CAS or Pen-and-paper: Factors that Influence Students' Choices

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This paper reports on a study of choices about the use of a computer algebra system (CAS) or pen-and-paper (p&p) by a class of seven Year 11 Mathematical Methods (CAS) students as they completed a calculus worksheet. Factors that influenced students' choices are highlighted by comparing and contrasting the use of CAS and p&p between students. Teacher expectation of students' use of CAS and p&p reveals that, even in a small class, the students' use of CAS and p&p sometimes differed from what was expected. The analysis here indicates that there are a variety of factors that influence students' decisions, including speed of calculation and accuracy of p&p work.

Background

In Victoria, Australia, a Year 11 functions and calculus course, called Mathematical Methods (CAS) (VCAA, 2010) has integrated CAS for more than a decade. Teachers and students working in classrooms with a computer algebra system (CAS) available face choices about the use of CAS or pen-and-paper (p&p) for teaching, and learning, mathematics and for solving problems. Teachers' choices about use of CAS may influence students' opinions about CAS (Artigue & Lagrange, 1997), which is not surprising given that a teacher will institutionalise acceptable techniques in their classroom. A CAS may "gobble up" (Flynn & Asp, 2002) intermediate steps that might normally be part of a p&p solution and when students are solving problems there is a choice to be made about p&p or CAS.

In a study of senior secondary students, Geiger (2008) found that access to CAS enabled students to solve problems beyond their p&p capability. Geiger, Galbraith, Goos, and Renshaw (2002) noted that students would use CAS when symbolic features were seen to expedite processes. Ball and Stacey (2005) found that in a group of five students there were differences in preferences for CAS use; the main use was for speed. One student used CAS to compensate for weak pen-and-paper skills and CAS was also used for checking answers. This study investigated the choices that seven Year 11 students made regarding CAS or p&p use when working on a calculus worksheet. Students' use of CAS for the problems is compared with their teacher's expectation of CAS or p&p use.

Methodology

Participants in this study were a teacher, Peter (all names are pseudonyms), with four years' experience teaching Mathematical Methods (CAS), referred to as MMCAS here, and his Year 11 class in a co-educational school in Victoria, Australia. Peter was invited to be involved in the study as he was known to the first-named researcher, and had previously expressed interest in participating in the study. His class of seven students had been studying MMCAS for 8 months at the time of data collection. Peter supported the use of CAS, and all students (excluding one student, Emily) owned their own CAS.

Data reported here was collected within a wider study examining influence of attitude on use of CAS in a class activity (see Cameron & Ball, 2014). Three research instruments were used to collect data: a worksheet, semi-structured individual interviews (teacher and students), and lesson observation notes. In consultation with Peter, it was decided that the

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focus for the worksheet would be calculus, the topic studied at that time. Students completed the worksheet after they had studied differentiation and anti-differentiation in class. Peter was consulted to ensure that the worksheet was an appropriate length for a 50-minute lesson. Six problems were discussed with Peter: three written by the first-named researcher and three from other sources. Peter requested minor modifications to the wording of original problems, resulting in the final six problems shown in Table 1. Two problems required the use of CAS to solve (problems 4 and 6); the other four problems could be solved using CAS or p&p.

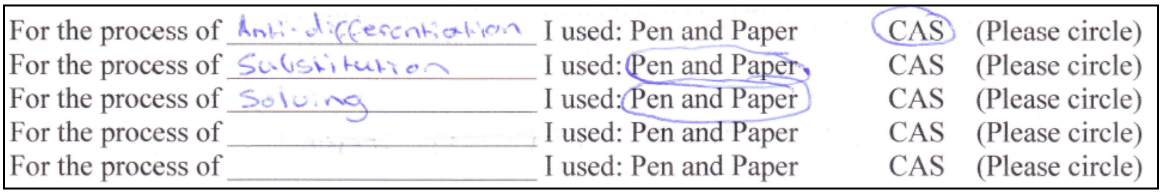
Table 1
Worksheet Problems

Problem	Statement of problem
1	Find $f'(x)$ if $f(x) = 95 + 2.7x + 4x^2 - 0.1x^3$
2	Use first principles to find the derivative of the function $f(x) = x^3 - 4x + 1$
3	An art collector purchased a painting for \$500 from an artist. After being purchased the value of this artist's paintings increase, with respect to time, according to the formula $\frac{dP}{dt} = \frac{3}{2}t^{\frac{3}{2}} + 20t + 100$, where P is the anticipated value of the painting t years after it is purchased. Find an equation that will give the price of the painting at a given time. Consequently, find the price of the painting 4 years after it was purchased.
4	Determine the anti-derivative of the function $f'(x) = \sin(2x + 1) + 7\cos(x)$
5	If $f(x) = k(x - a)(x - b)(x - c)$ what is the simplified form of the derivative $f'(x)$?
6	What are the coordinates of the stationary points for the function $f(x) = (4x + x^{-2})^2$??

Note – Problem 1 Flynn, Berenson and Stacey (2002); Problems 2, 4, & 6 First-named researcher; Problem 3 adapted from Flynn (2001); Problem 5 adapted from Flynn et al. (2002).

A semi-structured interview was conducted with Peter prior to students completing the worksheet. Interview prompts focussed upon his own use of CAS when teaching calculus, expectations of CAS use for these six problems, features of the problems that would contribute to CAS or p&p use, and difficulties that students may encounter.

Students had one 50-minute lesson to complete the worksheet. They worked individually or in groups and Peter interacted with students as they worked. While completing the worksheet, students self-reported (refer to Figure 1) both the process used (e.g. anti-differentiation) and the use of CAS or p&p. If students did not record any steps, but indicated use of CAS or p&p, the solutions were analysed to determine the steps used. The first-named researcher wrote lesson observation notes, without interacting with the participants. The focus of these notes was student/teacher and student/student interactions.



For the process of Anti-differentiation	I used: Pen and Paper	CAS	(Please circle)
For the process of Substitution	I used: Pen and Paper	CAS	(Please circle)
For the process of Solving	I used: Pen and Paper	CAS	(Please circle)
For the process of	I used: Pen and Paper	CAS	(Please circle)
For the process of	I used: Pen and Paper	CAS	(Please circle)

Figure 1. Self-reported CAS or p&p use – Jessica problem 3.

One week after completing the worksheet, individual semi-structured interviews were conducted with six students (Amy was absent). Interview prompts were based upon student responses to the worksheet and survey (not reported in this study). Interview transcripts were examined for comments to provide insight into students' choices of CAS or p&p.

Results and Discussion

Table 2
Methods Expected by Teacher and used by Students (adapted from Cameron & Ball, 2014)

		Peter	Sam		James		Emily		Simon		Jessica		Amy		Kate	
Problem	Key steps	Anticipated Method	Method	Correct	Method	Correct	Method	Correct	Method	Correct	Method	Correct	Method	Correct	Method	Correct
1	Differentiation	P	C	✓	C	✓	P	×	P	✓	C	✓	C	✓	P	✓
2	Expand $f(x+h)$	P	P	✓	P	✓	P	✓	P	✓	P	✓	P	✓	C	✓
	Simplify $\frac{f(x+h)-f(x)}{h}$	P	P	×	P	✓	P	✓	P	✓	P	✓	P	✓	P	✓
	Limit	P	P	×	P	×	P	✓	P	✓	C	✓	P	✓	P	✓
3	Anti-differentiation	P	N	×	N	×	N	×	N	×	C	✓	C	✓	C	✓
	Solve for c	P	N	×	N	×	N	×	N	×	P	✓	C	✓	C	✓
	Substitution	P	N	×	N	×	N	×	N	×	P	✓	C	✓	C	✓
4	Anti-differentiation	C	C	×	N	×	N	×	C	✓	C	✓	C	✓	C	✓
5	Expansion	R	N	×	N	×	N	×	P	✓	R		R		R	
	Differentiation	C	N	×	N	×	N	×	N	×	C	✓	C	✓	C	✓
	Simplification	R	N	×	N	×	N	×	N	×	N	×	N	×	N	×
6	Expansion	C	N	×	N	×	P	✓	N	×	R		R		C	✓
	Differentiation	C	N	×	N	×	N	×	N	×	C	✓	C	✓	C	✓
	Finding x co-ordinates	P	N	×	N	×	N	×	N	×	C	✓	C	×	C	×
	Finding y co-ordinates	P	N	×	N	×	N	×	N	×	C	✓	N	×	N	×

Table 2 provides a summary of the key steps for each problem (identified by the researcher prior to the lesson), the anticipated method (either CAS or p&p) identified by Peter and the methods used by students. Student responses for each step were coded using four categories: C (CAS was used to complete the step); P (p&p was used complete the step); N (No evidence that this step was completed); and R (CAS removed the need for the student to complete the step). Information is provided on correctness; a tick indicated that a step was completed correctly, whilst a cross indicated an incorrect response. A blank indicated that CAS removed the need to complete the step.

Problem 1

Peter expected his students to use p&p methods for problem 1, stating that “it would just be quicker to do it with p&p than type it into the calculator”. He elaborated: “if they [the students] recognise the function and the task looks easy to do, they will do it with p&p”. This may indicate that Peter expected his students to perform routine procedures, in this case differentiation of a function, with p&p. It is not unreasonable to expect Year 11 students to mentally calculate the derivative of a polynomial function and write the result, so it was not surprising that p&p methods were viewed as faster than CAS.

Emily, Simon and Kate used p&p (refer to Table 2) to solve problem 1, whilst Sam, James, Jessica and Amy used CAS; in interviews they gave a range of reasons for their choices. Kate, who solved the problem using p&p, made her choice based on speed, reporting that p&p was quicker than CAS. She stated, “I find myself a faster writer than pressing it into the calculator”, so syntax entry was viewed as time consuming. This concurred with Peter’s notion that students would use p&p methods for problem 1 for speed. Emily (who doesn't own a CAS, but borrowed Peter’s during the lesson) stated that she wouldn’t use CAS to differentiate, due to her lack of familiarity with syntax, as “I don’t really know what the buttons do and how to use [CAS features]”. It was not surprising that a student unfamiliar with CAS chose to use a p&p method for this problem and in this case, the problem was within the expected p&p range of students.

Some students chose CAS for speed, rather than p&p. Sam used CAS for speed and accuracy because “I knew that I’d probably get a more precise answer if I used my calculator and it’s quicker”. James, who also used CAS, valued its speed, stating that his CAS use is “… more as a time saver than anything”. Interestingly, James stated that “if you can solve it [a problem] by p&p then there’s not really a place for it [CAS]”. It seemed that James first considered whether or not he could solve a problem using p&p and only considered CAS if p&p wasn’t viable, citing speed as the determining factor. Even for this one step problem perceptions about speed of CAS and p&p varied between students.

Problem 2

Peter expected students to use p&p methods as this was how he taught differentiation from first principles in class and he anticipated his students would replicate his approach.

> Perhaps an overriding factor for students using p&p or calculator would be the initial way in class that they had been taught. If we started looking at the concept using p&p, this tends to be the way they [the students] respond to a questions and the same goes if we started off looking at a concept on the calculator.

Peter noted that “expanding can be done using the calculator” so for this problem, where calculus was the core mathematical focus, he supported use of CAS for expansion (a lengthy p&p process here).

Sam, James, Emily, Simon and Amy used p&p for problem 2 (see Table 2), so five out of seven students used the approach demonstrated (and expected) by Peter. Although Sam noted the speed of CAS for problem 1, she chose p&p for problem 2 as: “we would always do it [i.e., differentiation from first principles] with p&p”. Jessica and Kate used a combination of CAS and p&p to solve problem 2, while Amy (who worked with them) chose p&p only. The use of CAS and p&p seemed to be a personal choice, even for the students who worked in groups.

Students commented that they wouldn't always replicate what the teacher demonstrated and that speed of calculation played a factor in their decisions. Even though James solved this problem using p&p, he commented that if a problem "was going to take forever to do by p&p then it's going to really help if you can do it by the calculator". Whilst students appreciated the use of CAS to quickly complete problems, it is possible that because Peter had taught them to perform differentiation from first principles using p&p that they didn't realise, or perhaps thought it was unacceptable, to use CAS in this problem.

Jessica used p&p for substitution and CAS to find a limit commenting that p&p was needed to show working expected by her teacher.

> You can't just put it in [to the calculator] and go well there's your answer, because then they'll [the teacher] go, "well where's your working out?"

Jessica determined what was acceptable in mathematics classes by analysing Peter's teaching "I think the whole point of it [Peter's teaching], is to learn how to do it by hand [i.e., p&p] so then you understand where it's coming from". This insightful comment showed that students do determine what practices are institutionalised from their teacher.

From Table 2 we can see that Peter and Kate had the same preference for use of p&p for problems 1 and 2, except for expansion where Kate used CAS. In the interview, Peter identified expansion as a potential area for CAS use, despite expecting p&p, so their choices aligned here. Again we note the variety of choices, but in this problem student choices aligned well with Peter's expectation. This could be due to the nature of the problem, where Peter favoured p&p use to "show the process of first principles".

Problem 3

Peter expected use of p&p for problem 3. He noted: "CAS could be used just as easily to find the integral, particularly for working out the fraction part, which is where the students are likely to make the mistake". This suggested that CAS use may minimise the potential for errors that might appear in p&p working for this problem.

Emily, Sam and James asked Peter for assistance with problem 3, but neither they, nor Simon, gave a solution. Amy and Kate used CAS, whilst Jessica used a combination of CAS and p&p (refer to Table 2). Although problem 3 was not addressed specifically in the interview, Jessica stated that she completed "the easy stuff [on the worksheet] by hand" and used CAS for "the more complex things". The anti-differentiation required for problem 3 may have been viewed as complex by Jessica, whereas calculating the coefficient of integration and substitution could be perceived as easy tasks. It is possible that a belief that p&p was required to show working (discussed in problem 2) for multistep problems influenced Jessica's decision to use both CAS and p&p. Kate and Amy, who worked together, used CAS to solve problem 3 so in this case collaboration may have resulted in similar CAS use.

Problem 4

Peter stated, "Students would not know how to find the anti-derivative of this function [i.e., $f'(x) = \sin(2x + 1) + 7\cos(x)$] so I'd expect them to use the calculator". Students were anticipated to encounter difficulty here as "they've never seen those functions [in problems 4 and 6]" however he noted that "students that generally persevere at questions are more likely to use their calculators than those that are less confident, or give up easily". This suggested that confident students would persevere with solving problem 4 if they had CAS available. Although problem 4 was designed to lie outside the range of students' p&p skills (see Cameron & Ball, 2014), Peter believed that where "the problem involves functions they [i.e., the students] are uncertain about, they may use the calculator " to perform the anti-differentiation. In this case a CAS extends the range of problems that students can solve.

Sam, Simon, Jessica, Amy and Kate attempted problem 4 using CAS and four of them were correct. James and Emily did not provide working for problem 4, with James stating that he did not solve the problem as "the whole sine and cosine … confuses me". Conversely, Emily felt that she could have completed the problem "if I knew more about the CAS "; so one student viewed the mathematics as problematic, while the other had technical concerns. Both students sought assistance from Peter, but did not end up providing a solution. Although Peter believed that access to CAS would enable students to solve this problem, this was not the case for these two students. This highlights that the mere presence of CAS does not guarantee that students can use it to solve problems.

Jessica used CAS to solve problem 4 as the unfamiliar function perturbed her, stating, "It [the function] threw me off. I didn't think I'd be able to do it by hand, so I thought I'll just do it on the calculator and see what it says". Simon also chose to use CAS and suggested to the researcher that this problem was outside his p&p range stating "I don't think I'd done these problems before so I'd just use the calculator". Students can use CAS to solve problems outside their p&p skills when they have an understanding about what is required to solve problems of a particular type, which in this case involved anti-differentiation. Kate used her knowledge of inbuilt features of CAS, "you can do it for any problem with that part of CAS". It is possible that perseverance was a key requirement to solve this problem, as all students stated they were unfamiliar with the function, but as Peter expected, they applied their knowledge of CAS to solve the problem.

Problem 5

Peter expected students to use p&p to solve problem 5. However, he recognised "CAS would expand it [i.e., the function] for you and then find the derivative too" suggesting that a CAS based approach was possible. Peter did not identify that CAS could "gobble up" (Flynn & Asp, 2002) the step of expansion here to enable differentiation to be performed in one step. Peter believed that students might experience difficulty "because it [i.e., the original function] is factorised ", an unfamiliar problem format for his students.

Simon, Jessica, Amy and Kate were the only students who attempted problem 5. Simon expanded the function using p&p, but did not solve the problem. Simon explained that when he encounters a difficult problem he would most likely skip the problem, rather than persevere. This was illustrated on the worksheet where Simon started solving problem 5, (expanding the expression correctly) but struck difficulties in continuing to solve the problem, writing on his worksheet "I have no idea what I am doing". This provides an example to support Peter's contention that those who use p&p are less likely to persevere with difficult problems than CAS users. In this case a student using CAS could avoid the complexities encountered by Simon by finding a derivative in one step.

Sam, James and Emily did not complete problem 5, with James asking, "Have we ever seen questions like that?" Although some students may not attempt problems with unfamiliar functions, they do not necessarily form a barrier when students have CAS and know the mathematics, as evidenced above in problem 4.

Jessica, Kate and Amy used CAS for differentiation; CAS "gobbled up" (Flynn & Asp, 2001) the step of expansion here. In the interview Kate discussed her use of CAS to explore unfamiliar problems, "I just write it down … if it looks a bit weird then I'll stick it on the calculator and see what happens". This, along with Jessica's explanation of her approach to problem 4, suggested that unfamiliar functions can be less problematic for students when they understand what is required and can use CAS for calculation and exploration.

Problem 6

Peter expected students to use CAS for problem 6. CAS use was required here as the problem was outside students' range of p&p skills, with Peter noting that this problem would be within the range of p&p skills of Year 12 students, rather than Year 11: "I suppose in Year 12 you could use the rule to find the derivative; some p&p work could be used to find when the derivative is equal to zero".

Three students (Jessica, Amy and Kate) attempted problem 6 using CAS and Emily used p&p. Emily completed the problem up to the step where the use of CAS was required. Emily was not a CAS user, so it is not surprising that she did not use CAS. Kate used CAS to expand the brackets, as she stated that she does not like brackets. She then used the expanded form and "stuck it into the calculator because it knows how to do it for me". This contrasts with her use of CAS in problem 5 where she did not expand the brackets before using CAS to differentiate; students can make decisions problem-by-problem and also step-by-step. Jessica was able to correctly solve this problem using CAS, but required the support of Peter and her peers with syntax. Her first inclination when solving was to consider whether or not p&p was a viable option, "I thought I could do it by hand, but I didn't really know … I have no idea how to do it by hand". She then used CAS, stating "I threw in the original function to find the derivative on it and once I did that, I'm pretty sure I did all the steps on CAS". The CAS use by Kate and Jessica was quite different even though they were working together. As Jessica chose to differentiate the function using the format of the function provided on the worksheet, this removed the need to expand the function (a step where Kate used CAS, prior to differentiating).

Conclusion

It might be expected that in a class of seven students there would be some consistency of CAS use as there is the opportunity for the teacher to spend considerable time with individuals discussing use of CAS and p&p. In this study, we found that even with a small class of seven Year 11 students there were considerable differences in the choices that students made about CAS or p&p for solving common problems. Students' choices sometimes aligned with the expectations of the teacher, but this was not always the case, with students using CAS more than the teacher anticipated. This highlights the complexity of a CAS-active classroom, where students are solving problems in a variety of ways, not just the way that has been demonstrated by their teacher. Artigue (2002) commented on an "explosion of possible techniques" (p. 260) in a CAS-active classroom.

Students' made problem-by-problem and also step-by-step decisions about the use of CAS or p&p. This suggested that students evaluated problems based on the perceived personal benefits of CAS or p&p throughout a problem. Speed and accuracy appeared to be two key factors influential in students' choices. Different students cited either CAS or p&p as being faster for specific tasks, so this shows that speed is very dependent on facility with p&p skills as well as technical ability for using CAS. Although students based choice of CAS or p&p on speed and accuracy, these were not the only factors influencing their choices. Some students noted a preference to use the approach demonstrated by their teacher. Where Peter identified that his students could use CAS, the students tended to use CAS. This showed that Peter had insight into range of the students' p&p skills and also that CAS enables students to solve problems outside the range of their p&p skills; some students were willing to tackle unfamiliar problems with CAS. Where students have the conceptual understanding of the key mathematical principle in a problem (e.g., differentiation) they are able to use CAS to correctly answer the problem. This may have implications for teaching, as teachers can introduce more difficult problems using CAS; scaffolding students' conceptual understanding.

The findings reported in this paper highlight a diverse use of CAS and p&p in a small class and the complex way in which students make decisions regarding the use of CAS and/or p&p. The students in this study are undertaking a subject with CAS-assumed examinations at Year 12 level, so decisions about the use of CAS or p&p for speed, accuracy or to supplement their p&p skills are important ones to consider at Year 11.

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