Examining PCK in a Senior Secondary Mathematics Lesson

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Teacher knowledge, including Pedagogical Content Knowledge (PCK), continues to be the focus of research, with the general consensus being that PCK impacts upon teaching and learning. Much of the current research has focused on pre-service teachers and practicing primary teachers, with few studies focused on studying senior secondary teachers' PCK. Even rarer are studies which examine PCK from students' perspectives. This study investigates the nature of PCK as experienced in a lesson by a class of senior secondary mathematics students. The findings indicated that there were a number of PCK elements incorporated in the lesson and that these were noticed by the students.

Introduction

Effective mathematics teaching requires knowledge of mathematical content, knowledge of students' thinking, and knowledge of how to represent the content so that it makes sense to others (Hill, Ball, & Schilling, 2008). There has been substantial research into identifying and characterising the constituent parts of teacher knowledge including pedagogical content knowledge (PCK) (e.g., Chick, Baker, Pham, & Cheng, 2006; Krauss et al., 2008). PCK is knowledge about the way subject matter is transformed from the knowledge held by the teacher into the content of instruction. Shulman described PCK as an intricate blend of content and pedagogy that encompasses all that is needed to teach a subject or topic in a way that makes it comprehensible to others (1986).

It is generally accepted within the mathematics education community that PCK impacts upon teaching and learning (e.g., Ball, Lubienski, & Mewborn, 2001; Krauss et al., 2008). Most research into PCK has tended to focus on pre-service and practicing teachers in the context of primary mathematics (e.g., Rowland, Huckstep, & Thwaites, 2005) but comparatively few studies have examined PCK for teaching secondary mathematics (Matthews, 2013). Furthermore there has been little research into how multiple sources of evidence of PCK may inform us about the nature of this aspect of teacher knowledge. This paper focuses on investigating PCK within the context of a senior mathematics classroom by exploring the following research questions: What aspects of PCK does a teacher of senior secondary mathematics demonstrate in a lesson? To what extent are these aspects perceived by students as being helpful to their learning?

Review of Literature

Several frameworks have been developed to conceptualise the multi-faceted nature of mathematics teacher knowledge, including PCK (e.g., Ball, Thames, & Phelps, 2008). The domain map for Mathematics Knowledge for Teaching developed by Ball and colleagues delineates the boundaries of different categories of teacher knowledge and is widely cited in the literature on mathematics teacher knowledge. Some scholars however, have questioned if it is possible, particularly in practice, to precisely demarcate subject matter knowledge and pedagogical content knowledge in the context of teaching (e.g., Marks, 1990).

The framework for analysing PCK in mathematics teaching developed by Chick and her colleagues (e.g., Chick et al., 2006) gives a detailed inventory describing evidence for

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identifying key components of PCK within three broad categories. These include “clearly PCK”, “content knowledge in a pedagogy context” and “pedagogical knowledge in a content context” and represent the varying degrees to which content and pedagogy are intertwined without rigid delineation. Space prevents the inclusion of the entire framework but brief descriptions of some key PCK elements are given in Table 1. Some PCK elements relate to teachers’ knowledge of students’ existing conceptions about mathematical concepts, and others relate to knowledge of how to transform mathematics knowledge to facilitate learning (e.g., *Deconstructing Content to Key Components*).

Table 1.
A Framework for Pedagogical Content Knowledge. (Based on the framework in Chick and colleagues, 2006)

PCK Category	Evident when the teacher …
Clearly PCK	
Teaching strategies – general	Discusses or uses general strategies or approaches for teaching a mathematical skill or concept.
Student thinking	Discusses or responds to possible students’ ways of thinking about a concept, or recognises typical levels of understanding.
Student Thinking – Misconceptions	Discusses or addresses typical/likely student misconceptions about mathematics concepts.
Cognitive Demands of Task	Identifies aspects of the task that affects its complexity.
Knowledge of Examples	Uses an example that highlights a concept or procedure
Content Knowledge in a Pedagogical Context	
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept
Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident)
Methods of Solution	Demonstrates a method for solving a mathematical problem
Pedagogical Knowledge in a Content Context	
Assessment Approaches	Discusses or designs tasks, activities or interactions that assess learning outcomes

The framework enables close inspection of teachers’ PCK by applying it to data such as interview transcripts, written responses to items about teaching and learning mathematics content, and actual teaching episodes (Chick et al., 2006). As such, it provides an appropriate theoretical framework for the study discussed in this paper. Overlap among facets is also plausible. For instance, in discussing a method for solving a problem a teacher may show *Procedural Knowledge* as well as demonstrate evidence of having *Deconstructed Content into Key Components* (Chick et al., 2006).

Teachers often use examples in their classroom in order to illustrate key principles. The Chick et al. framework highlights examples as a facet of PCK. Other scholars (e.g., Krauss et al., 2008; Zodik & Zaslavsky, 2008) identify examples as integral to learning and teaching as they represent powerful learning opportunities for students. Examples refer to a particular case from a larger class from which one can reason and generalize (Zodik &

Zaslavsky, 2008). In fact it is not the specific example or even the answer that is the most important but the general principle illuminated by the example (Chick, 2009). In senior secondary mathematics, examples form a large part of classroom instruction; they were a feature of the lesson studied for this paper.

Methodology

This paper uses data from a larger study and explores aspects of PCK from the perspectives of a teacher, his students, and the researcher by examining one lesson in detail. A Grade 11/12 Mathematics Methods class from a large metropolitan secondary college in Tasmania participated. Mathematics Methods is one of the most demanding mathematics courses offered in Tasmanian schools. It is assessed by internal unit tests and a final external examination; the major topics are function study, differential and integral calculus, and statistics. Data presented in this paper focus on two examples involving optimisation, a practical application of differential calculus.

The participants were Mr Jones, a teacher of Mathematics Methods during 2014, and his 16-18-year-old students. Mr Jones has been a teacher of secondary mathematics for 25 years, including seven years at the senior secondary level. Of the 18 students enrolled in Mr Jones' class, 15 (six females and nine males) contributed data by participating in the focus group interview and/or completing a short answer survey. Teacher and student names are pseudonyms in this paper.

Data were collected during one lesson on applications of differential calculus. The 100-minute lesson focused on optimisation. The lesson was observed, video-recorded, and partially transcribed. At the end of the lesson a short-answer survey was completed by participating students, eliciting responses about the types of explanations and strategies that assisted them with their learning. A semi-structured audio-recorded focus group interview was also conducted, with five participants for 15 minutes, where students were asked to comment on aspects of the lesson that were particularly helpful for their learning. Mr Jones also participated in a 20 minute interview after the lesson. These approaches yielded three data sources for examining PCK: the researcher's notes on the lesson and accompanying video, student perspectives, and the teacher interview.

Data from the lesson observation, interviews, and surveys were transcribed and aligned with one of several teaching events in the lesson (e.g., the presentation of a solution to a particular example, the process of differentiating a function). Teacher actions during the class, student comments, and teacher interview comments were examined to see if they matched any of the PCK lesson descriptors (see right hand column of Table 1). The transcripts were read independently by each author, to ensure consistency. The multiple data sources linked to each lesson event were then examined for commonalities in PCK type. Of particular interest was the extent to which the multiple sources of evidence of PCK corroborated each other and what insights this provided about the nature of PCK.

Results and Discussion

This section begins with a brief overview of the lesson, followed by the presentation of results from multiple data sources linked to particular events in the lesson. Results are arranged in sections based on these lesson events. The teaching events described in each section have been classified using the categories from the Chick et al. framework for PCK (see Table 1). In some cases categories are grouped or paired (e.g., knowledge of examples/knowledge of assessment) to reflect situations where different aspects of PCK were clearly intertwined.

Lesson overview

Mr Jones began the lesson by providing an overview of the remaining content to be covered in differential calculus before the end of unit test in the following week. He forewarned the students that it would be "a frantic lesson" as this was the last lesson allocated to applications of differential calculus.

The instructional phase of the lesson focused on two optimisation (also called maximum and minimum) problems: "the bushwalker problem" (see Figure 1) and "the distance problem" (see Figure 2). Optimisation problems are the key focus of applications of differential calculus in the Mathematics Methods course, and involve practical situations in which students are required to minimize or maximize a quantity. The bushwalker and the distance problems involved obtaining a particular function and calculating its minimum value using calculus. Mr Jones demonstrated each example on the whiteboard starting with the bushwalker problem, modelling the written mathematical steps and explaining the process. At the conclusion of the presentation of the two examples students worked on similar problems.

A bushwalker can walk at 5 km/h through clear land and 3 km/h through bushland. If she has to get from point A to point B following a route indicated in the diagram on the right, find the value of x so that the route is covered in the minimum time. (Note: time = $\frac{\text{distance}}{\text{speed}}$)

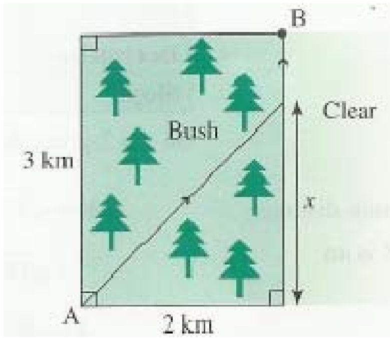

Figure 1. The bushwalker problem (Hodgson, 2013)

Find the minimum distance from the straight line with equation $y = x - 4$ to the point (1,1).

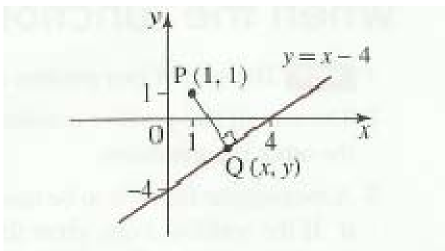

Figure 2. The distance example (Hodgson, 2013)

Lesson Events

Developing the functions. Mr Jones highlighted the development of the functions in each example, which he identified in the lesson as a key challenge in solving optimisation problems (*Cognitive Demand of Task*). He carefully unpacked the examples and drew attention to variables that would be crucial for the development of the appropriate function

(Deconstructing Mathematics into Key Components). For example, in the bushwalker problem, Mr Jones emphasised that distance x (see Figure 1) may not be labelled in an exam situation "and you would need to come up with it yourself; that this is a crucial missing distance" *(Deconstructing Mathematics into Key Components)*. Similarly, for the distance problem Mr Jones emphasised the idea that the point Q(x, y) must be expressed in terms of x only, that is (x, x-4), in order to develop a distance function with respect to x (*Cognitive Demand of Task*). Later in the post-lesson interview Mr Jones commented that "often some kids don't realise when and why they need to express one variable in terms of another even if it seems quite obvious" (*Knowledge of Students Thinking – Misconceptions*). While the students did not comment specifically on obtaining the functions, there was some evidence that they valued the way Mr Jones emphasised critical aspects of the examples: "It was helpful the way he used the board and some diagrams to show how to do certain things" (James; survey). Similarly, Alan commented during the focus-group interview: "His diagrams were, like, clearly set out … to show the different things; it makes it clearer in your mind".

Selection of examples. Mr Jones introduced the lesson by explaining to students why he had specifically chosen the bushwalker and the distance problems (*Knowledge of Examples*)

> We've spent a lot of time on area and volume problems but I don't want you to think that "that's it" for applications of calculus. It's probably the focus of my two problems today is to show you some of those other applications. All the function unknown ones we've done so far have been area and volume ones but there are other types that could pop up in the exam. (in-class comment)

He also highlighted the distance problem as a typical question for the non-calculator section of the examination, given that its "nice neat" answer could be obtained without the aid of the CAS calculator (*Knowledge of Examples/Knowledge of Assessment*).

> Remember how I said that "function unknown" problems are more calculator than non-calculator? Well, have we needed our calculator for this one yet? No we haven't, so that's why I wanted to do this one today, because it's the classic example of one that could be in the non-calculator section of the exam … because a lot of the other ones we've done have applied to realistic situations which don't end up being nice neat figures like the square root of eight, they could end up being something like 2.9564323… (in-class comment)

Mr Jones elaborated further on his choice of examples in the post-lesson interview.

> I wanted to give them an example of one that didn't require use of the calculator at all, because the nature of our course is that there is a calculator and a non-calculator section of the exam. So that was an important example because, I mean, I don't want to get too caught up in the exam, but in reality I have to be faithful to anticipating what sort of questions come up. (interview)

The interview data provided some further insight into Mr Jones' enacted *Knowledge of Examples/Knowledge of Assessment* in terms of the impact a high stakes examination has on teaching decisions, including the selection of examples. In one of his interview responses, Mr Jones expressed a tension between teaching the mathematics per se and teaching to the examination. The source of this tension was not discussed in the interview. While several students commented on the usefulness of the chosen examples, only one response mentioned the examination explicitly.

> The most helpful thing was when we went through the different types of questions on minimums and maximums. It helped me learn what types of questions I can expect on exams and tests. (Elizabeth; survey)

Highlighting a general method of solution. A common teaching strategy often used by Mr Jones was to encourage students to recognise the key processes involved in solving optimisation problems by directing their thinking through questioning (*Teaching Strategies/Deconstructing Knowledge into Key Components*).

> Mr Jones: Because we are looking for a minimum, what are we going to have to find eventually somewhere in this question? [class response: the derivative]. Yes, and then we make the derivative equal to? [class response: zero]. Good and then solve for? [Class response: x]. Good. And that should be an automatic reaction when we see the word "minimum" or "maximum"; [it] should be our trigger to say "right, that's our process". (*Teaching Strategies/Deconstructing Content into Key Components/Method of Solution).*

Further evidence of the teacher's knowledge of deconstructing ideas was apparent during the post lesson interview, particularly in relation to Mr Jones' emphasis on identifying the key steps involved in solving optimisation problems (*Method of Solution/Deconstructing Content to Key Components*).

> Mr Jones: This year I've probably identified key words in the question and making sure that they understand what the process is. When you are teaching a topic like that, these are the key steps you've got to do. So give them the framework I suppose and hopefully they can apply that framework to understanding other situations. This year I've really concentrated on that.

One student's response appeared to align closely with Mr Jones' focus on deconstructing the key components of the problem to provide a general method of solution: "The examples on the board helped me recognise when to do what (e.g., "when to look for a minimum or maximum and making $d'(x) = 0$"). (Lucy; survey). Other students tended to comment on specific aspects of the examples, such as the use of the distance formula in the distance example: "The example on the board finding minimum/maximum distance between points using the distance formula".

Close examination of evidence of PCK observed by the researcher and discussed by Mr Jones suggested that the deconstruction of the mathematics was limited to standard differentiation approaches. For example, a visual representation of the minimum values for each example could have been obtained using the CAS calculator, even though the examples had been selected specifically to be solved without the aid of technology. It would have been interesting to find out Mr Jones' reasons for omitting the graphs of the functions in each example, however this was not sought during data collection.

Algebraic skills. Mr Jones guided the class through the differentiation of the respective functions for each example step-by-step (*Procedural Knowledge*). Again he involved the students by asking strategic questions as demonstrated in the following lesson transcript based on the distance problem (*Procedural Knowledge/Teaching Strategies*). Note that Mr Jones referred to the surd and power forms of the distance function as $d(x) = \sqrt{2x^2 - 12x + 26}$ and $d(x) = (2x^2 - 12x + 26)^{\frac{1}{2}}$ respectively.

> Mr Jones: Before I can get the derivative of this function [points to the surd form of $d(x)$]. What form do I need to put it in Jessie? It's in surd form at the moment.
>
> Jessie: Power form.
>
> Mr Jones: That's right power form [rewrites the function $d(x)$ in power form]. Ok so to find the derivative $d'(x)$, what comes out the front Angela?
>
> Angela: Umm a half
>
> Mr Jones: That's right a half and then we multiply by what Ryan?
>
> Ryan: Oh umm the derivative of the bracket.

Mr Jones: Yes. The derivative of the bracket which is (4x – 12) and then multiplied by … what's the last bit Harry?

Harry: Umm the brackets to the power of negative a half.

Mr Jones: Yes good [completes the differentiation process to yield:

$d'(x) = \frac{1}{2}$ x (4x-12) x$(2x^2 - 12x + 26)^{-\frac{1}{2}}$]. Are we all right with that,? There's your process. OK, so we've got 4x – 12 in the numerator, and in the denominator we've got the 2. Remember that your negative-a-half [points to the expression $(2x^2 - 12x + 26)^{-\frac{1}{2}}$] moves to the denominator so we have $d'(x) = \frac{(4x-12)}{2\sqrt{2x^2-12x+26}}$. Now tell me if I've done too many steps at once there? OK, so the (4x – 2) and the 2 have stayed where they were and the bracket to the negative-a-half has gone underneath. Then I've just changed it from the power of a half to the square root.

There was also evidence that the students' noticed and valued Mr Jones explicit questioning about mathematical processes as suggested in Christopher's survey response: "The whiteboard examples were the most helpful. He engaged everyone in the class and you had to pay attention as he asked people for different values and numbers" (*Teaching Strategies*).

A similarly explicit approach was used to solve the bushwalker example. During the post-lesson interview Mr Jones also commented on his ongoing focus on consolidating students' algebraic skills (*Procedural Knowledge/Teaching Strategy*).

Even at this (almost) final lesson on differential calculus there might be gaps in their basic skills… Umm you might have noticed I asked a number of times what do we do when we have a square root, they know by now, they've got to convert it to a power, drum, drum, drum. Umm I don't know if that's effective or not but yeah. (Mr Jones; interview).

During the focus-group interview several students highlighted Mr Jones' step-by-step procedures as being particularly helpful (*Procedural Knowledge*).

Angela: When he did the steps on the board I could just look back to see how to do it.

Tom: Yes, helpful for differentiating square roots with more than one thing in it, like when there was x squared plus 2x and then the square root of all that and you had to differentiate it.

Danny: If you've copied one of his [Mr Jones'] examples down and you're at home and you get sort of one like it you can sort of match things up. You try and follow the same procedure with the different numbers and that can help you through.

Three students also commented that the step-by-step explanations helped them to learn to solve the "harder" optimisation problems: "His [Mr Jones'] step-by-step examples were very useful for the harder questions. It helped me to learn how to do the harder function unknown questions" (Dylan, survey). Similarly Emma commented that "The worked examples on the board with consistent pausing to further explain steps helped me gain an understanding of the work" (survey).

Conclusions

The results depict aspects of a senior secondary mathematics teacher's PCK based on evidence from three main data sources. The Chick et al. (2006) PCK framework provided a set of filters through which to examine elements of PCK that were observed or noticed and discussed by the teacher and students. The most prominent PCK categories identified in the data from the lesson were Teaching Strategies, Student Thinking, Cognitive Demand of Task, Knowledge of Examples, Method of Solution, Procedural Knowledge, and Deconstructing Content into Key Components (see Table 1). Broadly speaking, some categories tended to relate to an awareness of students' thinking about mathematical skills and concepts, and others focused on the mathematics itself and how it is transformed to make it comprehensible to others. Furthermore, these categories were often inextricably linked. For example, Mr Jones demonstrated both Procedural Knowledge and Knowledge of Student Thinking – Misconceptions during the differentiation of the distance function d(x) in that he was attentive to potential difficulties students may experience at each stage of the solution process.

The multiple sources of data tended to corroborate each other where particular lesson events were observed by the researcher and also discussed by Mr Jones and the students. For example, Mr Jones' step-by-step approach to solving the problems was particularly noticed and valued by the students. Similarly the use of questions to guide students' thinking about skills or ideas during the process of solving the problems was clearly observed by the researcher and mentioned by some students. Although the study is limited in that it was one account of a senior secondary mathematics lesson, it does provide insight into the nature of Mr Jones' PCK and its impact on students. Further studies that investigate PCK across different senior secondary mathematics topics and with different senior mathematics teachers would also add to the limited research in this area, and reveal if there are common aspects of PCK evident in teachers' work at this level.

References

Ball, D., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (Vol. 4, pp. 433-456). New York: Macmillan.

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education*, *59*(5), 389-407.

Chick, H. (2009). Choice and use of examples as a window on mathematical knowledge for teaching. *For the Learning of Mathematics*, *29*(3), 26-30.

Chick, H., Baker, M., Pham, T., & Cheng, H. (2006). Aspects of teachers' pedagogical content knowledge for decimals. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Proceedings of 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Prague: PME.

Hill, H., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372-400. doi: 10.2307/40539304

Hodgson, B. (2013). *Maths quest 12 mathematical methods CAS* (2nd ed.). Milton, Qld: John Wiley & Sons Australia.

Krauss, S., Brunner, M., Kunter, M., Baumert, J., Blum, W., Neubrand, M., & Jordan, A. (2008). Pedagogical content knowledge and content knowledge of secondary mathematics teachers. *Journal of Educational Psychology*, *100*(3), 716-725.

Marks, R. (1990). Pedagogical content knowledge: from a mathematical case to a modified conception. *Journal of Teacher Education*, *41*(3), 3-11.

Matthews, M. E. (2013). The influence of the pedagogical content knowledge framework on research in mathematics education: A review across grade bands. J*ournal of Education*, *193*(3), 29-37.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The Knowledge Quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255-281.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14. doi: 10.2307/1175860

Zodik, I., & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics*, *69*(2), 165-182. doi: 10.1007/s10649-008-9140-6