Exploring mathematical fluency: teachers' conceptions and descriptions of students

Katherin Cartwright
The University of Sydney
<Kcar8223@uni.sydney.edu.au>

Interviews can provide a window into what teachers think. This paper reports on findings from an exploratory study into teachers' conceptions of mathematical fluency. Focusing on stage two of the study, I discuss 3 of 17 teachers interviewed, analysing their conceptions and descriptions of students. Teachers spoke of students having understanding and multiple ways of thinking, and their ability to work through errors and transfer knowledge. This suggested fluency in mathematics as more than carrying out procedures. Viewing fluency as the result of having conceptual understanding, strategic competence and adaptive reasoning, would make it synonymous with mathematical proficiency.

Mathematical fluency can be defined and interpreted in many ways. The literature surrounding mathematics generally defines fluency as procedural or computational fluency (Kilpatrick, Swafford, & Findell, 2001; McClure, 2014; National Council of Teachers of Mathematics, 2014; Russell, 2000). The majority of research studies conducted focusing on fluency in mathematics use a definition of procedural fluency similar to the definition in Kilpatrick et al.'s (2001) conceptualisation of mathematical proficiency (Bass, 2003; Graven, Stott, Nieuwoudt, Laubscher, & Dreyer, 2012; Stott, 2013). Kilpatrick et al.'s (2001) model shows procedural fluency as one strand of proficiency along with adaptive reasoning, strategic competence, conceptual understanding and productive disposition. When looking at an Australian context and conceptualisation of fluency, Watson and Sullivan's (2008) definition as *mathematical* fluency is a broader term and is used as the definition of fluency described in the Australian Curriculum: Mathematics (ACARA, 2010).

Little research exists about practicing primary teachers' conceptions of mathematical fluency and how they describe mathematically fluent students. Research mainly centres on students' procedural fluency and its relationship to conceptual knowledge or on testing and improving [the speed] of their procedural fluency (Arroyo, Royer, & Woolf, 2011; Bauer, 2013; Ramos-Christian, Schleser, & Varn, 2008). Even though the term procedural fluency may describe other features of fluency, the use of the term *procedural* to describe fluency results in teachers interpreting procedural fluency at face value. This view of fluency can lead to a disconnect between the teaching of the procedure (the what), and the understanding of the concept (the why), of mathematics which need to be learned in unison (McClure, 2014). According to Watson and Sullivan (2008), fluency involves carrying out procedures flexibly, accurately, efficiently and appropriately *as well as having* "factual knowledge and concepts that come to mind readily" (p. 112). Their definition combines both the ability to readily perform the mechanics of mathematics (procedures) and the understanding of the mathematics being learned (concepts) providing a wider scope to focus on various aspects of fluency. Further research on mathematical fluency is required to provide insight into the complex nature of fluency beyond a mere process of memorising facts and quick recall.

The focus of this study was on exploring teachers' conceptions of mathematical fluency. For this research, the term *conceptions* was taken to be inclusive of both a

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teacher's beliefs and knowledge that they hold of the concept (Beswick, 2012; Thompson, 1992). Teachers' conceptions are highly dependent on their personal beliefs formed through life and educational experiences. Conceptions are also influenced by teachers' knowledge of mathematics, and of how mathematics is learned (Borg, 2003; Melketo, 2012) as seen in Figure 1. This model formed the theoretical framework for studying teacher conceptions.

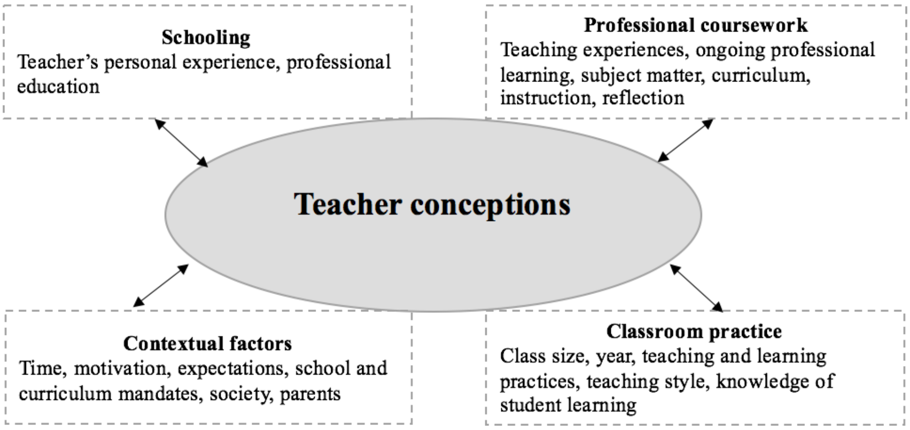

Figure 1. Teacher conceptions framework, synthesised from Borg (2003) and Melketo (2012).

The findings discussed in this paper aim to explore how teachers translate definitions from research of mathematical fluency by answering the following research questions: What knowledge and beliefs do primary teachers have about mathematical fluency? And, how do they describe mathematically fluent students?

Methodology

This qualitative study was designed to be exploratory in nature, aiming to gain a deeper understanding of teacher knowledge and beliefs by studying real-world settings inductively to generate rich narrative descriptions (Patton, 2002). An interpretive approach to research was taken during this study. A strength of using this approach is its emphasis on examining texts, such as written words, or conversations (Neuman, 2003). When interpreting a concept, people's beliefs, values and perceptions provide meaning and influence knowledge. This approach assisted in building rich local understandings of the beliefs and experiences of teachers and of the cultures of classrooms (Taylor & Medina, 2013).

The study was divided into two stages of data collection, involving a questionnaire and semi-structured interviews. Stage one, the questionnaire, involved the random selection of 300 NSW primary schools inclusive of both city, rural and remote locations. Teachers self-nominated to complete the survey that was sent to their school. The online questionnaire included background information questions, Likert-type items (dimensions of mathematics) and two open-ended response questions. The questionnaire was completed by 42 participants. At the completion of the questionnaire, the participants could remain anonymous or indicate their interest in participating in a follow up interview as stage two of the study. Of the questionnaire participants (n=42) 17 teachers agreed to be interviewed. These 17 teachers were representative of all teaching grades, Kindergarten (K)-Year 2 (n=7), Year 3-4 (n=5), Year 5-6 (n=2), K-6 (n=2) and one not specified.

Interviews can capture rich detail of the experiences and perspectives of those being studied (Lincoln & Guba, 1985). Semi-structured interviews allowed additional questions to be included based on general patterns of responses from the questionnaire data analysed. Each interview was audio recorded and later transcribed, using a unique code for each teacher. Questions focused on definitions, descriptions, connections and features of mathematical fluency and examples of students displaying these characteristics. The findings from the analysis of the questions regarding the descriptions and examples [of students' thinking] that teachers provided are the main focus of this paper.

Data Analysis

Thematic analysis provided an illustrative and exploratory orientation to the study (Guest, MacQueen, & Namey, 2012). Using both inductive and deductive coding as different layers of analysis allow codes to flow from the principles that underpin the research, and the specific questions one seeks to answer (Joffe & Yardley, 2004). Multiple opportunities emerged to analyse the data gathered. Similar to Clarke and Braun's (2017) thematic process phases, the analysis was undertaken in 6 steps: (1) questionnaire data summarised, (2) questionnaire data analysed, (3) identification of codes from questionnaire data, (4) interview questions refined based on questionnaire data, (5) interview data analysed for emerging themes and mapped to questionnaire codes, (6) searching for themes in the questionnaire and interview data mapped to the research questions, the Likert item dimensions of mathematics, and the teacher conceptions framework (TCF). Aspects of phase five and six of this thematic process are discussed below.

Results and Discussion

Initial analysis was conducted by highlighting key features of mathematical fluency that emerged within and across participant responses. Statements and quotes that directly related to the research questions, the questionnaire codes, the Likert item dimensions and the TCF were highlighted and added to spreadsheets for further analysis. In this paper, I report on the mapping to the questionnaire codes and the TCF.

In the open-ended response section of the questionnaire, teachers were asked to write three words to describe mathematical fluency and a short definition of mathematical fluency. The leading four words listed were: efficient, flexible, understanding and strategies. These features link closely to Watson and Sullivan's (2008) definition of mathematical fluency. In addition, teachers mentioned that mathematical fluency is inclusive of students' abilities to: use different/multiple pathways, make connections, communicate their reasoning, apply/transfer new learning and risk being wrong. These features were used as the initial codes for analysis of the interview data.

I have selected the analysis of three teachers and their interview responses. Their responses are typical of the 17 interviews and their descriptions of a specific student in their class were detailed. One teacher had 6-10 years teaching experience and the other two teachers had more than 10 years teaching experience. These teachers represent differing schooling grades (a K-2, Year 4 and a Year 5-6 class). They teach in a range of low and high SES (Social Economic Status) metropolitan areas, in schools with 18%, 45% and 81% Non-English-Speaking Background (NESB) students.

Interview Data Mapped to Teacher Conceptions Framework

The influencing factors represented in the TCF (Figure 1) were used as a lens for analysis. All TCF factors—teachers' own educational experiences, social context, classroom practices, and their knowledge of content and pedagogy—were evident across interviewee data.

When describing mathematical fluency, teachers used their classroom or student learning experiences to frame their responses. Teachers related fluency to their classroom practice stating:

> I teach fairly similarly, when I teach kindergarten and when I teach stage three, in that I have to go from the known to the unknown. 05_01K2

> To become fluent, you have to practice the skill, and then that will build up your known facts, that will help you then to solve problems in a variety of different ways. I think that - like explicit teaching, but also practice of skills, like games, and then open-ended tasks, and tasks that require them to then apply their skills. 14_0156

Teachers' knowledge of content and pedagogy came to the fore in their responses where a strong focus was placed on syllabus knowledge and the positive effects of professional learning they had experienced regarding mathematics. Some teachers associating beliefs to their own learning, for example:

> Early intervention programs have that ability for the students to learn how to reflect on their learning [which] has huge power. Because it doesn't just increase their fluency, and their accuracy. It gives them the ability to go, 'I made a mistake and this is where I think I made the mistake, and this is where I think I need to correct it'. 05_01K2

> They're working mathematically when they first kind of start to have a focus on those in the syllabus. I think it was an add-on. It was like, let's teach them all the mechanics as I like to call it and then we'll give them a problem at the end. I actually think it needs to go the other way. We need to be giving them doing and thinking and then teach them. 10_0105

Some respondents also described mathematical fluency and the importance of mathematics as a way to communicate within a social context. Such as:

> In some ways, you are looking for their ability to recognise the patterns and to make the connections. You want them to make connections to themselves and to the world and to other things that they've seen. 10_0104

> If you know how to solve something and you can't explain it to anyone else, then no one else is ever going to find out what it is. Fluency is a big part of communicating your knowledge. Because maths is always growing, and finding new things, and finding new ways. It's very important. 14_0156

Similarities arose in responses once the data were organised according to the TCF influencing factors. Examples included: making connections between fluency in mathematics and the real world (social contexts), identification of mathematical fluency as important (content/pedagogy knowledge), and mathematical fluency as a way of communicating knowledge (student/ classroom experiences). Features of mathematical fluency that teachers espoused were consistent with the initial questionnaire codes.

Descriptions of Students Mapped to Initial Questionnaire Codes

Teachers' conceptions of mathematical fluency were also mirrored in their specific examples of students they felt were mathematically fluent. Figures 2, 3 and 4 are excerpts from the three interviewees with definition codes and emerging themes identified within the text. Descriptions of students aligned to the teachers' beliefs and knowledge that was identified when the data were mapped to the factors in the TCF.

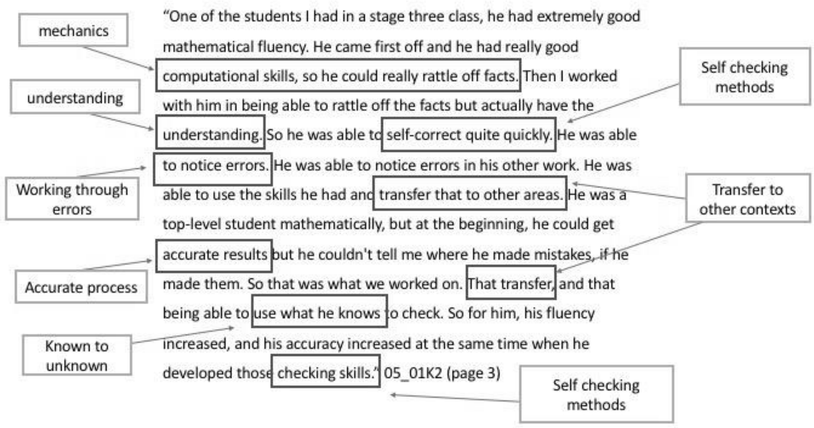

Figure 2. Excerpt from 05_01K2 interview mapped to questionnaire codes.

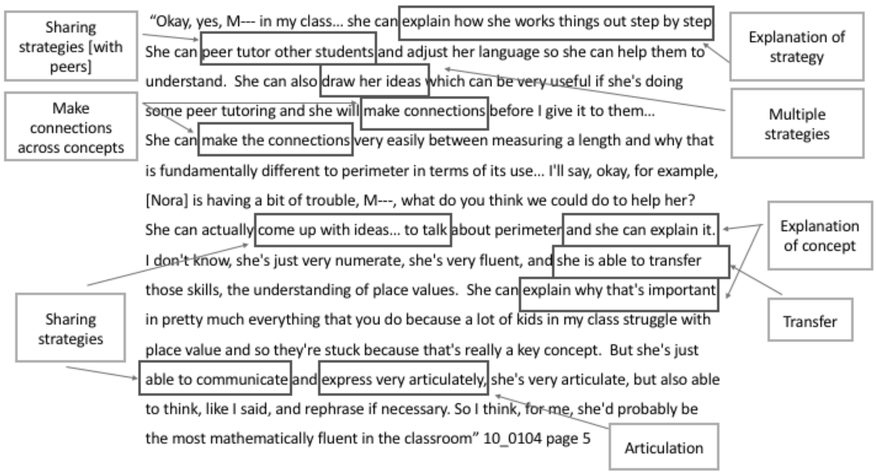

Figure 3. Excerpt from 10_0104 interview mapped to questionnaire codes.

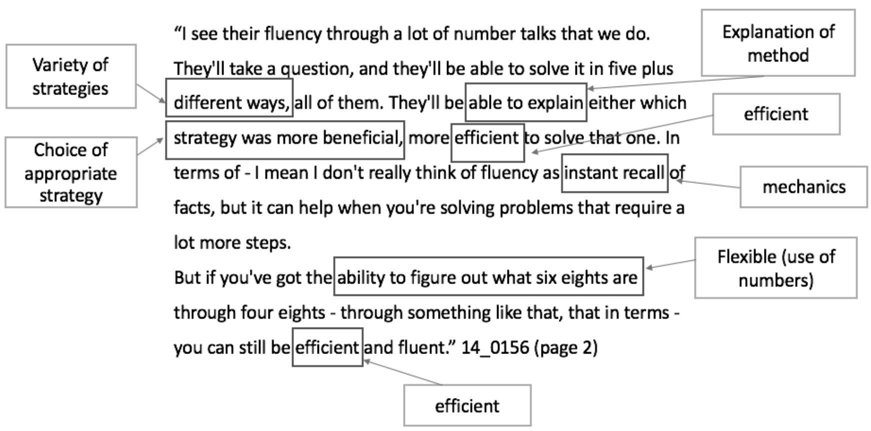

Figure 4. Excerpt from 14_0156 interview mapped to questionnaire codes.

Interviewee 05_01K2 referred to her teaching strategies when describing mathematical fluency and assisting students in moving from the known to the unknown. This conception is reflected through the description of her student's ability to transfer knowledge across areas and to transfer knowledge as a checking method. Interviewee 10_0104 emphasised the importance of making connections to the real world when mapped to social context in the TCF. Within the student description, this conception appears twice, making connections across areas, and making connections for other students. Interviewee 14_0156 references teaching practices such as number talks when describing her student, reflecting her knowledge of effective pedagogy. These descriptions richly illustrate what mathematical fluency may look like in the classroom. It is noted as a limitation of the study that although the interview data may be seen as a validation of teacher conceptions, a direct correlation of these conceptions to their classroom practice cannot be made.

Teachers' conceptions of mathematical fluency did include aspects related to procedures (efficient, instant recall, computational skills). However, these features appear to align more closely to Kilpatrick et al.'s (2001) other strands of proficiency, more so than the procedural fluency strand on its own. Support for this conclusion comes from analysing the context teachers provided when describing procedural features. Teachers referenced the procedural terms *in conjunction with* student strategies, understanding or, as a way of reasoning. Drawing on the initial definition codes from the questionnaire data and additional features that emerged from the interview data, similarities are visible. When these features were grouped, they noticeably aligned to three of Kilpatrick et al.'s (2001) five strands of proficiency: strategic competence, conceptual understanding and adaptive reasoning (see Table 1). Student confidence was also mentioned which links to the productive disposition strand of proficiency. However, teachers did not see confidence as a separate aspect of mathematical fluency but something that builds once students' strategies and skills develop.

Table 1
Fluency Characteristics Mapped to Kilpatrick et al.'s Strands of Proficiency (2001)

Strategic competence	*Conceptual understanding*	*Adaptive reasoning*
Variety of strategies/ ways Choice of strategy Accurate process (articulation) (Ease of) mechanics-*automaticity*	Making connections between concepts (known to unknown) Explanation of method Sharing strategies [with peers] (communicate)	Justifying strategy/method Transfer to other contexts or problems Self-checking method (reasonableness) Working through errors

Mathematical Fluency as Proficiency

Previous studies of *reading* fluency indicated "language researchers have offered countless different aspects that contribute to defining fluency as an overall oral proficiency in speech" (Götz, 2013, p. 1). Why has fluency as an *overall proficiency* not been applied to mathematics, as it appears from the data that there are many aspects that contribute to mathematical fluency. Common themes from the interviews addressed this question. The examples of student behaviours shared indicated the complex nature of fluency that stretched far beyond efficiency with procedural knowledge. It is clear that fluency, from teachers' perspectives, may be determined by a student's ability to apply, and demonstrate or transfer knowledge, for example, in problem solving tasks.

Conclusion

The depth to which the teachers explained their thinking and justified their ideas through student examples provided an insight in to the complex nature of fluency.

> I don't think of fluency as one thing. I think of it as a whole broader concept. I wouldn't call someone fluent if they could just apply an instruction. 14_0156

Kilpatrick et al.'s (2001) description of procedural fluency echoes the belief that separating procedures (skills) from understanding can have dire results, "students who learn procedures without understanding can typically do no more than apply the learned procedures, whereas students who learn with understanding can modify or adapt procedures to make them easier to use" (p. 124). Conversely, the Australian Curriculum (ACARA, 2010) and Kilpatrick et al.'s (2001) strands of proficiency both depict fluency as separate from (although intertwined with) understanding. Teachers in the questionnaire listed understanding as a feature of fluency, interviews comments also supported this view:

> I think if you're fluent in maths you're going to have the understanding with it. I think you can have the understanding without fluency but not the other way around. 10_0104

Figure 5 may be a more useful model for teachers in reflecting how mathematical fluency develops. From analysing the teacher descriptions, mathematical fluency is the result when students' strategies and ability to reason are concurrent with their conceptual understanding. This is consistent with Watson and Sullivan's (2008) description of mathematical fluency. This model puts forward the notion of fluency as a result, instead of one strand, of proficiency. Further research illustrating the nature of how understanding and fluency interact would be beneficial for teachers.

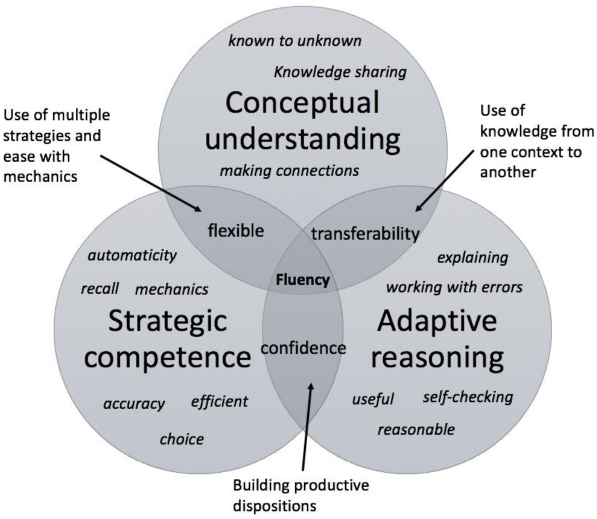

Figure 5. Reframing fluency model.

Rich descriptions of mathematical fluency have the potential to assist teachers in identifying aspects of fluency students possess, and aspects of fluency yet to be developed. The findings of these three teacher interviews assisted in discovering teachers' shared conception of mathematical fluency and identified features. Further research could enrich these descriptions and define when the characteristics are likely to be noted. Clearly identified features of mathematical fluency could also be researched when observing and assessing student conversations or work samples. A shared understanding of what we mean by 'fluency' is important if we expect teachers to assist their students to become fluent.

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