Pedagogical Content Knowledge for Teaching Primary Mathematics: A Case Study of Two Teachers

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This paper explores the usefulness of a framework for investigating teachers' Pedagogical Content Knowledge (PCK). Two primary mathematics teachers completed a questionnaire about mathematics and mathematics teaching, and were interviewed about their responses. These responses were then analysed using the PCK framework. The PCK held by the two teachers was found to differ in many ways, including the connectedness of their knowledge, and the specificity with which they discussed the mathematics involved.

Teacher knowledge has long been the subject of intense research, and the range of knowledge that teachers draw upon every day is vast — knowledge of content, of students, of curriculum, of pedagogy, of psychology. To examine one aspect of a teacher's knowledge in isolation is not only unrealistic, it is difficult. Nevertheless, in order to examine teachers' particular knowledge for teaching mathematics, it is also necessary.

Examining Pedagogical Content Knowledge

Shulman (1987) defined pedagogical content knowledge (PCK) as

> the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction (Shulman, 1987, p8).

This definition emphasises three aspects — content, pedagogy, and students — and the importance of the connections among them. Marks (1990) expanded this definition, making explicit the difference between "an adaptation of subject matter knowledge for pedagogical purposes" (p. 7) and what he termed *content-specific pedagogical knowledge*, or "the application of general pedagogical principles to particular subject matter contexts" (p. 7). Marks included a third category of PCK, which is knowledge that is a synthesis of content and pedagogy, rather than deriving more directly from one or the other.

Many aspects of knowledge included within PCK have been identified. Shulman (1986) emphasises knowledge of multiple ways of representing the content to students. Such knowledge relies on the teacher's understanding of the content, and has as its purpose the transformation of that content into a form that students will understand. Within this idea Shulman includes "illustrations, examples, explanations and demonstrations" (p. 9). He also includes "an understanding of what makes … topics easy or difficult" (p. 9) as part of PCK. This includes knowledge of students' typical preconceptions and misconceptions, and strategies for helping students reorganise their understanding. Van der Valk and Broekman (1999) identify five aspects of PCK: *pupil's prior knowledge*, *pupil problems*, *relevant representations*, *strategies*, and *student activities*, but do not suggest that these comprise a comprehensive PCK framework. Many authors emphasise representations (Graeber, 1999; Shulman, 1986; Van der Valk & Broekman, 1999). Others include knowledge of student thinking (Graeber, 1999; Marks, 1990; Van der Valk & Broekman, 1999), texts and materials (Marks, 1990), what makes a topic easy or difficult (Shulman, 1986, Henningsen & Stein, 1997), and teaching strategies (Graeber, 1999; Van der Valk &

Broekman, 1999). Ma (1999) used the term *profound understanding of fundamental mathematics* (PUFM) to describe the deep and well-connected understanding of elementary mathematics held by some teachers. Ball (1991) similarly highlights the importance of teachers knowing "the relationships among … topics, procedures and concepts" (p. 7). Ball (2000) describes "the capacity to deconstruct one's own knowledge into a less polished and final form, where critical components are accessible and visible" (p. 245) as important to teachers, and this is relevant to PCK.

The framework presented here (see Figure 1, from Chick, Baker, Pham & Cheng, 2006) draws on the above literature, although many aspects have been included after initial analyses of data from a larger project investigating teachers' PCK. The framework is divided into three parts: *Clearly PCK* includes those aspects which are most clearly a blend of content and pedagogy; *Content Knowledge in a Pedagogical Context* includes those aspects drawn most directly from content; and *Pedagogical Knowledge in a Content Context* includes knowledge which has been drawn most directly from pedagogy.

PCK Category	Evident when the teacher …
Clearly PCK	
Teaching Strategies	Discusses or uses strategies or approaches for teaching a mathematical concept
Student Thinking	Discusses or addresses student ways of thinking about a concept or typical levels of understanding
Student Thinking - Misconceptions	Discusses or addresses student misconceptions about a concept
Cognitive Demands of Task	Identifies aspects of the task that affect its complexity
Appropriate and Detailed Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams)
Knowledge of Resources	Discusses/uses resources available to support teaching
Curriculum Knowledge	Discusses how topics fit into the curriculum
Purpose of Content Knowledge	Discusses reasons for content being included in the curriculum or how it might be used
Content Knowledge in a Pedagogical Context	
Profound Understanding of Fundamental Mathematics	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept
Mathematical Structure and Connections	Makes connections between concepts and topics, including interdependence of concepts
Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident)
Methods of Solution	Demonstrates a method for solving a maths problem
Pedagogical Knowledge in a Content Context	
Goals for Learning	Describes a goal for students' learning (may or may not be related to specific mathematics content)
Getting and Maintaining Student Focus	Discusses strategies for engaging students
Classroom Techniques	Discusses generic classroom practices

Figure 1. Framework for analysing Pedagogical Content Knowledge
(based on Chick, Baker, Pham & Cheng, 2006).

In light of this background, this paper reports on an analysis of the PCK of two teachers using the PCK framework. It seeks to answer the following questions:

1. Were there noticeable similarities or differences in the PCK of the two teachers?
2. Was the framework effective in illuminating these similarities or differences?

Methodology

The present study is part of a larger project investigating the PCK of primary teachers for teaching mathematics. In that project, 14 Australian teachers completed a written questionnaire about mathematics teaching in their own time, and were interviewed about their responses. It was important to allow teachers to consider the questions before interview, in order to imitate teachers' real situations, in which they have time to review content and prepare teaching ideas before class. The two teachers examined here, Clare and Brian, were selected for this study because they initially appeared to have different levels of PCK, and it was hoped that the framework would illuminate those differences. At the time of the study, Brian (all names are pseudonyms) had over 20 years of experience, and was teaching Grade 5. Clare was a teacher of Grade 6 in her fifth year of teaching.

The questionnaire contained 17 items, and addressed mathematical problems, hypothetical classroom situations, and other issues related to mathematics teaching and learning. Only 5 of the questionnaire items are discussed here (see Figure 2). Items were chosen which enabled teachers to demonstrate their PCK, about which similar interview questions were asked, and which revealed detail about the teachers' PCK in initial analysis.

Subtraction Item	You notice a student working on these subtraction problems: $\begin{array}{r} 438 \\ -172 \\ \hline 346 \end{array}$ $\quad$ $\begin{array}{r} 5819 \\ -2673 \\ \hline 3266 \end{array}$ What would you do to help this student?
3/8 Equivalence Item	Write down three ways of convincing someone that 3/8 is the same as 37.5%.
Division item (Board of Studies, 2000)	This work was completed by a student: $4\overline{)8452}$ = 2113 $\quad$ $3\overline{)9438}$ = 3146 $\quad$ $5\overline{)585}$ = 117 $\quad$ $6\overline{)8406}$ = 141 $\quad$ $4\overline{)4248}$ = 162 a. What would you do to help this student? b. Explain *how/why* the division algorithm works. (Why *don't* we start dividing at the ones?)
Fraction item	A student submits this question and solution as part of his homework: $\frac{7}{10}+\frac{2}{5}=\frac{7}{10}+\frac{4}{10}=\frac{11}{20}$ a. What *does* this student understand? What does he *not* understand? b. How could you quickly convince the student that this answer is incorrect? c. What does the student need to learn/understand before he can complete questions of this type, and how would you help him achieve that understanding?
Paint-Mixing item	The following question was given to students: Some children are making pink paint by mixing together white and red. Lisa uses 4 spoonfuls of red paint and 10 spoonfuls of white paint. Eric uses 9 spoonfuls of red paint and 21 spoonfuls of white paint. a. Whose paint will be darker pink? Explain why. b. Is the following student statement correct? Explain. *Eric's is darker even though he has more white paint because he has even more red compared to the white.*

Figure 2. Questionnaire items used in analysis.

Results

Some areas of the framework were not brought out by the interview or contained little data and have been omitted to save space. In some cases these were categories teachers would be more likely to exhibit in the planning of an entire lesson, rather than in addressing particular problems or situations, as presented in the questionnaire.

Teaching Strategies. Both teachers demonstrated knowledge of teaching strategies, but Clare's responses tended to contain more detail, and she often suggested more alternative strategies. For example, in response to the fraction item, Brian suggested using a number line to model the sum of 7/10 and 4/10, and also suggested using a long rectangle, a pie, or MAB as alternative materials, although he offered no detail on these. He stressed the importance of the student understanding the concept, although he did not elaborate on what this concept was. Finally, he recommended lots of practice with examples. Clare also suggested using diagrams or concrete materials, such as a pie, a number line, or cutting up a piece of paper; however, her suggestions were more detailed. Firstly, she suggested demonstrating to the student that when you add 4/10 to 7/10 the result is more than a whole, so that the student could see that the given answer was incorrect. To reinforce this, she suggested showing the student 11/20 to compare with the correct answer. She also suggested showing the student that 3/10 must be added to make a whole from 7/10, and emphasised helping the student to understand the effect of the denominator on the size of the pieces. Although apparently similar, Brian and Clare's approaches actually differ in quality: Brian's, essentially, was to model the correct answer, whereas by starting with the emphasis on the incorrectness of the student's answer, rather than the correctness of the teacher's, Clare seemed to be trying to invoke a cognitive conflict in the student (Watson, 2002), after which she attempted to build conceptual understanding.

The responses to the subtraction item also demonstrated differences in their teaching strategies. Brian had two main suggestions: breaking the problem into parts, and using MAB to demonstrate that 7 cannot be subtracted from 3. He did not explain how to help the student bring the parts together again. Clare suggested using an easier example to re-explain the procedure of regrouping. She suggested the number 27 as a suitable example to explain the equivalence of 27 and 10 + 17, and detailed how she would use MAB to assist her explanation. A notable difference between Clare and Brian is in the detail: Brian only mentioned MAB, whereas Clare described, with an example, how to use these materials.

Brian's and Clare's responses to the other items did not reveal such striking qualitative differences, although small differences were observed.

Student Thinking. The clearest difference demonstrated by the student thinking categories was simply that Clare seemed to discuss her students' thinking far more often than Brian did, and displayed better knowledge of both general student thinking, and the thinking of her own students.

Clare's questionnaire response to the fraction addition item illustrates her tendency to be mathematically specific, and detailed. Clare suggested that the original student's error was because he did not "understand the concept of fractions as 7 out of 10 and 4 out of 10, because he has added the denominator." She further explained in interview that his error in the addition demonstrated that "he hasn't understood that two fifths *is* four tenths, he's just understood to times this by two, times that by two," arguing that his error reveals this lack of understanding. She then discussed errors she had seen in students' thinking about fractions, and attributed the origin of these errors to the way students had been taught and the experiences they had had in the classroom.

Clare also discussed the effect of previous teaching with the subtraction item, asserting that "they're not really understanding what they're doing … they're just doing that [completing the algorithm] because that's what they've been taught to do."

These examples illustrate, firstly, that Clare has a good understanding of the way students think about the content she teaches. They also highlight her tendency to consider the effect of various teaching strategies on the ways students will understand the content, and also to try to understand students' reasoning from their errors.

Brian's responses coded for this category tended to be much more superficial; for example, Brian acknowledged, but did not discuss, that the student in the fraction addition item knew how to find equivalent fractions, but not how to add them. However, some of Brian's responses suggested a lack of understanding of student thinking. For example, in his response to the subtraction item, Brian focused on using MAB blocks to demonstrate that 7 cannot be subtracted from 3. The error made by the student was to subtract all the smaller digits from the larger digits in the same column, regardless of which was in the minuend or the subtrahend. The student knew that 7 could not be subtracted from 3, and resolved this by subtracting 3 from 7 instead. The student either did not know about regrouping, or had forgotten how to regroup, but Brian did not mention regrouping at all.

In discussing the division item, Brian demonstrated a similar lack of understanding of the nature of a student's difficulty.

> They understand what they're basically meant to be doing because they got the first three correct, so it's only just this small little thing to hold the place with the other numbers, so they've got a basic understanding there, they're not far away, so to me it's not a big issue, and it should be something that should be able to be corrected fairly easily, because they've got the understanding of how to do division, they just don't quite know how to approach this thing where it doesn't go [Brian I1]

Brian recognised the error of not including the zero to hold place value in the answer, but dismissed this as "not a big issue". In fact, it is unlikely that the student had "a basic understanding"; it is more likely that the student had a grasp of the *procedure*, but little conceptual understanding. While Brian might have found it easy to correct such errors in his experience, he appears to underestimate the importance of the student's problem.

Cognitive demands of task. Many differences between Brian and Clare were evident here. Clare sometimes provided more detail than Brian, and often she mentioned aspects of a task that affected its cognitive demands, but that Brian did not mention at all.

Both Brian and Clare noted that the numbers in the paint-mixing item made the task difficult. Brian said he would not use "those bigger numbers", and suggested "2 and 5, 4 and 7" as alternatives. Clare noted that "14 and 30 aren't easily changed to equivalent", identifying that the difficulty lies in finding common multiples for these two numbers. The difference is in the detail: Brian claimed the numbers are difficult; Clare explained why.

Similar differences occurred on the subtraction item. Clare suggested that three-digit subtraction was difficult for this student, and suggested an easier example; she also emphasised regrouping. Brian also attributed difficulty to the number of digits, but his strategy, breaking the problem into parts, did not deal with the more difficult aspect, regrouping.

For the fraction addition item, Clare noted that the conversion 2/5 to 4/10 was a simple one, and that a student's success here was not strong evidence of understanding equivalence: he "might even have remembered it".

Profound understanding of fundamental mathematics (PUFM). This category revealed the clearest differences between Brian and Clare. Clare frequently discussed the mathematics underlying her instructional decisions, and in four of the items examined here demonstrated deep understanding of the content and the crucial aspects students need to understand. Brian's responses sometimes touched on similar ideas, but without the detail of Clare's responses, and so without demonstrating a similarly "profound" understanding.

In the 3/8 equivalence item, one of Clare's methods was to find 3/8 of 40, and demonstrate its equivalence to 37.5% of 40. This distinction between 3/8 as a number, and 3/8 *of an amount* demonstrates Clare's deeper understanding of this content, and she described making explicit use of it in her teaching. Clare's data contained many examples like this. In a description of another method for this item, Clare described using a pie cut into hundredths. This method is problematic, but Clare demonstrated an awareness of potential problems, and emphasised many important points: the need for the pieces to be of equal size; that halving three times is an easy, conceptually clear way to find eighths; and that the halving of a hundredth would be a difficult step for students that would need emphasis. Clare's responses to the subtraction, division, and paint-mixing items contained other examples where she demonstrated awareness of subtle, but important, points about content.

Brian had no such points noted for any items. He used 100 paddle pop sticks to model the situation, and although he mentioned that you would need "37 and a half" hundredths, he did not discuss how he would deal with this with his model. However, Brian did discuss the meaning of the 0.5 in 37.5%. Although this point is similar to Clare's, above, Brian only mentioned this when questioned about how he might respond to students having difficulty equating the remainder to the decimal. Clare, in contrast, was not prompted; rather, she suggested voluntarily that students would need to be guided through understanding this link.

Deconstructing content to key components. On the whole this category did not reveal significant differences between Brian and Clare. However, Brian's failure to identify regrouping as a critical component of the subtraction algorithm, as discussed earlier, suggests that Brian does not see either the difficulty or centrality of regrouping. In contrast, Clare's focus on regrouping, and the use of concrete materials to aid understanding of this process, suggesting that she recognises regrouping as the key component of the procedure.

Procedural knowledge. Little significant difference between the two teachers was noted in their knowledge of procedures. However, Clare's discussion of procedure suggested that she was better able to link procedure and concept than Brian. In the paint-mixing item, Clare suggested two methods of solving the problem, and indicated that she would use fractions for the method of comparing the amount of red paint to the total amount of paint, and ratios for comparing the amount of red paint to the amount of white. Brian also compared both the ratios 4 to 10 and 9 to 21, and the ratios 4 to 14 and 9 to 30, and he identified the first as being the amount of red paint compared to white, and the second as the amount of red compared to the total amount of paint. However, Brian used fractions for both methods, and he rejected the first, and could not explain why he believed the second method to be "more correct". Here, both teachers demonstrate competence in the use of procedures; however, in her use of ratio notation to describe a ratio of two different parts of a whole, and fraction notation to describe a ratio of one part to the whole Clare also demonstrated an awareness of subtle aspects of the use of fractions and ratios.

There were interesting responses to the request to explain why the division algorithm works. Both teachers had difficulty explaining the standard division algorithm. However, Clare was aware of her difficulty, and said so in her questionnaire and interview: "I actually find this difficult to understand and therefore explain." She then demonstrated an alternative division method (similar in process, if not in appearance, to long division), and explained this algorithm's working in detail. Brian, on the other hand, attempted an explanation of the standard method, but his unclear explanation fell back into explaining the procedure rather than the reasoning behind it. This may suggest the difference in Brian and Clare's procedural knowledge is more to do with *how* they know it, than how much they know. Brian appeared confident in his knowledge, yet could not explain the workings of this algorithm. Clare, on the other hand, seemed acutely aware that she could not link the standard procedure to its underlying concept, so much so that she chooses to teach an alternative with which she feels more comfortable. The implication is that when Clare feels that she does not understand the procedures she teaches, she seeks a remedy, whereas it may be that Brian teaches procedures he does not really understand, provided he can explain the process.

Methods of solution. Clare frequently suggested more than one method to solve problems, whereas the only time Brian did this (in the paint-mixing item), he rejected one of his possibilities in favour of the other. For example, in response to the 3/8 equivalence item, Clare suggested two different procedures, and a method using a diagram, whereas Brian only offered one procedural method. However, when prompted in interview to find a method for a visual learner, he was able to suggest finding 3/8 of 100 paddle-pop sticks, and 37.5% of another 100 sticks. In the paint-mixing and division items Clare also suggested more than one method of solving the problem.

Goals for learning. Both teachers occasionally referred to their learning goals for their students, but these goals differed in their nature. Brian made frequent reference to students getting "back to that understanding of the 'why'"; however, he rarely explained what concept it was that he felt was important, and when he did, he did not link this to any strategy for teaching. Clare rarely made such general statements; rather she tended to explicitly state the mathematical learning goals that she thought were important in relation to the specific topic. For the subtraction item she stated that she wanted students to know what was happening during regrouping; for the division item she mentioned the importance of students learning their "known facts" (division and multiplication facts); and she made conceptual understanding a focus of her response to the fraction addition item, stating that the student "needs to understand that 7/10 is seven out of 10 pieces," and that it is important for students to know that the two equivalent fractions really are the same.

Discussion and Conclusion

The results show clear differences in the PCK held by Brian and Clare. While they often suggested similar ideas when discussing the same topics, Clare's knowledge was richer and more detailed within certain categories, most notably *teaching strategies*, *student thinking*, *cognitive demands of task*, *methods of solution,* and *profound understanding of fundamental mathematics.* In particular Clare's discussion contained more specific reference to mathematics. Her discussions on student thinking revealed an ability to understand students' erroneous reasoning, whereas Brian generally only demonstrated the ability to identify the error. Clare's knowledge of *procedure* seemed to be better connected to the underlying concepts than Brian's. Finally, Clare seemed to be more explicitly aware of her mathematical *goals* for learning than Brian.

The differences observed between Brian and Clare's PCK are difficult to explain, given that Brian has more years' experience than Clare, and both teachers seem equally confident and passionate about teaching mathematics well. It is possible that they began with significant differences in the way they understood the mathematics taught in primary school. Clare frequently mentioned her father, who was a secondary mathematics teacher, as a source of ideas, and this may have contributed to Clare's PCK development. However, other factors may also explain these differences. Clare seemed to have a different approach than Brian to common difficulties that she encountered. She seemed more optimistic of the possibility of solving these problems, and appeared to take personal responsibility for it. She also seemed more likely to pursue a problem for which she had no solution, exploring ideas and possibilities. It is certainly unlikely that Clare's one-year teacher preparation course provided any more PCK than Brian's, especially as Brian had recently completed an extra year of study focusing on mathematics teaching. The cause of the observed difference in Brian and Clare's PCK is important to explore, both to inform the preparation of pre-service teachers, and for teachers' professional development.

Finally, the PCK framework proved a useful tool in exploring teachers' PCK. While there was some overlap in the examples used in different categories, this overlap was not superfluous, as each category provided a different focus for examining the teachers' ideas. The framework should also be trialled by using it to examine teachers in the classroom, as it is in practice that PCK really comes into play.

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