One-Third is Three-Quarters of One-Half

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This paper reports on part of a larger cross-sectional study of the development of students' quantitative concepts of fractions. In total, 1676 students in Years 4–8 were asked a series of questions designed to elicit their concept images of various fractions. Three questions asked the students to construct a regional model (parts of a circle) for one-half, one-third and one-sixth. The initial analysis of the evoked concept image of one-third revealed an unexpectedly high number of responses shading one-quarter of a circle. Further analysis suggested that what appeared to be one-quarter was frequently intended to be one of three parts, confirming that drawings frequently do not "speak for themselves". Area is not always the intended feature of regional models of fractions in students' concept images and it is argued that the "number of pieces" interpretation is a common response to regional models of fractions.

Mathematics teaching seeks to build on what students already know. When it comes to the teaching of fractions, the limitations of the "part of a whole" model of a fraction (Kerslake, 1986; Kieren, 1988) have been described. Many students appear to see fractions as two unrelated whole numbers—three-quarters is the whole number three written over the whole number four (Hart, 1989). In such an interpretation *a/b* denotes a two-number relationship, for example 3/7 means "three out of seven" (Brown, 1993). Children who view fractions this way may apply whole number strategies to fraction problems (Lamon, 1999; Mack, 1995; Streefland, 1993). Even questions using pre-partitioned regions designed to ascertain students' regional "parts of a whole" understanding may be subject to whole number interpretations. In Hart's (1988) study of 12 to 13 year olds, 93 per cent correctly shaded in two-thirds of the shape in Figure 1.

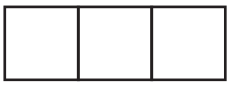

Figure 1. Hart (1988) test item: Shade in two-thirds.

Yet does this indicate a regional "parts of a whole" understanding? Hart notes (p. 69) that questions like this were almost invariably solved by counting the number of squares in the entire figure, counting those to be shaded and putting one number over the other. That is, pre-partitioned shapes may encourage a double count rather than an area-based interpretation of a regional model. As Kerslake has stated (1986), more attention needs to be given to the limitations of the "part of a whole" model of a fraction.

Thinking quantitatively about fractions depends on the concept images students have of fractions. Tall and Vinner (1981) describe a concept image as all of the cognitive structure in the individual's mind that is associated with a given concept, "which includes all of the mental pictures and associated properties and processes" (p.152). They refer to the evoked concept image as the portion of the concept image activated at a particular time.

In this way, seemingly conflicting images may be evoked at different times without necessarily producing any sense of conflict in a child.

The aim of this study is to examine students' evoked fraction concept images of one-half and one-third arising from their drawings and explanations. In particular, this study investigates the question, "How do students' drawings of regional models of one-half and one-third reflect their evoked fraction concept images?"

Method

Participants

Nine high schools, one central school and eleven primary schools were selected from across New South Wales government schools. The schools were selected to provide a good balance between metropolitan, regional and rural areas. Complete classes were chosen from Years 4, 5 and 6 in the primary schools and Years 7 and 8 in the high schools and the central school, including at least two classes in each of the larger schools. The classes were heterogeneous, including composite multi-age classes and ungraded, middle-band streamed classes as well as one class for talented students.

The result was a sample of 1676 students, comprising 331 Year 4 students, 330 Year 5 students, 331 Year 6 students, 342 Year 7 students and 342 Year 8 students. In total, 832 female students and 844 male students participated.

Procedure

Two series of fraction tasks, 30 questions in the first set and 7 practical tasks in the second set, were given to intact classes on separate days within one week, selected by the school to fit with other commitments, in July and August 2004. The school coordinators for the fraction assessment tasks were earlier provided with training in order to gain consistency in the delivery of the tasks.

Within the first series of fraction questions, students were asked to shade one-half, one-third and one-sixth separately in one of three identical circles (Questions 12–14). Question 23 asked them to indicate which is the bigger number, one-half or one-third and to explain how they knew. To reduce the likelihood of the notation *a*/*b* triggering a double count, this notation was not introduced until after Question 24. Up to this point, the fractions were written in words or spoken.

Coding always begins by attempting to describe what the student has recorded. The coded responses are then categorised to reduce the data. The creation of categories relies on the coded responses, which in turn are a reflection of the way the coder interprets the responses. Coding tasks designed to evoke students' constructions of fractions brings with it the challenge of coding in a way that allows the student's intent to be captured. Coding responses from the perspective of an adult's relatively sophisticated understanding of fractions can produce an artificial and sometimes inaccurate portrait of students' thinking.

The initial coding of Questions 12 to 14 viewed the responses through a traditional area model interpretation. Responses were initially coded according to the fraction of the total area shaded.

Consistently coding the area of a circle that a student had marked brought with it several challenges. Foremost was the need to appreciate the physical limits of drawing sectors by hand, combined with the difficulty of angle judgement. When estimating angles we tend to overestimate acute angles and underestimate obtuse ones (Carpenter & Blakemore, 1973). There also appears to be a spatial norm effect, where we generally see better in the horizontal and vertical orientations. Consequently estimates of angles depend upon their orientation. Given the range of ages and experiences of children in this study, error limits on the estimates for one-third and one-sixth were set at ±15°. This meant, for example, that a sector from 105° to 135° was coded as one-third unless the construction lines showed that the intent was to draw three-eighths.

The initial classification by area was followed by a conceptual analysis of the evoked concept images. The basic assumption of the conceptual analysis was that children's actions are always rational from the vantage point of their current understandings. From this perspective, children's unanticipated responses to questions are problems for the researcher to solve. The focus of the conceptual analysis was therefore on children's meanings—that is, on how they interpreted and attempted to solve the mathematical tasks they were presented with.

Results and Discussion

The percentage of students correctly shading one-half of a circle was very high with every Year 5, 7 and 8 student correctly answering the question. Two Year 6 students were incorrect, as were three Year 4 students; another two Year 4 students omitted the question. These results tend to confirm Hart's (1989) observation that one-half is not a typical fraction but "…seems to be an honorary whole number" (p. 216).

The responses to shading one-third of a circle were much more varied. The initial area-based coding of Question 13 is recorded in Table 1.

Table 1
Percentage distribution of fractions of a circle shaded to represent one-third (initial coding)

	Year 4 (N=331)	Year 5 (N=330)	Year 6 (N=331)	Year 7 (N=342)	Year 8 (N=342)
1/3	30	43	62	43	54
1/4	16	11	10	14	12
Segmented	12	15	7	10	6
3/4	10	7	3	4	3
Other	30	24	18	29	25
Omitted	2	0	0	0	0

Students applying a "number of pieces" interpretation of the regional model may attempt to create thirds as segments of a circle, as shown in Figure 2. Not all students consistently segmented both circles.

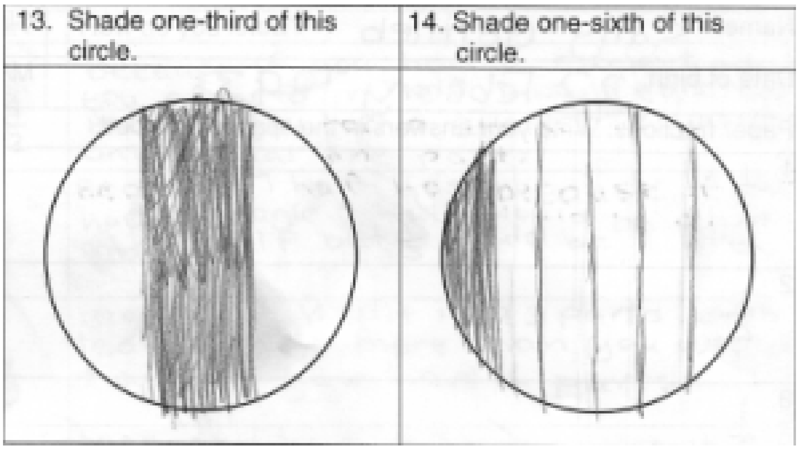

Figure 2. A Year 6 student's response coded as segmented.

The initial coding and classification by area brought to the surface several apparent anomalies that contributed to the conceptual analysis of the evoked concept images. The first was that the number of responses coded as "one-quarter" was surprisingly large, constituting over 12% in total of all responses. After the correct answer of one-third, this category was the next largest. This unexpected error also appeared to be relatively insensitive to grade level, with about 40 responses in each grade being coded as one-quarter.

The second, and perhaps more telling observation was that representing one-third as one-quarter did consistently use the shaded region. For example, the response shown in Figure 3 was initially coded one-quarter. Similarly, the attempt to create one-sixth of a circle was also initially coded as one-quarter. The shaded region, one-quarter of the circle, is the same in both questions. Clearly the student either believes that one-third and one-sixth are represented by the same area, or else area is not the intended focus of the student's response. It would seem that the student is focusing on the number of parts rather than the area of those parts. Viewed from an emergent perspective of sense making, by focusing on the number of parts the responses now appear to be consistent.

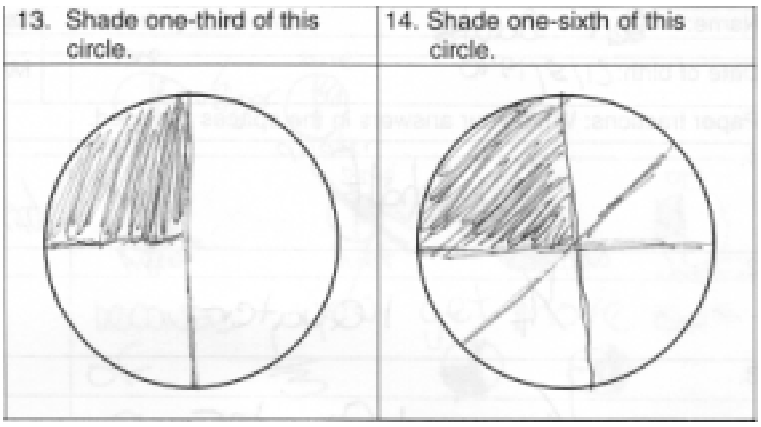

Figure 3. A Year 7 student's response initially coded as one-quarter.

As a result of this deliberation, all of the responses initially coded as 1/4 were recoded to determine how frequently the intent to display one-third as "one part out of three parts" was evident in the responses.

Table 2
Percentage distribution of fractions of a circle shaded to represent one-third (recoded)

	Year 4 (N=331)	Year 5 (N=330)	Year 6 (N=331)	Year 7 (N=342)	Year 8 (N=342)
1/3	30	43	62	43	54
Segmented	12	15	7	10	6
1/4	11	7	8	9	10
3/4	10	7	3	4	3
3 parts (1/2 & 2 1/4s)	5	4	3	5	3
Other	30	24	18	29	24
Omitted	2	0	0	0	0

The category of one part out of three parts (one half and two quarters) accounted for 4% of all responses. Most students shaded in one-quarter as the one part out of three but a small number shaded the one-half. As these students had correctly created one-half of a circle in the previous question, it is likely that their intent was to indicate "one part of three" for one-third.

The responses coded as one-quarter are now less in total than those coded as segmented, but the total is still quite high. It is worth noting that the category one-quarter contains constructions of two-eighths as in Figure 4, and other creations of one-quarter, presumably drawn to align with the common image of one-third.

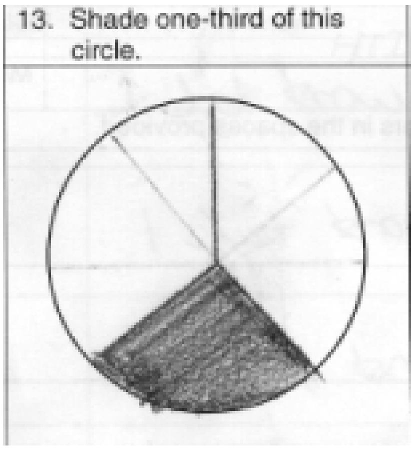

Figure 4. A Year 6 student's drawing coded as one-quarter.

The answers coded as three-quarters are also an unexpected response to drawing one-third. A further investigation of this category yielded an indication of how three-quarters could be a sense-making approach to recording one-third.

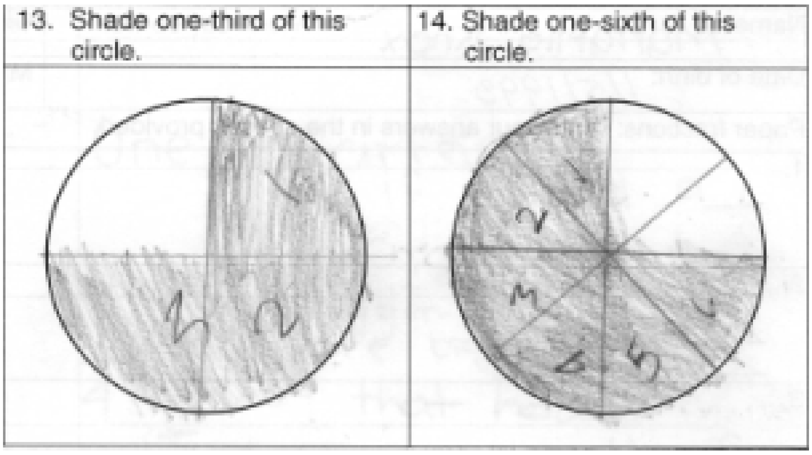

Figure 5. A Year 5 student's drawing identifying the number of parts.

In Figure 5, the student's numbered drawing suggests an attempt to indicate one-third as corresponding to three parts and one-sixth to six parts. That is, a "number of pieces" interpretation of the regional model that represents the number of pieces described by the denominator of a fraction could translate into drawing an area of three-quarters to represent one-third. Not only is this a "number of pieces" interpretation but it is also a "number of equal pieces" evoked image of a fraction. The response in Figure 6 is suggestive of the same interpretation.

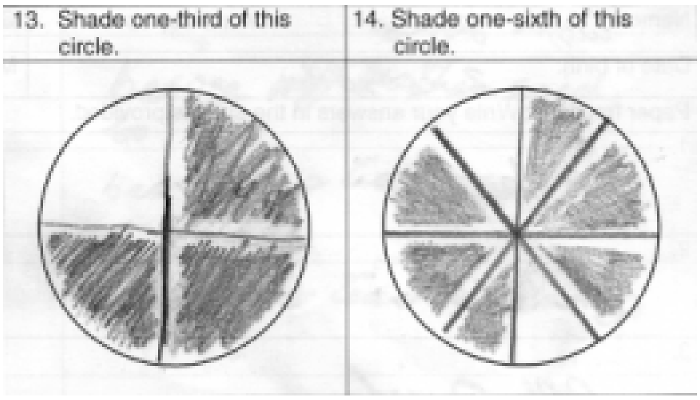

Figure 6. A Year 8 student using a "number of equal pieces" representation of a unit fraction.

A "number of equal pieces" interpretation could also be applied to those students who constructed three-eighths in response to the request to shade one-third of a circle. The total number of students who constructed three-eighths in response to Question 13 was 28, too few to justify a separate category. Although some students' constructions of three-eighths may be interpreted as a "number of equal pieces" image of a fraction, this interpretation may not always be accurate. Visually, this creation of one-third as three countable units (see Figure 7) has resulted in an image that is quite close to the correct sub-division of the area of the circle for one-third.

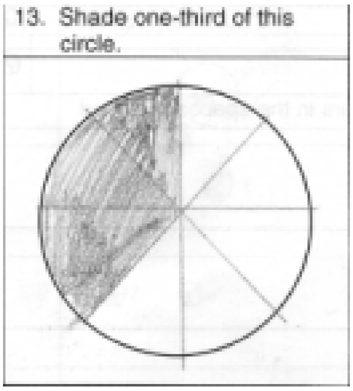

Figure 7. A Year 7 student's construction of three-eighths for one-third.

The visual similarity between three-eighths and one-third of a circle may be sufficient for some students to equate regional models. Students' explanations when comparing the size of two fractions on Question 23 suggest that they may have equated regional models as part of their concept images. A Year 8 student stated that "1/2 is bigger because 1/3 is one and a half quarters". That is, for this student three-eighths appears to have become an area equivalent for one-third. Responses to Question 23 also capture the belief that one-third is the same as three-quarters (see Figure 8).

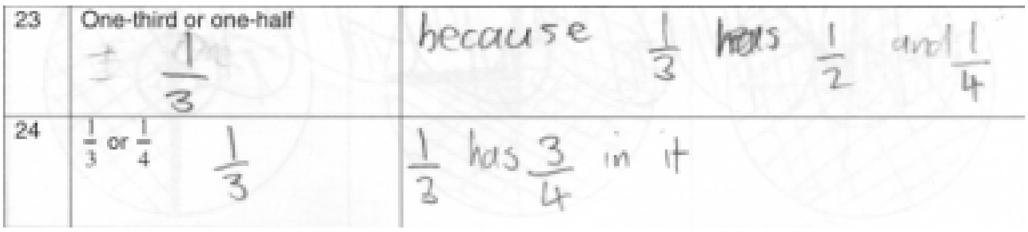

Figure 8. Year 8 student's explanation of which of two fractions is larger.

Conclusions and Implications

Students' evoked concept images for one-third, in responding to a request to shade one-third of a circle, demonstrated that the regional "part of a whole" model of a fraction is substantially incomplete in many students' understanding. A conceptual analysis of the evoked concept images of one-third showed that a "number of pieces" interpretation could result in segmenting a circle, constructing one-half and two quarters to represent three pieces and, constructing three-quarters or three-eighths to shade three "equal" pieces. Moreover, misinterpretation of the regional model of fractions can carry over into other contexts. For some students, one-third is identical to three-quarters or three-eighths.

Only some components of the regional "parts of a whole" model of fractions appear as elements of some students' evoked concept images. For some students, one-third is one of three parts, not necessarily all equal in area. For others, the three parts may be equal in area yet not create the whole. The number of parts, the equality of the parts and the total of the parts constituting the whole region are components of the regional model of fractions that appear disaggregated in some students' concept images. This focus on only some components of the regional model can result in students believing that one-third *is* three-quarters. This inappropriate "equality" may be supported in students' thinking by the existence of equivalent fractions.

Using pre-partitioned shapes in teaching and assessing can mask an incomplete or incorrect appreciation of fractions as relational numbers. Many teachers are unaware that students are adopting only *part of* a regional "part of a whole" model of fractions. That is, some students focus on the "number of pieces" named by a fraction and others the "number of equal pieces" named, without addressing the relationship between the area of the parts compared to the area of the whole region. One implication of this study is the need to reduce the dependence on pre-partitioned shapes in teaching and assessing fractions. Requiring students to create their own partitioning relies much more on the relation between the parts and the whole than using pre-partitioned shapes.

Using more "compare and contrast" questions in fraction lessons will also assist in reducing misconceptions. Questions such as "How is one-third of a circle like one-quarter of a circle, and how is it different?" can be used to strengthen the separation between concept images. Drawings of fractional parts of circles such as one-third and one-quarter in different orientations may also be used to determine which fractions are the same and which are different. The same process could be applied with different shapes. Misconceptions and an incomplete adoption of regional fraction models must be brought into the light before they can be challenged and discussed.

Students also need opportunities to make adjustments when partitioning shapes in using a regional model of fractions. Finding one-third of a region is much more difficult than finding one-half of the same region. Even asking the question, how you know that you have made one-third, requires students to justify the basis of their decisions. A focus on the number of parts, the number of equal parts or the equal areas making up the region, all become clearer when students are asked to justify their reasoning.

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