Real World Problems: Developing Principles of Design

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The provision of authentic applications, as distinct from artificial word problems, as vehicles for teaching students to apply their mathematics remains an unfulfilled need. This paper describes the generation and application of a framework that provides principles to inform such a purpose. The framework is grounded in classroom data produced by students solving modelling problems, and its use is illustrated through application to the design of a new problem. The development and application of the framework are on-going.

It is now more than thirty years since Henry Pollak (Pollak, 1969) challenged the mathematics education community to seriously engage with applications and mathematical modelling. Since that time some positive achievements (including in some Australian states), have been tempered by the knowledge that many challenges remain unfulfilled, or have been abandoned or emasculated. Large-scale initiatives such as the OECD Programme for International Student Assessment (PISA) have included items with applications and modelling content in its bank of items (Turner, 2004). An interesting range of application skills are sampled by the items, including the need to make assumptions, choose a mathematical approach, and interpret outcomes. The nature of the testing does not provide for extended modelling work, but student capacity to succeed on such items is indicative of modelling related skills, and it is interesting to note that the omission rates for such questions has been generally very high across countries, pointing to deficits in the confidence, as well as competence, with which students approach contextualised problems. If improvements in performance in the abilities that such measures aim to identify are to be achieved, these will be consequences of successful programs developed within classrooms. The potential context within which applications and modelling occurs is enormous, when one considers the relevance of mathematical activity to other discipline areas. Because the latter require also specific knowledge of specialist areas outside mathematics, the focus here is limited to situations that occur in the 'real life' of students. Given this, the overall purpose is to enable students to acquire skills for accessing their 'pure' mathematical knowledge in addressing problems relevant to their world, and more importantly to focus on how this can be successfully achieved.

The field itself occupies a distinctive space within mathematical education, for it represents a meeting place for mathematicians with a deep interest and involvement in educational issues, and teachers and educators committed to promoting excellence in mathematics for purposes other than performing on traditional coursework and examination tasks. As such, applications and mathematical modelling has a home at all levels of education (Elementary, Secondary, Tertiary) to which we should add Teacher Education. Periodically papers have surveyed the situation within the global field, endeavouring to paint a current picture of the state of play (e.g. Blum & Niss, 1991; Blum, Galbraith, Henn & Niss, 2006); Niss 2001). The recent book (Lesh & Doerr, 2003) provides by virtue of its content a similar service, from a mainly north American perspective. ICMI Study 14 (Blum et al., 2002) indicates the continuing significance of the topic at international level.

The more things change…

In revisiting historical roots we note that Pollak asked, not what commonly available word problems could achieve in terms of their contribution to curricular mathematics, but what they contributed to the capacity of students to apply their mathematics to problems outside the classroom, and what messages they conveyed about the nature of applications of mathematics. He concluded that they were almost all pseudo-problems, not really concerned with genuine applications, and involving skills far removed from those necessary to address problems based in the world outside the classroom. With this in mind, here is an example of a word problem taken from a contemporary source, circa 2004.

> *Example A:* A take-away food shop sells hamburgers, sausages, and pizzas. On one day the number of hamburgers sold was three times the number of pizzas, and the number of sausages sold was five times the number of pizzas. The number of hamburgers and pizzas sold was in total 176. How many of each type of food was sold?

While this problem is couched in the language of the real world there is no sense in which it represents how a vendor would make decisions essential to her/his livelihood - it does not show how mathematics is applied to enhance decision-making associated with real problems. The point here is that, despite all that has happened in the interim, this very issue (and problem type), a major motivation behind the article by Pollak more than 30 years ago, remains an issue today. As was stated then, such word problems have value in curricula and we can learn from advances made in the understanding of how students cope with them — but not in terms of applied mathematics. If they are to play an enabling role in helping students to apply their mathematics to real problems, it is essential to clarify those additional or different features that characterise examples that genuinely involve applications and modelling. Here is another word problem posing as an "application" taken from a contemporary curriculum source. It belongs to a family of problems classified as "whimsical" by Pollak, which he noted were greatly loved and probably did a lot of good — but not as applied mathematics.

> *Example B:* Two meatballs roll off of a pile of spaghetti and roll toward the edge of the table.
> One meatball is rolling at 1.2 m/s and the other at 0.8 m/s. They fall off the table and land on a $5000 Isfahan carpet. If the table is 1.2 m high, how far apart from each other do the meatballs land?

This scarcely requires comment in terms of real problem content — for example how many "impossibilities" are embedded in the wording? Within the contemporary scene it would seem fair to say that *word problems* continue to be widely construed as close relatives of *application* and *modelling* problems, with links made on the basis of semantic content, given that both word problems and modelling problems are couched in verbal clothes. In other ways however, as we have seen, the two usually differ markedly, specifically with respect to meaningfulness. Fortunately there are contemporary voices that continue to call for the use of more authentic problems.

> …the use of conditions that often make out-of-school learning more effective can and must be re-created, at least partially, in classroom activities. Indeed while there may be some inherent differences between the two contexts these can be reduced by creating classroom situations that promote learning processes closer to those arising from out-of-school mathematics practices. (Bonotto, 2006)

Problem Context

Palm (2006) points out that many (or most) real world examples used in education cannot be exact matches with external counterparts, so we now proceed to look at problems that represent, to a reasonable degree, genuine real-world activity — and how these can be sensibly represented in school-based applications. The subsequent emphasis in this paper is on student performance and problem design, and it addresses the following purposes:

1. Examining student performance on modelling problems to identify structural components with generic properties;
2. Using resulting insights to develop a framework for task design.

The following example, one of several real-world problems given in the course of a two-year program, provides material for subsequent analysis. The discussion below includes data illustrating how some students solved the given problem. Student comments, extracted from extended modelling reports, are shown in boxes, with essential steps undertaken by the students summarised in normal type. The context was a senior secondary college in which mathematical modelling formed a structured component of the year 11–12 mathematics course, for students bound for science or engineering at university.

Spacing Speed Bumps

This problem took advantage of the construction of a new entrance road to the college, which occurred while the students were undertaking their mathematics course. The entrance road diverged from a roundabout in the main road outside the college gate and extended to another roundabout by the administration block that fed into student and staff car parks. Speed bumps were to be placed along the stretch of road to control speeds within the college grounds.

The Problem

Students were given the problem of siting the speed bumps and required to provide a detailed report supporting their recommendations. (This was the level of generality with which the problem was set — no mathematical background was provided.)

1. Definition of the mathematical problem: Given only the general instruction above, one student group specified the problem as follows.

> To position speed bumps on the college's entrance road to keep speeds to a maximum of 30km/h and above all to keep safety to a maximum for students. For this purpose the entrance road has been defined as the road from the roundabout at the front of the college to just before the roundabout by the administration block (approximately 255 metres).

2. Initial factors identified by students: *Maximum speed allowed between bumps; Total length of entrance road; Maximum speed over bumps; Danger areas.*

The students used various means to obtain information deemed to be necessary for the project. The maximum speed was taken to be the speed limit within the college grounds, the length of entrance road was measured to have the value given in the problem specification, and the maximum speed over bumps was obtained from documentation of the type of bump to be used (designed to provide for a maximum speed of 8 km/h over the bump). Danger areas were identified as locations specific to the problem in terms of the college environment.

> The danger areas along the college's entry road are areas where speed bumps cannot be put due to safety reasons. There are four such areas.

‘Danger areas’ was a concept initiated by the students, and identified as the entrance to the college, a drain located 140 metres from the entrance that was deemed subject to cracking if subjected to extra stress, an area leading off the main roadway to car parks, and an area near bicycle parking bays where it was felt students should not be caused to swerve to avoid a speed bump. (Having initiated these considerations, decisions were made by the students following an examination of campus maps).

3. Finding location of first speed bump: The students set out to break the problem into successive parts associated with the respective bumps. If assumptions were needed these were addressed within specific calculations, rather than attempting to identify a global set of assumptions for the complete problem. For the first bump it was assumed that cars leaving the roundabout to go through the college gate would be travelling at about 30km/h.

Finding the time for an average car to slow down from 30 km/h to 8 km/h is found by going into the real world and taking an average car and doing a series of tests

The students provided a table from their (sedate) experiments using one of their own cars, which gave an average value of approximately 5.5 sec — cautious driving indeed! They then assumed motion would be rectilinear and used the formula $d = (v_i + v_f)t/2$ to estimate d as 29m from the front gate on the entrance road.

4. Finding location of second speed bump:

When finding the position of the second bump two processes have to be done. The first process is the accelerating from 8 km/h just as the car gets over the speed bump, to when the car is at maximum speed through the college, which is 30 km/h. The second process is the slowing down of the car from 30 km/h to 8 km/h where the next speed bump is…The car will be getting to a maximum of 30 km/h, then the driver will see the next bump and start to slow down…Finding the time for an average car to accelerate from 8 km/h to 30 km/h is found by again going into the real world and doing some tests with an average car.

The students again provided some test results in a table showing an average time of approximately 7.5 sec. (It was interesting that they did not assume a symmetrical result to the braking situation considered earlier). Again using $d = (v_i + v_f)t/2$ a value of approximately 39 m was calculated, and slowing was assumed to start immediately. The “slowing distance” was shown to be the same as that for the first bump (29m) so an estimated distance between bumps of 68m was inferred.

5. Finding the location of third and fourth bumps: These mirrored calculations for the second speed bump

6. Refinements and recommendations:

This step involves going back to the real world and seeing if the results of the maths fits the road and dodges the danger areas. Testing the first speed bump to see if 29 m from the front gate misses the danger areas and indeed it does…(similar for bumps 2 and 3)… However, the last speed bump lies on the fourth danger area. A suitable method of getting this speed bump out of the danger area is to move it forward 7 metres.

The implication that 4 bumps are consistent with a 255m stretch of roadway was implicit in the approach, but did not feature explicitly in the students’ testing of the solution obtained. What might also have been done here was to consider several acceleration and braking rates, to test the sensitivity of the suggested locations to a range of driving behaviours.

7. Testing the speed bump results: Two possible refinements were suggested: more severe speed bumps to decrease the minimum driving speed over the bumps, and

incorporation of the car park in the analysis, which would require additional bumps.

It will be clear that the problem solving process undertaken by the students contained features different from conventional approaches to teaching and learning in mathematics classrooms. Foremost among these was the interplay between mathematics and a problem context that was external to the classroom. This occurred for example through the seeking out of documentation enabling the intended maximum speed over a bump to be estimated, consultation of college maps, and most particularly experiments on acceleration and braking conducted using a student's car in a roadway. Most importantly, these actions were initiated by the students themselves. Additionally much of the modelling took place outside formal school hours, again initiated by the students through arranged meetings.

From Problem Solution to Design Principles

In this section some principles are articulated and illustrated based on data from the speed bumps problem (Table 1). See discussion below Table.

Table 1
Structure of Speed Bumps Solution

Principle	*Enactment in solution to speed bumps problem*
Principle 1: There is some genuine link with the real world of the students.	The proposed problem directly affected the students, and was located in their college.
Principle 2: There is opportunityto identify and specify mathematically tractable questions from a general problem statement.	This was demonstrated by the definition of the mathematical problem as written by the students.
Principle 3: Formulation of a solution process is feasible, involving the use of mathematics available to students, the making of necessary assumptions, and the assembly of necessary data.	*The students identified what they called *initial factors*, and set up a solution process that assumed equations of rectilinear motion. Data required in the solution process were obtained by consulting documentation, and by doing some trials with cars under road conditions.
Principle 4: Solution of the mathematics for the basic problem is possible for the students, together with interpretation.	*The students solved the basic problem for each speed bump in turn using appropriate assumptions (not always made specific), and interpreted the results appropriately.
Principle 5: An evaluation procedure is available that enables checking for mathematical accuracy, and for the appropriateness of the solution with respect to the contextual setting.	*The mathematics is easily checked, and the students tested and adjusted their recommendations in terms of the danger areas they had identified.*
Didactical principle: *The problem may be structured into sequential questions that retain the integrity of the real situation.* (These may be given as occasional hints at the discretion of a teacher, or used to provide organised assistance by scaffolding a line of investigation.)	*Sample questions for speed bump problem:* • *What speeds are important to consider in deciding the location of bumps?* • *What do you need to know in order to go about calculations for the first bump?* • *How is the situation between bumps different from that leading to the first bump?* •How can you check whether your recommendations are reasonable?

* It will usually be the case that some students will be unsuccessful. It is important to remember that these are principles for *problem design* not necessarily for individual *student success*, and that the latter require a different set of criteria.

It would be audacious in the extreme for any individual to claim to hold a definitive position with respect to structuring mathematical modelling problems, and this is certainly not the case here. However it is important to analyse and distil qualities that can be identified in problems that have proved successful, and to use such information in the design and testing of new problems, and indeed in the search for design principles. These principles (see Table 1) emerged, as student responses to a range of problems were viewed in the light of theoretical needs for a problem to be considered authentic. An application of these principles follows below, to illustrate key elements in the design of a modelling problem from a new situation. Such an enterprise is best viewed as work in progress, rather than as definitive, and can be seen as a contribution towards the development of a theoretically consistent approach to problem construction. It is desirable that principles should embody essentials, encompass the data, be theoretically consistent, and for practical purposes be limited in number - from these perspectives and that of the data, five were selected. In addition to these mathematical principles, one pedagogical principle has been added. (While a number of pedagogical principles will typically be used in teaching, this one is directed particularly towards the design issue).

Application to Task Design: EAN-13 Barcodes

Having developed principles based on student solutions as reference data, we now test these principles by application to the design of a new problem. In practice the principles are not applied sequentially but together form a working basis that is used to develop, critique, and refine an idea into a modelling problem. The example below has been designed for a current Australian initiative, for which student outcome data are not yet available.

The Problem Context

The purpose of barcodes is to enable instantaneous processing of information by computers, and their use in our society is now almost universal. Australia uses the European Article Numbering Code containing 13 digits (EAN-13), which is one of the most commonly used systems worldwide (Figure 1).

7 012345 678908

Figure 1. A sample barcode

The following codes have been taken from Supermarket labels.

930060112804 4 *(tomato sauce*)

930060118014 1 (*iodised salt*)

The left-most digit (zeroth digit shown as separated from the rest on labels) together with the next digit indicates the country of manufacture (e.g. 93 represents Australia and 94 represents New Zealand.)

The next five digits identify the manufacturer (these are both *Farmland* products)

The following five digits identify the particular product (*tomato sauce* and *iodised salt*).

The final number is a *check* digit. When the label is scanned, the barcode identifies the item, for which the price is stored in the retailer's database - the computer verifies that the check digit is correct before processing the number. If an error is detected the computer indicates accordingly. This can happen, for example, if a paper label on a can is damaged so that a digit is misread. A checkout attendant will then enter the barcode by hand, and this procedure is of course also subject to error.

The check digit works as follows. Using the first 12 digits in the code, the check digit satisfies the condition that:

3x(1^{st} + 3^{rd} +…+ 11^{th} digit) + 1x(0^{th} + 2^{nd} + 4^{th} +…+ 10^{th} digit) + check digit is divisible by 10. (Here 3, and 1 are referred to as *weights*)

The Problem Statement

Two types of error that barcodes are designed to deal with are mechanical errors and human errors. A mechanical error is caused for example by a machine misreading a digit on a damaged label. A human operator is prone to two types of error: mistyping a single digit, and transposing the order of adjacent digits. How effective is the barcode system in detecting such errors, and would other weights be as effective or better?

Table 2:

Application of Design Principles to EAN-Codes Problem

Principle	**Enactment in design of EAN - 13 codes problem**
Principle 1: There is some genuine link with the real world of the students.	The context is a part of the everyday experience of all students.
Principle 2: There is opportunity to identify and specify mathematically tractable questions from a general problem statement.	Suitable sub-questions are implied by the general problem: • What proportion of common errors will be detected by the check digit? • Is there a simpler set of weights that is as effective or better for this purpose?
Principle 3: Formulation of a solution process is feasible, involving the use of mathematics available to students, the making of necessary assumptions, and the assembly of necessary data.	The sub-questions require basic strategies of proof, and procedures that need only an understanding of integer arithmetic, including simple notions of divisibility. (Putting these together, as usual, increases the demand compared with the demand that would apply for each separately.)
Principle 4: Solution of the mathematics for the basic problem is possible, together with interpretation.	The solution of sub-questions can be addressed by students, using existing knowledge resources — here the successful completion of arithmetic procedures, and associated logic, and interpretation.
Principle 5: An evaluation procedure is available that enables checking for mathematical accuracy, and for the appropriateness of the solution with respect to the contextual setting.	Checking of mathematical answers is a feasible part of the procedure. Ideally the outcomes should also be tested in their real-world setting. (Given the number of students that have part-time jobs in commercial stores this is certainly a realistic possibility.)
Didactical principle: The problem may be structured into sequential questions that retain the integrity of the real situation. (These may be given as occasional hints at the discretion of a teacher, or to provide organised assistance by scaffolding a line of investigation - often helpful at the challenging specification stage in assisting applications of Principle 2.)	*Sample structuring questions (hints) for EAN codes problem:* • Is the check digit unique? • Which single digit errors will be detected by the coding method? • Which transposition errors will be detected by the coding method? • Will weights of 1 and 2 do as good a job as weights 1 and 3?

The context by itself provides nothing more than a potentially fruitful idea. The above table indicates how the principles are embedded in the design of this EAN code problem.

This is essential if the detail included in a problem statement is to provide a feasible modelling exercise. As one illustration of the elaboration characterizing the application of the various principles, an outline of single-error detection is given below.

Single error detection: If a digit 'a' is changed to 'b' the weighted sum will change by (b-a) if the weight is 1, or by 3(b-a) if the weight is 3 — noting the sign (direction) of the change does not matter. Errors will only go undetected if these quantities are a multiple of 10: 0,10,20… Now b-a = 0 requires b = a; and b-a = 10, 20. cannot be satisfied for unequal values of digits *a* and *b* that range between 0 and 9. Similarly 3(b-a) = 0 requires b = a. The next possibilities 3(b-a) = 10 or 20 do not give whole number values for a and b, and 3(b-a) = 30 cannot be satisfied for *unequal* values of a, b in the range 0 to 9. Hence a = b in all cases so the method gives a 100% detection rate.

Concluding Comment

The generation of problems presenting contexts for authentic applications of mathematics remains a significant need. Given an appropriate context, the challenge involves converting a fruitful idea into a suitably framed, accessible problem. For this purpose, principles of design need to be theoretically sound and practically workable. This paper has illustrated the creation and application of a design framework. The framework has been structured by analysing qualities embedded in successful solutions to problems at senior secondary level (illustrated by means of the speed bump problem). These properties when generalised have provided a framework of structural principles. The application of this framework in structuring a contextualised problem, so as to satisfy the associated principles, has been illustrated. The framework itself should be regarded as a work in progress, to be refined and improved through subsequent application and evaluation.

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