from arriving at correct solutions). Interviews revealed that most Thai interviewees thought that the x in the first pair of brackets in the equation $(x-3)(x-5)=0$ stood for a different value from the x in the second pair of brackets. A typical excerpt is given below – note that the first author was the interviewer and the student was a middle-performer in a high-stream class.

Interviewer: [*pointing to the different x’s in the equation* $(x-3)(x-5)=0$] Are those x’s the same variable?

Student: No. They are different.

Interviewer: So, what do you need to do to solve this equation?

Student: Use the substitution method.

Interviewer: You will use a substitution method. What are the numbers?

Student: Three and five.

Interviewer: Please show me your working.

Student: [*Wrote, on a piece of paper:* $3-3=0$, $5-5=0$, $0\times 0=0$] Three minus three equals zero. Five minus five equals zero. Zero multiplied by zero equals zero. It is a true sentence.

Interviewer: What is your answer?

Student: Three and five.

This interviewee was one of 11 interviewees (out of 18) who gave correct pencil-and-paper solutions to $(x-3)(x-5)=0$ but thought that the x’s took different values in the two parentheses. When checking their solutions they substituted $x = 3$ into $(x - 3)$ and $x = 5$ into $(x - 5)$ and concluded that since $0 \times 0 = 0$ their solutions were correct. Of the remaining seven interviewees, four seemed to have no idea how to solve the equation $(x - 3)(x - 5) = 0$, and for that matter did not seem to know what the instruction "solve the equation ..." actually meant. They did not realise that $(x - 3)(x - 5) = 0$ was likely to have two solutions. The other three interviewees gave correct solutions to $(x - 3)(x - 5) = 0$, and their pencil-and-paper responses and replies to questions during the interviews convinced the first author that they understood how the null factor law could be immediately, and appropriately, applied to solving quadratic equations in the form $(x - a)(x - b) = 0$.

Although 78 per cent of the Thai students gave correct solutions to $(x - 3)(x - 5) = 0$, interview data suggested that a majority of them were guided by a serious misconception when dealing with equations in the form $(x - a)(x - b) = 0$. They knew how to get correct answers, but did not know what their answers represented. This misconception was not confined to the $(x - 3)(x - 5) = 0$ task. With $x^2 - x = 12$, for example, some interviewees rearranged the equation as $x^2 - x - 12 = 0$, and then as $(x - 4)(x + 3) = 0$. Then, $x - 4$ and $x + 3$ were each equated to zero. However, these students were not sure whether the x in the "x^2 term" in the original equation represented the same variable as the x in the "x term" in the same equation (Vaiyavutjamai, 2004a).

Entries in Table 3 suggest that the Bruneian Year 10 students were more confused about the quadratic equation $(x - 3)(x - 5) = 0$ than were the Year 9 Thai students. So far as the US students were concerned, the 29 students were asked to respond, by indicating "True" or "False" to the four statements shown in the first column of Table 4. Entries in Table 4 relating to Statements 1 and 4 suggest that at least one-third of the US students were confused about the concept of a variable in the context of equations that are in the form $(x - a)(x - b) = 0$. Responses to Statements 2 and 3 support the conjecture that the US students who answered "True" for Statements 1 or 4 believed that the variable x assumed different values in the two sets of brackets. The authors intend to administer the true-false items to larger samples of students, and gather new interview data with respect to the issue of how students think about the concept of a variable in the context of quadratic equations, and indeed in higher-order polynomial equations, and in linear equations and trigonometric equations, in which the variable appears more than once in the equation statements.

Table 4

US Students' Responses to Written Questions Regarding the Equation $(x - 3)(x - 5) = 0$

Statement: Consider the Equation $(x - 3)(x - 5) = 0$:	Number of Respondents ($n = 29$) indicating that the Statement is ...	
	True	False
1. In the first brackets, x must equal 3, and in the second brackets, x must equal 5.	10	19
2. In the first brackets, x must equal 3, but in the second brackets, x can have any real number value.	3	26
3. In the second brackets x must equal 5, but in the first brackets, x can have any real number value.	3	26
4. This equation is equivalent to $x^2 - 8x + 15 = 0$, which is a quadratic equation with two solutions. Thus, with $(x - 3)(x - 5) = 0$, the x in the first brackets always equals 3, and the x in the second brackets always equals 5.	16	13

Concluding Comments

The above analyses of responses to two apparently straightforward quadratic equations suggest that in the Thai and Brunei subsamples more than 50 percent of the students were confused with respect to the concept of a variable as it manifested itself in the quadratic equations. The entries in Table 4 suggest that a sizeable proportion of the US subsample was also confused on the same matter. The results point towards the need to create a new item on the research agenda for the international mathematics education research community: if quadratic equations are to remain an important component of middle- and upper-secondary mathematics curricula, then research is needed to guide teachers about how students think about quadratic equations, and especially about what can be done to help teachers to improve their students' concepts of a variable in that context.

References

Chaysuwan, S. (1996). The construction of a diagnostic test on basic knowledge of algebra for Mathayom Suksa three students in the Bangkok Metropolis. Unpublished M.Ed thesis, Kasetsart University (Thailand).

Filloy, E., & Sutherland, R. (1996). Designing curricula for teaching and learning algebra. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 139–160). Dordrecht, The Netherlands: Kluwer Academic Publishers.

French, D. (2002). *Teaching and learning algebra*. London: Continuum.

Gray, R., & Thomas, M. O. J. (2001). Quadratic equation representations and graphic calculators: Procedural and conceptual interactions. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th Conference for the Mathematics Education Research Group of Australasia, Sydney, pp. 257–264). Sydney: MERGA

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

Kieran, C., & Chalouh, L. (1993). Prealgebra: The transition from arithmetic to algebra. In P. S. Wilson (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 119–139). New York: Macmillan.

Lim, T. H. (2000). The teaching and learning of algebraic equations and factorisation in O-level Mathematics: A case study. Unpublished M.Ed dissertation, Universiti Brunei Darussalam.

Mathematical Association (1962). *The teaching of algebra in schools*. London: G. Bell & Sons.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20–26.

Stacey, K., Chick, H., & Kendal, M. (Eds.). (2004). *The future of the teaching and learning of algebra*. Boston: Kluwer Academic Publishers.

Thomas, M. O. J., & Tall, D. (2001). The long-term cognitive development of symbolic algebra. In H. Chick, K. Stacey, J. Vincent & J. T. Zilliox (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 590 -597). Melbourne: University of Melbourne.

Vaiyavutjamai, P. (2004a). Factors influencing the teaching and learning of equations and inequations in two government secondary schools in Thailand. Unpublished PhD thesis, Universiti Brunei Darussalam.

Vaiyavutjamai, P. (2004b). Correct answers do not imply understanding: Analyses of student solutions to quadratic equations. In I. P. Cheong, H. S. Dhindsa, I. J. Kyeleve & O. Chukwu (Eds.), *Globalisation trends in science, mathematics and technical education* (pp. 153–162). Gadong, Brunei Darussalam: Universiti Brunei Darussalam.

Wagner, S., & Parker, S. (1993). Advancing algebra. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 119–139). New York: Macmillan.

Warren, E. (2000). Research in teaching and learning algebra. In K. Owens & J. Mousley (Eds.), *Research in mathematics education in Australasia 1996–1999* (pp. 161–180). Sydney: Mathematics Education Research Group of Australasia.

Warren, E., & Pierce, R. (2004). Learning and teaching algebra. In B. Perry, G. Anthony & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000–2003* (pp. 291–312). Flaxton, Australia: Mathematics Education Research Group of Australasia.