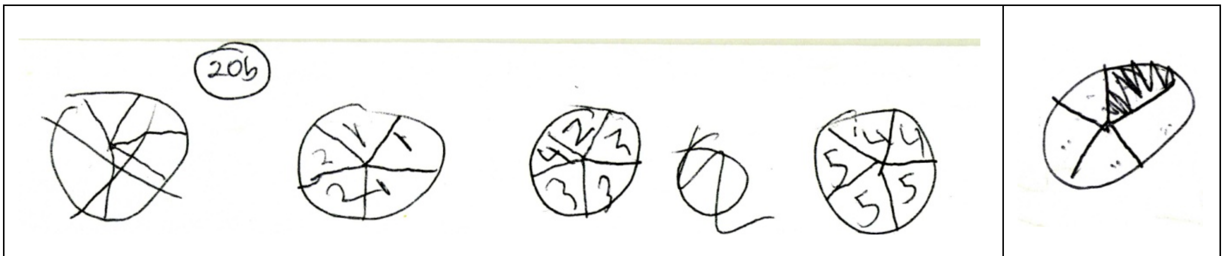
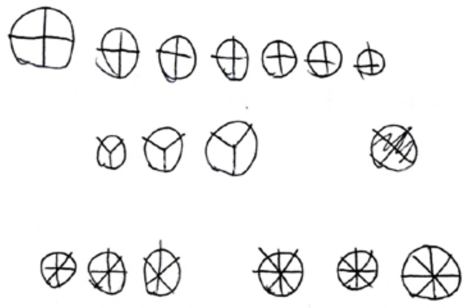


Meg: Three fifths | Ebony: $^3/_{15}$

*Figure 3.* Partitioning using French Division.

Similarly in Question C, some children used French division and divided all seven custard tarts into five parts and some successfully named one person's share as seven fifths or one and two fifths. Some children eliminated the wholes first and then divided each of the last two custard tarts into five, successfully naming this as one and two fifths. As in Question B, some children could do the partitioning but could not successfully name the share.

In Question B, Harry found the partitioning difficult with a circular area diagram. He divided a circle into six parts and after puzzling over more drawings he thought aloud, "there's no way you can cut it up into five pieces". Similarly, Brad (see Figure 4) tried partitioning the three custard tarts into thirds, eighths, and sixths (inaccurately) but could not find a partitioning that would enable fair sharing. For Question C, he partitioned the seven custard tarts into halves and then drew over that to partition them into quarters (see Figure 4). In both Question B and Question C, some children drew on partitioning and used repeated halving but could not create a number of parts that could be dealt out equally.

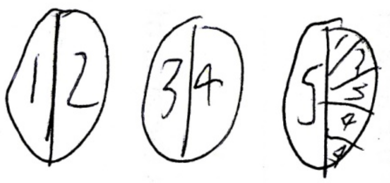


Brad: "Don’t know"

*Figure 4.* Unsuccessful partitioning in Question B.

### *Unit Forming*

Unit forming was drawn upon in these two questions when students made fair shares by combining a number of non-equal parts. They often also recruited partitioning in the execution of their strategy. Some students who divided all five custard tarts in half then partitioned the leftover half into five and drew on unit-forming to name the combined fair share. Of the 19 students who did this in Question B and made five fair shares out of two unequal parts, only one could name this correctly. Seth drew unit-forming (making a fair shares of two unequal parts) while recruiting partitioning (dividing the leftover half into five equal parts), and then used equivalence ($^1/_2$ = $^5/_{10}$) to name the share six tenths (see Figure 5).



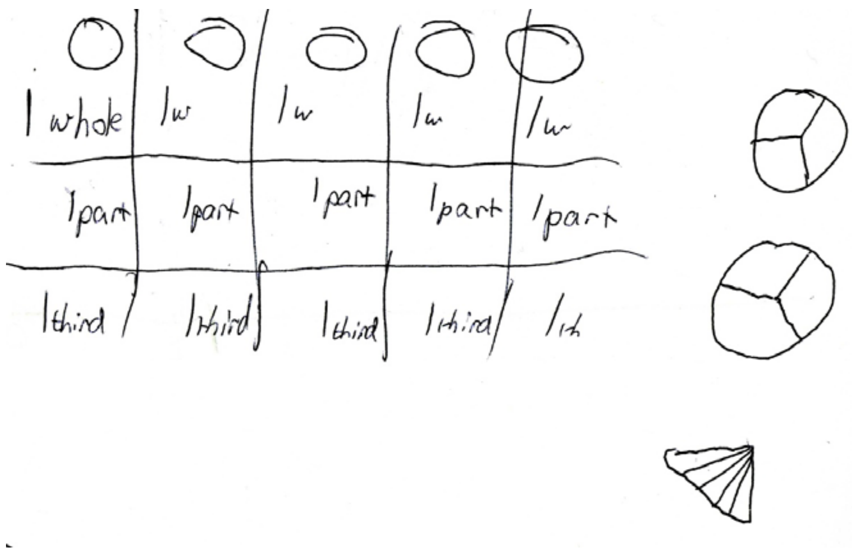
Seth: Six tenths

*Figure 5.* Successful unit forming in Question B.

Some children could describe the share, as a half and a fifth of the extra half, but did not have the equivalence skills to complete the renaming. Other children incorrectly described the share as a half and a fifth, unable to keep track of the unit.

On the other hand, one child, Felix, gave a reasonable estimate using unit-forming, but rather than recruiting equivalence knowledge, he visualising the two pieces combined and then drew on the concept of partitioning to estimate the size of that "part". He combined the half and one piece of the last part divided into five, and described this added together amount as “nearly two thirds”: this estimate (0.66) was very close to the exact value (0.6).

In Question C, some students drew on the concept of unit forming, for example, by eliminating the wholes, dividing the last two tarts into thirds, and then the leftover third into five parts. The equivalence knowledge needed to execute the naming of this share, made of combined amounts, was beyond the students who used this approach. However, one student, Courtney, was very logical with her representation (see Figure 6).




Courtney: one whole and one third and one fifth of the last piece

*Figure 6.* tracking unit-forming: sharing 7 tarts between 5 people.

One child, Ben, drawing on unit forming, made an estimate of his combined amounts, rather than calculate the fair share. After eliminating the wholes, he divided the last two custard tarts into halves. As this gave four parts, not five, he correctly reasoned that the fair share would be one and about three quarters of a half. His reasonable estimate of the non-whole number part, three quarters of a half ($^{3}/_{8}$ or $^{15}/_{40}$) was very close to the precise value of the part of one of the two leftover tarts ($^2/_5$ or $^{16}/_{40}$) that was needed to make a fair share when combined with a whole tart.

### *Equivalence*

As described above, students who drew on the concept of unit forming needed to recruit equivalence knowledge to name the overall fair share successfully.

## Conclusion

The present study demonstrated that the concepts of partitioning, equivalence, and unit-forming understandings were drawn upon by real students in different approaches to this quotient sub-construct task. The use of the terminology partitioning, equivalence, and unit-forming in describing the students' approaches to this quotient context has enabled a more detailed analysis of students' correct and incorrect responses to the task. The successful and unsuccessful responses described here that drew on partitioning, demonstrate that successful partitioning included imagining, making (diagram or mentally) and naming, sometimes non-contiguous, equal parts. Harry could imagine but not make partitions for fifths. Ebony could make but not name her partitioning when the parts were not contiguous. The results of the present study, particularly the infrequency of drawing on equivalence and the infrequency of using a fractions-as-division approach, demonstrate that the underlying concepts of partitioning, equivalence and unit-forming are still relevant in upper primary school.

A strength of the four-three-four model was its categorisation of unit-forming. This gives prominence to the many correct additive aspects of fraction understanding. Courtney's logical diagram (see Figure 6) is an example of Kieren's engineering reports (1988), and highlights that unit forming is a mathematically correct approach to quotient sub-construct fraction tasks that is not always fully executed. Using the concept of unit-forming, teachers and students could elaborate the restructuring of the whole in more sophisticated terms than a "part-whole" understanding in the five-part model allows. Similarly, the reasonableness of the estimates given by Felix and Ben, can be unpacked with greater depth in terms that more closely resemble their reasoning, by the use of the terminology of partitioning and unit forming. In Charles and Nason's study of Grade 3 children completing similar tasks, 12 strategies were reported but further categorisation under the term unit forming may have underscored the concept being drawn upon in their "repeated sizing strategy" (2000, p. 198).

The adoption of the four-three-four model would not be incompatible with the research already conducted using variations of the five-part model in the Kieren/Rational Number Project tradition. However, elaborating the categories of partitioning, equivalence (both already in use by teachers and students) and unit-forming would broaden teachers' pedagogical repertoire of part-whole instruction, as called for in the research literature on fractions. Equivalence has been a term in use with both teachers and students and so the concept has a recognised place in the curriculum. Extending the classroom vocabulary for both teachers and students, to include unit-forming would extend the explanatory power of the model to the students themselves.

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# References


Behr, M. J., Harel, G., Post, T. R., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York, NY: Macmillan.

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-125). New York, NY: Academic Press.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*, 323-341.

Charles, K. & Nason, R. (2000). Young children's partitioning strategies. *Educational Studies in Mathematics, 43*, 191-221.

Clarke, D. M., Roche, A., & Mitchell, A. (2011). One-to-one student interviews provide powerful insights and clear focus for the teaching of fractions in the middle years. In J. Way & J. Bobis (Eds.), *Fractions: Teaching for understanding* (pp. 23-32). Adelaide: The Australian Association of Mathematics Teachers.

Ginsberg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge, United Kingdom: Cambridge University Press.

Keijzer, R., & Terwel, J. (2002). Audrey's acquisition of fractions: A case study into the learning of formal mathematics. *Educational Studies in Mathematics, 47*, 53-73.

Kieren, T. E. (1980). The rational number construct: Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125-149). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. A. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Reston, VA: Lawrence Erlbaum.

Kieren, T. E. (1992). Rational and fractional numbers as mathematical and personal knowledge: Implications for curriculum and instruction. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 323-371). Hillsdale, NJ: Lawrence Erlbaum.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49-84). Hillsdale, NJ: Lawrence Erlbaum.

Kieren, T. E. (1995). Creating spaces for learning fractions. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 31-65). Albany: State University of New York Press.

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah, NJ: Lawrence Erlbaum.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. K. J. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 629-667). Charlotte, NC: Information Age Publishing.

Pantziara, M. & Philippou, G. (2012). Levels of students' "conception" of fractions. *Educational Studies in Mathematics, 79*, 61-83.

Mitchell, A. (2011). *Interpreting students’ explanations of fraction tasks, and their connections to length and area knowledge*. (Unpublished doctoral dissertation). Australian Catholic University. Fitzroy, Australia.

Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26*, 165-190.

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education, 14*, 307-317.