Problem Solving in the School Curriculum from a Design Perspective

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In this symposium, we discuss some preliminary data collected from our problem solving project which uses a design experiment approach. Our approach to problem solving in the school curriculum is in tandem with what Schoenfeld (2007) claimed: "Crafting instruction that would make a wide range of problem-solving strategies accessible to students would be a very valuable contribution … This is an engineering task rather than a conceptual one" (p. 541). In the first paper, we look at how two teachers on this project taught problem solving. As good problems are key to the successful implementation of our project, in the second paper, we focus on some of the problems that were used in the project and discuss the views of the participating students on these problems. The third paper shows how an initially selected problem led to a substitute problem to meet our design criteria.

Paper 1: Leong Yew Hoong; Toh Tin Lam; Quek Khiok Seng; Jaguthsing Dindyal; Tay Eng Guan; Lou Sieu Tee; Nanyang Technological University. *Enacting a problem solving curriculum.*

Paper 2: Jaguthsing Dindyal; Quek Khiok Seng; Leong Yew Hoong; Toh Tin Lam; Tay Eng Guan and Lou Sieu Tee; Nanyang Technological University. *Problems for a problem solving curriculum.*

Paper 3: Quek Khiok Seng; Toh Tin Lam; Jaguthsing Dindyal; Leong Yew Hoong and Tay Eng Guan and Lou Sieu Tee; Nanyang Technological University. *Resources for teaching problem solving: A problem to discuss.*

L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia.* Fremantle: MERGA.

Enacting a Problem Solving Curriculum

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In preparing teachers to enact a problem solving curriculum, the first two phases of professional development involve teachers attending sessions about problem solving and observing the teaching of problem solving to students. In this paper, we focus on the third phase: these teachers carry out the problem solving curriculum in their classes. We discuss how two teachers apply problem solving processes in their instructional practices.

In the research reported in this paper, we focused on implementing problem solving in a way that is in line with the Singapore mathematics curriculum (Ministry of Education/University of Cambridge Local Examination Syndicate, 2007). We wanted to elevate the position of problem solving from that of 'occasional visitor' to that of 'regular mainstay in usual mathematics classrooms'. This is, however, not a straightforward enterprise. One of the key challenges was to prepare teachers for this shift of emphasis in their teaching. In this paper, we examine how the teachers who had undergone the teacher development programme carried out the problem solving curriculum in their classes.

Background

The study reported here is part of a project known as M-ProSE – Mathematical Problem Solving for Everyone. The school we worked with is an independent Secondary school in Singapore. We have just completed three phases in the teacher preparation programme: (1) Sessions with teachers that were focused on teaching problem solving processes to teachers; (2) sessions with students (in an elective module) that were focused on demonstrating to teachers ways to introduce to students problem solving processes; and (3) observation of the teachers as they conduct the problem solving lessons, with a view to discuss changes for future implementations. Later phases will involve working alongside the teachers to infuse the problem solving approach across the broader curriculum.

In the first phase, one of the authors – hereafter known as the trainer – used problems as examples to help develop teachers' problem solving habits within Pólya's (1954) stages – Understanding the problem; Devise a plan; Carry out the plan; Check and Extend – while being aware of the influence of Schoenfeld's (1985) components of problem solving, which include Resources, Heuristics, Control, and Beliefs. In the second phase, he taught twenty-one Year 9 students who signed up for the (elective) course over ten lessons, each lasting one hour. The essential contents of the student module – the problems solved and the processes highlighted – were similar to the teacher module, but the pace, tone, and issues raised for discussion were adjusted to suit the needs of the students. We held post-lesson meetings with the teachers to discuss the lessons at regular junctures to gather ideas

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for improvement as well as clarify the instructional practices that were demonstrated. A more detailed report on the teacher preparation programme over the two phases is found in Leong, Toh, Quek, Dindyal, and Tay (2009).

In this paper, we focus on the third phase of the project where the teachers moved from the classroom where they *learn* about problem solving to the classroom where they *teach* problem solving. In Phase three, the school decided to offer the problem solving course as a compulsory module for the entire Year 8 cohort of the school, totalling 164 students. Three teachers were selected to teach the module. Due to the school's staffing constraints, only two of the three teachers — Raymond and William — attended the professional development programme over the two phases. There were also time constraints; instead of a module comprising ten one-hour sessions that we proposed, the school decided to cut it down to an eight one-hour introductory lessons on problem solving.

Data and Analysis

As the aim of this study was to examine how the teachers who have undergone the two phases of teacher development actually carried out the teaching of problem solving, the main source of data was derived from the classroom activities of Raymond and William. Video recordings and transcripts of these teachers' actions and instructions were used.

The focus of analysis is on how the teachers used each problem to help develop the students' problem solving abilities along the lines of Pólya's stages. For this paper, we have space only to report the analysis of how the teachers taught one particular problem — known by the team as the "sum of digits" problem:

Find the sum of all the digits of the numbers in the sequence 1, 2, 3, …, 9999.

Teacher Raymond

Raymond used a large chunk of Lesson 1 and some parts of Lesson 2 to cover the sum of digits problem. After introducing the problem, Raymond asked the students to work on the problem in pairs for 20 minutes. During that time, he walked around from pair to pair to monitor their work and to prompt the students towards productive directions. What was conspicuous in his teacher-pair conversations was his reluctance to provide students with answers they were looking for; instead, he asked questions that directed students' attention towards the process of problem solving. In particular, his prompts can be interpreted as an informal first introduction of Pólya's stages and heuristics to the students. Table 1 shows a sample of the language he used and the associated Pólya's processes that we interpret as implicitly intended.

Table1:
Sample of Raymond's Prompts when Interacting with Student Pairs

Raymond's language	Pólya's processes
"I suggest you read the question more carefully …"	Understand the Problem
"What are you trying to do from this step here?"	Devise a Plan
"Maybe you want to draw something to help you visualise it better?"	Heuristics
"How can you be sure that your solution is correct?"	Look Back – check
"Do you think you can find the general form for this kind of question? Let's say I don't add 1 to 9999, I add 1 to 99999 …"	Look Back – extend the problem

Raymond then used a whole-class instructional setting to formally introduce Pólya's stages in relation to the problem as well as to the attempts of the students at solving the problem that he observed. The presentation of a tight-linkage between Pólya's stages, the problem, and the students' Polya stages-like attempts is shown with some samples of Raymond's talk in Table 2. Under "Carry out the Plan", Raymond actually presented the solution by progressively enlarging the initial small problem: 1 to 9, 1 to 99, 1 to 999, then finally 1 to 9999. The entire segment of this part of the lesson lasted 41 minutes, with the bulk of the time spent in "Carry out the Plan" (26 minutes) and "Look Back" (9 minutes).

Table 2:
Raymond's Formal Introduction of Pólya's Processes in a Whole Class Setting

Pólya's processes made explicit	Relating to the problem	Relating to students' earlier attempts
"Let me first talk about the first step … Understanding the Problem".	"So you just sum up 1 to 9999 …"	"I noticed some of you finished quite quickly … [But] actually the question asks you to sum the digits of the numbers, not the numbers themselves."
"The next thing I want to talk about is Devise a Plan".	"I'll work on a smaller problem. … the simplest problem we can try to solve … we first try sum the digit of 1 to 9 instead …"	"I looked around. Some of you have devised some plans. One of the plan[s] I looked at is to identify a pattern."
"OK so now we Carry out the Plan to work on the smaller problem."	"This one should be 45. … So the second smaller problem will be … 1, 2 to all the way to 99."	"… which is quite a number of you did. … Some of you actually added wrongly but it's OK. Don't worry."
"OK, however, do you want to stop here? … We need to Look Back at our solutions. Let's look at the fourth stage …"	"OK, instead of just solving from 1 to 9999, we try to solve from 1 to whatever numbers of 9 …"	"OK [student] June can see the pattern 45×10×10 to the power of n minus 1 …"

Raymond rounded up the discussion on the sum of digits problem by once again explicating Pólya's four stages — this time with particular emphases that the stages need not be one-directional and that Stage 4 is something that students were not used to but was worth working on:

> And sometimes when you carry out the problem you realise, "Hey, I'm not solving the problem, it doesn't help." So what you should do is, right, you will cycle back to understand the problem again, ok?
>
> When the problem is solved already … we have to look back the solution … and try to look for a deeper understanding of this problem, alright? And hopefully, we can find a general solution, ok?

Teacher William

William used largely the same sequence of instruction as Raymond with the sum of digits problem. The difference in time allocation was most conspicuous in the first segment where William provided considerably less time for students to attack the problem on their own; instead, he allowed students to work on part of the problem *after* he set up the overall solution strategy. Table 3 shows the time allocation comparisons between Raymond and William across the lesson components. During the initial segment where students attempted to solve the problem, compared to Raymond (see Table 1), William's language – perhaps due to the lack of time – appeared to be more narrowly focused on 'Understanding the Problem' and getting the answers.

Table 3
Instructional Components and Time Allocations between Raymond and William

	Raymond	William
Students attempt to solve the problem	20 min	6 min
Teacher explicates Pólya's stages up to Stage 3: 1-99	12 min	12 min
Students work on remainder of "Carry Out the Plan"	Not done	4 min
Teacher completes "Carry Out the Plan"	20 min	27 min
Teacher introduces Stage 4	9 min	Not done
Teacher reviews Pólya's stages	5 min	Not done
Total time	66 min	49 min

William then proceeded to explicate Pólya's stages up to "Carry out the Plan". Like Raymond, he started from the smaller problem of 1-9 and went on to 1-99. Having demonstrated the strategy, he instructed the students to try "1-999" on their own before completing Stage 3 as whole-class demonstration. However, unlike Raymond, he left out Stage 4 and the segment on the review of Pólya's stages (see Table 3).

Discussion

From the point of view of teacher development for teaching problem solving, we take encouragement from the data that both Raymond and William clearly built in the Pólya's stages into their classroom instruction. In fact, the strong similarity in their overall approaches may indicate they had prior discussions on how to proceed with the problem in class and hence implies some deliberate 'buy-in' into the problem solving processes.

A preliminary broad-grained analysis admittedly does not do justice to the complexity of their classroom practices. Nevertheless, the data as presented in Tables 1-3 suggests that Raymond and William applied Pólya's stages to classroom-use quite differently. In brief, Raymond seemed to want students to actively own the problem and to focus on their own problem solving processes before offering them Pólya's stages and heuristics as a way to help them get 'unstuck'. He also emphasised the importance of going beyond the given problem (Look Back). In contrast, William appeared to be cautious about letting students work independently on the problem initially, preferring instead to provide the first steps towards the solution and asking students to follow along the same vein. Also, he focused more on answer-getting and did not introduce Pólya's Stage 4.

Seen through the lens of their pedagogical inclinations, William appears to prefer a more conservative approach of 'teacher demonstrate, student follow'. Pólya's stages were employed merely as add-ons to his instructional tool-set to fit into his existing instructional approach. As for Raymond, we are perhaps seeing the beginnings of the problem-solving approach having transformative effect in his way of teaching mathematics. The next phase of the teacher development programme and research should thus zoom-in on the causes of William's reservations and Raymond's openness.

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Problems for a Problem Solving Curriculum

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In this paper we highlight some of the problems that we used in our problem solving project. Particularly, we focus on the problems that students liked or disliked the most and look at some of their solutions to these problems.

The successful implementation of any problem solving curriculum hinges on the choice of *appropriate problems*. However, appropriate problems may have different connotations for different people. In the context of our problem solving project, we have used some problems on which we have collected some data from the students solving these problems. We hereby report, what the students perceived as the most important and least important problems and discuss some of their solutions to these problems. Due to space constraints only a few responses will be highlighted.

The nurturing of problem solving skills requires students to solve meaningful problems. Lester (1983) claimed that posing the cleverest problems is not productive if students are not interested or willing to attempt to solve them. The implication is that mathematical problems have to be chosen judiciously. It is clear that if the answer to a "problem" is apparent then it is no longer a "problem". Hence, the defining feature of a problem situation is that there must be some blockage on the part of the potential problem solver (Kroll & Miller, 1993). What is a problem for one person may not necessarily be a problem for another person. Schoenfeld (1985) clearly pointed to the difficulty in describing what constitutes a problem, "…being a problem is not a property inherent in a mathematical task" (p. 74). A problem is constituted in a threefold interaction among person(s), task and situation (e.g., time and place).

What should be the criteria for choosing good problems? Problems selected for a course must satisfy five main criteria (Schoenfeld, 1994, cited in Arcavi, Kessel, Meira, & Smith, 1998, pp. 11-12):

- Without being trivial, problems should be accessible to a wide range of students on the basis of their prior knowledge, and should not require a lot of machinery and/or vocabulary.
- Problems must be solvable, or at least approachable, in more than one way. Alternative solution paths can illustrate the richness of the mathematics, and may reveal connections among different areas of mathematics.
- Problems should illustrate important mathematical ideas, either in terms of the content or the solution strategies.
- Problems should be constructible without tricks.

- Problems should serve as first steps towards mathematical explorations, they should be extensible and generalisable; namely, when solved, they can serve as springboards for further explorations and problem posing.

Conscious of the fact that the choice of problems was critical in our problem solving project, we were guided by the following principles: (1) the problems were interesting enough for most if not all of the students to attempt the problems; (2) the students had enough “resources” to solve the problem; (3) the content domain was important but subordinate to processes involved in solving it; and (4) the problems were *extensible* and *generalizable*. Our approach in this project is design experiment (see Brown, 1992; Wood & Berry, 2003) whereby we are trying out materials and refining them for use in schools. One of the expected outcomes is also a set of well-designed problems for future use in mainstream schools. We wish to triangulate the views of students and the teachers together with our own ideas on these problems for selecting the best problems. However, here we focus only on the students’ views.

The Problem Solving Project

The study involved 153 secondary 2 students (Year 8) in a secondary school in Singapore spread over 6 different classes. The classes were taught by three teachers, each of them teaching two of the classes. Eight one-hour lessons were used to implement a specific problem solving curriculum in which the students had to use a “practical worksheet” (see Toh, Quek, Leong, Dindyal, & Tay, 2009) devised by our research team. We hereby report the students’ views on 13 of the problems that were used during the course as well as a few of their responses. After the course, the students were asked to rank the top three problems that they found most interesting and also to identify one problem that they found least interesting or which they disliked.

Surprisingly for the team the students ranked Problem 3 as the most liked and also as the least disliked problem (see Figure 1). The initial worry for the team was the nature of the context described in the problem. However, the students’ comments suggest that they understood the situation and looked at it solely from a mathematical perspective. From the students’ comments (see Figure 1), we can see that they liked the fact that the problem was interesting, intriguing, challenging or fun to solve. They also made positive comments about the fact they were applying mathematics to solve a seemingly *real-life* problem. While, the students’ comments about Problem 11 (given as a precursor to another problem and heavily dependent on symbolism) was expected, it was a bit of a surprise that they rated Problem 1 as one of the least liked problems. The students’ comments about this problem that they disliked the most include: tedious solution, lengthy calculations, too many steps, they do not understand, or it is too complex, etc.

A good question to ask is whether a problem liked by the students is necessarily a good problem to be included in the curriculum and accordingly whether a problem disliked by the students is one that should be excluded. Another question for discussion is whether those who like a problem are necessarily able to solve the problem and whether a problem they dislike is one that they necessarily cannot solve. What should be some basic criteria for deciding whether to include or exclude particular problems from the problem solving curriculum? How do we triangulate our views with the views of the teachers and the students about the appropriateness of mathematical problems?

Most Liked Problems	Least Liked Problems
Problem 2 You are given two jugs. Jug *A* holds 5 litres of water when full while jug *B* holds 3 litres of water when full. There are no markings on either jug and the cross-section of each jug is not uniform. Show how to measure out exactly 4 litres of water from a fountain. *Problem 3* Two bullets are placed in two consecutive chambers of a 6-chamber pistol. The cylinder is then spun. Two persons play Russian roulette. The first person points the gun at his head and pulls the trigger. The shot is blank. Suppose you are the second person and it is now your turn to shoot. Should you pull the trigger or spin the cylinder another time before pulling the trigger?	*Problem 1* Find the sum of all the digits of the numbers in the sequence 1, 2, 3, …, 9999. *Problem 11* The base 2 (binary) representation of a positive integer n is the sequence $a_k a_{k-1} a_{k-2} \ldots a_1 a_0$, where $n = a_k 2^k + a_{k-1} 2^{k-1} + a_{k-2} 2^{k-2} + \ldots + a_1 2^1 + a_0$, $a_k = 1$, and $a_i = 0$ or 1 for $i = 0, 1, \ldots, k-1$. Write down the binary representations of the day today and of the day of the month in which you were born. [Problem 11 was given as a precursor to another problem]
Students' comments on Problem 2: Fun and mind-boggling. It was interesting that I could solve the problem with algebra. Challenging and nice question! It can be used in everyday life to obtain a given amount of liquid without having the size of the jug available. *Students' Comments on Problem 3*: Very interesting, solution quite astonishing, fun to solve. It is interesting. Matter of life and death with probability. Problem 3 is the most interesting question I have encountered in the entire module as it involves probability and it is quite fun to solve as it is a life or death situation and involves loads of thinking together with listing out the possibilities. The problem is also spiced up with the element of a good plot for the question. Allows the entire class to discuss and argue about the possibility and chance of being shot. It shows the use of simple diagram heuristics to solve a complicated problem.	*Students' Comments on Problem 1*: It is very tedious. The solving of this problem is challenging and needs some time to think of a solution. The solution takes a long time to carry out. Too numerical and common. Too many steps needed. Takes a lot of counting and adding to see the pattern. The answer is very lengthy. Need too much steps. *Students' comments on Problem 11*: I do not have the resources to accomplish the problem. Too boring/complicated I do not understand the question. I don't understand it at all. Too complex, difficult to understand.

Figure 1. Most liked and least liked problems

Figure 2 below shows some students' solutions to these problems. More solutions to the problems would be highlighted in the symposium presentation.

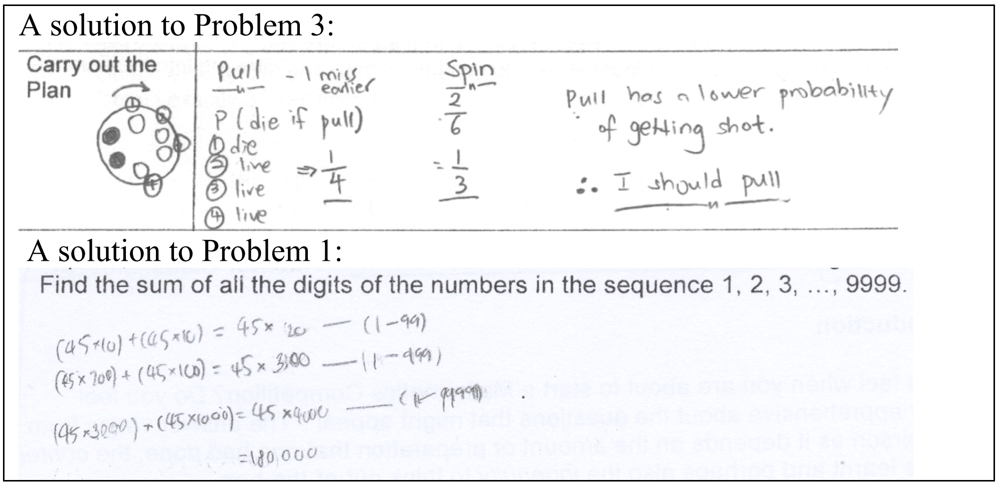

Figure 2. Students' solutions to most liked an least liked problems

Conclusion

The students' views on the problems used in the study provide a valuable source of information on the inclusion of certain types of problems or even for refining the problems. We wish students to be able to transfer what they learn during the problem solving process to other situations that they may meet within or outside mathematics. We also understand that we have to reconcile our own views with those of the students and the teachers in the project for making an informed choice of problems. We hope that the symposium will provide an opportunity for a fruitful discussion on this issue.

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Resources for Teaching Problem Solving: A Problem to Discuss

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In supporting teachers to enact a problem solving curriculum, resources are needed. In this paper, we show in the context of a design experiment, how an initial problem that we thought would fulfil the above criteria was subsequently replaced. The substitute problem is described in some detail.

In supporting teachers to enact a problem solving curriculum, resources are needed. In particular, the choice of problems is critical (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). In the second paper of this symposium, the basis for the choice of problems used in the Mathematics Problem Solving for Everyone (M-ProSE) project was presented: (1) the problems were interesting enough for most if not all of the students to attempt the problems; (2) the students had enough "resources" to solve the problem; (3) the content domain was important but subordinate to processes involved in solving it; and (4) the problems were *extensible* and *generalizable*. In this paper, we show, in the context of a design experiment (Middleton, Gorard, Taylor & Bannan-Ritland, 2006), how an initial problem that we thought would fulfil the above criteria was subsequently replaced.

The Initial Problem

Sum of Digits Problem

What is the sum of all the digits of the numbers in the sequence 1, 2, 3, …, 10n – 1?

This problem was initially chosen to be the exemplar in a resource book to be written for teachers implementing the problem solving module of the M-ProSE project. It had apparently the necessary richness for all four of Pólya's stages and the use of heuristics:

The sum required here is NOT 1 + 2 + … + 10n – 1. There is a need to pause and understand the problem, and not work out the sum of an Arithmetic Progression.

There is a need to really explore the problem by solving simpler problems and looking for patterns.

Suitable representations and diagrams can be very useful in carrying out a plan.

The problem can be 'expanded' in various ways, for eg. What is the sum of all the digits of the even numbers in the sequence 1, 2, 3, …, 10n – 1?

An elegant solution for the original problem can be obtained by considering all 10n n-digit sequences with each digit having a choice from 0, 1, 2, …, 9. Then there are exactly n10n digits. These digits are equally divided among 0, 1, 2, …, 9. Thus, there are exactly n10n ÷ 10 = n10n−1 of each digit 0, 1, 2, …, 9. Hence the sum = n10n−1 × (0 + 1 + 2 + … + 9) = 45 n10n−1.

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Unfortunately, we realised that the first solution needed mathematical induction. This clearly was not suitable for a Year 8 curriculum. We decided to modify the problem to:

> What is the sum of all the digits of the numbers in the sequence 1, 2, 3, …, 9999?

Sadly, the Year 8 classes who tried this problem ranked it very low. The students considered it too tedious and needed too many steps. One may overrule the objection that the problem needed too many steps because students do need to learn that *listing* often begins slowly but, done properly, a pattern is discovered and work proceeds very quickly after that. The practical worksheets for this problem showed that very few students could give a complete solution. Perhaps out of tiredness, most ended up with an unproved conjecture, not accepting that 'pattern is not proof'.

In the context of our design experiment, the problem then did not satisfy points (1) and (2) mentioned above, i.e., student interest and availability of resources. In the next section, we shall describe the replacement, the Lockers Problem, by summarising the main ideas in our treatment of the problem in the resource book.

According to Mason and Johnston-Wilder (2006), it is important to note that there are some distinctions associated with mathematical tasks at various levels: (a) the task as imagined by the task author; (b) the task as intended by the teacher; (c) the task as specified by the teacher-author instructions; (d) the task as construed by the learners; and (e) the task as carried out by the learners. There are bound to be mismatches between what the assigner wishes to achieve and what actually is achieved during the solving process. Although the research team was careful to ascertain if the problem satisfied the four points of reference, it has not been tried out by the students yet.

The Substitute Problem

Lockers Problem

> The new school has exactly 343 lockers numbered 1 to 343, and exactly 343 students. On the first day of school, the students meet outside the building and agree on the following plan. The first student will enter the school and open all the lockers. The second student will then enter the school and close every locker with an even number. The third student will then 'reverse' every third locker; i.e. if the locker is closed, he will open it, and if the locker is open, he will close it. The fourth student will reverse every fourth locker, and so on until all 343 students in turn have entered the building and reversed the relevant lockers. Which lockers will finally remain open?

Understand the Problem

The key parts of the problem to take time to understand would be: to 'reverse' a locker means to open it when it is closed and to close it when it is open; and, which lockers would a particular student act on?

Two useful *heuristics* to help understand the problem are to *act it out* and to *consider a simplified version*, can be used. Consider the problem when there are only 10 students. We list quickly the 10 lockers as 1, 2, …, 10 on a piece of paper. Now we act out what the 10 students will do, making a backslash '/' for open and a slash '\' over the backslash to obtain a cross '×' for close. After four students, we have:

1	2	3	4	5	6	7	8	9	10
/	×	×	×/	/	×/	/	×/	×	×

The problem (not the solution) looks clear now. If we want to, we could solve the simplified problem for 10 students. This may be our first plan.

[*Devise a Plan*] Our first plan is: we will *consider a simplified problem* for 10 students, and *look for patterns.*

[*Carry out the Plan*] Resource-wise, the problem solver must know multiples well. Continuing, we end with:

1	2	3	4	5	6	7	8	9	10
/	×	×	×/	×	××	×	××	×/	××

And thus we have a solution for our simplified problem as 'Lockers 1, 4 and 9 will remain open at the end'. Thus, our conjecture after carrying out the first plan is:

> The lockers that will remain open will be those whose number is a square, i.e. when there are 343 students, they are 1, 4, 9, 16, 25, 36, 49, …, 324.

[*Check*] Returning to the original problem, we check our conjecture for the next square that we did not actually act out, i.e., 16. We note that we need only to look at the first 16 lockers (out of the original 343) since students numbered 17 and above will not touch the first 16 lockers, in general Student *n* will not touch lockers numbered less than *n*. The conjecture looks good with the following result:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
/	×	×	×/	×	××	×	××	×/	××	×	×××	×	××	××	××/

[*Understand the Problem 2*] We ask a series of questions:

> What feature(s) of a square causes the corresponding locker to end up open? (Note that this is still a conjecture.)
>
> By *acting it out* slowly, why does a particular locker end up open (or closed)?

By looking closely at the *diagram* above, one eventually realises that if a locker is 'touched' an odd number of times, it will end up open; if it is 'touched' an even number of times, it will end up closed. Thus, we *restate the problem in another way*.

> What type of number has its corresponding locker touched an odd number of times?
>
> What feature(s) of a square causes the corresponding locker to be touched an odd number of times? (Note that this is still a conjecture.)

The series of questions involving the conjecture of squares, the lockers being touched an odd number of times, features of a square, distills to a more fundamental question which puts away the conjecture of squares for the moment.

> What feature(s) of a number causes the corresponding locker to be touched?

Eventually, we reach the understanding that a locker is touched only by students whose numbers are factors of the number of the locker. Thus, the number of times a locker is touched is exactly the number of the factors of its corresponding number. *Restating the problem in another way* again:

> What type of number has an odd number of factors?

The series of questions using heuristics, which twists, turns and focuses the original problem is crucial to better understand the problem before formulating a new plan of attack. The Lockers problem is very good for emphasising the importance of spending time to understand the problem.

[*Devise a Plan 2*] Plan 2 will now be to count the number of factors of some numbers and try to understand why a square has an odd number of factors while the others have even numbers. We will *look for a pattern*.

[*Carry out the Plan 2*] We will count the number of factors for 1 to 10 and then a few other random numbers when we spot a pattern.

Number	No. of factors	Number	No. of factors	Number	No. of factors
1 = 1	1	5 = 1 × 5	2	9 = 1 × 9 = 3 × 3	3
2 = 1 × 2	2	6 = 1 × 6 = 2 × 3	4	10 = 1 × 10 = 2 × 5	4
3 = 1 × 3	2	7 = 1 × 7	2	16 = 1 × 16 = 2 × 8 = 4 × 4	5
4 = 1 × 4 = 2 × 2	3	8 = 1 × 8 = 2 × 4	4	343 = 1 × 343 = 7 × 49	4

It is now quite clear that only squares have an odd number of factors!

[*Devise a Plan 3*] We will now write out a 'clean' solution focusing on showing that only the squares have an odd number of factors. We will do this by showing that the factors come in pairs.

[*Carry out the Plan 3*] An 'algebraic' proof was written out in the resource book.

[*Check and Expand 3*] We can check the solution for the locker 25 by *acting out* the visitors to Locker 25.

We are done for this problem but not done with the problem-solving process. It is a key feature of the model that the solver should try to 'expand' the problem even though it is solved. By expanding, we mean one of the following:

- finding other solutions which are 'better' in the sense of elegance, succinctness, or with a wider applicability
- posing new problems
- *adapting* by changing certain features of it (e.g. change some numbers, change some conditions, consider the converse)
- *extending* to problems which have greater scope
- *generalizing* to problems which includes the given problem as a special example

An alternative solution, sketched out in the resource book, uses prime factorisation to prove that a natural number is a square if and only if it has an odd number of factors.

We show here one of five problems that were posed in the resource book.

Adaptation: The *i*-th student reverses every locker whose number is a factor of *i*.

Sketch of solution: The multiple-factor relationship in the original problem is now reversed. Thus, we are interested in the number of multiples of a number *m*. The locker with number *m* will remain open at the end if and only if $\left\lfloor \frac{343}{m} \right\rfloor$ is odd.

Discussion

This paper shows the change that is implemented when design features are examined in a design experiment context. The four criteria points for choosing problems are quite stringent. To what extent can any of the criteria be relaxed? Additionally, is the Lockers Problem amenable for teachers to scaffold students' learning of problem solving?

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