Fractions as a Measure

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Simultaneous co-ordination of the referent unit, symbolic notation and pictorial representations are one aspect of understanding fractions as a measure. The notion that students must identify the referent unit, fractional parts of the referent unit, and apply it as an accurate method of measurement was investigated. Using different types of pictorial representations, students were required to represent the quantity being measured and conversely, identify the quantity represented. Four questions from a larger study that used fractions as a measure were examined. Nine students were interviewed to gain further insight into their thinking on these questions and their misconceptions identified.

Communication of common fraction ideas and principles relies on a precise system of symbols (inscriptions) and conventions that allows unambiguous and concise communication of mathematical ideas (Thompson & Saldanha, 2003). Fraction inscription comprises of a bipartite symbol in the form $\frac{a}{b}$, where a and b are whole numbers and $b \neq 0$ (Board of Studies NSW, 2002). Although the fraction symbol is devoid of physical meaning and context (Lamon, 2006) it has many interpretations including part/whole, quotient, measure, ratio and operator (Kieren, 1980). Although not independent, each interpretation allows rational numbers to be viewed from a different perspective thus providing a basis for developing an understanding of rational number (Kieren, 1980).

Fractions as a measure can be used to extend the whole number system (Hart, 1981). Fractional units are derived when the standard object or unit of measure is subdivided into smaller equal parts. These fractional units in combination with whole units of measure provide an accurate means of measuring (Skemp, 1986). A measurement task requires the "process of counting the number of whole units usable in 'covering' the region, then equally subdividing a unit to provide the appropriate fit" (Kieren, 1980, p. 236). The fraction inscription represents a quantity resulting from the construction of a reference unit from which situations are redefined in terms of that unit (Lamon, 1993; Olive & Steffe, 1980). The referent unit may be one of the International System of Units (SI) such as kilogram, metre and second or arbitrary unit such as a chocolate bar or pizza. Without the identification of the referent unit by the student, the successful application of fractions in a measure context is unlikely.

Reliable information about students' knowledge and understanding is crucial for teachers. It allows teachers to extend students' current level of understanding and address misconceptions students possess by focusing tasks and lessons on particular areas of concern (National Research Council [NRC], 2001). This paper explores students' understanding of the idea of fractions as a measure by examining the reasoning they employ to solve four fraction measure tasks. Student misconceptions in responding to these tasks were explored and implications for enhancing student learning provided.

Theoretical Perspective

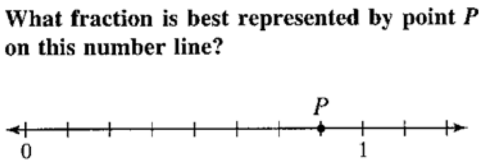
Mathematical models are mathematical representations used to convey, clarify, interpret and understand mathematical ideas (National Council of Teachers of Mathematics [NCTM], 2000). The term is quite nebulous, with descriptions such as representational models (Saxe et al., 2007), representations within contexts (Ball, 1993) or embodiments (Behr, Lesh, Post, & Silver, 1983) used to describe similar contexts. Number-lines and part-whole models are often used by teachers to convey fraction ideas (Ball, 1993; NCTM, 2000; Saxe et al., 2007). These different mathematical models illuminate different aspects of a mathematical concept (Ball, 1993; NCTM, 2000).

The use of mathematical models is supported by a representational system. This system comprises five interrelated elements: written symbols in the form of inscriptions and words, spoken language, hands on materials, pictures and real life situations (Behr et al., 1983). Explaining ideas and concepts often requires students to map different elements onto each other (NCTM, 2000). Words are employed as an intermediary to explain the link between mathematical symbols and mathematical models, typically illustrated using pictures and hands-on materials. Developing an understanding of fractions requires students to: (a) develop an understanding of the conventional models and representations used, and the ideas they capture (NCTM, 2000);

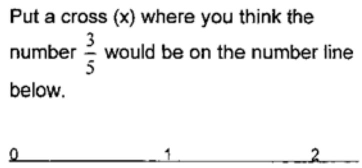
(b) comprehend the mappings across representations, and (c) understand the mappings within representations (Behr et al., 1983).

Influenced by their existing knowledge and experiences, students impart their meaning to teachers' representation of mathematical ideas. Students make sense of what they are taught by creating an internal, cognitive representation or mental model. These mental models provide a "workspace for problem solving and decision making" (Halford, 1993, p. 7). Based on tasks that elicit student thinking, student use of representational elements are the medium through which they demonstrate what they know and understand. Without examining students' work, interpretation of their understanding is impossible (Ball, 1993).

Number-lines are a mathematical model used to exemplify fractions as quantities and the measure interpretation. Conventional examples of number-line questions include Figure 1a, in which students are required to identify the quantity represented on a number-line or, locate a fraction on a number-line as in Figure 1b. Both examples require mapping across symbolic-pictorial representations. Students need to interpret the fraction symbol, recognise the properties of the number-line model, and how fractional quantities are represented.



(a) Identify the fraction (California Department of Education, 2008)

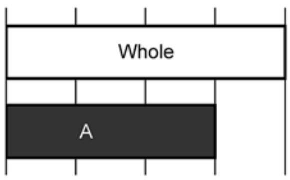


(b) Locate a fraction (Pearn & Stephens, 2005)

Figure 1. Fraction questions using a number-line representation.

Many students experience difficulties in locating fractions on number-lines. Number-lines are constructed with an arbitrary point chosen on a line as the origin. Points on the number line are subsequently labelled by their distance from the origin measured according to a referent unit chosen (NRC, 2001). When asked to place a proper fraction on a number-line, students often view the whole number-line, irrespective of its magnitude as a single unit instead of a scale (Ni, 2001). Placing proper fractions on a number-line labelled from 0 to 1 is also problematic. Fractions are often placed with disregard to any reference point or other known fractions. A lack of accuracy when dividing segments also results in the incorrect location of fractions (Pearn & Stephens, 2004).

Examples focusing on fractions as a unit of measure should not be constrained to number-line models. The idea of measure is closely related to the notion of part-of-the-whole using region and volume models (Kieren, 1980). Figure 2 shows variations of the region model (Watanabe, 2002), characterising mappings within pictorial representations. In the comparison representation (Figure 2a), the part and the referent unit are separate entities. The fraction quantity is constructed from the relationship between the explicit whole and the part to be measured (Watanabe, 2002). There is no confusion as to the referent whole. In contrast, the part-of-the-whole method (Figure 2b), the part is embedded within the whole and the referent unit (entire area) is implied in the diagram (Lamon, 2006). Like number-lines, the use of these representations requires a clear understanding of the referent unit (Ball, 1993; Kieren, 1980) and the peculiarities of each representation.



(a) Comparison representation for $\frac{3}{4}$

(b) Equivalent part-of-the-whole representation for $\frac{3}{4}$

Figure 2. Different pictorial methods of representing fractions using regions.

The fraction shaded represented in Figure 2b is typically considered $\frac{3}{4}$ when the implied referent whole is the entire area. Another view provides the answer $\frac{1}{3}$ represents the ratio of unshaded to shaded parts (Smith, 2002). The various referent units are cognitively different and operations with these units produce different results (Lamon, 2006). Understanding and interpreting the typical representations is one part of developing mathematical understanding.

Students often change their referent unit during problem solving. Olive and Steffe (1980) found that the surface features of pictures often confounded students. The following was observed of Karla a fifth grade student during a fraction lesson using computer software. "She had drawn two candy bars end to end… She partitioned each bar into six pieces… She now had a double candy bar consisting of 12 equal pieces. She issued the command F [6/12] and was surprised when only half of the first bar was filled (three of the 12 pieces)" (p. 61). In the process of constructing two single units which end to end appeared as one physical unit, Karla was not longer able determine the relationship between a fraction and its reference as her perceived referent unit changed.

Students develop an understanding of fractions as a measure when they are able simultaneously co-ordinate the referent unit, symbolic notation and pictorial representations of various fraction models. They are able to make connections between the different conceptual dimensions highlighted by the pictorial representations used to exemplify fractions as a measure (Ball, 1993; NRC, 2001). In this study, students' responses to two types of tasks were investigated. Using different types of pictorial representations of fraction models, students represented the quantity being measured and conversely, identified the quantity represented. By examining students' responses to these tasks, the types of reasoning students' employed, including: (a) the identification of the referent whole, (b) the perceived relationship between symbolic notation and pictorial representations, and (c) issues of interpretation and misconceptions, was explored.

Methodology

Participants

Six hundred and forty-six students in Years 3 to 6 from six co-educational Sydney primary schools participated in the larger study. Participant details appear in Table 1. Students who scored less than the 50th percentile in the Progressive Achievement Test in Mathematics (PATMaths) as compared to the norming sample (Australian Council for Educational Research [ACER], 2005) were identified as potential interview participants. After reviewing their fraction assessments, 45 participants were selected and interviewed from across all grades. Only nine interviews were reported in this paper. The suffix "G" or "B identifies the interviewee as a girl or boy.

Table 1

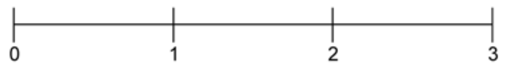
Details of Participants Completing Fraction Assessments and Interviews

	Average	AFUv1		AFUv2		
Grade level	**Age (years)**	**(*n*)**	**Interview**	**S2 (*n*)**	**S3 (*n*)**	**Interview**
3	8.65	68		104		521G 694B
4	9.72	93		82		
5	10.71	102	358G		84	663G 808G 813G 820B
6	11.61	34			79	625G 781G

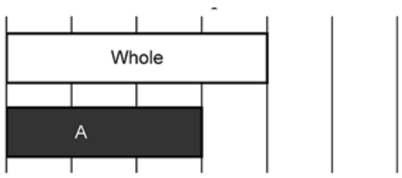
Instruments

The questions for the Assessment of Fraction Understanding (AFUv1, AFUv2S2 and AFUv2S3) were derived and adapted from various assessment instruments including: California Standards Test (California Department of Education, 2008), and Success in Numeracy Education program (Pearn & Stephens, 2005). AFUv1 was the pilot instrument. It was modified after pilot data analysis, which resulted in the creation of two versions: AFUv2S2 (grades 3 and 4), and AFUv2S3 (grades 5 and 6). Twenty common items were used to link all three assessments. Figure 3 shows the four measure questions analysed from the assessments.

10b. Put a cross (X) where the ½ would be on the numberline below.

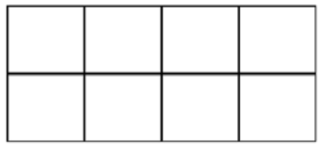


17. The white shape shows a whole bar and another bar is placed below it.bar is placed below it.



What fraction of the whole bar is A? What fraction of the whole bar is A?

14. In the rectangle, shade in enough small squares so that $\frac{3}{4}$ of the rectangle are shaded.



26  This rectangle represents one whole.

(a) What do the following represent altogether?

(b) Can you think of another name for the shaded fraction?

Figure 3. Four measure questions analysed.

Question details including representation, type of referent unit exhibited, mapping (s->p: represent quantity being measured given the fraction quantity/symbol; p->s: identify the quantity being measured), and the number of participants completing each question appear in Table 2. Question 10b was completed by participants that were administered AFUv2. A multiple choice version of the appeared in AFUv1 but was not analysed in this paper. Question 14 employed an equivalent, part-of-the-whole representation in which the number of small squares was a multiplicative factor greater than the denominator (Ni, 2001). The referent unit displayed in question 26 comprised of a subdivided referent unit.

Table 2

Question Details

	Representation			Features		AFU	
Question	**Num.-line**	**Comparison**	**P-of-W**	**Ref. unit**	**Mapping**	**v1(*n*=297)**	**v2 (*n*=349)**
10b	X			explicit	s -> p		X
14			X	explicit	s -> p	X	X
17		X		implied	p -> s	X	X
26		X	X	explicit	p -> s	X	X

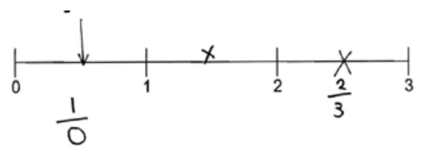
Procedure

All participants were administered a version of the fraction assessment and the third edition of the PATMaths (ACER, 2005), following standardised protocols. Each pencil and paper test was of 45 minutes duration. Calculators were not permitted and participants were asked to show all working in their test booklets and to attempt all questions. Interviews were semi structured with questions drawn from the fraction assessments. The researcher interviewed all participants individually. Participants were provided with concrete materials such as blocks and counters as well as pencils, ruler and paper. The interviews were videotaped to allow further analysis.

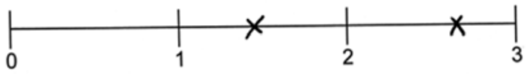
Results

The correct and common incorrect responses and percentages are reported for each question along with relevant interview data.

Question 10b: Eighteen percent of assessment participants completed the question correctly, with 57% placing a cross at half way between 1 and 2. Interview data confirmed that one reason for response was the dividing the entire number-line in half. But student 781 used the numerator and denominator as cues for placing the cross between 1 and 2, see Figure 4a. When asked to identify the cross placed between 2 and 3, the fraction $\frac{2}{3}$ was written below it. A similar procedure was employed when asked to identify the fraction shown by the arrow.



(a) Student 781's attempts at identifying fractions



(b) Student 813's response to locating ⅔

Figure 4. Student examples of placing fractions on a number-line.

Variations to student 781's strategy were observed. After placing the cross between 1 and 2, student 813 was asked to place $\frac{2}{3}$ on the number-line (Figure 4b). Explaining her method she said, "I'll look at the top number [points to the 2 in 2/3] See how it has the word two so I look… [moves finger along the number-line until she reaches 2] found 2 and look at the 3 as well. So I have to think alright if half is here [points to half way between 2 and 3] so the cross should be a little more higher [to the right of 2 1/2]". Not only did the numbers within the fraction provide cues for the location of the fraction, the fraction quantity assisted in positioning the fraction in the section identified.

Question 14: Forty-six percent of assessment participants answered the question correctly shading six squares, whilst 32% shaded three. Interviews were conducted to determine reasoning used to generate the response of 3. Student 358 placed counters in three squares of the top line of the rectangle. When asked to identify the remaining quarter, she pointed to the blank square on the top line. When asked how many squares make 1 whole, she replied, "um 4". Prior to circling her whole, she asked, "Like even the ones that are blank?" and drew a circle around the top 4 squares. Yet when asked after circling the whole, how many wholes were in the picture, she said, "um, 8 altogether". Confusion about how to interpret the picture, its relationship to a fraction and the referent unit was evident.

Some participants focused on the fraction numerator. Student 521 shaded three squares, but was confused when she looked at the shading and stated, "isn't it three eighths?". When probed whether she would have to shade some more for three quarters, she responded, "Three quarters oh. You do this in quarters [divides the last square shaded is divided into eighths]. Then you shade 1 then that 2 [points to first and second small square] and three quarters [points to the last small square divided into eighths]". Student 694, who also shaded three small squares when probed to examine the number of small squares said, "Ah there's 8. Oh no, it's not three". Using a rectangle divided into quarters (same size as the one divided into eighths), the student was able to shade three quarters of the whole rectangle. When comparing the area shaded in both rectangles, he realised another three small squares needed to be shaded. Three quarters represented using an equivalent pictorial representation was not a mental model student 694 possessed.

Both the numerator and denominator provided cues for student 813, "So I think if I colour three pieces there should be at least 4 left over. But if I colour 3 pieces there'll 5 left over. So I think I should colour in 4 and there should be 4 left over that's a half". When asked whether three quarters was more than a half, she responded, "um, I think no, don't know". Responses from these students suggest their lack of attention to the whole shape but a focus on the fraction symbol.

Question 17: Forty-five percent of participants answered the question correctly, whilst nine percent did not respond. Interview data suggests that two methods were employed to solve the question correctly. Firstly participants recognised the whole and it was divided into four parts. This was followed by: (a) recognising bar

A comprised 3 parts, hence three quarters of the whole, or (b) identifying the difference in size as one quarter missing, hence the shaded part was three quarters.

Student 813 gave the incorrect response of 3, explaining, "I think A is at least [counts number of lines that A crosses] um, 3 bars, like 3 wholes". When 813 was reminded of the size of the whole bar, the following response was given, "Like three thirds. Yep, because here's one missing [points to missing part of A]". This student recognised the significance of the three pieces but did not equate a piece to one quarter or make reference back to the whole. Another variation on this method was exhibited by student 781 who explained, "if you if you count the lines 1 2 3 4 [points to vertical dividing lines along length of A] it stops there. And that counts to 5 [circles the whole bar] the whole bar", hence the answer $\frac{4}{5}$. Reasoning was in part dominated by the surface features of the representation.

Question 26: Thirty-nine percent of participants completed the question correctly by giving the answer $1\frac{3}{4}$ or $\frac{7}{4}$, whilst 15% did not respond. Interview participants who answered the question correctly were able to keep in mind the representation of a whole. Student 781 stated, "Because there is one full shaded [pointing to first rectangle] and that's not full shaded [pointing to second] and it is only three shaded and four there". Identifying the fraction of each part of the referent whole is also necessary for this question.

The response of $\frac{7}{8}$ was provided by 12% of participants. During the interview, student 820 counted the number of parts shaded and altogether in both rectangles by ignoring the referent whole. Some interviewees counted the number of parts shaded (i.e., 7). In some instances the number of shaded parts in each rectangle was used to form a fraction. Student 663 explained, "Four over three… Because there is four [points to the first rectangle] shaded only three [points to the second] there". Using similar reasoning but ordering the rectangles differently, student 808 offered the answer $\frac{3}{4}$.

Discussion

Students' understanding of fractions as a measure is enhanced when they are able to co-ordinate simultaneously the referent unit, symbolic notation and pictorial representations. The interview data suggest that a number of issues impeded students' understanding of fractions as a measure. They generally concentrated on deciphering in isolation the pictorial representations or interpreting fraction notation. Students with some understanding were able to work with the symbolic notation and pictorial representation together, yet were unable to reconcile any discrepancies they encountered.

The interpretation of conventional pictorial representations of number-lines and area models posed a problem for some students. They focused on surface features (e.g., counting the number of parts or number of lines) to identify the fraction quantity. Although the referent unit was explicitly defined in Question 26, the most frequent incorrect response represented the shaded area of the combined rectangles. This was consistent with the findings of Olive and Steffe (1980) and further suggests that the notion of the referent unit is not a key feature of students' understanding of fractions.

Other students were unable to establish the relationship between the fraction symbol and pictorial representation. The numerator and denominator were used as cues for representing the fraction quantity as exemplified by the responses to Question 14. Although some students realised the discrepancy in their representation with the fraction quantity, guidance was required to re-interpret the equivalent part-of-the-whole representation.

The questions and the misconceptions they highlight provide teachers and researchers with a method of assessing students' understanding of the measure interpretation of fractions. This analysis, although limited by the type, number and content of questions selected, provides some suggestions for designing instruction. Firstly, mathematics is based on a system of conventions and symbols. Without a familiarity of the pictorial representations and an understanding of conventions of interpretation, communication of fraction ideas may be compromised. For example, students require a deep understanding of the number-line as a scale prior to its use as a mathematical model applied in the fraction context. Secondly, the notion of the referent unit both implicit and explicit needs consideration. Initial fraction instruction that only addresses proper fractions limits students' exposure to fractions as a measure. Realistic examples should be extended to cakes, pies and pizzas thus providing an avenue to understanding measurement in whole and part units. Finally, fraction symbol notation is one component of understanding fractions. It needs to be carefully linked to other representational elements and mathematical models for students to develop a deep understanding.

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