# Challenging Multiplicative Problems Can Elicit Sophisticated Strategies

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This paper reports on 13 Grade 3 students' approaches to *Equivalent groups* and *Times as many* multiplicative word problems. The findings are part of a larger study relating to children's development of multiplicative thinking. Of particular interest was the extent to which task level of difficulty influenced students' strategy choice. The results suggest a relationship between the level of difficulty and strategy choice: the more difficult the task the more sophisticated strategy choice.

One of the dilemmas teachers face when introducing operations to children relates to types and complexity of problems they pose. Textbooks often present tasks progressing from simple to complex, and likewise concepts are presented from simple to complex. This paper presents evidence to suggest that some students use more sophisticated strategies when presented with challenging problems, involving numbers considered beyond the factor structure determined by the curriculum for that particular grade. Conversely, easier problems prompt some students to use a less sophisticated strategy.

## Research Framework

Many authors have argued that multiplication is conceptually complex both in terms of the range of semantic structures (Anghileri, 1989; Greer, 1992; Kouba, 1989) and conceptual understanding (Clarke & Kamii, 1966; Steffe, 1994). Clarke and Kamii (1996) concurred with Steffe (1994) that understanding multiplication requires a higher level of abstraction than addition and greater demands on children as described by Steffe:

> For a situation to be established as multiplicative, it is necessary at least to co-ordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit (1994, p. 19).

Research has indicated that children as young as pre-school can solve a variety of multiplication problems by combining direct modelling with counting and grouping skills, and with strategies based on addition (e.g., Anghileri, 1989; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Clark & Kamii, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1997). Number triples within most of these studies were limited to small numbers such as (3, 4, 12), (3, 5, 15), (3, 6, 18), (3, 8, 24), (4, 5, 20), (4, 6, 24), and (5, 6, 30) from Kouba's (1989) study of Grade 1 to Grade 3 students. These studies identified the stages children move through in their transition from additive to multiplicative thinking. Some studies (Anghileri, 1989; Kouba, 1989; Mulligan & Mitchelmore, 1996, 1997) classified the strategies as either modelling (using physical objects, fingers, or drawings) or calculation (e.g., unitary counting, to skip counting, to additive strategies based on repeated addition to multiplicative strategies, such as known and derived multiplicative facts).

In their study of Grade 2 and 3 students Mulligan and Mitchelmore (1997) found that young children acquire a sequence of increasingly efficient intuitive models (defined as "an internal mental structure corresponding to a class of calculation strategies" p. 325)

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derived from the previous one. However, the intuitive models student employed to solve a particular problem was determined by the mathematical structure they imposed on it, rather than the mathematics inherent in the problem.

Research on Grades 4 to 6 students' solution strategies to whole number multiplication word problems involving more complex number triples, such as (5, 8, 40), (5, 19, 95), (13, 7, 91), (23, 4, 92) (Ell, Irwin, & McNaughton, 2004; Heirdsfield, Cooper, Mulligan, & Irons, 1999) found that although students' strategies progressed through a range of calculation strategies similar to those previously described, they did not necessarily consistently employ the more sophisticated strategies. In fact, Heirdsfield et al. (1999) indicated that the number triples and strategies available to the students influenced their strategy choice for solving word problems. Both studies (Ell et al., 2004; Heirdsfield et al., 1999) suggested the instructional effect and formal algorithm influenced students' strategy use.

Mulligan and Mitchelmore (1997) also found that the intuitive models children employed were influenced by the characteristics of the problem such as the size of the numbers, the multiples involved, or the extraneous verbal cues. As a consequence most students were not consistent with the intuitive models they used across the different problems. For example, many students who used repeated addition for problems involving small number triples seemed to experience a "processing overload" (p. 322) when attempted to use the same strategy for larger number triples. Conversely, some students who used a multiplicative operation for a problem involving small number triples were often unable to retrieve the number fact required for problems involving larger number triples and reverted back to repeated addition.

Three commonalities are evident from the literature presented relating to students' solution strategies to whole number multiplicative word problems. First, students' intuitive strategies progress according to their level of sophistication in the transition from additive to multiplicative thinking. Second, having mastered a more sophisticated strategy one cannot assume a student will employ it. Third, a range of factors such as the size of the numbers, multiples and instructional effect influences strategy use.

The purpose of the study underpinning the findings reported here, was to explore Grade 3 students' strategy choice across a range of multiplicative word problems, that involved numbers considered outside the factor structure stipulated by the curriculum. The study was informed by the work of Ell et al. (2004) and Heirdsfield et al. (1999), relating to Grades 4 to 6 students' approaches to two-digit by one-digit whole number multiplication word problems and extends the work of Mulligan and Mitchelmore (1999) who posed the conjecture "that students first learn a new strategy to solve problems where the situation is familiar and the relevant number facts are well known" (p. 327).

The questions guiding this study were: Are there students who use more sophisticated strategies, than they currently do, if the number triples are more complex? Are there students who use less sophisticated, than they currently do, if the number triples are simpler? In other words, is there a relationship between students' strategy choice and the factor structure of the problem?

## Methodology

This paper draws on one of the findings of a larger study, conducted from March to November 2007, of young children's development of multiplicative thinking. The study involved Grade 3 students (aged eight and nine years) in a primary school located in a middle class suburb of Melbourne. Thirteen students, representing a cross section of the class, were selected according to their mathematical achievement. A one-to-one, task-based interview was administered to the students to gain insights into and probe their understanding of and approaches to multiplicative problems. The findings of a subset of these results are reported in this paper.

### *Instruments*

The author developed a one-to-one, task-based interview on multiplication, consisting of three problems for each semantic structure identified by Anghileri (1989) and Greer (1992): equivalent groups and times as many. For each problem there were three levels of difficulty, rated as easy (E), medium (M) or challenge (C) from pilot testing. Number triples, considered outside the factor structure stipulated by the curriculum at this level, such as (7, 8, 56), (13, 8, 104), (8, 14, 112), (18, 4, 72), (8, 16, 128), were chosen for the challenge level of difficulty for two reasons. First, to gauge whether students attended to the structure of the problem or merely manipulated the numbers, as indicated by Mulligan and Mitchelmore (1997). Second, to identify whether students at this level are capable of solving problems involving harder number triples than commonly asked at this level, and do so using more sophisticated strategies. Visual cues were used in M1 (task 1, medium level of difficulty) as a context, M3 and C3 (task 3, challenge level of difficulty) to gauge whether students could replicate a collection, and M4 to provide students with a sense of the meaning of the times as many structure.

### *Interview Approach*

Each interview was audio taped and took approximately 30 to 45 minutes, depending on the complexity of the student's explanations. Responses were recorded and any written responses retained. The problems were presented orally, and paper and pencils were available for students to use at any time. Generous wait time was allowed and the researcher asked the students to explain their thinking and if they thought they could work the problem out a quicker way. Students had the option of choosing the level of difficulty to allow them to have some control and feel at ease during the interview. If a student chose a challenge problem and found it too difficult, there was an option to choose an easier problem.

### *Method of Analysis*

Initially, the researcher coded the students' responses as correct, incorrect, or non-attempt as well as coding the level of abstractness of solution strategies, informed by earlier studies (Heirdsfield et al., 1999; Kouba, 1989; Mulligan, 1992; Mulligan & Mitchelmore, 1997). For the purpose of this paper, the term abstraction refers to a student's ability to solve a problem mentally without the use of any physical objects (including fingers), drawings or tally marks. Where a student solved two problems for the one task (easy and medium), only the code for the more sophisticated strategy was recorded. The strategies chosen by the students presented in this paper are listed and defined in Table 1 according to the level of abstraction. Transitional counting is the only strategy listed that includes some form of representation; the other strategies indicate the students are abstracting. Students whose preferred strategy was multiplicative calculation or wholistic thinking were considered to be using multiplicative thinking rather than additive thinking.

Table 1
*Solution Strategies for Whole Number Multiplication Problems*

| Strategy | Definition |
|---|---|
| Transitional Counting | Visualises the groups and can record or verbalise the multiplication fact but calculates the answer to the problem using a counting sequence based on multiples of a factor in the problem. May use drawing or use of fingers or tally marks to keep track of the count. For example, "I counted by 6s four times using my fingers to help 6, 12, 18, 24", and recorded 4×6=24. |
| Building Up (counting by multiples) | Visualises the groups and the multiplication fact but relies on skip counting, or a combination of skip counting and doubling to calculate an answer. For example, "I know 4 sixes are 24 (skip counted by 6) and I need eights so I can double 24 and that's 48", and recorded 8×6=48. |
| Doubling and Halving | Derives solution using doubling or halving and estimation, attending to both the multiplier and multiplicand. For example, "4 times as many as 18. Double 18 is 2 times, double 36 is 4 times, so that's 72 stamps". This student doubled the multiplicand. |
| Multiplicative Calculation | Automatically recalls known multiplication facts, or derives easily known multiplication facts. For example, "I know 8 times 12 is 96 so I just added another 8 to get 104, and that's 13 eights." |
| Wholistic Thinking | Treats the numbers as wholes—partitions numbers using distributive property, chunking, and / or use of estimation. For example, one student rounded the number to nearest ten and then subtracted (compensation strategy) "I know 15 times 6 is 90 and then I took away 12, to get 13 times 6 and that's 78 15×6=90-12=78", another used distributive property, "If it was 8 boxes of 14 I know 8 times 10 and 8 times 4 so 80 and 32 is 112 (8×14=112, 8×10+8×4=112)." |

The examples accompanying each definition in Table 1 provide a guide to the classification of the students' solution strategies.

## Results and Discussion

The strategy choices of the 13 students on the six tasks are presented in two different tables. The first table (Table 2) provides the frequencies of strategies used by the students to solve the multiplication word problems pertaining to equivalent groups and times as many semantic structures. The main discussion of the results focuses on Table 3, which provides the strategy choice and task level of difficulty of each student. This closer look at the students' strategies provides evidence to support the argument that when challenged, students are capable of using sophisticated strategies.

The easy tasks were not included in Table 2 as no students chose them. For each task, students chose either medium (M), challenge (C) or in the case of equivalent groups extra challenge (ExC), so M1 in Table 2 refers to task 1 medium level of difficulty. The equations are included to indicate the number range used across the levels of difficulty.

The black vertical line distinguishes the use of abstracting strategies (to the right of the line), from those of some form of representation (fingers, recording), on the left. Students who consistently chose multiplicative calculation and or wholistic thinking were considered to be using multiplicative rather than additive thinking. The codes used in the table pertain to the strategies: Transitional counting (TC), Building up (BU), Doubling or halving (DH), Multiplicative calculation (MC), Wholistic thinking (WT). An asterisk indicates the use of visual cues, by the researcher when presenting the task.

Table 2
*Frequency of Strategy Choice for Equivalent Groups and Times as Many Tasks (n=13)*

| Semantic structure | Task No /Diff | Equations | TC | BU | DH | MC | WT |
|---|---|---|---|---|---|---|---|
| Equivalent groups | *M1 | 6×3 =18 | | 3 | | | |
| | C1 | 7×8 = 56 | | 6 | | 1 | |
| | ExC1 | 13×8 =104 | | | | 2 | 1 |
| | M 2 | 6×4 = 24 | | 3 | | | |
| | C 2 | 8×6 = 48 | | 2 | | 2 | |
| | ExC2 | 8×14 = 112 | | | | 3 | 3 |
| | *M3 | 5x4 = 20 | | 1 | | | |
| | *C3 | 6×7 = 42 | | 5 | | 6 | 1 |
| | Total strategy choice | | | 20 | | 14 | 5 |
| Times as many | *M4 | 3×6 = 18 | 1 | 2 | | | |
| | C4 | 4 ×18 = 72 | | 1 | 1 | | 8 |
| | M5 | 4×6 = 24 | 1 | 4 | | | |
| | C5 | 16× 8 = 128 | | | 1 | | 7 |
| | M6 | 6×$4= $24.00 | 1 | 4 | | 4 | |
| | C6 | 4×$3.50= $14.00 | | | | 2 | 2 |
| | Total strategy choice | | 3 | 11 | 2 | 6 | 17 |

From Table 2, it can be seen that multiplicative calculation (MC) and wholistic thinking (WT) were the preferred strategies of those who chose the challenge or extra challenge level of difficulty, with the exception of C1, in which building up (BU) was the preferred strategy. It may be argued that these are the more capable students and one would expect them to use multiplicative strategies. An alternative view is that enabling students to engage with number triples beyond what is commonly posed at this level prompts the use of more sophisticated solution strategies. Given students' familiarity with the equivalent groups semantic structure it was expected that multiplicative strategies would be the most frequently chosen strategy for C1.

In contrast, building up (BU) was the preferred strategy by those who chose the medium level of difficulty. It appears that the use of visual cues and the more familiar number triples influenced their strategy choice. One might infer from this that problems involving numbers within students' experiences prompt the use of simpler strategies.

In order to gain a deeper sense of the meaning of these findings, the strategies of individual students are presented in Table 3. The dots indicate task level of difficulty (one dot-medium level; two dots-challenge level and three dots extra challenge level of difficulty) and shading the strategy choice as follows: TC BU DH MC WT

In Table 3 the students are listed according to the frequency of sophisticated strategies and level of task difficulty chosen. For example, Sandy (listed first) consistently chose WT whereas Judy chose MC for four tasks and WT for two tasks. Sandy chose two extra challenge tasks (three dots) and four challenge tasks (two dots) whereas Judy chose one medium task (one dot). Gayle listed last, chose BU for each of the equivalent group tasks for both medium and challenge tasks and TC for the times as many medium tasks.

*Table 3*
*Strategy Choice and Level of Difficulty by Each Student Across the Six Tasks*

| | Equivalent Groups | | | Times as Many | | |
|---|---|---|---|---|---|---|
| Students | 1 | 2 | 3 | 4 | 5 | 6 |
| Sandy | ●●● | ●●● | ●● | ●● | ●● | ●● |
| Judy | ●●● | ●●● | ●● | ●● | ●● | ● |
| Jules | ●● | ●●● | ●● | ●● | ●● | ● |
| Annie | ●● | ●●● | ●● | ●● | ●● | ● |
| Mark | ●● | ●●● | ●● | ●● | ●● | ● |
| Sharne | ●● | ●● | ●● | ●● | ●● | ●● |
| Nigel | ●● | ●●● | ●● | ●● | ●● | ● |
| Bindy | ●● | ●● | ●● | ●● | ●● | ● |
| Danny | ●● | ●● | ●● | ● | ● | ● |
| Marty | ● | ● | ●● | ●● | ● | ● |
| Lewis | ● | ● | ●● | ●● | ● | ● |
| Lyle | ●● | ●● | ● | ● | ● | ● |
| Gayle | ● | ● | ●● | ● | ● | ● |

From Table 3 a clear division is evident between Bindy and Danny in relation to strategy choice and level of difficulty chosen. It is evident that BU is the preferred strategy of Danny, Marty, Lewis and Lyle regardless of level of difficulty chosen, whereas MC or WT were the preferred strategy of Bindy and those above in the table, who mainly chose challenge or extra challenge level of difficulty. An exception was task six, which related to money. All except Lyle and Gayle chose the challenge level of difficulty but some found it too challenging and so asked for the medium problem.

The students of particular interest, in relation to the research questions, are Jules, Annie, Mark, Sharne and Nigel who were identified as "middle group" when the thirteen students were chosen. These students generally chose challenge or extra challenge questions and consistently chose MC or WT for tasks involving harder number triples such as (16, 8, 128) but less sophisticated strategies on problems involving simpler numbers or those with which they were familiar (e.g., multiples of 6 related to their knowledge of Australian rules football), as was evident in challenge task 1 (2 dots). One might infer from this that students who use a binary operation for multiplication have the flexibility of choosing a less sophisticated strategy for easier problems.

Further evidence of this relates to the use of MC or WT by Jules, Sharne and Mark to solve task 5, challenge level of difficulty: "The Phoenix scored 8 goals in a netball match. The Kestrals scored 16 times as many goals. How many goals did the Kestrals score?" The following are abridged excerpts of their solution strategies.

> Jules: That's 16 times 8. It's 128. I started with 10 times 8 and that equals 80, then 6 times 8 is 48 then added them to get 128. 80 and 40 is 120 and 8 is 128.
>
> Sharne: I can halve 16 to get 8. I know 8 eights are 64 and another 8 eights is 128, because 8 times 8 and 8 times 8 is the same as 16 times 8.
>
> Mark: I know 12 eights are 96 and 4 eights are 32, then I just need to add 32 and 96.

Jules partitioned the 16 into ten and six using his place value knowledge and operated on each separately, using the distributive property. Both Sharne and Mark split the 16 into known facts to use as a starting point. Sharne halved the multiplier and operated on each separately, whereas as Mark split the problem up into 12×8 and 4×8. These examples indicate the students' ability to partition number and use the distributive property in order to solve problems mentally.

The number triples of the challenge and extra challenge tasks are similar to those used in studies for students in Grades 4 to 6 (Ell et al., 2004; Heirdsfield et al., 1999) and the evidence presented in this paper indicates that some Grade 3 are capable of solving problems relating to the different semantic structures involving numbers beyond what is expected at this year level and do so using sophisticated strategies.

One unexpected result was the use of WT by Marty and Lewis, two of the lower performing students for task 4, challenge level of difficulty: "Jamie collected 18 stamps. Jack collected 4 times as many. How many stamps does Jack have?" The following are abridged excerpts of their solution strategies.

> Marty: Ten, 4 times is 40 and eight 4 times is um 32. 40 and 32 is 72.
>
> Lewis: Four times as many as 18? 20, oh umm, so 4 times? 80, take away umm 8, is umm 72. I took away 10 first and added 2 onto 70, cause it's easier.

Marty partitioned the 18 into ten and eight and operating on each separately, showing an understanding of distributive property; whereas Lewis rounded the 18 to 20, a number that he could calculate mentally and then compensated by subtracting eight. These responses indicate their ability to use multiplicative rather than additive thinking when presented with a task involving number triples outside the factor structure stipulated by the curriculum at this level.

These findings are in contrast to the conjecture that "students first learn a new strategy to solve problems where the situation is familiar and the relevant number facts are well known" (Mulligan & Mitchelmore, 1997, p. 327).

## Conclusion

The findings of this study suggest that students at Grade 3, when challenged, are capable of engaging with problems at higher level of thinking, than would otherwise be the case. Second, students who used wholistic thinking were flexible in their thinking and realised that numbers could be split in a variety of ways. From these findings it is reasonable to infer there is a relationship between strategy choice and number triples: the more difficult the number triples, the more sophisticated the strategy choice; the easier the number triples, the less sophisticated the strategy choice. This is a key finding in that it is the reverse direction to what one would anticipate, as indicated by the literature, or that seems to be implied by most curriculum resources or texts.

A recommendation for teachers is to pose problems some of the time that extend children's thinking beyond what appears to be commonly the case. By doing so, teachers may gain insights into strategies, which students are capable of but have not demonstrated on simpler problems.

While acknowledging these results are different to those of earlier research, the author is aware there are possibly other explanations for these findings such as students' level of confidence, or their willingness to take a risk, and is sufficiently circumspect about these conclusions. These provide opportunities for further research in this area.

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