Modes of Reasoning in Explanations in Year 8 Textbooks

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Understanding that mathematics is not just an arbitrary collection of rules to follow is basic to good mathematics learning, but studies show that many classrooms exhibit little mathematical reasoning. In order to better understand the nature of reasoning in schools, this study examined the modes of explicit reasoning in the explanations, justification and proofs of several topics in four textbooks. Eight distinct modes of reasoning are identified, illustrations of these are given, and their characteristics are discussed.

Introduction

Understanding that mathematics is based on reasons and is not just an arbitrary collection of rules to follow is basic to any good mathematical education. Reasoning, explanation and proof at an appropriate level should therefore be a prominent part of learning mathematics. Unfortunately, this is often not the case. For example, TIMSS Video Study (Hiebert et al., 2003) looked for evidence of mathematical reasoning in lessons in a random sample of lessons from 8 countries. In the 87 Australian lessons, they found almost no lessons which explicitly contained formal or informal proof or verification. They also identified 'making connections' problems where there was some linking between mathematical concepts, facts or procedures. In total, 15 per cent of Australian problems were in this making connections category, a low figure but similar to that in three other countries. When the actual solutions presented in the class (by teachers or students) were analysed, only 2 per cent of the total number of problems exhibited evidence of *making connections*. More commonly, the public solution was to state a concept, use a procedure or just give the result. Together these results point to an absence of mathematical reasoning in the average Australian Year 8 mathematics class.

Given the importance of encouraging reasoning, this paper aims to understand better the modes of reasoning that are available in Australian mathematics classrooms. We present a preliminary study of the reasoning that is evident in mathematics textbooks. The purpose is to examine the type of explanations, justifications and proofs that are, or might be, presented to Year 8 students, thereby aligning with the Video Study age group.

Harel and Sowder (2007) provide an important recent resource on proof in mathematics teaching and learning. They use the term 'proof' to include both formal and informal justifications, verifications and explanations and qualify it as 'mathematical proof' for that acceptable to professional mathematicians. Proving is seen as "the process employed by an individual (or community) to remove doubts about the truth of an assertion" (p. 808) and consists of two parts: Ascertaining (removing one's own doubts) and persuading others. In the case of textbook explanations, we note that the three processes of finding an assertion to prove (usually discovering a rule), ascertaining its truth, and persuading others of its truth are all involved, often in just a couple of lines of the textbook.

Harel and Sowder report on extensive studies of students' attempts to prove, which generally highlight the difficulties observed around the world both with student learning outcomes and with formally teaching about proof. Many studies are in the context of teaching students to prove in Euclidean geometry, but there are also reports of widespread attempts to imbue all mathematics teaching with a spirit of mathematical enquiry. Through analysing students' proofs, Harel and Sowder developed the construct of 'proof schemes'. A proof scheme is "what constitutes ascertaining and persuading for that person (or community)" (p.809). They group proof schemes in three classes: External conviction (e.g., depending on the authority of a teacher or book), empirical (e.g., depending on evidence from examples) and deductive, which includes mathematical proof. They cite many studies which show that students' proof schemes frequently belong to the external authority or empirical classes, rather than the desired deductive class.

The Video Study reported that at least 90 percent of lessons in all the countries made use of a textbook or worksheet (Hiebert et al., 2003), so we decided to examine explicit reasoning in textbooks, whilst appreciating the important role of the teacher as a mediator between the text and the student. There are a few studies of the nature of proof, justification and explanation in textbooks. Reys, Reys, and Chávez (2004) compared traditional US textbooks with recent 'standards-based' textbooks. They note that rather than merely "covering" topics, standards-based textbooks emphasise teachers helping students to "uncover" important

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mathematical ideas. A recent study by Stylianides (2008) examined how proof is promoted in a popular US standards-based curriculum for middle grades. He found that about 5% of student tasks involved proof in Harel and Sowder's sense. Key concerns raised were the need to increase students' understanding of what constitutes a mathematically legitimate proof; how to reconcile the often competing considerations of students developmental trajectory and mathematical integrity (see also Ball, 1993) and how to provide teachers with sufficient support.

Methodology

For this preliminary study, we examined the 2006 best-selling Year 8 textbooks (textbooks A, B, C and D) in four states. Each was a clear market-leader in its state, selected to give a picture of the mathematical explanations presented to many Australian students. Textbooks A, B, C and D are a subset of those that the present authors used in an earlier study and are coded in the same way (Vincent & Stacey, in press).

We examined several topics from Number and Measurement. State differences mean that not every topic was in every textbook. For each topic, we examined all the explanations, justifications and reasoning presented explicitly. We did not examine implicit reasoning, either within worked examples, or required when solving exercises. In almost every case, we found that texts only presented reasoning in the process of deriving a formula or rule which was immediately illustrated with worked examples and practised.

Each explanation was examined very carefully to identify the nature of the reasoning that supported the critical steps of the argument. The mode of reasoning was identified, and from these examples, a list of the modes of reasoning was created. These modes of reasoning include the proof schemes above, but we do not use this label because they do not necessarily constitute formal or informal proving for the textbook author. Some may only be explanatory pedagogical devices. The purpose of this paper is to illustrate these modes of reasoning. Four topics are selected for presentation below. It is not yet known whether these modes of reasoning are typical of the modes of reasoning for other topics, in other textbooks or at other year levels, nor how comprehensive this list is.

Eight Modes of Reasoning Illustrated

Modes of Reasoning

The following sections illustrate the eight modes of reasoning that were identified:

- Logical deduction (in Harel & Sowder's deductive proof scheme)
- Deduction through guided discovery (in H&S deductive proof scheme)
- Deduction of a rule from a model (in H&S deductive proof scheme)
- Concordance of a rule with a model (in H&S empirical proof scheme)
- Property extension (in H&S empirical proof scheme)
- Empirical demonstration (in H&S empirical proof scheme)
- Analogy (unclear if in any H&S scheme)
- Appeal to authority (in H&S external conviction proof scheme).

Division of Fractions – Concordance of a Rule with a Model

Two of the textbooks, A and B, introduced division of fractions. The other 2 textbooks only provided revision exercises without any explanation. Textbooks A and B began with the definition of reciprocal of a fraction, in one book as the fraction obtained by inverting the initial fraction, and the other gave the mathematical definition of the number by which the initial fraction is multiplied to give 1. In both texts the single aim of the section appeared to be to introduce and justify the 'invert and multiply' algorithm, and to practise its use in a variety of cases (e.g., with mixed numbers). Additional aims are possible; for example to give meaning to the concept of division by a fraction. As described below, this received a little attention in the explanation and in the subsequent exercises, which each included less than 5 word problems.

The derivation of the rule in both texts proceeded in a similar way,

- first, drawing on a model to solve a carefully chosen division problem,
- second, demonstrating that the invert and multiply rule produces the same answer,
- third, stating the invert and multiply rule
- fourth, giving several ‘naked number’ worked examples and much practice.

We call this mode of reasoning ‘concordance of a rule with a model’. Meaning for division of fractions derives from a model of division. In both cases, quotition division was chosen as the model, presented in one case as the number of pizza quarters in half a pizza (giving the answer 2) and in the other as the number of half circles in 3 circles (giving the answer 6). These answers are then shown to correspond to the answers obtained by ‘invert and multiply”, which is then stated to be the rule to apply in future.

$$3 \div \frac{1}{2} = 6 \textit{ and } 3 \times 2 = 6 \qquad\qquad \frac{1}{2} \div \frac{1}{4} = 2 \textit{ and } \frac{1}{2} \times \frac{4}{1} = 2$$

There are variations between the textbook presentations. For example, textbook A presented a partition division example to supplement the quotition example, which certainly extends the meanings for division of fractions that students need to understand in order to identify situations where division of fractions is applicable. Textbook A also presented the rule as multiplication by the reciprocal, rather than invert and multiply, which has greater re-use potential in topics such as solving equations and ratio and proportion problems involving fractions or decimals.

In this ‘concordance of a rule with a model’ mode of reasoning the alignment of the answers obtained in the two ways is the essence of the explanation of why the result is true. It is possible to derive the rules by logical deduction based on the model, but this is not what the concordance mode does. For example, it is not difficult to argue that any number of pizzas can be cut into 4 times that number of quarter-pizzas; so that to divide by a quarter is always to multiply by 4; and that this argument applies for any whole number, not just 4. It is also simple to argue that to multiply by a quarter is to divide by 4 (both are taking a quarter of a quantity). In both cases, we see that division by a number (admittedly limited to whole numbers and unit fractions) is multiplication by its reciprocal. This different and more mathematical mode of reasoning we call ‘deduction of a rule from a model’.

Another characteristic of the ‘concordance of a rule with a model’ mode of reasoning is that the real world or diagrammatic model is used to provide some initial meaning for the operation of division. However, there is no attempt to use the model as a tool for thinking, problem solving or for understanding the meaning of the answers to the many practice examples following. Gravemeijer and Stephan (2002) note that “Usually, something is symbolized (‘model of’), and the symbolization is used to reason with (‘model for’)” (p. 159). In these fraction explanations, the model remains a ‘model *of* division’, but it never becomes a ‘model *for* division’. The rule has been shown to be in concordance with one or two instances of the model, but it is not used to give meaning to the operation of division beyond this, even though students often find the meaning problematic. This seems to be a lost opportunity, although a Ball (1993) illustrates, thinking with models is often not easy.

One reason for moving quickly to the division algorithm is that using models to give meaning to division of fractions is not straightforward. The quotition meaning of division works well for some divisions (e.g., those above), and the partition meaning (sharing) works well for division of a fraction by a whole number, but for a randomly chosen fraction division (e.g., 2/7 divided by 4/9) neither meaning is really satisfactory and it seems best to think of such divisions simply as the inverse of multiplication. This is a serious pedagogical limitation of the models available for fraction division, but unlike both textbooks A and B, we think that a stronger development of meaning for division is required.

Multiplication of Two Negative Integers – Five Modes of Reasoning

Ogden Nash (1902 – 1971), an American poet best known for light verse, wrote:

Minus times minus results in a plus,
The reason for this, we needn't discuss.

The three textbooks that covered negative numbers in Year 8 (A, B and C) fortunately disagreed with Mr Nash about this notorious result. Five different modes of reasoning were evident. Textbook A used two modes. The first was 'appeal to authority'. Students create a spreadsheet that multiplies numbers, use it to multiply directed numbers and observe the results of certain calculations. We call this 'appeal to authority' because the spreadsheet here is providing answers (e.g., -5 × -6 = + 30, -5 × + 6 = -30) which students believe because they trust the spreadsheet. As an aside, we note that this is a common use of technology, for example, in calculator supported investigations in both primary and secondary schools. It seems that the technology is better able to act as the authority than a teacher.

Textbook A also used a mode of reasoning that we call 'property extension', as did Textbook B. These arguments extend, to negative whole numbers, the observation made on positive numbers that as the multiplicand in a multiplication table changes uniformly, the product also changes uniformly. This initial observation is illustrated in column 1 of Figure 1. The subsequent columns outline the steps involved in demonstrating successively multiplication of the three combinations of positive and negative numbers. The information upon which this 'property extension' mode relies can also be seen in the spreadsheets and multiplication grids which the textbooks show. However, the distinction that we make with the 'appeal to authority' is that in those cases, one or a few individual results were considered separately without highlighting the numerical patterns which reflect the properties of number operations. Textbook A presented the argument in some detail (comparable to Fig. 1). However, the Textbook B argument was much abbreviated and possibly most students and some teachers would miss the reasoning that lies behind it.

Step 1	**Step 2**	**Step 3**	**Step 4**	**Step 5**
Observe that when multiplicand decreases by 1, the product decreases by 5	Extend this observation to negative multiplicands, to infer that - × += -	Extend the property of commutativity, and start a new table	Observe that as multiplicand decreases by 1, product increases by 3	Extend this observation to negative multiplicands, to infer that - × - = +
5 × 3 = 15	5 × 2 = 10	-3 × 5 = -15	-3 × 4 = -12	-3 × 2 = -6
5 × 2 = 10	5 × 1 = 5	-3 × 4 = -12	-3 × 3 = -9	-3× 1 = -3
5 × 1 = 5	5 × 0 = 0	-3 × 3 = -9	-3 × 2 = -6	-3 × 0 = 0
5 × 0 = 0	5 × -1 = -5	-3× 2 = -6	-3× 1 = -3	-3 × -1 = +3
	5 × -2 = -10	-3 × 1 = -3	-3 × 0 = 0	-3 × -2 = +6
	5 × -3 = -15	-3 × 0 = 0		-3 × -3 = +9

Figure 1. Extending properties of whole numbers to directed numbers.

Textbook B also used reasoning by analogy. Directed numbers were modelled by movement east or west on a film running backwards or forwards. The analogy was drawn between multiplication of directed numbers and apparent movement across the screen. We have called this reasoning by analogy, rather than from a model because it was only presented at a qualitative level and because the mathematical correspondence between quantities and the operations was not made clear. In particular, no numbers were multiplied, only signs as in 'positive × negative = negative'.

Textbook C adopted a different approach, drawing on an interpretation of – as 'taking the opposite'. Multiplication by a positive number is established as repeated addition. (In this textbook as in all the others, directed number arithmetic is almost exclusively done with integers.) So +2 × +3 is interpreted as 2 lots of +3 (+2 × + 3 = 3 + 3 = 6) and +2 × -3 is interpreted as 2 lots of -3 (+2 × -3 = -3 + -3 = -6). To this point the argument is using logical deduction, from a special case that is intended to be general. 'Seeing the particular in the general' is a common feature of teaching and learning mathematics, discussed by authors such as Mason and Pimm (1984). When the multiplier is negative (as in $-2 \times +3$), textbook C draws on the interpretation of the – sign as "the opposite of". Students are instructed to calculate $2 \times +3 = 6$ (2 lots of +3) and then they take 'the opposite', in this case -6. 'Doing the opposite' was introduced as part of the explanation of addition and subtraction of directed numbers, using the model of journeys along a number line. Since this is a valid model of addition and subtraction of negative numbers (although for various reasons, we believe it to be a difficult model for students to use), we classify the mode of reasoning here as 'deduction of a rule from a model'. The meaning of subtraction was given as "the subtraction sign between two numbers means *do the opposite of*" (e.g., move left instead of right). Ball (1993) discusses the difficulties of using a similar model to develop students' argumentation. Note that the argument presented in textbook C has two subtle mathematical flaws. First, it does not actually show that $-2 \times -3 = +6$ but instead shows that $-(2 \times -3) = +6$ which does not involve the multiplication of two negative numbers at all. Second, it confuses the subtraction operation (the binary operation) with the 'negative' operation (the unary operation taking the additive inverse). This example illustrates that as far as possible the classification of modes of reasoning is not concerned with mathematical correctness or completeness.

All the Australian textbooks tend not to use the presented model as a thinking tool but to replace it immediately by use of the rule in worked examples. This relates to a study by Mayer, Sims, and Tajika (1995). They compared teaching directed numbers in Japanese and US textbooks, noting that Japanese books made stronger connections between the models, words and calculations. This supported the Video Study findings (Hiebert et al., 2003), that found large inter-country differences in 'making connections' in lessons.

Area of a Trapezium – Logical Deduction and Guided Discovery

Textbooks A and C derived the rule by placing two congruent trapezia to form a parallelogram (Figure 2a), and then using the rule for the area of a parallelogram that had already been presented. This mode of reasoning is 'logical deduction'. The much abbreviated style of explanation (diagrams with few words) is common in textbooks, and is reminiscent of the 'proofs without words' enjoyed by mathematicians, as in Nelson (1993).

Textbook D used a different approach. Instead of deriving the rule and then setting exercises, this textbook placed two multi-step exercises leading to the area of a trapezium area within a problem set. Each exercise guided students to dissect a trapezium of specific dimensions and rearrange into shapes of known area (see Figures 2b and 2c). Textbook D further used this guided discovery approach for areas of kites and rhombuses. Only in a final section were the rules explicitly stated and practice exercises provided. This approach foregrounded the importance of students being able to find areas of polygonal figures of varied shapes by dissecting into areas of known shapes, rather than relying on memorised rules. We call this mode of reasoning 'logical deduction through guided discovery'. Although specific measurements are used in the Textbook D exercises, we judge that students are intended to see the generality in the particular, so do not class this as reasoning only from specific examples, as in Harel and Sowder's (2007) empirical proof scheme.

Area of a Circle – Deduction, Guided Discovery and Empirical Demonstration

Textbooks A, C and D all derived the formula for the area of a circle by dissecting a circle into sectors, rearranging them to form an approximate rectangle, and calculating the area of the rectangle, and hence the area of the circle (see Figure 3). We classify this mode of reasoning as logical deduction even though it requires very considerable refinement (especially related to the limit processes) to become a mathematically acceptable proof of the formula for the area of a circle. However, we judge that it functions well as a justification of the formula in this simplified version at Year 8 level.

Textbook C presented this argument alone, followed by exercises using the formula. Textbook A prepared students for this argument by preceding it with practical version of the dissection in Figure 3, where students cut a photocopy of a circular protractor into sectors, construct the 'rectangle' and hence find the area of the protractor. They were asked what would happen if the sectors were narrower, thereby acknowledging the limiting processes involved in the mathematical proof. We classified this activity as logical deduction, rather than empirical measurement, since we judged that it was to prepare students for the general argument.

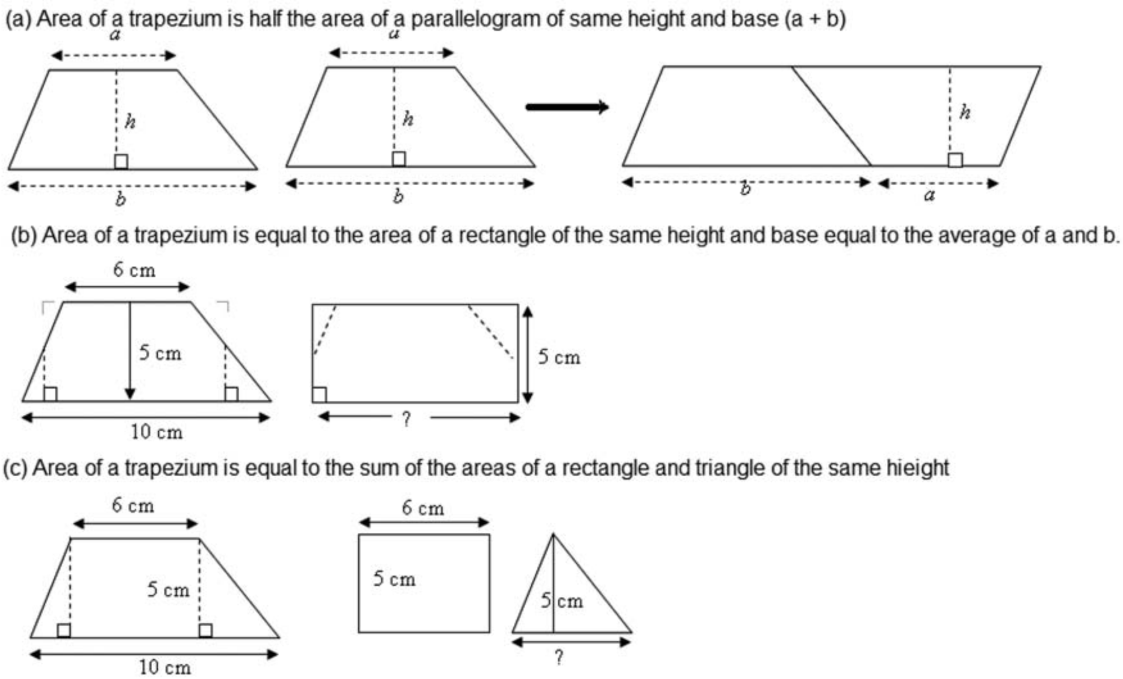

Figure 2. Three derivations of the area of a trapezium.

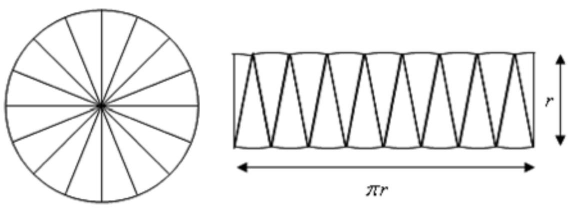

Figure 3. Area of a circle represented as area of a rectangle of equivalent area.

Textbook D presented a variety of approaches to justifying the area of a circle, approaching all through guided discovery. First, students find that the area of the circle must be between $2r^2$ and $4r^2$ (so approximately $3r^2$), by drawing circles inside and outside squares. This is followed by an *empirical* approach of placing a grid over the circle and showing 316 of 400 grid squares fall inside the circle (hence obtaining an area of $3.16r^{2.}$). Then students were guided through the explanation in Figure 3, supplying length, breadth and area of the rectangle themselves. Then, another dissection proof of the formula was also presented in guided discovery mode. Only then was the formula stated.

Conclusion

All of the textbooks made some attempt to explain each rule. Pleasingly, no textbook simply presented "rules without reason". On the other hand, almost always the sole purpose appeared to be to derive the rule in preparation for the practice exercises, rather than to use the explanations as a thinking tool. The explanations are, in general, very short with essential aspects of the reasoning unstated. Hence they are unlikely to stand alone, so students must rely on teachers to elaborate on the explanation provided. It is unlikely that all teachers can present these elaborations from the material provided, so this study further highlights the need for teachers' deep mathematical pedagogical content knowledge. Some of the electronic resources now being added to textbooks, including geometric dynamic demonstrations and templates for construction, are filling gaps. Nearly all of the explanations we examined were correct (rather than incorrect) although they were generally very curtailed and omitted basic reasoning (e.g., to state that a finding about one case also applies in general) as well as omitting difficult cases and subtle points.

Students encounter a considerable variety of the modes of reasoning in these explanations. At this stage of our investigation, it appears that the variety is between topics, more than between textbooks. An exception is that only one textbook used ‘deduction through guided discovery’ involving students actively in the deduction process. Four of the eight modes of reasoning are unacceptable from a logical and mathematical point of view: Reasoning by analogy, appeal to authority, empirical demonstration and concordance of a rule with a model. However, as illustrated above, all of these modes of reasoning might have a place in assisting students’ learning. The critical point for developing students’ mathematical reasoning is whether students understand that some modes of reasoning are indeed part of the acceptable range of modes of reasoning in mathematics, whilst others simply serve a local pedagogical purpose, such as helping them remember a rule.

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