MERGA 41

Mathematics Education Research Group of Australasia

Making Waves, Opening Spaces

Proceedings of the 41st Annual Conference of the Mathematics Education Research Group of Australasia

Edited by Jodie Hunter, Lisa Darragh, & Pam Perger

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Preface

This is a record of the Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia (MERGA), held at Massey University, Auckland, New Zealand. The Proceedings were published online at the MERGA website www.merga.net.au

The theme of the conference was "Making waves, opening spaces". This theme was specifically chosen by the conference organising committee in recognition of the ongoing need to address both equity and diversity across mathematical learning spaces. In particular, it acknowledges the need for us all to consider how our research and scholarship can potentially make waves and open spaces up for all learners. The keynote presentations addressed this theme with Associate Professor Robin Averill examining historical pedagogies towards opening spaces for all mathematics learners in culturally responsive ways. Dr Deborah Schifter challenged us to think of how we could open up structure for learners and move beyond a focus on computation in primary classrooms.

The conference included presentations of symposiums, research papers, short communications and round tables that covered a wide variety of topics related to mathematics education from a range of countries while maintaining a particular focus on the Australasian region. This also included a variety of settings such as early childhood, compulsory schooling sectors and tertiary institutions. All of the symposium and research paper submissions were blind reviewed by panels of mathematics educators with appropriate expertise in the field and either accepted for publication and presentation or presentation only. The published proceedings include the keynote papers, accepted research and symposia papers and the abstracts for research presentations, short communications, and round tables.

The Editorial Team would like to thank the authors for submitting to the conference and for their use of the MERGA template. We would also like to acknowledge and thank Review Panel Chairs and all the reviewers for their efforts in reading and providing constructive feedback. Ensuring the published papers meet high academic standards is an important and shared responsibility of the MERGA community.

At MERGA 41, the organising committee welcomed many participants from Australia and New Zealand as well as colleagues from the USA, Singapore, England, South Africa, Malaysia, and Indonesia. We hope that you enjoyed your visit to Aotearoa, New Zealand.

Jodie Hunter
Chair, Conference Organising Committee and Chief Editor

Lisa Darragh and Pam Perger (Editors)

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Examining Historical Pedagogies Towards Opening Spaces for Teaching all Mathematics Learners in Culturally Responsive Ways

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The literature on culturally responsive and culturally sustaining practice calls for ensuring educational experiences are relevant for, and compatible with the wider experiences and lives of students while strengthening their cultural identities. Shifts in practice are called for to help reduce inequities in mathematics and statistics learning opportunities and achievement. Drawing from research within and outside mathematics education, we will consider the potential of three pedagogical approaches consistent with those of many diverse heritage cultures – song, story-telling, and metaphor – for promoting engagement with, and learning of mathematics and statistics. Frameworks for culturally responsive practice will be used to examine whether enhanced use of these pedagogies may assist in engaging learners in holistic, caring, diverse, and mathematically and statistically productive ways, and in conceptualising new avenues for research.

The kids see it and they're laughing and they start doing it. (Teacher Aide)

Thank you very much for inviting me to give this talk. I would like to challenge us to consider various rationale for broadening the research and practices used for the teaching and learning of mathematics. I will draw from a wide research base and from my teaching in initial and in-service teacher education. I begin by exploring factors that impact on mathematics teaching pedagogies and calls for teaching that draw from a wide range of pedagogies and that appeals to emotional as well as to conceptual development. I then discuss culturally responsive education including considering learning as a holistic personal and social experience. Three pedagogical approaches – singing, storytelling, and metaphor – are then discussed in turn in relation to research into learning, broader effects, and fit with culturally responsive teaching. We consider ways that these pedagogies can enhance opportunities for learning, pleasure, identity-development, and mathematical achievement, particularly for marginalised learners. Along the way I will include examples from my practice and research.

Similar to the work of many other mathematics educators, I promote the use of problem solving, inquiry, manipulatives, representations, using good questions, cooperative learning, and so on. However, other aspects of my teaching are different to many. We sing. We tell and listen to stories. We respond to protocols that enable some to feel comfortable to participate. We explore ideas using the power of metaphor. We draw from historical and cultural contexts. Many appreciate experiencing these pedagogies and they appear useful for learning and motivation. I have used preparing this talk as an opportunity to search the literature for reasons why these three comparatively neglected pedagogies – singing, storytelling, and metaphor – may be useful for mathematics learning and engagement, and to help us consider the extent to which they are likely to be essential for students traditionally underserved to engage successfully with our discipline. Thank you for being here.

Experiences that started my using songs, storytelling, and metaphor in my own teaching of mathematics and mathematics education were in my work with bilingual (te reo Māori/English) primary initial teacher education mathematics methods classes and classes of second chance learners working towards senior secondary mathematics qualification-

based assessment. Working with these groups showed me the power of pedagogies less often used for mathematics learning for enhancing motivation to learn, engagement with mathematics, and learning. With the bilingual student teachers I was concerned to ensure my teaching was as inclusive of, and as responsive as possible to te Ao Māori/the Māori world. I studied te reo me tikanga Māori (wananga-based) and worked consultatively and in partnership with Māori colleagues. Research about our learnings from this time has been reported at MERGA conferences and elsewhere (e.g., Averill et al., 2009). The second chance learners had recently left school without qualifications and, although capable, came with low mathematical self-efficacy and low interest in mathematics. The ways in which the second chance learners had been taught mathematics in school had not worked well for them, so more of the same would be neither comfortable nor effective. I used the opportunities of working with these groups of students to find pedagogies they would find effective. We sang for every topic to help remember key ideas for assessments. We linked learning to student interest areas, told stories, and explored metaphor for developing understanding and retention of ideas.

Pedagogies and Learning – Engaging the Whole Self and the Collective

Goals, emotions, and self-efficacy impact on motivation (e.g., Ford, 1992). In our work with students we have many layers of concerns – how to help them develop conceptual understanding, to see mathematics as relevant to their own worlds, and to be excited by and feel they are confident, capable mathematicians. Varied teaching approaches chosen to suit learner interests and needs and real and relevant tasks enhance motivation to engage with and learn mathematics (Bobis, Anderson, Martin, & Way, 2011). However, many students experience a narrower range of pedagogies in their mathematics learning than in their other subjects and can experience predominantly teacher-directed learning with infrequent opportunities for student autonomy, discussion, practical work, and movement, and limited exposure to ways in which the learning content relates to real-life contexts (e.g., Hagan, 2017). This situation is despite our long-standing work in initial and in-service teacher education promoting student-directed and innovative teaching strategies rich in mathematical problem solving, argumentation, relevance, discourse, and exploration.

Bishop (1991) describes 'enculturation' of students into mathematics learning at a variety of levels – cultural, societal, institutional, pedagogical, and individual – reflecting that what we do in classrooms affects what students believe mathematics to be and how mathematics learning is acquired. Policy, assessment, research, and societal impacts on mathematics education over time have resulted in conflicting messages about what mathematics learning looks like and how it occurs, with many of these impacts helping to foster 'impersonal learning' such as textbook and teacher-led approaches (e.g., Walls, 2010). Mathematics learning environments have been affected by the 'crowded' curriculum, an increase in the directed nature of curricula and provided resources (e.g., numeracy projects), increased parental, societal, and governmental scrutiny of numeracy and mathematics teaching and learning, in part stemming from high stakes international assessment, and increased focus on assessment at many levels of schooling (Gonzales, 2009; Neyland, 2010; Walls, 2010). Many teachers have experienced effects of such constraints on the pedagogies they use, resulting in enhanced focus on cognition and achievement and reduced attention to more personal, cultural, and creative dimensions of mathematics learning, as the:

> ...relentless quest for higher standards and curriculum coverage which dominated this period [1990s and 2000s] may well have obscured the personal and affective dimensions of teaching and learning

> and fostered a mindset characterised more by compliance and conformity than curiosity and creativity. (Pound & Lee, 2011, p. ix)

Pedagogical approaches consistent with meeting external policy and societal expectations have not always sat well with teachers' own philosophical views (English, Hargreaves, & Hislam, 2002), and there are increasing expectations from policy makers and mathematics and indigenous education researchers for creative teaching that supports links with the community and its cultures in a spirit of partnership (e.g., Education Council, 2017; Ministry of Education, 2011; National Council of Teachers of Mathematics, 2000). There are also repeated calls for research into links between pedagogical considerations and approaches and student affect (Attard, Ingram, Forgasz, Leder, & Grootenboer, 2016) to support our understanding of the effects of teachers' pedagogical choices on student learning. Examples from our own field of mathematics education show a varied and collective desire for teachers to move beyond focussing only on cognitive aspects of mathematics learning and traditional mathematics classroom patterns. Next we consider just a few of these examples which illustrate thinking about mathematics learning as a holistic, creative, and collective activity.

Askew (2012) refers to teaching as an improvisational activity and transformative mathematics teaching as paying attention to learning as a collective activity within cooperative learning environments, acknowledging that the social context can support or hinder learning. Similarly, Ernest (2011) draws attention to the importance for students of the places and ways that they learn, and the social context for mathematics learning:

> A growing body of research is suggesting that not only *what* students learn, but *where* they learn it is important. In other words, it is not only the *content* of learning that matters, but also the s*ocial context* of that learning that counts. (p. 126, italics in original)

Lakoff (2008) describes thought as a physical process, with the body and brain interconnected by how thoughts are processed by neurons and neural pathways. Neyland (2010) pleads for a return to education as a place for curiosity and wonder as primary motivators, seeing learning environments as places for creativity and 'full-bodied' knowledge (acknowledging learning as involving mind, emotions, and body) where learners can experience surprise with 'comical spirit'. Such a view of learning as being experienced holistically is shared by many indigenous cultures, who give:

> ...much higher priority to the *wisdom tradition* of thinking, reflecting, doing, and being. When that view is normal for Indigenous students, imagine how disappointed they must feel when studying mathematics and discover that the subject is *only* about an *intellectual tradition* of thinking, a much narrower view of education. (Aikenhead, 2017, p. 84, italics in original)

Reflecting on the excitement and motivation of Year 8 students after working with them on a topic beyond official curriculum content (the history of mathematics), Brahier (2011) also conveys the importance of creating emotive responses to mathematics learning:

> ...learning mathematics transcends acquiring skills and even developing conceptual understanding – ...beyond these goals, my role as a mathematics teacher is to spark the interests of my students and excite them about mathematics and its usefulness in our world… (p. 4)

To consider the role of emotion in learning in general a little further, enthusiasm and excitement are known to support engagement, learning, and memory (Eliot, 1999; Rhodes, 1988), and Goleman (1996) goes further to state that learning *requires* emotional engagement. Emotions, attitudes, beliefs, and values are all also believed to be interlinked with mathematical learning (e.g., Goldin, 1987, 1992).

A study that demonstrates well the power of emotional engagement for learning carried out in Auckland schools found that the energising uplifting atmosphere and connections of emotion, cognition, psyche, and spirit created between performers and their audience is essential for the learning of New Zealand Tongan students (Manu'atu, 2000). Such activities and feelings also help form a sense of shared experience and community and help us persist when we meet challenging tasks.

Owens (2015) sees pedagogical change as necessary for enhancing mathematics learning opportunities of indigenous students. In an Australian-based study involving teachers and community, she found that partnership with the community and teachers valuing students' family and cultural heritage resulted in increased warmth and communication between school and community, which in turn impacted on the curriculum and teaching approaches used. Owens reported decolonisation of teachers' thinking and approaches as their cultural understandings developed, over time, through the teachers and communities:

> Learning about different approaches, trying appropriate changes, experiencing change in student attitude, behaviour, and mathematical performance, and through affective experiences that unsettle their current ways of acting and thinking. (p. 59)

In Owens' project, students' cultural identities were enhanced through inclusion of art, dance, and telling stories, enabled by the increased communication with community and the wider professional development. Teachers' beliefs about effective teaching and the goals of their work changed during the project. They reported using more narrative, non-verbal communication, and action learning, including learning with small groups outside the classroom. Increasingly during the professional development, students identified as aboriginal, and with mathematics positive changes in relation to student identity development similar to those found in the Kotahitanga professional development project in Aotearoa New Zealand (Bishop, Berryman, Tiakiwai, & Richardson, 2003).

Culturally Responsive and Sustaining Pedagogies for Mathematics

In light of the calls for moving beyond the cognitive focus in mathematics learning, we take a look into some of the work focussed on teaching indigenous and marginalised learners. When students do not see themselves reflected in the classroom curriculum, they can find it difficult to engage (Barton, 2008; Lunney Borden, 2013). A broad array of research calls for and describes pedagogies that are culturally 'relevant' (Ladson-Billings, 1994), 'responsive' (Gay, 2010), 'specific' (Irvine, 2002), and 'sustaining' (McCarty & Lee, 2014; Paris, 2012; Paris & Samy Alim, 2014). Attending to 'cultural brokering' and 'border crossing' (Aikenhead, 1997) and 'diversity pedagogy' (Sheets, 2005) are also advocated for effective teaching of indigenous and other learners served less well by traditional western classroom practices. However, while there are increasing calls for creative, innovative, interactive, and adventurous teaching for engaging indigenous learners (e.g., Ka'ai, 2012) and increasing literature in the area of culturally responsive pedagogy, there is little to date that focusses directly on the potential of our pedagogies of singing, story-telling, or metaphor for provoking holistic perspectives in mathematics learning.

Marginalised and indigenous people, cultures, and languages are frequently "in subordinate positions in schools and curricula, with national priorities frequently determined by the needs and aspirations of the majority" (Meaney, Edmonds-Wathen, McMurchy-Pilkington, & Trinick, 2016, p. 159). Ways that have been described to reflect students' lives and cultures in mathematics teaching include examples such as ethnomathematics research (e.g., Barton, 1996; Civil, 2002; d'Ambrosio, 1985; Lipka et al., 2005; Nicol & Archibald,

2009; Sterenberg et al., 2010; Wagner & Lunney Borden, 2015), using groupwork with consensus decision-making (Sullivan, Jorgensen, Boaler, & Lerman, 2013), and rich investigative tasks (Averill, 2018). However, drawing mathematics activities from culturally-based experiences (e.g., Leonard, 2008) is predicated on educators having deep knowledge of the cultures of their students (e.g., Sterenberg, 2013; Trumbull, Nelson-Barber, & Mitchell, 2002), understanding how to incorporate such ideas suitably, and being willing and able to shift their own cultural beliefs, values, and practices to encompass approaches new to them (Whitinui & Kaiwai, 2012). Insight that cultures develop and change over time (Battiste, 2002), and that all students are different within and across cultural groups is also needed. Fortunately, strong examples of research into culturally responsive pedagogies and practice exist in our Merga community (e.g., Howard, Perry, Lowe, Ziems, & McKnight, 2003; R. Hunter & J. Hunter, 2018; Jorgensen, 2018; Jorgensen, Sullivan, & Grootenboer, 2012) but many of the ideas in this powerful work are yet to be widely reflected in practice.

There has been critique that the work on culturally relevant/responsive/specific pedagogy focusses on helping indigenous and ethnic minority learners manage in Eurocentric educational settings and does not go far enough in challenging and changing the status quo towards greater equity of opportunity and liberation (Watts, Williams, & Jagers, 2003). Martin and McGee (2009) call for framing mathematics education in ways that address historical oppression and achieve liberation, leading us to reconsider what we teach as mathematics and the contexts and pedagogies we use to teach it. There is also growing recognition that learning is acquired in different ways depending in part on learners' cultural heritage/s and that pedagogical approaches, compatible with culturally-linked pedagogies and ways of being and doing, are important for improving access to mathematical thinking and achievement (e.g., Aikenhead, 2017; Gay, 2010; Owens, 2015; Rhodes, 1988; Sterenberg, 2013). For example, in the American context, Martin and McGee (2009) advocate for mathematics education that is "worthy of being experienced as part of African-American children's development as full human beings" (p. 208) and Lee (1994) calls for African-centred pedagogy which "positively exploits and scaffolds productive community and cultural practices" (p. 297). In a Canadian example, policy states that a "variety of teaching and assessment strategies is required to build upon the diverse knowledge, cultures, communication styles, skills, attitudes, experiences and learning styles of [Aboriginal] students" (Alberta Education, 2006, p. 3). Despite the useful work already done in the areas of indigenous and culturally responsive mathematics education, further ways to enhance our teaching are needed to ensure equity of access to mathematics learning and achievement – ways that involve full-bodied knowledge, enthusiasm, excitement, the creative, and improvisation. To inform this work, suitable research is also necessary.

Achieving equitable teaching of mathematics requires that teachers learn, experiment, and reflect. Teaching for equity and for social justice that enables strengthening of students' cultural identities requires a shift for many teachers in their views about their own identity, their teaching identity, and their identity as a teacher for social change (Gonzales, 2009). Many of the strategies advocated within the culturally responsive and sustaining teaching literature require deep cultural knowledge, knowledge that takes time, effort, opportunities, and commitment to acquire. In multicultural contexts such educator learning is even more demanding.

There are many frameworks suitable to ground exploration into pedagogical practices for mathematics. For example, one of the cultural competencies for teachers of Māori students (Ministry of Education, 2011) intended to lead to Māori students enjoying

educational success as Māori by improving New Zealand teachers' responsiveness to this group, ako, is being used alongside cogenerative dialogue to develop mathematics teaching practices and enhance engagement and learning in a secondary school setting (Saunders, Averill, & McRae, in press). The competencies (Ministry of Education, 2011) draw from concepts from the Māori world in relation to communication, relationships, care, socio-cultural awareness, and teaching and learning, portraying learning as a partnership between students, their whānau (family), the teacher, and school. Another indigenous model, the whare tapa wha (four-sided house) (Durie, 1998), originally described to assist thinking about the holistic nature of health and well-being, has also been used to examine learning environments for responsiveness to Māori students (Averill, 2012). Durie's model identifies physical, emotional and cognitive, social, and spiritual aspects as four interrelated aspects of health and well-being. The cultural competencies and the whare tapa whā frameworks have similarities with those described for indigenous and minority learners outside of New Zealand in relation to the importance of partnership between teachers and community and their emphasis on the cognitive domain being intertwined with broader holistic dimensions (e.g., Aikenhead, 2017; Doolittle & Glanfield, 2007; Lunney Borden, 2013; Owens, 2015).

Further research tools for exploring the learning of marginalised students include Fraser's (2013) theoretical lens of redistribution, recognition, and participation, understanding indigenous knowledge systems as ways of 'knowing', 'being', and 'doing' (Martin, 2003), and considering learning environments and pedagogies in relation to community values, such as Pasifika values of belonging, inclusion, respect, leadership, reciprocal relationships, service, spirituality, family, and love (Ministry of Education, 2013). Such research is challenging and regardless of the framework or methodology used, essential is deep consideration of the nuances and complexities of challenges for equity, social justice, and culturally responsive and indigenous education in relation to aspects such as culture, context, and collaboration (Ismail & Cazden, 2005; Leonard, Brooks, Barnes-Johnson, & Berry, 2010; Vale, Atweh, Averill, & Skourdoumbis, 2016). The nature and scope of these and other models that encompass learning as a culturally-located and holistic experience and draw from non-Eurocentric worldviews also help show that broadening our mathematics pedagogical practice base is important for marginalised learners. They may provide useful frameworks for considering research into the use for mathematics engagement and learning of the focus pedagogies of this talk.

Next, in light of the ideas above about mathematics learning, motivation, and culturally responsive teaching, we consider the affordances for mathematics learning of singing, storytelling, and metaphor, pedagogies shared across many cultures. As a brief aside, we can think more generally about the extent to which these pedagogies may support effective learning communication. Alda (2017) describes in an engaging way how research into communication can inform how we understand ways of creating community, developing empathy, and encouraging listening. For example, doing things together in time, be it as simple as tapping the desk in time with one another, can build a sense of community and empathy for one another. Alda talks of the power of improvisation techniques for communication, describing the same 'yes, and' technique discussed by Askew (2012) in relation to how teachers can best respond to student questions, difficulties, and progress. Singing, storytelling, and metaphor may open more spaces for communication strategies than are enabled by traditional classroom pedagogical approaches. Let's consider these spaces as we consider what these pedagogies can offer.

Singing, Storytelling, and Metaphor as Pedagogies for Mathematics

> The arts provide a continuous thread that connects generations through imagery, movement, and voice … Through the process of making art, we engage in an experiential activity that affects us internally, touching upon emotion and thoughts while also offering a tangible object that serves as a source of inspiration. (Dalton, 2015, p. 131)

Music is thought to be the biological and cultural starting point for the field of mathematical thinking (e.g., Egan, 1991), with rhythm, tune, and understanding of time, key ideas drawn from music and dance. Music is multi-sensory and supports the development of language, concentration, and self-discipline (Pound & Lee, 2011), with music therapy an established field that explores how music can improve cognitive, psychological, and emotional well-being. Using music in mathematics learning is highly motivating and can help strengthen links between students' in and out-of-school-lives (Pound & Lee, 2011). A wide range of sing-along YouTube clips with a mathematical focus exist and are used in classrooms. Research into their use and effectiveness for learning, and their use in contrast to teacher or student-generated or led singing is timely.

Making music with others enhances group cohesion and understanding, strengthening the sense of the collective. Singing in particular is known to help language development, memory, concentration, self-discipline, motivation, and understanding (Good, Russo, & Sullivan, 2015), and the melody and rhythm of song can enhance short and longer-term memory of native and foreign text (Good et al., 2015). How music can be used to enhance learning has been explored in some disciplines (e.g., memory, language learning, literacy, listening, and comprehension), but there has been little such research to date in relation to mathematics learning.

In addition to assisting cognitive development, there can be social and emotional benefits of singing with others. An Australian study into the meaning and importance of group singing to singers (Judd & Pooley, 2014) is among many finding that regular singing with others is a joyful activity that fosters a sense of belonging, promotes well-being and camaraderie, develops cognitive skills, and is life enhancing (e.g., Beaton, 1995; Mellor, 2013; Pearce Launay, Machin, & Dunbar, 2016). Such researchers call for greater understanding of the benefits of singing and wider acceptance of singing as a valuable activity (e.g., Judd & Pooley, 2014). Singing can also have physiological and neurological benefits, particularly when the song has associated actions, such as kapa haka. Although the mechanism by which it occurs is not yet well understood (Pearce, et al., 2016), studies have found that across age groups and participant types, group singing can create chemical changes in the body, reducing singers' stress and enhancing their immune system (Beck, Cesario, Yousefi, & Enamoto, 2000; Fancourt et al., 2016; Kreutz, Bongard, Rohrmann, Hodapp, & Grebe, 2003). Singers also report that singing with others is relaxing, uplifting, helps them cope with stress, and enhances their well-being. Singing helps participants to express emotions and facilitates mood changes (Davidson, 2004; Hays & Minichiello, 2005). Singing also helps us listen to one another and together create a shared experience.

Singing with others is common to many cultural groups including Māori, Pacific groups, Blackfoot (an indigenous group in Canada), (e.g., Hemara, 2000; Sterenberg, 2013), as well as many western cultures (e.g., Scottish, English, Welsh, Irish, French…). In a New Zealand study, student interviews showed that participating in kapa haka had a range of benefits for students including fostering students' identity, self-worth, cultural connectedness, group connectedness, and learning success (Whitinui & Kaiwai, 2012). Students reported that kapa haka is fun, exciting, and challenging and instils in them the desire to aim high and achieve. Similarly, African American students were more motivated in mathematics learning when

music was used (Albert, 2000) and students have sought to use rap as a way of providing evidence of their learning (Leonard, 2008).

Given what we have just heard about the benefits of singing, it would be strange and sad not to sing together now. Using an example from our initial teacher education programme, we will sing an action song designed to help us think about measurement concepts while learning measurement terms in te reo Māori, the indigenous language of the place of this talk (Figure 1). The tune we will use is a French one, *Frère Jacques*.

Whānui, whānui
Whāiti, whāiti
Teitei, poto, iti, teitei, poto, iti
Tāwhito, tāwhito

Figure 1. Measurement action song.

Looking to our second focus pedagogy, storytelling is another historic pedagogy common to many cultures and oral traditions. Storytelling has served both as entertainment and a teaching pedagogy. Storytelling is a powerful way to share ideas, illustrate a point, generate curiosity, emphasise learning objectives, and to "promote reflection, engage creative thinking and imagination, stimulate meaning making, awaken insight, and pose critical questions for discussion of perplexing issues faced in today's world" (McDonald, 2009, p. 181). Storytelling appeals to imaginations and emotions, connects people, and creates mental images of the listener's making (Goral & Gnadinger, 2006). As humans, we love story. Stories give us a way to draw from familiar contexts to cope with complex and abstract ideas (Devlin, 2000; Paley, 1990) and help develop mathematical understanding (Haven, 2007). The power of storytelling is shown by recent Pacific education leadership conferences, where research presentations are now told in story form and conventional research presentations have been abandoned. Explorations in science education have shown the value of storytelling for developing understanding of, learning about, and remembering scientific concepts (e.g., Barker, 1997, 2004; Gilbert, 2001; Miller & Osborne, 1998; Solomon, 2002).

Stories enable us to identify our own feelings, strengths, and insecurities and consider them towards change. An example that strongly exemplifies this is a chapter about changing attitudes to learning mathematics from fear and loathing to positivity and enjoyment (Yaffee, 1996) that we have used for many years at the start of our initial teacher education mathematics courses. We have found this story-based chapter generates strong discussion and elicits powerful responses from our student teachers as they identify with their own mathematics learning experiences and reflect on the type of mathematics teacher they wish to be. The power of storytelling for engaging students with learning is also emerging from a study into exploring Pasifika values in practice, as indicated in the following quote from a Māori teacher aide working with a New Zealand Samoan teacher in a New Zealand primary classroom with a high proportion of Pasifika children:

> [When I was at school] it was like cold, you know, you did this, this, this, whereas now I think you need to engage more with your children… [the teacher is] really good, she gets very emotional sometimes you can hear her yelling … and she's not yelling at the kids she's getting excited, like about a story they've got to use, getting them to use their imagination and it can be quite loud, but that's alright, that's ok because the kids see it and they're laughing and they start doing it…

Storytelling has been used to explore, understand, and develop our work as teachers and teacher educators from early childhood to tertiary settings, as stories provide rich contexts for exploring the intricacies and dilemmas of teaching (e.g., Carter, 1993). Calls for more use of storytelling in mathematics have been made for some time (e.g., Pound & Lee, 2011; Whitin & Wilde, 1995). Explorations into using storytelling in relation to mathematics learning include studies showing that storytelling can enhance kindergarten children's learning of geometrical concepts (Casey, Erkut, Ceder, & Young, 2008) and early primary children's understanding of place value (Goral & Gnadinger, 2006). Stories can be told by the teacher or by students. In Butterworth and Lo Cicero's (2001) study, four- and five-year-old children were asked to tell a story; the stories were then used to present mathematical concepts (addition, subtraction, multiplication and division) using the real-life situations from the children's lived family experiences.

Oral storytelling is common to many cultural heritages and has been advocated as a pedagogy for multicultural settings (Schiro, 2004). Many have described ways of using mathematical aspects of stories in story and picture books to help develop mathematical thinking (e.g., Perger, 2010; Wilburne & Keat, 2011), particularly in early childhood and primary settings. Oral storytelling can be an even more personal activity than reading stories, as the teller has greater flexibility to match the content and development of the story to their audience, enabling support for developing the intended thinking and concepts and greater communication by way of eye contact, body language, and facial expression.

For Māori, Pasifika, and many others, stories and legends are a way of conveying information, community expectations, and ways of understanding their world and one's place within it (e.g., Mutonyi, 2016; Rameka, 2012). For example, traditional Māori narratives are:

> …fundamental to Māori symbolism, culture and world-views. They serve as an illustration of culture, reflecting the philosophy, norms and behavioural aspirations of the people. They highlight current social practice, and present a model for people to aspire to. The Māui traditions provide a culturally authentic way to reorientate and interact with the world as Māori. They therefore contribute to our perceptions of Māori in New Zealand society today, and can provide legitimate pathways for future schooling change and development. (Rameka, 2012, pp. 119-120)

Looking to research about learning, Mutonyi's (2016) study of Ugandan secondary school science students found that stories, proverbs, and anecdotes drawn from the students' everyday world and cultural context helped them understand science concepts. Nicol and Archibald (2009) worked in partnership with the Haida Gwaii, an indigenous group in Canada, drawing mathematics learning from traditional community stories.

It seems that more attention has already been given in mathematics education research and writing to metaphor than to singing and storytelling. Metaphor is another ancient and long-standing teaching and learning tool, with research indicating the power of metaphor for expanding the mind, developing critical thinking, categorisation, and memorising (Low, 2008). Metaphor can deepen understanding and enable underlying meanings to be more easily grasped (Lakoff & Johnson, 1980). Metaphors enable learners to generate inferences and testable predictions. They can motivate learners and allow tailoring of teaching to individual needs (Duit, 1991; Low, 2008). Metaphor, analogies and similes are widely used to help learning in various fields (e.g., science education (Mutonyi, 2016), language education (Low, 2008), computer science (e.g., Colburn & Shute, 2008; Dufva & Dufva, 2016), and teacher education (e.g., exploring the role of the teacher; e.g., Askew, 2012)). Metaphor can be used in deliberate ways to help students clarify and connect new concepts to their existing schema (e.g., Browne, 1992; Dagher, 1995). Metaphor has also been used

to understand students' dispositions to mathematics, such as by considering student responses to completing sentence starters like "if mathematics was a food, it would be…" (e.g., Cai & Merlino, 2011).

Lakoff and Johnson (1980) help us realise that, whether or not we are aware of it, much of our language and teaching of mathematics has a metaphorical base, as linking abstract ideas with physical activity aids sense-making. They describe that deliberate and unintentional use of metaphor is always present in mathematics teaching and learning (e.g., flipping a coin as a random event, using container schema to consider operations and number lines and movement along them to represent using number systems, using objects to assist in considering manipulation of variables). Neyland (2004, 2009) also drew metaphorical parallels to help explain how we can work with mathematics facts and ideas, describing mathematics as a creative form of improvisation, playing within and around structure. Our challenge is to ensure unintended metaphors and those intended as bridges to understanding are in the experience of all students and that they enhance rather than limit understanding (Low, 2008).

We can consider two types of metaphor, the first grounded in our everyday experience (e.g., life is a journey), and the second, more complex or elaborate, linking ideas to existing schema, framework, or theories (Aubrey, 2009). Metaphors assist learning as they can give visual images which stick in the mind, enabling neural-binding and concept blending (Duit, 1991; Fauconnier & Turner, 1998; Lakoff, 2008), and they invite interrogation of the idea the metaphor focusses on. González (2013) found that using metaphor along with an image assisted students to remember and apply geometrical theorems to solve a problem and prove a new theorem, suggesting the use of metaphor may facilitate students' retrieval of their geometrical learning for later work.

Looking to mathematics teacher education, we have found using metaphor useful for developing students' conceptual understanding as well as to describe their development as teachers of mathematics in several of our methods courses (Anderson, Averill, Easton, & Smith, 2005; Averill et al., 2009). Metaphor was used in several ways in this work – as an overarching theme for the methods course, to link mathematical ideas across the focus mathematical content areas (measurement, algebra, and number), and through a practical culturally-based activity of making a physical representation of their learning in a tukutuku-style panel (traditional woven design panel).

Now we have briefly explored each pedagogy separately, we consider them used together. Singing with others and story-telling have known positive cognitive, physical, psychological, and social effects (Mellor, 2013; Pearce et al., 2016), factors important for learning and likely to enhance motivation, pleasure, and engagement of learners. These and metaphor tap into teachers' and students' creativity and emotional responses. Music, dance, narrative, and physical activity all enhance enjoyment, which can lead to increased student engagement (Paquette & Rieg, 2008; Sandberg, 2009). Several studies have explored using combinations of music and narrative for learning. For example, Colwell (1994) found an increase in kindergarten students' attention and participation levels when big story-books were first introduced using song, finding that children made more eye contact with books and participated verbally more than the control group with no music. Kouri and Telander (2008) found richer vocabulary was used by children in retelling stories when music was used in narrative-based learning than in the non-music control group.

In a Māori worldview, similar to the worldviews of some other indigenous groups, pedagogies are interconnected and interdependent. For example, Hemara (2012) describes that waiata (song), ancient kōrero tawhito (stories), whaikōrero (speeches) and whakataukī

(proverbs), “none of which exist in isolation” (p. 127), support understandings of whakapapa, a fundamental principle of Māori culture.

Conclusions and Ngā Wero

In summary, there is much research that shows it is past time to broaden our approaches to the teaching and learning of mathematics. Preparing this talk has helped me reflect on some perspectives on why using singing, storytelling, and metaphor to develop mathematics learning can be appreciated and effective for cognitive, emotional, psychological, and social engagement, can enhance classroom communication, and can fit with culturally responsive and sustaining practice. We need to engage every learner in holistic, culturally sustaining, and innovative ways. There is useful and diverse research that indicates that greater use of singing, storytelling, and metaphor has the potential to broaden and strengthen our students’ experiences of learning mathematics, leading toward more full-bodied ways of learning and knowing. These powerful pedagogies with the potential to breathe life, warmth, and creativity into our work and to enhance health, well-being, and a sense of community are inherent in many heritage cultures, yet they are infrequently experienced by many learners of mathematics. We need greater understanding of the impacts of purposeful use of singing, storytelling, and metaphor on mathematics learning, affect, and motivation to learn across mathematics learning contexts.

In aiming to improve experiences of indigenous and marginalised students, care is needed to ensure sensitivity and respect are paramount, that pedagogies and research are neither tokenistic nor respond to stereotypes, and that both are developed in consultation and partnership with students and the community. Given the mismatch in many settings between the heritage cultures of mathematics teachers and their students, particularly for students historically and currently underserved, suitable research is vital to ensure the voices of those often underserved are not only heard but are sought in suitable ways and are understood.

Persistent inequity in mathematics learning opportunities shows that we must seek to make big waves to disrupt thinking to enhance and supplement commonly found mathematics teaching practices. The ideas in this talk are presented as starting points for considering new directions in broadening our students’ experience of learning mathematics. However, it is clear that there is evidence from a range of sources that singing, storytelling, and reflecting on metaphor in and beyond traditional classroom settings have the potential to open spaces for more learners to enjoy learning mathematics, and potentially to increase teachers’ sense of pleasure, satisfaction, purpose, and community. In using these pedagogies, we help protect the treasures handed down to us, and enhance the creativity, flavour, and colour of our teaching, and the feel, nature, and ownership of our learning environments. Evidence indicates that using these pedagogies well will strengthen the cohesion, and social, physical, spiritual, and emotional aspects of our learning communities, while promoting stronger desires to be engaged in, and learn mathematics.

Evidence suggests that we can feel confident to open up a wider range of pedagogies for mathematics learning, collectively powerfully aligned with the pedagogies of many of our cultural heritages. Doing so will certainly give us stories to tell, metaphors to draw from, and songs to sing. Research into our efforts will ensure we are well-informed and confident in doing so.

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Looking for Structure: Moving Out of the Realm of Computation to Explore the Nature of the Operations

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Although K-6 mathematics focuses on calculation procedures for adding, subtracting, multiplying, and dividing, the meaning of the operations may fall into the background. Indeed, many common errors stem from confusion about which structures apply to which operation. Early algebra, with its emphasis on recognizing and expressing mathematical structure, has the potential of rooting out such errors. This paper presents a teaching model designed to bring students' attention to the unique structures associated with each operation as they notice, articulate, illustrate, and prove generalizations that contrast the behaviour of different operations.

Third-grade teacher, Alice Kaye, posts the following pairs of equations for her class to examine and asks her students what they notice about the problems. (Alice Kaye is a pseudonym, as are the names of her students.)

7 + 5 = 12 7 + 6 = __	7 + 5 = 12 8 + 5 = __
9 + 4 = 13 9 + 5 = __	9 + 4 = 13 10+ 4 = __
What do you notice? What's happening here?	

Ms. Kaye acknowledges to the students that the numbers are not challenging for them. The purpose of the discussion is not to solve the problems, but to consider what is going on between the pairs of equations and to state their ideas clearly and convincingly. The class fills in the blanks, and then begins to talk about what they notice. Students mention that one number changes, one stays the same, and the last number in the equation changes, too. The discussion goes on like this for a few minutes, until one child, Evan, says, "Since 9 + 4 is 13, 9 + 5 *has* to be 1 more than 13."

This discussion launches for the class an exploration of generalizations about the behavior of the operations. In this case, students are investigating what happens to the sum when one addend is increased by 1. They will work together to state a conjecture, represent it, and justify why it must be true for all whole numbers. They will go on to consider what happens when analogous changes are made with other operations. Such examination of structure—which brings *operations* to the fore and leaves *numbers* in the background—is an aspect of early algebra.

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Introduction

In extant practice, the teaching of calculation in the elementary and middle grades tends to focus on procedures for producing correct results. Although students realize there is a different procedure for each operation, the distinction among operations may fall into the background. Confusion about the operations frequently results in consistent procedural errors. Indeed, common errors in subtraction or multiplication can be interpreted as an application of structures that apply only to addition.

For example, consider such errors as these:

- 35 – 16 = 21 Decompose the numbers into tens and ones; subtract the tens (30 – 10) and subtract the ones (6 – 5); add the results (20 + 1). The correct answer is 19.
- 35 × 16 = 330 Decompose the numbers into tens and ones; multiply the tens (30 × 10) and multiply the ones (5 × 6); add the results (300 + 30). The correct answer is 560.

The basic approach behind these errors is related to a strategy that works for addition: To add 35 + 16, decompose the numbers into tens and ones; add the tens (30 + 10) and add the ones (6 + 5). The sum of the results provides the correct answer, 51. Students who make the errors illustrated above may be thinking of their correct addition strategy as the way *numbers* work, rather than how *addition* works. For that reason, despite a teacher's corrections, students often continue to apply the incorrect procedure, and the error is likely to persist unless its underlying foundation is examined.

The realization that each operation has its own set of properties and behaviors is fundamental in understanding mathematics. Implications apply not only in the elementary grades as students study whole numbers, fractions, and decimals, but also in later years to make sense of operations on new classes of numbers, such as integers or complex numbers, and on algebraic expressions. Indeed, we see confusion about the operations similar to those illustrated above carry over into errors frequently made by algebra students:

- $(a + b)^2 = a^2 + b^2$ (instead of $a^2 + 2ab + b^2$)
- $2(ab) = (2a)(2b)$ (instead of $(2a)b$)

In making these errors, students may be applying the same symbol pattern as the distributive property, ignoring how addition and multiplication must come into play.

Early algebra, with its emphasis on recognizing and expressing mathematical *structure* (Kieran, 2017 et al.), has the potential of rooting out such errors. A focus on the behavior of addition, subtraction, multiplication, and division helps students come to see an operation not exclusively as a process or algorithm, but also as a mathematical object in its own right (Kieran, 1989; Sfard, 1991; Slavitt, 1999). As the operations become salient, seen as objects with a set of characteristics unique to each, students are positioned to recognize and apply their distinct structures.

Key to this work is encouraging students to notice and express regularities in the number system and to verbalize in general terms the strategies they apply in their calculations. As students move to a level of generalization, they engage in metacognitive acts (Cusi et al., 2010). In the language of Malara and Navarra, students "substitute the act of calculating with looking at oneself while calculating" (p. 2002, p. 230).

Also key to this work are representations—diagrams, physical objects, or story contexts—that embody relationships among quantities defined by the operations (Warren & Cooper, 2009; Moss & McNab, 2011). For example, addition may be represented as the joining of two sets and subtraction as comparison or removal. An arrangement of equal

groups or an array can represent multiplication or division. Such representations are used to explain how and why general strategies must work. As illustrated in the classroom examples below, images of the operations become thinking tools for students that they can call upon to reason about arithmetic symbols.

Phases of an Investigation

Over the last fifteen years, I have been working with a team of researchers and teacher collaborators to investigate the learning of students when teachers focus attention on the structure of the operations (Russell et al, 2011; Schifter, 2011, 2018; Schifter et al, 2007). As part of our research, we created a teaching model comprised of five phases of investigation (Russell et al, 2017).

I. Noticing regularity. Students examine pairs or sequences of related problems, equations, or expressions and describe patterns they notice.
II. Articulating a claim. Based on the patterns noticed in phase I, students work individually and collectively to write a conjecture clearly enough so that someone not in the class could understand it.
III. Investigating through representations. Students represent specific instances of the claim with manipulatives, diagrams, and story problems.
IV. Constructing arguments. Students extend their representations to explain why the conjecture must be true for all (whole) numbers.
V. Comparing and contrasting operations. Once a claim has been verified, students consider the question, Does this work for other operations? This leads them back to phase I, to consider a set of problems, equations, or expressions that illustrate an analogous claim that could be made for another operation, and then to go on to phases II to IV, eventually to prove a second conjecture.

Phase I: Noticing Regularity

As illustrated in the opening of this paper, an investigation begins with students examining a set of related problems, equations, or expressions and describing what they see. When Evan offered his insight, "Since 9 + 4 is 13, 9 + 5 *has* to be 1 more than 13," Ms. Kaye recognized the importance of Evan's statement with its assertion of necessity and asked the class to break into pairs to discuss it. When the class gathered for further discussion, his classmate, Pamela, said, "I was just wondering. How did Evan come up with the idea he had? Because these are not just everyday ideas that you come up with every day."

Evan responded, "I'm not really sure. I just know it. It kind of seems obvious to me, so I didn't think to think about it before."

Most classrooms are likely to include both students like Evan for whom such generalizations are so obvious as to be invisible—they never realized there was something to pay attention to—and students like Pamela, who had never looked for such patterns before. Pamela and many of her classmates were coming to see that noticing patterns in the number system is an important kind of mathematical activity.

Whether an individual student is for the first time finding patterns in the number system, is searching for language to describe the pattern he or she has noticed, or is beginning to articulate the mathematical relationships that underlie the pattern, all students in the class are engaged in the same task. The entire class benefits from the range of questions and ideas, although students may each learn something different from the very same discussion.

Phase II: Articulating a Claim

Noticing regularity leads to articulating a claim that a relationship holds for a set of numbers. As students begin to explore a generalization, there is a tendency to declare that *it* works, without specifying what *it* is. The class may appear to agree, but it isn't necessarily clear that they are talking about the same idea. Thus, in the teaching model, students are challenged to come up with language to state the generalization they have noticed.

Toward the end of the lesson introduced above, Ms. Kaye asked students to work individually or in pairs to either write a statement that described what was going on or come up with more examples of the same phenomenon. The next day, she posted additional numerical examples along with the following student statements.

- In the first column, if the number goes up, the answer goes up.
- The first number goes up by 1 and the second number stays the same, so the last number goes up by 1.
- One number grows by 1, so the sum grows by 1.

Among the goals of the lesson is that students learn the language of generalization. Yet, the objective is not to provide students with the most precise and concise statement of a special case of the associative property of addition. (Represented in algebraic notation, the class is working on: $a + (b + 1) = (a + b) + 1$.) Rather, students use their own language and work together to create a statement that is clear enough that someone outside the class would understand. This is one aspect of what Malara and Navarra (2002) call "algebraic babbling." Analogous to the way children learn natural language, students learn to communicate in algebraic language by starting from its meaning and through collective discussion, verbalization, and argumentation, gradually become proficient in syntax.

Note that in the students' statements above, the first two have no mention of the operation involved; in the third statement, the word *sum* implies addition. As mentioned above, as students begin this work, numbers tend to have greater salience and the operations fall into the background. In facilitating the discussion in which the students collaborate to produce a "class conjecture," the teacher points out aspects of their statements that need attention, such as specifying that two numbers are added.

The teacher may also introduce technical vocabulary and symbols as the need arises, especially to clarify referents. For example, if students use the word "number" to refer to different objects in their claim, the teacher might ask students how someone reading their words will know what they're referring to. She might suggest such terms as "addend" or "sum" so that students hear more precise language and begin to use it themselves.

By the end of Ms. Kaye's lesson, the class agreed on the following statement as their conjecture: *In addition, if you increase one of the addends by 1 and keep the other addend the same, then the sum will also increase by 1.*

Phase III: Representing

In the elementary grades, representations in the form of physical models, drawings, diagrams, number lines, arrays, and story contexts can be tools for reasoning, communicating, and constructing arguments. Representations that embody the relationships defined by the operations allow students to examine *why* the symbol patterns they identified work. They help students develop their own internal logic and connect the words of their conjecture to images of the operation as well as its symbolic representation.

In Ms. Kaye's class, students worked in pairs to create representations that illustrate the relationships in the generalization being explored. One such representation is shown in Figure 1, in which the action and meaning of the operation of addition is represented as the joining of stacks of cubes. There are several additions: the joining of the light gray and dark gray stacks; the joining of the single white cube to the dark gray stack; and the joining of the single white cube to the light gray and dark gray stacks.

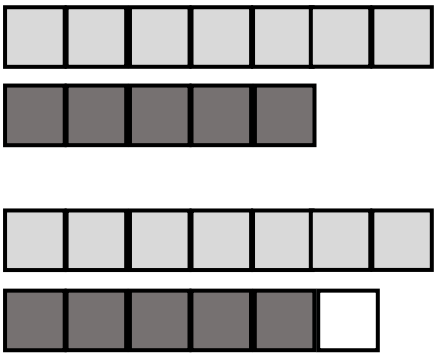

Figure 1. A representation of the class conjecture.

When the class came together to share their work, Ms. Kaye prompted the discussion of each representation by asking a set of questions designed to help students make connections to the elements of their claim. At this point, the discussion referred to specific numbers: 7 + 5 = 12 and 7 + 6 = 13.

- Where do you see 7 and 5?
- Where do you see the sum?
- How is addition represented?
- Where do you see the 1 that was added to 5?
- How does your representation show the sum increased by 1?

Although the discussion still refers to specific numbers, these questions emphasize how the representation shows the operation. The questions may seem straightforward, but they take students into deeper understanding of the representation and often uncover confusions the class needs to work through about the mathematical relationships they are attempting to represent. As they answer the same questions for a variety of representations, students recognize what is common across all of them and see how the same relationship can appear in different forms.

Phase IV: Constructing an Argument

The notion of proof is new to most elementary students, and some may not understand the purpose of it. The goal at this phase of the learning is to deepen students' understanding of the conjecture, to have them consider what it means to make a claim for an infinite class of numbers, and to engage in discussion of the representations that could be used to prove the conjecture.

For example, Andrew, who created the representation shown in Figure 1, explained that the stacks of cubes could stand for any numbers. "This number [points to the first row of light gray cubes] plus this number [second row of dark gray cubes] equals the sum. Then …

the same number here [third row of light gray cubes] plus the same number plus one [points to the last row of dark gray cubes and the white cube] equals the sum plus one."

Andrew's explanation is independent of the number of light gray or dark gray cubes in his stacks. The representation is not just about specific quantities but can be used to show what happens with any whole numbers. Andrew makes this clear by referring to each stack as "this number" rather than naming the actual number of cubes in the stacks; that is, the light gray and dark gray stacks can represent any two whole number addends. Finally, the representation shows *why* the conjecture must be true. When the white cube is added to the dark gray stack (1 is added to an addend), in that same action the combined light and dark gray stack increases by one cube (the sum increases by 1).

Again, especially because these ideas are new to students, they work through the variety of arguments constructed by members of the class. As students compare different representations and identify commonalities, the common mathematical structure at the basis of each becomes prominent.

Phase V: Comparing and Contrasting Operations

Even with the emphasis on addition in the previous discussions, if students have not had experience thinking about the operations as objects, each with its own properties, they frequently think of generalizations as about *numbers*, rather than about an *operation.* That is, when they notice a generalization, they assume the same number patterns will hold, whether they insert the symbols, +, -, ×, or ÷. Often, after they investigate a generalization about one operation, they are surprised to discover that the same pattern does not hold for other operations. However, as soon as they try to apply the same conjecture to another operation, they may run into counterexamples and realize that, for this other operation, they must look for another regularity and state a different conjecture. Because one of the goals of this work is for students to understand more deeply the structure that is particular to each operation, it is important to explore sets of related generalizations that highlight these differences.

Ms. Kaye began this second investigation in the same way she began the first. She presented her students with pairs of equations and asked what they noticed.

7 × 5 = 35 7 × 6 = 42	7 × 5 = 35 8 × 5 = 40
9 × 4 = 36 9 × 5 = 45	9 × 4 = 36 10 × 4 = 40
What do you notice? What's happening here?	

After some discussion, she asked students to work individually in response to this prompt: *In a multiplication problem, if you add 1 to a factor, what will happen to the product?*

Multiplication was new to the class and not all students at first noticed the regularity evident in the problems. The following are some of the statements written by students who could see the result of adding 1 to a factor:

- The number that is not increased is the number that the answer goes up by.

- The number that is staying and not going up, increases by however many it is.
- I think that the factor you increase, it goes up by the other factor.

Because some of the students were not yet able to see the regularity, Ms. Kaye decided to postpone work on a more precise statement of the claim and, instead, to have the class investigate the equations with representations. In this way, students would develop a stronger image of multiplication and could then recognize the pattern in the arithmetic symbols.

In order to scaffold the investigation, Ms. Kaye gave the following assignment.

> Choose which of the original equation pairs you want to work with. Write a story for the original equation; then change it just enough to match the new equations.
>
> Then do one of these:
>
> - Draw a picture for the original equation; then change it just enough to match the new equations.
> - Make an array for the original equation; then change it just enough to match the new equations.
>
> Example: Original equation: 7 × 5 = 35
>
> New equations: 7 × 6 = 42, 8 × 5 = 40

Students worked in pairs and then came together to share their work, each pair presenting their story problem and picture or array. Elise and Maria's story was, "There are 7 jewelry boxes with 5 jewels in each box. There are 35 jewels in all." For this, they drew a diagram that showed 7 groups of 5, as shown in Figure 2.

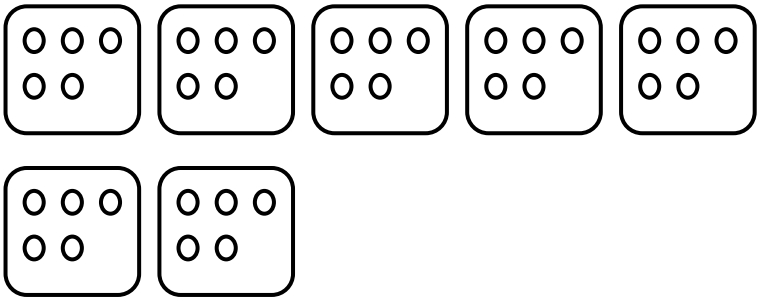

Figure 2. A representation of 7 × 5.

To change the story to 8 × 5, they said there was one more jewelry box, increasing the total by 5. They showed the new jewelry box by coloring it, as shown in Figure 3.

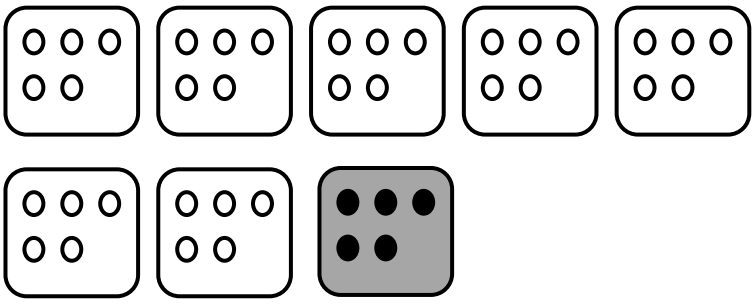

Figure 3. Comparing 7 × 5 and 8 × 5.

To change 7 × 5 to 7 × 6, the girls said that 1 more jewel was added to each box. They indicated the additional jewels in their picture by coloring a dot in each group. When 5 increased to 6, there were 7 additional jewels, one for each box, as shown in Figure 4.

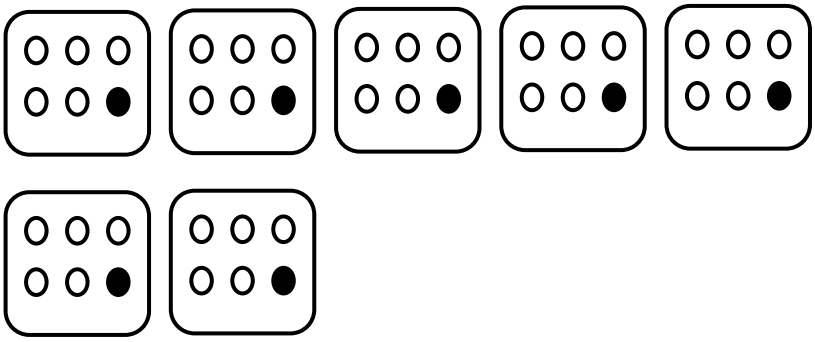

Figure 4. Comparing 7 × 5 and 7 × 6.

Many of the students had story contexts and diagrams like Elise and Maria's. Some of the students presented arrays to show what happens when you add one row or one column. With each pair's presentation, the class was provided a new representation of the concept, solidifying the ideas as they saw them expressed in different stories, different pictures, and different arrangements of cubes.

Throughout the presentations, the class worked on stating what was taking place: When 1 is added to 7, the product increases by 5, the size of one group. When 1 is added to 5, the product increases by 7 because each group increases by 1. By the end of their discussion of different representations, they had developed a more general class conjecture: *When the first factor increases by 1, the product increases by 1 group equal to the second factor. When the second factor increases by 1, each group is 1 larger, so the product increases by the number of groups, the first factor.* (In general in the United States, the first factor designates the number of groups and the second factor, the size of each group.)

Now the class had engaged in two investigations: 1) what happens to the sum when 1 is added to an addend? and 2) what happens to the product when 1 is added to a factor? However, students may remember these two investigations as distinct, unrelated activities and not necessarily reflect back on the difference between the two operations. For that reason, after students presented their arguments about multiplication, Ms. Kaye asked the

class to reread their addition and multiplication conjectures and think about their arguments. "What is different between our two conjectures?"

The question prompted students to articulate how multiplication differs from addition. When adding two numbers, the addends refer to the same unit: 7 + 5 = 12 might mean 7 jewels combined with 5 jewels to make a group of 12 jewels. However, when multiplying two numbers, the factors refer to different units: 7 × 5 might mean 7 boxes each containing 5 jewels, to make 35 jewels altogether.

Impact on Student Learning: Focus on Operations versus Focus on Numbers

The design of the teaching model was built on the insights and creativity of the collaborating teachers, working with the research and development team, to investigate how to approach generalizations about the operations in elementary classrooms. Using the model as a basis, the researchers wrote eight lesson sequences (Russell et al, 2017), each consisting of about twenty 15- to 20-minute sessions, to be taught over several weeks in addition to regular math instruction. Each sequence focused on a pair of analogous generalizations for two operations, for example, how to form equivalent addition expressions versus how to form equivalent subtraction expressions, or, as illustrated in the example above, the result of adding an amount to an addend versus the result of adding an amount to a factor. Each collaborating teacher field tested two of the eight lesson sequences, one in the fall semester, the other in the spring. The generalizations to be explored depended on the grade level, second through fifth grades. The fourth- and fifth-grade sequences included generalizations that extended the domain from whole numbers to fractions or decimals.

From the project in which the lesson sequences were developed and field tested, as well as a prior project focused on related ideas, three data sources provide evidence of the impact on student learning of lessons that focus on the behavior of the operations: 1) documentation of the field tests of the lesson sequences, along with teachers' reports, 2) a set of interviews with individual students of the collaborating teachers, and 3) a written assessment of students whose teachers participated in an online professional development course.

Documentation of Field Tests and Teacher Observations

Based on observation and documentation of the field tests, staff and collaborating teachers identified four key areas of student learning: noticing regularities and making conjectures, developing mathematical language, engaging in mathematical reasoning, and recognizing that the operations have meaning, properties, and behaviors (Russell et al, 2017). Specifically, with regard to the last area, teachers reported on changes of how students engaged in their regular mathematics lessons. Many teachers wrote about how students had become curious about the number system and pursued questions they posed for themselves.

> During our regular math instruction, we spent time noticing and discussing patterns we may not have paid any attention to prior to our work [with the lesson sequences]. Students used language like, "I can prove they're related!" and "Let's write a conjecture for that!"

> Because of the emphasis on reasoning about mathematical ideas and noticing regularity [in the lesson sequences], I find students referencing that all the time. For example, while doing a page [from the student activity book], students will often write notes on the side like, "This is just like our conjecture…" Sometimes, students ask questions of each other like, "But will that always work?" Sometimes, they just have a more systematic approach to reasoning through ideas.

A fourth-grade teacher, whose class explored what happened when their conjecture about multiplication was extended from whole numbers to fractions, wrote,

> I used to feel like students learned one way of multiplying, and then I would show them how to multiply differently when they were introduced to fractions and decimals. Now students are thinking of multiplying as something that stays the same and looking for the ways it plays out with new classes of numbers.

Furthermore, teachers reported, the lesson sequences engaged the range of students. Students who were not typically the fastest, most vocal, and confident math students became central to this work.

> Students are more likely to notice patterns and to feel confident to share their ideas as well as ask questions about things they don't fully understand. The level of confidence, a freedom to speak their minds, and their ability to speak intelligently about mathematical ideas has greatly improved.

> Over the course of these sequences I have seen students more willing to share their thinking and students more willing to work harder to answer their own questions through the use of representations and manipulatives. They also become risk takers. Someone would throw out an idea and others would then add on. Sometimes the language was hard to unscramble, but they all felt confident just to contribute even if it only started the conjecture. This really helped to carry into regular math class. Students who were timid and shy seem to find their voice after we worked in the routines sessions.

There were opportunities for struggling students to continue to work on creating representations, and at the same time more advanced students pushed themselves to think more deeply about refining a conjecture or developing a proof. All of these students were strengthening their understanding of the unique mathematical structures of the operations.

Interview Data

As part of a teaching experiment with twelve field test teachers of grades 2 to 5, one-on-one interviews were conducted with three students from each classroom representing the range of learners, characterized in terms of strong, average, or weak in grade-level computation. In one strand of the interview, students were given pairs of subtraction problems (for example, 10 – 3 = 7; 10 – 4 = ?) that illustrate a structural property not explored in the instructional sequences: *Given a subtraction expression, if the second term (the subtrahend) increases by 1, the difference decreases by 1*. Students were asked to describe what they noticed, come up with other pairs of problems that illustrate the same feature, state a conjecture, and use a representation to explain why the conjecture must be true.

In the analysis of interview data (Higgins, in preparation), one of the dimensions that distinguished students' conjectures was "salience of the operation: the degree to which students attend to the behavior of the operation versus focus almost exclusively on the numbers when drawing generalizations and articulating conjectures." Some students articulated generalizations that were fundamentally about the operation: "When you have the same numbers, once you subtract more, you'll have less. And if you subtract less, you will have more." Other students showed no evidence of attention to the operation: "The numbers in the middle, you just add 1. Then the answer you take away 1. The first numbers are the same."

In the database (n = 36), in interviews conducted at the beginning of the year, lack of salience of the operation was found for close to half the students. After having worked on lesson sequences that explored a different set of generalizations about the operations, the percentages improved. At each grade level, more students explicitly referenced the operation or talked about what they were noticing in operation-specific terms. The operation was no longer just part of the background but became something that students realized they needed to attend to when articulating what they were noticing.

Written Assessment Data

In a project prior to developing the lesson sequences, the team designed an on-line professional development course (Russell et al, 2012 b) with the following goals: to help teachers understand and look for structural properties implicit in students' work in number and operations, bring students' attention to such properties, and support students to articulate, represent, and create mathematical explanations of the properties. The first year the course was offered, pre- and post-course assessment data was collected from 600 students of 36 participating teachers and, as comparison, 240 students from 16 non-participating teachers in the same school systems (Russell, et al., 2017). Items included those in which students were asked to explain why they think two expressions are equal. Student responses were coded for the type of explanation they provided: a) no explanation; b) a computational explanation; or c) a relational explanation, that is, an explanation that refers to mathematical structure. For instance, to explain why 9 – 5 and 10 – 6 are equal, a student could carry out both computations, showing that each expression equals 4, or the student could give a relational explanation: e.g., "Since 9 is 1 less than 10 and 5 is 1 less than 6, the difference is the same." In the post-intervention assessment, students of teachers in the Participant Group provided significantly more relational explanations than in the Comparison Group.

The assessment also asked students in grades 3 to 5 to write a story problem for a given multiplication expression. In the posttest, 74% of the Participant Group (n = 475) produced a correct story, but only 48% of the Comparison Group (n = 180). Students of grades K to 2 were asked to write a story problem for a subtraction expression. Although the Participant Group (n = 128) showed significant progress from pretest to posttest—from 28% correct to 74% correct—the difference with the Comparison Group (n = 60) was not significant.

Conclusion

Data from all three sources suggest that lessons in which teachers draw students' attention to the distinct structures of each operation help to make the operations a salient object in students' mathematical experience. Teachers reported that implementation of the lesson sequences shifted the culture of the regular mathematics classroom, in which students exhibited curiosity about the operations, knew how to explore their own questions, and recognized their conjectures in their calculation work. The interview data demonstrate how, when students notice patterns across calculation problems, they recognize the pattern as related to the structure of a given operation. The written assessment data reveal that, once students have an opportunity to explore and represent properties of the operations, they have a better understanding of contexts that are modeled by the operations and rely on structures to explain the equivalence of arithmetic expressions.

However, within the interview data, even after the intervention, there were still students at each grade level that produced conjectures in which the operation was invisible. In the post-intervention written assessments, there were students who continued to rely on computation to prove the equivalence of two expressions and students who could not create a story problem for a given arithmetic expression. To make the operations salient objects in *all* students' mathematical experience requires persistent effort.

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From Research to Practice: The Case of Mathematical Reasoning

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Mathematical proficiency is a key goal of the Australian Mathematics curriculum. However, international assessments of mathematical literacy suggest that mathematical reasoning and problem solving are areas of difficulty for Australian students. Given the efficacy of teaching informed by quality assessment data, a recent study focused on the development of evidence-based Learning Progressions for Algebraic, Spatial and Statistical Reasoning that can be used to identify where students are in their learning and where they need to go to next. Importantly, they can also be used to generate targeted teaching advice and activities to help teachers progress student learning. This paper explores the processes involved in taking the research to practice.

Introduction and Theoretical Background

A capacity to solve unfamiliar problems and reason mathematically is a desired goal of mathematics education at all levels. Defined broadly in the *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment & Reporting Authority [ACARA], 2015) as a "capacity for logical thought and actions", mathematical reasoning has a lot in common with mathematical problem solving, but it also relates to students' capacity to see beyond the particular to generalise and represent structural relationships. This ability is a key aspect of further study in mathematics and thereby further studies in science, technology and/or engineering (Wai, Lubinski, & Benbow, 2009).

While the importance of problem-solving and reasoning are clearly recognised and valued in the ACM, there is little evidence that these are a focus of teaching and learning in schools. Results from large-scale research studies (e.g., Siemon, 2016; Siemon & Virgona, 2002) and international assessments (e.g., Thomson, De Bortoli, & Underwood, 2016; Thomson, Wernert, O'Grady, & Rodrigues, 2016) have consistently shown that Australian students in Years 4 through 9 experience considerable difficulty solving unfamiliar problems and explaining and justifying their mathematical thinking. Perhaps this is not surprising given that the mathematics texts used at this level tend to focus on relatively low-level, repetitious exercises that are unlikely to be conducive to the development of either deep understanding or mathematical reasoning (Shield & Dole, 2013). Clearly a focus on all of

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the proficiencies is needed but this is a challenge in an environment where "fluency is disproportionately the focus of most externally set assessments" (Sullivan, 2011, p. 8).

Teaching informed by quality assessment data has long been recognised as an effective means of improving mathematics learning outcomes (e.g., Black & Wiliam, 1998; Goss, Hunter, Romanes, & Parsonage, 2015; Masters, 2013). It is also evident that where teachers are supported to identify and interpret student learning needs, they are more informed about where to start teaching, and better able to scaffold their students' mathematical learning (Callingham, 2010; Clarke, 2001). As Wiliam, (2006, p. 6) stated

> What we *do* know is that when you invest in teachers using formative assessment … you get between two and three times the effect of class size reduction at about one-tenth the cost. So, if you're serious about raising student achievement … you have to invest in teachers and classrooms, and the way to do that is in teacher professional development focused on assessment for learning.

At the time, the terms 'assessment *of* learning', 'assessment *for* learning' and 'assessment *as* learning' were being used to draw attention to the different purposes of assessment (e.g., Earl & Katz, 2006). Since then, Wiliam (2011) and others (e.g., Masters, 2013) have blurred this distinction to recognise that any "assessment functions formatively to the extent that evidence about student achievement is *elicited*, *interpreted*, and *used* by teachers, learners, or their peers to make decisions about the next steps in instruction" (Wiliam, 2011, p. 43, our emphasis).

Referred to as *targeted teaching* in the context of the *Scaffolding Numeracy in the Middle Years* (SNMY) project (Siemon & Breed, 2006), the process of eliciting, interpreting and using assessment evidence to inform subsequent teaching and learning requires valid assessment tools, evidence-based learning progressions, professional learning, and the flexibility to use classroom time effectively (Siemon 2016). Consistent with Wiliam's (2006) observations, targeted teaching has been shown to lead to effect sizes well beyond what would otherwise be expected. For example, a 2013 study exploring the use of SNMY materials for multiplicative thinking in 28 Australian secondary schools, used matched data from 1732 students across Years 7 to 10 to show that the average achievement of students grew above an average effect size of 0.6. This result indicates an influence beyond what might have been expected, although the results varied considerably between schools, (Siemon, 2016).

The demonstrated efficacy of adopting a targeted teaching approach to multiplicative thinking, prompted the design of the *Reframing Mathematical Futures II* (RMFII) project (see Siemon, 2017). The aim was to build a sustainable, evidence-based, learning and teaching resource to support the development of mathematical reasoning in Years 7 to 10 that could function formatively in the way described by Wiliam (2011). That is, to inform a deeper, more connected approach to teaching mathematics that recognises and builds on what learners already know and takes them beyond low-level skills and routines.

This paper builds on the body of work presented at MERGA 40 that outlined the rationale, aims and methodology of the RMFII project and described the processes involved in developing and testing the draft learning progressions for algebraic reasoning (Day, Stephens, & Horne, 2017), spatial reasoning (Horne & Seah, 2017), and statistical reasoning (Watson & Callingham, 2017). Our focus here is on the practical implications of this work which we will do by exemplifying how the elicited evidence of students mathematical reasoning (the research) was translated into a form that teachers can use to better understand what that evidence means and, importantly, how they might use the inferences drawn from the evidence to inform a targeted teaching approach to mathematical reasoning (the practice).

Methodology

For the purposes of the RMFII project, mathematical reasoning was defined in terms of three core elements:

i. core knowledge needed to recognise, interpret, represent and analyse algebraic, spatial, statistical and probabilistic situations, and the relationships/connections between them;
ii. an ability to apply that knowledge in unfamiliar situations to solve problems, generate and test conjectures, make and defend generalisations; and
iii. a capacity to communicate reasoning and solution strategies in multiple ways (i.e., through diagrams, symbols, orally and in writing).

A design-based research approach was used as the intent was to "directly impact practice while advancing theory that would be of use to others" (Barab & Squire, 2004, p. 8). Thirty-two secondary schools from each State and Territory with the exception of the Australian Capital Territory participated in the project. One teacher from each school was supported to work with up to 6 other teachers in their school to trial the mathematical reasoning assessment tasks and activities. From 2015 to 2017, approximately 80 teachers, and 3500 students in Years 7 to 10 were involved in the project. Project schools were visited at least twice a year by a member of the research team and residential professional learning opportunities were provided on an annual basis. An additional 1500 or so Year 5 to 10 students from other schools participated in the trialling of the assessment tasks.

The research plan was designed in terms of three overlapping phases. Phase 1 used rich tasks and scoring rubrics to test the hypothetical learning trajectories derived from the literature for each reasoning strand. Rasch modelling (Bond & Fox, 2015) was used to analyse the data and inform the development of Draft Learning Progressions for algebraic, spatial and statistical reasoning. Phase 2 focussed on the preparation, trial and use of multiple assessment forms both to validate the forms and to test the Draft Learning Progressions. This phase also included the analysis of student and teacher on-line surveys, and the development of teaching advice and professional learning modules to support a targeted teaching approach to mathematical reasoning. The final phase of the project is focussing on the development and publication of project outcomes and reports. This paper will focus on a key part of Phase 2, the development of teaching advice from the analysis of student responses to the final assessment forms.

By the end of the third round of assessment, it was evident that the scales produced as a result of the Rasch analysis were stable. At this stage, specialist members of the research team met as appropriate to interrogate the student responses located at similar points on the scale to decide whether or not there were qualitative differences in the nature of adjacent responses with respect to the sophistication of the mathematics or mathematical reasoning involved and/or the extent of cognitive demand required. This process established cut off points between Zones and supported the development of broad descriptions of the characteristic behaviours evidenced at each Zone to serve as interpretations.

Using a process established in the SNMY project (Siemon, Breed, Izard, & Virgona, 2006), the next step in generating the teaching advice was to consider the question "If students located in this Zone are doing …, what is needed to help them move to the next Zone?" Rasch modelling allows both students' performance and item difficulty to be measured using the same unit and placed on an interval scale (Bond & Fox, 2015). Student performances are located at the point on the scale (marked by '#' in Figure 1) where they have more than a 50% chance of gaining the score required for the items located below that

point but less than a 50% chance of scoring at the level required for items located above that point. This means that there are some aspects of the behaviours identified within the relevant Zone that need to be consolidated and established to deepen students' understanding and others that need to be introduced and developed to progress their learning to the next Zone.

Given the strong research base for using low threshold high ceiling tasks in mixed ability groups (e.g., Sullivan, 2011), and feedback from project school teachers that they wanted to explore more effective and engaging ways of teaching, the research team focussed on identifying rich tasks that would address a range of learning needs across a number of Zones.

Results

The approach and findings are exemplified for spatial reasoning. The variable map for spatial reasoning produced as a result of the Rasch analysis is shown in Figure 1. Item responses are ranked from easiest (bottom of the map) to most difficult (top of the map). Those items at the same or very similar levels of difficulty were interrogated to identify similarities or differences in the reasoning required. Responses exhibiting similar levels of reasoning were grouped together to form eight relatively discrete, hierarchical Zones. For example, GTILE2.3 indicates a correct response (coded as 3) to an item (GTILE2) that requires students to minimise the perimeter of a rectangular tiling feature made up of 36 square tiles and identify its dimensions. It is located in Zone 7 alongside GRECT2.4 that required students to correctly identify all 6 rectangles in a display of 12 polygons (all but one a quadrilateral) and explain their reasoning (coded as 4).

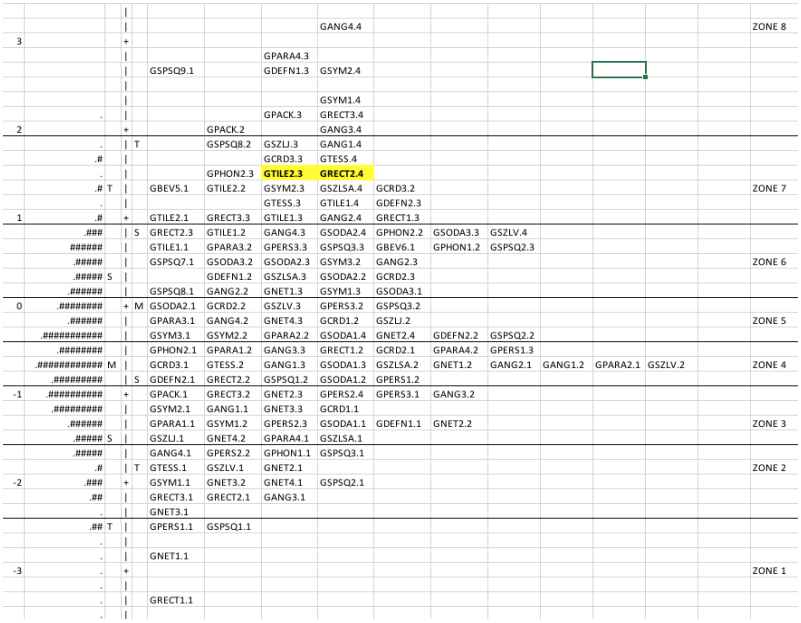

Figure 1. Excerpt from the variable map for spatial reasoning for MR1 and MR2 (n = 1041).

Having agreed on where the Zone boundaries would be located, broad descriptions of the behaviours evident within each Zone were developed and used to consider the teaching and learning implications. Table 1 gives an example of the broad description for Zone 3 of the Spatial Reasoning Learning Progression (left hand column) with related advice for teachers about the types of activities needed to consolidate the learning and move the

students forward on the right. The italicised text indicates the big organising ideas for spatial reasoning. The activities referred to in the Teaching Implications column are available to teachers via a drop box or from indicated websites.

Table 1
Example of Teaching Advice for Zone 3 of the Spatial Reasoning Learning Progression.

Zone 3 Behaviours	Teaching Implications
Hierarchy and properties Uses one or two properties (insufficient) to explain reasoning about shapes (e.g., triangles and quadrilaterals). Beginning to coordinate multiple information sources, but justification limited to using part of the information (e.g., check net to see if it will make a cube). Makes and names familiar 2D shapes, but may not recognise right angles, parallel lines, or properties in non-standard representations. Represents 3D objects in limited ways (e.g., may show only part of the object). Sees objects and groups of objects as a whole but has difficulty in analysing components independently. *Transformation and location* Visualises objects mostly from own perspective Uses coordinates in first quadrant only. Beginning to manipulate visual images and coordinate information. *Geometric Measurement* Demonstrates awareness of measurement attributes. Uses one or two attributes (in-sufficient) to explain their reasoning about measurement (e.g. considers length but forgets impact of width/height) Beginning to be aware of volume and capacity and the relationship between length, area and volume.	**Consolidate and Establish**: *Hierarchy and properties* Provide experiences in different contexts where students explain their reasoning about shape identification (e.g., 'Feely Box'; 'Property Chart' [nrich.maths.org]). Find/identify shapes presented in non-standard orientations using one or two specific properties. Construct specific shapes with compass and straight edge and/or 'Geogebra' using properties. Draw 3D objects from different perspectives and build objects from different perspective drawings. *Transformation and location* Identify 2D shapes that have been transformed under simple reflections and rotations. Use different maps to identify features from coordinates; place items on maps given coordinates for both street and Cartesian maps *Measurement* Order shapes and objects by area and volume and justify choices. Recognise and identify specific angles such as right angle, straight angle and reflex angle. **Introduce and Develop:** Identify parallel lines and right angle in the environment and in diagrams. Use correct geometric language such as diagonal, rotation, perpendicular. Justify answers working in groups to encourage language use. Create an illustrated class chart of geometric language. Examine families of 2D shapes and 3D objects, describing what is the same and what is different. Give directions on a map of their local area using N S E & W and perspective of traveller Introduce formal units of length and use them to calculate of perimeter, area and volume explaining solutions. Explore relationships between length, perimeter, area and volume

The *Feely Box* uses a cardboard box with holes covered by cloth on opposite sides so that a student can put both hands in the box but not see the contents. Thin cardboard 2D shapes or 3D objects are placed in the box – one shape/object at a time. One student feels the shape/object in the box. Groups of students ask questions to which they receive an answer of "yes", "no", "I don't understand, please ask in another way", or "I don't know, please tell me how I could find out". Groups in the class take turns at asking questions until they think they can draw the shape. Discussion centres around how they know and what would be good questions to ask and why. The challenge can be made simpler or more difficult by the nature of the shapes/objects in the box or by restrictions on the questions that can be asked. For example, questions that contain "is it like …?" or the use of names of shapes or objects can

be banned. While this activity is particularly good for Zone 3 it also can support learning in the preceding and later Zones as shown in Table 2. Zone 3 has been omitted as it is described above and only behaviours and teaching implications relevant to the *Feely Box* activity have been included from the Zones 2, 4 and 5.

Table 2
Example of how the Feely Box can be Utilised Across Zones to Support Mixed Ability Teaching.

Zone	Specific Behaviours	Teaching Implication
2	Identifies familiar 2D shapes in situ and as part of simple solids. Beginning to represent 3D objects and uses some related language. Shows awareness of some properties that discriminate shapes. Beginning to use geometric language accurately but cannot coordinate, manipulate/ or check sufficiency of information.	**Consolidate and Establish:** Explore shapes in environment using geometric language to explain and justify their identification. Identify a range of 3D objects and identify some of their features (e.g., square faces on cube) Draw simple 3D objects so that the features are identifiable. **Introduce and Develop:** Use geometric properties of shapes when discussing and justifying their choice of shape names (group discussion is encouraged).
4	Recognises relevance of properties in more complex shapes. Uses some geometric language but has difficulty using all properties or only focuses on one aspect. Recognises some conditions for a shape (e.g., square), but may not attend to all relevant information; has difficulty explaining reasoning. Does not yet recognise necessary and sufficient conditions. Know names of some 3D objects (difference between prism and pyramids). Shows incomplete reasoning in geometric situations.	**Consolidate and Establish:** Explore properties of 2D shapes, including different types of triangles and quadrilaterals. Identify shapes from sets of properties (e.g., *What's my Shape*? It has 2 right angles and at least one pair of parallel lines). Develop language such as diagonal and regular. Investigate families of polyhedra and identify features that relate to the names (e.g. prisms and pyramids). Use a variety of representations of 3D objects including nets, isometric and perspective drawings (in this activity drawings). **Introduce and Develop:** Reason about geometric situations (e.g., discuss good questions and how to justify choices). Describe all properties of a family of shapes/objects.
5	Uses either properties or orientations to reason in geometric situations, and to identify classes of shapes. Recognises parallel lines in non-standard representation. Uses relevant geometric language. Recognises and uses appropriate information to solve problems. Identifies and recognises relevance of multiple representations. Beginning to use sufficient conditions, but unlikely to recognise redundancy (e.g., describes all properties of a square). Uses more complex language in specific context but has difficulty with an integrated explanation.	**Consolidate and Establish:** Explore similarities and differences between shapes. Extend the identification of 2D shapes using properties to include angle and diagonal properties, justifying their choices (depending on the complexity of the shapes in the box). Explore classes of triangles and quadrilaterals, identifying properties. Given one or two properties, identify all possible types of shapes (a pause in the questioning to ask what is it you know now and what are some possible shapes – with reasons). Identify possible 3D objects from a group of properties (again stopping with partial properties to identify possibilities). **Introduce and Develop:** Construct own understanding of the hierarchy of quadrilaterals. Use geometric properties to argue in a variety of situations. Identify lines of symmetry and rotational symmetry on a variety of shapes (this can arise if questions about symmetry are encouraged).

The use of activities such as *Feely Box* with the whole class allows students to be extended from their current knowledge base. The encouragement of discussion and justification within the groups is critical in allowing all students to develop ideas further. The activity also focuses on all three overarching big ideas in spatial reasoning – visualisation, language, and discourse and representations, in this case drawing.

Discussion and Practical Implications

It is often claimed that educational research does not usefully inform the work of teachers or lead to sustained improvements in practice at scale. The RMFII project set out explicitly both to involve teachers in the research and to provide useful, evidence-based materials for teachers that could be translated to practice at scale (Cobb & Jackson, 2011). The decision to focus on algebraic, spatial and statistical reasoning across Years 7 to 10 was ambitious but felt necessary to provide the sort of evidence and resources needed to support a significant and sustained change in practice away from low-complexity, procedural exercises to teaching based on a deeper understanding of the big idea and the connections between them (Sullivan, 2011). Of course, the risk in this is that the grain size is large, and the descriptions of the different Zones may overgeneralise and possibly mask the very particular difficulties that some students might have. It is important therefore that learning progressions are understood for what they are – they do not imply a single, one-way path to learning. Nor are they exhaustively definitive. The descriptions at each Zone are better understood as highly probable behaviours that provide some guidance as to how to interpret or make sense of similar but unreferenced behaviours.

The commitment to work with teachers 'where they were at' (e.g., they could choose assessment tasks and teaching activities relevant to what they were teaching), meant that they were more likely to provide feedback and make suggestions as to how tasks/activities could be improved. Teacher feedback was particularly valuable in refining the scoring rubrics to clarify ambiguities and better reflect the language used by teachers. The tasks and items also proved valuable in generating discussion among teachers. While the content of many of the tasks and items addressed the curriculum, many went beyond this to address the big ideas identified in the literature. These tasks and items prompted rich discussions in the professional learning sessions and helped deepen teachers' knowledge of the mathematics and its connection to other aspects of mathematics.

Mathematical proficiency is a key goal of the *Australian Curriculum: Mathematics* (ACM). Described in terms of understanding, fluency, problem solving and reasoning, each proficiency is characterised in terms of the content descriptors at each level of the curriculum. For example, at Year 8 reasoning "includes justifying the result of a calculation or estimation as reasonable, deriving probability from its complement, using congruence to deduce properties of triangles, finding estimates of means and proportions of populations" (ACARA, 2018). There is little advice beyond this to indicate exactly what might be involved in developing mathematical reasoning or the sort of difficulties students might experience in deducing, justifying and/or explaining their thinking.

Given that the "variability at the classroom level is up to four times greater than at the school level" (Wiliam, 2006, p. 36), it makes sense to work with teachers to build an evidence-based resource that elicits information about student learning in relation to important mathematical ideas and processes – in this case, mathematical reasoning - and provides research informed advice about how to use that information to inform teaching.

A design-based research approach was used by the RMFII project to develop, test and refine learning progressions for algebraic, spatial and statistical reasoning. This involved iterative rounds of assessment and the use of Rasch modelling (Bond & Fox, 2015) to scale the items used from easiest to most difficult in each of the three reasoning strands. The evidence that this produced was then used to identify and flesh out eight relatively discrete levels of increasingly sophisticated reasoning. Referred to as *Zones* to reflect Vygotsky's (1978) notion of the Zone of Proximal Development, the behaviour evidenced in the zones was then used to develop teaching advice that indicates what needs to be consolidated and established and what needs to be introduced and developed at each Zone. The practical implications arising from these research-based outputs[1] are described below.

Evidence-based Learning Progressions. Although originally focused on Years 7 to 10, the assessment trials in non-project schools have shown that the learning progressions[2] are relevant for Years 5 and 6 as well. One of the most valuable practical aspects of the learning progressions is that they identify the big ideas that underpin each content strand of the ACM. Not all content descriptors in the ACM are equal and the identification of big ideas and the connections between them can assist teachers make more informed decisions about curriculum priorities. Another is that they provide teachers with a clearer idea about where students are in their learning and where they need to go to next in relation to the big ideas. By showing how reasoning develops in each area over time, the learning progressions effectively provide a road map that helps teachers navigate the curriculum content areas of the ACM in a way that supports a deeper, more connected approach to teaching mathematics in Years 5 to 10.

Valid Assessment Forms. Well over 88 tasks were developed, trialled and validated to create the learning progressions. Tasks generally comprised more than one item and scoring rubrics for each item were provided to reflect the definition of mathematical reasoning used in the project. The tasks generally enabled all students to make a start and provided opportunities to display their reasoning. For example, the Hot Air Balloon task requires students to (i) construct a graph from a table of values (time vs height), (ii) determine how long the balloon stayed at or above 250 metres, and (iii) identify when the balloon was at 400 metres and explain their reasoning. The tasks with their component items were presented as Forms with 5 to 7 tasks per form. Mixed Forms (tasks from two areas) and Standard Forms (tasks from one area only) were trialled to explore reasoning both within and across strands. Feedback from project schools suggested that they would be more interested in standard forms. As a result, four Standard Forms for each strand have been developed together with the associated scoring rubrics. Maximum score totals are different for each Form to prevent the inappropriate use of raw scores. This necessitated the provision of a Raw Score Translator for each Form that can be used to locate students on the respective learning progression for mathematical reasoning. The Forms can be used as pre-tests to determine where students are in their learning with respect to the relevant learning progression and the information derived from this can be used to inform planning and teaching. A parallel Form can then be used as a post-test to determine if there has been a qualitative shift in student behaviour and to provide feedback on the effectiveness of what was planned and taught.

Research informed teaching advice. The evidence that underpins the learning progressions was used to develop broad descriptions (i.e., interpretations) of what students are able to do and what they may find difficult at each zone of each learning progression. This in turn supported the development of targeted teaching advice for each zone that is focused on consolidating and establishing the content and reasoning evident in the behaviours associated with that zone as well as introducing and developing the key ideas,

strategies and forms of mathematical reasoning needed to progress to the next zone. An example of the teaching advice for one zone in the spatial reasoning learning progression is provided in the paper. Given the demonstrated efficacy of reform-oriented pedagogical practices at this level (e.g., Boaler, 2006), a key consideration in preparing the teaching advice was to include a range of indicative, rich tasks, investigations and/or problems (e.g., the *Feely Box* task) that can be used with mixed ability groups to address aspects from more than one zone. Many of these multi-zone activities have been drawn from existing, well known resources such as maths300 (http://www.maths300.com). For example, *Mountain Range Challenge* (adapted from *Unseen Triangles*, lesson 20 maths300) uses the context of a mountain range to explore a visual growing pattern based on equilateral triangles. It is referred to in the teaching advice for Zones 3, 4, 5 and 6 of the algebraic reasoning learning progression.

Professional Learning. Wiliam (2006) emphasised the critical importance of professional learning in sustaining an evidence-based approach to teaching and learning mathematics. Annual residential and regular online professional learning sessions were provided throughout the project. Among other things, the sessions explored what was involved in algebraic, spatial and statistical reasoning, and how this could be supported through the use of rich tasks in mixed ability groups. In partnership with AAMT, many of these have been developed into a series of online professional learning modules, the aim of which is to support school-based, teacher learning communities to understand, explore and use the resources provided by the RMFII project to make better, more informed decisions than they might have made otherwise about what to teach and how they might teach it to more fully engage students in the enterprise of learning mathematics.

Notes:

1. It is anticipated that all of the outputs from the RMFII project will be available from mid 2018 via the Dimensions Portal being developed by the Australian Association of Mathematics Teachers
2. The final forms of all three learning progressions will be included in a forthcoming book to be published by Sense

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Symposium: Stimulating Proportional Reasoning through Engaging Contexts

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Proportional reasoning is fundamental to successful operation with many topics in the primary school curriculum, including fractions, decimals, place value, ratio, proportions and percentages. The literature continually documents students' difficulties with these topics and by extension, their limited proportional reasoning capabilities. Research into proportional reasoning has a long history and continues to generate strong interest. Why is this type of reasoning so elusive and why is it so difficult to develop?

In this symposium, our aim is to continue to emphasise the importance of proportional reasoning and its pervasiveness throughout the school curriculum and to share alternative ways to promoting students' proportional reasoning capabilities. The development of proportional reasoning is underpinned by multiplicative thinking. Our concern is that multiplicative thinking in primary schools is too often thought of in terms of repeated addition leading to "equal groups", multiplication facts, and algorithms. Many teachers are not aware of the potential to support students' proportional reasoning in terms of rate and multiplicative comparison. To address the theme of this each researcher has critically reflected on how meaningful problems can serve to build conceptual understanding of proportionality.

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Creating Opportunities for Multiplicative Reasoning Using "Elastics" as a Context

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The representations, tools and task discussed were designed as a response to a pedagogical challenge: How can the "times as many" idea of multiplication be investigated meaningfully? The task involves experimenting with different types of elastic to test their "stretchiness". Reasoning during the task, with the affordances offered by the tools, learners as young as 11 years old were able to reason in terms of the multiplicative comparison. We discuss what mathematical insight, activity, and understanding is available to learners via engagement with the task.

Here we report the trialling of a task intended to stimulate proportional reasoning, particularly the idea of multiplicative comparison. Concepts underpinning multiplication are complex and the *times as many* aspect of multiplication is challenging for students to learn and for teachers to teach. However, the idea is important for students' understanding of ratio, proportion, fractions and scale. Usually students are expected to demonstrate an understanding of these topics in the secondary years. We believe that the foundational thinking for these mathematical concepts can be laid in the upper primary school.

Background

The central importance of exemplification in mathematics is at the heart of this study. We agree with Sfard (1991) in connecting the genesis of mathematical knowledge with the process of coming to know. Sfard saw examples as raw material for generalizing processes and conceptualizing new objects. The mathematical example we offer students is intended to illustrate the concept of *times as many* and uses an "investigative approach in which learners experience the mathematisation of situations as a practice, and with guidance, abstract and re-construct general principles themselves" (Bills et al., 2006, p.1-128).

Multiplicative reasoning is vital for children's mathematical development. It is not simply a generalisation of additive reasoning; multiplicative reasoning requires a qualitative shift in understanding (Vergnaud, 1983). Research indicates that many students rely on additive reasoning when the problems require multiplicative reasoning (Anghileri, 2001). Current teaching practices may be unintentionally reinforcing this additive thinking rather than challenging it (Downton & Sullivan, 2013; 2017). The *times as many* idea, first described by Greer (1989) is a multiplicative comparison. It is considered difficult for students to learn because the idea is linguistically and conceptually hard. However, this type of comparison is present in everyday life.

The research question was: Does measuring the stretch of elastic provide an investigative context that leads students to conceptualise the situation as a multiplicative relationship?

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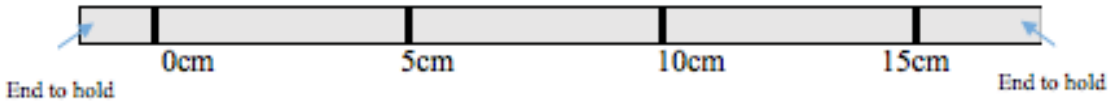
Method

We characterised the study as design research because it was: interventionist; iterative; process oriented; utility oriented, practical in a real context; and theory oriented (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006).

Fifty-six 11-12 year-old students in Years 5 and 6 in a Victorian primary school participated in two separate mathematics classes. Each one hour lesson was taught by the second author. The two classroom teachers were briefed in advance, and were present when the lesson was conducted. Thus, three experienced teachers listened and observed as the students experimented and searched for patterns in their results.

The problem: *How stretchy is elastic?* was posed introducing four types of elastic: shirring elastic, hat elastic, 5mm wide elastic, and rubber elastic, as we called them. We displayed the names and a 20-25cm sample of each to give students the vocabulary necessary to distinguish between them. Students were invited to form groups of two or three and collect their materials: each elastic sample length of approximately 20 centimetres, a metre rulers/tape measure, pencils, and strips of paper. This gave students a chance to handle the elastic samples and think a little about them. We then set up the "test" method demonstrating marking the one elastic in 5 centimetre sections as shown:



We demonstrated stretching the elastic as far as possible, taking care not to over-stretch it so that it could not spring back into shape. We asked the students to experiment and use a metre ruler to find out where the original marks stretch to. We encouraged students to keep records of their results, to swap with their partner and repeat the experiment. Students were expected to consider their findings and explain what they noticed. The lesson concluded with verbal reports from three teacher-selected groups of students.

Data were collected using classroom observations, work samples and video. Mathematical conversations (Cheeseman, 2009) were held with students as they worked, photos were taken of work in progress and finished reports. In addition, video was taken of verbal reports. We looked in detail at the students' finished written work to analyse student responses to the task. We treated the written work as representing the group's thinking.

Results

We began with the evidence on paper and using a grounded theory approach, put the formed three broad categories: work showing clear evidence of *times as many* thinking; recordings that presented raw data in systematic ways with experimental results potentially showing a multiplicative comparison but with no evidence of *times as many* thinking; and recordings showing only final lengths. Each of these categories will be illustrated in turn.

Category 1. Clear evidence of times as many thinking

Times as many thinking is evident in Figure 1. Although symbols are invented and not entirely consistent, the top left quadrant reveals that these students noticed that the mark they made at 5 cm was equal to 15 cm when the hat elastic was stretched, was a factor of three. The 10 cm mark became 28 cm, which was approximately or about three times the original (A 3x). Throughout the recording a capital A is consistently used to denote the approximate nature of the multiplicative relationship.

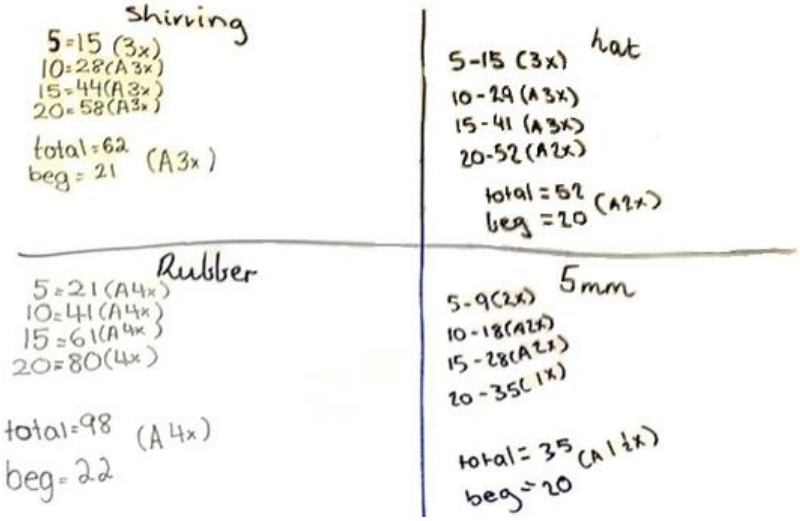


Figure 1. An example of Category 1. Times as many thinking

Some students conducted a very careful and precise test of each of the elastics and recorded their results in a clear and logical manner (Fig 2). Whether they could not see any patterns in the data due to the error margins in the measurement, or whether they did not look for patterns is unknown.

Category 2. Systematic experimental results with no evidence of identified patterns.

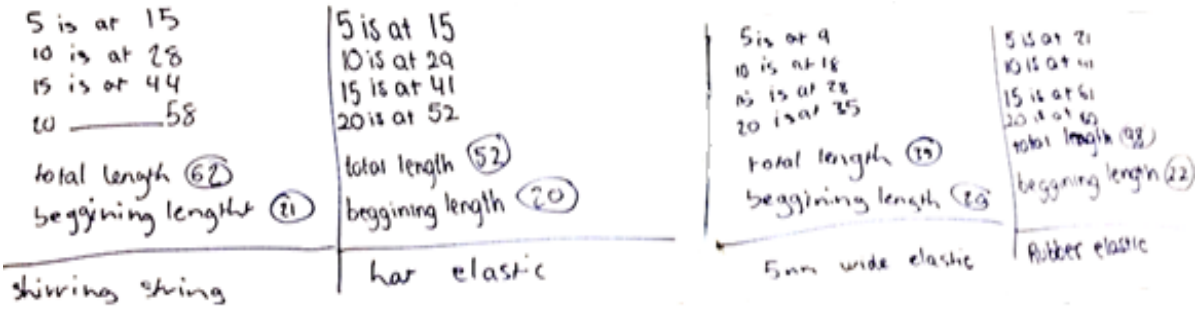


Figure 2. Category 2. Evidence of experimental results but no multiplicative thinking.

Category 3. Results focused on a comparison of maximum length stretched

Work samples in category 3 showed a student focus on the maximum length to which each elastics sample could each be stretched. These students had apparently transformed the problem from, "How stretchy is the elastic?" to "How far does the elastic stretch?" This seemingly small change of wording changed the focus from the features of the elastic to the greatest length that can be attained. This thinking is illustrated by the following report:

> We measured the elastics one at a time and marked them each multiple of 5 … on the elastic. The next step was to stretch the elastic and see how far it went. Then we saw what was the last mark we did on the elastic and saw how far the elastic stretched and we worked out the difference.

Some work samples in this category saw the stretch of the elastic as an additive action where their calculation involved subtracting the original length from the final length.

Discussion and Implications

We cannot say exactly how many students have begun to develop emerging concepts of multiplicative comparison because in groups of students it is sometimes not clear which of the individuals has which concept. What we can definitely say, based on this experiment, is that some students 10-12 years of age can deduce *times as many* relationships in sets of data. Our observations suggest 13 (23%) of the 56 students in the classes we observed looked for multiplicative patterns in their results.

This was the first time, as far as we could ascertain, that the students were offered the opportunity to consider times as many relationships at school. We think that with follow-up learning opportunities, early *times as many* concepts could be established and possibly initiated for other students. We are keen to experiment with potential ideas for further learning. For example, because results the students collected were meaningful and accurate we would use them with the whole class by displaying them and challenging the students to search for patterns in the figures. In this way the finding of *times as many* ideas could be made explicit. Another possible follow-up lesson could be devised using different elastics samples or rubber bands to test the "stretchiness" using the same experimental methods.

The *times as many* aspect of multiplicative reasoning is not presented to students as often as it might be. We wonder whether contexts that exemplify the concept are difficult to find. We encourage teachers to think of other everyday situations that might serve to help students to conceptualise multiplicative comparison ideas of multiplication.

We recommend a sequence of lessons on the idea of "stretch factors". As we think that several mathematical examples would likely serve to establish multiplicative relationships for some of the students who were thinking additively and a sequence of investigations would consolidate the learning for students who have developing times as many concepts.

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Sharing the Cost of a Taxi Ride

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We aimed to explore the extent to which a challenging yet accessible real world financial context where two people stand to gain from sharing a taxi ride might stimulate students' mathematical exploration and discovery related to multiplicative thinking and proportional reasoning. Data were collected from 37 Year 5 and 6 students (10-12 years of age) in a Catholic primary school in suburban Melbourne. The findings reveal that the majority of students had some intuitive understanding of how to solve a financial problem that involved rates, and at least half of them used either proportional reasoning or multiplicative thinking. We argue that, given the right problem context, upper primary school students can be encouraged to engage in proportional reasoning earlier than the Australian Curriculum requires.

Given the increasingly challenging economic conditions and financial products and services we face, the need to prepare young people to make informed financial decisions is a topical priority for schools and teachers. Money and financial mathematics features explicitly in the Australian Curriculum (AC) Mathematics (M). There are a number of everyday financial contexts that require multiplicative thinking and proportional reasoning that might be meaningfully explored in the upper primary years of schooling. These include sharing costs like a restaurant bill, transport and accommodation in ways that are fair, and accounting for fluctuating monthly expenses over the course of an annual budget.

Multiplicative thinking is conceptually complex and yet the intended curriculum does not reflect this complexity. While problem solving and reasoning are two of the four AC:M proficiency strands, ratio and proportional reasoning are not suggested within the AC:M until Years 7 and 8. The actual term proportional reasoning is not stated until Years 9 and 10, where a need to "interpret proportional reasoning" is specified (ACARA, 2015, npn).

Meanwhile, teachers seem to have difficulty finding productive approaches to teaching all but the simplest multiplicative "equal groups" ideas (Downton, 2010). Related to this, various studies have found that Year 7 and 8 students' difficulties in solving problems involving fractions, decimals, ratio and proportion are attributable to a reliance on additive reasoning when multiplicative reasoning is required (e.g., Hilton, Hilton, Dole, Goos & O' Brien, 2012). Others have argued that students' lack of proportional reasoning is directly related to their limited experience with different multiplicative situations, including rate and ratio (see Greer, 1988).

With this situation in mind, we aimed to examine the extent to which a challenging yet accessible real world financial problem might stimulate students' exploration and discovery related to multiplicative thinking and proportional reasoning. Our research question was: In what ways do 10-12 year old students formulate and employ mathematics when solving a real world financial problem that involves sharing costs?

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The Research Design

We will present an aspect of ongoing classroom research in a small Catholic primary school in suburban Melbourne, Australia. Data were collected in the school's two Year 5 and 6 composite classrooms. The first author presented a 60 minute modelled lesson exploring a task where two people share a taxi ride. The task deals with ideas of rate and was presented as follows:

> Catching a taxi
>
> The taxi fare is $3 flagfall (what you pay when you get into the taxi) and then $1.50 per km after that. It does not matter how many people are in the taxi.
>
> Mike and Matt decide to share a taxi because they are going in the same direction but to different houses. The journey to Mike's house is 20 km, then a further 30 km to Matt's house.
>
> How much should each of them pay for the taxi? Explain why your suggestion is fair for both people.

The fact that the characters Mike and Matt are travelling different distances means that sharing the cost of the trip evenly may not be the fairest solution. We considered the task an appropriate choice for examining further the ways and means by which real world problem contexts can stimulate student mathematical exploration and discovery, particularly in terms of proportional reasoning.

The two lessons were audio and video recorded. Students' worksheets, were collected at the end of each lesson. Across the two classes, there were 37 student participants.

The OECD PISA 2012 mathematical literacy assessment framework (OECD, 2013) served as useful framing for data collection and analysis. The framework depicts a modelling cycle involving three mathematical processes that students apply as they attempt to solve problems - formulate, employ and interpret (OECD, 2013). These mathematical processes might be understood as stages of realistic or applied modelling through which a real-world problem is solved (Stacey, 2015). First, the problem solver identifies or formulates the problem context mathematically. This involves making various assumptions to simplify the situation. In so doing, the problem solver shifts the problem from the real world to the mathematical world (OECD, 2006). Next, the problem solver employs mathematical knowledge, skills and reasoning to obtain mathematical results. This usually involves mathematical manipulation, transformation and computation, with and without physical and digital tools. Finally, the problem solver interprets the mathematical results against the problem context. This involves the problem solver evaluating the adequacy and reasonableness of the mathematical results, shifting them back to the real world (OECD, 2006).

Student worksheets were analysed for the purpose of categorising how students formulated and employed mathematics. Hence, we examined the thinking evident in the response, but also the mathematical strategies used. We were also interested in the explanations students gave about why their suggestion was fair for both people, as these explanations revealed insights into how students interpreted their solutions against the problem context. Using a grounded theory approach (Strauss & Corbin, 1990) four categories for formulating the problem emerged. These are presented below, from most to least sophisticated:

A. Some students perceived the journey as taking place in two parts, but suggested that as the men shared the first 20km of the distance, they should also share the

cost of that leg of the journey. In this scenario, Mike and Matt would pay $16.50 each to travel the first 20km and Matt would pay an additional $45 to travel the next 30km alone, meaning a total of $16.50 for Mike and $61.50 for Matt. This approach showed a sophisticated grasp of the problem context, as well as proportional reasoning.

B. Some students perceived the journey as taking place in two parts, with Mike paying $31.50 to travel 20km and Matt paying $46.50 to travel the additional 30km. In this case, there is no benefit for Mike in sharing a taxi, but there is a saving for Matt.

C. Some students perceived that Mike and Matt should pay separately based on the distance travelled. In this scenario, students typically suggested that the flagfall should be shared evenly. Here, Mike would pay $31.50 to travel 20km; and Matt would pay $76.50 to travel 50km. Unchecked, such a misconception would result in a windfall for the taxi driver.

D. Some students calculated the total cost of the journey ($78) and divided this by two. In this scenario, Mike and Matt would share the cost evenly, paying $39 each. Here, there is no benefit for Mike in sharing a taxi – in fact he would pay more than if he was to travel home alone.

Within each of the above categories, three categories of strategies for employing mathematics were readily able to be identified: additive thinking; multiplicative thinking; and proportional reasoning. A fourth category - no documented strategy - was applied to student worksheets where there was an answer, but no mathematical working. It is important to note that this category signals the possibility of mental computation.

Results and Findings

Table 1 presents the two levels of categorisation described above: how students formulated the problem (rows); and how they employed mathematics (columns). Of the 37 student participants, three students noted more than one solution. As each of these solutions was considered separately, a total of 39 responses were categorised and tabled. Four incidences of student mathematical error were noted.

Table 1
Analysis of the Way the Problem was Formulated and the Mathematical Thinking Employed

	Way mathematical thinking was employed				
Way of formulating the problem	Proportional reasoning	Multiplicative thinking	Additive thinking	No documented strategy	Total
A	3	4	0	0	7
B	2	6	3	5**	16
C	1	4*	3	4	12
D	0	1	1*	2	4
Total	6	15	7	11	39

* indicates one error in computation

The student worksheets revealed that 90% of students were able to correctly calculate an answer based on how they mathematised the problem. It was evident that the multiplicative

nature of the reasoning required to find a solution was clear to the students. Twenty-one students (54%) used proportional reasoning (6) or multiplicative thinking (15), with a total of seven using repeated addition as a way of finding a solution. It is possible that those with no documented strategy (11) also used these methods, but their records were unclear. We can say that the problem was largely understood as a multiplicative situation.

During our presentation, these results, supported by examples of student worksheets, will be presented and discussed with the intention of arguing that the task, lesson structure and pedagogical architecture encouraged proportional reasoning at a younger age than it appears the Australian Curriculum: Mathematics.

Conclusion and Implications

The findings suggest that the majority of students could formulate a real world financial problem that involved proportional reasoning and employ mathematics in a way that reflected the approach they selected. Further, at least half of the students used proportional reasoning or multiplicative thinking, which suggests that Years 5 and 6 students can not only attend to rate tasks such as this, but some can appropriately apply proportional reasoning. Catching a taxi seemed to provide an appropriate hook to introduce proportional reasoning – a concept that does not appear in the Australian Curriculum: Mathematics until Years 7 & 8 – to upper primary school students. The findings suggest that, given the right challenging yet accessible real-world problem context, upper primary students can explore and discover more complex mathematical tasks than curriculum writers, task designers and teachers might assume.

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Authentic Numeracy Contexts for Proportional Reasoning – the Case of the Seven Summits

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This paper presents a case study of one primary school teacher's journey of realisation about the importance of proportional reasoning for numeracy. Through immersing students in a rich numeracy investigation to meaningfully compare the world's tallest mountains, this teacher reflects on authentic contexts and hands-on experiences for promoting and enhancing students' multiplicative thinking. This study included analysis of interview data, classroom observations and student artefacts against a rich model of numeracy that served to emphasise the power of meaningful contexts for promoting multiplicative comparison.

Introduction

Proportional reasoning is one of the most commonly applied mathematics concepts in the real world (Lanius & Williams, 2003). Adjusting measures of ingredients in a recipe, adding sugar for the perfect cup of coffee, estimating the time to travel when found in a traffic jam, choosing the right food storage container when saving left-overs, calculating percentage discounts on sale items, are some everyday tasks that require proportional reasoning. As proportional reasoning is required in so many everyday situations, it is essential to numeracy (Dole, Goos, Hilton & Hilton, 2015). However, students' difficulties with proportional reasoning tasks are well-documented (e.g., Lamon, 2007).

In the absence of knowledge of ways to promote proportional reasoning, teachers may revert to skill-based approaches that may serve to hamper students' proportional reasoning development and capacity to use proportional reasoning in complex and unfamiliar situations. Tasks requiring proportional reasoning are a continual stumbling block for so many students in many areas of the curriculum, which suggests the need for a broad-spectrum, multi-pronged strategy for action.

Theoretical Framework

The theoretical framework that guided the research reported in this paper draws from two fields of maths education research: (1) proportional reasoning, and (2) numeracy.

The essence of proportional reasoning is an awareness of how two quantities are related in a multiplicative sense. The American Association for the Advancement of Science (AAAS) (2001) identified two key components of proportional reasoning: Ratios and Proportion (parts and wholes, descriptions and comparisons, and computation) and Describing Change (related changes, kinds of change, and invariance). Lamon (2007) outlined central core ideas for proportional reasoning as rational number interpretation, measurement, quantities and covariation, relative thinking, unitizing, sharing and

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comparing, and reasoning up and down. These two sources highlight the encompassing nature of proportional reasoning and the fact that it is more extensive than simple rules or calculation procedures, and certainly more than promoting multiplication as repeated addition. This theoretical framework has underpinned the design of tools for assessing middle school students' proportional reasoning (see Hilton, Hilton, Dole & Goos, 2016).

A rich model of numeracy has been proposed by Goos (2007). The model highlights five elements of numeracy as comprising mathematics knowledge, use of tools, positive dispositions, a critical orientation, and grounded in context. The model has been found to support teachers in designing rich learning tasks to promote numeracy (Goos, Geiger & Dole, 2013). Drawing on this theoretical framework, this project aimed to answer the following research question: to what extent can a rich model of numeracy and a broad conceptualisation of proportional reasoning support teachers in building curriculum knowledge for proportional reasoning?

Design and Approach

This paper reports on a single case study of a teacher who participated in a large project that involved middle school teachers from five school clusters over an extended period of three years. In this project, we designed a professional development (PD) program to build teachers' awareness of the pervasiveness of proportional reasoning throughout the curriculum. Teachers tailored and trialled teaching sequences on ideas and suggestions presented at the PD. The researchers visited project teachers' classrooms in-between the PD sessions and offered support, advice, and encouragement. The case study reported here draws from interview data (ID), classroom observations (CO) and student artefacts (SA) to describe one teacher's journey of developing awareness of the pervasiveness of proportional reasoning and how engaging learning experiences can support all learners in developing proportional reasoning capabilities.

Results

Luke (pseudonym) is a teacher of a composite upper primary school class of 27 students. The school is located in a rural community. Prior to commencement in this project, Luke had planned to teach a unit based around the seven summits (the highest mountain peaks in each of the seven continents) drawing on his personal interest in mountain climbing. He commenced this unit with students "undertaking some basic mapping and activities involving coordinates". Initially he felt that his students had a "pretty good understanding of how to use scale, but ratio, they didn't understand" (ID). Luke elaborated that he had provided students with some mathematical exercises where the scale was indicated as 1cm:1km. The students successfully completed conversion exercises to determine the length between particular places based on this scale. Luke's comment was in relation to students' conceptualisation of magnitude of the scale in which they were working. This was evidenced when he referred to a map of Australia and Oceania that was located on the classroom wall. The scale was presented as 1:15 000 000 (CO). Luke reflected on how he attempted to make this scale meaningful to the students: "I explained to them in a very poor way, that according to the scale on that map that Australia is 15 000 000 times bigger than the image of it on the map. Of course no one can visualize that." At the end of the lesson, Luke pondered how he might assist students to "visualise" the ratio.

Luke's next (Art) lesson focused on scale drawings of the human body. At the beginning of the lesson, Luke used the word "proportion" and drew students' attention to the structure

of the human body. Students paired with a partner and compared the length of their arms, noting where their arms finished, and whether the arms were longer or shorter than other parts of the body. They were then instructed to sit with their partner and draw each other as life-like as they could. Many students expressed frustration with their drawing as the "proportions were wrong" (CO). At the end of the lesson, the drawings were in a rather crude form. Noting students' frustration with their drawings, Luke asked the students to measure the height of the person on their drawing and to make a calculation of the actual size of the person. Students readily determined that the size of the paper was approximately 20 cm. One student stated that the picture would need to be enlarged 20 times to be lifesize. Many other students readily agreed until there was growing realisation that "twenty times twenty - they're not that big" was not an appropriate scale-factor. The mathematical behaviour exhibited by the students was exciting to Luke: "they were estimating a ratio, then calculating it and then refining it…Some kids in my class are not into estimating at all, they won't do it, they just feel that there is too much room for going wrong" (ID). Students then began to spontaneously engage in undertaking multiplicative comparisons: "And from there we looked at their drawings and they actually worked out just looking at the height from head to toe, they worked out an actual correct scale for that drawing. We picked one part of the body and that part of the body was 1cm on the paper, so then it must therefore equal a certain amount in real life" (ID). There was a new sense of industry in the classroom from this point. Students used rulers, tape measures and calculators to take measures of body parts and to then draw them on the page. In efforts to increase the realism of their drawings, many students were seen to sketch and then to erase sections of their work, and then to redraw elements after making further measurements or observations of their partner (CO). Luke reflected on how students would measure each other's noses and then compare this length to the nose drawn on the paper. They quickly saw "for example, the arms might be in proportion but then the nose was almost as big as an arm in real life." Luke recounted how he saw students "slapping their heads" and exclaiming "oh no, the eyes would be this big in real life". Luke described how he saw students taking measurements of different parts of their partner and selecting a scale of 1 to 7. Luke noted the pride in which the students viewed their second sketch compared to their first: "they compared their drawings and talked about their first sketch against their refined drawing and they were quite proud. They were telling me how terrible they are at drawing, but it was almost like the scale and ratio had given them a formula for drawing more accurately." The students were very keen to pin their refined drawings around the room.

It was from this experience that Luke directed the learning to the end-goal he had from the start: the seven summits. Luke found that the students had little trouble in representing the mountains to scale. After introducing and discussing the seven summits (and students exploring further via the internet), Luke instructed students to create a triangle from an A4 sheet of paper that would represent Mt Everest. The students measured the height of the triangle as 18.5 cm. "Then they got the height and divided it by the scale to give them the measurement of the height of each mountain. They reasoned that if Mt Everest at 8848 m scaled to 18.5cm, then the scale was 1 cm: 478 m." The result was a graphical representation of the seven summits, each mountain a different coloured triangle, all neatly line up as a sequence of triangles of descending heights.

Luke could see the rich numeracy experiences that he had created for the students and how the activities built on one another to continue to fuel students' interests and learning:

> yeah so initially it started out as our SOSE and Science, but then health because we started talking about pulmonary oedema and hypoxia the kids had no understanding of what these meant but just this

week the kids finished off a dictionary of all of the different mountaineering terminology, equipment and the conditions or ailments that affect them. So it ended up going into our health curriculum because the kids would then argue about whether hypoxia and pulmonary oedema were diseases, because you can't catch it, so how are they diseases? So it's been a really good term, when I first walked in and told them that we were going to be doing it for a term, none of them cared.

Discussion and Conclusion

In analysing the lesson sequence from a numeracy perspective (Goos, 2007), we can see the richness of these experiences. The sequence was grounded in *context*, with Luke's personal interest in mountain climbing fuelling and generating continued student interest in natural phenomena. The students applied their *mathematical knowledge* to the context, making mathematical estimations and reasoning and justifying their calculations. They used *tools*, not only via the use of measuring instruments and calculating devices, but also through the creation of the visual representational tool of the seven summits. They were developing positive *dispositions*. They were clearly enjoying the learning in which they were engaged. They also took risks in calculating and estimating and sharing their ideas, rather than seeking confirmation from the teacher at every step. They were developing a *critical orientation*, not only through their growing awareness of health issues and mountain climbing, but on reviewing the reasonableness of their results. The lesson sequence was also a multi-directional approach to developing ratio and scale and proportional reasoning. The purpose for scale and ratio emanated from the task, and all students were seen to build their confidence in relation to dealing with scale and large numbers. When discussing their mountain representation pictures, the students would readily discuss how they approached the calculations and could confidently discuss the magnitude of the mountains. Of most interest was the complex scale factor of 1 cm : 478 m that was confidently discussed by all students (CO).

This case study has presented one teachers' journey of realisation about the power of a multi-dimensional approach to proportional reasoning through a rich numeracy investigation. With the end product in mind - the seven summits - the teacher did not revert to a skill and drill lesson of scale and ratio. In fact, early lessons of this type were regarded to be of minimal value for the end-goal: an appreciation of the size of the largest mountains in each continent. This case study serves to remind us of the value of non-sequential approaches to developing multiplicative thinking and proportional reasoning, through learning experiences that are inclusive of all learners.

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Teachers' Perceptions of Students' Development of Multiplicative Thinking

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Having an understanding of the key ideas underpinning multiplicative thinking is critical for future learning beyond the primary school years. The shift to multiplicative thinking can be challenging for both students and teachers due to its multifaceted nature. This paper reports on a pilot study of professional learning in schools that identified multiplicative thinking, an area of concern. We sought to explore in situ professional learning (school-based) within 14 primary schools across a six-month period. Our findings suggest that in situ professional learning had a positive impact on teachers' mathematical content knowledge and pedagogical content knowledge.

In the current political climate, there is increased pressure on teachers to improve student-learning outcomes in mathematics education. In particular, there is a concern regarding the number of students in Years 5 to 8 who rely on additive thinking to solve proportional reasoning problems when multiplicative thinking is required and those who cannot distinguish whether a task requires additive thinking or multiplicative thinking (Van Doreen, De Bock, & Verschaffel, 2010). This may be attributed to an emphasis in the early and middle primary years on multiplication as repeated addition, equal groups and arrays. Alternatively teachers' limited understanding of the complexity associated with the development of multiplicative thinking and their knowledge of the different multiplicative structures may be contributing factors.

Theoretical Framework

A recurring theme in the literature is that multiplicative thinking is a crucial stage in students' mathematical understanding, the basis of proportional reasoning, and a necessary pre-requisite for understanding algebra, ratio, rate, scale, and interpreting statistical and probability situations (e.g., Hilton, Hilton, Dole, Goos, & O'Brien, 2012). Some scholars argue that the difficulties associated with students' lack of proportional reasoning are related to their limited experiences of different multiplicative situations such as multiplicative comparison (times-as-many) and rate/ratio (e.g., Greer, 1988) or to their reliance on additive thinking when multiplicative thinking is required (e.g., Van Doreen et al., 2010). Greer (1988) suggests that students need to engage in multi-step contextual problems that include more complex numbers so that the appropriate operation cannot be intuitively grasped.

In relation to professional learning models, research suggests that professional learning for teachers needs to be situated in realistic contexts as part of the on-going work in schools, in contrast to one-off models of professional development (Bruce, Esmonde, Ross, Dookie, & Beatty, 2010). Teachers are seen as learners and schools as learning communities (Clarke & Hollingworth, 2002). Bruce et al., (2010) support Clarke and Holllingworth's notion of

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professional learning (PL) being embedded in classroom experiences and practices within the school context, and argue that such professional learning is characterised as occurring in sustained and iterative cycles of planning, practice and reflecting. Dole, Clarke, Wright, and Roche (2008) engaged teachers in a focused professional learning program on teachers' understanding of proportional reasoning. They found that although there were marginal differences in teachers' proportional reasoning, teachers had the language to discuss proportional reasoning, and could articulate the difference between additive and multiplicative thinking.

Informed by the research literature, a pilot PL program focused on developing teachers' knowledge of multiplicative thinking was situated within each participating school. The study aimed to address the research question: What is the impact of an in situ, spaced, professional learning on teachers' pedagogical content knowledge for developing multiplicative thinking in their students?

Method

The purpose of this mixed methods study was to examine the perceived impact of an in situ PL program on teachers' pedagogical content knowledge related to multiplicative thinking. We characterised this pilot study as an effectiveness study (Bruce et al., 2010) as it studied PL opportunities for classroom teachers within their own setting and measured their pedagogical content knowledge (PCK) through the use of an online survey, administered pre and post the PL.

The structure of the professional learning (PL) was informed by the abovementioned research. The research team, led by the first author, developed five 90-minute PL modules with co-researchers (Teaching Educators) from a New South Wales Catholic Education System. Each module focused on an aspect of multiplicative thinking and pedagogy, and included challenging tasks, professional readings and between session classroom tasks. The co-researchers facilitated the PL at participating schools across terms two to four, and provided in classroom support in Years 3 and 4, due to the identified need and high proportion of students still reliant on counting based strategies.

Fourteen primary schools (approximately 230 participants: classroom teachers, specialists, lead teachers, and leadership teams) across the diocese agreed to participate in this research, as multiplicative thinking was their PL priority. The data collection instruments included teacher online surveys, focus group interviews and teacher reflective journals. The data reported here pertain to one open response question from the online teacher survey: How do you believe students develop multiplicative thinking?

All responses were entered into a spreadsheet, coded and categorised through the analysis of data using a grounded theory approach (Strauss & Corbin, 1990). If a teacher wrote multiple ideas, each was coded as a separate response. The first two authors independently coded the teachers' responses using open coding to identify key themes.

Results and Discussion

Table 1 shows seven themes developed from the data analysis and teachers' illustrative responses to the aforementioned question. Pre PL 37% of respondents and post 25% believed that students develop multiplicative thinking by using some form of representation that leads to the development of abstract thinking. There is a noticeable shift in responses post the PL from a focus on aspects of general pedagogy (themes 1, 4, and 7) to focussing on aspects relating to multiplicative thinking (themes 2, 5, and 6).

Table 1

Percentage of Responses Relating to How Students Develop Multiplicative Thinking

Theme	Pre (n= 244)	Post (n= 236)	Illustrative of comments written by teachers
1.Materials and representations moving to abstract thinking	37	25	Pre: By working with concrete materials, partial models, to abstract thinking. Post: Build up multiplicative foundation, move from visualising arrays to abstract thinking and reasoning.
2. Moving from additive to multiplicative thinking	12	22	Pre: From additive thinking to applying known facts. Post: Use of arrays, and times as many that encourage multiplicative thinking and reasoning strategies such as known and derived facts that shift their thinking.
3. Relationship: multiplication and division	4	10	Pre: Knowing link between multiplication & division Post: When engaging with problems/tasks that require thinking about inverse operations.
4. Engage in real life problems and open tasks	30	12	Pre: Being exposed to real life problems. Post: Engage in real life multiplicative tasks and multi step word problems and open tasks that encourage MT
5. Use of multiplicative language	4	11	Pre: Experience the language of 'groups of', 'arrays' Post: Opportunities that expose them to multiplicative language such as commutativity, times as many.
6. Experiencing multiplicative structures	0	13	Pre: Provide 'groups of' and 'arrays' activities. Post: Regular experience with challenging problems relating to arrays, times-as-many, allocation and rate.
7. Teacher demonstration and practice	13	7	Pre: Teacher modelling strategies, and practice times tables. Post: Having strategies shared by students and reinforced by teachers and through practice of a variety of questions.

Prior to the PL 80% of responses related to general pedagogical approaches to mathematics, compare to 44% post the PL. In contrast, 56% of responses related to multiplicative thinking post the PL, which was more than double that of the pre PL (20%). This appears to suggest that the program challenged existing ideas about students' development of multiplicative thinking and resulted in a shift in teachers' perceptions.

We anticipated a reduction in a procedural approach to learning (Theme 7) and using materials (Theme 1). While there was some reduction as a result of the PL it is evident that these views are strongly held, particularly in relation to use of materials to support students' shift to abstract thinking and teachers wanting to do explicit demonstration.

Teachers became increasingly aware that students' development of multiplicative thinking is linked to shifting from additive thinking and counting based strategies (Theme 2, Table 1). Many responses indicated that some powerful and engaging tasks facilitated the transition from additive to multiplicative thinking. Nick, a Year 4 teacher, recorded the following in his reflective diary after exploring the carrot patch task with his students.

> Having to imagine the missing carrots in the array was powerful and the kids were using distributive property and the language of arrays, partitioning, factors and multiples.

The biggest shifts related to themes four and six. While we were initially surprised that there was a decline in teachers' focus on the importance of engaging students in real life problems and open tasks, we realised that teachers' experience of the different multiplicative structures (rectangular array, rate, ratio, and times-as-many) had a major impact on their own learning. Sophie, a Year 3 teacher, recorded this entry in her diary.

> The language of times-as-many was challenging for students initially but once they had more experience with tasks like this, I saw a shift in the strategies they used and they were using multiplicative language and making connections between multiplication and division.

Concluding Comments

The PL provided teachers with a range of rich and challenging tasks using everyday relevant content related to arrays, rate/ratio, and times-as-many that teachers then explored with their students in the classroom. Making links to proportional reasoning in the modules when exploring teachers' and students' solution strategies to rate/ratio was critical. Teachers realised that primary school students can engage in tasks such as these and do so using proportional reasoning and multiplicative thinking. They saw the tasks as a major source of their learning and understanding of the complexity of developing multiplicative thinking. Teachers also recognised that developing such tasks was their greatest challenge when planning for learning, and indicated they need further support in this area. The findings suggest that providing in situ targeted professional learning over a sustained period of time that requires teachers to implement the learning with their students improves their knowledge of multiplicative thinking and proportional reasoning. It equipped these teachers with ways to support their students' development of multiplicative thinking using rich learning experiences relating arrays, rate/ratio and times-as-many. However, they still require on-going support and PL to embed the practices and deepen their understanding.

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Symposium: Multiplicative Thinking: Enhancing the Capacity of Teachers to Teach and Students to Learn

Multiplicative thinking is a key aspect of primary and middle school mathematics and is considered to be a predictor of students' capacity to progress beyond basic mathematical learning. It is characterised by a complex set of connecting ideas about which teachers need to have a broad and deep understanding. The study on which this symposium is based began in 2014 in Western Australia. It has involved over 1900 primary school students of ages 9 to 11 years, approximately 120 teachers, and 16 schools. This symposium presents an overview of the project and then focuses on the New Zealand phase of the project. Assessment of students' multiplicative thinking in the form of a written quiz and semi-structured interviews enabled teachers and researchers to identify students' knowledge and understanding of multiplicative concepts and led to the structuring of a targeted teaching program over several months. Parallel pre and post quizzes were used to investigate the extent of student learning that occurred. A highly significant increase in student attainment was noted. The use of manipulative materials to identify the extent of students' multiplicative thinking was also investigated through semi-structured interviews. Teachers' content knowledge was explored with particular emphasis on the use of student tasks targeting specific aspects of multiplicative thinking. It was found that teachers became more confident in teaching multiplicative concepts, showed a greater awareness of connections between ideas, and demonstrated a growing awareness of the importance of explicit mathematical language.

Keywords

NCON – Number Concepts

PRIM – Primary

PRDE – Professional Development

TPER – Teacher Perceptions

KNBO – Knowledge Building and Organization

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Multiplicative Thinking: Developing a Model for Research and Professional Learning

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The study on which this symposium is based began in 2014 in Western Australia. It has involved over 1900 primary school students of ages 9 to 11 years, approximately 120 teachers, and 16 schools. It began as a research project with a semi-structured interview as the initial data gathering instrument. A written multiplicative thinking quiz (MTQ) was developed to gather large data sets more efficiently. From the data a model for multiplicative thinking was developed and the MTQ was refined. During 2017 data were gathered from schools in Plymouth (UK), Dunedin (NZ), and Victoria. A professional learning program was also developed and papers in this symposium report on the development and use of the instruments, the multiplicative model, data gathered from students, and the professional learning program.

Background

Multiplicative thinking is a critically important aspect of mathematics. It underpins much of the mathematics learned beyond the middle primary years, informs the understanding of proportional reasoning, ratio, and statistical sampling, and is an important component of algebraic reasoning (Siemon, Breed, Virgona, Dole, & Izzard, 2006). It is well documented (Clark & Kamii, 1996; Siemon et al., 2006) that students who do not develop the ability to think multiplicatively find it difficult to move beyond primary school mathematics. It is also of concern that much mathematics teaching is procedural in nature whereas students need assistance to develop a conceptual understanding of the structure of the mathematics (Warren & English, 2000; Young-Loveridge & Mills, 2009).

Effective teaching of multiplicative concepts lies at the heart of success in terms of student learning. There are several aspects to this, one being a broad and deep mathematical content knowledge and a varied pedagogical content knowledge on the part of the teacher. Second, pedagogies need to be explicitly focused on specific aspects of concepts designed to develop a connected understanding of key content. These pedagogies involve questioning, demonstration, discussion, reasoning, investigation, and interaction between all students and the teacher (Askew, 2016). Third, procedural fluency and conceptual understanding must be developed alongside one another with conceptual understanding informing the use of procedures (Hiebert & Grouws, 2007). Finally, an understanding of the structure of the mathematics involved in the development of multiplicative thinking is seen as being critical (Warren & English, 2000).

The following definition of multiplicative thinking is based on the work of Siemon, Breed, Dole, Izard, & Virgona (2006) and Siemon, Bleckley, & Neal (2012). Multiplicative thinking is demonstrated by an ability to:

- Work flexibly with a wide range of numbers including very large and small whole numbers, decimals, fractions, ratio and percentage;
- Work conceptually with the relative magnitude of whole and decimal numbers in a range of representations, demonstrating an understanding of the notions of ‘times bigger’ and ‘times as many’;

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- Demonstrate a conceptual understanding of the multiplicative situation, the relationship between multiplication and division, numbers of equal groups, factors and multiples, and the various properties of multiplication; and
- Articulate a conceptual understanding of a range of multiplicative ideas in a connected way with explicit language and terminology (Hurst, 2017).

Development of the project

This project on children's multiplicative thinking began in Western Australia in 2014. To date, researchers have worked with teachers and students from 16 schools in Perth, Western Australia; Dunedin, New Zealand; and Plymouth in the United Kingdom. Approximately 120 teachers and 1900 students have been involved. The key finding from the project so far is that primary school students have the capacity to think multiplicatively and make multiplicative connections. However, whilst all students have demonstrated an understanding of some aspects of multiplicative thinking, only a handful have shown an understanding of how different aspects relate to and inform each other. To this end, the researchers developed a suite of nearly two hundred multiplicative thinking tasks which form part of a professional learning package. Data gathering instruments have also been developed and progressively refined over four year life span of the project. These comprise a Multiplicative Thinking Quiz (written) and a series of semi-structured interviews.

The model for multiplicative thinking and the written instrument were informed by one another. Initially, a comprehensive semi-structured interview was used to gather data. However, while the interview generated very rich data, it was time consuming. Hence, the written Multiplicative Thinking Quiz (MTQ) was developed. This could be administered to a whole class in less time than it took to interview one student. The original model was based on several ideas – numbers of equal groups and the inverse relationship between multiplication and division, the notion of 'times bigger', the properties of multiplication, and the development of written methods for calculating. The MTQ questions were based on these ideas. It became apparent that several sets of connecting ideas existed and a new model was constructed.

A model for multiplicative thinking

Four sets of connecting ideas were identified and have been termed Connections 1, 2, 3, and 4. The MTQ questions are clustered into four sections, each of which corresponds to one set of connecting ideas. Connections 1, based on the multiplicative situation, is shown in Figure 1.

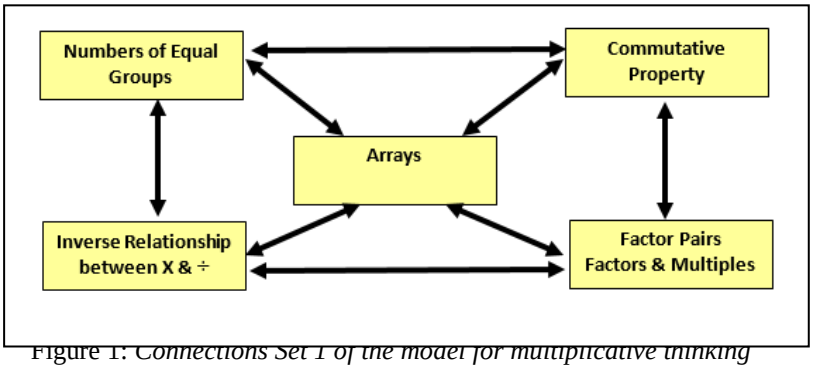

Figure 1: *Connections Set 1 of the model for multiplicative thinking*

Other sets of connecting ideas that comprise the model are based on the following ideas:

- Connections 2 – place value partitioning, and the distributive property of multiplication
- Connections 3 – the 'times bigger' notion, extended number facts, and movement of digits across places.
- Connections 4 – ratio and proportional relationships

Unfortunately, the scope of this paper does not enable the presentation of the complete model.

The Multiplicative Thinking Quiz (MTQ)

The MTQ contains 18 questions, each of which is designed to gather data about one set of Connections 1, 2, or 3. At the time of writing, quiz questions for Connections 4 had not been written. Typical questions from the MTQ based on Connections 1 include the following:

- What is the answer to 4 × 3? Using dots, crosses or something similar, draw a picture to show what 4 × 3 means.
- What is the answer to 8 × 7? What do the 8 and the 7 tell you? Write a story problem about 8 × 7.
- Write as many factors of 30 as you can? How do you know they are factors of 30?
- Which of the following will give you the same answer as 6 × 17? [16 × 7] [7 × 16] [17 × 6]. How do you know?

Parallel versions of the MTQ were developed to enable it to be used as both a pre-teaching and post-teaching assessment.

Targeted semi-structured interviews

The format for interviewing students has changed from a full interview covering the complete model for multiplicative thinking to shorter interviews that target a particular idea. Students' understanding of a specific aspect identified from the MTQ can then be probed more fully. One such idea targeted in this symposium is the use of materials to demonstrate or explain what is happening in multiplication. Students are asked to work out the answer for a one digit by two digit multiplication example, such as 7 × 15. They are then asked to show what is happening by using sets of bundling sticks, some of which are pre-bundled in tens. If children are unable to calculate say 7 × 15, they are offered an easier task, such as 4 × 9.

Teacher professional learning

The project has a professional learning model aimed at developing teachers as action researchers. Teachers are initially asked to analyse some student work samples taken from previously completed interviews and quizzes, to reflect on what the samples indicate about a student's mathematical understanding, and suggest what interventions are needed to help the student progress. This is followed by discussion and sharing of responses. The teachers are engaged in professional learning based on the multiplicative thinking model, trained to administer the MTQ, and to record the students' responses. Spreadsheets have been developed for this purpose and guidelines for recording data are also provided to ensure that the data recorded are reliable and consistent. For example, in relation to the above question about factors (Write as many factors of 30 as you can? How do you know they are factors of 30?), teachers are asked to record how the factors are written. That is, has a student written

them in the format 1, 2, 3, 5, 6, 10, 15, 30, randomly as 5, 10, 2, 30, 1, 6, 15, 3, or in pairs (1, 30) (2, 15) (3, 10, (5, 6).

The model was initially implemented over a period of five months during 2017 with some 16 teachers from the Dunedin area of New Zealand. Teachers were supported throughout by regular visits from 'academic critical friends' who facilitated discussions and reflections in cluster meetings. Particular aspects of multiplicative thinking were identified from the initial data gathering and specific tasks were chosen from a bank of tasks created for this project. Teachers and their academic critical friends used student work samples gained from these tasks as the basis for discussion and further planning. Teachers compiled reflective notes throughout the five months and participated in a whole-day final debriefing professional learning session, one aim of which was to reflect on the data gathered from the parallel MTQ and to consider gains made by students.

Conclusion

The project has generated a lot of data that suggests very strongly that children have the capacity to think multiplicatively but that their knowledge is fragmented and unconnected. Askew et al. (1997) conducted some seminal research about the most effective teachers of numeracy having a 'connectionist' orientation. Results from this project suggest that this is the case and that teachers can be assisted to hold their content knowledge in a more connected way in order to be able to make connections explicit for their students. Early indications are that the professional learning model has changed teachers' practices about teaching multiplicative thinking and the data gathered by the teachers involved indicate that these changes in practice are translating into improved outcomes for students.

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Growth in Students' Multiplicative Thinking: Evidence from the Data

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This paper examines the learning by students who were participating in a project designed to promote multiplicative thinking. We used parallel pre and post quizzes to investigate their learning of connections within the multiplicative situation, place value partitioning, the distributive property, times bigger, extended number facts and movement of digits across places. Overall there was a highly significant increase in student attainment with a very large effect size. Items in the quizzes that related to students' explanations of the mathematics showed particularly large gains in attainment.

Multiplicative thinking is much more than just repeated addition and involves understanding of the multiplicative situation, including relationships between multiplication, division, fractions, factors, multiples and products. It also involves understanding of the commutative, associative and distributive properties, the ability to work with the relative magnitudes of large and small numbers, and use of appropriate language (Hurst, 2017; Siemon, Bleckley, & Neal, 2012). This view of multiplicative thinking places great importance on the connections between related ideas and conceptual understanding rather than just procedural knowledge. Multiplicative thinking is a fundamental component of proportional reasoning, fractions, decimals, ratio, statistical sampling and algebraic reasoning (Siemon, Breed, Virgona, Dole, & Izzard, 2006), and is therefore extremely important for learning mathematics. Crooks, Smith, and Flockton, (2009) found that only 56% of New Zealand Year 8 students could correctly calculate 39×6 and only 26% of Year 4 students could correctly calculate 19×4. For the Year 4 students who showed their working, half used an additive rather than a multiplicative strategy. In this part of the study the research question addressed is, "What was the impact on student achievement of engagement in tasks designed to promote multiplicative thinking?"

Methods

Twenty-three teachers from schools in the Dunedin area of New Zealand originally applied to participate in the project, but only 16 provided full sets of student data. Two hundred and forty-two students from these sixteen classes consented to take part in the study and completed both pre and post assessments. The students were predominantly from Years 5 and 6 and nine of the classes were composite Year 5 and 6 classes (ages 9 to 11). However, nine percent of the students were from Years 3, 4, 7 or 8 because seven of the classes were multi-level. The students were taught by their teachers, who were participating in the *Multiplicative Thinking Project* during Terms 2 and 3 of 2017.

Diagnostic assessment data, in the form of a written quiz, were gathered at the start of the study. The quiz assessed not only students' knowledge, but also their language and their explanations of the mathematics. There were three sections to the quiz: Connections 1 - the multiplicative situation; Connections 2 – place value partitioning and the distributive

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property of multiplication; Connections 3 – times bigger, extended number facts and movement of digits across places.

During Term 2 and early Term 3 the students engaged in tasks related to multiplicative thinking. All teachers used at least twelve of the provided tasks with their students. Most teachers used more than one lesson for some of the tasks and some teachers used more than twelve tasks to meet the learning needs of their students. All teachers used some of the *Bags of Tiles* tasks discussed in the results below. At the end of Term 2 or early in Term 3 a parallel written quiz was administered to provide summative assessment data.

Diagnostic assessment

There was a huge range of achievement in the pre quiz, with students attaining total scores between 0 and 49 out of a maximum of 60 correct or acceptable answers (M=24.5%, SD=19.3%). There was also considerable variability between items, with the number of students answering the item correctly ranging between 6 students (2.5%) and 189 students (78.4%).

There was little difference between students' percentages of correct or acceptable responses in the three different sections of the pre-quiz (Table 1).

Table 1
Students' scores on pre-quiz sets of connections

Quiz section	Description	Mean %	S D %
Connections 1	The multiplicative situation	21.0	14.4
Connections 2	Place value partitioning and the distributive property of multiplication	24.4	21.0
Connections 3	Times bigger, extended number facts and movement of digits across places	20.7	22.6

The items with the highest frequencies of correct responses were related to knowledge of multiplication facts and performing calculations, e.g., 8×7 (67.8%), 6×17 (42.3%) and 16×10 (59.5%). The items with the lowest frequencies of correct or acceptable responses were related to students' explanations of the mathematics: e.g., description of 8×7=56 in terms of factors and/or multiples (9.1%); explanation of an alternative strategy to place value partitioning for 6×17 (6.2%); and explanation of 16×10=160 in terms of powers of 10 and/or digit movement (7.9%).

The diagnostic assessment information informed the teachers' choices of tasks and approaches to teaching. Each teacher chose different tasks to address the learning needs of their students but all teachers were focused on developing students' understanding of multiplication rather than learning facts and procedures.

Tasks used and links to the sets of connections

Within the limited scope of a symposium paper it is not possible to describe all the tasks used by the teachers. However, four of the original twelve tasks provided were in the *Bags of Tiles* series and were based on students using 2cm×2cm square plastic tiles to construct arrays and addressed objectives from Connections 1 and 2. In each task students were provided with a given number of tiles, e.g., 24, and asked to make an array with no tiles left

over. The different arrays were compared and teachers modelled the descriptions using correct mathematical language of factors, multiples and products. Each *Bags of Tiles* task was focused on a different issue, including the commutative property of multiplication and division, relationships between fractions and division, square numbers, prime numbers, and the distributive property of multiplication and division. The overarching goal of these tasks was to provide students with an understanding of the rich connections within the multiplicative situation. The use of this meaningful concrete representation was intended to move students from a perception of multiplication and division as separate facts and procedures to an appreciation of the multiplicative situation as a coherent whole

Overview of students' results

There was a highly significant difference in student attainment on the sixty parallel items in the pre-quiz (M=14.7, SD=11.6) and post-quiz (M=30.1, SD=13.1), using a paired t-test t(241)=23.7, p=0.000. The effect size of this comparison was extremely large (Cohen's d=1.52) and it can be concluded that participation in the project had a major impact on students' multiplicative thinking.

This improvement in attainment was observed in all three sets of connections (Table 2). Paired t-tests revealed highly significant differences in students' scores on all three sets of connections and Cohen's d demonstrated large or very large effect sizes.

Table 2
Pre-/post comparisons of sets of connections

Quiz section	Mean (pre)	S D (pre)	Mean (post)	S D (post)	Paired t(241)	Cohen's d
Connections 1	4.8	3.3	10.1	3.9	20.5*	1.31
Connections 2	2.7	2.3	4.7	2.5	14.2*	0.92
Connections 3	6.6	7.2	14.4	8.2	18.6*	1.20

* p=0.000

Main themes of improvement

A more informative picture of the improvement of students' multiplicative thinking was obtained by examining the items on which there were the largest increases in the number of students answering correctly.

Table 3
Frequency of correct responses

Quiz section	Item(post-quiz in parentheses)	Pre-quiz	Post-quiz	Increase
Connections 1	*In the number sentence 7×5=35 (9×7=63), identifies which numbers are factors	41	161	120
Connections 1	Represents the number fact 4×3 (5×4) as a multiplicative array	54	158	104
Connections 3	*Identifies 400 (700) as 10 times bigger than 40 (70)	75	177	102

Connections 1	Writes an appropriate story for 8×7 (7×6)	87	189	102
Connections 3	*Writes "÷10" or "×0.1" to transform 30 into 3 (60 into 6)	89	183	94
Connections 1	Identifies 144÷6 (135÷5) and 144÷24 (135÷27) as inverses of 24×6=144 (27×5=135)	103	182	79
Connections 2	Calculates 6×17 (9×14) using mental calculation	102	165	63

* Similar items with similar increases have been omitted for brevity

It can be seen from Table 3 that there were substantial increases in the number of students giving correct responses to questions requiring appropriate use of mathematical language, use of arrays to represent multiplication, interpretation of times bigger, inverse operations, and use of the distributive property. During Workshop 1 all of these categories had been identified by the teachers as aspects of multiplicative thinking that they wished to address. It is also interesting to note that students' knowledge of the basic fact 8×7 (7×6) increased from 164 students to 213 students, even though the teachers had not been focusing on basic facts.

Conclusions

There were substantial gains in achievement on all assessment items, clearly demonstrating that students' knowledge and skills related to multiplication and division increased through engagement in the tasks. There were particularly large gains in achievement on items related to students' explanations of the mathematics, rather than on items related to facts and procedures. The tasks used in the project provided opportunities for students to make connections, use mathematical language, use concrete materials as tools, and engage in worthwhile tasks. Anthony and Walshaw's (2009) summary of effective pedagogical practices argues that providing the opportunities described above are components of good practice and so it is perhaps not too surprising that the students made such substantial progress in their multiplicative thinking.

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Using materials to support multiplicative thinking

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This paper reports on a small aspect of a large multiplicative thinking project, which first started in 2014 in Western Australia. The research question 'How does the use of materials impact on multiplicative thinking?' is a focus of this iteration, which took place in sixteen classes in New Zealand schools. This paper evidences the benefits of using materials when solving and communicating problems, and describes some issues that arise through unintended consequences of using materials.

Using materials in New Zealand classrooms

Materials, which encompass concrete materials, manipulatives, equipment and fingers, have been widely used in New Zealand classrooms for some time. The introduction of The Numeracy Development Project (NDP, 2001) was an impetus to a dramatic change in teachers' mathematics practice in New Zealand schools. Part of the NDP was to promote quality teaching with tools and materials and this was pivotal to the success of the students' achievements. Materials and professional development were provided to help teachers understand the conceptual development in students' thinking and to offer them an effective model for teaching strategic thinking in number. Today's classrooms are very different from the latter half of the 20th century with rooms now reflecting the value of mathematics in the environment, with students discussing mathematics and using materials (personal communication, Peter Hughes, February, 1999). The picture looks great but a question does arise. Do the teachers and students have the same understandings about the purpose of the materials or are they working at a tangent with their differing thoughts?

Solving problems with materials is nothing new; indeed materials have been used for centuries and many cultures had some form of counting using materials. The benefits of materials for learning mathematics, over many decades, are lauded in researched or synthesised papers written with the focus of materials in mind (e.g., Kinzer and Stanford, 2014; Higgins, 2005) and more with the use of materials to develop multiplicative thinking as the focus (Boaler, 2017; Jacob & Mulligan, 2014). Anthony and Walshaw (2007) support the use of tools and representations as one of the ten principles of effective pedagogy in mathematics (p. 23). Black (2013) backs up that idea as she espouses the use of materials as "a powerful tool to support sense making, mathematical thinking and reasoning when they are used as tools to support these processes rather than adjuncts to blindly following a taught procedure to arrive at an answer" (p. 5).

There is some critique around the use of materials. Black (2013) questioned whether the materials are used in a way where they are "perceived as being central to the early development of mathematical ideas especially for children aged under 11" (p. 3). This aligns with Piaget's well-known concrete operational stage. When researching the use of materials in some schools in the NDP in NZ, Higgins (2005) questioned the use of materials used in a procedural way and suggested the schools followed an algorithmic way of learning, in contrast to the consistent messages in the NDP. In her paper Higgins, whilst positive about

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the use of materials, queried the "purpose of the equipment designer and the teachers' purpose in using the equipment" (p. 95). Both do not necessarily coalesce with the best outcomes. Furthermore using 'hands-on' materials does not necessarily mean students have their 'minds on' developing the mathematical concepts the materials are designed to engender.

Using materials to support multiplicative thinking

Materials are particularly useful to support children's multiplicative thinking. NCTM (2017) provided a comprehensive list of how materials can be used in mathematics classrooms. The following eight resonate particularly with aspects of multiplicative thinking:

- distinguishing patterns—the foundation for making mathematical generalisations;
- understanding the base-ten system of numbers;
- comprehending mathematical operations such as addition, subtraction, multiplication, division;
- recognising relationships among mathematical operations;
- engaging in problem-solving;
- representing mathematical ideas in a variety of ways;
- connecting different concepts in mathematics; and,
- communicating mathematical ideas effectively.

Methodology

To answer the question 'How does the use of materials impact on multiplicative thinking?', we used a design research methodology (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006) because we were engineering and systematically studying the students' learning related to multiplicative thinking after furnishing their teachers with professional development on a multiplicative model and a set of related tasks. The set of tasks were structured to include instructions and suggestions for pedagogy, including the use of materials (see Offen & Ingram, this issue for further information). A model for multiplicative thinking was used as a signpost to ascertain the multiplicative understandings they have in regard to four sets of connections (See Hurst, this issue for detail). Data was drawn from sixteen teachers and their 242 (Y3-Y8) consenting students in schools in urban Dunedin, New Zealand. Data was collected through assessments of children's multiplicative thinking, teacher feedback and teacher progressive feedback when using a set of multiplicative tasks provided. In class, when the students worked on the tasks, the teachers made a wide variety of materials available to the students. A further source of data was that 43 students from Years 3 – 8 were interviewed early in the study to elicit student thinking, particularly multiplicative reasoning, to gauge the connections between their calculations and materials. Each teacher was asked to nominate a high, middle and low achieving student from their class. During the interviews the students were asked to represent problems using bundles of sticks. The length of the interview depended on the level of questions the student answered. Once they showed evidence of lack of understanding the interview was immediately stopped. The data relating to materials was qualitatively analysed to explore students' representation of multiplicative thinking with materials.

Results

First, students' use of materials during the interview is related to their multiplicative thinking. Then, how materials were used during tasks to develop multiplicative thinking is explored through the teachers' perceptions.

Using bundles of sticks to represent 23×4

Twenty of the 43 students interviewed reached the stage when they were asked to solve 23 x 4. All but one student answered correctly. Most students used place value partitioning mentally or thinking through jottings. One student used compensation (25 x 4) – (2 x 4) successfully. Two used an algorithm. If the 19 students had sat a pen and paper test they would have been marked as correct. However, there were differing responses when asked to demonstrate a representation for 23 x 4 using bundles of sticks and only six were successful. The two students who used an algorithm were not successful.

Most students seemed unfamiliar with modelling mathematics using bundles of sticks. This inability to demonstrate representations was alarming as classrooms have many materials to support student thinking since the NDP and there is an expectation students should be very familiar with representing mathematics through materials. The students sampled indicated that, generally, there is a mismatch between the purpose of the use of materials and students' representation of multiplicative thinking. This reinforces the idea of Higgins (2009) who questioned whether the purpose of the materials was the same as the purpose teachers had in mind.

All students who successfully solved 23 x 4 were asked to represent the problem using bundles of sticks. There was a range of ways the student cohort failed to model the problem when using materials. For example, Student A simply modelled the answer of 92 with 9 tens and 2 ones rather than using the materials to support multiplicative reasoning during the process of problem solving. Student B incorrectly modelled the answer of 92 using 9 ones and 2 ones rather than 9 tens and 2 ones. Student C laid out 4 ones separate from 2 tens and 3 ones, and then laid out 9 ones and 2 tens to represent the answer. Student D did not see the purpose of materials and said "I could do it easily in my head but I thought you wanted to know all that", even though he correctly modelled the problem. Interestingly all of the six students who modelled 4 groups of 2 tens and three ones were also able to link the notion of times bigger to 400 x 23**.** A typical answer was "23 x 4 is 92 so 400 x 23 is 9200 because 400 is 100 times bigger than 4".

Using materials with the tasks

During the study the teachers saw the benefits of using materials to explore and demonstrate multiplicative ideas. Although there was some confusion with which materials to use for a particular situation, evidence from the teachers' feedback showed the use of materials benefitted the students' multiplicative thinking as they presented the tasks with more of an onus on discovery and investigation. This was particularly evident when the teachers provided children with square plastic tiles to create arrays.

> Using arrays is an extremely powerful way to show relationships. (Carol)

> Tiles were particularly helpful for visualising number sentences and all their related facts. (Diane)

In final feedback teachers realised previously there was a lack of using materials in developing multiplicative thinking in their classrooms and resolved to make better use of the tools.

Realised I wasn't using them often enough. (Ginny)

Will use more materials in my teaching so students can discover the answers for themselves. (Diane)

A change in the teachers' attitudes towards using materials in those ways was a pleasing result. However, one teacher clung to the idea "it was great to use materials to *show* the children" (Lisa), confirming Higgin's (2005) concern about using materials in a procedural way.

Students' multiplicative thinking improved dramatically in this project (see Linsell, this issue for details). There is evidence that the use of materials supported students' understanding of the multiplicative aspects particularly the aspects of place value partitioning, times bigger, the relationship between multiplication and division, communicating mathematical ideas effectively, and connecting different concepts. The more the children worked with materials, the more their confidence grew and the more risks they were likely to take. Unfortunately, the length of this paper limits reporting further results (see Hurst & Linsell for further results, sent for review).

Conclusion

Materials are a useful tool to support children's mathematical learning, and particularly pertinent to this project, their multiplicative learning. Through their involvement in this project, the teachers saw the benefit of using materials in the classroom and using appropriate materials with relevant tasks. The tasks provided by this study were designed so that, when used with materials, mathematical connections were woven in a sense making tapestry, confirming the tasks are so much richer when used with materials to support children's multiplicative thinking. If materials are in constant use in a classroom it is hoped that children will become self-motivated to choose the appropriate materials that will support their thinking. There has to be a consistent understanding of what the use of materials stand for and the teachers' flexibility and knowledge of how to use them. These elements have to work in harmony for effective practice and student growth in developing robust multiplicative thinking.

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Teachers' perspectives of tasks designed to promote multiplicative thinking

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In this paper, teachers' perspectives of their involvement in a multiplicative thinking project are explored in terms of their growth and their use of a set of specifically-designed tasks with their classes. Teachers reported growth in their own content and pedagogical knowledge. Furthermore, the use of the tasks gave them more confidence in teaching multiplicative thinking strategies and demonstrated the connections between aspects of multiplicative thinking. Teachers also identified a growing awareness of the use of explicit language and the importance of using materials from engaging with the tasks in this project.

Being multiplicative is a vital component of being able to think mathematically (Siemon, Breed, Virgona, Dole, & Izzard, 2006), yet students, and teachers, find working multiplicatively can be very challenging (Young-Loveridge, 2007). To best support children's multiplicative thinking, teachers need strong content and pedagogical knowledge (Ball, Hill & Bass, 2005), i.e., a connected understanding of the multiplicative situation, and effective pedagogy such as questioning, demonstration, discussion, reasoning, investigation, and interaction (Askew, 2016). Teachers need to source tasks that support children to grow their conceptual understanding (Sullivan, Clarke & Clarke, 2013) through deep thinking about the mathematical ideas (Anthony & Walshaw, 2009).

In earlier research Hurst (2017), working with various others, developed a model for multiplicative thinking based on four sets of connecting ideas (see Hurst, this publication). In this iteration, we sought to grow teachers' understanding of this model through professional development and the provision of a set of tasks related to the multiplicative situation to use in their classrooms. This paper reports on the teachers' perspectives of their own mathematical and pedagogical growth through the use of these tasks.

Method

This research can be considered 'design research' because, by giving suggestions for the content and structure of the tasks, a particular form of learning was being engineered and systematically studied (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006). This research was iterative in that it drew on learning from previous projects with Australian and English schools (e.g., Hurst, 2017).

Sixteen New Zealand teachers of Year 5-8 students attended two professional development days, where teachers learnt about the connections between aspects of multiplicative thinking and the importance of making these connections explicit with their students. Further to those days, teachers were supported by regular cluster meetings with the researchers; 'academic critical friends' who facilitated discussions and reflections.

The teachers chose tasks from a set of suggested tasks to use with their Year 5-8 students. The tasks were specifically designed to connect students' understanding across specific aspects of the multiplicative model. These tasks, which were variously sourced, were re-structured to include suggestions of: how to administer the task; what to look for in students' actions and responses; how to phrase questions and how to develop the task, depending on

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children's responses. Further to the task instructions, each task also contained teacher notes that explained the mathematical concepts, links to specific aspects of the multiplicative model and the mathematical language targeted by the task. A feature of each task was students were required to explain their thinking using concrete materials.

Data were drawn from the progressive feedback teachers gave during professional development days, field notes taken during the cluster meetings, a teacher questionnaire at the beginning of the project and further reflections on these questionnaires at the end of the project. Analysis of the qualitative data was based on grounded theory (Strauss & Corbin, 1998), where the data were coded into categories of themes that developed, which were related to the teachers' growth in knowledge, and how their pedagogy changed, namely the connections they made between the different aspects of multiplicative thinking, the use of materials, and finally, the mathematical language they used when teaching multiplicative strategies. The results, presented in the next section, are evidenced by representative quotes from the teachers. Pseudonyms have been used to protect the identity of the teachers.

Results

We know that students learnt more about the multiplicative situation (see Linsell, this publication). There was also qualitative evidence that both the teachers' mathematical content knowledge and pedagogical content knowledge related to multiplicative thinking were enhanced through this research, and change in their teaching practice.

Growth in knowledge

Within the data, 86% of teachers referred to an increase in their knowledge related to multiplicative teaching as a result of their participation in the project. The emphasis some teachers placed on the connectivity between the multiplicative aspects demonstrated that teachers understood the big idea of the multiplicative model.

> [My] own knowledge of multiplicative thinking has improved. (Danielle)
>
> I have learnt about strong relationships between all principles in multiplicative thinking. (David)

This growth in content knowledge had an flow-on on teachers' increased confidence in their teaching of the multiplicative situation and the use of the tasks.

> [Being in the project] helped me with my own understanding – what is important to teach re: multiplicative thinking. (Rose)
>
> [Being in the project] I have deepened my knowledge of how to teach mult/div thru [sic] using different materials and the tasks provided. (Stacey)
>
> The [tasks provided] scaffolding for teachers. (Lucy)

Indeed, the teachers suggested the structure of the tasks, the accompanying instructions and the teacher notes gave teachers confidence in teaching multiplicative thinking strategies with their students. Importantly, the teachers reported they felt more confident when teaching multiplicative strategies when using these tasks.

Furthermore, the cluster meetings and discussions with teachers from other schools meant teachers could reflect and grow their understanding of the mathematical content they were teaching. The teachers had the opportunity to discuss how they were using the tasks and ask questions. This meant that teachers had support in interpreting the tasks and their purpose, as well as getting ideas from each other.

Change in teaching practice

The professional development and the structured nature of the suggested tasks also changed the teachers' practice, especially in the areas of making connections, using materials, and using language to the multiplicative situation. It should be noted first, however, that some teachers continued to rely heavily on the task instruction, even when tasks were adapted for a different purpose. For example, tasks that were intended for smaller numbers were also used for problems with larger numbers. This caused children spending an inordinate amount of time organising the tiles individually and not enough tiles available for children to manipulate. This meant, in some cases, teachers reverted to abstract thinking or procedural steps before children had mastered the conceptual understanding. In contrast, teachers who were focused on the intent of the task, rather than the task itself, either used a different task, or adapted the materials accordingly. For example, when the problems used larger numbers, teachers used grid paper or dotty arrays.

Making connections. The connections the teachers made between aspects of the multiplicative situation changed their teaching practice.

> I'm making connections and not teaching [multiplicative strategies] in isolation. (Danielle)

> I've learnt to make connections between fractions, decimals, ratios, percentages and mult/div wherever possible. (Joanne)

The tasks helped teachers make links within and between the four connecting ideas of the multiplicative thinking model and other mathematical concepts. For example, some teachers noted the tasks fed naturally into learning about measurement, and especially area. For others, the natural link was with other aspects of multiplicative thinking. It was also noted in feedback, that as well the tasks lending themselves to natural connections in multiplicative thinking, they also discouraged multiplicative concepts being taught in isolation. Furthermore, teachers who planned to use the tasks for a three-week block as part of the project continued to use them for a term or more.

Using materials. The sense of making connections was enhanced by the visual nature of the tasks, and the use of materials.

> It all makes sense. I was going to teach fractions later in the year, but I could actually see how the tasks and the materials would just naturally flow into this. (Joanne)

The teachers used a variety of concrete materials to support their children's multiplicative thinking (see Holmes, this publication). The tasks highlighted for teachers that "even older kids benefit from using materials" (Robert). It was a surprise to the majority of teachers that initially, students who could accurately solve problems procedurally were unable to offer either a conceptual explanation or demonstrate the nature of the task using materials.

> I knew they could answer the questions correctly, but when I asked them to explain to other students, they stuck to step-by-step procedures and couldn't explain why those procedures were in place or show how they worked using the tiles. (Joanne)

After teaching using the tasks and the materials, Joanne noted that children were able to "demonstrate what was happening in their heads". Having children demonstrate using materials showed what they didn't know, as much as it showed what they did know, which was helpful for the teachers identifying misconceptions and determining next steps. Another key element of children being able to demonstrate what was 'happening in their heads' was their use of specific mathematics language to explain their ideas.

Using the language of mathematics. Teachers commented on the growth of the specific language of multiplicative thinking by students as a direct result of teachers using the tasks.

Each task explicitly emphasised multiplicative language specific to the task (for example; factor, multiple and product). Feedback from teachers showed explicit use and teaching of the specific language enabled children to more clearly describe their thinking.

> I always thought that if they could do the maths, the language wasn't too important, but I could see kids had more confidence and clarity explaining the concepts when they knew the language to use. (Jane)

> I could tell the kids were really excited to be using real 'mathsy' words, and they used them whenever they could. (Lucy)

From a pedagogical viewpoint, teachers found the structure of the tasks with explicitly targeted language reinforced or alerted them to how important the language of mathematics is for children to explain their thinking.

Conclusion

The teachers responded positively to their participation in the multiplicative thinking project. Their content and pedagogoical content knowledge grew, and this had a positive affect on their teaching practice, particularly in the areas of making connections, using materials and using specific language. The professional development in this project not only met these criteria but also provided tools and a structure to implement the professional development. Teachers' overall conclusion of the professional development component of the project was they felt empowered and confident when teaching multiplicative strategies using the tasks. Mathematical language was enhanced and teachers noted students were more confident when explaining their thinking when they knew the correct language to use. Furthermore, the tasks enabled teachers to make connections of how multiplicative thinking linked naturally to other areas of mathematics.

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Symposium: CHOOSE**MATHS** – an Australian Approach to Increasing Participation of Women in Mathematics

Overview and Individual Contributions

The underrepresentation of women in Science, Technology, Engineering and Mathematics (STEM) in Australian is well known throughout the educational pipeline and in STEM careers. Girls have a lower average performance in mathematics, and fewer young women participate in the higher levels of mathematics in senior secondary school, in STEM degrees and in the STEM-related workforce. To address this underrepresentation of women in STEM and in particular in mathematics, the BHP Billiton Foundation has been funding Choose Maths, a 5-year initiative, since mid-2015 in collaboration with the Australian Mathematical Sciences Institute, the national institute for mathematics education.

The Choose Maths team has 18 staff, including eight full-time mathematics teachers, the Outreach Officers, who work with 120 schools across Australia. Choose Maths also focusses on Career Awareness, a Women in Mathematics Network which includes Mentoring for young women, Teacher and Student Awards, and statistical research. An advisory committee oversees the work of the team.

In this research symposium we consider different aspects through the mathematical pipeline and into the workforce as they relate to gender.

Inge Koch: *Attitude towards Mathematics and Confidence in Mathematical Ability of Students – Can it Change?* presents survey instruments and results of student interventions of Year 5 to Year 9 students that were conducted in 120 schools across Australia in 2017. The effectiveness of the interventions, which focus on growth mindset ideas and year-appropriate mathematical activities, is shown for the more than 2300 students in Year 5, and the differences between the pre- and post-survey results of boys and girls are highlighted.

Ning Li: *Gender Gaps in Participation and Performance in Mathematics at Australian Schools 2006 – 2016* looks at the difference of male and female students' performance in mathematics tests, and their participation in mathematics subjects in Years 11 and 12, when mathematics is no longer compulsory. In both areas female students score lower than male students. These results are complemented by teachers' opinions on factors that are most influence students in their subject choices,

Gilah Leder: *Mathematics, gender, and careers* reviews the participation of women in the workforce and starts with potential reasons for the lower participation of women in senior mathematics classes that have been presented in psychology and related disciplines. Leder ask the question of what influences the choice of career of young men and women, relates male and female teachers' surprisingly different ratings of the level of mathematics required for different career pathways and examines the occupational pathways by gender.

Janine McIntosh, AMSI, and Helen Forgasz, University of Monash, have agreed to chair the session and to be discussant respectively.

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Attitude towards Mathematics and Confidence in Mathematical Ability of Students – Can it Change?

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We study students' confidence in their mathematical ability and attitude to mathematics before and after an intervention in 120 schools in Australia. The 2017 Choose Maths intervention measures the effect of growth mindset ideas and targeted mathematical activities in students in Years 5 to 9. The analysis of the pre- and post-survey responses shows: boys are more confident and have a more positive attitude than girls, there is positive change in both domains, and the change for girls is much larger than that for boys.

Introduction

Australian primary and secondary students show similar performances across different national and international tests such as NAPLAN, PISA and TIMSS: on average boys outperform girls in numeracy, while girls outperform boys in literacy at every year level. Almost twice as many boys participate in Year 11 and 12 intermediate and advanced mathematics courses as girls, that is, in the years when students in Australian schools can choose different levels of mathematics including none (Li & Koch, 2017).

It is too simplistic to assume that girls' participation in Year 11 and 12 mathematics courses is lower as a consequence of their lower average performance. TIMSS and PISA results (Mullis *et. al.*, 2015; Thomson *et. al*., 2017) demonstrate clearly that students' economic background has a much stronger influence on mathematics performance than gender. However, the effect of gender is not negligible, and it is important to examine the causes for the lower performance and lower participation of girls.

Based on our understanding and belief that a more positive attitude to mathematics and increased confidence in one's own ability are positively correlated with more enjoyment and engagement in the subject and that the latter are expected to have a positive effect on performance, we focus on attitude and confidence of students with regards to mathematics.

In this paper we discuss results of surveys of more than 4800 students which we conducted as part of the Choose Maths Outreach in 120 Australian schools throughout 2017. We report students' attitudes towards mathematics, and confidence in their mathematical ability. Informed by the changes observed in the data, we comment on the potential for change. A better understanding of underlying processes affecting mathematics performance will inform if and how we can change students' confidence, attitude, engagement and ultimately performance regarding mathematics.

The Choose Maths Outreach Component

Choose Maths has eight experienced primary and secondary teachers -- Outreach Officers -- who work in 120 primary and secondary schools across Australia. They provide professional development for the local teachers, conduct teacher surveys and student surveys and engage with students, their parents and teachers (Koch & Li, 2017; Li & Koch; 2017). Principals of the participating schools participate in Choose Maths with the conviction that

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their teachers' increase in confidence and competence through involvement with Choose Maths will have a flow-on effect on students' engagement and performance.

To study attitudes and confidence of students with respect to mathematics Choose Maths developed annual intervention strategies, described in more detail below, for Year 5 to Year 9 students. We obtained ethics approval for these interventions through the University of Melbourne in late 2016 and conducted a pilot study involving about 300 Year 5 and 300 Year 8 students in Term 4, 2016. Following analysis of the pilot survey data, we modified the original intervention strategies and survey instruments, and, in 2017, collected survey data from more than 4800 students in Years 5, 6, 8 and 9.

Here we focus mostly on the Year 5 and Year 8 interventions conducted in 2017. The Year 5 cohort represents the largest sample – about 2300 students. The Year 8 data from about 1360 students are included to show that the changes observed in primary school students are also evident in the secondary students' data. The Year 5 data form a baseline for comparisons with Year 5 cohorts in 2018 and subsequent years; and assessment of the changes of the Year 5 students in their later school years.

Classroom Intervention and Survey Instruments

The Outreach Officers conducted the intervention classes with the local teacher present. Each intervention consists of a pre-survey, a presentation on growth mindset ideas (Boaler, 2015), a mathematical group activity appropriate for their year level and a post-survey. Each intervention class presents a snapshot in time. Due to time and organisational reasons, it was not possible to measure the effect of the intervention a few months later again. Interventions and surveys in 2018 and in later years will allow a follow-up. The questions for the pre- and post-survey and admissible responses are shown in Table 1.

Table 1 *Survey questions 2017*

	Pre-survey	Responses	Post-survey	Responses
Q1	It is okay to feel confused about maths	Agree/ Disagree	It is okay to feel confused about maths	Agree/ Disagree
Q2	Girls and boys can learn maths equally well	Agree/ Disagree	Girls and boys can learn maths equally well	Agree/ Disagree
Q3	Sharing tasks with others helps me to understand maths better	Agree/ Disagree	Working with others on the task today helped me understand this maths better	Agree/ Disagree
Q4	When I think about maths I would describe myself as	Very confident/ Confident/ Neutral/ Not Confident	After the lesson today, I feel	Very confident/ Confident/ Neutral/ Not Confident
Q5	When I think about maths I feel	Enthusiastic/ Somewhat Enthusiastic/ Neutral/ Bored	After the lesson today, I feel	Enthusiastic/ Somewhat Enthusiastic/ Neutral/ Bored
Q6	I have a maths brain	Agree/ Disagree	My brain allows me to learn new maths	Agree/ Disagree

We collect the answers in the pre- and post-survey using *Plickers* cards (see https://www.plickers.com/). The answers are collected with the Outreach Officer's mobile phone. We record the gender of the students, and the students use the same Plickers card for the pre- and post-survey as this allows us to record and study the change in their responses as a consequence of the intervention activities.

A growth mindset presentation explains how the brain learns and introduces the '*power of YET*': 'I can't do fractions **yet**'. The Year 5 group activity required students to create geometric shapes and use language to describe the shape, so the other members of the team could construct the identical shape without seeing it. This activity focused strongly on the interplay of language and mathematics and made students aware that the language of mathematics must be very precise. The Year 8 activity focussed on discovering and generalising patterns which will ultimately lead to quadratic equations.

Analysis of Year 5 and Year 8 Student Surveys

Table 2 Percentages for each response category in Q4 for Year 5 and Year 8

	Y5 n conf	Y5 neutral	Y5 conf	Y5 v conf	Y8 n conf	Y8 neutral	Y8 conf	Y8 v conf
Boys pre	8.0	23.6	36.5	31.9	12.1	35.8	34.0	18.0
Boys post	7.8	13.7	33.3	45.0	11.7	29.5	32.9	26.0
Girls pre	9.2	32.6	40.7	17.5	14.7	43.5	32.7	9.1
Girls post	6.5	21.8	35.7	36.1	8.4	34.7	41.0	15.9

Notation used in the table: Y5 = Year 5; Y8 = Year 8; n conf = not confident; conf = confident; v conf = very confident.

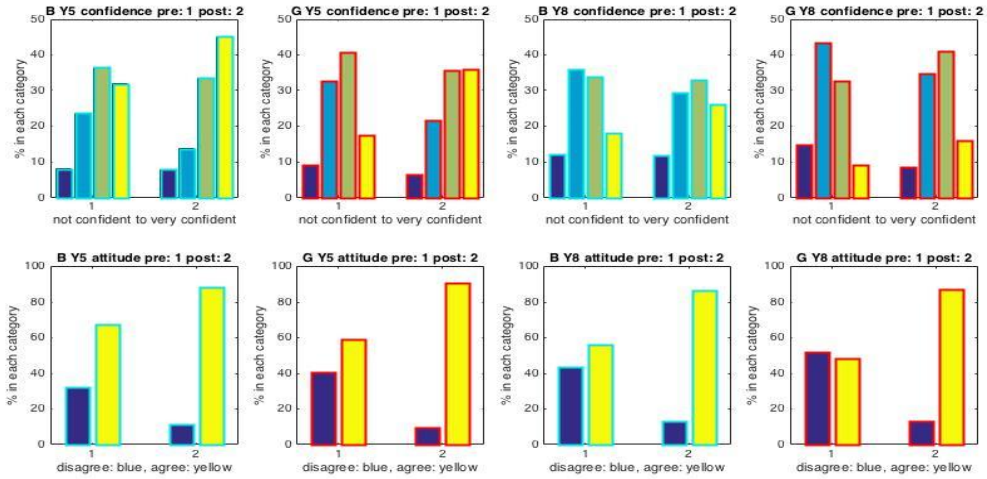

Figure 1. Change in confidence and attitude Year 5 and Year 8.

The proportion of boys in the sample is about 46% across all years. There are more girls than boys in the sample, as some of our schools are single-sex girls' schools. The results for the Year 6 and Year 9 cohorts are similar to those reported below.

As can be seen in Table 2, the results from Q4 and, the pattern of change from pre- to post-survey, are similar for Year 5 and Year 8 students, but the percentage of confident and very confident students decreased for the higher school year. The responses to Q4 show an increase of the very confident students: 13.1% (resp. 8%) for boys and 18.6% (resp. 8.3%) for girls in Year 5 – with the Year 8 results in brackets – while the other three response groups, and in particular the 'neutral' group, decrease. For girls the changes are bigger than for boys; the not confident group for girls shrinks by about one third and is smaller than that for boys in the post-survey, although the girls started with a higher not confident percentage than the boys.

Figure 1 shows the change in confidence and attitude in the form of histograms, separately for boys – with blue edging -- and girls – with red edging. In each panel the first block of bars – four in the top row and two in the bottom row – refers to the pre-survey, and the second block of bars in each panel refers to the post-survey. The Year 5 data are shown in the first two panels and the Year 8 data follow in panels three and four in each row. Percentages of responses in each category are shown on the vertical axis.

The top panels in Figure 1 refer to the change in confidence, Q4: the four differently coloured columns are given in the same order as in Table 2: not confident, neutral, confident, very confident. The bottom panels refer to change in attitude, Q6. The dark blue bar shows the percentage of 'disagree' responses and yellow refers to 'agree' responses. For the changes in positive attitude, Q6, we find: boys show a 21.1% in Year 5, and a 31.5% in Year 8 and girls show a 31.3% in Year 5 and a 38.8% in Year 8, that is, about one third of girls changed their attitude as a result of the intervention activities.

In Q4 and Q6 we note that the change due to the intervention is particularly large for girls, and overall the results suggest that students' confidence in and attitude towards mathematics is not fixed but can be affected and changed in a positive way.

Final Words

Survey results of classroom interventions of more than 4800 students in Years 5, 6, 8 and 9, which comprised a pre-survey, mathematical activities and a post-survey during one lesson, show that students' confidence in their mathematical ability and their attitude to mathematics can change through intervention – with change occurring in a positive direction. The larger change particularly for girls is encouraging and there is hope that growth mindset approaches and appropriate teaching methods will lead to longer-lasting effects which allow students to become more confident and ultimately perform better.

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Gender Gaps in Participation and Performance in Mathematics at Australian Schools 2006-2016

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How do boys and girls differ in voluntary mathematical studies in Years 11&12? Do boys and girls perform differently in standardized mathematics exams? What factors affect students' decisions to choose or not choose mathematics? This document updates the previous literature using recent data from various sources. It is found that between 2006 and 2016 participation in Year 12 mathematics has been stable for both boys and girls, with the boys' percentage being higher than girls', both being shifted away from advanced mathematics. Students' previous achievement has been recognized by the teachers as an important influential factor for students' decisions to continue studying mathematics in senior high schools.

Participation rate in mathematics in senior high school is a basic indicator for the progress of mathematics education, the quality of the prospective labor market, and the future economic competence. In Australia, mathematics is not compulsory in senior high school. The participation rate determines the supply pool for many university courses, which may affect gender balance in the STEM workforce (Roberts, 2014). Previous research findings show the existence of a gender gap in mathematics enrolments of Year 12 students between 1990 and 2004 (Forgasz, 2006 Sec 1.1). A few years passed since the call for action to encourage females into STEM disciplines (Office of the Chief Scientist, 2012). What is the current situation?

Students Taking At Least One Mathematics Subject

The typical age of Year 12 students in Australia is between seventeen and eighteen years. Persons in the age group of 17-18 form the Year 12 *potential population*, whose size can be estimated by the average number of 17 or 18 year olds in Australia (Li & Koch, 2017). According to Australian Bureau of Statistics (ABS data series 3101059, Table 59), between 2006 and 2016 the sizes of the Year 12 potential population, displayed as solid lines in Figure 1, have grown from 141344 to 151698 for boys and from 134330 to 143083 for girls. Data on Year 12 enrolments (Barrington & Evans 2017) indicate that each year, on average, one third of the boys and one fifth of the girls in the potential population did not study Year 12 between 2006 and 2016. While there were 7015 to 9128 more boys in the potential population each year, during this period 7381 to 13357 more girls enrolled in Year 12 each year. A restructuring of the secondary curriculum in Western Australia led to a half-cohort reduction in the state in that year, evident from the dips in 2014 enrolments in Figure 1. The extra number of boys, or the gender gap, in the Year 12 potential population has shown a decreasing trend. In contrast, the extra number of girls, or the gender gap, in the Year 12 actual population has shown an increasing trend between 2006 and 2016.

Mathematics subjects are offered to Year 12 students at various levels of difficulty. A student who takes any of these subjects is referred to as a mathematics student. Between

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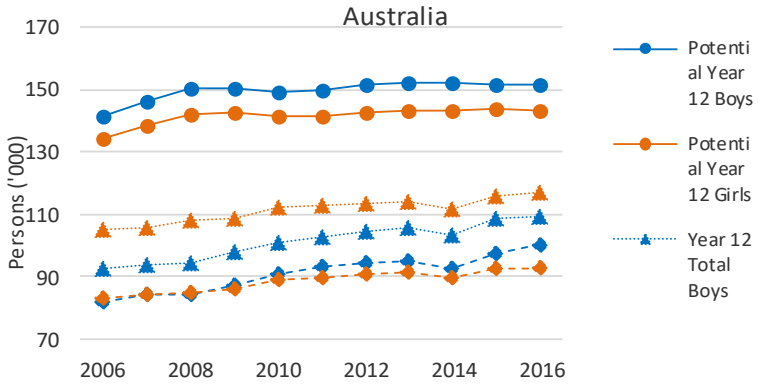
2006 and 2016 the total number of Year 12 mathematics students has been growing proportionally to the total number of Year 12 students, for both girls and boys. Each year, despite more girls enrolled in Year 12, fewer girls than boys chose mathematics, being evident from the long-dashed lines in Figure 1. Moreover, the difference between male and female mathematics students has been widening over time in the period.



Data sources: ABS data series 3101059, Australian Demographic Statistics, Table 59 (Estimated resident population by single year of age, Australia); Year 12 enrolments data (Barrington & Evans 2017).

Figure 1. Year 12 potential, actual, and mathematics populations, 2006 – 2016

Elementary, Intermediate, and Advanced Mathematics Students

Based on the definitions by Barrington and Evans (2016, 2017), the elementary level mathematics subjects involve little or no calculus, and are not intended to provide a foundation for any future tertiary studies involving mathematics (Forgasz, 2006). On the other hand, the intermediate and advanced mathematics subjects meet the minimum requirement for tertiary studies in which mathematics is an integral part of the discipline. By estimating the overlap of students concurrently taking elementary and non-elementary *subjects*, Barrington and Evans (2017) estimated the number of *students* taking elementary subjects only. The data reveal that the yearly increments of mathematics students between 2006 and 2016 are mainly due to increments in elementary mathematics students. Over time, students were shifting away from advanced towards elementary subjects, for both boys and girls. It is found that in Year 12 between 2006 and 2016 (Li & Koch, 2017)

- Each year, on average, at least twice as many boys and girls enrolled in elementary mathematics as in intermediate mathematics; four times as many boys and seven times as many girls enrolled in elementary mathematics as in advanced mathematics.
- The percentage of elementary mathematics students has increased by 15% for boys and by 6% for girls in the period.
- In contrast, the percentage of intermediate mathematics students has decreased by 12% for boys and by 10% for girls.
- The percentage of advanced mathematics students has decreased by 12% for boys and by 10% for girls.
- Girls were, on average, at least 43% less likely than boys to study advanced mathematics.
- The percentage of Year 12 advanced mathematics girls appears to have a mild increase from 6.6% to 7.0% monotonically over the period between 2012 and 2016.
- The girl to boy ratio within advanced mathematics students has decreased from 2006

to 2014, but has increased since, and reached 6:10, the highest in the last decade.

Performance in Standardized Mathematical Tests

Students' average scores in mathematics tests in the National Assessment Program — Literacy and Numeracy (NAPLAN), the Programme for International Student Assessment (PISA), and the Trends in International Mathematics and Science Study (TIMSS) are displayed in Table 1, blue for boys and red for girls respectively. They show that boys outperformed girls in every year level and all tests that have been conducted. Li & Koch (2017) also find evidence that girls outperformed boys in reading in the above tests most of the time, that the gender difference in reading is larger than the gender difference in mathematics, and that the performance varies more among boys than among girls.

Table 1

Average scores of students' mathematics tests in NAPAN, PISA, TIMSS, by gender

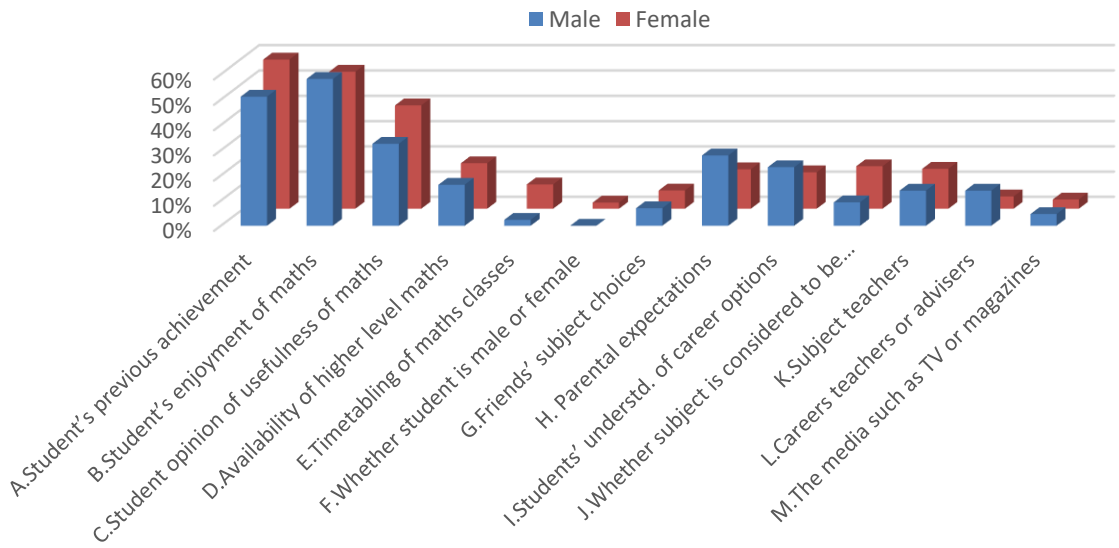
			1995	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
NAPLAN	Year 3	Boy										401	398	398	403	400	399	405	402	407
		Girl										393	390	393	394	391	395	398	394	397
	Year 5	Boy										482	493	494	493	492	492	493	497	497
		Girl										470	481	483	482	485	479	482	488	489
	Year 7	Boy										552	549	553	550	544	547	550	546	552
		Girl										537	538	543	539	532	537	541	539	547
	Year 9	Boy										587	592	591	589	590	590	593	596	593
		Girl										578	586	579	577	578	577	582	587	585
PISA		Boy		540			527			527			519			510				494	
		Girl		527			522			513			509			498				486	
TIMS	Year 4	Boy	496				500				519				519				522	
		Girl	493				497				513				513				513	
	Year 8	Boy	507				511				504				509				506	
		Girl	511				499				488				500				504	

Source: NAPLAN National Report 2008 – 2016. Mullis et al (2015), TIMSS 2015 International Results in Mathematics. Thomson et al (2017), PISA 2015: Reporting Australia's results.

Teachers' View about Factors influencing Students' Decisions to Choose or Not Choose Mathematics in Years 11 & 12

Factors that may potentially affect students' decisions to continue studying mathematics in Years 11 & 12 are obtained from a survey of mathematics teachers (Li & Koch, 2017), and are displayed along the horizontal axis in Figures 2. The teachers expressed their opinions by selecting one box from five choices 'Strongly Disagree', 'Disagree', 'Neither Agree Nor Disagree', 'Agree', and 'Strongly Agree' for each factor. The percentage of 'Strongly Agree' responses is displayed along the vertical axis in Figure 2.

The teachers reported that students' previous achievements in mathematics and students' enjoyment of mathematics are the most influential factors to students' decisions in the subject selection. The next most influential factors, as reported by the teachers, are students' perceptions of the usefulness of mathematics, followed by parental expectations, students' views of career options with Mathematics, whether the subject is regarded to be easy, the subject teachers, and the media.



*Data source: CHOOSE**MATHS** Teacher Survey 2016*

Figure 2. Number of Year 12 elementary, intermediate, and advanced mathematics students, by gender 2006 – 2016

Final Words

Girls are less likely to choose mathematics when they have the option not to, and girls on average perform less well than boys on standardized tests. According to teachers' opinions, students' previous achievements and enjoyment in mathematics are important factors regarding whether students chooses mathematics in Years 11& 12. There seems to be little data of Australian students on their thoughts in the process of subject selection. Nonetheless, effective teaching practices must be identified and used in classrooms to encourage students', particularly girls' participation in mathematics. It is also important to show students career opportunities involving mathematics. It is crucial for teachers to show the fun and wonder of mathematics to motivate and maintain students' intrinsic interest in mathematics.

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Mathematics, gender, and careers

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Gender differences in mathematics learning continue to attract attention – from educators, researchers, and stake holders. The genesis of this topic and early research findings are outlined briefly. Contemporary occupational participation data are provided, generally and for those with a sound mathematics background. Teachers' beliefs about the mathematical pre-requisites for selected occupations are also presented.

Mathematics is generally recognized as a critical component of the school curriculum and as a gatekeeper to many tertiary pathways and career opportunities. Historically, mathematics has been considered to be a male domain, that is, an area more suitable for males than for females. "There are perhaps only three or four women until the nineteenth century who have left behind a name in mathematics. Women were lucky to receive any education at all" (Mckinnon, 1990, p. 347). Over time, and as schooling became more widely accessible, it was recognized that females, particularly those in a sympathetic social environment and from a financially comfortable milieu, could cope adequately with the mathematical curriculum demands imposed on males (Clements, 1979). Yet small but persistent gender differences in mathematics achievement, typically in favour of males have continued to be reported.

Gender and mathematics learning – a snapshot of research

A number of findings emerged from the early research work. On average, females' achievement levels were found to be lower than males, particularly when it came to solving challenging mathematics problems. When mathematics was no longer compulsory, females' participation rates were lower than males. Females' views were found to be less functional regarding future success than those of males, on a range of affective/attitudinal measures about mathematics and about themselves as mathematics learners. At the same time it was regularly emphasized that, when observed, gender differences were small compared to much larger within-group variations.

Recurring differences in mathematics learning in favour of males have continued to be reported including: achievement in post-compulsory mathematics courses, on certain content domains and topic areas, and among high-achieving students (e.g., Li & Koch, 2017; Andreescu, Gallian, Kane, & Mertz, 2012; Leder 2011, 2009).

Multiple models and explanations have been put forward to account for the small yet persistent gender differences in mathematics achievement. Different theoretical and value-driven perspectives have been used to shape and guide research on gender and mathematics learning. Most of the models proposed contain a range of interacting factors, both personal and environmental. Included among the latter are the school culture, social mores, and the values and expectations of peers, parents, and teachers. "It is important to note", wrote Eccles (1986, p. 15) "that any discussion of sex differences in achievement must acknowledge the problems of societal influence". Else-Quest, Hyde, and Linn (2010) argued that "considerable cross-national variability in the gender gap can be explained by important

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national characteristics reflecting the status and welfare of women" (p. 125). Leder (2017) reported that for mathematically able females, more than able males, societal expectations might serve as a barrier to continued participation in mathematics and eventual career intentions.

Why not do mathematics?

However, not all students, whether male or female, necessarily aim for intensive study or proficiency in mathematics. Congruent explanations for turning away from mathematics are found in different theoretical frameworks (e.g. Damarin, 2000; Francis, 2010). The latter argued that some female students in particular struggled to achieve a "'balance' between sociability and high achievement to avoid being 'othered' as a 'boffin' or 'swat'" (p. 31). Within the psychological literature, and within the framework of expectancy-value theory of achievement motivation, the fear of success or motive to avoid success construct has been used to highlight a dilemma considered relevant to high-ability, high-achievement oriented females – those who are capable of, and aspire to success, but at the same time are concerned about the negative consequences that may accompany this success. Success in a male-dominated employment area could be such a situation (see e.g., Leder, 2017).

What influences the choice of occupations which are pursued by males and females in the Australian workforce? Of the myriad of issues that could be examined several are considered here: the gender profiles of different occupations, the occupational choices of mathematical science graduates, and the views of mathematics teachers about the level of mathematics required for different occupations.

Composition of the Australian workforce

Using the Australian and New Zealand Standard Classification of Occupations [ANZSCO], jobs can be clustered into eight major occupational codes, with each further divided into five hierarchical levels bundled together on the basis of the similarities of occupations with respect to skill level and skill specialization. The major groups are: Managers, Professionals, Technicians and Trades Workers, Community and Personal Service Workers, Clerical and Administrative Workers, Sales Workers, Machinery Operators and Drivers, and Labourers. Of these, Professionals is the largest group, followed by Clerical and Administrative Workers, and Technicians and Trades Workers. Educational qualifications vary within and across the groups. In the most highly skilled groups, Managers, Professionals, and Technicians and Trade Workers, more than 70% of workers have post-school qualifications. In contrast, less than half of the workers categorized as Labourers, Machinery Operators and Drivers, and Sales Workers hold any post school qualification (Australian Government, 2017).

Gender composition of the Australian workforce

More detailed inspections of recent collections of occupational data reveal different gender profiles for different occupations. "The Australian labour market is highly gender-segregated by industry and occupation, a pattern that has persisted over the past two decades" (Workplace Gender Equality Agency [WGEA], 2016, p. 2). For males, the three most common occupational codes, technicians and trade workers, professionals, and managers, are the same as those listed for the full workforce. For females, however, professionals, clerical and administrative workers, and community and personal service workers are the largest categories. Examples of starkly different levels of male/female participation in

different industries, based on 2016 census data, include Health Care and Social Assistance (F: 78%; M: 22%), Education and Training (F: 71%; M: 29%), Mining (F: 14%; M: 86%); Construction (F: 12%; M: 88%) (WGEA, 2016). The career directions of those drawn to mathematical studies, that is, those who have completed a mathematical science degree are the focus of the next section.

Mathematical science graduates, pathways by gender

For many years Graduate Careers Australia [GCA] surveyed newly qualified higher education graduates. In 2015, well over 100,000 graduates (38% males and 62% females) responded. Among the respondents there were 750 graduates in the field of mathematics. Of these, two-thirds were males. The Office of the Chief Scientist (2016) also reported somewhat older, but still relevant gender related data. In 2011 there were more than 25,000 individuals in Australia with a degree in mathematical science. The majority of these (61%) were males. The employment pathways of the graduates were described as follows:

> The top three industry divisions that employed Mathematical Sciences graduates were Education and Training, Professional, Scientific and Technical Services, and Financial Services (24, 20 and 15 per cent, respectively)…. There were more males compared to females employed in all industries of employment except Health Care and Social Assistance. (Office of the Chief Scientist, 2016, p. 150)

Thus gender differences in participation in more advanced levels of mathematics education continue, with more males than females engaged in courses. Occupational fields in which females were found to outnumber males mirrored those reported for the larger workforce. What those involved in the teaching of pre-university mathematics think about the mathematical demands of selected occupations is described next.

Teachers' beliefs about mathematical pre-requisites for selected occupations

As part of a larger survey, administered to 620 mathematics teaching staff in 85 schools, Li and Koch (2017) collected information for 14 occupations about the level of mathematics thought to be needed: university mathematics, year 12 mathematics, year 10 mathematics, and basic mathematics skills. For six of the occupations at least 70% of both the male and female teachers considered university mathematics to be necessary. For each of these a higher percentage of females than males believed this to be the necessary pre-requisite – see Table 1.

Table 1
Occupations requiring university mathematics – teachers' ratings

Occupation	% males	% females
Biologist	72	81
Computer scientist	89	97
Economist	94	94
Finance advisor	78	83
Pilot	83	89
Secondary school teacher	78	83

Adapted from Li and Koch (2017)

A small number of the occupations listed were thought to require only basic mathematics. Again gender differences were found. As a group, the females identified five such areas: chef (6% thought this); farmer (6%); lawyer (3%); retail sales worker (8%), and

health worker (3%). Among the males only one of the occupations was assumed to need only basic mathematics: retail sales worker (11% considered this). It is not easy to determine whose judgements about the level of mathematics required in the different occupations are the more accurate, nor the extent to which the students are aware of, or are influenced, by these views.

Final words

As noted at the outset, mathematics is widely thought to be a gatekeeper to tertiary pathways and career opportunities. Data presented in this paper serve as examples of the persistence and extent of gender linked occupational participation, for the workforce at large and for those in mathematics related areas. Gender differences in post school mathematics courses enrolments, and in teachers' assessment of the mathematical requirement for different occupations have also been presented. Options to counter the flow-on effects of the gender differences highlighted here, as well as those found more broadly, certainly warrant further exploration.

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Symposium: Young Children's Transition to Mathematical Drawing

The set of papers comprising this symposium report selected aspects of separate research projects. To address the theme of this symposium - *Young children's transition to mathematical drawing* – each researcher has critically reflected on children's drawings to highlight the diversity in the characteristics of those drawings, and the ways in which they represent (or do not represent) mathematical concepts or processes. In doing so we seek to problematize the expectation that most children will 'naturally' develop drawing skills during pre-school and the first few years of school, that are effective for representing and communicating mathematical meaning.

Although 'drawing' is only specified a few times as a necessary form of representation in Australian Curriculum (Foundation to Year 2), the expectation is emphasised more strongly through the student work samples provided as illustrations of performance standards. For example, of the twelve 'Satisfactory' work samples provided for Year 1 in the Australian Curriculum website (ACARA, 2014), seven of the tasks required drawn responses. The drawings include pictorial representations of quantities, operations and problem solutions, as well as more formal diagrams (number line, graph and a map indicating routes, directions and informal distances). In an assessment situation, what is actually being assessed, the child's mathematical understanding or their drawing skills?

Our concern is that a substantial number of children struggle to develop the required drawing skills, and that many teachers are not aware of the need to explicitly support the development of mathematical drawing. The purpose of this symposium is to draw on some existing research to argue the case for further research that can inform early-years classroom pedagogy designed to obviate the potential learning barrier experienced by many children because of their under-developed drawing skills.

ACARA. (2014). Work sample portfolio: Year 1 Satisfactory. Retrieved 2 December 2017 http://docs.acara.edu.au/curriculum/worksamples/Year_1_Mathematics_Portfolio_Satisfactory.pdf

Paper 1: Jennifer Way. *Two birds flew away: The 'jumble' of drawing skills for representing subtraction Pre-school to Year 1.*

Paper 2: Sarah Ferguson, Jill Cheeseman, & Andrea McDonough. *Children's drawings can be windows into mathematics learning.*

Paper 3: Joanne Mulligan. *Interpreting children's drawings as indicators of mathematical structural development.*

Paper 4: Amy MacDonald & Steve Murphy. *Children's representations of clocks at the start of school.*

Chair/Discussant: Janette Bobis

Key words: EARC Early childhood; REPR Representations; PRIM Primary

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 97. Auckland: MERGA.

Two Birds Flew Away: The 'Jumble' of Drawing Skills for Representing Subtraction Pre-School to Year 1

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This paper contributes to the Symposium: *Young Children's Transition to Mathematical Drawing*, by revealing the diversity of drawn responses to a simple subtraction story. The drawings created by 104 children (aged 4.5 to 8 years) showed an expected age-related progression of representational skill. However, the drawings also revealed the difficulties children encountered in representing the dynamic operation of subtraction, and the persistence of diversity in drawing forms across all age levels.

A naturally developing medium of representation for children is drawing. From around the age of four-years children naturally begin to explore the use of drawing, in iconic, symbolic and even emergent mathematical ways (Machón, 2013). However, the natural development of *drawing as personal expression* is interrupted by the intervention of adults around the time of school entry. Children are, rather suddenly, expected to produce drawings that have specific meaning, represent mathematical concepts and processes, communicate their thinking to others, and make use of formal structures and conventions.

Some of the earliest mathematical drawings expected of young children are the depiction of a quantity (group of items), and the combining (addition) and separation (subtraction) of groups. Children's number sense begins well before school entry, with most toddlers and pre-schoolers able to recognize quantities of two to four items, even before they master the process of counting – the visualising skill known as subitising. Similarly, awareness of 'more' and 'less', and the comprehension of informal addition and subtraction of very small quantities, is typical in children of 3 to 5 years. However, the development of children's representational drawing has been less thoroughly researched (Bobis & Way, 2018; MacDonald, 2013).

This paper explores the drawings created by children (aged 4 to 8 years) to represent a subtractive scenario conveyed through a simple story. The purpose is to reveal the variety in such drawings, and to explore similarities and differences across the age range. The set of drawings has been extracted from a larger study, *Emerging Mathematical Drawings*.

The research is framed by a representational theory for learning mathematics (Goldin & Kaput, 1996). From the representational perspective, the critical importance of representations lies in the fact that mathematics essentially consists of *ideas* that are neither directly visible nor tangible, that is, abstract. These representations exist internally (such as mental images, concepts and relationships), and can also manifest as self-created external representations (such as movements and gestures, drawings, models or verbal descriptions). Potentially, we can infer children's internal representations from the external representations they produce (Goldin & Shteingold, 2001). However, children need assistance to connect their representations (both internal and external) to mathematical concepts in more explicit ways – a process often referred to as 'mathematising' (Ginsburg, Lee & Boyd, 2008). The focus of this study is on young children who have little or no experience of explicit coaching in mathematising their drawings (Pre-school), and those who have begun such experiences in formal schooling.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 98-101. Auckland: MERGA.

Method

The site for the 2017 study was a state primary school with an attached Preschool, in the metropolitan area of Sydney. The class teachers for three Preschool, Kindergarten (Foundation), and Year 1 classes (total 9 classes, 104 children), used a script provided by the researcher to ask the children to create a drawing.

Teacher Script:

Listen to this little story. Then I'm going to ask you to draw what happened.

Five birds perched in a row along the top of a fence. Two birds flew away.

Repeat the story, then ask them to '*draw what happened in the story'*.

The decision was made to exclude a final question (e.g. How many birds were left?) to avoid emphasis on just the remaining group, and so encourage attention to the dynamics of the story. Each drawing was labelled with a code indicating the class and child, and their age in years and months. Analysis of the drawings took the form of repeated sorting, attending to similarities and differences in specific features, until groupings emerged. Both pictorial features (e.g. birds, fence) and mathematical features (e.g. groupings, number of 'birds') were observed. Other features such as arrows to depict movement were also noted. Further examination of the groupings led to refinement of the sorting, then clustering to form four broader categories. The categories were named and described, and sub-categories noted. The child's class, age and drawing category code were entered into a spreadsheet to facilitate sorting and sequencing to search for patterns and progressions.

Findings

Types of Drawings

Only the four broad categories of drawings are reported here, and have been arranged in sequence from the least coherent to most coherent in terms of mathematical representation of the 'subtractive story'.

Category 1: Scribble. Twelve drawings (12% of 104 drawings) were incoherent, in that there was no apparent representation of the story or depiction of number. This included to two blank pages. Four drawings were literally 'scribble', consisting of seemingly random swirls and lines (see Figure 1). Another seven drawings showed some form or structure, but no recognisable features of the story were discernible (see Figure 2).

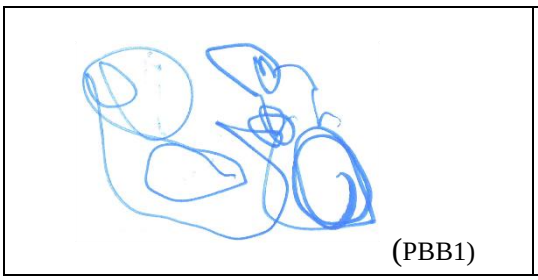

Figure 1. Category 1 - Scribble without form.

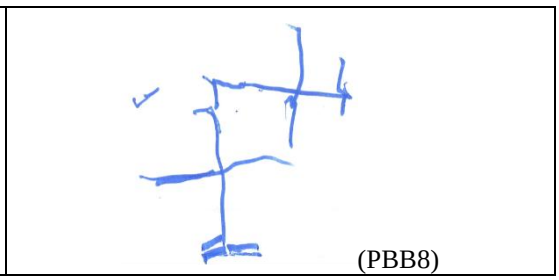

Figure 2. Category 1 - Scribble with some form.

Category 2: Picture. This category contained 21 drawings (20%) that showed the fence and/or birds from the story, but the neither the number of birds nor the number of groups, connected to the quantities in the story. For example, four drawings showed only one bird, seven drawings showed six birds, and one drawing contained 12 birds (Figures 3).

Category 3: Partial Story. These 27 drawings (26%) focused on a particular part of the numerical story by showing one group of birds - either five birds in a single group, or only the two birds that flew away (Figure 4). Some of the drawings included the fence, others did not.

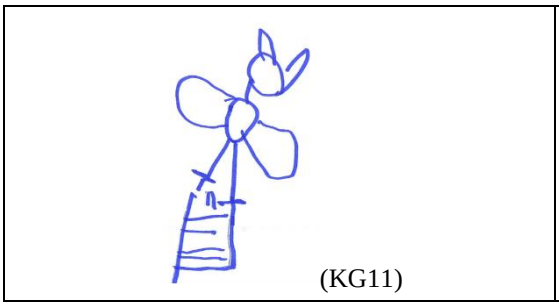

Figure 3. Category 2 – Picture of story element.

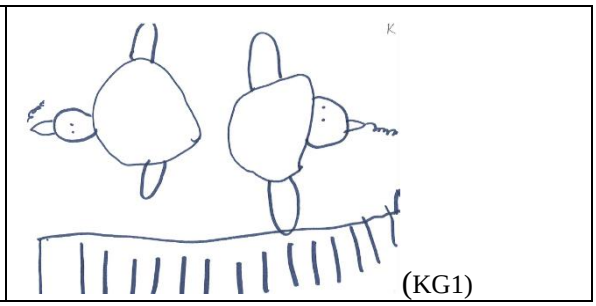

Figure 4. Category 3 – Partial story using number.

Category 4: Partition and solution. All 44 of these drawings (42%) clearly represented the partitioning of a group of five into sub-groups of three and two to reveal the 'solution to the problem'. However, there were several distinct ways of showing the separation. Twenty-eight children drew five birds with two birds clearly positioning above the fence, or crossed out, or identified by upward arrows (See Figure 5). Five children drew just the three birds that were left. Some children captured two events in the story, either with two separate drawings, or with five birds sitting on the fence *and* two flying away (Figure 6).

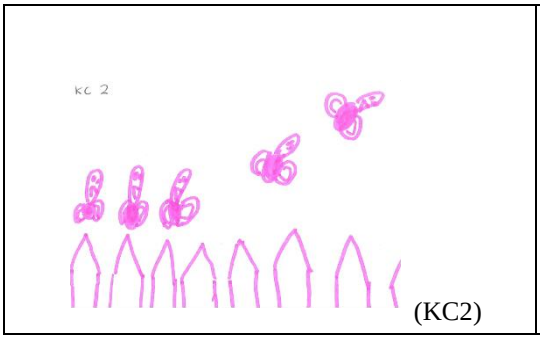

Figure 5. Category 4 – Partition of 5.

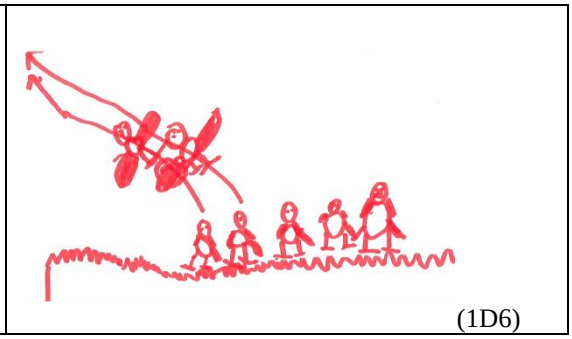

Figure 6. Category 4 – Partition, story events.

Age Groups and Drawing Types

Age groups spanning one year were used, but, as there were only seven children older than 7 years, they were included with the preceding age group (See Table 1). Percentages have been used to facilitate comparisons in both the table and the graph (Figure 7).

Table 1. *Percentage of each Age Group Who Produced each Category of Drawing*

Age	Number of Children	Drawing Type: Percentage (number)			
		Scribble	Picture	Partial Story	Partition
4 to <5 years	27	33 (9)	22 (6)	26 (7)	19 (5)
5 to <6 years	41	2 (1)	29 (12)	32 (13)	36 (15)
6 to 8 years	36	5 (2)	8 (3)	19 (7)	67 (24)
Total	104	12 (12)	20 (21)	26 (27)	42 (44)

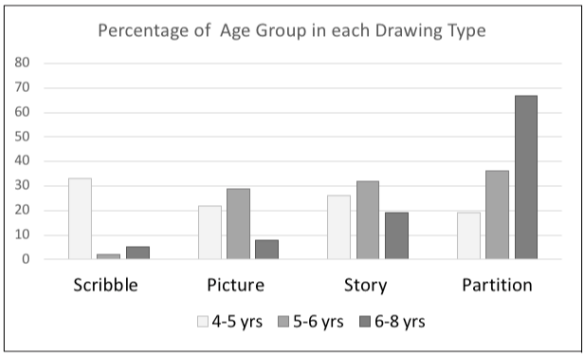

Figure 7. Distribution of age groups across drawing categories

Children from all three age groups produced drawings in all four categories. An increase with age in drawings depicting the partitioning process of subtraction, is clearly apparent. The 4-5 age group is well-spread across all categories, and the 5-6 age group is spread almost equally across the Picture, Story and Partition categories.

Discussion and Conclusion

Both the story and the operation of subtraction are dynamic processes, but a 'one scene' drawing is a static representation. Therefore, the task is quite challenging for young children. The mathematical content of the bird story aligns with the NSW Syllabus (BOS, 2012) expectations for Kindergarten (approx. 5 years). However, the cognitive challenge is increased by having to listen to and process the story information, think of what to draw, and then draw it. It is important to remember that a child may understand both the story events and the mathematics, yet be *unable to draw a representation* of their understanding.

The findings suggest that during the first year of school, many children successfully begin the transition from 'drawing as personal drawing expression' towards mathematical representation of partitioning, yet a substantial number of children are still struggling with the transition midway in their second year of school, when they are expected to use more sophisticated mathematical representations such as empty number lines.

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Children's Drawings can be Windows into Mathematics Learning

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Using the PPELEM (Pupil Perceptions of Effective Environments in Mathematics) drawing and description tool, 208 children in their first two years of school portrayed themselves learning mathematics well. The responses provided insights for teachers and researchers into children's perceptions of mathematics learning and often revealed a detailed and accurate recall of mathematical events, the locations in which they occurred, the people who were involved, and the mathematics learned.

Children's perspectives of learning are an area of interest for both researchers and teachers who seek insights into children's thinking and feelings about learning experiences (Christensen & James, 2000). However, there are difficulties associated with seeking such insights from young learners.

Drawing holds promise as a research tool for investigating young children's thinking as it can reveal growing conceptual understanding. For example, the Pengelly (1985) "Draw a clock" task demonstrated how children aged 3 to 7 years could represent their understanding of time through drawing. MacDonald and Lowrie (2011) found that children aged 4 to 6 years could represent their growing understanding of length through *drawing telling* as Wright (2007) termed it, which allowed children to "create and share meaning using both verbal and non-verbal modes" (MacDonald & Lowrie, 2011, p. 8). Drawing can also shine light on more affective aspects of children's experiences such as student views on the nature of mathematics (e.g., Solomon & Grimley, 2011) and students' perceptions of changes in education such as the role of the teacher (e.g., Haney, Russeo, & Bebell, 2004). Einarsdottir. Dockett and Perry (2009) used drawing and the child's related narrative to gain insights into how children aged 4 to 6 years viewed starting school. It is apparent that children's drawings and associated narratives can be a window for teachers and researchers into children's understandings and how a child thinks and feels about learning.

Pupil Perceptions of Effective Learning Environments in Mathematics (PPELEM) is a drawing and description instrument developed by McDonough (1992, 2002) to discern student perceptions of effective learning environments in mathematics. McDonough and colleagues have used it as a research tool with students in Years 1 to 6 (e.g., McDonough & Pavlou, 1994) and used an adaptation of PPELEM with teachers (Ferguson, 2011). PPELEM potentially provides insights into preferences and needs of respondents.

In the context of a research project titled *Fostering Inquiry in Mathematics (FIiM)*, PPELEM was used to investigate perceptions of effective mathematics learning situations held by 208 children of 5-7 years. Teachers of Foundation and Year 1 classes in three Victorian schools had been experimenting with adding open-ended activities, problems and investigations to their teaching programs. The children's PPELEM data were intended to complement classroom observation data and testing of children's competencies.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 102-105. Auckland: MERGA.

Method

Teachers collected PPELEM responses by giving the following instruction with pauses between sentences, “I am going to ask you to draw a picture. First, I would like you to close your eyes. Think of times when you have been learning maths and choose one time and place when you are learning maths well. Make a picture of that time in your head. Think about who was there and what was happening.” Following the completion of the drawing, children were asked to describe the situation they had portrayed, either through writing or transcription by the teacher.

Analysis and discussion

The PPELEM responses were analysed in relation to a framework of categories. Initially we used categories developed by McDonough (1992) and adjusted them as necessary. After coding a sample of five PPELEM drawings and descriptions, the authors refined and adjusted the categories to suit the responses of young children with the final analysis categorising children’s portrayal of location, people and interactions, tools, and mathematical content. We coded a sample of 20% and found an inter-rater reliability agreement of 86.3%.

Several themes emerged from our analysis of 208 drawings and accompanying annotations. We noted that 5-7 year-old children had clear memories of events, and largely pictured themselves participating in mathematical activity and engaging in mathematics with others: classmates, teachers and parents. The clarity of children’s memories was a striking feature in the data. For example, one child wrote “I was counting in 10s on the calculator. After 100 I thought it was 200 but it’s actually 110”. The calculator had allowed her to discover for herself the next number in the sequence and surprise and excitement were evident as she recalled this event. In another example, a child said “I can count to eleven now using my hands. I use ten fingers and one more.” Another child described some mathematics from the past and could recall both what he did and who he worked with: “I like measuring because you will get to measure stuff like table, chairs everything and it was not today. It was long, long ago, very long. I was measuring with Joanna and Sui”. This supports the findings by Cheeseman (2008) who found that young children could recollect often with clear detail, events from mathematics lessons.

Children’s responses often showed them actively participating in mathematics, using manipulatives and undertaking activities. Mathematical manipulatives such as countable objects and geometric materials featured most often (66 times), and 51 drawings included representations such as number lines, number charts, tens frames and cards. Fifty responses gave examples of counting activities (including 27 of skip counting) and objects used for counting were drawn by 34 children. As counting is a focus of the *Australian Curriculum: Mathematics* in Foundation, it receives much attention in the first two years of school. Children have many opportunities to count collections, to structure material to count them efficiently and to count to solve early operations, with this reflected in some children’s responses including, “When we counted the days of school I learnt how many days we’ve been at school”. However, the children saw mathematics learning in broader terms than counting and numbers as they drew materials for geometry (10), measurement (8), probability, pattern, time, money, “sharing” and adding (coded as Other manipulatives: 12).

The people the children chose to represent provided some interesting insights. In 62 drawings there was only the child in the picture. Of course, we did ask them to draw a time when *they* “were learning maths well” so it is unsurprising that children drew themselves.

This may indicate that children feel they are "doers" of mathematics. However, it was notable that in a further 57 drawings the child drew themselves interacting with other children. Mathematics was seen as something done alone and with others.

Adults had an important role also. Both parents (33) and teachers (31) were portrayed interacting with the child. An interesting finding was how infrequently (4) children drew their mathematics learning happening in a class setting. Learning something with a teacher was often portrayed as learning individually with their teacher. For example, Figure 1:

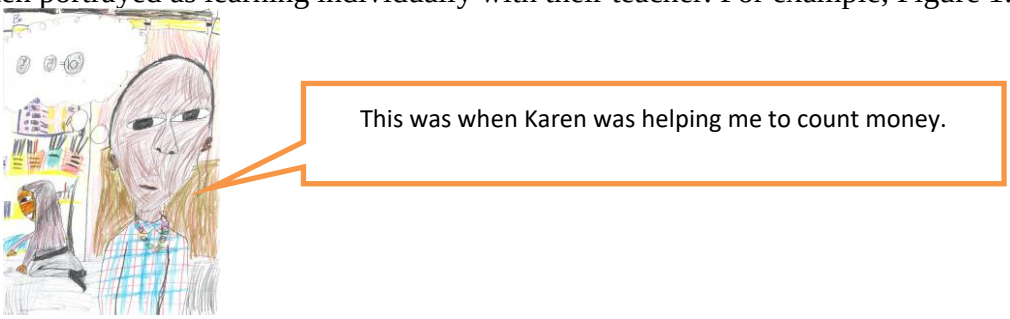

Figure 1. Teacher in the background helping to count money.

Both mothers and fathers were drawn interacting with children and doing mathematics together. Sometimes the whole family was in the picture and it was often clear that the parents were encouraging their child by challenging them to remember facts or to write numbers. In all but two drawings the parents were interacting with the child. In a particularly memorable drawing a five-year-old boy drew himself in the category we called "Location other". He was in a limousine on the way to his mother's wedding trying to make her feel happy by counting to her by threes. Most of the drawings in the "Location other" category were in more everyday places such as shops and supermarkets and at the beach (11) but in each drawing the family was involving the child in practical mathematics. For example, Lucy drew a time at the supermarket (Figure 2):

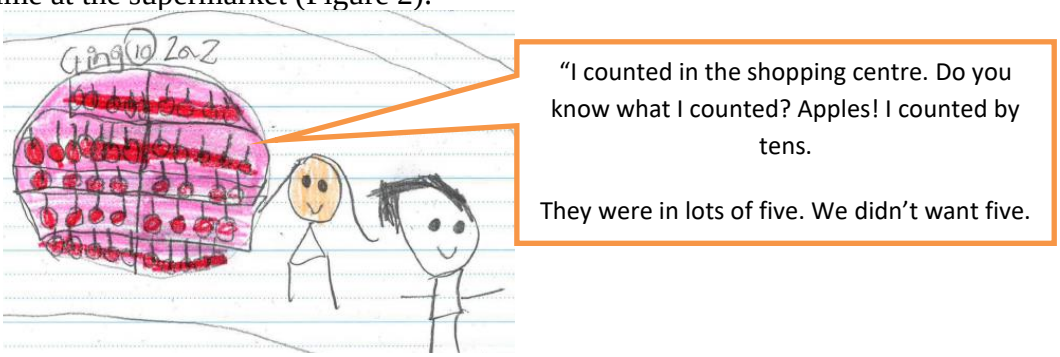

Figure 2. Lucy counting apples in the supermarket.

Impact of PPELEM – a classroom teacher's perspective

The theme that stood out for me from the PPELEM drawings and descriptions completed by my class was the prominence of calculators. The children were able to clearly tell me what they were learning when they used the calculator such as "I learnt that 800 plus 800 is 1600!" said with delighted surprise. Using the calculators to skip count and investigate patterns was one of the suggested tasks from the FIiM project in which I was involved.

Clearly this task resonated with the children who recalled with photographic clarity participating in the task, could articulate clearly what they had learnt, and communicated their enthusiasm and engagement with using calculators.

My reflections about the children's PPELEM responses featuring calculators led to the following conclusions: the children had certainly learnt and explored important mathematical ideas as a result of using the calculators; the calculators allowed them to investigate numbers which we had not previously talked about in the classroom such as larger numbers and negative numbers; and their engagement was high when using calculators. Further reflection on the children's engagement led me to believe that the individual choices the calculators allowed the children to make gave them a sense of agency and control which they enjoyed. As a result of the PPELEM responses I used calculators far more than in previous years when teaching young children and also made them available for children to use during any 'free time' they had.

Conclusion

Drawing tools like PPELEM have the power to provide insights and windows into children's thinking about mathematics learning. The results from young children's drawings show that young children are capable of recalling in detail past mathematical events and communicating what they learnt from them. Children largely drew themselves actively participating in mathematics and engaging in mathematics with others: children, teachers and parents. This in turn influenced their teachers as they gained insights into the contexts, tasks and tools that had an impact on their students' learning of mathematics.

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Interpreting Children's Drawings as Indicators of Mathematical Structural Development

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This paper contributes to the Symposium: *Young Children's Transition to Mathematical Drawing*, by providing an analysis of children's drawings of number patterns comparing spontaneous and task-based situations. Data drawn from the *Reconceptualising Early Mathematics Learning Project*, involving 153 Kindergarten children (aged 4.5 to 6 years) were analysed for five increasing levels of structural features. Children's spontaneous drawings of patterns and their self-constructed representation of a number sequence elicited at a task-based interview, achieved 0.82 consistency. The analysis exemplifies that children's self-initiated drawing, and the process of creating these offers reliable and authentic evidence of their developing conceptual structures.

The analyses of children's drawings of their mathematical ideas and solutions to tasks have played an important role in much of the research in early childhood education over the past decades (Worthington & Carruthers, 2005). Children's drawings have featured for example, in the analysis of early inscriptions of number (Hughes, 1986), in studies on story problems (Carpenter, Moser & Bebout, 1988) and the counting sequence (Thomas, Mulligan & Goldin, 2002), and more recently in studies of early algebra, and pattern and structure respectively (Brizuela, 2004; Mulligan, English, Mitchelmore, & Crevensten, 2013). Curriculum developers and professional development programs have also promoted broadly the development of representational thinking through children's drawings and justifications often portrayed as 'work samples'.

Recent shifts in theoretical approaches based on 'embodied action' have re-directed attention to the role of drawings as more than artifacts that are used to assess what children have learned, "*re*presentations that reveal their cognitive schema— what they 'know' about geometry, such as their cognitive capabilities, spatial awareness, and conceptual understanding" (Thom, 2018). Thom and McGarvey (2015) conceived children's mathematical drawings as both acts and artifacts where the act of drawing serves as a means of developing awareness of concepts and relationships rather than being a product of that awareness. Although new research is directing attention to the analysis of the embodied process of drawing, there remain few studies that provide analyses of both the process and the artefact or product, along with the child's explanation and sense making of the process. Ideally, intensive and systematic use of digital recordings would be required to capture longitudinal evidence of developing conceptual structures.

In this paper, I raise the question of how to effectively elicit and interpret children's drawings as authentic indicators of their mathematical development. The distinction is in whether the child initiates and creates the mathematical features depicted in the drawing or whether the drawing is a reaction to, or a replication of an imposed mathematical model, tool or graphical representation. The purpose of analysing and describing structural features of different types of drawings for the same individual is to provide a more coherent and reliable basis for scaffolding children's mathematics learning.

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Background to the study

Several studies of young children's development of pattern and structural thinking in mathematics have been based on exploratory work analysing children's images (self-initiated drawings) and how these reveal critical developmental features of mathematical conceptual development. Thomas, Mulligan and Goldin (2002) analysed drawings (and explanations) of the counting sequence 1-100 from a sample of 172 children from Grades K to 6, and 92 highly able children. Children's understanding of the baseten system was reflected in a wide variety of iconic, pictorial and notational recordings showing how representational systems for numbers may change through a period of structural development to become eventually powerful, autonomous systems. The analysis highlighted the importance of identifying whether the child's drawing reflected a static or dynamic view of numerical sequences.

A series of studies followed (see Mulligan, 2010), informing the development of the Pattern and Structure project, aimed at developing and validating an interview-based assessment of structural development and evaluation of a pedagogical program to promote early awareness of patterns and emergent generalisation. This paper reports one aspect of the analysis of children's representations drawn from the *Reconceptualising Early Mathematics Learning study,* (see Mulligan et al., 2013).

Method

An intervention program focused on developing mathematical patterns and structures across a wide range of concepts was trialled with experimental groups over the entire first year of schooling. Children's responses to interview-based assessment tasks (the Pattern and Structure Assessment [PASA]) and structured tasks, included drawing using paper and pencil. Opportunities for children's spontaneous constructions and drawings of patterns were also integrated into the program.

A representative sample of the drawings and accompanying exemplars of digital recordings of the drawing process from each student for each group of learning experiences were collected and analysed for qualitative differences by the research team. For illustration, the paper provides an overview of a comparative analysis of drawings from 153 children (aged 4.6 to 6 years) of the numerical sequence "9, 10 and 11" along with one example of their spontaneous drawing of a mathematical pattern. The assessment interview task aimed to capture children's self-constructed representations of number for features of pattern and structure such as equal groups or array structure. Another pre-interview task asked the children to draw a mathematical pattern— "anything that shows me clearly what a pattern is". Data was collected at one interview point prior to commencement of the intervention program by trained researchers. The children were required to using pencils and A3-sized paper for recording. Thus, the exemplars are limited to a 'shapshot' of the child's conceptual structure of number and pattern.

Analysis and Discussion of Findings

Recordings of the numerical sequence task were analysed for features of pattern and structure consistent with previous coding for one of five levels of structural development: pre-structural, emergent, partial, structural and advanced structural. Table 1 shows the percentage of drawings categorised by level and a description of the drawings' features.

Table 1.
Drawings of the numerical sequence 9, 10 and 11 by structural level (n=153)

Structural Level	% Drawings	Description
1. Pre-structural	13	Numerals 9, 10 and/or 11 drawn randomly without any representation of order or quantity
2. Emergent	26	Objects, marks or icons randomly drawn to represent the quantities 9, 10 and 11 often accompanied by symbols correctly represented.
3.Partial structural	35	Partially formed groups, rows or arrays drawn with dots, marks, pictures or icons in order, accompanied by correct numerals; or incomplete representation of the sequence
4. Structural	19	Groups of objects, dots, marks or icons correctly forming arrays such a 3 x 3 and systematically ordered with correct use of numerals
5. Advanced	7	All structural features shown with evidence of extending and decomposing the pattern of representations (such as "3 x 3 and two more is 11") and/or creation of application of the numerical pattern such as 19, 20, 21.

The majority of drawings fell into emergent and partial structural levels consistent with other assessment data on the same group of students.

Table 2.
Percentage of spontaneous drawings of patterns by structural level (n=153)

Structural Level	% Drawings	Description
1. Pre-structural	19	Icons, marks or other idiosyncratic features place randomly without evidence of pattern as repetition
2. Emergent	23	Objects, marks or icons randomly drawn to represent a simple repetition but without any consistent spatial structure
3.Partial structural	32	Objects, marks or icons drawn consistently in order to represent simple repetitions usually with an incomplete unit of repeat
4. Structural	20	Groups of objects, dots, marks or icons correctly forming units of repeat to represent simple or complex repetitions including border patterns
5. Advanced	6	Structural features shown with evidence of extending and symbolising the pattern, and expressed as a generalised 'rule'

Table 2 provides a similar analysis of children's spontaneous drawing of a pattern. There were more drawings categorised at the pre-structural level (19%) compared with the numerical sequences at the same level (13%). However these findings were consistent for the emergent and partial structural level for both tasks. Analysis of individual patterns of response found a 0.82 level of consistency.

The levels of structural development depicted by the drawings provided partial evidence of the child's developing mathematical structures. In combination with analysis of assessment data and a range of other drawn and verbal responses over time a more coherent profile of development can be built. In the present study an overall level of Awareness of Mathematical Pattern and Structure (AMPS) was measured and described. Supporting evidence from the drawings was critical in the formation of a reliable measure of AMPS. Although the analysis did provide explicit indicators of developing structures it could not be assumed that the origins of these representations were created entirely by the child or what

influences impacted on the child's imagery. The question remains whether children re-configure or transform images from their interaction with the real world to represent what they see and mean. Further, depictions in the drawings may have reflected a response that their classroom teacher encouraged or expected, even though the children were at the early stage of formal schooling.

Limitations and Implications

The exemplars analysed in this report present artefacts or products as drawings produced at one point in time. These indicated the presence or absence of important developing structural features such as equal grouping, partitioning, array structure and unit of repeat. Although videos of the patterning process were collected in this study it was not feasible to record and analyse the drawing process for each child. What we did capture were examples of the process of drawing conceptual structures for a representative sample for each conceptual topic over the course of the program. Despite the valuable insights that can be gained from explicit interpretation of drawings, particularly the authentic exemplars self-initiated by the child, drawings in themselves do not provide a coherent picture of children's developing mathematics. Goldin cautions that the researcher must consider that they can only ever make inferences about the child's external representations (drawings) as accurate representations of their internal structures.

In connection and aligned with Way (this symposium) when children begin formal schooling there seems to be a transition from spontaneous self-initiated drawing, and using a broader and unrestricted range of media, to teacher-directed more formal drawings of mathematical ideas or situations, often limited by the size and shape of the media and tools for expression. This paper raises the issue that there remains a stark difference between children's drawings that are 'pedagogically imposed' and those that are spontaneously-elicited images of mathematical ideas. Raising this awareness with both researchers and professionals is critical for further reliable use of drawings as evidence of mathematical development.

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Children's Representations of Clocks at the Start of School

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This paper contributes to the Symposium: *Young Children's Transition to Mathematical Drawing*, by examining the extent to which children are able to draw the structural features of a clock at the start of school. The drawings were produced by 132 Kindergarten children in their first six weeks of primary school. The drawings showed that the majority of the children started school with the ability to represent the structural features of a clock (numbers, hands, partitioning).

Background

Time is often seen as a difficult topic by teachers and children throughout primary school (Burny, Valcke, Desoete, & Van Luit, 2013). There is also relatively little research around young children's understandings of clocks. A seminal study was that of Pengelly (1985), who asked children aged 3 to 7 years to create a clock face using a range of materials. Pengelly suggested that children's understanding of the clock face progresses through five developmental stages: 1. Early impressions of a clock; 2. Awareness of the numerals on a clock; 3. Awareness of the importance of the twelve numerals; 4. Partitioning of the twelve numerals becomes significant; and 5. Recognition of minute markers. More recently, Smith and MacDonald (2009) examined the clock drawings of 4 to 6 year olds and noted, in particular, a fixation on the role and movement of the hands of a clock - a finding that challenged Pengelly's developmental sequence, which did not include a focus on hands.

Despite children's early understanding of clock faces, the *Australian Curriculum – Mathematics* (ACARA, 2017) only expects children to be reading clock faces at the conclusion of Year 1, when aged 6 to 7 years. There is no mention of clocks in the curriculum for the Foundation year, just a requirement to sequence familiar events in time. This study examines the extent to which children are able to use drawings to represent the structural features of a clock at the start of school. Specifically, this study considers three key structural features – numbers, hands, and partitioning – and the sophistication with which children are able to represent these features in their mathematical drawings.

Method

This study was part of a wider project undertaken with 132 children who had just commenced Kindergarten (Foundation) at two primary schools in regional NSW. The data were collected within six weeks of the children starting school. The children were simply asked to "draw a clock"; no further instructions were given. Once the drawing was completed, children were invited to explain what they had drawn and the drawings were annotated with this narrative. Only two children chose to draw digital clocks, with the rest of the sample drawing analogue clocks. For this study, analysis is based on features of analogue clocks as represented in the drawings only – independent of the accompanying narrative. The coding was based on three structural features of an analogue clock: numbers, hands, and partitioning. The drawings were coded according to the degree of sophistication of these three features evident within the drawing, as shown in Table 1.

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Table 1
Coding of the structural features of an analogue clock

Numbers	Hands	Partitioning
1. No number representation	1. No indications of hand(s)	1. No partitioning
2. Some number representation	2. Indication of hand(s)	2. Developing partitioning
3. Numbers in sequence	3. Two equal length hands	3. Partitioning
4. Numbers 1-12 in sequence	4. Two (or three) differentiated hands	

Results

The analysis revealed that the majority of children represented one or more structural features of a clock. Only 14 children (11%) were classified as not representing any of the three features. Of these 14, one drew a digital clock (but with no numbers represented), and one chose to draw a cuckoo clock. The remaining 12 drew a vaguely circular form, but with no clearly discernible structural features of a clock. 22 children (17%) represented, to some degree, one of the features. 46 children (35%) represented two features, while the other 50 children (38%) represented at least some indication of all three features.

Representation of numbers

Only 17 children (13%) did not make some representation of numbers in their drawings. These children did, however, represent at least one other feature, such as the hands (Figure 1). 42 children (32%) represented numbers in some form; usually through the use of dots or dashes (Figure 2), or through identifiable numerals. The majority (57 children; 43%) not only represented numerals, but also represented these in a sequence (Figure 3). Often these sequences extended beyond the number 12. Finally, at the highest level of sophistication, 16 children (12%) clearly represented the numerals 1-12 on their clock face (Figure 4).

Figure 1. No representation of numbers.

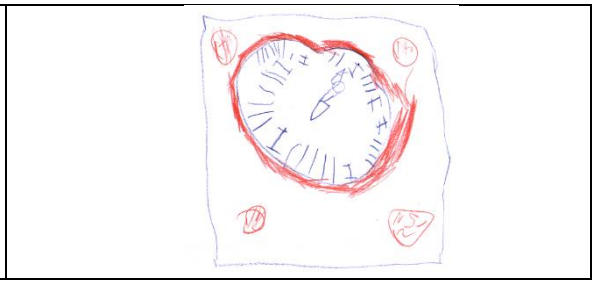

Figure 2. Representation of numbers.

Figure 3. Representation of a number sequence.

Figure 4. Representation of numbers 1-12.

Representation of hands

The majority of the children gave some indication of clock hands in their drawing; although, 45 children (34%) did not represent hands in any way (Figure 5). Of those who did represent hands, most (37 children; 28%) used marks to indicate the position of at least one hand; sometimes more than three hands were evident (Figure 6). 20 children (15%) represented two hands of the same length, with no differentiation between hour hand and minute hand (Figure 7). Nearly a quarter (30 children; 23%) of the drawings clearly represented dimorphic hands (Figure 8), with some children also including a seconds hand.

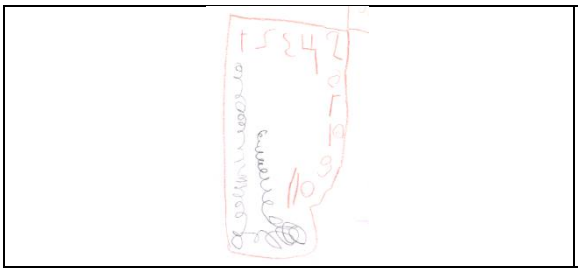

Figure 5. No representation of hands.

Figure 6. Indication of hands.

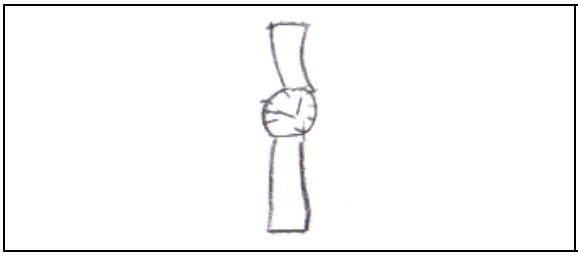

Figure 7. Two equal-length hands.

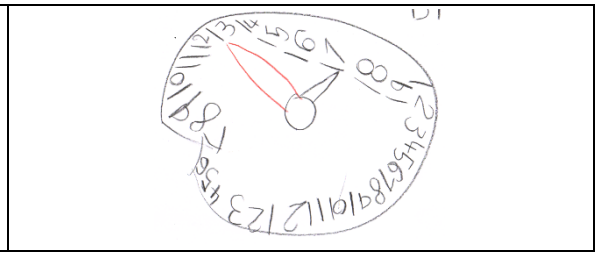

Figure 8. Two (or three) differentiated hands.

Representation of partitioning

Drawings were classified as having no partitioning evident if the numerals/marks were placed haphazardly, or around an arc of the clock face (Figure 9). This was characteristic of most of the drawings, with 70 children (53%) coded as not representing partitioning. Inversely, partitioning *was* evident in nearly half of the sample. 48 children (36%) showed a developing sense of partitioning. Drawings were coded as "developing" when there was an attempt to evenly place numerals/marks around the clock face (Figure 10). Some responses also showed a need to "fill the face", i.e. have numerals/marks all the way around the clock face. In instances where the children stopped at 12 (or another number, i.e. 19), attempts were made to "fill the gap" with scribbling, colouring, or the placement of the hands in the space left over. Finally, there were 14 children (11%) who clearly represented partitioning of 12 numerals/marks around the clock face (Figure 11).

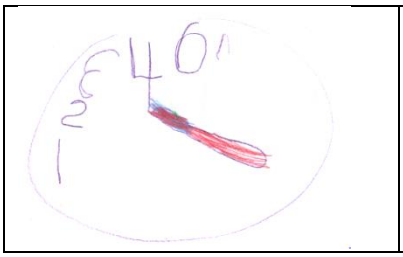

Figure 9. No partitioning.

Figure 10. Developing partitioning.

Figure 11. Partitioning.

Discussion and Conclusion

The results showed that the majority of children start school with some ability to represent the structural features of a clock (numbers, hands, partitioning), with 117 children (89%) representing at least one structural feature in their drawing.

It was logical that children who did not represent numbers in any way were also classified as not representing partitioning. Only 19 children (14%) represented the numbers only, with no indication of hands or partitioning. 87 children (66%) represented hands, consistent with Smith & MacDonald's (2009) finding that many children recognise the hands as a feature of clocks. Interestingly, there was a relationship evident between number sequencing and developing partitioning, with 31 children demonstrating these two categories. Typically, these children continued or repeated a number sequence to continue their partitioning of numerals right around the clock face. Encouragingly, five children represented all three features at the highest level of sophistication.

Our analysis suggests that children's ability to represent clock structure does not progress linearly, as posited by Pengelly (1985). While some form of number representation is necessary to demonstrate partitioning, some children showed partitioning without any number sequence. Some children demonstrated clock hand differentiation without any number representation, while others represented the numbers 1 to 12 without drawing hands or demonstrating any partitioning. The drawings suggest that different children attend to different features of clocks, and thus have different developmental journeys.

This study demonstrates that many children arrive at school with a sophisticated understanding of clock features; yet, the *Australian Curriculum – Mathematics* (ACARA, 2017) makes no explicit mention of clocks for children just beginning school. This is consistent with international research that indicates a mismatch between the intended mathematics curriculum for the first year of school and children's mathematical ability when starting school (Perry, MacDonald, & Gervasoni, 2015). This presents a risk of these students becoming bored or disengaged with mathematics upon school entry.

The children's drawings present a number of opportunities for mathematical development in the first year of school. For example, clock drawing is means of supporting children's writing of number sequences in a meaningful context. Children can also be supported to develop skills in partitioning and spatial representation. These skills also lend themselves to the representation of other mathematical concepts, such as division and fractions. The "draw a clock" task could easily be utilised in Kindergarten classrooms as a means of ascertaining a foundation for further mathematics learning in the first year of school.

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Preparedness to teach: The perspective of Saudi female pre-service mathematics teachers

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This study investigates how well Saudi pre-service teachers feel prepared to teach mathematics at secondary or middle schools through an interview-based exploration. The participants, a sample of 16 female mathematics pre-service teachers, were near the end of the final year of their 4-year education degree. Key findings show that these graduate teachers felt prepared in teaching methods and strategies, but less prepared about some aspects, namely classroom management, lesson preparation, and integration of technology. Findings from this study will contribute to the current drive to improve teacher and teaching quality, including initial teacher education in Saudi Arabia.

Being well prepared for teaching is a key outcome of initial teacher education (ITE). There is a positive connection between preparation and teacher quality and subsequent student achievement in schools (Hattie, 2012). Recent calls for improvements in learning outcomes for diverse learners have focused on teacher quality (Lim, 2011) with ITE being seen as "an ideal site for increasing teacher quality, providing it is subject to reform" (Ell & Grudnoff, 2013, p. 79). Within Saudi Arabia, concerns about student mathematics achievement, fuelled by low TIMSS data in 2007, has likewise put the spotlight on teacher quality. Analysing educational reforms that focus on only specific parts of the education system, Alghamdi (2013) argues that more efforts are needed to address styles and theories of classroom instruction and their impact. This call is backed by studies of teacher quality and its relationship to mathematical achievement in Saudi schools (Al-bursan & Tighezza, 2013) that suggest that researchers need to look at teacher-related factors including how well pre-service teachers (PSTs) are prepared to teach.

In this paper, we report on part of a doctoral study investigating Saudi PSTs' sense of preparedness to teach mathematics at secondary or middle schools. Specifically, we report findings from interviews that explored PSTs' sense of preparedness to teach by addressing two research questions: (i) How do PSTs define or describe being prepared to teach? and (ii) How do PSTs perceive their level of preparedness to teach mathematics?

Literature Review

The influence of quality teaching is undisputable, with the likes of Darling-Hammond (2006) and Hattie (2012) claiming that the biggest influence on student outcomes is attributed to teaching quality. With regard to ITE, studies note that well-prepared graduates are likely to outperform those who are not and are more likely to have better student outcomes and remain in teaching for longer (NCATE, 2006). In contrast, poorly prepared teachers disrupt the learning environment (Mitchell, Marsh, Hobson, & Sorensen, 2010) and leave teaching at high rates (Darling-Hammond, Chung, & Frelow, 2002). However, ensuring the preparation of quality teachers is challenging, with ITE program design involving "a range of complex and even controversial issues" (Cochran-Smith & Power, 2010, p. 6), each of which may impact on effectiveness. For example, PSTs need to be well prepared in content knowledge (CK), specialised pedagogical content knowledge (PCK), and pedagogical knowledge (PK) (Ponte & Brunheira, 2001). In addition, to knowing and

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being skilled in a range of pedagogical approaches to help all students learn (Kraut, 2013), a well-prepared graduate should also be self-confident in their knowledge of students, know how to choose appropriate materials and strategies, understand how different contexts affect education (NCATE, 2006).

Many studies claim that PSTs' beliefs about teaching and learning influence their practices, but few consider how these beliefs might impact on perceptions of preparedness (Buehl & Fives, 2009; Leong, 2012). In looking to assess levels of preparedness, Kraut (2013) found that preparedness meant different things to different PSTs. Some studies have found that PSTs' sense of preparedness is strongly related to their perceived levels of teacher knowledge. For example, beginning teachers in Leong's (2012) study considered that CK was the best indicator of good mathematics teaching and helped them feel more confident with lesson planning and explaining concepts in different ways. Connected to PTS' sense of preparedness, Buehl and Fives (2009) found that CK and PK were identified by PSTs as the most important components for effective teaching. However, other studies (e.g., Balatti & Rigano, 2011) found that PSTs did not consider CK to be so important, possibly because they took CK for granted. Rather, the PSTs in Balatti and Rigono's study mentioned characteristics such as the ability to relate to students, organisational skills, communication skills, and using creative learning tasks, real-life examples, and student-centred teaching strategies as important indicators of preparedness.

Confidence in PK, especially behaviour management, is another area that contributes to feelings of preparedness. O'Neill and Stephenson (2012) found that Australian PSTs felt only somewhat prepared regarding their ability to manage misbehaviour. Although somewhat confident in their ability to use a variety of behaviour management strategies they tended to use only a few strategies (e.g., praise and encouragement). In contrast, Cabaroğlu's (2012) study in Turkey, found that PSTs mostly used reactive strategies (e.g., shouting and threatening). We need to be careful not to assume that this contrast reflects differences between Western and non-Western nations as Roble and Bacabac (2016) found that mathematics PSTs in the Philippines were confident about using a wide range of behaviour management strategies.

Studies that seek to quantify PSTs' sense of preparedness typically use measures of teacher efficacy. Darling-Hammond et al.'s (2002) exploration of teachers' sense of preparedness noted that the strongest predictor of beginning teachers' preparedness was their sense of efficacy. They found that PSTs who felt better prepared were more likely "to believe they could reach all of their students, handle problems in the classroom, teach all students to high levels, and make a difference in the lives of their students" (p. 15). In contrast, PSTs who felt underprepared were "more likely to feel uncertain about how to teach some of their students and more likely to believe that students' peers and home environment influence learning more than teachers do" (p. 15). Likewise, Clark (2009) found that PSTs' feelings of preparedness and associated teaching efficacy were important indicators of how well they felt they would be able to cope with the daily challenges of the classroom and how successful they will be in their teaching careers.

Previous studies that have explored PSTs' sense of preparedness across a range of curriculum areas (e.g., Clark, 2009; Darling-Hammond, 2006) found that most ITE graduates felt adequately prepared to teach and rated their ITE as effective for preparing them for their careers. Anthony et al. (2008) reported that ~87% of graduating secondary teachers felt well prepared or very well prepared to begin teaching. However, when digging deeper, studies have found that PSTs, despite their overall sense of preparedness, reported feeling less confident in some areas. For example, in Anthony et al.'s study PSTs felt less

prepared in assessment and monitoring of student progress, responding to students' diverse needs, inclusive educational practices related to Māori, and communication and working with parents. Rodie (2011) found that PSTs were less confident about planning assessments, writing reports, communicating with students and other teachers, standing in front of the class, preparing teaching resources, and dealing with misbehaviour. Similarly, Koehler, Feldhaus, Fernandez, and Hundley (2013) found that PSTs felt less prepared about classroom management and meeting students' psychological needs.

There is a notable absence of studies related to PSTs' sense of preparedness in Saudi Arabia, but studies in other developing nations, such as Ghana (Agyei, 2012) and Kenya (Ng'eno, Githua, & Changeiywo, 2013), point to PSTs' concerns about the use of technology. Unfamiliarity with ICT, low accessibility, and a lack of infrastructure mirror the situation in Saudi Arabia (Alshehri, 2012).

In studies involving prospective mathematics teachers, concerns about levels of CK are more likely to be expressed by primary or intermediate teachers than specialist secondary teachers. For example, in Lim's (2011) study many U.S middle school PSTs felt poorly prepared due to inadequate CK. Likewise, in an Australian study, Hine (2015) noted that 60% of upper primary and middle school PSTs felt unconfident about their levels of CK. A study Ben-Motreb and Al-Salouli (2012), investigating the PCK of 40 Saudi PSTs, expressed concern that many PSTs were unable to explain the concepts they were teaching, show connections between/among different knowledge strands, or demonstrate how mathematics relates to daily life. Other Saudi studies involving middle school teachers (Al Nazeer, 2004) and elementary school teachers (Khashan, 2014), also found that most PSTs had a more procedurally based CK rather than a profound understanding of mathematics.

Methodology

Conducted in Saudi Arabia, the findings reported in this paper draw on interviews with 16 female mathematics PSTs in the final year of their 4-year undergraduate program at a university-college of teacher education. At the time of data collection, the participants had recently completed their practicum, which took place in intermediate and/or secondary school over a period of ~4 months.

In the larger study, data were collected via interviews and a questionnaire. Interviews, the focus of this paper, were carried out face-to-face or by telephone. Taking approximately 30 minutes, interviews were selected as an appropriate means for exploring the participants' perceptions of situations and their constructions of reality related to their experiences in learning to become a teacher. Following a pre-planned protocol, the semi-structured nature of the interviews allowed the interviewer to probe participants' responses. The recorded interviews were transcribed into Arabic then translated to English for analysis. Manually sorting and categorising the data allowed the researcher to understand the topic's complexity and become familiar with the data. The data were analysed through summarisation, coding, and derivation of themes through applying description and conceptualisation analysis. The segments were named through a process of inductive coding to represent the data as distinct themes, sub-themes or categories.

Findings

Definitions of Preparedness to Teach

Four themes emerged from PSTs' explanation of preparedness to teach: levels of pedagogical knowledge and skills; levels of specialised and curriculum knowledge; feeling confident and gaining experience; and teacher attributes related to the 'good' teacher.

A sense of efficacy in different aspects of pedagogical knowledge and skills was the most frequent (n=13) framing of preparedness. Of note was that seven of the descriptions focused on capabilities related to classroom management—including time management, interactions with students, and behaviour management. For example, PST11 reported that "it is important for the teacher to follow a method to manage the students and it is essential to have respect between the teacher and her students". Lesson planning and good preparation was mentioned by three PSTs (e.g., PST13 specified "being fully prepared to prepare the lesson content and objectives"). These three PSTs also mentioned the importance of being able to "enthuse the students about the lesson". Effective delivery of content, linked to clear step-by-step explanations and illustrations and familiarity with and the ability to apply different teaching methods (e.g., using teaching aids and motivating students), was also used to define preparedness by ten interviewees.

Content and curriculum knowledge was the second theme (n=12). Here PSTs described preparedness as "having a good understanding of the subject content", "the ability to apply multiple mathematical representations", or "being fully versed in understanding the content of the mathematics curriculum". Of note was the lack of reference to pedagogical content knowledge (PCK) in relation to preparedness. Only two aspects of PCK were mentioned: "familiarity with the teaching strategies that are specific for mathematics" and "making students like the subject and not forcing memorization".

Feeling confident and gaining experience were used to define preparedness to teach by seven PSTs. Becoming 'good' at teaching was related to positive experiences in their practicum. For example, PST4 noted that "being able to improve your teaching from the beginning of practicum period to its end," meant she felt prepared. However, in contrast to other PSTs, PST16 noted that "reading lots of books" helped her feel prepared.

For some PSTs, their definition of preparedness included teacher attributes such as "having a strong personality", being "strict" or "patient", "not complaining about students' questions because mathematics needs further explanation and clarification", and caring about students by "avoiding choosing difficult questions to include in the exam questions that have not been presented to the students previously".

The picture that emerges is that being prepared for these PSTs comprised having sufficient PK to help them know how to manage the classroom and feel confident. Preparedness also involved having sufficient CK and familiarity with the curriculum, with PCK being less important in shaping their descriptions of preparedness. However, as many of the interviewees pointed out, familiarity with the curriculum and tools and developing teaching expertise comes with experience. Not surprisingly then, these PSTs affirmed the importance of the practicum.

Sense of Preparedness

When asked to identify the areas that they felt most confident or prepared in five themes emerged: teaching methods; classroom management; lesson preparation and explanation; knowledge of mathematics and the curriculum; and self-confidence.

Feeling prepared about applying different teaching methods was noted by half the interviewees. These PSTs reported feeling confident about using different teaching methods to deliver mathematical content to ensure students' understanding. They also felt prepared about using different teaching methods to adapt to the abilities of diverse learners, to involve all students to participate, to support students' positive relationship with mathematics, and to develop students' mathematical thinking skills. In addition, some PSTs indicated a sense of confidence in their ability to use teaching aids such as concept maps and manipulatives and worksheets to check students' understanding. Only two PSTs expressed that they felt well prepared to link mathematical concepts to reality by using examples from daily life. They elaborated how using a variety of strategies (e.g., playing, teacher role-playing, and cooperative learning) helped them support students' understanding and motivate and engage them in lessons. Feeling prepared about using technology (e.g., PowerPoint and display sketches and images related to the lesson) was reported by only three PSTs.

Nine PSTs reported a sense of being prepared in classroom management. Discussions around classroom management typically included behaviour management techniques simultaneously focused on rewards and punishment. For example, PST10 reported using:

> rewarding methods such as giving gifts for the disciplined students and creating competition between the groups of students. Also using the style of punishment for the students who did not do their homework by deducting marks.

Seven PSTs discussed how they felt well prepared in aspects of lesson preparation and explanation, especially preparing lessons in advance, organizing the blackboard, delivering mathematical information, and using step-by-step explanations.

Familiarity with the mathematics curriculum and CK was noted by five PSTs as an indicator of their sense of preparedness. Regarding the curriculum, PST1 noted: "I have the ability to answer a student's question from another curriculum for the following grade that I have not taught, because I have a background in mathematics as a university student". Feeling prepared in CK was reported as "I have enough knowledge in mathematics".

Feeling confident was or a strong leader was another area described by four PSTs (e.g., "I am confident of my knowledge and the information that I have", "I have a strong leadership personality").

Although the PSTs indicated that they felt prepared in some areas, their explanations indicated that that they were aware of their need to become even more prepared ("I hope to be able to strengthen my strengths"), particularly in classroom management, using a range of teaching strategies and technology, and linking mathematics to reality.

Feeling Less Prepared

Within the category of feeling less prepared four themes emerged: classroom management; content and curriculum knowledge; lesson preparation and explanation; and integrating technology in teaching mathematics.

The majority of the interviewees (n=14) reported feeling less prepared in classroom management. Concerns included difficulty managing a large number of students and difficulty adjusting the narrow and crowded classrooms for implementing student-centred teaching strategies. For example:

> I am not prepared at all in classroom management, controlling/managing the classroom, and organizing the blackboard; I haven't arranged it well at all. (PST15)

> When a student asked me a question, I became distracted from the lesson and I lost control of the classroom. (PST14)

> Most of the teaching strategies that we have studied cannot be applied because the class time finishes, but the strategy has not yet been completed. (PST13)

Managing behaviour was a central concern for most PSTs. Several PSTs reported instances of lack of patience and difficulty in controlling their anger. For example, PST12 noted:

> I do not know how to deal with the naughty students and I have difficulty with that especially with middle school students, because I do not like to deal with them by screaming and giving orders.

Content and curriculum knowledge was less often mentioned by PSTs as an issue. However, five PSTs expressed a desired to be more versed in CK in general and problem-solving strategies in particular, and more familiar with the curriculum across grade levels. PST15 noted ways that she was addressing this shortcoming as follows:

> I am acquainted with the maths curriculum for middle school, but I was not familiar with maths curriculum for secondary school, so I was searching on the Internet to be more familiar with it.

Four PSTs noted specific aspects of lesson preparation and explanation as areas of feeling less prepared. Elaborations included concerns about lesson planning, engaging the students, delivering lessons for student understanding, and using mathematical expressions and symbols.

> I need to be a bit more experienced in explaining the lesson and being able to deliver its content more easily without confusing the students. In addition, I faced difficulty in planning and arranging the lesson content, as I was unsure about what I should present first. (PST6)

> I used to teach using the vernacular (informal language) during my teaching and did not use the mathematical expressions and mathematical symbols. (PST7)

In addition, two PSTs noted that they felt less prepared to connect mathematics with real life, reporting that going forward they were relying on learning about these aspects online.

> I was trying as much as possible to connect mathematics with reality and other sciences, but I was afraid that the students did not understand. I could link the lesson sequences by flowing on to the results of scientific experiments in chemistry and physics, but there were too many mathematics lessons that I could not link to reality. (PST12)

Surprisingly, integrating technology into teaching was mentioned in relation to 'being less prepared' by only one PST. However, it was evident from interview responses that PSTs felt they needed to be more prepared in going beyond using PowerPoint and that integrating technology in mathematics was not sufficiently covered in their ITE.

> I faced difficulty integrating technology. It was not a weakness, I have not learnt how to integrate technology in teaching. We only learned about using PowerPoint presentations. (PST4)

In discussing areas where they felt less prepared it was apparent that although the PSTs reported knowing different teaching strategies they found that in practicum they could not implement them properly because of time, space, and behavioural management constraints. The picture that emerges is that PSTs had gained PK in an academic sense from their ITE course, but found that applying these in reality was harder than expected. This highlights the importance of the practicum for helping PSTs feel prepared.

Discussion and Implications

As in other studies (e.g., Anthony et al., 2008; Clark, 2009; Rodie, 2011), the PSTs generally expressed an overall sense of preparedness to begin teaching, espoused through their knowing about different teaching methods and strategies. However, they defined being prepared to teach mostly in terms of 'having' teacher knowledge, especially CK and PK,

suggesting that ITE enabled them to feel prepared in this respect. Interestingly, aspects of PCK, an aspect more closely related to 'practice' was seldom mentioned in the interviews. Their definitions were consistent with the findings of many studies (e.g., Buehl & Fives, 2009; Kraut, 2013; Leong, 2012), though they conflicted with the perceptions of PSTs in Balatti and Rigano's (2011) study who did not consider CK to be important.

Affirming a theory/practice divide, it appears that PSTs found the application of different pedagogical strategies when on practicum was harder than expected. In this respect, classroom management proved especially challenging and somewhat disrupting to their trialling of more non-traditional student centred teaching approaches. This finding agrees with Koehler et al. (2013), and O'Neill and Stephenson (2012) who found that the PSTs felt prepared regarding CK but less confident about classroom management, especially behaviour management.

In accord with Balatti and Rigano (2011), the interviews revealed that teacher characteristics concerning confidence and the ability to relate to students were important aspects of PSTs' sense of preparedness. Affective aspects of PSTs' feelings of preparedness, noted in expressions of confidence, control, and ability to form relationships with students remind us that knowledge exists in a "dynamic relationship between social, psychological, material, and embodied realities" (Ord & Nutall, 2016, p. 357). However, a sense of preparedness related to these attributes, reflecting a desire to enact a student-centred rather than a teacher-centred approach, were mixed, with PSTs identifying a continuum of strengths. On the positive side, several PSTs expected that this was an area that they would continue to develop expertise in once teaching in the classroom.

While the findings affirm the importance of the practicum for helping PSTs feel more prepared, the lack of explicit reference to PCK by the PSTs suggests scope for ITE methods courses to make the link between PK and PCK more explicit, as suggested in practice-based reforms (Hunter, Anthony, & Hunter, 2015). Indeed learning the work of teaching could include a focus on management of learning, more so then the management of behaviours. In examining PSTs' feelings of preparedness to teach mathematics, this study has provided further insight into the preparation of secondary and middle mathematics teachers in Saudi Arabia. Informing our understanding about improvements needed to ensure quality mathematics teaching these findings suggest that greater links are needed between the teacher education programmes and the sometimes contradictory realities of the classroom.

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Dialogic Practices in the Mathematics Classroom

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Teaching mathematics involves a lot of talking, and dialogic practices are central to most pedagogical practices in mathematics classrooms. Furthermore, for mathematical processes such as 'reasoning', 'explaining' and 'mathematical thinking' to be developed, there is a need for rich and robust dialogic interactions in the classroom. In this paper we investigate the dialogue in a *typical* Year 5 mathematics lesson by analysing the transcript using two different analytical frameworks. While the analysis showed that there were many interactions with nearly half being student turns, it was also evident that almost all the exchanges followed an *Initiation-Response-Feedback* pattern, with a high degree of teacher control. Furthermore, there was little evidence that the dialogic pedagogies of the lesson promoted student development in the mathematical processes. Thus, we content that there is a need to understand the dialogue of mathematics pedagogy, and its impact on students' broader mathematical learning.

Introduction

In this paper we investigate the relationship between dialogicality in primary school mathematics teaching and student learning as it is experienced in classrooms. There is growing evidence advocating the importance of dialogue-rich interactions for student learning and engagement in classrooms, albeit not a great deal in mathematics. Research in primary schools addressing the impact of instructional dialogues in mathematics classrooms is lacking (Anderson, Chapin & O'Connor, 2011). Additionally, researchers and educators lack a framework for teacher-self assessment analysing the impact of their dialogic strategies on student's learning of mathematics (Hennessey et al., 2016). In this paper we look at one mathematics lesson through a range of analytic frameworks that establish the nature of mathematical dialogues experienced in the lesson. Through this process we hope to first, provide some preliminary insights into the relative value of each framework for this purpose; and second, to gain some initial understandings of the repertoire of dialogic practices used in mathematics pedagogy.

Literature Review

To ground the investigations presented in this paper, we briefly review the literature related to dialogic pedagogical practices, and the learning of mathematical process.

Dialogic Pedagogies in Mathematics

Educational research across the globe overwhelming suggests that dialogic approaches to instruction provide an educationally productive environment that promotes student learning and engagement (Alexander, 2017). Moreover, current research has shown that the nature and influence of pedagogy in classrooms is comprehensively and persistently dependent on the dialogic patterns at play in the sequential flow of teacher-student exchanges

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in lessons (Edwards-Groves & Davidson, 2017). Dialogicality involves repertoires of classroom talk and interaction that promote student participation (Sedova, Sedlacek, & Svaricek, 2016); and as found by Edwards-Groves and Davidson (2017) include questioning by teachers *and* students that provoke thinking, extended responses involving justifications and elaborations, critical evaluation of ideas, and explorations of different perspectives. Nevertheless, observational studies strongly indicate that these features are by no mean firmly embedded in classrooms around the world (Alexander, 2017; Skidmore, 2006). Instead, the Initiation-Response-Feedback (IRF) (Mehan, 1979) identified as the default pattern of classroom pedagogical talk remains dominant in classrooms (Skidmore, 2006). The IRF is centred on closed, leading questions with "low cognitive demand" (Sedova et al., 2016, p.14). Even more significant, is that less is known about dialogicality in mathematics instruction (Anderson, Chapin & O'Connor, 2011).

Research has shown that the significance of dialogic pedagogies is the capacity for teachers to open up classroom exchanges to enable students more time and opportunities for engaging in substantive productive discipline talk. Indeed, dialogicality in lessons focuses on tuning into others' perspectives and the continuous collective construction of knowledge through sharing, listening actively, critiquing, problem-solving, questioning, extending and reconciling contrasting ideas. Importantly, these forms of talk are cumulative and often make links between past and future learning or to wider contexts beyond the immediate interaction. More fully developed pedagogical dialogues have not only been shown to assist student's thinking and learning (Mercer & Littleton, 2007), but are also pivotal for developing students' content knowledge in mathematics through oral language use in discussions (Anderson et al., 2011). Yet, teacher understanding of dialogic approaches across the disciplines is limited (Hennessy, Dragovic & Warwick, 2017).

Dialogue-rich instructional strategies have been shown to be a high-leverage pedagogical tool for both constructing subject knowledge and as a valued process clearly linked with the development critical thinking and productive learning and the connection making between and within subject disciplines (Kazepides, 2012). What is striking is that the research worldwide reporting on the educational potential of participating in dialogues have not resulted in substantial changes in teaching. Rather, studies have consistently shown that in today's classrooms, discourse remains dominated by monologic teaching (Reznitskaya & Gregory, 2013). Further to this, and despite growing international evidence for the educational value of student-student and student-teacher dialogues, researchers and teachers lack an analytic framework for making sense of the form and function of dialogic approaches to instruction (Hennessy et al., 2016).

These issues have particular significance in mathematics education, particularly when it is widely accepted and evident in curricula across the world, that mathematical processes are an integral and important aspect of learning mathematics. While these 'processes' are multi-faceted and variously labelled, commonly they include aspects like *reasoning*, *explaining* and *thinking mathematically* (Clarke, Clarke, & Sullivan, 2012). Research implies that dialogue-rich pedagogical practices are valuable for enabling students to develop mathematical processes, and as such it is important to understand how they are enacted in mathematics classrooms, and specifically how these connect to the development of skills, knowledge and dispositions related to reasoning, explaining, thinking and communicating processes.

The Study

The transcript used in this paper is drawn from a larger corpus of recorded, transcribed and analysed lessons gathered as part of a broader funded four-year critical ethnography investigating educational practices in primary schools (see Kemmis, Wilkinson, Edwards-Groves, Hardy, Grootenboer, & Bristol, 2014). The study was conducted in six purposively selected schools in two regions of Australia in the states of New South Wales and Queensland. The particular lesson transcript used in this article was recorded in a Year 6 primary classroom in a rural school; students are 11 and 12 years of age. The lesson was organised as a whole class mathematics lesson focused on decimal fractions; it continued for approximately 50 minutes in a timeslot before the lunch break.

Data Analysis

For this paper, the data was systematically analysed using two different analytical frameworks for studying classroom dialogues: i) *Engaging Messages* (Munns, 2007); and, ii) the *Teacher Scheme for Educational Dialogue Analysis (T-SEDA)* (Hennessy et al., 2016). Because there is very limited analysis of teacher talk and dialogue in mathematics classes, these schemes are used to show any common themes, factors or concerns across the frameworks, and then to ascertain the affordance and limitations of each scheme.

The Engaging Messages framework (Munns, 2007) describes discourses of power and messages of engagement that form part of classroom pedagogy. The identified discourses of knowledge, ability, control, place and voice evolved from the work of Bernstein (1996) and were shaped into a broader framework for engagement by the Fair Go Project (NSW Department of Education and Training, 2006) prior to being adopted for specific use in mathematics by Attard (2011). Although originally intended as an observation framework, for the purpose of this paper it has been adapted as an analytical tool to interrogate the dialogue in the given classroom scenario against each of the individual messages in order to identify its potential value as a tool to assess dialogic strategies.

The Teacher Scheme for Educational Dialogue Analysis *(T-SEDA)* was developed by scholars from Cambridge University (Hennessy et al., 2016) to delineate the substantive nature of the turn-by-turn interaction patterns in lessons and to analyse the extent to which particular teacher talk moves enable student participation in learning episodes. The T-SEDA framework is intended for teacher professional development and is a modified version of the SEDA piloted in primary science classroom settings in the UK and Mexico. The T-SEDA relies on systematically coding talk moves according to 10 identifiable communicative acts categorised into 10 clusters (noted in Table 1 below).

Analysing the Dialogic Practices in a Mathematics Lesson

In this section we will present two analyses of the same lesson transcript on decimal fractions using the two analytical frameworks outlined. These are now presented in turn.

Analysis using "Engaging Messages"

The transcript of the Year 6 lesson on decimal fractions was coded against the five engaging messages of knowledge, ability, control, place and voice. Although the transcript contained a significant number of interactions (a total of 490 teacher and student verbal exchanges), only 15 examples of engaging messages were identified against the framework. The verbal interactions between the teacher and students appeared to be evenly balanced in

instances (student turns recorded 236 times, teacher turns 254 times). However closer inspection of the transcript revealed a pattern of closed questioning that illustrates a heavy emphasis on a 'question and answer' structured as the IRF (Mehan, 1979) exchange structure rather than turn-taking that enabled extended student turns conversation or allowing student-student discussion. This pattern indicates a high level of teacher control that has implications for the production of the engaging messages of knowledge, place, and voice. Although continuous interaction is identified as an important element of an engaging classroom (Attard, 2014), the teacher-driven nature of the dialogue did not appear to promote engaging messages in this lesson.

Five examples of engaging messages that represent discourse relating to knowledge were identified. These examples included statements from the teacher that incorporated 'we' rather than 'I' or 'you' statements. However, the following sequences of interaction sees the teacher reverting to a more negative message of knowledge where the knowledge is controlled by the teacher: "...but I told you at the start that it is a hundredths chart". Other comments coded against the message of knowledge were more positive, demonstrating an intention by the teacher to assist students make connections within their learning, linking previous knowledge. In one instance the teacher's comment indicated a valuing of the students' contributions and knowledge: "He's simplified it, that's the term that we use when we break it down to a smaller amount. Has he simplified it as far as it could go?"

Messages of ability occurred twice during the lesson, encouraging the students to consider themselves as capable. For example, "I don't think this will take you very long because I think you've got a pretty good grip on it". Comments such as this indicate a strong knowledge of students' abilities, an important element in establishing positive pedagogical relationships towards student engagement (Attard, 2014).

The transcript revealed five messages relating to place, promoting feelings of belonging and ownership over learning and providing learning activities that assisted students in making connections in their learning. This incorporated the use of an interactive whiteboard and ensuring all students were given opportunities to use it, and the use of an online tutorial to present a different representation of the mathematical concept being learned. The teacher also made several positive affirmations of students' abilities with statements such as this: "I think you've got a pretty good handle on it".

Although there were attempts by the teacher to include all students in the talk amidst the 500 verbal exchanges in the lesson, the quality of the dialogue did not appear to promote student voice. As discussed earlier, the question/answer pattern of dialogue did not promote discussion amongst students and while the teacher carefully crafted students into his questions with prompts such as, "Let's talk about why Lizzie might have thought that it could have been 1.18", he was still the dominant voice in the classroom, steering the discussion and not allowing the communicative space to be shared equally among students.

The use of this framework as an analytical tool revealed the interconnected nature of the messages and illustrated how the quality of the dialogue can be linked to the mathematical processes that form our mathematics curriculum. The messages seemed to unintentionally hinder mathematical reasoning, problem solving, and communication, yet the closed nature of the responses appeared to provide some opportunities for students to build fluency and some level of understanding amongst students.

Analysis using the "T-SEDA"

The T-SEDA framework was also used to examine the transcript of the Year 6 lesson. T-SEDA makes it possible to delineate the prevalence and particular kind of teacher and

student initiated talk moves (noting it was adapted in this analysis to also delineate the frequency of turns initiated by students as well as teachers). Table 1 presents the frequency (presence or absence) of each dialogic code as applied to each speaker's turn; examples from the lesson are provided to offer a distinctive sense of the particular dialogic move.

Table 1
T-SEDA Frequency of Instances

T-SEDA Dialogic Code	Frequency of Instances		Lesson examples (what was said) Tch (teacher) – S (student)
	Teacher Initiated	Student + (SI student initiated)	
i) IEL - invite elaboration, invite others to build on or clarify ideas	12	3 (SI)	(Tch) "what do you notice Emily's done that's different when she says the number (14.658) compared to the numerals after the decimal point?"; "if it's tenths, what do we visualise what we're doing when we look at tenths?"; "what's he done to turn it into nine over fifty?"; (SI) "Is it base ten?"; (SI) "but you don't need the zero because it's a?"
ii) EL - elaborate ideas, clarify or extend an idea	45	7 (S – in response to a teacher prompt or question)	(Tch) "we can express 2.18 as a mixed fraction"; "he's simplified it, that's the term that we use when we break it down to a smaller amount"; "because it's a non-significant zero"; (S) "It's a fraction that has a whole number and a fraction"; (S) "He's broken it down to decimals"; (S) "[Halving] is dividing by two"
iii) Q - querying, questioning, disagreeing with or challenging other ideas	4	6 (SI)	(Tch) "What thinking's behind - that and it's not 1.18, but I can understand why you thought that Laura. What thinking's gone behind her making that suggestion?" "Do you agree or disagree?" (S) "Why doesn't the whole number change, I thought it should be like 1.08, why doesn't the whole number change?"
iv) IRS - invite reasoning	8	0	(Tch) "why is she right? Why is she correct in putting 0.01 there?"; "Why was that wrong Meg?"
v) R - make reasoning explicit	2	1 (SI)	(Tch) Right, OK, because that's what we've got there. 658 thousandths. There's a subtle difference between how you write it and say it"; (S) "I was thinking about the next number coz"
vi) CA - coordinate ideas and agreement	1	2 (SI)	(Tch) "You didn't used to think that though did you?"; (S) "that was hard work and now they're easy"; (S) "I made a mistake, 'coz when you learn about decimals fractions I thought if you went up by one but now I know"

vii) RD - reflect on activity	1	0	(Tch) “what was the point of that activity? What was it getting you to show Lizzie?”
viii) C - connect ideas to past or future	15	1 (S)	(Tch) “So let’s revise what we’re doing in maths and what we’ve learnt about so far through the last part of last week”; “we did greater than less than earlier in the week, so let’s see if our knowledge of that has been attained?”; “can you see similarities between this chart to the hundred chart that you would have used in year 1, year 2, Kindergarten, all other years?”; (S) “I wasn’t here yesterday”
ix) G – guide the talk, activity and thinking	2	0	(Tch) “Shh, let Meg think”; “think about what they did when you’re comparing the decimals you’ve got to add?”
x) E - express or invite other relevant ideas; respond directly to a question within an IRF	147 + 17 (related to managing a task; e.g. we’re gonna do this sheet)	216	(Tch) “We’ve got the same thing for whole numbers, then we have a what?”; “The first place value is?”; “How would you say that number Elsie?”; “take out a red pen to mark with” (S) “decimal point”; “because it’s part of a whole”; “it’s ten pieces”; “he halved it”; “no”; “a tenth”; “14.658”. ***Note:** the student turns were direct responses elicited by a teacher question as part of an IRF exchange
TOTAL	254	236	

Close inspection of Table 1 indicates a significant variation between the particular kind of teacher talk moves and the student talk. According to this measure, the E (Express or invite) category was the predominant talk move, whereby it was evident that the teacher in this classroom produced almost all invitations and elaborations (E, EL, IEL); closer scrutiny of the transcript revealed these were generally in the IRF exchange pattern. This move almost always required to the students to provide a known-answer response to a teacher question; this occurred 216 times from a possible 236 student turns in the lesson. Here the teacher would initiate a sequence with a question (I) (e.g. “What do we call our number system?”), then a student (generally nominated by the teacher) would respond (R) (e.g. “Arabic number system”), and the teacher (in the next turn) would provide feedback (F) (e.g. “That’s right, okay”) mostly in relation to the correctness of the response.

In this lesson, it appeared that this turn structure in fact limited the possibility for students to contribute a more extended turn or to initiate a question or invitation themselves. This finding appears to be counter to a dialogic approach whereby student-initiated turns (questions, elaborations or invitations) signify a more dialogic classroom (Edwards-Groves & Davidson, 2017). The frequency of elaboration codes (EL) appear moderately high in this lesson, but it is clear that the teacher produced most of these turns. Instances of guiding (G) and reflecting (RD) turns were not often encountered in this lesson but these instances were almost exclusively produced by the teacher. There were minimal instances of co-ordinating (CA) connecting students to past lessons, experiences, concept or activities (C) by a teacher preformulation. One of the surprising findings was that in a mathematics lesson such as this, reasoning (IRS, RE) moves did not appear more often.

Discussion, Implications and Conclusions

The analysis of pedagogical dialogues has been the focus of research across the globe attempting to understand the nature and role of classroom talk for learning and teaching. It is evident that the two analytic frameworks used in this paper offer some insight into the particular patterns of interaction that constitute this mathematics lesson. While there is not scope in this paper to give a comprehensive analysis of the lesson, there are four notable features from both analytic schemes that raise matters related to classroom dialogue and mathematical learning. First, it was evident that the IRF pattern of talk dominated the lesson; this turn structure appeared to limit the scope for students to develop and indeed produce evidentiary talk and mathematical reasoning beyond providing a predetermined known-answer response. Second, both the Engaging Messages framework and the T-SEDA showed that the teacher controlled the dialogue and, even on occasions when a student did initiate a turn, it was mainly directed to and mediated by the teacher, and predominantly related to clarifying queries related to task completion. This is interesting in light of findings by Edwards-Groves and Davidson (2017) that showed dialogic pedagogies actively and overtly promote student-initiated questioning, extended student-student sequences and turns focused on making meaning of the substantive content of the lesson. Third, allied to the preceding two points, there was no evidence that the lesson dialogue promoted mathematical reasoning or significant explanation to any significant degree. In fact, there was no evidence of the utility and development of any mathematical processes at all. Fourth, it is evident that although these analyses are primarily concerned with the sociality of classroom management as expressed through verbal communication, they do not specifically show, however, how the talk facilitates deeper mathematical understandings by students. Added to this, both systems do not enable a nuanced description of the interaction sequences across the lesson as these pertain to mathematics learning. This means that some distinctive features of the dialogue are overlooked or remain implicit; for example, closer inspection of the full transcript reveals that students rarely built on each other's turns, also many of the teacher's turns were multi-unit, meaning they were extended turns that invited, reflected on and/or questioned a student response *as well as* elaborated a concept more fully (in the form of explicit instruction).

While this analysis was only of a single lesson, for the researchers involved it is in many ways a typical, or at least common type of mathematics lesson. So the concern here is not the occurrence of the teacher-directed IRF classroom pattern of talk (which has an important utility in managing and organising students in lessons), but it was the distinct absence of its connection to mathematics itself that we argue distorts the students experience of it in practice. In fact, it is the prevalence of this turn structure that limits students' capacities for developing deep mathematical knowledge and producing extended turns aimed at deepening reason (for example) about specific mathematics concepts. Given the apparent importance of rich dialogic pedagogies for promoting deep mathematical thinking (Mercer & Littleton, 2007), there seems to be a need for research and development in this space. Obviously there is a need for research beyond one lesson to provide greater understanding about the nature of classroom dialogue in classes in different sites, at different levels, and on different topics. But also, there is a need to understand how different dialogic pedagogies might enable the development of skills, knowledge and dispositions in the mathematical processes. Furthermore, while there is scope for research and development related to dialogic pedagogical practices, perhaps there is evidence here of a deeper issue related to the culture of mathematics classrooms and teacher identity. Specifically, these analyses revealed a strong sense of teacher control. While that may have just been a feature of the single lesson

analysed, it is unlikely if more diverse and richer dialogic pedagogical approaches will be effective without a cultural shift that sees control of classroom talk shared in some way with the students.

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The R in the ELR Process: Reflection and the Emotions of Pre-Service Teachers. A Case Study

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Reflective practice in mathematical teaching improves teaching skills and confidence. This paper investigates affect-based critical moments as a reflective practice for pre-service teaching. An embedded case study is presented by one regional university as a discovery into the *reflective* phase of the Enhancement-Learning-Reflection process to uncover the types of emotions and themes from student chosen critical moments. An analysis of these critical moments found students' expressed mainly positive emotions combined with the negative emotion of anxiety. Themes emerged around pedagogical content knowledge: teaching strategies; student thinking; and appropriate mathematical content knowledge.

While not all students will enjoy learning mathematics, a key to confident mathematics students is confident teachers (Cotton, 2013). Research has shown that teacher confidence in their own mathematical abilities, or lack of it, can have a powerful effect on students (Laursen, Hassi, & Hough, 2016). Students taught by teachers with positive beliefs about mathematics generally have positive views about mathematics and confidence in their ability to do, learn, enjoy and discuss mathematics (Uusimaki & Nason, 2004).

There is evidence in the field of mathematics that many teachers lack mathematics confidence (Beswick, Ashman, Callingham, & McBain, 2011). Such studies show that teachers lack confidence in their own mathematical ability and/or their ability to effectively teach mathematics-related curriculum. Studies have also shown that preservice teachers (PSTs) regularly report feeling both a lack of preparedness to teach mathematics at the level they will be qualified to teach (Beswick et al., 2011), as well as mathematics anxiety (Boyd, Foster, Smith, & Boyd, 2014). A consequence of the increasing proportion PSTs entering teacher education programs with negative beliefs and anxieties about mathematics is a relative increase in the number of teachers who have confidence issues when it comes to teaching mathematics. From a school student perspective, the declining level of confidence of those teachers working in the mathematics field has been linked to the declining standards and interest in mathematics exhibited by Australian school students (Lyons & Quinn, 2015; Roberts, 2016), thus indicating intervention is required at the teacher preparation level.

Background

In an effort to address teaching confidence issues in the areas of mathematics and science, as well as the related lack of student interest in these subjects, a preservice teacher (PST) education program targeting teaching confidence has been developed collaboratively by six partner universities belonging to the Regional Universities Network (RUN). The program, titled "It's part of my life: Engaging university and community to enhance science

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and mathematics education" (Woolcott, Scott, et al., 2017), was trialled and refined through multiple iterations at each partner university. A core component of the program is the use of collaborative relationships that connect PSTs with university-based science and mathematics researchers and education specialists. The aim of this nexus is twofold: first, to facilitate the development of improved mathematics or science-related teaching confidence of PSTs through repeated contact with expert mentors who provide high quality mathematics, science and related pedagogical content knowledge (PCK); second, to provide a learning environment in which PSTs develop the skills and confidence required to then themselves be able to create a supportive classroom environment for students.

A key structural component of the "It's part of my life" program is the Enhancement-Lesson-Reflection (ELR) process. The processes engage PSTs in collaboratively developing a lesson (with their expert mentors) that has a focus on a 'real life' local problem or issue; then teaching that lesson to a classroom of high school students; and finally collaboratively analysing the experience with the expert mentors and other PSTs involved in the process. The ELR process is utilised by each university as part of the It's part of my life program; however each university is investigating how the process may be used to improve the education of preservice teachers in different STEM disciplines and scenarios. In this study the PSTs are studying to become high school mathematics teachers and the STEM teaching scenario utilised is mathematical modelling. While early evidence already suggests the ELR method is effective in its ability to positively affect PST confidence (Axelsen, Galligan, & Woolcott, 2017; Woolcott, Whannell, et al., 2017; Yeigh et al., 2016), the focus of this paper is on the integral role of the reflection component of the ELR process in developing mathematics confidence and mathematics teaching confidence.

The lesson and mathematical concepts to be taught were decided and planned by the PSTs in conjunction with their assigned expert mentor. The lesson had to involve a real-world problem, to be solved using open-ended mathematical modelling: devise a group-generated formulation to the presented problem; discuss assumptions and variables; develop a mathematical solution; model possible solutions; and interpret the real world meaning with further model refinement (Stillman, Galbraith, Brown, & Edwards, 2007). Due to the fluid nature of modelling, a definite answer is not possible and thus PSTs had to be prepared for a range of different mathematical scenarios.

Using reflective practice

The importance of preparing thoughtful, reflective mathematics teachers is well recognised in the literature and arguably many of the benefits associated with reflective practice may also contribute to the development of greater teaching confidence. Reflective practice comes in various forms; however a commonly used technique to evoke reflection involves identifying and subsequently reflecting on a certain moment which may have: evoked an emotion (Yeigh et al., 2016); challenged one's own assumptions or made one think differently (Ng, Widjaja, Chan, & Seto, 2012); offered an opportunity to offer an insight to students or change the direction of the lesson (Stockero & Van Hoest, 2013); been transformative in its impact on the education of students (Bedeian, 2007); and/or been seen as critical by the individual (Tripp, 1993). The identification of critical moments that occur during classroom engagement are useful for not only guiding PSTs' reflections on their teaching experiences, but also for exploring accompanying emotions that coincide with the identified moment. Exploring emotions are important because novice teachers and PSTs are often affected by their emotions, particularly when these emotions are triggered by their perception of how well – or more often how poorly – a lesson is progressing (Kilgour,

Northcote, & Herman, 2015). Research also shows that developing an understanding of one's emotions is fundamental to the professional development of confidence in teacher training (Yeigh et al., 2016).

In relation to mathematical teaching, several authors have identified that reflection is essential to effectively facilitate the learning of mathematics (Maree, 2009; Posthuma, 2012) and mathematical teachers can improve their subject and pedagogical knowledge through reflective processes (Sowder, 2007; York-Barr, Sommers, Ghere, & Montie, 2006). Reflective practices in mathematical teaching is important as it allows teachers to shift their thinking from a teaching focus to a learning focus and observe the mathematical thinking that is required for the given situation (Taylor, 2005).

Research Aims

As part of the ELR process being reported in this paper, the PSTs engaged in a collaborative reflection of their mathematics teaching experience, which centered on affect-based critical moments. In order to help the PSTs to develop an understanding of their emotions and the role their emotions played in influencing both their perceptions of their teaching experience and related teaching confidence, the reflection process required the PSTs to identify critical moments that evoked an emotional response – defined here as *affect*. Critical moments are those moments in teaching that are chosen due a self-determined emotion or affect and provide a basis for discussions in reflective self-evaluation of teaching performance. Integral to the reflection of the critical moment is the emotions the PST associates with that moment; that is *how* they felt while they were teaching and *why* they felt that way. To better understand how the use of affect-based critical moments in the teaching of mathematics can be used to improve PST pedagogical confidence, this paper examines the critical moments being selected by the PSTs in their reflections, and considers the questions:

- Are there common themes among the emotions being identified by the PSTs when engaged in reflective practice of critical moments; and
- How do the use of these critical moments and the ELR refection process contribute to developing teaching confidence and/or mathematical confidence?

Method

The ELR process involves engaging PSTs in multiple, repeated sessions that focus on learning and planning (enhancement), teaching (lesson), and feedback and reflection (reflection) (Axelsen et al., 2017; Woolcott, Scott, et al., 2017). This paper reports on findings from one of the six regional universities which focused on mathematics teaching; and focusses on the reflection process.

Participants and Data Collection

Data was collected in 2015. Nine PSTs participated in the ELR process. The participants were 2nd, 3rd and 4th year students studying to become middle- or high-school mathematics teachers. The PSTs presented their lessons to Year 9 and 10 students from local high schools attending on-campus sessions, run in both Semester 1 and Semester 2. A total of 25-40 students participated in each session, which included ninety minutes of problem solving time. When teaching the lesson the PSTs were assisted by an expert mentor (a university mathematics lecturer and/or a practicing mathematician).

Critical Moments

The PSTs' teaching lessons were video-recorded and, following a viewing of the recording within one week of the completion of the lesson, the PSTs were required to nominate six affect-based critical moments during which they felt they had experienced a significant emotional response (either positive or negative). Two moments were chosen from the beginning of the lesson, two from the middle and two from the end. The rationale of choosing moments from throughout the lesson was to ensure a range of emotional states were selected as PSTs (and indeed teachers) often experience different emotional states at the beginning of their lesson (e.g. anticipation, anxiety) compared to those they feel at the end of the lesson (e.g. relief, satisfaction, disappointment) (Yeigh et al., 2016).

To reflect on these critical moments and to investigate whether the emotions the PSTs think they display in their identified critical moments are perceived differently by people observing the critical moment, the PSTs were required to re-play the recordings to a group of observers that consisted of their mentors and other PSTs. After watching the six critical moments, each of the observers responded to those moments using an emotional diary (Ritchie et al., 2014; Yeigh et al., 2016) and any emotions that were observed were discussed. The idea of this reflection process and asking observers to respond to identified critical moments was to highlight to the PSTs that, while they may have felt certain emotion while teaching, such as anxiety, this may not have been obvious to observers. By reflecting on their emotions, the idea was to help the PSTs build teaching confidence by accepting their negative emotions, which may occur due to, for example, lack of confidence.

The emotion diary displayed the emotions: excitement/enthusiasm; happiness; enjoyment; pride; anxiety/worry; frustration; disgust/contempt; annoyance/irritation; disappointment; embarrassment; interest; and confidence. There was also an `other' section to place other emotions, however this wasn't utilized by the PSTs. This protocol was developed to allow PSTs to contemplate and reflect on the emotions they associated with each critical moment (Ritchie et al., 2014; Yeigh et al., 2016).

Observers were required to consider how strongly they felt the PST displayed each emotion during a critical moment, and this was recorded using a 1-5 scale that represented the intensity of the emotion, where 1 was 'not observed' and 5 was 'strongly observed' (OR 'emotion was highly obvious'). The PST whose critical moments were being evaluated (labelled the Teaching PST from here forward) was also asked to use the emotional diary to similarly evaluate how strongly they had felt certain emotions during the critical moment. As part of the group reflection, the emotions observed by the group were then compared to the emotions the Teaching PST had recorded feeling during the critical moment. This discussion was important for highlighting differences between emotions felt and emotions displayed, and for PSTs (both the observing PSTs and the Teaching PST) to reflect on and learn from their emotional responses to the teaching of mathematics.

Post-teaching session audio-recorded debrief

At the conclusion of a full iteration of the ELR process, after all PSTs had taught a lesson and engaged in a reflection of that lesson, the PSTs were asked to individually reflect on their experience in an audio-recorded debrief. During this debrief the PSTs were asked to discuss aspects of the ELR process including: how the ELR process may have impacted on their confidence; how watching a video of the lesson and determining critical moments impacted on their teaching and mathematical confidence; and what they learnt from the process from a pedagogical content knowledge (PCK) perspective. This debrief allowed

PSTs the chance to contribute additional comments on the reflective process without the constraints of following a protocol. The purpose of the debrief was to explore further the role using critical moments can play in developing PST mathematical and teaching confidence, as well as helping the PST develop more positive emotions towards their teaching.

Results and Discussion

The most commonly mentioned emotion by the nine Teaching PSTs across each of their six critical moments was anxiety/worry, a negative emotion. This emotion was mentioned by all of the PSTs in relation to several of the moments they each had chosen. The next most commonly mentioned negative emotion was frustration. The other negative emotions were not widely discussed. With regards to the positive emotions, the most commonly mentioned emotions were excitement and happiness. Again, these were mentioned by all of the Teaching PSTs with regards to at least one of the critical moments they had identified.

When the reflection group was asked to evaluate the critical moments identified by each Teaching PST (using the emotional diaries) the results were quite different. While the Teaching PSTs ranked their negative emotions highly – that is, they felt they displayed high levels of anxiety or worry for example – the other people involved in the reflection process indicated that the Teaching PST had not looked as, for example, anxious as the Teaching PST had perceived. For example for one PST the mean response for the emotion anxiety/worry was 3.20 (s = 1.47) however the mean of the observants was 1.96 (s = 1.07). This was similar across most PSTs.

For the positive emotions, the opposite occurred: while the Teaching PSTs commonly ranked themselves low in how they felt they had displayed more positive emotions, the others involved in the reflection tended to think the Teaching PST had shown stronger outward displays of that emotion. For example for another PST the mean response for the emotion happiness was 2.50 (s = 0.55) however the mean of the observants was 3.95 (s = 0.32). This was similar across most PSTs.

The largest difference between observers and Teaching PSTs perceptions was for the emotion anxiety/worry. This indicates that while PSTs might feel a high degree of anxiety while teaching a mathematics lesson, they have learnt to mask those feelings in front of their students. Indeed this is an important skill to possess as a mathematics teacher as it has been shown that teachers who show mathematic anxiety are more likely to make their students feel anxious about maths (Laursen et al., 2016). The reflection process is thus important for helping PST realise they are able to disguise negative emotions, such as anxiety and frustration, and they should also be commended for this ability. It is also an important time for being able to highlight the positive role an outward display of positive emotions can have in helping their students experience mathematics in a positive manner.

Post-teaching session audio-recorded debrief

Several themes emerged from the debrief sessions. The main themes to emerge were:

All PSTs felt the reflection session had contributed positively to how confident they felt when teaching their classroom lesson. As articulated by one PST:

> With building my confidence I could see (from the enhancement sessions) that unforeseeable problems are inevitable and that you can still basically have a successful session.

Compared to ‘prac’ placements where reflections regarding teaching experiences often rely on individual self-reflection’ following an experience, the ELR reflection process was

considered a more superior process for the fact that the PSTs had to sit and watch a recording of their teaching and consider the emotions they were experiencing and/or showing. It was therefore an important teaching method for helping the PSTs to consider the role their emotions played while teaching mathematics. As articulated by one PST:

> The video component of this process was the best thing; you actually sit back and watch what you are doing. You cannot possibility do that on prac. In prac you are so caught up in the classroom with student's emotions and responses that you don't get a moment to reflect and watch your own teaching.

An important lesson from the ELR reflection process was the realisation that there is a discrepancy between how the PSTs feel and the emotions they display while teaching. This realisation thus contributed to the PSTs feeling more prepared for their classroom teaching experience. As articulated by one PST:

> In our heads when we thought we were anxious, we actually didn't come across as that in the video, and more importantly others didn't notice it either…. on prac we don't do this; it is all reflect on how we think we went (instead of actually observing ourselves on video).

From a PCK perspective, the reflection in the ELR process, and in particular the requirement to identify and discuss affect-based critical moments helped the PSTs to identify elements that they may need to focus on or improve. The most common themes related to PCK were:

- The need to obtain student interest by presenting the mathematical problem right at the start of the session. Most of the PSTs were concerned that the students would not engage and consequently the lesson would be a disaster. A few students were anxious about potential non-engagement, however most felt excitement as the students became engaged (PCK: teaching strategies and student thinking).
- PSTs typically identified as a critical moment the time/s when they felt 'caught out' by the mathematics behind the problem being presented, describing the emotions they felt as anxiety (or in some cases panic). For example, one PSTs needed help with converting from square meters to square kilometres. Another ran out of materials 30 minutes ahead of their scheduled time and thus had to `make up' mathematics on the spot. Such experiences were important for Teaching PSTs to understand what concepts are required in order to teach mathematics more effectively and to develop a deeper understanding of the mathematical knowledge required for teaching (PCK: Content knowledge in a pedagogical context).
- Another moment across the PST teaching experiences that was identified as a critical moment for the anxiety the PSTs felt was when students were not grasping the mathematical concepts or concept of the problem. Understanding the cognitive demands of their students and the appropriateness of questions, particularly in understanding the underlying mathematics and connections in topics, is a very important skill for mathematics teachers (PCK: cognitive demands of a task and appropriate representations).
- From a positive emotion perspective, the most common theme for positive emotions was when students were connecting the problem. PSTs discussed feeling excitement and happiness when their students demonstrated understanding or interest in the problem, or when they became involved in animated group discussions. For the PSTs this is important as it allows them to gain an appreciation of students' ways of thinking about certain concepts and the levels of their understanding (PCK: Student thinking).

Conclusion

The ELR teaching method was trialled across six regional Australian universities. This paper reported on some of the results of one university's 2015 reflective iteration of the program.

With regards to the research questions explored in this paper, it was shown that there are common emotions PSTs associate with teaching mathematics; the strongest of these being anxiety. Importantly however, the study showed that while the PSTs felt anxiety or felt they displayed anxiety while teaching a mathematics lesson, other people (i.e. other PSTs and their mentors) observing were less likely to observe displays of anxiety. Here the use of critical moments and engaging with video recordings in reflective practice is important for helping to demonstrate to PSTs that while they may feel certain negative emotions and indeed feel they are displaying those emotions, often those feelings are being professionally and well contained. This intervention therefore helps PSTs to build a certain level of confidence for the realisation that their teaching 'performances' are often better than they perceived. Arguably, individual self-reflection following a teaching session, without the use of critical moment analysis, group reflection and the ability to re-view video recordings of one's teaching does not allow such detailed examination of emotions and the role these play in teaching confidence. Particularly in areas such as mathematics, the development of positive emotions related to teaching content are integral to helping address those problems related to teaching confidence in this subject area.

Even though the paper presents a snapshot of the emotions displayed during the reflective process from one iteration of the ELR process at one university, similar results have been reported from the other five universities. This thus indicates that from the ELR process the PSTs are able to learn how to use their affective states to assess their own emotions (Woolcott, Scott, et al., 2017; Woolcott, Whannell, et al., 2017). While the results are not generalizable and there are limitations with the sample size and scope in the data, the purpose of this paper was to show the effectiveness of the reflective phase of the ELR process in helping PSTs to improve their confidence in mathematical teaching and to learn from the process to help PSTs in the future.

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Beginning teachers learning to teach mathematics through problem-solving

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New Zealand curriculum documents have referred to mathematics as a problem-solving endeavour for more than 25 years. Although an intended curriculum focus, problem-solving seems to be an aspect of mathematics that many beginning teachers are not familiar with. This research follows three beginning teachers in their first year of teaching as they incorporate problem-solving within their mathematics programmes. Data shows that familiarity with a structure for mathematics lessons that fosters problem solving and reasoning (Sullivan, Walker, Borcek, & Rennie, 2015) supported beginning teachers' subsequent successful efforts to teach a problem-solving lesson.

The New Zealand primary classroom is a multi-faceted, complex context in which teachers are required to educate children in mathematics. In recent decades there has been a focus on problem-solving in mathematics in New Zealand curriculum documents. For example, Mathematics in the New Zealand curriculum (Ministry of Education, 1992) advocated for the use of a problem-solving approach, and the current New Zealand Curriculum (Ministry of Education, 2007) also explicitly refers to mathematics as problem-solving, an approach expected for all levels of schooling. However, although problem-solving has been an intended curriculum focus for more than twenty years, it has often been overlooked in many mathematics classrooms (Holton, 2009).

Research with pre-service teachers indicates that experiencing a problem-solving approach within their teacher education programme enables the beginning of envisaging how such mathematics pedagogies could be enacted in future practice (Bailey & Taylor, 2015). In this current research study, a group of beginning teachers who experienced problem solving within their teacher education programme have been followed into the classroom to explore what enables and constrains them in adopting a problem-solving approach for the teaching and learning of mathematics.

Literature review

Mathematics is a social, constructive and creative human endeavour (Mason, 2008; Solomon, 2009), with problem-solving an integral part of this discipline (Schoenfeld, 2007; Liljedahl, Santos-Trigo, Malaspina & Bruder, 2016). Mathematics is neither separate to one's self, nor is it finite, but a creation that is "never finished, never completed" (Barton, 2008, p. 144). Moreover, mathematics is created by communication between people (Barton, 2008), and involves experimentation, observation, abstraction and construction.

For more than 25 years New Zealand curriculum documents have encouraged the teaching of mathematics as a social, constructive endeavour with a focus on problem-solving. In the most recent document (Ministry of Education, 2007) mathematics and statistics are presented as an active endeavour, with learners expected to be creating, exploring, investigating, justifying, explaining, communicating and making sense (Ministry

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of Education, 2007). The lists of mathematics achievement objectives also highlight problem-solving, prefacing the objectives for every year of schooling (year 0-13) with the statement, "In a range of meaningful contexts, students will be engaged in thinking mathematically and statistically. They will *solve problems* (emphasis mine) and model situations. ..." (Ministry of Education, 2007).

Mathematical problem-solving is a term that is interpreted in different ways, by teachers and texts alike. In this research mathematical problem-solving refers to "the solution of problems, the method of which is not immediately obvious to the potential solver" (Holton, Anderson, & Thomas, 1997, p. 3). It differs considerably from traditional "triple-x" mathematics lessons of teacher explanation followed by examples and exercises (Foster, 2013). In a problem-solving approach, students develop a conceptual mathematical understanding in the process of creating their own strategies to solve problems. In such a classroom mathematics is regarded as creative, imaginative and includes an emphasis on the communication of emerging ideas and concepts (Boaler, 2016; Ministry of Education, 1992, 2007) rather than being a solitary experience with the teacher being the dispenser of mathematical knowledge.

Despite problem solving being at the heart of mathematics and a problem-solving emphasis in curriculum documents this vision of mathematics, and mathematics teaching and learning has not been consistently adopted (Holton, 2009). For many beginning teachers their previous experiences of learning mathematics are more likely to be aligned with lessons such as those described as "triple-x" (Foster, 2013) lessons. As pre-service teachers however, they may begin to encounter a different conception of mathematics with a likely focus on core practices of ambitious teaching. Problem solving is central to ambitious teaching and refers to supporting all learners to develop conceptual understanding, procedural fluency, strategic competence and adaptive reasoning to solve authentic problems (Lampert & Graziani, 2009; Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010). Research has shown that *experiencing* problem solving firstly situated as a learner is an important first step towards pre-service teachers learning about mathematics as a problem-solving endeavour (Bailey & Taylor, 2015). Cavanagh and McMaster (2017) suggest such experiences need to be further supported by having opportunities to observe experienced teachers teaching problem solving lessons, followed by engaging in co-teaching a problem-solving lesson.

In the late 1990s a research project specifically investigating teachers' learning about teaching mathematics by problem-solving concluded that with more widespread use of problem-solving there will be challenges for teachers and a need for more professional development (Holton, Anderson, & Thomas, 1997). Cavanagh and McMaster (2017) echo this sentiment, writing, "a problem-solving approach to teaching mathematics presents a major challenge for many PSTs [pre-service teachers] in primary education" (p. 48). Sullivan, Walker, Borcek and Rennie (2015) comment that even though there is agreement about the importance of incorporating problem-solving and reasoning into mathematics teaching there is limited specific advice about how to do this. They propose a defined structure for lessons that support children's learning when using tasks intended to prompt problem solving and reasoning.

A Structure for Lessons that Fosters Problem Solving and Reasoning

The lesson structure suggested by Sullivan, Walker, Borcek and Rennie (2015) comprises four phases described as the 'Launch', an 'Explore' phase, a 'Summary' phase

followed by a 'Consolidation' phase where additional experiences are posed to consolidate the learning activated by the initial task.

Posing the task is a critical aspect of structuring a problem-solving lesson. Two key tasks are establishing a common language, so the task is interpreted appropriately, and deliberately maintaining the cognitive demands of the task (Sullivan, Walker, Borcek, & Rennie, 2015). During the 'Explore' phase children work individually or in small groups, with the teacher thoughtfully walking around the desks. A key aspect of this phase is the teacher having already anticipated ways that different students might respond to the challenge by pre-planning questions/tasks that differentiate the experience. It is suggested this is done by the provision of enabling prompts and extending prompts. Enabling prompts involve "reducing the number of steps, simplifying the complexity of the numbers, and varying the forms of representation for those students who cannot proceed with the task" (Sullivan, Walker, Borcek, & Rennie, 2015, p. 44). It is important to note that this is done with the explicit intention that the children subsequently return to work on the initial task. Extending prompts are offered to students who "complete the original task quickly which ideally elicit abstraction and generalisation of the solutions" (Sullivan, Walker, Borcek, & Rennie, 2015, p. 44).

In the 'Summary' phase the way student activity on a problem/task is reviewed, including solutions and strategies, needs to be carefully managed. Sullivan, Walker, Borcek and Rennie (2015) in referring to the work of Smith and Stein (2011) describe the key elements as:

> Selecting responses for presentation to the class and giving those students some notice that they will be asked to explain what they have done;
>
> Sequencing those responses so that the reporting is cumulative; and
>
> Connecting the various strategies together (p. 45).

The last phase, consolidating the learning, involves posing a task similar in structure and complexity to the original challenging task. Some elements of the original task remain the same while other aspects change to help the learner avoid over generalisation from solutions to one example.

Building on Sullivan, Walker, Borcek and Rennie's (2015) research with teachers who "were a mix of age and experience, although skewed toward being more experienced" (p. 46) this article investigates the use of the lesson structure by beginning teachers. This is one aspect of an ongoing research project following a small group of beginning teachers through their first two years of teaching, with a focus on their learning to incorporate problem-solving within their mathematics programmes.

The Context, Data Collection and Analysis

In this qualitative action research project, data were gathered from three beginning teachers who responded to an invitation to be involved in the project at the end of their one-year graduate diploma in primary teacher education. The beginning teachers (hereafter called teachers) were teaching a diverse range of year levels at three different schools. Julia (pseudonyms have been used for all names) was teaching year 0-2 children at a small rural school; Charlotte, year 5-6 children at an urban city school; and Reine, year 7-8 children, at another small rural school.

Action research has an emphasis on the participation and collaboration of all involved in the research. My role has been that of a facilitator of this small group contributing my experience as a mathematics educator and researcher. A facilitator acts as a co-participant

within an inclusive network, aiming to improve practice, challenge and reorient thinking, and transform contexts for learning (Locke, Alcorn, & O'Neill, 2013). Congruent with the principles of action research activities and procedures were negotiated throughout the year in a co-constructed, responsive and emerging way as the research evolved.

The first step in the process was a one-and-a-half-hour focus group discussion at the beginning of the year before teaching had begun. The aim of the first discussion was primarily determined by me as facilitator and included inviting the teachers to reflect on problem-solving aspects of the mathematics education paper (completed as part of their teacher education year); brainstorm how they might introduce this approach in their first year of teaching; and identify and plan what actions might be needed to support their trialling of this approach in their first year of teaching. It was agreed during this first focus group discussion that some workshops would be useful to support their learning and reflections. Three of these were subsequently held, each for three hours, at the end of terms one, two and four (the beginning teachers opted for these to be held during school holidays). It was also decided that I would visit and observe the teachers teach a problem-solving lesson in term three. All focus group discussions, workshops, lesson observations were audio-taped and transcribed.

Data from the workshops was analysed using an emergent analytical approach (Strauss & Corbin, 1994; Borko, Liston, & Whitcomb, 2007) as the year of research unfolded. Transcripts were read and re-read with notes taken as particular issues and themes emerged. These notes not only constituted data analysis but also provided direction for subsequent workshops. During the analysis of the first workshop the teachers all expressed concern about catering for the diverse range of learners within their classes. They pondered that it would be useful to have 'extensions' ready to give those children who solved problems more quickly. It also became apparent that they needed to know the problems they were using more thoroughly and understand in more depth how a problem could be planned to cater for a range of children. In response to this finding I searched for a resource that might be helpful to share during the second workshop, and in the process located the article by Sullivan, Walker, Borcek and Rennie (2015). The specifics around sharing ideas from this article, the teachers' responses (during workshops two and three – workshop three was primarily a feedback session about the year, and planning for the next year) and brief reference to observations of all three teaching a problem-solving lesson, constitute the bulk of the data presented below.

The Second Workshop

The professional learning in the second workshop began with a discussion based on the teachers' problem-solving experiences in term two. During the first workshop we had co-constructed a format for a problem-solving session (like that proposed by Sullivan, Walker, Borcek, & Rennie, 2015). During the second workshop I asked the teachers what had worked well, what they found easy and where they encountered challenges. We then discussed the structure for a problem-solving lesson as suggested by Sullivan, Walker, Borcek and Rennie (2015) (referring to a one-page handout constructed by the author based on the four phases described by Sullivan, Walker, Borcek and Rennie (2015)). The teachers then engaged in solving a "river crossing" problem (an algebra problem based on the linear relationship $y=4x+1$) as the author deliberately modelled the sequence of phases suggested by Sullivan et al. (2015). During this experience teachers varied their role between 'being teachers' thinking about the suggested structure in relation to their previous teaching experiences and being in the role of mathematical problem-solvers. After the "river crossing" problem was

solved (with two of the three having begun a consolidation task) time was spent discussing their experience making specific links to each of the four phases. Another one-page handout was provided, giving examples that linked the "river crossing" problem to each of the four phases. At the end of each workshop there was an opportunity for written reflection about what had been learned, and what might be tried during the following term.

Results and Discussion

An analysis of the discussion from workshops two and three yielded two key findings pertaining to the use of the lesson structure (Sullivan, Walker, Borcek, & Rennie, 2015): the first being that the teachers found the structure helpful for their learning about how they might teach mathematics through problem-solving. The structure, appeared to enable two of the three teachers, Charlotte and Reine, to engage more fully in teaching mathematics through problem solving. Julia had already gained some experience through teaching problem-solving lessons one day a week for most weeks in terms 1 and 2. The second finding was that examining the structure alongside the first-hand experience of solving a problem appeared to engender personal reflection for improving their future practice. All teachers were able to identify what they needed to change to more successfully conduct problem-solving lessons in the future.

Problem-Solving Lesson Structure Helpful

During the first workshop one of the tasks was co-constructing a 'list' of useful tips for teaching a problem-solving lesson: it was envisaged this could become a 'guide' for teaching mathematics through problem-solving. As the facilitator I envisaged the teachers would contribute ideas based on their problem-solving experiences from term one. Julia had been trialling problem solving lessons approximately once a week. Charlotte and Reine had attempted to use a couple of problems at the beginning of term one and not since then. The compiled 'list' read as follows (NB. The list was co-constructed with ideas contributed by all participants hence some aspects of the list have a range of ideas):

Begin lesson with teacher reading through the problem with whole class.

Give individual thinking time. Maybe answer, clarify some questions from children about the problem.

Organise class to work on the problem individually or in pairs or small groups up to 3. If children, choose to work independently it's OK for them to also have 'talking-time'. Maybe have a table for those who choose to work independently.

Think carefully about how to visually present the problem. Maybe not too many problems. Maybe have the piece of paper folded in half with the beginning of the problem on the first side, and extension questions on the second.

Offer equipment but don't be specific, so children are doing thinking from the start. Allow choice.

Say to children: If you know what to do, go and begin. And then maybe work with those who still have questions (but make sure you don't start solving the problem for them!).

Group children according to teacher choice OR mixed ability OR might try letting children group themselves.

Need to teach group behaviour. E.g. Set up expectation that teacher can ask anyone. Shift the focus from the answer.

Have the same problem for all children and know how to extend the problem.

Once children are working, teacher observes the groups, and asks questions.
Monitor groups. Choose when to extend. Check they all really understand the answer. Avoid the temptation to rush on. E.g. have they solved the problem in different ways?
Use scrapbooks and groups record how they solve it.
Whole class feedback. Think, pair, share for older children. Maybe share between two groups. Pick random or pick specific groups/individuals.

It was during the second workshop the structure for a problem-solving lesson proposed by Sullivan, Walker, Borcek and Rennie (2015) was shared, experienced and analysed. Consistent with the finding in Sullivan et al.'s (2015) study that teachers found the lesson structure useful and achievable, these *beginning* teachers experienced likewise. From the beginning of the discussion all were able to make connections to each of the four phases, and even offered some suggestions building on those offered in the article by Sullivan et al. (2015). For example, Reine suggested having a two-stage launch process. He said, "I wonder if it's like two parts – like you just initially clarify words you're not sure of… and then you go a bit deeper after the start ...". Reine was referring to a possible need for further clarification once the children had had an opportunity to work with materials and begin solving the problem. At the end of the second workshop after the teachers had engaged in solving the "River Crossing" problem and analysed their experience against the structure, they were asked what the main thing they had learned was. All of them referred to the structure specifically identifying the enabling and extending prompts as useful.

> Reine: That you can give the entire class a problem, you've just got to have a plan, [plus] your enabling and extension prompts.
>
> Charlotte: Yeah, I think the main thing is that using the same one – it's important and it's useful for your whole class to be working on the same thing. And kind of [how] easy it is to have enablers and extenders to make sure that everyone feels successful. And, that approaching it, and how to present it to your class. Yeah.
>
> Julia: … I think around that planning. Yeah ensuring that I plan, and trialling the problem, planning it well so that we've got enabling and extending prompts. But the other thing I'm going to try and make sure I do is how they're recording what they're learning.

During the next term all three teachers were observed teaching a problem-solving lesson. All three lessons delighted the respective teacher, with them noting the prolonged engagement of the children, the resulting learning and being able to cater for all learners with the one problem. In Charlotte's words, "it really worked".

During the final workshop of the year feedback was sought about which aspects of the research throughout the year had been most useful. All three teachers identified the lesson structure proposed by Sullivan, Walker, Borcek & Rennie (2015). Julia said "I think for me, it was talking about what a problem is. And the framework [referring to the lesson structure] of how to present a lesson that way". Reine commented,

> I like the framework. So, from start to finish, how you go through that whole lesson. So how you set it up. And then you go through the phases. Then off that I like the, the prompts that we went through…. knowing where, where you could go, if they're like, 'What do I do?' And then like if they get it too easy, then where can you go? So, you've got all these little avenues.

Charlotte said,

> Definitely that the – like the problem that you've experienced the problem – that whole like, setting it out. Yeah. And at the start I didn't – I guess I didn't use that very well, or that much. And then that time that you came and observed I planned it that way. And it really helped, and really worked. So, I found that really useful. And I think it was, it was really useful for me and my class. 'Cause they really understood. And just – I think also making sure that you know, yeah, like all the ins and outs of a problem. So where could they go? What do they... What do you need to know? What do they need to know? And all that stuff.

It is interesting to note that during term two (after workshop one where we compiled the 'list') Julia had once again been trialling problem-solving lessons approximately once a week, whereas Charlotte and Reine had not made any further significant attempts. In contrast following the second workshop all three teachers were keen to be observed and to trial the lesson structure they had experienced and analysed. Despite the similarities between the compiled 'list' which all the teachers had contributed to, and the lesson structure, it is possible the defined and focused nature of the four phases in Sullivan, Walker, Borcek & Rennie's lesson structure (2015) was more effective. It is also likely that aligning the lesson structure against the first-hand experience of solving the "River Crossing" problem facilitated more understanding of the lesson structure. During the first workshop we had aligned the generated 'list' against a problem for new entrant children (in an effort to respond to a previous question from Julia who was teaching new entrants). Maybe because the mathematical content did not constitute a genuine 'problem' for the teachers, connections were not made at that initial stage.

Identifying Needed Changes in Problem-Solving Teaching Practices

Examining the lesson structure alongside a first-hand experience of solving a problem also enabled all three teachers to reflect on and identify what aspects of their practice they needed to change to more successfully conduct problem-solving lessons. Charlotte explained that she had previously not known how to extend the problems, explaining,

> I think that's what I'm kind of missing out is the... Like these kind of extensions, write it down as a general equation, write it as an equation. You know? Like I think I'm skipping that, 'cause they go, 'Oh yeah, but it's this many trips.' But then the generalising and the recording of it....

Julia realised that she had probably been under-planning. She said, "I feel like at times I've been underprepared with the problems I've used. Like I haven't spent enough time thinking about – or tried it, or spent any time thinking about what the kids might do with it". Reine agreed commenting that, "yeah sometimes I just print it all out the morning before". This is an important learning for these teachers, given Sullivan, Walker, Borcek and Rennie's (2015) similar finding that teachers who are more successful in terms of improvement in students' responses are those who did the task/problems before the lessons. It seems likely that being able to utilise the lesson structure to reflect on their current practice has potential for improving the teachers' future practice.

Summary

Given this is a small study care needs to be taken with making claims and generalising. However, it appears that for at least this small group of beginning teachers, providing and analysing the lesson structure (Sullivan, Walker, Borcek, & Rennie, 2015) in conjunction with a genuine problem-solving experience facilitated their learning. This finding has implications for teacher educators in that beginning teachers need to not only have first-hand experience of solving problems, but also opportunities to analyse their experience against the lesson structure. The lesson structure also appears to offer a reflective framework against

which the teachers could identify what needs to change and improve in their current problem-solving teaching practice.

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Using Technology in Mathematics: Professional Development for Teachers

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The Ghanaian mathematics curriculum expects teachers to adopt technologies as an instructional tool to assist students to learn mathematics relationally. Teachers' dispositions (knowledge, beliefs, and attitudes) towards technology are critical in translating the curriculum intention into practice. This paper presents teachers' initial dispositions related to technology integration and their views about a professional development model. In this model, they worked in teams to develop and enact GeoGebra-based mathematics lessons with support from expert, exemplary materials, and demonstration lessons. The results indicate that the model of professional development is promising in engaging teachers in technology integration.

The mathematics curriculum for Ghana's senior high school students (15-17 years) emphasises the need for teachers to assist students to use computers, calculators, and spreadsheets to develop mathematical concepts, and to investigate and solve real life problems (CRDD, 2007). Though not explicitly stated, the general objectives outlined in this curriculum suggest the use of these technologies mediate constructivist teaching and learning approaches where students are guided to use tools to explore mathematics concepts relationally. However, lack of subject-focused technological knowledge and skills impede teachers' ability to use technologies in their classroom (Agyei & Voogt, 2012).

Considering the technology adoption policy in the mathematics curriculum for senior high schools and the fast-changing nature of technology, a necessary condition for effective implementation of these technologies is the teachers' dispositions (knowledge, beliefs, and attitudes) related to the use of technology. Teachers need to have knowledge about technology, content, curriculum, teaching approaches, classroom management, learners and their characteristics, assessment of student learning, and ways to evaluate instructional approaches (Webb & Cox, 2004). Consistent with this argument is the Technology, Pedagogy, and Content Knowledge (TPCK) theoretical framework proposed by Mishra and Koehler (2006). Mishra and Koehler espoused that teachers need a well-developed integrative knowledge of technology, pedagogy, and content for effective technology adoption in the classroom. A professional development model, where teachers are engaged in technology-oriented activities is one of the key steps to enhance teachers' knowledge and skills to use technologies to teach mathematics (Koehler, Mishra, & Yahya, 2007; Koellner, Jacobs, & Borko, 2011).

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Theoretical Underpinnings

The TPCK framework guided the content and structure of the professional development model proposed in this research study. The tenets of the TPCK framework espoused that the knowledge and skills which the teachers are required to adopt depend on the contextual understanding of the interconnected ideas between technology, pedagogy, and content knowledge. Technology knowledge (TK) is the knowledge about how to use ICT hardware and software and associated peripherals. Technology content knowledge (TCK) is the knowledge about how to use technology to represent and create the subject content in different ways. Technology pedagogy knowledge (TPK) is the knowledge to choose a technological tool based on its fitness for the learning activity. Technology pedagogy content knowledge (TPCK) is the knowledge of using technological resources based on their appropriateness to teaching and learning tasks of specific subject content (Mishra & Koehler, 2006).

Hechter, Phyfe, and Vermette (2012) argue that "the application of technological, pedagogical, and content knowledge principles should be understood under the broad contexts of school environments, individual teachers' previous experiences, and epistemological beliefs about teaching and learning" (p. 141). Although Mishra and Koehler (2006) acknowledged the relevance of context (which include teachers' working environment and their personal orientations-prior experiences, beliefs, and attitudes) in effective implementation of technology, this component is less explored in most of the TPCK research studies (Chai, Koh, & Tsai, 2013). For teachers to appreciate the constraints and maximise the affordances of technology, it is argued in this paper that their technology integration knowledge needs to be developed alongside their beliefs and attitudes related to the use of technology (Ertmer & Ottenbreit-Leftwich, 2010). Attitude is defined in a psychological sense to encompass teachers' affective, cognitive, and behavioural attitudes related to technology integration. The affective attitude is the feeling (either comfortable or dislike) the teachers have about the use of technology (e.g., technology makes me feel comfortable). Teachers' cognitive attitude is the belief (either personal or pedagogical) about technology importance (e.g. Using technology in class will make my students learn independently). Teachers' behavioural attitudes is their intention and willingness to use technology in future (e.g., I would rather use technology to illustrate mathematical concepts to students than chalk-board illustrations) (Christensen & Knezek, 2008; Albirini, 2006).

Teachers' dispositions can be enhanced if they work collaboratively in design teams that involve at least two teachers from the same or related disciplines (Koehler, Mishra, & Yahya, 2007). These design teams come together on a regular basis with the common aim to redesign and enact their common curriculum. Empirical studies have demonstrated the significance of this approach in developing teachers' knowledge, beliefs, and attitudes. Kafyulilo, Fisser, and Voogt (2015) adopted teacher design teams as a professional development approach for developing technology knowledge and skills among in-service science teachers. They supported teachers with demonstration lessons, exemplary materials, and expert assistance. The pre-and post-test measurements indicated significant improvement in teachers' technology integration knowledge and skills. In a similar study, Agyei and Voogt (2012) found that four pre-service teachers developed their TPCK better when they worked in groups of two to design spreadsheet-based lessons and subsequently taught those lessons to their peers.

From the foregoing discussion, it appears that teachers develop their technology integration knowledge (TPCK) and beliefs and attitudes through a professional development model involving design and enactment of lessons by design teams, provision of expert's

support, use of specific mathematics software applications, and use of exemplary materials. Hence, this professional development model was adapted in this research study to achieve two purposes: to gain an understanding of teachers' initial disposition of technology integration; second, to examine their views about the professional development model where they worked in groups to develop and teach technology-based mathematics lessons. GeoGebra was proposed for use in this professional development model because it is open software and is readily available and user friendly. It also has the potential to enhance teachers' knowledge and skills of using technologies to mediate constructivist teaching approach where students are engaged in higher-order mathematical thinking (Prodromou, Lavicza, & Koren, 2015).

Method

To achieve the two purposes stated above, the following research questions were addressed:

- What are teachers' dispositions (knowledge, beliefs, and attitudes) related to technology integration in mathematics?
- What are the views of the teachers about the professional development model where they are working in groups to develop and teach technology-based mathematics lessons?

This research study employed case study design rooted in an interpretive approach where eleven (two females and nine males) teachers in the mathematics department of a Senior High School in Ghana voluntarily participated in a one-year professional development model involving the use of technology in mathematics at their own school. The age of the participant teachers ranged from 25 to 45, and their teaching experience ranged from 1 to 20 years. The participant teachers' knowledge, beliefs, and attitudes about the use of technology in mathematics were explored through semi-structured interviews, self-report questionnaires, lesson plans developed by the teachers, and lesson observation. A questionnaire and a semi-structured interview approach involving informal conversation and interview guide were used to collect preliminary data on teachers' demographic information and their experiences of using technology to teach mathematics. The teachers were interviewed individually. Each interview lasted not more than 45 minutes and it was audio recorded. The initial understanding of teachers' technology disposition helped the first author adapt a professional development model to their needs.

A workshop was designed where teachers were introduced to the concepts of the TPCK framework and learning technology by design. This provided the teachers with the theoretical grounding of including technologies in their pedagogical decisions as well as appreciating the constraints and affordances of using technologies in their working environment. The teachers were introduced to geometric, constructing, and algebraic tools in GeoGebra. This provided them with the specific technological knowledge they required to integrate technologies into their classroom practices. The teachers were supported with exemplary materials and expert's assistance. The exemplary materials were in the form of GeoGebra tutorials (which provided the teachers the introductory knowledge and skills of using the geometric, algebraic, and constructing tools in GeoGebra), exemplary lesson plans (which served as basis for pedagogical reflection and replication), and the GeoGebra online community (which served as a platform to borrow existing mathematics lesson for classroom use). The first author provided expert support to the teachers. He developed four GeoGebra-based lessons, two of which he taught as a demonstration. The teachers worked in design teams of two to three members to develop GeoGebra-based mathematics lessons. They first

taught those lessons to their peers and subsequently expected to teach those lessons to students in their actual classrooms. The teachers were engaged in discussion after each teaching rehearsal to critically reflect on the best knowledge, classroom practices, and teaching experiences (Koellner, Jacobs, & Borko, 2011).

A TPCK self-report questionnaire was adapted from Schmidt, Baran, Thompson, Mishra, Koehler, and Shin (2009), which is reported as having a Cronbach alpha reliability coefficient ≥ 0.8 across the various TPCK constructs. The items on teachers' affective attitudes, cognitive attitudes, and behavioural attitudes were adapted from Albirini (2006), which is reported as having a Cronbach alpha reliability coefficient of 0.9. Teachers' perceived benefit of learning technology by design team approach was determined through self-report questionnaire adapted from Koehler and Mishra (2005). All the items on the questionnaires consisted 5-point Likert scale (strongly disagree = 1, to strongly agree = 5), where 1 is interpreted as the lowest possible score which represents a very strong negative response, and 5 is the highest possible score which represents a very strong positive response about technology integration in mathematics.

Mean scores and standard deviations were employed to determine the level of teachers' technology integration knowledge, beliefs and attitudes related to technology usage in mathematics teaching. The audio-recorded interviews were analysed to identify themes and patterns in relation to the theoretical background, purpose, and the research question of the study. The interview data was triangulated with the questionnaire and the observation data. In cases where teachers' views were quoted in this report, their names were altered.

Results

Teachers' Initial Beliefs, Attitudes, and Knowledge Related to Technology Integration

The results in Table 1 indicate that at the outset of the professional development model, the participant teachers had relatively high notions of the effectiveness of technology in mathematics education. All teachers indicated that technology could enable new instructional approaches where students can concretise mathematical ideas through dynamic pictures and videos. Joseph highlighted, "using technology makes teaching simple as students can visualise the mathematics concepts and make out their own meaning".

Table 1
Teachers' Perceived Beliefs and Attitudes Related to Technology Integration

Perceptions	Example	Mean (SD)
Personal belief	Using technology in class can raise student performance.	4.40 (0.39)
Pedagogical belief	Using technology in class will make my students learn independently.	3.81 (0.49)
Affective attitude	Technology makes me feel comfortable.	4.15 (0.40)
Cognitive attitude	I think I need technology in my classroom every day.	3.95 (0.39)
Behavioural attitude	I would rather do things by technology than with my hand.	4.73 (0.24)

With regards to teachers' knowledge of technology integration (TPCK), the results in Table 2 indicate that majority of the participant teachers have a relative limited knowledge of technology integration. For instance, more than half of the teachers were less aware about the existence of teaching and learning mathematics software such Derive, Geometer's Sketchpad, Inspiration, Macromedia Authorware, Green Globe, GeoGebra and Graphmatica. At most they used the software such as Microsoft Word for typing end of term examination questions and SPSS, Matlab and Microsoft Excel for recording students' continuous assessment. For instance, Setho narrated "I know Excel and SPSS. I also know GeoGebra which have not use it before. I use SPSS and Matlab for my assessment and evaluation and not purposely for classroom instructions". This is an indication that after a decade of introducing technology as a medium of instruction in the national mathematics curriculum, teachers are yet to use it meaningfully to orchestrate mathematics lessons.

Table 2
Teachers' Perceived Technology Integration Knowledge in Mathematics Teaching

TPCK construct	Example	Mean(SD)
Technology knowledge (TK)	I know how to solve my own technical problems.	2.74 (0.53)
Technology content knowledge (TCK)	I can find and evaluate the resources that I need for my mathematics learning.	2.91 (91)
Technology pedagogy knowledge (TPK)	I can choose technologies that enhance the teaching approaches for a lesson	3.14 (0.80)
Technology pedagogy content knowledge (TPCK)	I can teach lessons that appropriately combine mathematics concept, technologies and teaching approaches.	3.06 (0.65)

Teachers' Views about Professional Development Model

The responses from the teachers indicated that the design team approach, expert support demonstration lessons, and exemplary materials adapted in this professional development model had a great impact in developing their technology integration knowledge, beliefs, and attitudes. The results in Table 3 indicate that teachers held strong view that working in design teams is time efficient (as they share roles among themselves), it helps them collaborate with others (they had ownership of the GeoGebra-based lessons they designed), and it provided them opportunity to learn (unique talents are tapped from each individual).

Table 3
Teachers' Perceived Benefits of Learning Technology by Design Team Approach

Components of design team	Mean(SD)
Appreciation about the collaboration	4.23 (.24)
Learning opportunities	4.61 (.38)
Development of technology integration knowledge	4.42 (.48)
Opportunity for collaboration	4.32 (.52)
Time efficiency	4.23 (.39)

Expert support was crucial during the lesson preparation stage by the teachers. A point in case is the lesson on trigonometric ratios designed by team three. The group initially struggled with how to insert a textbox that will animate and at the same time show the values of the trigonometric ratios as any of the vertices of the right-angled triangle is dragged. This challenge was resolved upon expert support and the teachers were able to develop and use the diagram in the GeoGebra window to gradually unfold the content of trigonometric ratios to students.

The demonstration lessons and teaching rehearsals afforded the teachers opportunity to put their perceived technological knowledge, pedagogical beliefs, and attitudes into practice. For example, Philo was able to use GeoGebra to scaffold student geometric learning and thinking. Using GeoGebra, Philo was able to build on student's previous ideas on the area of a circle to establish that the volume of a cylinder is $\pi r^2 h$ and that of the cone is a third of $\pi r^2 h$ (see Figure 1). Setho appropriated the GeoGebra tools: point, line segment, polygon, angle measure, slider, textbox, export graphics, and formatting to create a simulated object that helped students conceptualise the orientation of an object when it is rotated either clockwise or anticlockwise about the origin (see Figure 1). The teachers indicated using GeoGebra changes their instructional approach to a more student-centred approach where students co-operatively construct mathematics concepts.

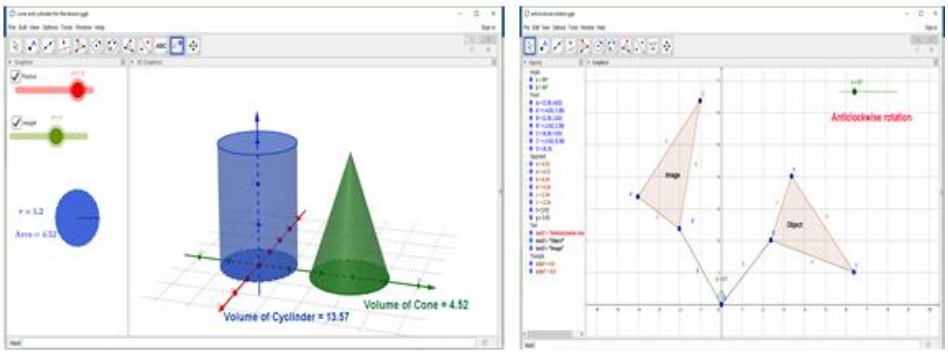

Figure 1. GeoGebra learning environment conducted by Philo and Seth.

It was observed that providing teachers the opportunity to work in the GeoGebra environment, influenced their confidence and willingness to use technology in the classroom. The teachers were observed to be eager learning how to integrate technology in mathematics. All the 11 in-service teachers, having experienced GeoGebra support lessons, admitted that teaching in such a technological environment was interesting and enhanced students' attention and participation. Thus, perceived pedagogical importance of a particular technology motivates teachers to apply it in their classroom decisions.

> …we were motivated to design a lesson on circle theorem because that topic challenges students in examination. We saw GeoGebra as a powerful software which we can use to help students to understand the properties of circle theorem. (Patrick)

With regards to the exemplary materials, the teachers in the study acknowledged that the exemplary materials (GeoGebra tutorial, exemplary lesson plan, and GeoGebra online community) provided them with practical insights for preparing and enacting technology-based lessons. For example, the teachers indicated that the GeoGebra tutorial was helpful because it provided them the basis to learn the GeoGebra tools (geometric, constructing, and algebraic tools) which they applied in the lessons they developed. The exemplary lesson

plans stimulated teachers' reflection and further broadened their understanding of integrating technology into their classroom practices. The GeoGebra online community provided opportunity for teachers to adopt existing mathematics lessons for their classroom instructions. The teachers pointed out that to be proficient in the use of GeoGebra requires commitment and continuous practice.

Discussion and Conclusion

The purpose of this research study was twofold: first, it was to gain insights into teachers' initial dispositions related to the use of technology in mathematics teaching; second, it was to examine teachers' views about the professional development model where they worked in groups to develop and teach technology-based mathematics lessons.

With regards to teachers' disposition related to technology use, the results indicate that teachers have positive beliefs about the importance of technology in the classroom. The teachers have high intentions and willingness to employ technology into their pedagogy because they believe it can make students learn mathematics concepts meaningfully and independently. The teachers acknowledged the pedagogical importance of technology, yet admitted they had not been able to use it in the way they hoped because they had limited knowledge and skills to apply the technology in their classrooms. These findings support the results of earlier studies reporting on Ghanaian mathematics teachers' technology integration knowledge (Agyei & Voogt, 2012). Agyei and Voogt reported that Ghanaian mathematics teachers do not have the needed knowledge and skills to apply technology in mathematics. The results of our study elaborate the existing literature that teachers' competence of blending their technology knowledge with their existing pedagogical and content knowledge is critical in successful implementation of technologies in the classroom (Mishra & Koehler, 2006; Webb & Cox, 2004). Though teachers' values and beliefs about the importance of technology is necessary, their technology competence in terms of knowledge and skills seems to play a key role in technology integration.

With regards to the second purpose of this research study, the results indicate that teachers are very responsive to a technology integration programme that involves pedagogical activities. The teachers acknowledged that coming together as a team to design technology-based mathematics lessons had a remarkable influence on their TPCK development. The design teams are an essential approach for not only stimulating and supporting teachers to learn but it makes them active consumers of technological tools, initiates them to be designers of technology resources, and make them claim the local ownership of technology resources (Agyei & Voogt, 2012). It offered them the opportunity to share classroom experiences. Demonstration lessons adopted in the professional development model provided teachers the basis for replication and stimulated them to learn from their own practice. It was noted that the initial challenges teachers experienced when designing technology-based lesson were resolved upon expert support. This is an indication that the "provision of scaffold tasks to teachers and the opportunity to collaborate with experts and peers enhances teachers' learning" (Kafyulilo, Fisser, & Voogt, 2015, p.677).

It was noted in this study the introduction of GeoGebra offered the teachers the opportunity to put their technology integration knowledge as well as their personal and pedagogical beliefs, confidence, and willingness to use technology into practice. The teachers in the study reported that using technology changed their instructional approach to a more student-centred approach. Teachers were able to align their instructional objectives and activities for the learners to negotiate and construct their own mathematics concepts. It was observed that teachers used multiple situations created in GeoGebra to assist students'

conceptions of properties of circle theorem, solid geometry, and clockwise and anticlockwise rotation of an object about the origin. Also, it was observed that GeoGebra has the potential of accommodating individual teachers' busy schedules by adopting existing instructional lessons from the GeoGebra online community. The exemplary GeoGebra-based lessons developed and implemented by the first author initiated teachers' reflection and further broadened their understanding of subject matter for integrating technology into their classroom practices. In spite of the benefits of using GeoGebra, it requires commitment and continuous practice to be proficient in its use.

In summary, the results from this study provide strong evidence that teachers' may have high values and beliefs about technology importance, but they require technology knowledge and skills for effective implementation. The professional development model where teachers work in design teams to develop GeoGebra-based mathematics lesson with support from expert, lesson demonstrations, and exemplary materials offers an appropriate approach for teachers to be engaged in technology integration.

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Identifying Practices that Promote Engagement with Mathematics Among Students from Disadvantaged Backgrounds

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Students' post-school options are limited if they have not completed mathematics subjects involving the study of calculus. This paper employs a sociocultural approach to identify institutional practices that might promote sustained engagement with these subjects among students from disadvantaged backgrounds. Drawing on interviews with mathematics teachers and the school guidance counsellor, six themes emerged as potential institutional practices contributing to increased enrolments: curriculum organisation across year levels, staffing of mathematics classes, culture of the Mathematics Department, STEM program, and provision of appropriate tasks and resources.

Students whose post-school aspirations involve university qualifications in areas such as engineering, health sciences, economics or agricultural sciences need to complete mathematics subjects involving the study of calculus in the final two years of schooling. These subjects, often described as Intermediate and Advanced Mathematics subjects, are therefore crucial for opening up individual life chances. Students from disadvantaged backgrounds are over-represented among those who do not meet national and international benchmarks in mathematics (e.g., Thomson, De Bortoli, & Underwood, 2017). They are also under-represented in mathematics related degree programs at university (Ainley, Kos, & Nicholas, 2008). These findings suggest that students from disadvantaged backgrounds are less likely to choose to study Intermediate and Advanced Mathematics subjects than their peers from more privileged backgrounds. Identifying ways to promote students' aspirations and engagement in learning mathematics may be one way of broadening post-school options for students from disadvantaged backgrounds.

In Australia, there is a national trend of declining participation in Intermediate and Advanced Mathematics subjects (Barrington & Evans, 2016). Despite this trend, some schools located in low socioeconomic areas have achieved increased enrolments in these subjects. This paper reports on some preliminary findings from a study that is investigating practices that may have contributed to improved engagement and enrolment in Mathematics B (the Intermediate Mathematics subject offered in Queensland) by students from disadvantaged backgrounds. The aim of the paper is to identify practices that appear to be effective in promoting sustained interest and engagement in mathematics involving the study of calculus, via a case study of one school located in a low socioeconomic area.

Background and Context

Science, Technology, Engineering, and Mathematics (STEM) disciplines have a pivotal role in making Australia globally competitive (Office of the Chief Scientist, 2014). To undertake tertiary studies in these disciplines, students need to successfully complete

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mathematics subjects involving the study of calculus in their final two years of schooling (Intermediate or Advanced Mathematics subjects). The importance of studying these subjects extends beyond the STEM disciplines because they provide the necessary mathematics foundation for an extremely broad range of professions, and therefore open up life chances. From 2006 to 2015, the percentage of Year 12 students who enrolled in Advanced Mathematics and Intermediate Mathematics decreased from 10.6% to 9.6% and from 21.8% to 19.2%, respectively (Barrington & Evans, 2016). This downward enrolment trend is more concerning when enrolments in these subjects in 1995 were 14.1% and 27.2%, respectively (Barrington, 2011).

The issue of declining enrolments in Advanced and Intermediate Mathematics subjects is exacerbated among students from disadvantaged backgrounds. Students from linguistically, culturally, and socioeconomically disadvantaged backgrounds are disproportionately represented among those who fail to meet the benchmarks in the National Assessment Plan – Literacy and Numeracy (NAPLAN) and Program for International Student Assessment (PISA) numeracy tests. In PISA 2015, for example, only 37% of students in the lowest socioeconomic quartile met the national benchmark (as agreed in the *Measurement Framework for Schooling in Australia*) compared to 76% of students in the highest socioeconomic quartile (Thomson et al., 2017). One of the implications of these findings is that few students from disadvantaged backgrounds are likely to undertake higher levels of mathematics in their final two years of schooling, thereby limiting their career choices. This paper addresses following research question: What institutional practices might contribute to promoting sustained interest and engagement in Mathematics B among students from disadvantaged backgrounds?

Theoretical Framework

Rogoff's (1995) three planes of analysis were adopted to construct a person-in-context framework for analysing students' subject choice decisions within overlapping personal, social, and institutional levels.

At the personal level of analysis, *engagement and motivation* constitute one set of affective factors influencing students' aspirations towards mathematics (Watt & Goos, 2017). These factors are underpinned by psychological constructs such as beliefs, identity, self-image, anxiety, emotions, and attitudes (Lomas, Grootenboer, & Attard, 2012). A second set of affective factors derives from *students' perceptions of their ability and achievement*, as these perceptions are related closely to identity and self-image (Sheldrake Mujtaba, & Reiss, 2014).

At the social level of analysis, *teacher beliefs and practices* in mathematics classrooms are strongly related to student engagement (Attard, 2014), confidence and belonging (Darragh, 2013). In several studies, teachers were regarded as the main source of encouragement for students to continue with secondary mathematics in post-compulsory settings (e.g., Noyes, 2012).

At the institutional level of analysis, the influence of *educational structures and processes* on student academic aspirations has been the subject of international debate (e.g., Reiss et al., 2011). Major barriers identified include shortage of qualified teachers (Hobbs, 2013), and the political nature of public discussion around contested curricular aims (Noyes & Adkins, 2016).

The *institutional plane* is foregrounded in this paper in an attempt to identify educational structures and processes that may contribute to increasing participation of students in this subject. Although the focus is on the institutional plane, it is acknowledged that interactions

a student has with teachers, fellow students, guidance counsellors, and parents (*social plane*) along with development of affective traits and identity (*personal plane*) contributes to their motivation to study Mathematics B.

Research Design and Methods

The study was conducted in 2017 and employed purposeful sampling to identify schools located in low socioeconomic areas in Queensland that have recorded a sustained increase in enrolment in Mathematics B over the period from 2012 to 2016. A sustained increase in enrolment in Mathematics B was considered to have occurred if the ratio of enrolment in 2016 to that in 2012 was greater than one and the total school population was relatively stable. A further constraint was that the subject enrolment for both years was greater than ten. Data for this paper are drawn from one of the three participating schools to illustrate the approach taken in the study and identify effective practices in this school.

Marigold State High School (pseudonym) was in a low socioeconomic area on the outskirts of large metropolitan city with 82% of the student population in 2017 from backgrounds in the bottom two quartiles for socio-educational advantage (https://myschool.edu.au/). Local issues identified by the guidance counsellor that might impact on student engagement with schooling included high unemployment, mental health issues, drug and alcohol dependency, and intellectual disability among parents. Student performance on numeracy in the NAPLAN for Year 9 in 2017 was lower than, but not significantly different from, the Australian average. The student population was approximately 1,500, of whom 10% were from Indigenous backgrounds and 11% were from a language background other than English.

Participants were five mathematics teachers (including the Mathematics Head of Department), the school guidance counsellor, 13 students across Years 10 and 11 nominated by their teacher because they are planning to study Mathematics B (Year 10 students) or had already commenced studying this subject (Years 11 students). The mathematics teachers and guidance counsellor participated in individual semi-structured interviews that lasted between 16 and 32 minutes. Interviewees were asked about their backgrounds, information about mathematics in the school (e.g., how classes are arranged, advice given to students, and strategies they think have led to an increase in enrolments in Mathematics B). Students participated in semi-structured group interviews with fellow students from the same class that lasted between six and 17 minutes. Questions focussed on students' past experiences of mathematics and factors that had influenced them to study Mathematics B. Interviews were audio recorded and transcribed. Additional data included lesson observations of Year 10 mathematics and Year 11 Mathematics B classes taught by the interviewed teachers and school documents relating to subject selection. As the *institutional plane* is foregrounded in this paper, data collected from the mathematics teachers and guidance counsellor are drawn on to identify institutional practices that might contribute to the observed increased enrolment in Year 12 Mathematics B from 33 students in 2012 to 42 in 2016.

An inductive content analysis was employed to identify educational structures and processes within the institutional plane that may contribute to increased participation in Mathematics B. This analysis was initially conducted for the interview responses from the Mathematics Head of Department (HoD), then applied to responses from the other teachers and guidance counsellor. Categories were refined or expanded to accommodate any different responses.

Institutional Practices Contributing to Increased Enrolment in Mathematics B

Six themes emerged as institutional practices that might contribute to increased enrolments in Mathematics B at Marigold State High School: *curriculum organisation across year levels*, *staffing of mathematics classes*, *culture of the Mathematics Department*, *STEM program*, and *provision of appropriate tasks and resources*. These factors (summarised in Table 1) are discussed in turn to build a rich picture of the approach taken at the institutional level to increase students' motivation to study Mathematics B.

Table 1
Summary of Institutional Practices Contributing to Increased Enrolment in Mathematics B

Institutional practice	Description of practice
Curriculum organisation across Year levels	A clear pathway from Year 7 to Year 10 to prepare for Mathematics B in Year 11
Staffing of mathematics classes	Strategic choice of teachers for mathematics classes, and mentoring for new teachers
Culture of the Mathematics Department	A collaborative environment that extends to the classroom
STEM program	A well-structured STEM program to encourage participation in senior science and mathematics subjects
Provision of appropriate tasks and resources	Inclusion of problem solving tasks in each lesson and use of resources to promote engagement and consolidation.
	Access to a wide range of resources provided by the Mathematics Department

Curriculum Organisation Across Year Levels

Students with a potential interest in studying Mathematics B are provided with a clear pathway from the time they enter the school in Year 7 (since 2015), although there is also flexibility. A single Year 7 Extension class is based on student performance in Year 5 across literacy and numeracy in NAPLAN: "predominantly upper two bands in reading ... about 50% of them are upper two bands in numeracy" (Mathematics HoD). There are two similar extension classes in Year 8 where the "upper two bands" students are together for English, Mathematics, Science, Social Science and Physical Education, and two mathematics extension classes in Year 9 and Year 10. The Year 10 Extension Mathematics classes follow the Year 10A Australian Curriculum: Mathematics (ACARA, n.d.). The Mathematics HoD uses a range of information to select students for extension classes, and described how she selected students for the Year 9 Mathematics Extension class:

> I use as many pieces of information as I can. Parents have the opportunity to request it. Teachers' recommendations. But also, on NAPLAN data and I also run an end of semester, Year 8 test and an end of year, Year 8 test and I use that data as well as their subject data achievement scores as well. I don't want them streamed within themselves. I want to have students in there who have achieved at upper two bands in year 7 but also students who are just below that. (Mathematics HoD)

As many students as possible are given the opportunity to follow a pathway that prepares them to study Mathematics B. For example, the current Year 10 cohort is smaller than other cohorts because of the introduction of the preparatory year in Queensland in 2007, which effectively increased the school starting age by six months resulting in what is sometimes described as the "half cohort". Despite the small cohort, Marigold State High School has maintained two Year 10 Extension mathematics classes:

> … with the small cohort I've still kept two extension classes, even though our data would suggest I shouldn't, because I want to give as many kids the opportunity to excel at Year 11 as possible. I sort of had one and a half classes worth, so I thought let's put a few more kids in. Give them a different environment. Give them the opportunity to see what is possible. (Mathematics HoD)

The Mathematics HoD also recognises that some students' career aspirations change in Year 10 and allows students to change from the extension class to the core class and vice versa:

> Yes, and even now, in second semester, Year 10, there's movement happening now that SET plans are happening. [Senior Education and Training Plans are prepared by all Year 10 students in Queensland as part of the subject selection process for their final two years of schooling.] I've got some kids - students who want to move from core to extension. (Mathematics HoD)

This flexibility extends to subject choices at the end of Year 10:

> I don't have any of those hard rules that some other schools have about having to get a B in extension or having to get a B in core or anything to get into Maths B. I don't have those rules and I've had several students in the past who've proven it's worth not having those rules. (Mathematics HoD)

Staffing of Mathematics Classes

All the teachers who teach mathematics at Marigold State High School, except one, are qualified mathematics teachers. The Mathematics HoD is strategic about how she allocates teachers to classes. She ensures that those teaching senior classes also take junior classes "because I believe that in the junior school, that's where it starts, the thinking about what they are going to do in Senior" and that new mathematics teachers are mentored:

> I strategically place them [new mathematics teachers] on year levels where there is an experienced person to show them the ropes … Then that person - a new person, you know, it influences all their teaching further down the line.

An additional criterion is employed in allocating teachers to extension classes:

> I try to have my senior teachers also on junior school share the extension teachers around - extension classes around amongst people who I consider to be, I guess, passionate about mathematics. That's it. I think that's a criterion to take an extension class, that you're passionate about mathematics. ... You're on there because you love mathematics. (Mathematics HoD)

Culture of the Mathematics Department

There as a belief amongst those interviewed was that students at Marigold State High School were capable of studying Mathematics B but their choice to study this subject and be successful was often influenced by other factors:

> Oftentimes students don't choose it [Mathematics B] or don't succeed at it, not because they don't have the basis from Year 10. It's because of their home situation or their part-time work situation or the influence of their friends that prevents them from passing Maths B. ... To me it's a lifestyle decision rather than a "I just don't get the work" decision. (Mathematics HoD)

> I think access to technology is a sticking point for us so it's good that we have stuff to lend out that they don't have to be disadvantaged based on socioeconomic status. They like to have technology

> that they can still be using, and we can't make the assumption that they're just going to have it or be able to get it, because money is such an important thing here. (Teacher 2)

The atmosphere in the mathematics staffroom is positive and this is seen as by one teacher as being transferable to the classroom: "I think because it's such a positive environment to be in as a staff member, it's easy to bring that positivity to the classroom and then that rubs off on the kids" (Teacher 2). One possible outcome of the culture of the Mathematics Department extending to the classroom is the building of supportive classroom environments in which students are encouraged and work collaboratively in order to learn from each other:

> Those that are having some success, if they have a bit more and then they realise they're in a class with other people that are having success, that can also help them sort of support each other I think too. I think sometimes the ones that are in the core classes, if they're bright or doing well, don't have that link with the other students that - where they help each other and so on … I always think that you do need to have that sort of culture I suppose, in the classroom, that you're working at that higher level. (Teacher 3)

STEM Program

The HoD Learning and Teaching initiated a STEM program at Marigold State High School several years ago with a view to encouraging students to study science and mathematics in the final two years of schooling. The program includes a STEM day for Year 6 students from feeder schools, a Year 8 STEM day, a Year 9 STEM camp, and a robotics program. This focus on STEM is supported by the selection process for the Year 8 "upper two bands" class.

> Any students that have shown a desire or an interest in STEM subjects in particular. If they're - maybe they might be slightly - not as academic as some of their peers but they've really shown an interest in robotics or something, then we might try to get them into the extension class if we can. (Teacher 3)

One teacher described the importance of the STEM program in the following way:

> We're from a really low socioeconomic area here, and a lot of the kids from our school, parents haven't even finished high school … their parents don't know about the opportunities that are there. If we get them opportunities to go and do all these different things, then they think, "Oh I can do that" - because they're bright kids, they just don't know what they're capable of. You give them those experiences, and they just excel. (Teacher 5)

Marigold State High School has experienced increased enrolments in science subjects over the last few years, and currently has two chemistry classes and a physics class in both Year 11 and Year 12. At the time the study was conducted (August), there were 20 students in the Year 11 Physics class. The increase in enrolments in Mathematics B was attributed by one teacher to the increased interest in science: "They're choosing [Mathematics B] because 'I want to do Physics in 11 and 12 and go to uni and follow in the sciences, and to do that I'll need that level of maths to support me'" (Teacher 3).

Provision of Appropriate Tasks and Resources

The approach to teaching mathematics at Marigold State High School is strongly led by the Mathematics HoD, and includes some problem solving in every lesson:

> We've really stuck with having - trying to have warm ups that involve some reasoning and problem solving in them each lesson. … If we're doing a past NAPLAN question, for example, students try to do it themselves and work with a partner and compare answers and see how they've worked it out. Those types of discussions amongst students are happening in most classes, whether they're core or extension. (Mathematics HoD)

The use of "hands on" activities, games and other resources such as Maths 300 were encouraged to both consolidate and extend learning. One of the teachers described how activities were embedded in some of the Year 7 units:

> Like the algebra unit in Grade 7 has all these board games added in, like do this lesson and play this board game, so you just grab it from the storeroom and go. The algebra lessons are really fun because there's all these board games and activities that you can be doing that make it more interesting for the kids. (Teacher 2)

Teachers at the school designed these units *for* the students at the school: "A lot of the stuff is already there for us and we've designed it ourselves so it's suitable for our context" (Teacher 2). This teacher attributed the increase in enrolments in Mathematics B in part to the activities employed in the junior extension classes:

> It tends to be I think because the teachers are enjoying teaching the maths in an interesting way, the kids are enjoying the maths and seeing that they can do well in it, so Maths B seems like a natural choice. (Teacher 2)

Marigold State High School appears to be reasonably well equipped with certain types of resources for teaching mathematics, such as those mentioned earlier. Access to appropriate technology is important for students studying Mathematics B and Mathematics C (Advanced Mathematics). Graphics calculators are widely used in these subjects but many students at the school are unable to afford a scientific calculator or basic resources, let alone a graphics calculator:

> I have 10 scientific calculators all numbered so I don't lose any I have a big box – I learnt that from [Teacher 1] – full of pens and pencils. With some of our kids it's a big deal that they've made it to school, I'm not going to yell at them about not having a pen or a calculator. I'd rather just have the stuff I can lend them so that they can access the curriculum, have the learning. (Teacher 2).

The Mathematics HoD has tried to ensure that access to technology does not prevent students from choosing to study Mathematics B:

> The P & C bought a couple of class sets [of graphics calculators] and the rest I've paid through years of just getting class sets. They're in boxes in the library and the teachers borrow out. That's for Year 10s. The 11 Maths Bs - everyone Maths Bs and Cs has one borrowed through the textbook hire scheme in Year 11 and Year 12 Maths B. I don't want a student think they can't do Maths B because a parent can't afford a $200 calculator. (Mathematics HoD)

Concluding Remarks

There is widespread debate on what are the most significant factors influencing students' academic aspirations and decisions regarding subject choices. The value of Rogoff's (1995) person-in-context perspective is that it highlights the important linkage between students' individual cognitions and embedded contextual influences at different levels – personal, social, and institutional. Student's academic aspirations, manifested through subject choices, are seen as being socially constructed through prolonged negotiation involving multiple sources of social/institutional influences and personal cognitions that are interdependent and mutually constitutive.

The case study reported here focused on factors within the institutional plane of analysis and identified a number educational structures and processes that have been highlighted in the literature. For example, the staffing of mathematics classes was organised strategically so that both junior secondary and extension classes would have access to experienced, qualified mathematics teachers. Similarly, a high degree of flexibility in curriculum organisation allowed as many students as possible to follow a pathway into Intermediate and

Advanced Mathematics. The default position in this school seemed to be one of high expectations and a belief that students are capable of excelling in mathematics, if given the opportunity. Clearly, there are elements of Rogoff's (1995) social plane of analysis at work here, seen especially in the positive culture of the Mathematics Department and the problem-solving approach to mathematics pedagogy. This observation highlights the inseparability and interdependence of the three planes of analysis – although only one, at the institutional level, was foregrounded in our case study. Analysis of data from other participating schools will further illuminate the significance of these interlocking factors affecting students' aspirations towards studying mathematics.

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Pre-Service Teachers' Difficulties with Problem Solving

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This paper reports the results of an investigation into the ways pre-service teachers engaged in structured problem solving as part of their first-year mathematics education course. The purpose of this study was to determine the preferred problem solving strategies of pre-service teachers and the types of difficulties they experienced. The written discourse of 179 pre-service teachers indicated difficulties with being able to articulate the strategies they used in their solution processes. The results also showed that pre-service teachers did not readily use models and relied largely on numerical procedures.

The proficiency strands in the Australian Curriculum: Mathematics describe how content is explored and developed to provide a meaningful basis for the development of mathematical concepts. Problem solving is one of the proficiency strands characterised as "…the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively" (Australian Curriculum, Assessment and Reporting Authority [ACARA], n. d.). A central activity of mathematics teaching and learning is to develop the ability to solve a wide variety of problems (Stacey, 2017).

Success in problem solving is influenced by students' beliefs about the nature of mathematics learning (Schommer-Aikins, Duell & Hutter, 2005). Prior experience shapes the amount of time and effort that will be invested in a problem (Schoenfeld, 1985). Other affective factors such as motivation, perceived personal control, perceived usefulness of mathematics (Schoenfeld, 1989) and maths anxiety (Ramirez, Chang, Maloney, Levine & Beilock, 2016), shape how a student engages in problem solving. A lack of exposure often results in many students having difficulties planning and applying procedures when faced with non-routine problems (Mataka, Cobern, Grunert, Mutambuki & Akom, 2014). Students have come to separate the mathematics they know and experience in their classrooms from the discipline of creativity, problem solving, and discovery, about which they seldom experience (Schoenfeld, 1989).

Problem solving can be considered as a set of skills worthy of instruction in its own right. However, developing instructional models for problem solving is a difficult process. Some instructional models reduce the complexity of mathematical tasks and students' opportunities to grapple with content and misrepresent the flexible and non-routine nature of problem solving (Boaler, 2001; Stein, Grover, & Henningsen, 1996). Creating the 'right' instructional context, and providing the appropriate kinds of modeling and guidance, is challenging for teachers (Schoenfeld, 2016). Teachers also often provide a rationale for avoiding problem solving based on arguments that the curriculum requires students to master facts, procedures and algorithms (Wilson, Fernandez & Hadaway, 1993). One consequence of experiencing mathematics in such a way is that students learn that answers and methods to problems will be provided to them and are not expected to figure out the methods for themselves (Schoenfeld, 2016).

Nonetheless, the fundamental goal of mathematics instruction is to develop transferable skills, knowledge, and abilities that students can apply to tasks not explicitly covered in the curriculum (Wilson, Fernandez & Hadaway, 1993). "Effective teaching is dependent on

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 162-169. Auckland: MERGA.

presenting students with engaging tasks for which they make their own decisions on solving strategies, rather than following procedures (Sullivan, 2011, p. 64)". Students who are able to solve problems think critically within instructional models that emphasise thinking processes above mathematical content procedures (Snyder & Snyder, 2008).

> Mathematics instruction should provide students the opportunity to explore a broad range of problems and problem situations, ranging from exercises to open-ended problems and exploratory situations. It should provide students with a broad range of approaches and techniques (ranging from the straightforward application of the appropriate algorithmic methods to the use of approximation methods, various modeling techniques, and the use of heuristic problem solving strategies) for dealing with such problems. (Schoenfeld, 2016, p. 32)

This study seeks to determine the type of strategies pre-service teachers used and difficulties they experienced when solving problems that involve fractions. Studies into pre-service teachers' problem solving ability are scarce, mainly because of the difficulties in accounting for the type of instructional approaches they encountered as students (Mataka, Cobern, Grunert, Mutambuki & Akom, 2014). Developing pre-service teachers' problem solving skills by providing them with the necessary tools that they can later utilise is important because they will be responsible for cultivating these skills in their own students. Teachers need learning opportunities to develop their own content knowledge and skills to solve mathematical problems themselves (Sullivan, 2011).

Problem Solving Heuristics

Solving problems requires a base content knowledge of mathematics and a repertoire of problem solving heuristics. Early researchers identified heuristics as essential methods for guiding the systematic discovery of mathematical proofs (Neth & Gigerenzer, 2015). In problem solving, heuristics can be considered somewhat synonymously with terms such as strategies, approaches, methods and techniques used in the context of doing mathematics. Efforts to teach novices must take into account that problem solving processes and heuristics develop slowly over time (Lester, 1994).

Many formulations of problem solving frameworks depict Polya's four stages of *understanding the problem, devising a plan, carrying out the plan,* and *looking back*. These stages are often seen as a series of linear steps with an emphasis on getting answers rather than teaching students how to think (Wilson, Fernandez & Hadaway, 1993). Polya's stages are actually cyclic in nature involving passing through one stage, going back and checking before proceeding on to a possible solution path (Mataka, Cobern, Grunert, Mutambuki & Akom, 2014). Reconsidering and re-examining solution processes and results is an important step in consolidating knowledge and developing skills to solve problems. Students need to understand that their thinking and the strategies they use in obtaining a solution are just as important as getting the correct answer. According to Lester (1994), teaching students about Polya's framework does little to improve students' abilities to solve problems. What is important is that teachers value problem solving as part of a systematically planned instructional program where students solve many problems and learn to communicate their thinking.

Teaching problem solving involves exposing students to particular strategies. Producing drawings, for example, allows a problem context to be 'seen' and modelled which in turn facilitates problem solutions (Bakar, Way & Bobis, 2016). Providing occasions for mathematical modeling engages students in learning situations that develop a deeper, conceptual knowledge of mathematics (Boaler, 2001).

Structured Problem Solving

Structured problem solving is a powerful way of developing mathematical concepts and skills – a major instructional approach in Japanese mathematics lessons (Takahashi, 2006). Students work on a problem individually before sharing their solutions with others. The teacher leads a whole class discussion to allow students opportunities to share and learn from each other and encouraging them to think about problems, highlighting that there is often more than one solution process (see Figure 1). Students think more deeply about mathematical content when they are exposed to problems they haven't previously been shown how to solve, challenging them to find their own solutions and justify their reasoning (Sullivan, Askew, Cheeseman, Clarke, Mornane, Roche & Walker, 2015).

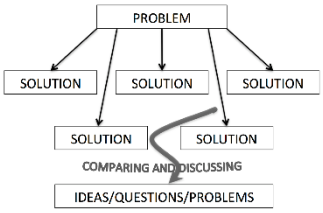


Figure 1. Structured problem solving (Takahashi, 2006, p. 39).

Method

This study involved 179 first-year pre-service teachers across 7 tutorial groups from one university in Melbourne. Problem solving was a common feature of their course content employing a structured problem solving approach (Takahashi, 2006) where problems were attempted on a weekly basis. Their solution processes were discussed in classes as a means of identifying particular problem solving heuristics and emphasising the stages of Polya's framework. In addition, pre-service teachers engaged in a 2-hour tutorial exploring further problems as small group tasks with a specific focus on *The Manchester Warehouse* problem (Booker, Bond, Sparrow & Swan, 2014) where they were guided through a range of problem solving approaches.

> The manchester warehouse was having a sale on beach towels. On Monday, it sold 1/3 of its beach towels, on Tuesday it sold 1/2 of what was left from Monday, and on Wednesday it sold 3/4 of what was left from Tuesday. If 3 beach towels were not sold, how many beach towels did the warehouse have when the sale started?

The Fashion Warehouse problem below is an adaptation of *The Manchester Warehouse* problem developed by one of the researchers and was used to assess pre-service teachers' problem solving ability at the duration of their course.

> Problem solving is a process that involves analysing the problem, exploring means to a solution and trying various solution strategies. Provide two strategies that can be used to solve the following problem.
>
> The fashion warehouse was having a sale on sunglasses. On Tuesday, it sold 1/5 of its sunglasses. On Wednesday, it sold 1/2 of what was left. On Thursday, it sold 3/4 of what was left from Wednesday. If 6 sunglasses were not sold, how many sunglasses did the Fashion warehouse have when the sale started?

Pre-service teachers' written work samples were analysed for the effective use of problem solving methods and the application of numerical approaches to the inherent fractional content in the question. The initial evaluation considered the proportion of pre-

service teachers that could solve the problem using two problem solving strategies. The analysis also examined the effectiveness of their strategies in achieving a correct mathematical solution. The process of mathematizing involves explaining one's actions and choices using a common mathematical discourse (Sfard, 2008). Further analysis therefore examined aspects of pre-service teachers' written discourse for effective specification and use of problem solving heuristics. The data was summarised into tables using percentages and pseudonyms are used throughout the data analysis.

Results

Pre-service teachers used a range of strategies, including a combination of numerical calculations (fractions or percentages), whole number trial and error methods, discrete or region models, tables, and generating and using algebraic equations to solve the problem. Table 1 shows the number of effective strategies employed by pre-service teachers. Two different strategies would have been graded at the highest level for this task.

Table 1
Use of Problem Solving Strategies

Number of effective strategies	% of responses
Two different strategies	25
One strategy	15
Ineffective strategies	45
Unrelated mathematical discourse	12
No attempt	3

This data shows that 40% of pre-service teachers were able to solve the problem using at least one strategy. More than half could not produce a mathematically acceptable solution. Joanne's response is similar to the 12% of pre-service teachers that provided unrelated mathematical discourse indicating little understanding of the problem solving strategies applicable to the problem (see Figure 2).

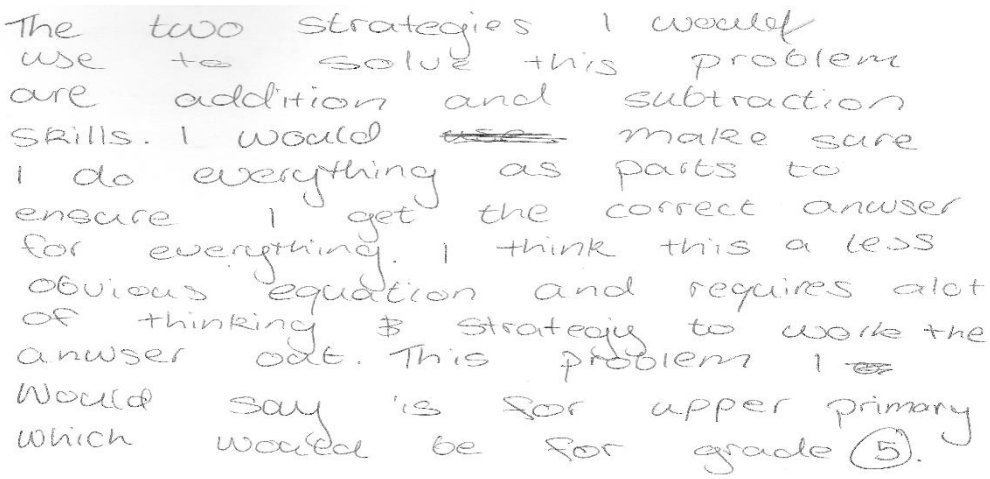

The two strategies I would use to solve this problem are addition and subtraction skills. I would ~~use~~ make sure I do everything as parts to ensure I get the correct anwser for everything. I think this a less obvious equation and requires alot of thinking & strategy to work the anwser out. This problem I ~~w~~ would say is for upper primary which would be for grade 5.

Figure 2. Joanne's discourse about effective problem solving strategies.

A detailed analysis of the problem solving strategies used by pre-service teachers revealed numerical approaches as well as distinct problem solving heuristics. Table 2 shows

the effectiveness of each of their chosen strategies whether pre-service teachers specified them or not.

Table 2
Effectiveness of Problem Solving Strategies used in Written Submissions

Strategies	Correct solution	% of responses minor errors	Incorrect solution
Calculations using fractions	24	9	13
Calculations using percentages	3	1	1
Whole number (trial & error)	12	2	9
Discrete model	7	1	2
Region model	12	3	2
Used tables	2	2	3
Algebraic	3	1	2

Only 24% of pre-service teachers confidently worked with fractions without having to convert them to percentages or use whole number approaches. Paul's submission is indicative of the difficulties most of them had with applying fractions (see Figure 3). As in Paul's case, when novices arrive at a numerical answer they are usually satisfied and rarely see if the answer makes sense (Heller, Keith & Anderson, 1992).

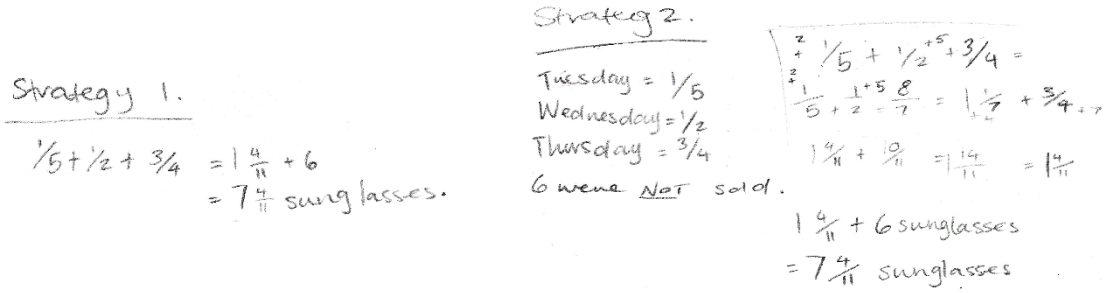


Figure 3. Paul's application of fractions.

When analysing the cohort's ability to name and described their chosen methods only 13% of them were also able to correctly specify the two strategies and effectively use them to solve the problem. Conversely, 27% of them specified at least one strategy but could not effectively use them to solve the problem as shown in Table 3.

Table 3
Specification of Problem Solving Strategies

Number of strategies specified	Number of effective strategies	% of responses
2	2	13
2	1	6
1	1	9
2	0	9
1	0	18

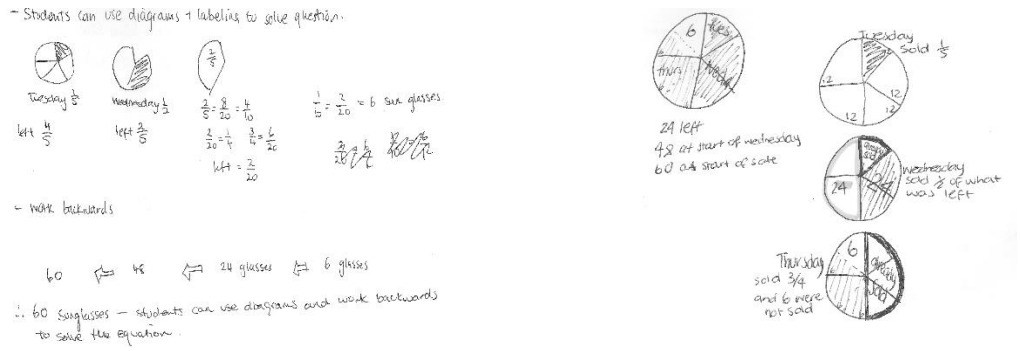
While 45% of pre-service teachers could not specify a strategy (Table 1), 9% of them nevertheless used problem solving approaches to work out a correct answer to the problem. These pre-service teachers may not have recalled the specific terminology to describe their approaches but could demonstrate a solution process.

The problem solving strategies specified by pre-service teachers were grouped into four main categories. Table 4 shows these strategies and the proportion of pre-service teachers applying them effectively in their written mathematical discourse.

Table 4
Written Discourse and Effective Use of Problem Solving Heuristics

Problem solving heuristics	% of effective responses
Working backwards/reverse	21
Using a diagram/model	12
Using an equation/algebraic	1
Process of trial and error/elimination	8

Problem solving strategies are not always distinct and are often combined. For example, the strategy of working backwards can be demonstrated through using numerical approaches, algebraic equations as well as a model. Mary's solution indicates the naming of and combining of strategies to effectively solve the problem (see Figure 4). However, the majority of the cohort did not specify combined strategies even though most of their correct solutions used these combinations. Peter's solution is an example of how several of the cohort effectively used problem solving strategies but were unable to use mathematical language to describe their thinking and solution processes.



Mary's solution Peter's solution

Figure 4. Mary and Peter's specification and use of problem solving strategies.

Sheree may have known how to solve the problem intuitively or by other means but her solution involving the manipulation of fractions indicated a creative use of mathematics but also the need for appropriate instructional intervention (see Figure 5).

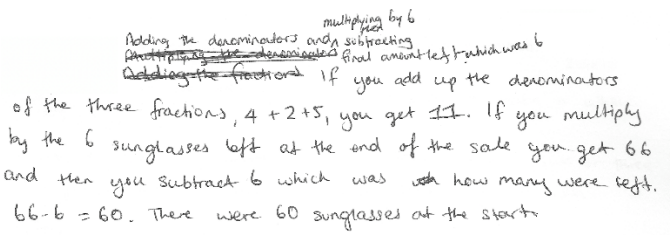


Figure 5. Sheree's manipulation of fractions.

Discussion and Conclusion

Analysis of pre-service teachers' written discourse indicated several gaps in their ability to accurately describe their thinking processes and apply fractional concepts. Being able to communicate one's actions in a common mathematical discourse is akin to thinking mathematically (Sfard, 2008) and is an important part of being a proficient problem-solver (ACARA, n. d.). Despite weekly and intensive exposure to structured problem solving tasks modelled by tutors, 60% of the cohort could not provide a correct solution to the Fashion Warehouse problem. Of the 55% of the cohort who specified the strategies they intended to use, less than half were successful.

A lack of mathematical content knowledge of future primary teachers (Sullivan, 2011), especially with fractions (Chinnappan & Forrester, 2014), present challenges for education programs. Only 19% of the cohort used either discrete or region models successfully, indicating a reliance on procedural methods by the majority. Pre-service teachers tend to have a procedural understanding of fractions and are less likely to develop conceptual knowledge for fraction problems (Tirosh, 2000). Of the 50% who used fractional manipulations to solve the problem, and may have also used models as Mary did (see Figure 4), less than half did so successfully. An additional 12% successfully applied trial and error processes involving whole number manipulations. Polya argued that trial and error is a legitimate, but often undervalued, solution method as mathematics is dependent on guessing, insight, and discovery (Schoenfeld, 2016; Wilson, Fernandez & Hadaway, 1993). However, choosing whole number methods above fractional methods by some pre-service teachers may indicate a lack of confidence when working with fractions.

Several implications can be made to build up prospective teachers' capacity for problem solving and addressing mathematical content knowledge especially with fractional concepts. More attention is needed in addressing pre-service teachers' proficiency in problem solving. Instructional approaches can improve their performance provided they have explicit instruction and practice in implementing problem solving strategies (Heller, Keith & Anderson, 1992).

One limitation of this study is that variations in instructional approaches and time taken by tutors may have affected some pre-service teachers' engagement in problem solving. The influence of the instructor on student performance (Mataka, Cobern, Grunert, Mutambuki & Akom, 2014) and perceived value of problem solving (Schoenfeld, 2016) are major considerations for future instructional models. Further, measuring growth in pre-service teachers' problem solving ability and mathematical knowledge cannot be easily achieved due to the absence of data on these two measures prior to the commencement of their course.

This paper offers a starting point for further theorisation and investigation of teachers' knowledge in problem solving and appropriate instructional models that support their learning. Teaching task-specific heuristics has been shown to effectively enable students to form problem solving plans (Wilson, Fernandez & Hadaway, 1993). Questions about how best to develop problem solving ability in students and how future teachers can be helped to become better problem-solvers and, therefore, better teachers of mathematics, are potential directions for future research.

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Changes in Students' Mathematical Discourse When Describing a Square

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Developing students' geometric reasoning skills is dependent on the quality of task designs and the role of the teacher. The purpose of this study was to apply Sfard's (2008) interpretive framework to analyse changes in students' mathematical discourse. This paper reports on the results of an investigation into the ways one class of Year 7 students communicated their understanding of a square. The results showed that students grappled with the necessary elements involved with describing a square leading to several misconceptions about its key attributes, and raises questions about task designs and the teacher's role in developing geometric reasoning.

The current educational interest in science, technology, engineering, the arts and mathematics [STEAM] in Australia presents opportunities for richer connections of mathematics with other learning areas. Geometry is a significant strand of mathematics as it can be applied across mathematics and to other disciplines. Geometry helps develop students' spatial reasoning skills and abilities to solve real-world problems (Marchis, 2012). Lowrie, Logan and Ramful (2016) found a strong relationship between students' spatial reasoning and mathematics performance, highlighting the importance of promoting spatial reasoning in the Australian Curriculum.

Research indicates that many students have difficulties engaging in tasks that require visual, logical, and deductive thought due to a lack of spatial and geometric reasoning ability (Marchis, 2012; Oberdorf & Taylor-Cox, 1999). Commonly, students experience difficulties recognising geometrical shapes in non-standard orientation and formulating accurate definitions (Marchis, 2012), due to a lack of exposure to geometric vocabulary (Oberdorf & Taylor-Cox, 1999). Equally concerning is that teachers often retain the same misconceptions and misunderstandings of geometric concepts from their own schooling (Cunningham & Roberts, 2010; Fujita & Jones, 2006; Marchis, 2012), unaware of their own students' difficulties (Canturk-Gunhan & Cetingoz, 2013), and making it unlikely that they would provide learning experiences for extending their students' geometric reasoning.

Reasoning with Shapes

Geometry begins with perception and imagery - an ability to visualise with a 'picture in the mind' (Clements, 1982). Visualisation is vital for communicating geometric concepts both verbally and non-verbally at all levels of geometric reasoning (Battista, 2001). Visualisation involves generating a mental image, whether static or dynamic, and understanding that an image depicts visual or spatial information (Presmeg, 2006). Visualisation, therefore, is a complex process involving imagery, with or without a diagram, to organise information into meaningful structures that are important in guiding the analytical development of a solution to geometric problems (Fischbein, 1993).

Geometric reasoning develops from processes of recognising and manipulating mental objects and the relations among those objects (Lowrie, Logan & Ramful, 2016). In geometric reasoning, what is important is to have a sense that because a shape has certain properties, other

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properties must also be true. It is important for students to be able to deduce facts by interpreting the geometric information that they 'see' in their minds (Fujita & Jones, 2006). A specific geometric diagram embodies the attributes of a class, providing students with prototypes. Prototypes in geometry are generalised representations having common visual characteristics and are useful for simple manipulations. However, prototypes are limited references to geometrical concepts having internal constraints of organisation and do not support hierarchical, inclusive definitions (Presmeg, 2006). Students need to be able to explore shapes by 'seeing the parts' – a notion that Owens (2003) referred to as *disembedding*. An image is no longer a 'picture in the mind' but rather images are abstract, malleable, less crisp, and are often segmented into parts.

Diagrams are an essential component of geometric reasoning (Dreyfus, 1991). The effective use of diagrams as a communicative tool for high school students necessitates an understanding of the universal mathematical signifiers used to indicate particular properties on a figure (such as a square in a corner ∟ for a right angle, or the use of arrowheads $\rightarrow\!\!\leftarrow$ for parallel lines). Diagrams are as powerful as definitions (Tall & Vinner, 1981). However, students prefer to rely on visual prototypes rather than verbal definitions when identifying and classifying shapes as they typically remember prior experiences with diagrams presented by their teachers (Cunningham & Roberts, 2010).

Definitions serve the dual role of identifying a category to which a shape belongs, and indicating how it might be distinguished from other objects in that category. *Concept definitions* are word formations used to specify that concept, and a *concept image* is the total cognitive structure that is associated with the concept (Fujita & Jones, 2006), including all the mental pictures and associated properties and processes (Tall & Vinner, 1981). Fischbein (1993) defined the notion of a *figural concept* – a square, for example, is a concept as well as a geometric figure. Many secondary teachers expect a one-way process for concept formation, that is, "…the concept image will be formed by means of the concept definition" (Vinner, 1991, p. 71). Consequently, their students tend to use partitional definitions creating difficulties with logically connecting 'new' information with what they have been previously taught.

Sfard's Interpretive Framework for Mathematical Discourse

The discourse used in the classroom has a significant influence on what and how students learn mathematics (Ferreira & Presmeg, 2004). Analysis of student discourse is an important aspect in understanding students' interpretations of tasks, as well as their ability to communicate geometric concepts (Berenger, Barkatsis, Seah, 2017). According to Sfard (2008), mathematical discourse is exhibited by four inter-related components. These are:

Keywords – Shapes are described and defined in distinctly mathematical ways. How a shape is seen and interpreted by a student is revealed by their use of *keywords*.

Visual mediators –As part of the communication process that helps define shapes and their properties, visual objects that are operated on are known as *visual mediators*.

Narratives – A sequence of expressions or statements used to frame descriptions of objects, either spoken or written, are known as *narratives*. Narratives are subject to rejection or acceptance as deductive accounts of an endorsed consensus.

Routines – Specific repetitive patterns characteristic of creating and substantiating narratives about shapes form *routines* of mathematical discourse.

Mathematics discourse is made distinct by the tools of *keywords* and *visual mediators* giving rise to *narratives* and possible *routines* one applies to shared practices of reasoning, arguing, and symbolising while communicating particular mathematical ideas (Cobb, Stephan,

McClain & Gravemeijer, 2010). Convincing others through a common discourse is a necessary component in the meaning-making process of geometry (Berenger, Barkatsis, Seah, 2017). Students need to be able to connect learned facts to construct logical arguments as endorsed mathematical discourse (Sfard, 2008). Conversely, student misconceptions are revealed by difficulties in formulating mathematically acceptable descriptions or definitions. Their narratives are therefore subject to rejection.

Sinclair and Yurita's (2008) application of Sfard's interpretive framework with secondary teachers working in a dynamic geometric environment [DGE] revealed changes in their use of *visual mediators* and *narratives* to perceive and reason about mathematical objects with their students. Few studies, however, have used this framework to analyse students' reasoning with geometric concepts. One study by Seah, Horne and Berenger (2016) found that middle year students had limited ability to use *keywords* to formulate accurate and complete *narratives* such as definitions. In a related study, Berenger, Barkatsis and Seah (2017) found that Year 8 students experienced difficulties aligning *keywords* and *narratives* to *visual mediators* when describing 2-dimensional shapes.

Method

Students in one Year 7 class in an inner suburban secondary school in Melbourne were given two written tasks. **Task A** asked students *What is a square?* The teacher instructed them to record as much as they knew, to work individually, and did not allow discussion before or during the task. Questions such as *"can we draw a picture?"* were not allowed as a means of ensuring that students did not prompt each other through questioning.

After responses had been collected, the teacher conducted a 30-minute teaching episode to assess current student thinking and reinforce mathematical concepts drawn out by the task. This session allowed students to state known facts about a square as the teacher listed them on the whiteboard. She drew several squares and labeled geometric properties according to student responses. The teaching episode was recorded to assist the analysis of the discursive features of the teacher's communication. To assess students' retention of key ideas explored in the teaching episode, **Task B** was conducted one week later requiring students to draw a square and list its properties.

The purpose of both tasks was to understand how students in Year 7 think and communicate about, what the researcher anticipated as, a familiar geometric shape. The teacher assessed students' use of *keywords* and categorised them according to definitional properties, transformational relationships, formal property-based reasoning, and hierarchical properties in relation to a square. In this study, Sfard's interpretive framework was used to analyse changes in students' mathematical discourse about the square concept. Analysis of students' written discourse considered their use of *keywords, visual mediators* and *narratives* providing the basis for what they knew and communicated about a square, as well as what they learned about a square as a result of the teaching episode.

Results

The results from both tasks are grouped and reported together in terms of *keywords, visual mediators,* and *narratives* used by students. Student misconceptions about squares are also reported. Segments of the teaching episode that occurred between tasks are presented indicating some of the teacher's actions impacting on student learning. It is not possible to report on *routines* requiring well-defined discourse patterns over time.

Keywords

Michelle's response is representative of the way most students listed known facts about a square identifying 4 sides, 4 corners, and other properties, but without reference to right angles or use of any visual mediation (see Figure 1).

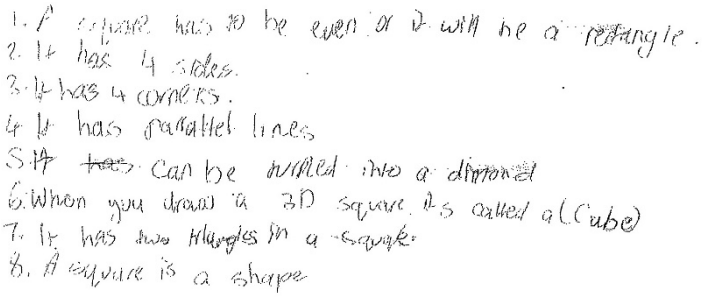


Figure 1. Michelle's response to Task A.

In Task A, initial analysis of keywords showed that 25% of students specified 4 sides of equal length, and 10% specified right angles. Only 5% of students provided both conditions for a square. After the teaching episode, in Task B, 29.4% of students stated the two necessary conditions for a square. There was also an increase in the use of 'new' terms of parallel and symmetry. These results are indicated in Table 1.

Table 1
Keywords Used to Describe a Square

Category of keywords	% of responses before teaching episode (Task A)	% of responses after teaching episode (Task B)
Definitional		
2D	70.0	29.4
4 sides	55.0	41.2
4 lines	20.0	17.6
4 edges	10.0	11.8
4 equal sides	25.0	29.4
even sides	10.0	11.8
4 corners	80.0	100
4 right angles (90º)	10.0	29.4
Transformational		
symmetry	5.0	35.3
Formal property-based reasoning		
parallel	5.0	29.4
Hierarchical		
rectangle	10.0	0
Other		
Personal (eg. dice, grids)	45.0	0
3D reference to cubes	10.0	0

Further examination of the changes in keyword usage indicated a decrease in the proportion of students referring to a square as being 2-dimensional, and a decline in personal references (ie. dice, cubes). These changes may be due to the structure of Task B asking students to draw a square before listing its properties, or as a result of the teaching episode after Task A where some of the critical attributes of a square were highlighted.

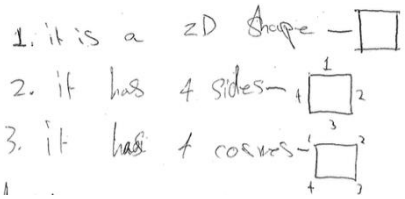
Visual Mediators

Table 2 shows that before the teaching episode, no students indicated the necessary and sufficient properties of a square on a diagram. General shape outlines were produced by 25% of students where they used personal signifiers of arrows and numbering to indicate sides of equal length (see Figure 2). After the teaching episode, only 5.9% of students used correct mathematical signifiers when depicting a square. Diagrams used by students did not always match their accompanying narratives (see Figure 3).

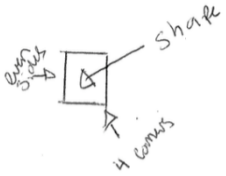
Table 2
Visual Mediators for a Square

Type of visual mediator	% of responses before teaching episode (Task A)	% of responses after teaching episode (Task B)
No diagram	65.0	0
Incorrect diagram	10.0	0
General shape (no signifiers)	25.0	88.2
Right angle signifiers	0	5.9
Equal side signifiers	0	0
Both angle and side signifiers	0	5.9

In the first instance it was not automatic for students to include diagrams nor was it seen as necessary when describing a square. The lack of accurate diagrams of squares with signifiers after the teaching episode indicated an ongoing problem with students' use of visual mediators to indicate key geometric properties other than its general shape.



Fadi: It has right angles in its corners and it has one face



Mary: That a square has 4 even sides. It has 4 corners. It's a shape, can be 3D or 2D

Figure 2. Sample of visual mediators and personal signifiers used to describe a square and accompanying narratives before teaching episode (Task A).

Sharee: It has four corners. All corners are about a 90º angle and a square also has a straight angle. It has 4 sides and has parallel lines. All sides are equal length.

Abdul: 4 corners, 4 sides, 4 90º angles

Figure 3. Sample of visual mediators used to describe a square and accompanying narratives after teaching episode (Task B).

After the teaching episode, almost every student used rulers to produce neat diagrams of squares. However, students were unable to retain information conveyed to them about using diagrams to communicate geometric properties such as equal sides and right angles even if they accurately listed the necessary properties of a square. Instead, students had retained the importance of neatness emphasized during the teaching episode.

Narratives

Analysis of written narratives revealed imprecise thinking about squares. The types of misconceptions recorded as shown in Table 3 indicated a large proportion of students made reference to a 3-dimensional object (cube or box) despite having also referred to it as a quadrilateral or 2-dimensional shape. Other misconceptions relate mainly to orientation.

Table 3
Students' Misconceptions when Describing a Square

Common misconceptions	% of responses
2-dimensional version of a cube, box, 6 faces	45.0
is three-quarters of an A4 page	15.0
made up of 2 triangles	5.0
rotated becomes a diamond	5.0
stretched to become a rectangle	10.0
vertical and horizontal	5.0
Other	15.0

Michelle incorrectly stated that a square "…can be turned into a diamond…a 3D square is called a cube…has two triangles…" Michelle's response indicated multiple misconceptions about a square, and was also detected in the work of three other students.

The Teaching Episode

The teaching episode conducted between Task A and Task B provided an opportunity for the teacher to assess student understanding of geometric concepts, and to emphasise keywords and the significance of diagrams. The teacher supported student responses through her questioning to draw out descriptions from students (see Figure 4).

Student A: [a square] has a right angle
Teacher: how would I show it here? (*invited the student to add to the diagram on the board*) Teacher: what does a right angle mean? (*drawing out further meaning*)

Figure 4. Dialogue from teaching episode for indicating a right angle.

The teacher folded a square piece of paper to help define *diagonal* and *symmetry* concepts. Students were encouraged to use hand gestures to connect ideas of horizontal and vertical symmetry and parallel lines. Gestural forms of communication are relevant to the discourse narrative (Ferreira & Presmeg, 2004; Sfard, 2008) (see Figure 5).

Student B: A square has parallel lines.
Teacher: What does that mean?
Student B: …go in the same direction… (uses his hands to motion movement in one direction)

Figure 5. Dialogue from teaching episode for indicating parallel lines.

The teacher's questions challenged students to provide more detailed responses thus building an exhaustive list of properties of a square. She later drew a rotated square on the board (see Figure 6) and asked the students what it was.

Teacher: So, a rhombus. A diamond. What are you telling me?
Student C: Same thing.

Figure 6. Teacher's rotated square and dialogue.

The teacher wrote *Diamond* above the object as well as *a rhombus*. The teacher's acceptance of 'same thing' indicated her own misconception in relation to orientation and conveyed to her students, that is, a diamond, a rhombus and a rotated square are all the same. Both the teacher and her students believed that the non-critical attributes of an object, such as its orientation, are important in its *concept definition*.

Discussion and Conclusion

Geometric reasoning is characterised by the use of specific keywords and visual mediators giving rise to endorsed narratives. The analysis of students' use of keywords, visual mediators and narratives indicated significant gaps in their ability to describe the necessary properties of a square and to determine what is sufficient. This claim is substantiated by only 29% of students being able to list 4 equal sides and 4 right angles yet none depicted this accurately on their diagrams despite this being modeled by the teacher.

Application of Sfard's interpretive framework revealed that students do not accept visual images as powerful aspects of geometric discourse. The use of visual mediators improved marginally after the teaching episode with most students depicting general shapes without the use of signifiers for equal sides and right angles, due partly to an over-emphasis on neatness rather than the critical attributes of a square. If, as Dreyfus (1991) and Presmeg (2006) suggested, students might generate visual images but have a basic reluctance to use them to communicate geometric concepts, then it could be conjectured that students' prior experiences of reasoning with shapes were restricted to basic recognition and memorisation activities. These activities are often characterised by listing facts without emphasis on the need for mathematically acceptable visual mediators.

This study raised several questions about geometric task designs and the role of the teacher. Students were asked *What is a square?* in Task A. Such open-ended tasks are commonplace and have merit in understanding the extent of student knowledge. However, the focus on listing facts to reason about shapes, emphasised in Task A and the teaching episode, placed weight on written narratives above visual mediation hampering students' ability to discern and articulate the minimal properties needed to describe, and therefore define, a square. Further, students may not have understood the purpose of the tasks nor found them engaging, suggesting a lack of exposure to non-routine geometric tasks. This was indicated by their inability to retain key concepts presented to them, and implying the need for newly learned concepts to be reinforced through further teaching activities.

How teachers question, listen, and respond to their students is crucial in their understanding of mathematics (Ferreira & Presmeg, 2004). This study identified several misconceptions and difficulties with geometric concepts stemming from students' imprecise or personal concept images of squares. The teaching episode indicated concerns about the teacher's content knowledge as it shed light on her own misconceptions with the square concept, and provided

an explanation for how students might develop similar misconceptions in the first instance. This study was limited to a snapshot of teaching and learning in one classroom, and indicates a direction for future research into effective teaching approaches to sustain geometric reasoning.

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What can be learned from teachers assessing mathematical reasoning: A case study

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Recently, mathematical reasoning has grown in prominence in curriculum documents and professional learning programs. However, the assessment of reasoning actions continues to be an elusive task for many teachers. Research has shown that many primary teachers focus only on explaining. This case study examines the salient behaviours of two Year 6 primary teachers employing the *Assessing Mathematical Reasoning Rubric*. Results indicated the teachers gained deeper insights into the diverse nature of reasoning through the employment of rubric. Therefore, it provides teachers with a vehicle for a more nuanced examination of reasoning beyond explaining and is a launching pad for lesson planning.

The emphasis on reasoning in curriculum documents is reflected in the reasoning focus of professional development for pre- and in-service teachers through: demonstration lessons (Herbert, Vale, Bragg, Loong, & Widjaja, 2015); and workshops (Hilton, Hilton, Dole, & Goos, 2016); teachers' use of reasoning language (Clarke, Clarke, & Sullivan, 2012); peer-learning-teams (Herbert & Bragg, 2017); and, mathematics teacher educators modelling of reasoning focused lessons in primary classrooms (Livy & Downton, 2018).

Previously, we noted teachers focused on reasoning as *explaining* (Herbert et al., 2015). This focus on *explaining* disregards the multifaceted nature of reasoning. Therefore, we developed the *Assessing Mathematical Reasoning Rubric* (See Figure 1) to foster a more nuanced examination of children's reasoning. This paper specifically explores the research question, "What can be learned from teachers' employment of the Assessing Mathematical Reasoning Rubric?"

Literature review

Teachers often grapple with the nuances of mathematical content and do not have strategies for helping their students to recognise or utilise it to solve problems (Hilton et al., 2016). Extending this thinking beyond content to proficiencies, if teachers struggle with understanding reasoning, then it may be difficult for them to teach it effectively. This section outlines the background literature which informed this current study, including a discussion of reasoning, and its assessment.

Mathematical Reasoning

In mathematics, reasoning is viewed as "the glue that holds everything together, the lodestar that guides learning" (Kilpatrick, Swafford, & Findell, 2001, p. 129). It "involves making, investigating and evaluating conjectures, and developing mathematical arguments to convince oneself and others that the conjecture is true" (Goos, Vale, & Stillman, 2017, p. 37), thus allowing students to go beyond routine procedures towards an appreciation of the interconnected, logical and meaningful aspects of mathematics (Mata-Pereira & da Ponte, 2017). These views of mathematical reasoning are consistent with the Australian Curriculum: Mathematics (AC:M) which states:

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 178-185. Auckland: MERGA.

> Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising. (ACARA, 2017, Key ideas, para. 5).

Analysing is described in the AC:M to occur when "students explain their thinking, … they adapt the known to the unknown, … transfer learning from one context to another, … and when they compare and contrast related ideas and explain their choices" (ACARA, 2017). Students generate specific cases or examples to satisfy the conditions of a problem drawing on prior knowledge, such as recalled facts, to construct examples or cases (Vale et al. 2017). Examples are compared and contrasted to form conjectures explaining similarities and differences between examples (Jeannotte & Kieran, 2017). Mason (2003) recommended that teachers use prompts "such as 'What is the same and what different about…?'" (p.24) to support learners to connections between cases or examples.

Forming conjectures and generalising are essential components of the teaching and learning of reasoning (Lannin, Ellis, Elliot, & Zbiek, 2011). Furthering the reasoning action of analysing, generalising identifies commonalities across cases, extending beyond the original case (Kaput & Blanton, 1999). Lannin, et al. (2011) merged conjecturing and generalising to proffer four key understandings of generalising: (1) developing statements [forming conjectures], (2) identifying commonality and extending beyond original cases, (3) recognising a domain for which the generalisation holds, and (4) "clarifying the meaning of terms, symbols and representations" (p. 12).

Justifying is more than explaining "what", including "why" (Vale, et. al., 2017) to verify a claim (Sowder & Harel, 1998). A mathematical justification is a logical argument based on accepted procedures, properties, concepts, and mathematical ideas (Mata-Pereira & da Ponte, 2017). As students' complexity of reasoning grows, they are able to offer a mathematically and sound logical argument to support a claim (Jeannotte & Kieran, 2017).

Despite the complexity of reasoning, teachers mainly focus on *explaining*. Clarke, Clarke, and Sullivan (2012) found nearly all 104 teachers surveyed regularly used explaining, with less use of other reasoning words. Therefore, there is a need to extend teachers' awareness of a broader range of reasoning actions. This paper reports on the efficacy of the *Assessing Mathematical Reasoning Rubric* in assisting teachers to gain a deeper view of reasoning, than merely explaining, by interrogating the data collected in two post-lesson discussions where two teachers utilised the rubric to assess the reasoning capacity of their students.

Assessing Reasoning

Assessment in mathematics is the process of examining evidence about student learning to reveal student knowledge and skills (Heritage, Kim, Vendlinski, & Herman, 2009) and to plan for subsequent action with a goal to improve their student' conceptual understanding (Binkley, et al., 2012). In our larger study exploring teachers' knowledge of reasoning and enriching their understanding of reasoning through a professional learning program, it was noted that primary teachers struggle to define, recognise, and implement reasoning (Loong, Vale, Herbert, Bragg, & Widjaja, 2017). Consequently, without an understanding of the complexity of reasoning it is challenging to notice and thereby assess when reasoning takes place.

Figure 1. Assessing Mathematical Reasoning Rubric. Version 1 (Herbert & Bragg, 2017).

	Analysing	Forming Conjectures and Generalising	Justifying and Logical argument
Not evident	• **Does not notice** numerical or spatial structure of examples or cases. • Attends to **non-mathematical** aspects of the examples or cases.	• **Does not communicate** a common property or rule for pattern. • **Non-systematic** recording of cases or pattern. • **Random** facts about cases, relationships or patterns.	• **Does not justify**. • **Appeals** to teacher or others.
Beginning	• **Notices** similarities across examples • **Recalls** random known facts related to the examples. • Recalls and **repeats patterns** displayed visually or through use of materials. • Attempts to **sort** cases based on a common property.	• Uses body language, drawing, counting and oral language to **draw attention** to and **communicate**: o a single common property o repeated components in patterns. • **Adds** to patterns displayed verbally and/or visually using diagrams or through use of materials.	• **Describes** what they did and why it may or may not be correct. • **Recognises** what is correct or incorrect using materials, objects, or words. • Makes **judgements** based on simple criteria such as known facts. • The **argument may not be coherent** or include all steps in the reasoning process.
Developing	• **Notices** a common numerical or spatial property. • **Recalls, repeats and extends** patterns using numerical structure or spatial structure. • **Sorts and classifies** cases according to a common property. • **Orders** cases to show what is the same or stays the same and what is different or changes. • **Describes** the case or pattern by labelling the category or sequence.	• **Communicates** a rule about a: o *property* using words, diagrams or number sentences. o *pattern* using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition. • **Explains** the meaning of the rule using one example.	• **Verifies** truth of statements by using a common property, rule or known facts that confirms each case. May also use materials and informal methods. • **Refutes** a claim by using a counter example. • Starting statements in a **logical argument** are correct and accepted by the classroom. • **Detecting and correcting errors** and inconsistencies using materials, diagrams and informal written methods.
Consolidating	• **Notices more than one** common property by systematically generating further cases and/or listing and considering a range of known facts or properties. • **Repeats and extends patterns** using both the numerical and spatial structure. • Makes a **prediction** about other cases: o with the same property o included in the pattern.	• **Identifies** the boundary or limits for the rule (generalisation) about a common property. • **Explains** the rule for finding one term in the pattern using a number sentence • **Extends** the number of cases or pattern using another example to explain how the rule works.	• Uses a correct **logical argument** that has a complete chain of reasoning to it and uses words such as 'because', 'if...then...', 'therefore', 'and so', 'that leads to' ... • **Extends the generalisation** using logical argument.
Extending	• **Notices and explores** relationships between: o common properties o numerical structures of patterns. • **Generates examples**: o using tools, technology and modelling o to form a conjecture.	• **Communicates** the rule for any case using words or symbols, including algebraic symbols. • **Applies** the rule to find further examples or cases. • **Generalises** properties by forming a statement about the relationship between common properties. • **Compares** different symbolic expressions used to define the same pattern.	• Uses a **watertight logical argument** that is mathematically sound and leaves nothing unexplained. • **Verifies** that the statement is true or the generalisation holds for *all* cases using logical argument.

Although it is known that student outcomes are improved when rubrics are used (Panadero & Jonsson, 2013), little is known about how the use of existing rubrics for assessment may build teachers' knowledge. So, to support teachers with the complex task of assessing reasoning the Assessing Mathematical Reasoning Rubric (hereafter also referred to as the "rubric"), with five levels (Not Evident; Beginning; Developing; Consolidating; and, Extending) for the three reasoning actions: Analysing; Generalising; and, Justifying (see Figure 1), was developed. Dot points in each cell are intended to assist teachers to identify a student's level for each of the reasoning actions.

This rubric was developed by a team of academics at Deakin University as one aspect of the resources created for the reSolve: Mathematics by inquiry (Australian Government Department of Education and Training, 2017). It was trialled and refined by teachers at four Victorian primary schools, using it to assess the reasoning demonstrated by their students during specifically designed reasoning lessons.

Methodology

A case study provides ‘‘an intensive, holistic description and analysis of a single instance, phenomenon or unit’’ (Merriam, 1988, p. 21). It is “an empirical inquiry that investigates a contemporary phenomenon within its real-life context; when the boundaries between phenomenon and context are not clearly evident; and in which multiple sources of evidence are used” (Yin, 1984, p. 23). A case study is used to “explore those situations in which the intervention being evaluated has no clear, single set of outcomes” (Baxter & Jack, 2008, p. 548). In this paper, case study is being employed as it was intended, by utilising this approach a deeper understanding may be gained of specific issues associated with teachers’ awareness of the diversity of actions encompassed in the broader term ‘mathematical reasoning’ through the assessment of students’ work samples in the post-lesson discussion facilitated by the rubric.

Description of the case

The participants in this case study are two Year 6 teachers and their children (approximately 50 children) engaged in the classroom enactment of the painted cube task at a suburban primary school on the outskirts of Melbourne. They were chosen to assist the researchers understand about the challenges in assessing mathematical reasoning because of their involvement in trialling tasks and resources for the reSolve project: Assessing Mathematical Reasoning.

Painted Cube Task

The painted cube task (Driscoll, 1999) (Figure 2) was adapted to provide children with the opportunity to share and debate their algebraic thinking as they searched for patterns and generalisations. The multiple layers of the problem offered incremental developments in the sophistication of the children’s reasoning (Koellner, Pittman, & Frykholm, 2008/2009).

Imagine a cube made up of 27 smaller cubes (3 x 3 x 3). Imagine that you dip the cube in paint. If you now separate it into 27 small cubes, you will notice that some of the small cubes are painted. Which small cubes have been painted on 3 sides, on 2 sides, on 1 side, and not painted at all – and how many are there?
Fill in the grid for a 3 x 3 x 3 cube.
Consider and complete the grid for different size cubes, 2 x 2 x 2, 4 x 4 x 4, 10 x 10 x 10, etc.
Create a rule for predicting the answers for larger cubes without counting all the small cubes n x n x n ?
Describe the patterns that you see. What changes, and what stays the same?

Figure 2. The painted cube task.

Analysis

The post-lesson discussion was audio-recorded. The work samples and their assessment via the rubric were collected. Transcripts were jointly read to establish common coding consistent with the *Assessing Mathematical Reasoning Rubric*. The findings arising from the data analysis are presented in the following section in narrative form.

Results

This section presents the results of the analysis of the transcript of the post-lesson discussion. It is structured according to the reasoning actions: Explaining, Analysing, Generalising, and Justifying. A total of 47 statements were recognised as related to reasoning actions in the transcript of the post-lesson discussion. The statements were further coded into Explaining (7); Analysing (11), Generalising (23) and Justifying (6). The examples below are illustrative of how the teachers utilised the rubric to assess students' work samples, thus demonstrating their attention to a wider range of reasoning actions than explaining.

Explaining

As is typical of what other researchers have noted, the teachers did talk about "explaining" as a reasoning action. For example:

> Lee: He was trying to explain his formula at the end yeah. He wasn't very clear though.
>
> Rosie: So, she was explaining that in here

While teachers did refer to explaining there were also many instances of their focus on other reasoning actions as they attempted to use the rubric to assess their students' work samples.

Analysing

Both teachers noticed the students' analysing and were able to articulate their interpretation of their students' actions. In the quote below, Rosie has noticed the attention paid by students to the pattern related to the number of unpainted cubes. This evidence indicates that she can see the students are analysing this problem, i.e., noticing a common property, describing the pattern and exploring relationships between the examples they are generating.

> Rosie: They started to have a look at the pattern of the cubes not painted at all and looking at the connection between this column and then the total number of small cubes.

The next two quotes demonstrate Lee's iterative contemplation of the levels in the Analysing column of the rubric. In this way, the rubric's wording assisted him to notice analysing in his students' work. Firstly, he reviews the 'Consolidating' description 'Makes a prediction about other cases with the same property'.

> Lee: I'm just looking at "Consolidating" now. I can see for sure that they can predict.

He recognises that this student has met that level's indicator but may also meet the indicators of a higher level. Lee reviews the next level in the Analysing column 'Extending' where he reads 'Numerical structures of patterns'.

> Lee: So, I'm just going to move down to "Extending" and just see if they fit that. "Numerical structures of patterns," yes, I think that's evidence by the actual formulas they've written out.

In the next quote, Rosie is grappling with the idea of what constitutes analysing. Her paraphrasing of the words in the rubric demonstrates she is building her language related to reasoning. This is different from Lee's use of the words in the previous quote where Lee is reading the rubric dot points verbatim, whereas, Rosie is embedding the wording from the dot point in her articulation of analysing.

> Rosie: I've got her as "Consolidating" in Analysing. But I wasn't sure whether to put her in "Developing" or "Consolidating" because I suppose she made a prediction that it would work with

> any other numbers, but she didn't really elaborate on that. She didn't use different examples, she just used what was already here.

Later in the post-lesson discussion we notice Lee beginning to appropriate the words into his understanding and expression of analysing.

> Lee: So, she [student name] has begun to find the pattern but she's doing them one at a time I've noticed instead of going down. So that indicates to me that she's not maybe going any further with the pattern. She's doing one at a time still whereas with [a different student] you can see that he's actually gone [filled the column] all the way done.

The quote indicates Lee's deepening understanding of the nature of analysing as he compares these two students, noticing that generating further examples in a pattern is considered a higher level of Analysing.

Generalising

As with Analysing, teachers used the words embedded in the rubric to assist in assessing their students' level of Generalising. Below Lee and Rosie expressed their evaluation of their students' work in terms of the rubric's language.

> Rosie: He can "explain the meaning of the rule using one example", and he can add to the pattern, and he can "communicate a single property and repeated components".

The teachers demonstrated their growing awareness of the nature of reasoning in their noticing of their students' generalising, for example:

> Lee: So, I think for this one he is actually using algebraic symbols here. That's evidenced by the actual formulas they've written out. He's actually explained the formulas for the first 2 columns.

> Rosie: So, they noticed that you've got 8 cubes with a 2 by 2 by 2 and then in a 4 by 4 by 4. That's how many cubes that aren't painted. So, they started to notice that connection. They've just begun to make that connection and come up with a formula

The rubric assisted Rosie to evaluate the complexity of the students' generalising capabilities through a comparison of two students' work samples.

> Rosie: I've put her [one student] for "Developing" in Generalising because she was talking about the rules and the patterns. … Definitely not in "Extending" because she [another student] didn't talk about other examples.

Justifying

The results revealed that justifying, whilst not the focus of this task, was identifiable in the students' reasoning actions as noticed by these teachers. This is evident in Rosie's articulation below:

> Rosie: Using a "logical argument that has a complete chain of reasoning" and yeah, she used the words, "just, because, if, then, therefore". That's why I would sort of put her in the middle of those two [levels]. Definitely not in "Extending" because she didn't talk about other examples. … When I asked them about it they were trying to work out the rule.

The words Rosie used, are embedded in the Consolidating level for Justifying. These words indicate that she is using the language of the rubric to identify justifying and to categorise this student's reasoning. Of note, Rosie is moving beyond describing this student's reasoning as "explaining".

Discussion and Conclusion

Clarke, et al. (2012) uncovered primary teachers' focus on explaining as the key action of reasoning observed and promoted in their classrooms. With this concern, regarding the limitation in viewing the role of reasoning, in mind, our professional learning goal was to develop and utilise a rubric to shift teachers' understanding, enactment, and assessing of reasoning actions from explaining to encompass Analysing, Generalising, and Justifying. In learning from the teachers' employment of the rubric, we noted that whilst explaining, rightly, was articulated during the teachers' feedback on their students' actions, evidence of teachers' recognising the complex nature of reasoning was apparent. The use of the rubric pushed the teachers beyond explaining and allowed them to notice the students' developing arguments, conjecturing, generalising, and convincing others (Goos, et al., 2017). While we acknowledge that the main focus of the task was to generalise, pleasingly, the teachers were able to capture examples of their students' exhibiting the actions of analysing and justifying, thus, we witnessed in the teachers' appraisal of the students' actions the interconnectedness of reasoning (Mata-Pereira & da Ponte, 2017).

Not surprisingly, we would anticipate that using a rubric specifically designed to examine multiple actions of reasoning would result in the teachers' noticing the selected actions in the rubric. However, explaining is a feature of the rubric, and yet encouragingly was the number of accurate examples of the other reasoning actions within the rubric which the teachers were able to articulate, without returning to the holdall of "explaining". Prior experiences with similar Victorian teachers noticing reasoning had resulted in them describing complex reasoning actions predominantly as explaining (Herbert et al., 2015). Therefore, the rubric offered the teachers a nuanced vocabulary to describe the reasoning actions they were witnessing, thus leading to a deeper understanding of these reasoning actions.

In the course of creating the *Assessing Mathematical Reasoning Rubric*, we were concerned that the heavily detailed rubric would deter teachers from employing the rubric. Thus, we have developed a briefer, less-detailed rubric. However, as a result of our investigation in this study of the usefulness of the detailed rubric, one implication for further research is to reconsider how to balance the effectiveness of the shorter rubric as a tool for quickly assessing students' reasoning versus the detail-rich Assessing Mathematical Reasoning Rubric which offers more breadth in supporting teachers' noticing of the complexity of reasoning.

The Australian Curriculum: Mathematics (ACARA, 2017) and others (Goos, et. al, 2017; Kilpatrick, et al. (2002) in describing reasoning as multifaceted, encourages teachers to facilitate learning with tasks that reflect the complex nature of reasoning. The Assessing Mathematical Reasoning Rubric, in this case study, appears to be successful in providing teachers with a tool to notice and assess the complex nature of reasoning exhibited by their students.

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Conspiracy in senior school mathematics

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Research across five countries has identified inability to pay attention to mathematical detail – the discipline of noticing – is an issue in senior secondary school mathematics teachers. The test and questionnaire completed by an Australian cohort further identifies a reluctance to employ non-routine questions in assessments, with teachers concerned about damaging the trust relationship they enjoy with their students. As teachers fail to demonstrate strong ability in non-routine written test questions themselves, this paper questions whether there exists a 'conspiracy' between teachers and their students to avoid scrutiny of conceptual understanding.

Introduction

In mathematics education, systematic reflective thinking may involve interrogation of practice both inwardly and outwardly (Mason, 2002). Inward reflection may be practical, a consideration of such matters as classroom management and appropriate delivery of the curriculum. This may lead to a metacognitive reflection, as the practitioner searches for the causes of their confidence in the techniques they are applying and a search for their personal assumptions and abilities. At a higher stage again, Mason (2002, p. 17) identifies "social-reflection", a more outward-directed critique of the values which impose upon the teaching situation.

A key concept in reflective thinking is intentional noticing. Mason (2002) relates an anecdote about pianist Artur Rubinstein deliberately choosing to not use a certain finger in a concert, just to be more aware of his playing. In teaching practice, self-noticing can be practiced in respect of gesture, how conversation is initiated or terminated, or of the things the practitioner chooses to note down in writing. Underlying Mason's approach is the idea that, through conscious practise, noticing and ultimately teaching performance can be improved.

This study seeks to apply Mason's (2002) concepts of intentional noticing to assessment of mathematics. Incorrect responses to mathematical questions can result from overlooking aspects of the problem. For example, in calculus a local maximum may be obtained by differentiation when, for the defined domain, the global maximum may be greater and optimal. Overlooking discontinuities in functions is another difficulty in the mathematical performance of school students. In the opaque language of Examiners Reports, a statement such as "Standard questions involving calculus, logarithms and the exponential function were well attempted" (School Curriculum and Standards Authority, 2018, p. 1) can be taken to indicate that the unmentioned non-standard questions were not subject to sufficiently close attention.

This paper recounts research by Klymchuk (2014) on the phenomenon of lack of attention in mathematics students and in their teachers. It describes replication in an Australian setting of Klymchuk's study and presents an account of the self-knowledge of the Australian teachers involved. Klymchuk's conclusion is that "Solving non-routine, non-standard questions would better prepare students for the real world. Enhancing their own and

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their students' discipline of noticing by paying attention to details can also be a useful addition to teachers' professional development." (p. 69). This paper explores whether these sentiments apply in an Australian situation and reports the self-interpretation of the actors involved. It explores whether the intentional noticing of Mason (2002) may be a means of improving the mathematical performance of teachers and students in Australia.

Theoretical Framework and Literature Review

Intentional noticing appears under several names in educational research. Schön (1987) favoured the term "reflection", but partitioned it into such categories as "reflection-in-action". Mason (2002) offers further varieties of reflection, examining intention and motivation. Noticing is treated as a study of student misconceptions in works including Ryan and Williams (2007). More recently, mindfulness in mathematics is a theme of Boaler (2016), developing work by Dweck (2007).

Duit, Treagust and Mansfield (1996) note that from a constructivist perspective there is a symmetrical relationship between teacher and school students, both being partners in a communication and trying to obtain an idea of the understanding of the other. With this in mind, it is important to examine the self-perceptions of the teachers, not just the performance of the students, and also the joint behaviour and understandings of teachers and students.

To move from habitual or mechanical patterns in teaching practice, Mason (2002) recommends teachers undertake a series of exercises to develop increased sensitivity. These include mirroring gestures, listing key words for a lesson and introspection about achievement of desired intentions in interactions with students. This same intentional improvement in performance is possible in mathematics, claim Meyer, Falkner, Sooriamurthi and Michalewicz (2014). Their solution is exposure to carefully-selected mathematical puzzles where close attention is needed.

> Puzzle-based Learning is rapidly becoming a bigger and bigger part of the curriculum as there is no guarantee that a traditional education will provide students with enough practise and experience to develop problem-solving skills. The rapidly changing face of employment and technology means that the problems that we train people to solve today are probably not the problems they will be solving in ten years. When our current education system tends to favour highly focused learning of rigid approaches to predictable problem sets, there is no guarantee that our students will be flexible enough and resilient enough to cope with open-ended problems with no guaranteed solution. (Meyer et al., 2014, p. 4)

Research originally undertaken by Klymchuk (2014) in New Zealand was then was replicated by his associates in three other countries: Hong Kong, Germany and Ukraine. Klymchuk describes the New Zealand group as "experienced upper secondary school mathematics teachers"; the Hong Kong group as "secondary school mathematics teachers"; the German group as "experienced school mathematics teachers"; and the Ukraine group as "[Tertiary] Year 3-4 mathematics students training to become secondary school mathematics teachers with the majority having had teaching experience as part of their training" (p. 64). Klymchuk distinguished two clusters: in the New Zealand and German groups "roughly half of the participants were disappointed and embarrassed while the other half were more positive and saw the opportunity for improvements", whereas the Hong Kong and Ukraine groups "the vast majority were very disappointed and uncomfortable". Klymchuk indicated "The difference between the two clusters might be due to culture" (p. 67).

Klymchuk (2014) designed a test and questionnaire to explore the role of reflective thinking in mathematics assessment and to search for the underlying reasons for incorrect answers. The test questions – as given in Appendix A, with solutions – are described by

Klymchuk as "provocative" in the sense that, although they appear routine, each of the seven questions contains a non-routine 'catch' which relies on conditions and constraints within the mathematics. For example, the third question asks respondents to "Solve the equation $ln(x^2 + 17x - 18) - ln(x^2 + 5x - 6) = 0$". The initial impression may be that this question will succumb to standard mathematical manipulations, however the domain of the logarithm function is restricted to numbers greater than zero, therefore no real value of x satisfies the equation.

Mason's (2002) "discipline of noticing" involves paying attention to such detail as conditions, constraints, locality, properties and relationships. It was expected that the various 'catches' in Klymchuk's test would result in some participants obtaining incorrect answers, even though some countries were represented by experienced mathematics teachers. In practice, Klymchuk (2014, p. 63) reported that "The results of the test were startling – the vast majority of the participants gave incorrect answers to most questions in the test."

Methodology

In order to replicate the research undertaken by Klymchuk (2014) in an Australian setting, an account of the exact conditions under which the test and questionnaire were administered was obtained. The English language version of the test and questionnaire were adopted verbatim. The Australian survey participants were delegates at a November 2017 conference of secondary mathematics teachers, thereby ensuring all were practicing secondary mathematics teachers. The participants were invited to take part in a conference session entitled "An experiment which may change your teaching practice" with this session abstract: "We will do a test which contains some routine questions and some trick ones. Some are calculus questions. We will discuss the answers and consider the implications for our teaching." The expectation was that the mention of calculus would dissuade attendance by mathematics teachers whose experience was only of junior secondary classes.

Sixty teachers attended, completing the 15-minute test and a short questionnaire about their test results after the solutions had been discussed. In order to maintain conformity with the research in other countries, the wording of the session abstract, test questions, solutions and questionnaire prompts were used verbatim as reported in the original study (Klymchuk, 2014), with the same procedure and timing. The seven-question test was distributed as a write-on paper while a 15-minute countdown timer was visible on a projector screen. Although participants were seated adjacent to each other around large tables, no discussion was permitted and no such cooperation was observed.

In Western Australia three of the senior school mathematics courses have summative examinations to which school-based assessments are moderated: Mathematics Specialist, Mathematics Methods and Mathematics Applications. These are collectively titled the ATAR [Australian Tertiary Admission Rank] courses. The test questions discussed in this paper are based on content from ATAR mathematics courses. As evident from their test responses, in four cases the teachers demonstrated no acquaintance with the content of the ATAR courses, only with algebra. Although they were attending a conference for secondary mathematics teachers, these four may teach only junior classes and / or may be teaching out-of-field without a tertiary mathematics background.

Mathematics Test Results

The percentage of correct answers for each question, calculated against the number of completed test papers, is presented in Table 1. The percentages are a comparison of the number of correct solutions for the question, marked either fully correct or incorrect, against the country cohort size n. Klymchuk (2014) is the source of the non-Australian data.

Table 1

Percentage of Correct Solutions in Non-Routine Senior Secondary Mathematics Questions

		Question number						
Country	*n*	1	2	3	4	5	6	7
Australia	57	2	54	33	4	28	9	28
Germany	10	0	60	30	NA	20	0	0
Hong Kong	26	23	12	27	19	12	15	12
New Zealand	14	0	7	21	7	0	8	0
Ukraine	8	26	0	19	31	15	12	0

In the Australian study, 60 people were present to undertake the test, but only 57 test papers were received. Of these, there were 17 teachers who failed to obtain any correct solutions. It was evident from blank responses to questions that many of the Australian teachers had little idea of how to approach the questions at all, and the various 'catches' proved difficult even for those who were able to make a start. As in the research in other countries, no information was gathered directly from participants about their extent of training in mathematics nor length of experience as mathematics teachers.

Several misdirecting factors may have contributed to low scores. Despite indication in the session abstract that "some" trick questions would be included, in fact what Klymchuk (2014) refers to as 'catches' existed in every one of the seven questions. Some indicative questions may be seen in Appendix A. In Q1 the phrasing "height dropped on the hypotenuse" may have been unfamiliar to some participants. Q3 and Q5 demanded solving where there is no solution. Q4 directs participants to "prove the identity" of an equation which is not an identity at all. Q6 and Q7 ask participants to "find" items which actually do not exist. Such directly-phrased instructions may have been taken by the Australian cohort as indication that the task is feasible. Also, a typographical error (the symbol for "equals zero" was omitted) existed in Q5 until corrected verbally during the test.

Questionnaire Results

When the test papers were collected in, solutions were distributed and discussed and a follow-up questionnaire was distributed. 51 completed questionnaires were received. The questionnaire paper contained just three prompts: "What are your feelings after you have learnt about the correct solutions to the test questions?", "What are the reasons for not solving all test questions correctly?" and "Would you make any changes in your teaching practice after doing the mini-test. If so – what changes? If not – why?". Fifty-one completed questionnaires were received. They are reported here without regard to the prompt as the written responses often were not specific to a single prompt. The themes which emerged are

frustration, self-awareness, a trust relationship between teachers and students, and a perception that procedural understanding is prioritised by the examination system.

The responses were phrased politely but they do give an idea of the degree of exasperation and concern felt by some of the teachers: "Annoyed at being tricked by some obvious things", "I've been knocked down a peg or two. While doing the questions I felt confident in my ability and did spot a couple of the conundrums but it turns out that most of them got past me", "We were stitched up followed by great learning opportunity", "Ashamed because I did not get the correct answers", "I think this is an 'immoral' test. I don't believe you should ask someone to prove something is true when it isn't", "Make sure questions work and don't be nasty", "I didn't pay attention to the fundamental assumptions for a concept!", "I was too busy doing the mechanics of the questions and didn't take time to look carefully at all parts of the questions presented to see if they truly exist", "I would like to re-sit the test so that I can check if I have learnt from errors", "Tricked", "A little foolish, surprised", "Frustration, disappointment", "Not paying attention", "Excitement, satisfaction, curiosity". Responses such as "Not paying attention" reflect Klymchuck's (2014) subtitle "drawing attention to a lack of attention", but other responses do not. Many Australian respondents did not identify in themselves a lack of attention to mathematical detail, but rather they identified improper questions – questions which did not accord with the expectations of the respondents because they incorporated difficult and unexpected elements. These are elements which required noticing of detail. The Australian cohort are best associated with the New Zealand and German cluster, where there is divergence of response, rather than the Hong Kong and Ukraine cluster where subjects focused strongly on deficiencies in their own performance. This may indicate "social-reflection" values as identified by Mason (2002) or "culture" as identified by Klymchuk (2014).

In the Australian cohort, a dozen respondents (24%) in their questionnaire comments noted their own lack of knowledge, often coupled with indication of the reasons: "Lack of practice – have not studied / applied maths at this level for 12 years", "I had forgotten some concepts which I haven't used in a long time, as I don't teach methods & specialist at the moment", "I am not maths trained", "Didn't know the topic to that depth", "I failed to see each question had underlying principles or assumptions that I did not remember".

The word "trust" featured in six responses (12%). It was used to indicate that teaching and assessment is a bond between teachers and students. "Students should be able to trust the questions", "When students are doing a test they shouldn't be looking for a trick all the time. It is a matter of trust.", "[I have] basic trust test questions are mostly correct!!!", "I ignored whether the solutions were possible, i.e. I trusted the questions were "real" and [solutions] existed", "Trust. Didn't check all solutions. I did pick up some issues and became suspicious", "I didn't even think to check whether or not the triangle was possible as there was a "trust" that the triangle is real if the examiner is asking for an area. ... Q5 I showed that there are no solutions but crossed out my working, again having a "trust" in the examiner's questions". In a similar vein, other responses included: "In my teaching practice my students are given only problems which are possible to solve. They follow the script.", "I would hope that the majority of my questions are already checked and are workable as they are." "At school we don't tend to pose impossible questions", "I feel a teacher giving these questions would be unfair". At the conclusion of the session, one teacher approached the investigator to share that such 'trick' questions are inappropriate for school students because they "destroy the trust of my students". She encourages young teachers to work the examination questions themselves so that no impossible questions are presented. Another

teacher privately intimated that he always tells the students there are no 'trick' questions in the test, otherwise the students waste too much time looking for them.

One factor which figured prominently in the responses was that the teachers saw preparation for examinations as the pre-eminent priority: "Exams always have questions which make sense, so why teach them beyond the process?", "I am worried that in an assessment they will become absorbed by looking for the trick and waste precious time, as the assessments they do, do not have trick questions.", "We are trying to get them to be successful in their WACE [West Australian Certificate of Education] exams after all." Twenty-five respondents (49%) used the near-equivalent words "process", "mechanics", "methods", "procedure", "algorithms", "routines" and "strategies". These terms were used to indicate that the test-taking activity was viewed as a predictable pathway. Gaining familiarity with this pathway was preparation for examinations. But, as one teacher noted, "It is very easy to follow a rule / algorithm / formula, but unless you have the understanding 'why' you cannot see when there may be no solution".

A need for students to "question the questions" was recommended by 27 respondents (53%) who indicated that they will make use of such 'trick' questions, but only with their most able students for two respondents. Conversely, 23 respondents (45%) indicated it is contingent on teachers to ensure all questions have feasible solutions. These respondents did not see value in 'trick' questions which call for greater scrutiny of the conceptual underpinnings of the content, or they felt such considerations should not be included in assessments.

Discussion

The key observation about the overall results in Table 1 is that many teachers were unable to answer the questions. In many of the test questions it was not the 'catch' which caused Australian respondents to fail the question, it was lack of application of the routine methods of solution. In Question 5, for example, 28 of the 57 teachers did not employ the Intermediate Value Theorem nor any other productive means of attack. It was not that the function contains a discontinuity which was problematic: the teachers were unaware of basic function analysis technique in the first place.

Subtle linguistic cues may influence teachers in different cultures and operating in different languages. For some people it may be discourteous to respond "this cannot be done" to a question posed in a university-warranted test. Australian teachers may have little experience in mathematical questions which have no solution – or a multiplicity of solutions – whereas the format may not be unconventional in other countries.

If senior secondary mathematics assessments contain few questions which explore full understanding, the questions presented to "trusting" students must be routine, the content pre-negotiated and expected. Students who are not exposed to the risk of losing their self-confidence are therefore not being challenged in assessments to demonstrate more than procedural competence. This suggests there may be a conspiracy between teachers and their students to avoid coverage of true conceptual understanding in senior school mathematics tests and examinations.

The idea of 'conspiracy' in relation to the assessment of mathematics in Australia has appeared before. The architect of the current mathematics curriculum, Professor Peter Sullivan, writes on the use of open-ended tasks:

> One of the major constraints that teachers experience when utilising such tasks is that many students avoid risk taking and do not persist with the challenges that are required in order to complete the task. And teachers are sometimes complicit in this avoidance strategy. Desforges and Cockburn (1987), for

> example, reported on a detailed study of primary classrooms in the United Kingdom and found that students and teachers conspired with each other to reduce the level of risk for the students. (Sullivan, 2010, p. 38)

Although Sullivan (2010) discusses primary students, his point is that teachers can be complicit in an avoidance strategy.

The local mathematics curriculum (School Curriculum and Standards Authority [SCSA], n. d.,) states in the Rationale of each of the ATAR curricula that "For all content areas of the ... course, the proficiency strands of the Year 7–10 curriculum continue to be applicable and should be inherent in students' learning of the course. These strands are Understanding, Fluency, Problem-solving and Reasoning ...". The curriculum requirements include that students "interpret mathematical information and ascertain the reasonableness of their solutions to problems", a phrase repeated in the Learning Outcomes section in every Unit of the ATAR curricula. Senior secondary students therefore should encounter content such as boundary conditions, discontinuities and limitations to definitions. Provision exists within senior secondary mathematics school-based assessment to provide investigation of such topics as the limitations on and counter-intuitive properties of functions. For all three ATAR courses the curriculum (SCSA, n. d.) notes of the school-based Investigation tasks "This assessment type provides for the assessment of general inquiry skills, course-related knowledge and skills, and modelling skills." If teachers have only a procedural understanding of mathematics themselves, the school-based assessments they provide may just mimic examinations and fail to include these wider skills.

The question arises whether the ATAR examinations themselves could do more to allow "paying attention" as Klymchuk (2014) phrases it. In the 2017 Mathematics Methods examination (School Curriculum and Standards Authority, 2017) Q9 requires calculation of a linear model but later allows students to reject the model on the basis of patterns in residuals; Q10 asks students to select appropriate "equation(s)" where four are offered and two of them are appropriate; Q12 has marks for students declaring a given interpretation is incorrect because "cause is not established"; Q14 "find any point of inflection" has no point of inflection because the function is undefined at the only value of x where the second derived function is equal to zero. Unlike Klymchuk's questions, the examination does not direct students to find solutions which do not exist, and there is little of the "noticing" advocated by Mason (2002). An in-depth study of Australia's senior secondary mathematics teaching and examinations is beyond the scope of this paper, but the examination cited provides very mild support, if any, that the questions allow students to employ Mason's "noticing".

Conclusions

The Australian teachers had difficulty recognising 'trick' questions, and many expressed a disinclination to employ such questions in their own teaching. This study reveals that a significant number of teachers are unable to tackle hard mathematics questions on a routine basis in the first place. Avoidance of difficult material is not in the best interests of students and defensive responses by teachers may indicate a need for increased provision of in-service professional learning. Senior secondary teacher preparation courses mandate tertiary-level mathematics units. But tertiary mathematics does not constitute the material taught in secondary schools. Passing a tertiary mathematics unit does not ensure deep understanding of concepts in secondary school mathematics, an area in which secondary mathematics teachers may be wanting.

Australian teachers greatly value the trust relationship they enjoy with their students. However, this seems to deter some teachers from utilising questions which call on students to demonstrate deep conceptual understanding and confident exhibition of self-belief. Greater use of puzzle-based learning and intentional noticing may well prove advantageous in Australian schools.

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Appendix A

Indicative "provocative" questions posed by Klymchuk (personal communication, March 13, 2017).

2. Find the domain of the function $y = f\big(g(x)\big)$ if $f(x) = x^2 + 1, \quad g(x) = \sqrt{x-2}$

3. Solve the equation $ln(x^2 + 17x - 18) - ln(x^2 + 5x - 6) = 0$

5. Show that the equation $\frac{x^2+\sqrt{x}+1}{x-1} = 0$ has a solution on the interval $[0, 2]$

6. Find the derivative of the function $y = ln(2sin(3x) - 4)$

Using apps for teaching and learning mathematics: A socio-technological assemblage

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This paper reports on an aspect of a research project that examined the use of apps in primary school mathematics programmes. It reports teacher and student perspectives on how they used a range of digital tools within the apps to solve problems. We consider the interplay between the affordances of the mobile technologies, including multi-representation and haptic, with other social and pedagogical aspects, and how the assemblage of social and technological entities influences the ways that teachers might integrate apps into their mathematics programmes.

Introduction

Mobile technologies (MT) are ubiquitous in our social and occupational landscapes. Their low instrumentation and ease of operation, together with the potential to facilitate interaction, offer potential for enhancing the teaching and learning process. This interaction focuses primarily on touch and visual elements, making them intuitive for learners. Coupled with MT in educative settings is the use of educational apps. These can vary in quality evoking questions regarding the appropriateness of the content and pedagogical approaches of some apps (e.g., Philip & Garcia, 2014), but, with MT relatively prevalent in classrooms, their potential to enhance mathematical learning requires examination. Previous research has suggested that the affordances of digital technologies, including MT, have the potential for offering fresh approaches to engage with mathematical concepts and processes, and for re-envisaging aspects of mathematical education in both primary and secondary settings (e.g., Borba & Villareal, 2005; Calder, 2011). National curricula refer to using digital technologies to: enable new learning environments, facilitate shared learning and enhance opportunities to learn (e.g., Ministry of Education, 2007).

In this paper we explore the potential of using apps in school mathematics programmes by examining teacher and students' views on the use of iPad apps. We also consider how an assemblage of social and technical elements can enhance the mathematics teaching and learning experience. We present data from a larger study on the use of apps, in two primary school settings. The aim of the full project was to undertake a co-inquiry with teachers into the ways MT might enhance student learning in mathematics. Through discussion with the teachers, a question arose in relation to the use of apps, including those used for screen casting to facilitate mathematical thinking. In particular, the features of the apps were examined in conjunction with the dialogue and other social aspects. It was suggested that in order to examine the learning experience provided by the apps and how teachers might enhance the mathematical learning, we needed to consider the inter-relationships between the learner and the digital medium through an assemblage of social and technical entities involved in a learning experience. The question examined in this paper was: How might

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teachers utilise the assemblage of social and technical entities to enhance the mathematics learning process, when using apps?

Affordances

Building on the notion of affordance as the inter-relationships between the learner and the environment (Gibson, 1977) we acknowledge how the digital medium exerts influence on the students' approach, whilst the students' existing knowledge guides the use of the technology. Hence, the learning experience is fashioned in distinctive ways. For example, an affordance often associated with digital environments is the aspect of multiple representations, the ability to link and interact with visual, symbolic, and numerical representations simultaneously in a dynamic way (e.g., Calder, 2011; Sharples, Taylor, & Vavoula, 2007). Meanwhile, apps that enable screen casting, the digital recording of the computer screen, along with video and audio recording, introduce a further modal affordance in creating an aural representation of student thinking that can be listened to. This allows students to record individual or group presentations of mathematical processes, strategies and solutions, using a dynamic combination of tools and visual representations.

Other research contends that the affordances of digital technologies, coupled with the associated dialogue and social interaction, may facilitate students learning to pose problems and create explanations of their own (Sandholtz, Ringstaff, & Dwyer, 1997). They allow students to model in a dynamic, reflective way with other learners, mediating the language evoked through interaction with each other, the digital media and the mathematical ideas, and hence influencing the learning experience (e.g., Calder, 2011). Assemblage, as a theoretical perspective, acknowledges this inter-relationship of the multi-representations possible with apps, together with the mathematics and social interaction.

Socio-technological assemblages

Delanda's (2006) assemblage theory explored how inter-relationships are merged to form a social complexity. The social complexity as a whole emerges from heterogeneous parts and the properties of the whole emerges from the interaction between the parts. We relate Delanda's notion of assemblage with other theoretical perspectives suggestive of collectives. For example, Borba and Villarreal's (2005) perspective saw understanding emerging from the reconciliation of re-engagements of the collectives of learners, media, and environmental aspects with the mathematical phenomena. Borba and Villarreal contend that, as each engagement re-organises the mathematical thinking and initiates a fresh perspective, it in turn transforms the nature of each subsequent interaction with the task. The digital media influences the engagement and ensuing dialogue in particular ways, which, with self-reflection or further dialogue with others, transforms the learners' perspective (Borba & Villarreal, 2005). The learners then re-engage with the task from this new perspective.

It has been suggested that MT offer a socio-material bricolage for learning (Meyer 2015). Meyer envisaged interconnected systems where resources interact with knowledge that is socially distributed. A range of people, communities and sites of practice might be influential in assisting student learning. Meyer (2015) used the term socio-material bricolage to describe the "ecological entanglement of material and social aspects of teaching and learning with technology" (p. 28).

The notion of bricolage suggests that there is a mutually influential collective of tools and users affecting the dialogue, learning experience, and mathematical thinking, in particular and personalised ways. For example, when students collaborate on a task, they incorporate input from the wider class, school and "home" communities, while also drawing from the broader underlying discourses, such as political or socio-cultural elements that influence their pre-conceptions about the task and mathematical activity. De Freitas and Sinclair (2014) discussed thought as being distributed across both social and physical environs and influencers. We consider that thought evolves in a complex material and social milieu. When screen casting their strategy and solution(s) the students might incorporate a range of digital, visual, and concrete material resources in mutually interdependent ways. All of this activity has associated social elements, both immediate interaction as well as the drawing forth of the underlying discourses. The resulting process is not just the accumulation of the various 'bits', but also a new mesh of the social and material elements.

From these various theoretical perspectives, the whole, or in the case of this paper the learning experience, becomes the articulation of discursive and non-discursive elements of objects and actions (Delanda, 2006). However, a key distinction is Delanda's proposal that all entities and relationships, whether social or non-social, are ontologically and epistemologically indistinct. As such, knowing or understanding within the learning experience is no longer a means of representing or reflecting on new knowledge but one of interacting with and creating new knowledge. Social assemblages may be codified through language, whereas non-social or technical may not. However, the very use of the technical can be seen as expression and this is illustrated through the multi-representational and multi-modal affordances of the iPad. The learner may select representations as a way of expressing and creating knowledge; the learner may also use hand actions with the touch interface of the iPad screen, again as a way of expressing and creating knowledge.

Despite distinctions, these perspectives of merging learners with digital media within a situated learning experience, point to the notion of an assemblage of digital and social elements which we term a socio-technological assemblage. In this paper we report on teachers' and students' perceptions when using mathematics apps and investigate how they perceived the learning opportunities presented by the apps. Some considerations for teachers' practice are then indicated, in relation to teaching and learning mathematics and to the notion of socio-technological assemblages.

Methodology

The research for the larger two-year project used an interpretive methodology related to the building of knowledge and the development of research capability through collaborative analysis and critical reflection of classroom practice and student learning. The research design was aligned with teacher and researcher co-inquiry whereby the university researchers and practicing teachers work as co-inquirers and co-learners (Hennessy, 2014), In the first year of the project three teachers, all experienced with using MT in their programmes were involved in the study. The schools were situated in a provincial city. One teacher taught a year-4 class (7 and 8-year olds) in a school that used a BYOD approach, while the other two teachers team-taught in a combined year-5 and 6 class (9 to 11-year olds) in a school with one to one provision of iPads (80 students in total). In the second year, 12 teachers across a range of age groups and experience with using apps were involved. Data, obtained through different sources(focus group interviews, classroom observations, interviews with teachers, and blogs), were analysed using NVivo via a mainly inductive or grounded method to identify themes. While the researchers identified the initial themes and

codes, refinement occurred through joint critical reflection between teacher practitioners and academic researchers in research meetings. One theme identified in the research meetings was socio-technological assemblages.

In this paper we present data from teacher interviews, class observations, student focus group interviews and individual blogs in order to investigate their views in relation to student learning. We consider the ways that socio-technological assemblages may help to examine how the use of apps influenced the students' interactions with mathematical ideas. The student blogs were obtained partway through the first year of the project, while the teacher interviews and student focus group interviews were carried out at the beginning and end of both years.

Results and Analysis

Teacher Data

The data were relatively cohesive, in terms of apps being influential in the learning process, regarding the connection between the use of the apps, other technologies such as concrete materials, and the dialogue and social interaction that engagement with them evoked (Meyer 2015). For instance, students were observed using the iPad to investigate a problem in context, then using counters and rods, all the time interacting with each other and the range of tools. They used an empty number line in the app and a white table for story boarding the screencast of their strategy and solution. This was then loaded into a *Google classroom* site that the teacher could access for review and feedback.

One teacher, Brad, saw this tapestry of material and social elements as an ecosystem:

> Brad: There's a really big app eco system – I don't think there's many other devices that you can program on the iPad and then program robots and record your voice and make videos and all that stuff – it's a very rich ecosystem.

There were also instances where concrete materials were used in conjunction with apps. For example, Joy talks mainly of an assemblage of material elements. However, the associated social aspects, including the relationships and interaction between students, teacher, school community and the broader societal discourses are inherent in the activity described:

> You might do something with those Cuisenaire rods… those plastic things… there's also an app that would do it as a lesson and then there's an app that actually has the rods in it, so the kids can go away and practice moving them around the screen after they've done it with you physically… so there's a nice connection.

One teacher commented on the direct interface of the iPad screen, suggesting that the students were interacting more directly with the content of the mathematics:

> Like a physical object that they're interacting with.

The teacher further explained how apps involving screen casting for recording students' strategies were powerful agents in learning as the students were,

> creating something...explaining their own thinking, creating their own content, their own language.

This teacher comment points to the notion of personalisation of the learning. The students are creating their own content and language, hence differentiating the experience and learning to some extent.

Another teacher noted how screen casting enabled less confident students to explain their thinking in a "nonthreatening environment" with

> no teacher staring at them, no other kids waiting for them to hurry up. They're in a safe place where they can just record their thinking without any pressure.

Student Data

In the student blog data, references were made to the features and affordances of the apps and iPads, how these influenced the learning experience and the ways understanding might emerge. Several blog entries referred to the multimodal affordances of screen casting.

> You can record your learning and you can see what stage you are working on and: Instead of writing in our book we can just record our voices and upload it to Google classroom!

The use of the *Multiplier* app also illustrated the haptic affordance as the students used their fingers to draw out the matrix and had both visual (including colour coding), and numeric representations simultaneously linked together. It was a multi-levelled problem-solving environment with a tap used during the experimentation and review stages:

> You tap on which one you think is correct.

The use of programming apps with Sphero robots made strong physical and visual connections too, especially in geometry.

> We used this app (Tickle) to learn about making shapes, angles and vertices.

Students made further comments related to a mixed use of pedagogical media. For example, the students were comfortable moving between their iPad and more traditional media:

> I can still switch back to my book easy and it's still easy to use apps.

The students recognized the same potential for using a mixture of technologies at the appropriate time for their learning:

> Tim: Sometimes I make a plan (on paper) to work out my word problem, then I can put the pictures on and record my answer on the iPad.

Key viewpoints from the student blogs referred to the dynamic multi-modal representations, hand actions and the haptic, and exploration. Students also referred to mediation through programming and the use of different pedagogical media.

These key viewpoints were reiterated in the student interviews. The students talked of videoing themselves doing maths and recording their working. As one said,

> It's just like making a movie for maths.

The use of multiple modes simultaneously supported this student in expressing his thinking. The opportunity to pause, reflect, and edit recordings also appeared to be significant in supporting students in expressing their thinking.

> The cool thing is that you can actually pause it and then think about what you're going to do.

Other students referred to the opportunity to use the different visual and dynamic representations on the iPad and how these introduced them to new strategies.

> I learnt how to use the reversing strategy on the number line.

Students also referred to opportunities for collaboration with their peers and how they worked on a mathematical idea together.

> I like working with my friends and then recording our voices like working out an equation together.

Collaboration also provided the opportunity to explore and experiment with mathematical ideas:

> Luke asks me to work with him because we like to help each other out and solve things – so if we don't get something we try and work it out.

Students also contested ideas and processes when collaborating on a problem using *Minecraft*, on a single iPad. From the observational data:

Aaron	Okay, 5 lots of 5 blocks
Zac	Yep, 5 blocks
Don	Shall we use a line? (He indicated where the 5 blocks might go on the screen)
Zac	No, not 5 blocks up!
Aaron	Yes, you need to use it there.
Don	Yeah, there.
Zac	Is it? No, this one (pointed to the screen)
Aaron	You need the 5 blocks across and going up (indicated on the screen)
Zac	Oh yeah, yeah now I see.

Here, Zac's understanding of the solution and the process changes through the discussion related to the group's direct interaction with the app. It was the visual tension evoked from touching the screen, and the immediate impact from that action, that initiated the dialogue and also enabled, in conjunction with the dialogue, the transition in Zac's understanding. In this way the learning through apps took place within interconnected groupings of digital elements and the social aspects that they evoked. The responses from students suggested further viewpoints in relation to the use of multiple modes in expressing and creating their thinking, visual and dynamic images in learning key concepts, and peer collaboration in sharing ideas and in exploring or working our new knowledge.

Discussion

Teacher and student responses acknowledged the potential of the iPad in manipulating objects dynamically onscreen. They spoke of acting directly with the object and referred to tapping or drawing on the screen. The screen-casting feature was seen to introduce multiple modes and representations as students worked simultaneously with dynamic visual recordings (drawing, manipulating digital tools, and writing symbols and words), along with speech, to create a dynamic aural-visual representation. The coding app *Tickle* was used to connect numeric and symbolic representations in the coding with the physical movements of the *Sphero* and the creation of geometric shapes. Although the movements were mediated by the coding process, the students commented on the connections between the movements and their learning. Furthermore, the teachers and students referred to collaborative working with their peers. They indicated how ideas could be shared and worked on together again. The students also mentioned how non-digital as well as digital technologies were used together. All these elements draw forth from the underlying preconceptions and discourses that each individual's socially situated context evokes.

In sum, teacher and student data were suggestive of inter-relationships between the multi-modal affordances of the iPad, along with other non-digital entities including peer interaction and other pedagogical media. These inter-relationships are interpreted through the notion of an assemblage where social and non-social become merged.

In relation to Delanda's (2006) assemblage theory, the learning experience is viewed as a social complexity constituted of heterogeneous entities. Students' comments were suggestive of social assemblages such as the use of verbal language when communicating with each other or voice recording. However, students also communicated through tapping on the screen or in sharing a document. Students also referred to use of hand actions when using *Multiplier* or *Tickle*. As such, the technical materiality, that is the multi-modal affordances of the iPad, were used by the students to communicate and express ideas. Social and non-social could be seen to merge in line with Delanda's theory, and the learning experience became a means of interacting with and creating new knowledge in ways that were determined by the features of the iPad as well as through other media and communication.

Borba and Villarreal's (2005) perspective focused on engagement within a collective of learners, media and the environment. Engagement re-organises thinking and provides fresh perspectives for re-engagement. The students suggested opportunities to interact in collaborative ways to "work it out" and experiment. The students had opportunities to pause recordings in order to reflect before engaging further with the media. In order to interact with the mathematical ideas, the students drew on existing knowledge and affective dispositions to engage with the mathematical ideas through, not just the iPad, but through a range of social interactions that evoked interpretations or understandings that were negotiated further (Calder, 2011). The teachers noted that the recordings were a way for the students to show their thinking processes when solving a word problem. It appeared that, through pausing and editing, the students took time in preparing and perfecting their recordings. They were able to reflect on what had been said and think about what to say next. Here the students were influenced by the iPad, which they then influenced.

From both theoretical perspectives the multimodal affordances of the iPad can be seen to provide new entities for social and technical to merge as an assemblage or a collective within a learning experience. So, in answer to our initial query about how teachers might utilise the assemblage of social and technical entities to enhance the mathematics learning process, we would suggest that, teachers need to be aware of the potential for the various representations possible through using apps to be connected through engagement. As well, the potential for that engagement to evoke social interaction and collaborative thinking. The content and nature of the screen-casting recordings were seen to merge the multiple modes of verbal expressions with drawings and symbols. Students created their own ways of expressing their knowledge. Furthermore, some students developed these recordings collaboratively and acknowledged opportunities that enabled them to share and negotiate their knowledge in conjunction with the multi-modal affordances. Such recordings compiled individually or collaboratively, would seem to illustrate the notion of a socio-technological assemblage that could influence the mathematical understanding of those that created them and those that viewed them.

Conclusions

Previous research has suggested that MT offer affordances that can reshape the learning experience. In this paper, we aimed to consider how teachers might utilise the assemblage of social and technical entities to enhance the mathematics learning process, when using apps. The idea of an assemblage suggests that the same mathematical phenomena can evoke different ranges of social and technical entities when approached through alternative pedagogical media, and so the resulting learning experiences, constituted by the merging of these different ranges of social and technical influences, will differ. Teachers might consider

the ways that they use apps in their mathematics programmes. Are they including screen-casting apps that allow the students to use an assemblage of technical and social elements to both explore and communicate mathematical thinking? The research suggested that they emphasise the collaborative aspects that using apps evoke, including the potential to stimulate the contestation and validation of ideas and processes. Also, that they give opportunity for students to move seamlessly between material and digital resources. They should consider giving opportunity for coding, which might mediate the mathematical thinking. They might also utilise apps that include haptic and aural affordances as well as the more commonly recognised ones, such as linking multi-representations. As tools, apps have considerable potential for teachers to reshape the learning experience and offer students new ways to engage with mathematics. This is consistent with expectations for teachers' programmes as expressed in national curricula. Most importantly, we found that the quality of the teachers' pedagogy and practice was more influential than the quality of the app.

We acknowledge the support of the Teacher and Learning Research Initiative.

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Exploring mathematical fluency: teachers' conceptions and descriptions of students

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Interviews can provide a window into what teachers think. This paper reports on findings from an exploratory study into teachers' conceptions of mathematical fluency. Focusing on stage two of the study, I discuss 3 of 17 teachers interviewed, analysing their conceptions and descriptions of students. Teachers spoke of students having understanding and multiple ways of thinking, and their ability to work through errors and transfer knowledge. This suggested fluency in mathematics as more than carrying out procedures. Viewing fluency as the result of having conceptual understanding, strategic competence and adaptive reasoning, would make it synonymous with mathematical proficiency.

Mathematical fluency can be defined and interpreted in many ways. The literature surrounding mathematics generally defines fluency as procedural or computational fluency (Kilpatrick, Swafford, & Findell, 2001; McClure, 2014; National Council of Teachers of Mathematics, 2014; Russell, 2000). The majority of research studies conducted focusing on fluency in mathematics use a definition of procedural fluency similar to the definition in Kilpatrick et al.'s (2001) conceptualisation of mathematical proficiency (Bass, 2003; Graven, Stott, Nieuwoudt, Laubscher, & Dreyer, 2012; Stott, 2013). Kilpatrick et al.'s (2001) model shows procedural fluency as one strand of proficiency along with adaptive reasoning, strategic competence, conceptual understanding and productive disposition. When looking at an Australian context and conceptualisation of fluency, Watson and Sullivan's (2008) definition as *mathematical* fluency is a broader term and is used as the definition of fluency described in the Australian Curriculum: Mathematics (ACARA, 2010).

Little research exists about practicing primary teachers' conceptions of mathematical fluency and how they describe mathematically fluent students. Research mainly centres on students' procedural fluency and its relationship to conceptual knowledge or on testing and improving [the speed] of their procedural fluency (Arroyo, Royer, & Woolf, 2011; Bauer, 2013; Ramos-Christian, Schleser, & Varn, 2008). Even though the term procedural fluency may describe other features of fluency, the use of the term *procedural* to describe fluency results in teachers interpreting procedural fluency at face value. This view of fluency can lead to a disconnect between the teaching of the procedure (the what), and the understanding of the concept (the why), of mathematics which need to be learned in unison (McClure, 2014). According to Watson and Sullivan (2008), fluency involves carrying out procedures flexibly, accurately, efficiently and appropriately *as well as having* "factual knowledge and concepts that come to mind readily" (p. 112). Their definition combines both the ability to readily perform the mechanics of mathematics (procedures) and the understanding of the mathematics being learned (concepts) providing a wider scope to focus on various aspects of fluency. Further research on mathematical fluency is required to provide insight into the complex nature of fluency beyond a mere process of memorising facts and quick recall.

The focus of this study was on exploring teachers' conceptions of mathematical fluency. For this research, the term *conceptions* was taken to be inclusive of both a teacher's beliefs and knowledge that they hold of the concept (Beswick, 2012; Thompson, 1992). Teachers' conceptions are highly dependent on their personal beliefs formed through life and

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educational experiences. Conceptions are also influenced by teachers' knowledge of mathematics, and of how mathematics is learned (Borg, 2003; Melketo, 2012) as seen in Figure 1. This model formed the theoretical framework for studying teacher conceptions.

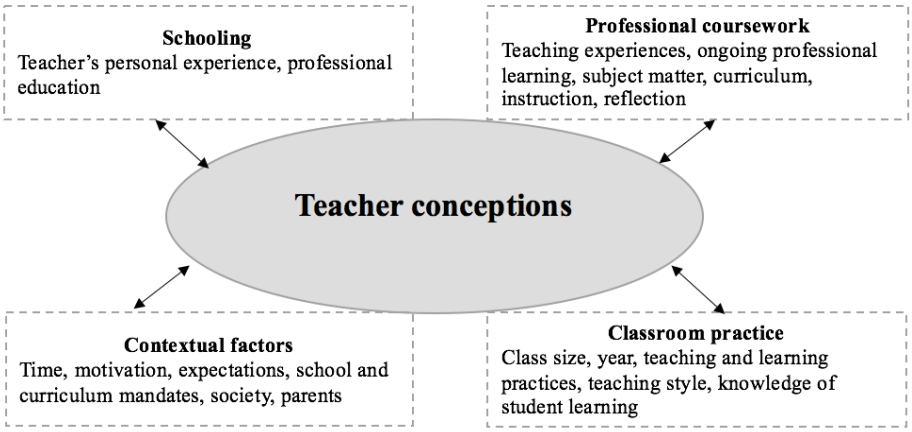

Figure 1. Teacher conceptions framework, synthesised from Borg (2003) and Melketo (2012).

The findings discussed in this paper aim to explore how teachers translate definitions from research of mathematical fluency by answering the following research questions: What knowledge and beliefs do primary teachers have about mathematical fluency? And, how do they describe mathematically fluent students?

Methodology

This qualitative study was designed to be exploratory in nature, aiming to gain a deeper understanding of teacher knowledge and beliefs by studying real-world settings inductively to generate rich narrative descriptions (Patton, 2002). An interpretive approach to research was taken during this study. A strength of using this approach is its emphasis on examining texts, such as written words, or conversations (Neuman, 2003). When interpreting a concept, people's beliefs, values and perceptions provide meaning and influence knowledge. This approach assisted in building rich local understandings of the beliefs and experiences of teachers and of the cultures of classrooms (Taylor & Medina, 2013).

The study was divided into two stages of data collection, involving a questionnaire and semi-structured interviews. Stage one, the questionnaire, involved the random selection of 300 NSW primary schools inclusive of both city, rural and remote locations. Teachers self-nominated to complete the survey that was sent to their school. The online questionnaire included background information questions, Likert-type items (dimensions of mathematics) and two open-ended response questions. The questionnaire was completed by 42 participants. At the completion of the questionnaire, the participants could remain anonymous or indicate their interest in participating in a follow up interview as stage two of the study. Of the questionnaire participants (n=42) 17 teachers agreed to be interviewed. These 17 teachers were representative of all teaching grades, Kindergarten (K)-Year 2 (n=7), Year 3-4 (n=5), Year 5-6 (n=2), K-6 (n=2) and one not specified.

Interviews can capture rich detail of the experiences and perspectives of those being studied (Lincoln & Guba, 1985). Semi-structured interviews allowed additional questions to be included based on general patterns of responses from the questionnaire data analysed.

Each interview was audio recorded and later transcribed, using a unique code for each teacher. Questions focused on definitions, descriptions, connections and features of mathematical fluency and examples of students displaying these characteristics. The findings from the analysis of the questions regarding the descriptions and examples [of students' thinking] that teachers provided are the main focus of this paper.

Data Analysis

Thematic analysis provided an illustrative and exploratory orientation to the study (Guest, MacQueen, & Namey, 2012). Using both inductive and deductive coding as different layers of analysis allow codes to flow from the principles that underpin the research, and the specific questions one seeks to answer (Joffe & Yardley, 2004). Multiple opportunities emerged to analyse the data gathered. Similar to Clarke and Braun's (2017) thematic process phases, the analysis was undertaken in 6 steps: (1) questionnaire data summarised, (2) questionnaire data analysed, (3) identification of codes from questionnaire data, (4) interview questions refined based on questionnaire data, (5) interview data analysed for emerging themes and mapped to questionnaire codes, (6) searching for themes in the questionnaire and interview data mapped to the research questions, the Likert item dimensions of mathematics, and the teacher conceptions framework (TCF). Aspects of phase five and six of this thematic process are discussed below.

Results and Discussion

Initial analysis was conducted by highlighting key features of mathematical fluency that emerged within and across participant responses. Statements and quotes that directly related to the research questions, the questionnaire codes, the Likert item dimensions and the TCF were highlighted and added to spreadsheets for further analysis. In this paper, I report on the mapping to the questionnaire codes and the TCF.

In the open-ended response section of the questionnaire, teachers were asked to write three words to describe mathematical fluency and a short definition of mathematical fluency. The leading four words listed were: efficient, flexible, understanding and strategies. These features link closely to Watson and Sullivan's (2008) definition of mathematical fluency. In addition, teachers mentioned that mathematical fluency is inclusive of students' abilities to: use different/multiple pathways, make connections, communicate their reasoning, apply/transfer new learning and risk being wrong. These features were used as the initial codes for analysis of the interview data.

I have selected the analysis of three teachers and their interview responses. Their responses are typical of the 17 interviews and their descriptions of a specific student in their class were detailed. One teacher had 6-10 years teaching experience and the other two teachers had more than 10 years teaching experience. These teachers represent differing schooling grades (a K-2, Year 4 and a Year 5-6 class). They teach in a range of low and high SES (Social Economic Status) metropolitan areas, in schools with 18%, 45% and 81% Non-English-Speaking Background (NESB) students.

Interview Data Mapped to Teacher Conceptions Framework

The influencing factors represented in the TCF (Figure 1) were used as a lens for analysis. All TCF factors—teachers' own educational experiences, social context, classroom

practices, and their knowledge of content and pedagogy—were evident across interviewee data.

When describing mathematical fluency, teachers used their classroom or student learning experiences to frame their responses. Teachers related fluency to their classroom practice stating:

> I teach fairly similarly, when I teach kindergarten and when I teach stage three, in that I have to go from the known to the unknown. 05_01K2

> To become fluent, you have to practice the skill, and then that will build up your known facts, that will help you then to solve problems in a variety of different ways. I think that - like explicit teaching, but also practice of skills, like games, and then open-ended tasks, and tasks that require them to then apply their skills. 14_0156

Teachers' knowledge of content and pedagogy came to the fore in their responses where a strong focus was placed on syllabus knowledge and the positive effects of professional learning they had experienced regarding mathematics. Some teachers associating beliefs to their own learning, for example:

> Early intervention programs have that ability for the students to learn how to reflect on their learning [which] has huge power. Because it doesn't just increase their fluency, and their accuracy. It gives them the ability to go, 'I made a mistake and this is where I think I made the mistake, and this is where I think I need to correct it'. 05_01K2

> They're working mathematically when they first kind of start to have a focus on those in the syllabus. I think it was an add-on. It was like, let's teach them all the mechanics as I like to call it and then we'll give them a problem at the end. I actually think it needs to go the other way. We need to be giving them doing and thinking and then teach them. 10_0105

Some respondents also described mathematical fluency and the importance of mathematics as a way to communicate within a social context. Such as:

> In some ways, you are looking for their ability to recognise the patterns and to make the connections. You want them to make connections to themselves and to the world and to other things that they've seen. 10_0104

> If you know how to solve something and you can't explain it to anyone else, then no one else is ever going to find out what it is. Fluency is a big part of communicating your knowledge. Because maths is always growing, and finding new things, and finding new ways. It's very important. 14_0156

Similarities arose in responses once the data were organised according to the TCF influencing factors. Examples included: making connections between fluency in mathematics and the real world (social contexts), identification of mathematical fluency as important (content/pedagogy knowledge), and mathematical fluency as a way of communicating knowledge (student/ classroom experiences). Features of mathematical fluency that teachers espoused were consistent with the initial questionnaire codes.

Descriptions of Students Mapped to Initial Questionnaire Codes

Teachers' conceptions of mathematical fluency were also mirrored in their specific examples of students they felt were mathematically fluent. Figures 2, 3 and 4 are excerpts from the three interviewees with definition codes and emerging themes identified within the text. Descriptions of students aligned to the teachers' beliefs and knowledge that was identified when the data were mapped to the factors in the TCF.

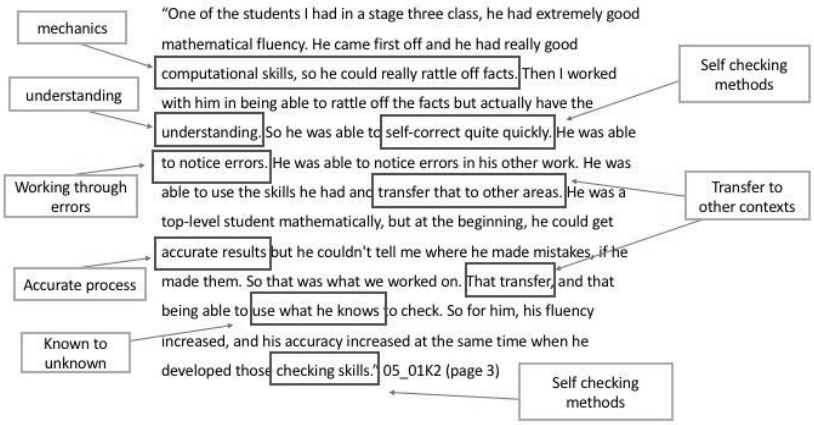

Figure 2. Excerpt from 05_01K2 interview mapped to questionnaire codes.

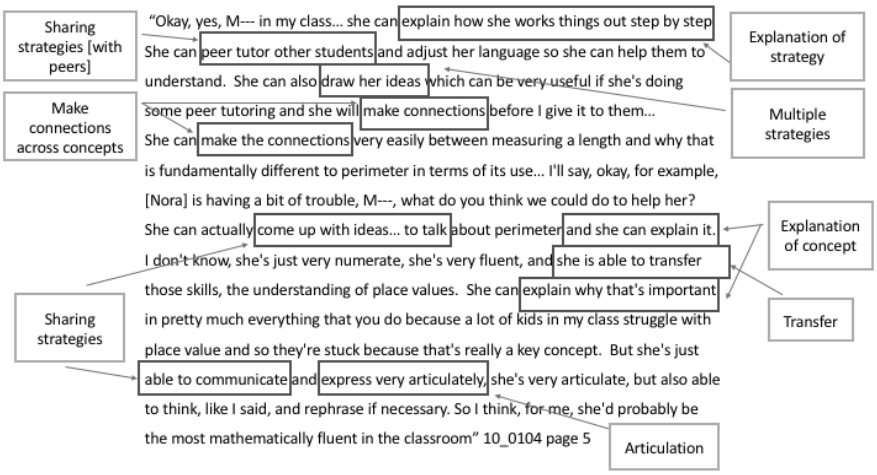

Figure 3. Excerpt from 10_0104 interview mapped to questionnaire codes.

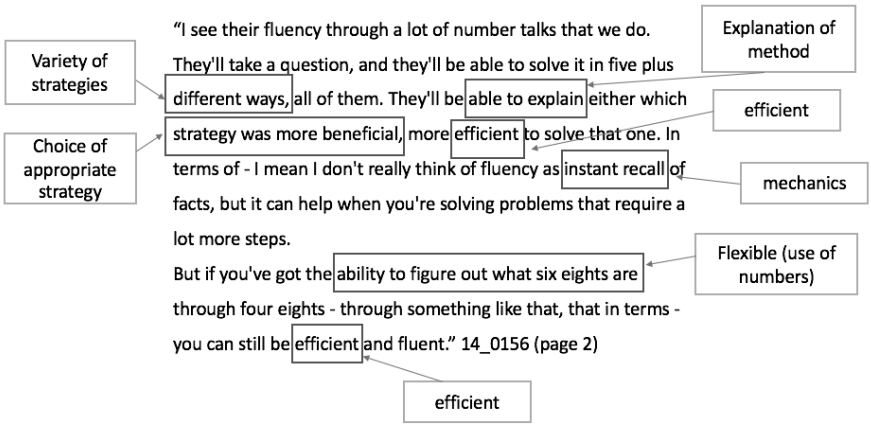

Figure 4. Excerpt from 14_0156 interview mapped to questionnaire codes.

Interviewee 05_01K2 referred to her teaching strategies when describing mathematical fluency and assisting students in moving from the known to the unknown. This conception is reflected through the description of her student's ability to transfer knowledge across areas and to transfer knowledge as a checking method. Interviewee 10_0104 emphasised the importance of making connections to the real world when mapped to social context in the TCF. Within the student description, this conception appears twice, making connections across areas, and making connections for other students. Interviewee 14_0156 references teaching practices such as number talks when describing her student, reflecting her knowledge of effective pedagogy. These descriptions richly illustrate what mathematical fluency may look like in the classroom. It is noted as a limitation of the study that although the interview data may be seen as a validation of teacher conceptions, a direct correlation of these conceptions to their classroom practice cannot be made.

Teachers' conceptions of mathematical fluency did include aspects related to procedures (efficient, instant recall, computational skills). However, these features appear to align more closely to Kilpatrick et al.'s (2001) other strands of proficiency, more so than the procedural fluency strand on its own. Support for this conclusion comes from analysing the context teachers provided when describing procedural features. Teachers referenced the procedural terms *in conjunction with* student strategies, understanding or, as a way of reasoning. Drawing on the initial definition codes from the questionnaire data and additional features that emerged from the interview data, similarities are visible. When these features were grouped, they noticeably aligned to three of Kilpatrick et al.'s (2001) five strands of proficiency: strategic competence, conceptual understanding and adaptive reasoning (see Table 1). Student confidence was also mentioned which links to the productive disposition strand of proficiency. However, teachers did not see confidence as a separate aspect of mathematical fluency but something that builds once students' strategies and skills develop.

Table 1
Fluency Characteristics Mapped to Kilpatrick et al.'s Strands of Proficiency (2001)

Strategic competence	*Conceptual understanding*	*Adaptive reasoning*
Variety of strategies/ ways	Making connections between concepts (known to unknown)	Justifying strategy/method
Choice of strategy		Transfer to other contexts or problems
Accurate process (articulation)	Explanation of method	Self-checking method (reasonableness)
(Ease of) mechanics-*automaticity*	Sharing strategies [with peers] (communicate)	Working through errors

Mathematical Fluency as Proficiency

Previous studies of *reading* fluency indicated "language researchers have offered countless different aspects that contribute to defining fluency as an overall oral proficiency in speech" (Götz, 2013, p. 1). Why has fluency as an *overall proficiency* not been applied to mathematics, as it appears from the data that there are many aspects that contribute to mathematical fluency. Common themes from the interviews addressed this question. The examples of student behaviours shared indicated the complex nature of fluency that stretched far beyond efficiency with procedural knowledge. It is clear that fluency, from teachers' perspectives, may be determined by a student's ability to apply, and demonstrate or transfer knowledge, for example, in problem solving tasks.

Conclusion

The depth to which the teachers explained their thinking and justified their ideas through student examples provided an insight in to the complex nature of fluency.

> I don't think of fluency as one thing. I think of it as a whole broader concept. I wouldn't call someone fluent if they could just apply an instruction. 14_0156

Kilpatrick et al.'s (2001) description of procedural fluency echoes the belief that separating procedures (skills) from understanding can have dire results, "students who learn procedures without understanding can typically do no more than apply the learned procedures, whereas students who learn with understanding can modify or adapt procedures to make them easier to use" (p. 124). Conversely, the Australian Curriculum (ACARA, 2010) and Kilpatrick et al.'s (2001) strands of proficiency both depict fluency as separate from (although intertwined with) understanding. Teachers in the questionnaire listed understanding as a feature of fluency, interviews comments also supported this view:

> I think if you're fluent in maths you're going to have the understanding with it. I think you can have the understanding without fluency but not the other way around. 10_0104

Figure 5 may be a more useful model for teachers in reflecting how mathematical fluency develops. From analysing the teacher descriptions, mathematical fluency is the result when students' strategies and ability to reason are concurrent with their conceptual understanding. This is consistent with Watson and Sullivan's (2008) description of mathematical fluency. This model puts forward the notion of fluency as a result, instead of one strand, of proficiency. Further research illustrating the nature of how understanding and fluency interact would be beneficial for teachers.

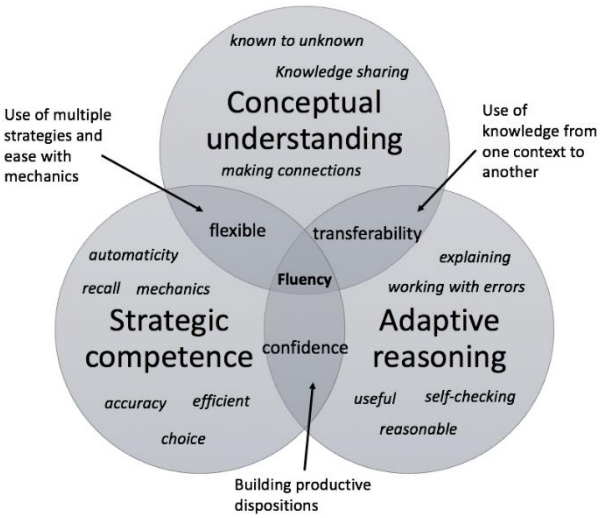

Figure 5. Reframing fluency model.

Rich descriptions of mathematical fluency have the potential to assist teachers in identifying aspects of fluency students possess, and aspects of fluency yet to be developed. The findings of these three teacher interviews assisted in discovering teachers' shared conception of mathematical fluency and identified features. Further research could enrich these descriptions and define when the characteristics are likely to be noted. Clearly identified features of mathematical fluency could also be researched when observing and

assessing student conversations or work samples. A shared understanding of what we mean by 'fluency' is important if we expect teachers to assist their students to become fluent.

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Teachers' Perceptions of Obstacles to Incorporating a Problem Solving Style of Mathematics into their Teaching

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Despite recommendations to incorporate mathematical problem solving into the practice of primary teachers there is little evidence of the widespread acceptance of such advice by Early Years teachers. Understanding teachers' perceptions of the obstacles they encounter when incorporating mathematical problem solving into their teaching can shed light on the matter. Survey responses of 22 teachers of Foundation and Year 1 across three Victorian schools indicated that the initial obstacles teachers perceived were those concerning children, teaching pedagogy, planning, resources, tasks and time.

The future will rest on a foundation of applied mathematical, scientific and technological knowledge of today's children. Jonassen wrote that, "problem solving is generally regarded as the most important cognitive activity in everyday and professional contexts" (2000, p. 63). In addition, as Gagne stated, "the central point of education is to teach people to think, to use their rational powers, to become better problem solvers" (1980, p. 85). For Polya, considered the father of mathematical problem solving, mathematical epistemology and mathematical pedagogy are deeply intertwined (Schoenfeld, 1992) and educators ideally look for "authentic problem-solving situations in which children behave as mathematicians and have opportunities to develop mathematical power" (Baroody, 2000, p. 61). The ability to solve problems is a fundamental life skill and develops naturally through experiences, conversations and imagination (Cheeseman, 2009; Geist, 2001). Teachers are critical in creating rich mathematics learning experiences for children at school and in helping them to "make sense of problems and persevere in solving them" (NGA Center, 2010, cited in Blair, 2014). In the Australian Curriculum: Mathematics (Australian Curriculum Assessment and Reporting Authority, 2016) problem solving is one of four fluency strands that are interwoven across all the mathematical content strands at every level of school. Yet while teachers are often aware of the potential of young children as problem-solvers in their own lives, they are uncertain about how to harness that potential in mathematics classrooms. Blair (2014) found that inquiry-based learning has not found its way into daily teaching practice.

Problem solving is not new. Yet children seldom work on engaging and challenging mathematical problems in Early Years primary classrooms. It is important to understand this situation and to understand the obstacles that teachers believe prevent them from incorporating mathematical problem solving into their practice.

Incorporating new elements into existing teaching is always a complex process. Jackson et al. (2015) developed an empirically grounded theory of action for instructional improvement in mathematics. These authors maintained that five interrelated components were necessary to support "ambitious teaching": materials and instructional guidance; teacher professional development and collaborative meetings; job-embedded support for teachers' learning; school instructional leadership; and school system leadership. While the research project reported here was not a large-scale project like that of Jackson and his colleagues, this project incorporated four components of Jackson's theoretical framework with the use of supportive curriculum materials and instructional guidance, teacher professional learning days out of the classroom,

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collaborative meetings, and school-based support for learning. The intention was to stimulate and support ambitious teaching of mathematical problem solving with young children. The nature of obstacles teachers perceived they faced when they make changes to their teaching to incorporate problem solving, is the subject of this paper.

Method

A design research project that was: an intervention in the real world, iterative, process-oriented, useful for its users in real contexts, and theory-oriented (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006) was the context in which the data reported here were collected. The project entitled *Fostering Inquiry in Mathematics* (FIiM) connected mathematics education research and teacher professional development with 22 teachers. Underpinning the project was a history of research evidence of the effectiveness of teacher professional growth stimulated by innovative mathematical problem solving materials (for example, Clarke, 1997). Challenging tasks (Cheeseman, Clarke, Roche, & Wilson, 2013; Clarke & Clarke, 2004) which took a problem solving approach to mathematics in the Early Years of school were provided to teachers to be trialled in classrooms. In addition, research describing features of highly effective teachers of mathematics with young children (Cheeseman, 2010; Clarke et al., 2002) was emphasised in the professional development component of FIiM to enhance teachers' practice and lead to improved learning by children. Pedagogical approaches to problem solving and investigations in mathematics were raised with teachers to emphasise the importance of teacher noticing (Mason, 2011), engaging in mathematical conversations (Cheeseman, 2015), and encouraging curiosity and persistence (Cheeseman & McDonough, 2016). The outcomes of the project were evaluated in terms of children's learning, and teacher feedback. Children's learning was observed in classrooms, and tested by clinical interview. Children's drawings were also collected to reveal their dispositions to learning mathematics. Teachers were encouraged to reflect on any changes in their practice and on effects on children's learning of mathematics. Data related to these evaluations of project outcomes appear elsewhere (for example, Ferguson, Cheeseman, & McDonough, 2018).

In the first stage of the project 22 mathematical problem solving activities focused on the Number strand of the curriculum in Foundation and Year 1 (ACARA, 2013) were written by the author as lesson ideas suitable for children 5 to 7 years old (Cheeseman, 2017). After three months in the project, teachers completed an online survey that asked about aspects of the research. The question reported here asked participants to "list any obstacles you have experienced trying to incorporate a problem solving style of mathematics in the classroom".

Teachers' written responses were analysed using a grounded theory approach (Strauss & Corbin, 1990). Qualitative data examined during the research project were used to build a theoretical view of the situation under study. Six categories emerged from the data concerning children, teaching pedagogy, planning, resources, tasks and time. These categorised data were given to the teachers for their review and feedback.

Findings and Discussion

Table 1 shows how each idea expressed by the teachers was categorised. The table is structured by the frequency of response. The findings will be discussed later under four sub-headings: children; teachers and pedagogical issues; planning matters; and time and resources.

Reaction of the Children to the "New"

Teachers in the FIiM project noted that young children who are new to school have just grasped the general school routines in mathematics. The view was expressed that problem

Table 1
Categorisation of Obstacles Teachers Encountered Implementing Problem Solving Tasks

Category	Responses of teachers (n = 24)
Children (8)	At this age, children are self-important and rightly so, but it does present a problem if they do not learn from each other.
	Students in the early years are new to problem solving in Numeracy.
	Overwhelming for some students- not used to failing and being a risk-taker in their mathematical learning
	As our students are Foundation, they still require a large amount of explicit teaching to support their knowledge and understanding.
	Children that need structure struggle with some of these activities.
	If students don't have certain skills needed for the task that can make it difficult for students to complete the task successfully.
	For some kids it can be too open
	Getting children to challenge themselves
Teachers and pedagogy (6)	Lost interest among some students who want to only explain their own reasoning.
	As they get started, it is hard to know when to step in and get a child going and when to hang back and see how they go.
	Our team is still experimenting with how to offer engaging, hands-on learning experiences that support and extend critical thinking but ensures that their knowledge is sound and correct.
	Behaviour
	One to one time with students and juggling 26 students each session. Conversations are the key elements and it is difficult to deeply engage with all of the students. Focusing on small groups is good, but it's hard not getting to all students.
	Experience with problem solving as a teacher to be able to execute in the classroom
Planning (3)	We are planning units of work in Mathematics and the inquiry tasks can seem a little separate but we are working on making them more central and linked to the curriculum demands we have when planning. I can see this improving already.
	Trying to link the tasks to our current planner and fitting them in.
	Trying to incorporate tasks with my other lessons that we have previously planned.
Resources (3)	We do need to improve the resources available to students so that they are reinforcing basic skills e g. number lines, counting grids, calculators so that students can make decisions for themselves about what will be the best way to solve the problem and what they need to do.
	Lack of resources (a school issue that we are working on)
	None really, maybe just lack of resources.
Tasks (2)	Some tasks too closed and students completed easily and quickly.
	Had to modify some tasks for the range of learning abilities in my class
Time (2)	I think it is about allowing an extended amount of time for students to truly inquire and try many different ways to solving these tasks.
	Setting up: Would love to organise double numeracy sessions

solving approaches can involve new routines and expectations. For example, one teacher commented: “Students in the early years are new to problem solving in Numeracy.” This comment reflects the perception that “new” can be difficult for children, yet many young children are also very excited by trying new things. In fact problem solving is not “new” to children starting school who problem-solve with mathematical ideas in their lives before school. Many examples could be given to illustrate ways in which young children’s lives are rich in problem solving where children make decisions about number, position and size. Often children’s prior-to-school experiences are in authentic measurement contexts (Cheeseman & Pullen, 2017). Clements and Sarama noted that children “learn to mathematize their informal

experiences by abstracting, representing, and elaborating them mathematically" (2011, p. 968). Research shows that young children have intuitive and informal capabilities in both spatial and geometric concepts, and numeric and quantitative concepts (Bransford, Brown, & Cocking, 1999).

Two teachers wrote comments that reflected their views of children as learners, "Children who need structure, struggle with some of these [problem solving] activities." In addition, "For some kids it can be too open." These comments raise important questions about the role of the teacher and the responsibilities of the learner. Many young children expect to be told exactly what to do by adults, especially teachers. Therefore, the expectation that they have to think things through for themselves takes time to establish in a classroom. Similarly, the view that some children will struggle raises the fact that there are different views of the need for struggle with mathematics. For some teachers of young children, it is important that mathematics is easy and fun, whereas for others it is important for children to concentrate and to persist so that they experience the joy of solving a problem after a struggle. In the FIiM project, I was encouraging children to welcome challenge and, as one teacher said, "This can take time. Getting children to challenge themselves, for example, during the "Count How Many" tasks - most children just resorted to the "easy" way of counting by ones."

As a group of professionals experimenting with teaching practice, we were also encouraging children to communicate their mathematical thinking. Teachers in the project were aware that young children are often egocentric and more interested in talking about themselves and their ideas than listening to those of their classmates. As one teacher put it: "Some students who want to only explain their own reasoning lost interest. At this age, children are self-important and rightly so, but it does present a problem if they do not learn from each other." Another teacher observed and commented "[children were] finding the answer but unable to explain their reasoning - impatient with [their] peers in group tasks and taking over rather than explaining." These comments reflect the personal and social skills required of children in collaborative problem solving where they are asked to cooperate, to speak, and to listen.

Other obstacles to the implementation of problem solving and investigations related to the demands that are made of teachers.

Teacher Factors and Pedagogical Issues

The teacher's role was part of the professional development. FIiM teachers were asked to think about and discuss the role of the teacher in a problem-solving classroom. Teachers were encouraged to: "establish the problem, maintain the mathematical focus, lead without telling, support and shape ideas, and help children to a solution" (Cheeseman, 2017, p. 3).

I acknowledge the role of the teacher in a problem-solving classroom is something that takes time and some experience to work out in practice. For example, a perceived obstacle noted by one respondent was a lack of "experience with problem solving as a teacher - to be able to execute it [the role] in the classroom." In particular, it takes determination to resist the temptation to direct the children's thinking or to tell them what to do rather than to lead without telling. Letting young children struggle to get started on a problem is something new for many teachers. One respondent noted the judgement needed by teachers: "As they [children] get started, it is hard to know when to step in and get a child going and when to hang back and see how they go."

The selection of a task was another difficulty, in addition to concentrating on their classroom behaviours, teachers were struggling to identify the "match" between the suggested tasks and the apparent learning needs of their children. The range of mathematical experience of young children is large (Young-Loveridge, 1989) and on entry to school children's mathematical knowledge, while substantial, is varied (Clarke, Clarke, & Cheeseman, 2006). Teachers adapted tasks to suit their children. For example, "Had to modify some tasks also had

to modify for the range of learning abilities in my class." This need to adapt tasks was seen as an obstacle.

It is interesting to examine the thinking behind another teacher's comment that it is "Overwhelming for some students - not used to failing and being a risk-taker in their mathematical learning." Perhaps in some ways this remark reflects a hesitancy to let children struggle. Or maybe it is acknowledging that it takes time for children to build persistence and resilience. It raises questions about how we support children to become more willing to take risks and more aware that effort is what is expected of them (Dweck, 2007). In the teacher's comment, not completing a task or finding a solution was equated with "failing". This raises the question: What are teachers' expectations of tasks? The same respondent went on to comment: "Some tasks [were] too closed and students completed [them] easily and quickly." Does a task have to be "just right"? Not too hard and not too easy? How is that achieved? In planning for problem solving, teachers were asked to consider launching a problem "for which the solution is not immediately apparent" (Baroody & Wilkins, 1999, p. 63). So, rather than selecting a problem that the children *can* do, teachers make sure they find a task that the children *cannot yet* do. Even then, for some students the task may not be a problem at all and may be completed easily; for other students the problem may be too difficult to get started. To address the range of thinking in the children, each task in the collection of FIiM materials had a suggested *enabling prompt* and *extending prompt* (Sullivan, Mousley & Zevenbergen, 2006). Perhaps this differentiation technique was new to many teachers. It is true to say that a range of mathematical thinking is found in mathematics classrooms and creates difficulties and obstacles for many teachers.

The place of problem solving in a balanced program was raised by one respondent:

> As our students are Foundation, they still require a large amount of explicit teaching to support their knowledge and understanding. As such, our team is still experimenting with how to offer engaging, hands-on learning experiences that support and extend critical thinking but ensure that their knowledge is sound and correct.

What does "a large amount of explicit teaching" imply about teachers' views of how children learn mathematics? The idea that if teachers explicitly tell children information they will have it "correct" seems embedded in this quote. How representative this view is I cannot say. Nor can I say exactly what "explicit teaching" means to this respondent. It might be interesting to ask teachers what their children know about mathematics on entry to school and how they think children learned this knowledge. In this way, it could also be possible to understand teachers' theories of learning.

A long-held quandary expressed by teachers of problem solving was raised by the comment: "If students don't have certain skills needed for the task that can make it difficult for students to complete the task successfully." Skills before problems or problems before skills, is the chicken and egg dilemma. Both cases can be argued. If children cannot yet count rationally (Gelman & Gallistel, 1978) how can they solve problems involving quantifying? Alternatively, what is the point of learning to count unless you need to count objects reliably? In a way, there is no "right" answer to this dilemma. Perhaps one approach is to do both: have children practice their skills through games and activities while presenting them with problems that are contexts in which to use their growing skill sets.

The idea that children need to be explicitly taught mathematics before they tackle problems was expressed to researchers in the *Encouraging Persistence Maintaining Challenge Project* (EPMC) (Sullivan et al., 2014) across the mid to upper primary years and into Year 8 level. It seems that many teachers are reluctant to offer children mathematical experiences first then discuss the mathematics and elicit the resultant learning.

One approach to introducing "challenging tasks" in the EPMC (Cheeseman, Clarke, Roche, & Walker, 2016) was to begin the problem of the day with an introduction that clarified the

meaning of the problem then the students were expected to struggle to find a solution. Perhaps a question to ask in the current research is: Are young learners able to meet problems the same way?

The role of the teacher in a problem-solving classroom involves *letting the children go* as is said in the vernacular, to get on with their thinking. At the same time, teachers need to monitor children to ensure that individuals can make a start after a reasonable time and, after they make progress towards a solution, listen to their mathematical thinking. Often this "between table teaching" (Clarke, 2004) happens with individual children in a conversational style (Cheeseman, 2018). Teachers manage such "dialogic teaching" (Wood, Nelson, & Warfield, 2001) in different ways. For many teachers the balance between extended conversation with a few children and fleeting interactions with others is hard to reconcile. One example of an obstacle was the difficulty of spending time with all:

> One-to-one time with students and juggling 26 students each session. Conversations are the key elements and it is difficult to engage deeply with all of the students. Focusing on small groups is good, but it is hard not getting to all students.

Time and Resources

The shortage of resources can be seen as an obstacle to incorporating new approaches to mathematics as exemplified by the comment: "Lack of resources [is] a school issue that we are working on." In addition, time and teaching materials for problem solving were mentioned, as was the need to integrate new problems into planned units of work:

> I think it is about allowing an extended amount of time for students to truly inquire and try many different ways to solving these tasks. We do need to improve the resources available to students so that they are reinforcing basic skills e.g., number lines, counting grids, calculators so that students can make decisions for themselves about what will be the best way to solve the problem and what they need to do so. At the moment we are planning units of work in mathematics and the inquiry tasks can seem a little separate but we are working on making them more central and linked to the curriculum demands we have when planning. I can see this improving already.

Experimenting with new learning materials and teaching approaches is not as straightforward as it may seem. In many schools, planners are constructed by teams of Year level teachers in advance of the school term. While planners create a sense of direction and an overview of intended learning, they also produce some inflexibility in the program. This constraint can be a limitation to a change initiative based on implementing teaching materials.

In Conclusion

Due to the limited number of participants in this study, the findings cannot be claimed to represent the profession broadly. However, the results are indicative of teachers' perceptions of obstacles faced when incorporating problem solving into their classroom practice.

The results presented here indicate that some teachers' views of young children and their learning of mathematics are obstacles to using investigative or problem solving tasks. Some teachers viewed 5 to 7 year-old children as passive recipients of "correct" mathematical knowledge. These views may reflect a transmissive theory of learning but this is implied rather than supported by evidence gathered during the research process. The view that young children need to be told what to do in mathematics may also be an obstacle to teachers' willingness to expect young children to solve problems in mathematics. In fact, the selection of tasks that require children to struggle, persist, and make independent decisions, requires teachers to view young children as active learners who can initiate mathematical thinking.

Other obstacles to incorporating problem solving are pedagogical matters. Posing problems in the mathematics classroom takes pedagogical knowledge and skill. The role of the teacher is different in problem solving classrooms as the traditional pattern of demonstrating a skill to

children then having them apply the skill is overturned. The pedagogical skill to: establish the problem, maintain the mathematical focus, lead without telling, support and shape ideas, and help children to a solution is complex and requires skill and judgement. These skills develop over time with experience. It can be rather threatening for some teachers to be placed in the situation where they feel like a novice teacher again.

In addition, reference was made to pedagogical advice to Early Years teachers advocating "explicit teaching". Whether "explicit teaching" means expository "teacher tell" approaches to the teachers surveyed cannot be determined but the term does seem to imply that inquiry by children is insufficient. It may be that system-wide pedagogical directives can generate uncertainty for teachers.

Some teachers pointed to organisational and management matters in primary schools that create obstacles to changed practice. The recognition that time, support, flexibility of planning, and resources are needed when experimenting with teaching practice was apparent in their responses. These factors could be considered as necessary in any educational change.

Understanding the views of teachers can be the first step towards overcoming their perceived obstacles. Convincing teachers of the value of experimenting with investigative problems in their classrooms may build their trust in young children's mathematical potential. With experience and success with problem solving tasks, teachers' pedagogical knowledge will be built. As was seen from some teachers' responses, the first place to start was with support, resources and planning because these organisational obstacles were relatively straightforward to overcome.

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"Plot 1 is All Spread Out and Plot 2 is All Squished Together": Exemplifying Statistical Variation with Young Students

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The idea of variation is a foundation of statistical reasoning, and many curriculum documents, including the *Australian Curriculum*, include variation in the learning required for the primary years. In this paper, we consider the design of activities that can exemplify the idea of variation for young students and investigate how students can use graphs to support discussions about variation. The use of appropriate contexts and the provision of physical experiences of the phenomena seemed to help students make sense of graphical representations and allowed them to discuss how variation was exemplified in the graphs.

Many phenomena involve attributes that are different under different circumstances. For example, men are, in general, taller than women (here the height attribute varies across two groups); within a group of women the heights of individuals will vary (here the height attribute varies within a group); and for a particular woman, her height may change over time (and here the height attribute varies temporally). The field of statistics, which has issues of variation at its heart, provides tools for dealing with such phenomena. The *Australian Curriculum: Mathematics* (*ACM*) acknowledges the role of variation within the Statistics and Probability strand and, beginning at Year 3 level, includes reasoning with data that involves "interpreting variations in the results of data collections and data displays" and continues through to allowing for the variation that might be present when determining lines of best fit for a scatter plot in Year 10 (Australian Curriculum, Assessment and Reporting Authority, 2018).

With variation a core concept for statistical understanding, it is important to investigate what activities allow students to learn about variation as a data phenomenon, and how certain representations can exemplify variation in meaningful ways. For young students it is vital to identify activities that build a foundation for understanding variation. The study reported in this paper had, as its focus, an examination of the kinds of experiences and activities that will allow students to make sense of variation, and whether or not students can make sense of and use representations that might help exemplify variation.

Background

Statistical Variation

As Moore (1990) pointed out, without variation there would be no need for statistics. Traditionally in the school mathematics curriculum, however, expectation, based on averages, has received more attention. Shaughnessy (1997) suggested this may have been related to the formula for the arithmetic mean being much easier to calculate than the formula

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for the standard deviation. Although much early research in statistics education followed this line (e.g., Strauss & Bichler, 1988), when classroom research began, it became evident that children's basic appreciation of variation emerges before their appreciation of expectation (Watson, 2005). That the *ACM* now recognises this, and includes reference to variation in year 3, puts pressure on primary teachers to provide meaningful experiences that exemplify the concept. Nevertheless, although the curriculum talks about variation for young students, it does not make clear the scope of what might be learned about with respect to variation.

There are three types of variation situations that are of interest in this study. The first, variation within a group, concerns situations where there is a single group and an attribute/variable that varies among cases in the group. An example of this might be measuring how far students' paper planes can fly: some will fly only a short distance while others fly further. The second situation is variation across groups, where there are different groups and one or more attributes/variables that are being considered and there are differences from one group to another. Comparing how much pocket money year 6 students earn with how much pocket money year 3 students earn is an example of such a situation. It should be noted that there will almost certainly be within-group variation as an additional phenomenon in these cases. The final situation is variation with time, in which an attribute may vary as time passes. As an example, consider how the height of a bean plant changes with time. This phenomenon may also arise in conjunction with within-group and across-group variation. These types of variation are familiar to those who study formal statistics at the tertiary level (e.g., Moore & McCabe, 1993); the question is whether they can be exemplified and comprehended in the primary years.

Exemplifying

To help students learn general principles they are often given examples that are specific instantiations of the principle, with the expectation that the general ideas will become evident as they work with the specific example/s. For instance, to learn about outliers and how they affect the mean, a teacher might have students examine a certain set of house prices as a specific illustration. As discussed in Chick (2007), one of the critical roles for a teacher is to choose and use examples that allow students to learn the intended principle. This involves being able to identify or design for key affordances (Gibson, 1977) within the chosen example and then use it in the classroom, so that the example succeeds in exemplifying the principle. Achieving this can be difficult; Chick and Pierce (2012) showed that preservice teachers struggled to design lessons that effectively used a specific data set that had many affordances for teaching general statistics principles.

In addition to the issue of using situations to exemplify general concepts, there is another kind of exemplifying that is relevant for this paper. This concerns how students exemplify the evidence underpinning assertions that they make. Among studies with upper primary students, Watson and Moritz (1999) found that students could use supplied graphical representations to exemplify or support their assertions about differences between two classes' maths scores. Chick and Watson (2001) examined how students could use their created graphs to exemplify claims they made about the data, finding that some students could exemplify the situation with sophisticated representations that they interpreted effectively. There has been less work in this area with younger students.

Research Aims

This paper focuses on some of the issues surrounding the design of scenarios that allow exemplification of the concept of variation, and how to make the concept visible to young students. In addition, the report explores whether students can recognise that graphs might exemplify the variation in phenomena they have experienced, and how they talk about what they see in the scenarios and the graphs.

Scenarios for Exemplifying Variation

The two scenarios to exemplify variation used for this report were devised in the first year of a 4-year research project using data modelling to enhance STEM in the primary curriculum. Previous research (e.g., Watson, 2005) suggested it was essential to make variation explicit for students, a phenomenon they experience daily in many diverse ways.

Licorice scenario

The licorice scenario was adapted from earlier work with teachers and young children (Watson, Skalicky, Fitzallen, & Wright, 2009). Students used Play-Doh™ to make licorice sticks in two ways: by hand and with a Play-Doh™ Extruder. The requirement was that the sticks be 8 cm long and 1 cm in diameter. Before students began making the sticks with this "factory" and by hand, the word "variation" was introduced, and examples from the students' experiences discussed. To compare across the two ways of making the sticks, all sticks (three of each type per child) were weighed and their masses recorded on sticky-notes. Within each method, the variation in the masses was discussed, with students giving reasons for its occurrence (e.g., care with the ruler, reading the scale carefully). The researchers then needed to represent the information visually to reinforce the variation within each method's data and the difference in the variation between the two methods, for students who had, until then, only experienced bar graphs as a form of data representation.

Heat scenario

The second activity was based on the concept of heat in the Year 3 Science curriculum, using the measurement of temperature and elapsed time, which were new experiences for most students (Fitzallen, Watson, & Wright, 2017). The concepts covered were more complex than for the Licorice activity. Insulated and non-insulated plastic cups were filled with hot water and placed in a trough; measurements of the temperature in the two cups were taken every 5 minutes for 30 minutes; 10 minutes from the start, ice water was added to the trough and its temperature was also measured for the remaining 20 minutes. Students recorded the temperatures in a table and described the change in workbooks. The question for the researchers was how to represent the variation present for the students who had no experiences with Cartesian graphs; the resolution is shown in the Results section.

Method

Participants

The activities were carried out in two year 3 classes (students about 9 years old) in a parochial school in Tasmania, with data collected from 48 students for Licorice and 49 students for Heat. The teachers taught the lessons for the activities with notes provided by

the researchers. Four members of the research team (including the second and third author) were present at each activity to assist with the materials and supervision of the group work.

Data Sources

The comments about students' interactions with the actual physical situations—making licorice and observing changes in temperature—are based on the field observations of the research team (including the second and third authors). For the licorice scenario, the students had a workbook with guiding questions for their observations during the activity. After making the sticky-note plots shown in Figure 1, one workbook question asked them to list the differences between the two plots. Their responses provided the data for some of the results. Data for the heat scenario came from transcripts of class discussions recorded on video. These class discussions had input from both teachers and researchers.

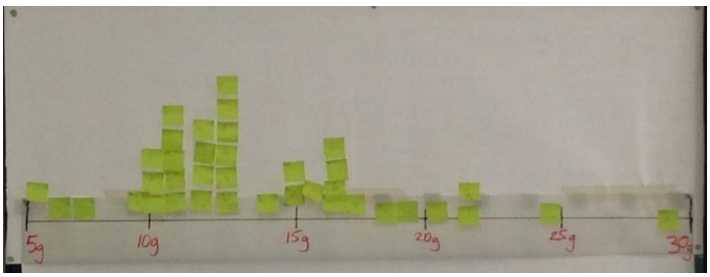
Results

For each scenario, we will first report on the students' reactions to the tasks. We then consider how they described and interpreted the variation from graphical representations.

Licorice scenario

Some students were already familiar with the Play-doh™ Extruder and there was great interest in the activity. Students were very careful in making the hand-made licorice sticks, trying to achieve even thickness and equal length. Students used electronic scales to record the masses of the licorice sticks, which provided the data for their discussion comparing the factory-made and hand-made situations.

As seen in Figure 1, stacked "dot" plots were used, with students selecting a sticky-note with one of their data values and placing it on the plot. The teacher began by asking for the heaviest and the lightest masses and used them to determine the scale on the axis. As this was a new experience for the year 3 students, guidance was needed at the start, both in working out the distances between the labels and in creating the stacks.



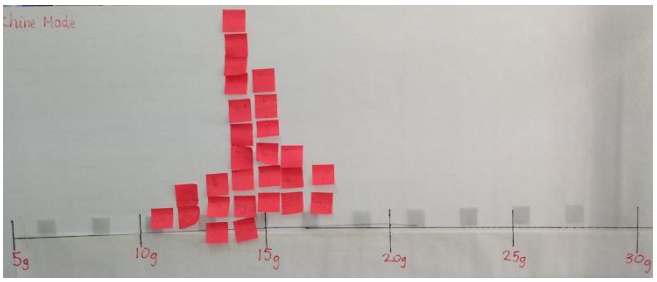


Figure 1. The stacked sticky-note plots, showing the mass data from the hand-made licorice sticks (top) and the factory-made licorice sticks (below).

The students were asked to list any differences between the two plots in Figure 1. One student did not clearly refer to the graphs to discuss the variation, but certainly identified the variation that he/she saw between the groups [spelling errors and punctuation have been corrected in all quotations]:

> The ones that we made with our hands were very different but the ones the machine made were a little bit different.

Other students talked about the spread of the values in the graphs, and the way these varied, using age-typical language that captured the contrasting ideas of "spread out" and "compressed", as illustrated by a sample of such responses below.

> The hand-made class plot is very spread out with the masses. As for the factory-made class plot, all the masses were mainly the same but are a bit different.

> The hand-made one was spread out and the factory-made one was stacked on top of each other.

> Plot 1 is all spread out and Plot 2 is all squished together.

> The first one [hand-made] is like all around the place and the second one is straight in the middle.

> They all are spread out on the handmade one, and all are bunched in on the machine-made one.

Another set of comments additionally referred to specific values from the data, often giving an indication about modal values or the range.

> The one with the machine most were the same, the others were different. When you make it handmade it is bigger. Handmade one has 28 as the highest.

> One was in between 10g and 16g and the other was in between 6g and 28g. The machine is more accurate.

> There are more different weights in the handmade one. Most people have 14g on the machine-made one. Nobody on the machine-made one had 5g and one person on handmade did.

Some students described the graphs in terms of buildings and described the variation in terms of the shapes of the sets of buildings.

> The handmade is like a city with homes spread out and factory-made is the same except the factory-made is with tall buildings, not many but tall buildings.

> (1) 14 is more common in machine-made. (2) In machine-made they are more closer together than handmade. (3) Handmade is spread out and machine is more close together. (4) Handmade is like a city but machine-made is like a tower.

Finally, some students seemed to have picked up on the language used by the teachers, describing the differences between the graphs using the word "variation" itself.

> Factory-made had a larger typical number. Hand-made had more variation in their mass.

> There aren't many variations with the factory-made licorice and there is a large variation with the hand-made licorice.

> The plot with the machine-made things go straight up. Machine-made things don't have much variation.

In the quotations above it is evident that students not only noticed that the two groups were different, but that it was the contrasts in distributions—or the differences in the ways each group varied within the group—that characterised the difference. Moreover, students were able to refer to features of the graphs to exemplify the variation between the groups.

Heat scenario

The heat scenario was much more complex than the licorice one, with variation evident over time, across the three conditions, and across the data from different student groups. Because students had used thermometers to collect the data, it was decided to provide a plot made up of thermometers (a y-axis) for each time (x-axis) that measurements were made (e.g., see Figure 2). Students finally transferred the data to the graph in Figure 3.

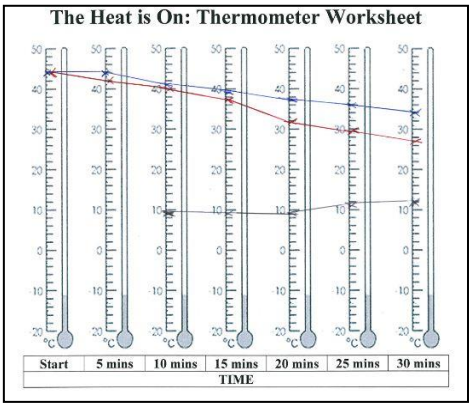


Figure 2. Graph created to show the temperature variation over time for one student group: top line is insulated cup, middle is plain cup, and bottom line is the ice water in the trough (note the use of repeated representations of a thermometer to record the temperature values over time).

The class plot of the data from each group (Figure 3) served two purposes. First, it was a way to include all students in the summary, as each group member put up the dots for one set of measurements from the group. Second, it showed the "between" group variation across groups. Although there was discussion about differences among the groups—for example, the number of ice cubes making some water mixtures colder or judging the time exactly for the measurements—the main aspects of variation that students discussed were changes over time and the difference between the insulated and non-insulated cups.

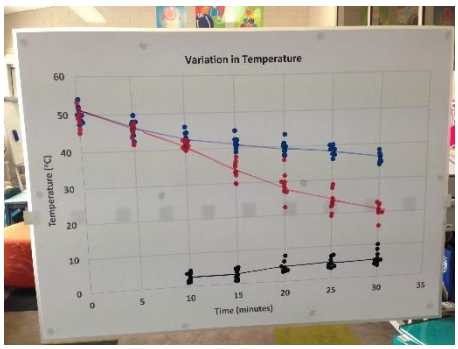


Figure 3. Graph showing the results of the heat experiment for all the groups, showing variation among conditions (top line: insulated cup, middle: plain cup; bottom: ice water), over time, and across groups (dots).

Students were able to talk about the variation that they noticed over time, together with the variation that they noticed across the conditions (plain cup, insulated cup, ice water), and

could do so in reference to the graph. This is illustrated by the following quotations from the class discussion in response to various teacher and researcher questions.

> The blue [Fig 3, top line, insulated cup] has gone … staying … and a bit lower. And the red [middle line, plain cup] has gone straight down and the same bit and the cold ice has gone same and up.
>
> That the black – the black line with the – the ice water was the most consistent […] And the non-insulation was the most [long pause] – had the most variation.
>
> That there's a lot more variation [for the red line] than the blue line.
>
> It [the red line] goes down a little bit more than the blue.

When questioned about the variation across students' groups (resulting in different temperatures for the same time and conditions), students offered plausible explanations.

> Because from 10 minutes to 30 minutes the ice could have – there could have been more ice in one to make it colder and less ice when they get melted.
>
> Because of variation if he [the researcher adding hot water to the cups] came around, tipping the water in the thing [trough] he could have […] put a little bit more in one and less in the other.
>
> Some of the water could have been out of the [heating jug] longer than other parts – other could have been a bit warmer than the one that you poured out first – could have cooled down a bit quicker.
>
> Also, because are sitting near the windows and cool air comes …

The students' comments indicate that they had noticed the obvious changes with time, and the differences in cooling for the plain cup and the insulated cup (and the trough). In addition, they were able to recognise that there was variation across the data from different groups' experiments and offer sensible reasons for this.

Discussion and Conclusion

For those experienced in reading graphs it is obvious that the situations exemplified variation: Figure 1 clearly shows variation within the groups and a difference in variation between the two groups (hand-made and factory-made); Figure 2 shows variation among three conditions (plain cup, insulated cup, and iced water) as well as variation with time; and Figure 3 shows, in addition, the variation among data obtained by different groups of students (the different dots around each time point for a given condition). The careful design of the activities and emphasis on variation as the key focus of the activities also allowed this variation to be evident to year 3 students. Despite their young age, they could notice variation over time, variation among situations, and, very specifically in the licorice task, variation (i.e., differences) in the within-group variations present in each of the two groups/situations. Students were also able to notice and explain the variation across different groups of students in the heat scenario. Their awareness of these aspects of variation was governed by the variation-inherent activities and the in-class discussions that they had. They were able to use their own and newly-learned language to describe the differences that they saw.

Interestingly, later statistical work often focuses on differences in means, and the importance of potential differences in the variation among groups gets lost in the formulas. In the licorice activity the expected "typical" value was the same for each method; here the difference between the factory-made and hand-made cases really is made evident by observing the differences in variation between the two groups. There are many statistical situations where it is the variational difference that matters; this research demonstrates that even young students are capable of noticing and making sense of such differences.

The success of the activities relied not only on the exemplifying power of the activities themselves but on the way in which discussions were scaffolded (see Chick, 2007, for more

about teacher knowledge for successful implementation of examples). Importantly and additionally the graphs could be and were used to exemplify the phenomenon. This required a clear focus for the teachers on variation as a learning outcome. There can be risks associated with the implementation of such activities where the focus can diminish to the "fun" of the activities and fail to progress beyond this to the variation and graphical ideas. Chick and Pierce (2012) suggest that, in teaching, there can be a tendency to do the active hands-on part of the work, but not have a deep consideration of the concepts. In this case, however, the tasks seem to have been implemented in a way that allowed exemplification of a number of different types of variation, and also empowered students to exemplify or give evidence of the variation that they observed. Although the tasks had complex, multi-faceted aspects most of the students were able to talk about variation in meaningful ways and saw variation as a concept that allowed them to describe change and differences in data-rich situations.

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Solution of word problems by Malaysian students: Insights from analysis of representations

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Within the broad area of whole number operations, understanding and solving word problems continues to be an important area of inquiry. In the present study we draw on the framework of representational fluency to examine conceptual and procedural understandings that are exhibited by a group of Malaysian seven-year-olds as they attempted to solve 2-digit addition and subtraction word problems. Preliminary results show that the range of representations that were constructed by the participants as they searched the problem space was limited as was their ability to translate representations. Implications of these findings for further work about using representations are discussed.

Introduction

In its discussion about skills and mathematics competencies, NCTM Principle and Standards of School Mathematics (2000) has called for increased attention children's understanding of whole numbers and the use of this understanding to interpret and solve word problems. The solution of word or story problems is an important part of most primary and early-childhood mathematics curriculum. Despite recent instructional advances in practice, this area of mathematic learning continues to present considerable challenges to many students because multiple steps are involved in the solution process. Students have to read and understand the text. They will then have identify key parts of the text that are relevant to decoding the problem and developing the solution. In order to make these series of steps in the solution process explicit, we need a tool that is sufficiently context sensitive. In the present study, we draw on the framework of representation to track how students negotiate the word problem-solving environment.

The aim of our larger study is to document the range of representations that young children could construct and the level of fluency they exhibit in articulating the links among these representations. In this report we provide preliminary data about the representational range as a cohort of children attempted to solve two problems that involved 2-digit numbers.

Background to the study and research problem

Understanding whole numbers and operations are fundamental to children's number sense and their ability to apply those skills in making judgements and doing calculations. Studies of students' problem-solving competence with numbers and operations tend to identify a broad range of computational strategies that students could use. Counting up, Doubles and Bridging to 10 are examples of strategies that children could use when solving addition and subtraction problems. While the use of a particular strategy is indicative of students' competency with operations, such information is not useful for a) understanding

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why that strategy was preferred, b) why the strategy works and c) how to assist student when they apply the strategy incorrectly. We argue that the latter issues can be tackled head on by asking students to construct multiple representations of their strategies and reason the links between these representations – representational fluency. Analysis of strategy use from a representational fluency perspective can be expected to provide deeper insight in the lack of computational flexibility that has been reported as a continuing problems with children (Perry & Dockett, 2008).

In her study of Chinese and American teaching of elementary mathematics, Ma (1999:112) commented that

> 'being able to calculate in multiple ways means that one has transcended the formality of the algorithm and reached the essence of the numerical operations -- the underlying mathematical ideas and principles. The reason that one problem can be solved in multiple ways is that mathematics does not consist of isolated rules, but connected ideas. Being able to and tending to solve a problem in more than one way, therefore, reveals the ability and the predilection to make connections between and among mathematical areas and topics'.

Thus, the examination of alternative models of a solution or a solution strategy constitute a productive line of inquiry.

Over the past decade, considerable research ground has been covered in examining strategies used by students when performing the addition and subtraction, computations with non-contextualised problems (Blöte, Klein & Beishuizen, 2000; Russell, 2000; Verschaffel, Luwel, Torbeyns & Dooren, 2009). However, computations with contextualised problems introduce another layer of processing that involve interpreting and translating the text. In a more recent work on computational skills and problem solving, the added cognitive demands posed by the extra layer of linguistic information that has to be processed in word problem was acknowledged by (Fuchs, Fuchs, Hamlett, Lambert, Stuebing & Fletcher, 2008). According to Fuchs et al., (2008), while abstract computation problems are 'set up' for solution, word problem require students to analyse the text in order to identify missing elements and generate an equation that embodies the relationship between the given and missing elements. Thus, there is a need to unpack the relations between different elements in the problem environment of word problems.

Within the broad field of whole number concepts and operations, students' understanding and solution of addition and subtraction word problems have been the subject of sustained inquiry (Fuson, 1992; Verschaffel & De Corte, 1997). In an extensive review of literature, Verschaffel, Greer and De Corte (2007) commented the complexities underlying children's ability to solve arithmetic word problems must be examined both from a cognitive and socio-cultural perspective. The goal of our larger study is to gain insights into the effect of Malaysian culture in moulding students' conceptual understandings and problem-solving ability with arithmetic word problems. In so doing we aim to examine the range, flexibility and connectedness of representations that Malaysian children use in solving the above category of problems. In the report here we present selected data relevant about the type of representations and evidence of translation among representations.

Theoretical framework

Representations and fluency

The construct of representation has been in currency among researchers for a considerable time. Defined broadly, representation refers to the depiction or portrayal of a mathematical concept or entity. This definition suggests that a mathematical concepts can

be given multiple representations. Indeed, the ability to construct multiple representations have been argued to provide a powerful window into children's depth of understanding (Schoenfeld, 1985; Barmby, Harries, Higgins, & Suggate, 2009). In the present study, we use the term *representation* to refer to 'external (and therefore observable) embodiments of students' internal conceptualizations (Lesh, Post & Behr, 1987: 34). In so doing we draw a distinction between internal and external representations. Internal representations reside in the long-term memory, and as such, are not observable. The manifestation of content and structure of internal representations in an observable mode constitutes external representation. *Representational fluency*, on the other hand, is indicative of translations between and within modes of representations. *Representational fluency* has been shown to be critical in building students' mathematical understanding (Goldin & Shteingold, 2001; Kaput, 1989).

Consistent with the above framework, the generation of representation is a precondition for demonstration of representational fluency. Thus, studies of representation fluency need to generate rich data about types of representations in the first instance before the question of fluency could be entertained. For example, a concept could be represented in different modes: tables, texts/verbal descriptions, graphs, symbols/notations and concrete/pictorial. For example, the addition of two whole numbers can be represented on a number line or as total of two groups of coins. Further, by relating the different components within and between representations, students could demonstrate the fluency there in. Skemp (1976) argued that connections among and within representations could provide insight into students' *relational* and *instrumental* understandings.

Research Questions

The following questions guided data generation and interpretation:

What is the range of representations that Malaysian children use in the solution of word problem involving addition?

What is the range of representations that Malaysian children use in the solution of word problem involving subtraction?

Methodology

Design

This was a descriptive study aimed at observing, recording and analysing participants' responses to a series of word problems.

Participants

Twenty six students from three regular intact Malaysian mathematics classrooms participated in the study. The age of children ranged from 7-8 years. All students were fluent in the Malaysian language – Bahasa Malaysia. The children had completed topics on addition and subtraction operations before the commencement of the study.

Tasks

In developing tasks that would assist us generate data relevant to the research questions, we were guided by two key design principles: a) the tasks were sensitive to the generation of multiple problem representations by the children and b) addition and subtraction problems had *part-part-whole* and *change* structures respectively (Carpenter, Fennema, Franke, Levi

& Empson, 1999). We chose problems with structures that were relatively easy as our focus was on representation and fluency. We selected 6 items from a pool of 2-digit addition and subtraction word problems from texts books that were used in Malaysian primary schools. The problems were anchored in Malaysian real-life contexts so that children could better relate to the meaning underlying the numbers. These problem were given to regular classroom teachers for their comment about the wording and authenticity of contexts. In this paper, we report students' responses to the following two problems (Table 1) which are English translations of the original problem which were in Bahasa Malaysia.

Table 1 – Focus Problems

Addition Problem	My brother went to a grocery store. He bought 8 apples and 7 oranges. How many pieces of fruit did my brother buy?
Subtraction Problem	Mr Smith has 13 computers in his store. He sold 5 computers. How many computers does Mr Smith have in his store now?
Subtraction Problem	Sally keeps 51 chicken. She sold 24 of the chickens at the market. Find the number of chicken that Sally has now?
Addition Problem	The Columbus Zoo has 36 seals and 47 penguins. Find the number of animals in the seal and penguin pools at the zoo.
Addition Problem	123 people visit the Butterfly exhibit on Monday and 98 people visit the Butterfly exhibit on Wednesday. How many people visit the Butterfly exhibit on both days?
Subtraction Problem	Alina bought a pair of shoes and a bag for 143 dollars. If Alina had 200 dollars to pay for these, how much change will she receive?

The researchers analysed problems in Table 1 with the view to generating representations that are based on students' learning experiences that were guided by the Malaysian national mathematics curriculum. This exercise yielded three broad categories of representations of word problems: verbal, visual, symbolic and/or algorithmic. These representations were given to the regular classroom teachers for comment before using it in our analysis.

Procedure

The primary source of data was one-to-one interviews that were conducted within the school environment during regular school hours. Prior to the interview, the teachers had briefed the students about purpose of the study and its value is helping them learn mathematics. The tasks were presented in Bahasa Malaysia, the medium of instruction in Malaysian national schools. Children were asked to solve the problem. Once they had produced a solution, they were asked to solve the problem in a different way. Following their second solution, children were encouraged to provide a third solution. When they commented that they could not think of other solutions, our final prompt was to get them to explain the relationships among the solutions that they had generated.

Concrete aid such as counters, graph papers, Unifix cubes and base-10 mats were provided for children to use. Our expectation was that the availability of multiple aids would assist them to formulate a range of solutions. The maximum duration of the interview was 30 minutes with most students completing the task within 20 minutes. All sessions were video-taped and transcribed for analysis. Our first level of data analysis involved the identification of frequency of three major categories of problem representations.

Results

Types of representation

Consistent with our conceptualisation of representational fluency, our first aim was to generate data about representation type before the question of fluency could be addressed. We report the percentage figures for representation types for addition subtraction problem (Table 2).

Table 2: Problem Representation

Representation		Addition Problem (%)	Subtraction Problem (%)
Verbal		8	0
Visual	Counters/Unifix cubes	25	21
	Number line	10	6
Symbolic/Algorithmic		46	62

Table 2 shows that in both problems, the Symbolic/Algorithmic representation dominated students' search in the problem space. The columns add to 89% because there were cases of students whose representations could not be coded as one the three above. A closer analysis of the algorithms indicate that children tended to favour the use of vertical algorithms which are commonly taught in their classroom. It is encouraging to note that a high percentage of representations that involve visuals was used in representing both the problems. As expected, a low percentage of the children could represent the problem verbally.

Representational fluency

Representational fluency, we have argued above, involved the construction of links among and between representation. Figure 1 shows an example of representation by one of the children, Ali (pseudonym) for the Addition problem (36+47) by using counters.

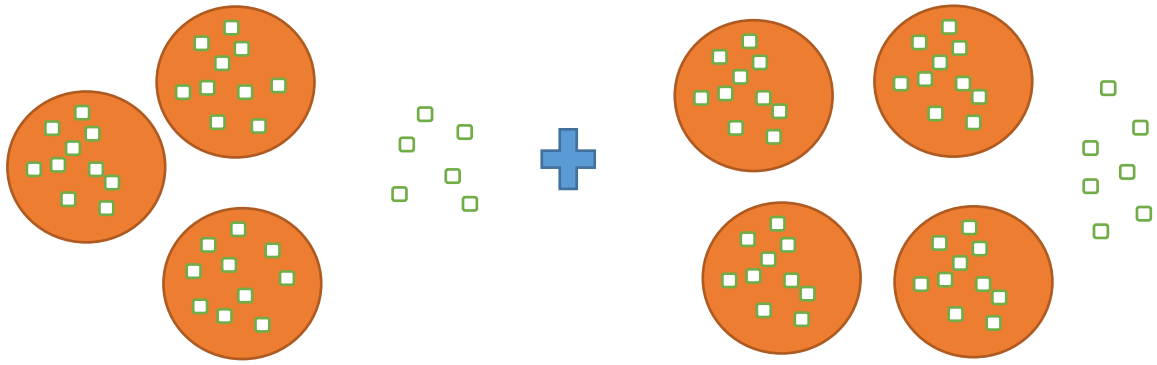

Figure 1

The above representation by Ali is indicative a number of key elements of his representation. Firstly, he was able code the problem as involving addition operation (use of addition symbol). The way the counters have been used suggests that Ali understands the base-10 system, where 36 is decomposed as 3 tens and 6 ones. He went on the work out the total as seven tens and 13 ones which was regrouped into eight tens and 3 ones (83). The above visual representation was then related to his vertical algorithm for adding 36 and 47. In being able to reason about the links between the components in Figure 1 with his vertical algorithm, he displayed representational fluency and translation between two representations. However, only 9% of the students could explicitly relate representations.

Discussion

The aim of the study was to analyse word problem-solving competencies by documenting preliminary evidence of problem representation and representational fluency among a cohort of Malaysian primary school children. Our assumption underlying this study was that a representationally-based analyses of students' problem-solving performance can be expected to provide insightful information about students' sense-making and understanding. In making progress with the given set of problems, children had to work with a number concepts within each representation and the shift they had to make between representations was not trivial. Representational fluency requires that children understand the conceptual underpinning of the problem and associated operations. Concepts such as place value and groupings are essential elements of such an understanding.

The predominance of use of Symbolic/Algorithmic representation was not totally unexpected as this mode is privileged in regular classroom instruction in Malaysia. The Malaysian mathematics curriculum is, by and large, examination driven. Success in these examinations require children's ability to produce correct answers rapidly. Algorithms as a form of representation are valuable tools but children need to understand the conceptual basis of such representations, a point that was highlighted analysis of procedural knowledge by Star (2005).

The results of this early study are encouraging in that we see evidence of children generating visual representations of solution of both the addition and subtraction problems. Counters were the main concrete aid that children used to depict ones and tens for the numbers. The processes of decomposition and regrouping of numbers with counters during the construction of representation provided windows into children's understanding of the

decimal system. Malaysian culture, in the main, does not value or support talk in formal and informal contexts. We note, this aspect of their culture in the relatively low verbal representations.

In framing the research questions of the present study, our theoretical orientation was representations and fluency. The construct of representations provides a sensitive lens through which to better understand the interplay of socio-cognitive processes that underpin Malaysian students' understanding and solution attempts of word problems. As Kaput (1989) suggested, each representation of a problem could highlight or hinder an aspect of students' understanding. Thus, the opportunities for students to construct multiple representations of word problems will increase the likelihood of generating data that reveals not only the depth of students' understanding but also the interaction between socio-cultural and cognitive elements during the course of solution development. Toward that end, our future work will aim to develop word problems that are germane to the generation of multiple representations.

This is a preliminary report which forms part of our on-going larger research the aim of which is to understand Malaysian children's representations and associated flexibility or fluency. There are number of variables within a problem that could impact on the level and quality of representations including symbols, language and problem structure (Carpenter et al., 1999; Kintsch & Greeno, 1985). Our future work in this space will examine these factors more closely. Additionally, in the current report we have not provided data on the question of fluency which involves students reasoning among representations. As this is a pilot study, we are guarded in making general claims about representational fluency among Malaysian children that is based on limited data. However, the methodology and an initial data analytic procedures does provide directions for future debated and discussion for future work in this area.

The results of the study has implications to promote teachers' instructional agenda in their mathematics classrooms. It would seem that current priorities are mainly concerned with getting children to perform operations and produce correct answers to problems. There is a need to shift this algorithm-driven approach to engaging children that supports experimentation and experience with multiple problem representations (Lewis, 1989), and investigate why representations are related. Teaching for both skill development and conceptual understanding is crucial - these are learned together, not separately. But teaching in a way that helps students develop both understanding and efficient procedures is a complex and challenging task (Lawson & Chinnappan, 2015; Chinnappan & Forrester, 2014). Such an enterprise requires that teachers develop a sophisticated understanding of the basic mathematical ideas that underlie representational fluency, use tasks in which students develop these ideas, and recognize opportunities in students' work to focus on these ideas.

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Orchestrating Mathematics Lessons: Beyond the Use of a Single Rich Task

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Teachers have several challenges when designing and implementing mathematically-rich tasks, and hence, these tasks are not prevalent in many mathematics classrooms. Instead, teachers often use *typical problems*, such as standard textbook tasks and examination questions, to develop students' procedural fluency. This begs the question of whether, and if so, how teachers can think about, and use these typical problems differently to develop conceptual understanding. In this paper, we report findings drawn from a two-year design-based research project and highlight two teaching vignettes to illustrate how typical problems were used to orchestrate instructional activities. Our findings suggest three important principles for teachers to consider when using typical problems.

Orchestrating discussions around mathematically-rich tasks (Grootenboer, 2009) can potentially enhance students' learning experiences by providing opportunities for students to reason about, and communicate their mathematical ideas (Smith & Stein, 2011). To support teachers in developing their competencies in this high-leverage teaching practice, Smith and Stein (2011) proposed five inter-dependent practices—anticipating, monitoring, selecting, sequencing, and connecting—which hinge on a single high cognitive-demand task. However, the use of high cognitive-demand tasks in the classrooms remains relatively infrequent, and often problematic. For example, Kaur (2010) suggested that teachers in Singapore may prefer to use standard examination-type questions, or what we termed as *typical problems* (Choy & Dindyal, 2017) during day-to-day teaching. This is not surprising because teachers in an examination-driven education system, such as Singapore, may believe that it is "important to prepare students to do well in tests than to implement problem-solving lessons" (Foong, 2009, p. 279). Furthermore, mathematically-rich tasks have a high entry-point for students, and teachers have to provide additional support or prompts for students (Sullivan et al., 2014). These factors limit the use of rich tasks as teachers may find these tasks time-consuming and pedagogically challenging to implement. While acknowledging the importance of mathematically-rich tasks in mathematics lessons, we also wonder whether typical problems have a role to play to develop conceptual understanding, and if so, how can these problems be used to orchestrate instructional activities? In this paper, we draw data from a bigger study to describe two teaching vignettes of an experienced teacher, Alice (pseudonym), and highlight how she had used typical problems differently to develop relational understanding (Skemp, 1978).

Orchestrating Instructional Activities

Instructional activities refer to the ways in which "teacher, content, and diverse students would interact within work on authentic problems, how materials of instruction would be used, how the space would be arranged, and how the teacher would move around the room" (Lampert & Graziani, 2009, p. 493). As argued by Lampert, Beasley, Ghousseini, Kazemi, and Franke (2010), these interactions form part of the core high-leverage teaching practices

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 234-241. Auckland: MERGA.

needed to enact *ambitious teaching*, which focus on responding to and building on students' answers as they work on problem-solving tasks. The idea is to *co-construct* instructional explanations and dialogues with students (Lampert et al., 2010). Co-constructing mathematical conversations with students places tremendous demands on teachers, and teachers would need support to do this work. A key strategy to break down the complexity of this classroom practice is then to use *routines* to make some of this work "automatic", as highlighted by Leinhardt and Steele (2005, p. 142).

Lampert et al. (2010) see these routines, particularly the exchange routines (Leinhardt & Steele, 2005), as the foundation of ambitious teaching. They highlight three exchange routines—*call-on, revise, and clarification*—to be one of the means that make the work of orchestrating discussions learnable by novice teachers (Lampert et al., 2010; Leinhardt & Steele, 2005). First, the *call-on* routine seeks to invite students to respond to a problem, and is followed by an exchange involving analysis of, justifications for, and critiques of ideas by the other students. Next, the *revise* routine facilitates students to rethink ideas put forth by their peers before they explain their new ideas. Finally, the clarification routine seeks to understand the source of any confusion regarding the ideas discussed.

Another important aspect of preparing teachers to orchestrate discussion is recognise that certain aspects of this teaching expertise can be planned. To this end, Smith and Stein (2011) suggested an instructional sequence, where teachers use five practices around a single rich task in which students attempt, present, and discuss the mathematics embedded in the task. The crux of orchestrating discussions is to purposefully select students' work so that both important mathematical ideas as well as common misconceptions are addressed. This provides opportunities for the teacher to share useful alternative strategies that were not presented by the students. The motivation for carefully selecting and sequencing responses is to lay the groundwork for the teacher to connect these different responses to important mathematical ideas. By directing students' attention to the connections between different strategies, and by shifting their focus from solutions to mathematical ideas, teachers can begin to support students' efforts in understanding the concepts targeted in the lesson (Smith & Stein, 2011).

The idea of focusing on connections reflect a *connectionist* orientation (Askew, Rhodes, Brown, Wiliam, & Johnson, 1997), which emphasises connections within mathematics when teaching numeracy. According to Askew et al. (1997), a connectionist numeracy teacher strikes a balance between a transmission orientation and a discovery orientation, and is more likely to be more effective in the classrooms. Although these beliefs pertain to the teaching of numeracy, we have found these beliefs useful in explaining the teaching practices of the experienced teachers in our study, who had exploited the use of typical problems to orchestrate discussions.

In our earlier paper (Choy & Dindyal, 2017), we described how Alice, an experienced teacher, orchestrated a mathematically productive discussion (Smith & Stein, 2011) by carefully attending to students' answers to a typical problem before she asked for volunteers during the whole class discussion. Alice's orchestration of instructional activities differed from the five practices in two important ways. First, Alice used a selection of four contextual questions on matrix multiplication, taken from past-year examination papers. This stands in contrast to Smith and Stein's idea of using a single rich task for the lesson. Second, although Alice's way of orchestrating discussion seems to reflect the five practices, Alice interjected to explain the connections in between the different solutions, instead of connecting the solutions at the end of the presentation. This provided opportunities for her to emphasise the *connections* between matrix multiplication and arithmetic to provide meaning to matrix

operations in between students' presentations. Hence, Alice kept the concept in focus and ensured coherence in the discussion by *co-constructing the explanations* for the different approaches with her students (Lampert et al., 2010). In this paper, we build on our previous paper to describe two more vignettes, taken from different lessons conducted by Alice, to build up a more complete picture of how typical problems can be used to develop conceptual understanding.

Methodology

The data reported in this paper came from a larger study on orchestrating learning experiences in a secondary school mathematics classroom in Singapore. We used a design-based research approach to develop a toolkit for our teachers as a means of supporting their orchestration of learning experiences, as well as to develop a theory about teachers' productive noticing (Choy, Thomas, & Yoon, 2017) in the context of orchestrating learning experiences. Details on how we worked with the teachers can be found in Choy and Dindyal (2017). Besides audio-taping the pre-lesson and post-lesson discussions, we also made video recordings of the lesson and collected lesson artefacts used during the lessons. The recordings were transcribed, and segments related to the major divisions of the lessons were identified and analysed. The findings were developed through identifying themes related to the five practices, as envisioned by Smith and Stein (2011), and the notion of routines used in orchestrating instructional dialogue (Leinhardt & Steele, 2005). In this paper, we examine the instructional activities of Alice, a Senior Teacher at Coventry Secondary School (pseudonym), which is a government-funded school. As a Senior Teacher, which is an official appointment in the school, she has demonstrated strong content knowledge, familiarity with the national curriculum, and strong pedagogical content knowledge. For each of the vignettes, we will describe briefly the context of the lesson, and highlight how Alice's orchestration of instructional activities using typical problems reflect a connectionist orientation towards teaching.

Two Teaching Vignettes

Techniques of Differentiation

This lesson for Secondary Four (Grade 10) students focused on developing procedural fluency in differentiating real-valued functions of one variable using one, or more of the following formula: (a) the "basic" rule ($\frac{d}{dx}x^n = nx^{n-1}$); (b) Chain rule; (c) Product rule; and (d) Quotient rule. On the surface, Alice's lesson appeared to focus solely on developing skills, but closer examination reveals that she was deliberate in the selection of the functions for differentiation. In particular, Alice wanted her students to develop the reasoning skills to determine which of the rules is most efficient for a given function. As each of the functions lend themselves to be differentiated using a variety of methods, her choice of functions afforded opportunities for students to focus on the structure of the given function. In the following exchange, we see Alice's discussion with the class after students had worked through a series of questions together in groups.

294 Alice: So, I want to look at a few questions, like this one. (Proceeded to write the following on the board)

$$y = \frac{(2-3x^2)^2}{\sqrt{2-3x^2}}$$

		Ok what comes to your mind? I asked that group (pointing to a group of students) just now, what comes to you mind when you first see this? Ok so initially, they told me it's u/v, right? u/v, ok? So, if you use quotient rule, right? Ok, it's correct la I never say it's wrong, you can still do it. However, you can do it in a simpler way if you represent it in another manner. (Pointed to Student S5) So, [Student S5], come and write down. How would you represent it? Just write down quickly. How you re-write the y function?
295	S5:	Ok. (Walked to the whiteboard and wrote the following: $y=\left(2-3x^2\right)^{3/2}$
296	Alice:	Yes, ok, so he has represented this function like that. Ok, so he has represented it like this. Once he has represented it like this, you see that it is what rule? Which rule can apply? (Inaudible answers by students) Ok, then you can see that you can apply chain rule easily, instead of the quotient rule, ok alright?

Alice then followed up with another function $y=\frac{1}{\sqrt{2}x}$ and called on Student S6 to write a representation for the class.

297	Alice:	So, another question, I see some interesting representations of this. [Student S6], how do you simplify this initial y function? What do you re-write it as first, before you differentiate?
298	S6:	(Walked to the whiteboard and wrote the following) $y=\frac{1}{\sqrt{2}x}=\left(2^{1/2}x\right)^{-1}$
299	Alice:	Ok, then you intend to use?
300	S6:	Chain rule.
301	Alice:	Chain rule, ok, alright, thank you. [Explained why Student S6's expression is right.] … Is there another group that wrote it differently? Anybody wrote it differently? … (after a while) Ok, alright, [Student S7] has written it like that: (wrote the following) $y=\frac{1}{\sqrt{2}x}=\frac{1}{\sqrt{2}}x^{-1}$ If a function is written like that, what rule would you use?
302	Students:	Normal rule.
303	Alice:	Basic rule, correct or not? This is you're a (referring to the constant), this is your x to the power of minus 1, so straight away you can use this, which is a very simple, basic rule. … Can? [Student S8], [Student S9], can you see this? Ok, this is different from this ah. We're not saying that the chain rule is incorrect, we're saying that if you can represent it like that, it becomes the basic rule. Let's do another one. (Wrote the following expression) $y=x\sqrt{x}$ …

In the preceding vignette, Alice invited a few students to share their ideas with the class with the purpose of directing students' attention to the structure of the given function. For example, in the case of $y=\frac{1}{\sqrt{2}x}$, we see that Alice was aware of the different ways students might have perceived the function ("I see some interesting representations of this."). We can infer that Alice was purposeful in the selection of Student S6 and Student S7 to drive home the point that it is important to consider different possible "representations" of the same function. In many ways, Alice's teaching reflected a connectionist orientation, in which "more efficient methods are offered" and "discussed the sort of contexts where different representations would be used" (Askew et al., 1997, p. 30). Furthermore, there are instances

of the five practices, such as monitoring (Line 297), selecting, sequencing (Lines 298 to 301), and connecting (Line 303) in this short exchange.

Standard Deviation Lesson

In this lesson for Secondary Four students, Alice used a sequence of examination-type items to get students think more deeply about the concept of standard deviation. In particular, she adapted an EXCEL worksheet for students can manipulate a data set to explore statistical diagrams (such as histograms). Students had about 40 minutes to work through the nine items while Alice circulated the various groups to offer prompts and assistance when requested. The vignette, which follows, centres around the discussion on Question 3 as shown in Figure 1.

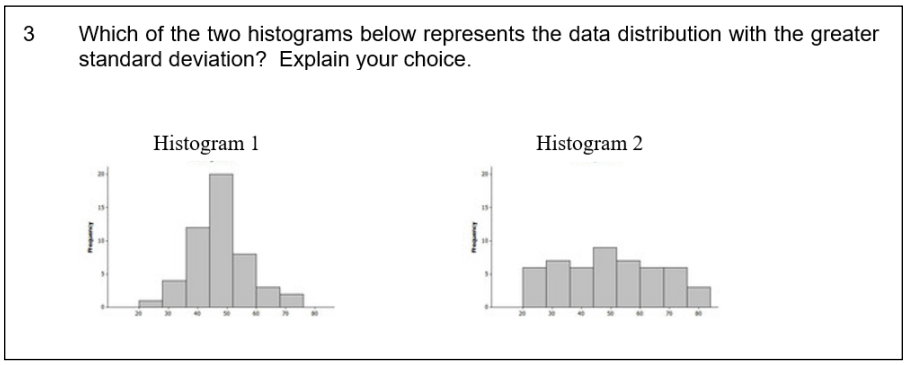

Figure 1. Question on Histogram and Standard Deviation.

248	Alice	Ok maybe uh [S8] can tell us, which histogram you chose for question 3.
249	Student S8	We chose histogram 1.
250	Alice	Yeah, can you tell us why you chose histogram 1? Why 1?
251	Student S8	Greater spread of data.
252	Alice	Why is there greater spread?
253	Student S8	There's a lot of variation.
254	Alice	There's a lot of variation, ok?
255	Student S9	Because histogram 2 everything is almost the same…
256	Alice	Ok, what does standard deviation measure?
257	Students	Spread
258	Alice	It measures spread. So, for this ah, where do you think the mean is? Somewhere... Where's the mean? Because standard deviation, we are measuring the deviation from the mean right? So where do you expect the mean to be, roughly?
259	Students	In the middle.
260	Alice	Centre ah, ok so let's say you have it here, ok, alright? So how does that give you more deviation from this centre? [S10], what do you think?
261	Student S10	[inaudible]
262	Alice	Huh? [S11] what do you think? Why is there more deviation from the mean as compared to this?
263	Student S11	The data is clustered around the middle.
264	Alice	Which one?
265	Student S11	[inaudible]
266	Alice	The data is not clustered. Which one is not clustered? Which diagram are you talking about? This one? (Pointed to Histogram 1) But you compare to this, you see ah, ok? Standard deviation measures spread ah, you look at this, this one, you got high frequencies of data at the 2 ends, correct or not? ...

Alice demonstrated the clarification routine (Lampert et al., 2010) to understand students' erroneous thinking about the notion of spread (Lines 249 to 265). Although she could have revealed the answer earlier, Alice withheld her explanation until she got a sense of what students were confused about. By listening to her students, Alice realised that students did not pay attention to the idea that the heights of the histogram bars refer to frequencies and tried to emphasise that idea in her explanation (Line 266). After her explanation, she called on another student, S12, to give his reasoning because she had a brief discussion with him during the seatwork. By engaging Student S12 to give his comments, Alice provided a platform for her students to understand the justification for the correct answer—Histogram 2. To make the explanation more accessible to students, Alice decided to use the EXCEL worksheet to show how the standard deviation relates to the shape of the histogram. Hence, we can say that Alice use of discussion routines and spreadsheet around the typical problem to make connections between explanations, concepts, and representations reflects her connectionist orientation.

Orchestrating Instructional Activities Using Typical Problems

In many ways, Alice's use of typical problems to orchestrate instructional activities for a day-to-day lesson is not unique but is commonly practised by the experienced teachers in our study. Her discussion moves around typical problems to develop relational understanding—knowing how and why (Skemp, 1978)—suggest new affordances for typical problems, beyond its use to develop procedural skills. Notwithstanding the limitations of a case study, we think that Alice's case provides an existential proof for how typical problems can be used differently to develop a relational understanding of mathematics. Furthermore, we argue that teachers can better tap the affordances of typical problems when they adopt a connectionist approach to teaching.

Affordances of Typical Problems

Alice's use of typical problems, and other teachers in our study, highlight new affordances of typical problems in developing a relational understanding of mathematics. As we have highlighted earlier in this paper, typical problems are widely used in mathematics classrooms to develop procedural fluency. Often, these problems are used with a transmission approach to teaching (Askew et al., 1997), or what some may termed as "drill and practice". However, Alice use of typical problems suggests a more balanced view of what these problems can afford beyond developing an instrumental understanding of mathematics. As many of the typical problems are narrowly focused on one or two instructional outcomes, they provide an excellent avenue to direct learners' attention to specific features of the target concept. For example, in the vignette on *Techniques of Differentiation*, Alice used a sequence of questions to highlight the importance of examining the structure of the functions given before deciding on the appropriate technique for differentiation. In the *Standard Deviation* lesson, we see how Alice used a single typical problem to highlight the relationship between the standard deviation and the statistical diagram representation of a given data set. Furthermore, typical problems also lend themselves to be modified slightly to open up its solution space, so that teachers can discuss different solutions and the connections between these solutions (Choy & Dindyal, 2017). In all these cases, Alice could have simply used the problems in a transmission approach by highlighting the solutions and the procedures needed to solve the problem. Instead, we see how she had noticed productively (Choy et al., 2017) about the affordances of typical

problems, and had orchestrated discussions around these problems by making explicit the connections between the problems and the concepts taught.

A Connectionist Approach to Teach Mathematics

A key aspect of Alice's use of typical problems lies in the connections she made when orchestrating instructional activities in class. As we have already seen in Choy and Dindyal (2017), Alice tried to connect different students' solutions by highlighting the connection between arithmetic and the method of matrix multiplication, and the connection between the solution methods in relation to the use of matrices to represent information in a systematic manner. In this way, Alice tried to highlight the "links between different aspects of mathematics", which reflect a strong connectionist orientation (Askew et al., 1997, p. 32). Similarly, we see that Alice made connections between representations of mathematics as she attempted to clarify students' thinking about standard deviation in the *Standard Deviation* vignette. In addition, referring to the vignette on differentiation techniques, Alice used a series of focused discussions to support students in making sense of the efficiency of different differentiation techniques by considering the structure of the given functions. Hence, we see Alice's use of typical problems to make connections within mathematics as an extension of the connectionist orientation to teach mathematics. We believe that it is the connectionist mindset adopted by Alice, and other teachers in the study, which made it possible for teachers to exploit the affordances of typical problems to teach mathematics in a more relational way (Skemp, 1978).

Partial or Rapid Cycles of Five Practices

Another aspect of Alice's orchestration of instructional activities, which brought forth the affordances of typical problems, is how she directed the discussions during her lessons. As discussed in each of the vignettes, we see some elements of Smith's and Stein's (2011) five practices in the way Alice orchestrated the discussions. In the *Differentiation Technique* vignette, we see Alice monitored students' answers, selected, sequenced, and connected their responses to highlight the thinking behind the choice of differentiation rules to apply. Here, Alice's lesson differed, in terms of structure, from that envisioned by Smith and Stein (2011) in the plurality of tasks within the same lesson, punctuated by several more rapid successions of the same discussion moves: monitoring, selecting, sequencing, and connecting. This structure was made feasible by Alice's choice to use typical problems, which generally take a shorter time to complete. Moreover, there were times when Alice did not use all the practices. Instead, she employed rapid but partial cycles of the five practices to discuss a modified typical problem, as in the case of the *Standard Deviation* vignette. Smith and Stein (2011) highlight the five practices as inter-dependent moves that hinge on the use of a single high cognitive demand task for the lesson. However, our data suggest that partial or rapid cycles of the five practices can be used effectively with typical problems to emphasise the connections between mathematical ideas and representations.

Concluding Remarks

This paper explores the possibility of using typical problems for developing conceptual understanding and highlights how Alice orchestrates discussions around typical problems. More specifically, Alice recognised the affordances of typical problems and exploited them effectively through partial or rapid cycles of the five practices, to help connect students' thinking to the mathematical ideas embedded in the typical problems. Given the time

constraints in Singapore and other examination-oriented systems, typical problems offer a way to strike the balance between a transmission orientation and a discovery orientation for teaching mathematics. Like Alice, connectionist teachers can better initiate and sustain productive mathematics discussions, which can potentially support students in developing a relational understanding of mathematics. The question now is not whether typical problems can be used, but rather, how teachers can be supported to notice more productively the affordances of typical problems and orchestrate instructional activities around them.

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An argument to engage really young children in mathematics

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Although it is now more commonly acknowledged that young children can engage with mathematics, it is not as commonly acknowledged that really young children (such as those aged under three years) can engage with and think mathematically. One of the reasons for this is the inability of adults to recognise mathematics in what these young children do. This paper explores the natural curiosity and engagement young children have with mathematics and discusses everyday activities within which mathematics is evident. It then suggests three lenses that adults (such as educators, family, and caregivers) can use to help them identify mathematical ideas. In providing these lenses and encouraging adults to recognise the mathematics evident in young children's everyday activities, the argument extends to maintaining opportunities for young children to willingly engage in mathematical thinking in their everyday activities over the imposition of more formalised and academic experiences in early childhood.

Really young children (toddlers up to age three and referred to as 'young children' from this point forward) can engage in mathematical activities and in mathematical thinking (Department of Education, Employment and Workplace Relations [DEEWR], 2009). Kinnear and Whittmann (2018) state research into the mathematical understandings of children shows that these young children engage with mathematics and have mathematical knowledge, their social environments impact on their "mathematical potential" (p. 20). Appropriate mathematical education in early childhood has significant and long-term positive effects on academic success and mathematical thinking.

Young children engage with mathematical ideas in their everyday lives (Greenberg, 2012) and likely do not have the aversion to anything considered 'mathematics' that adults do (Korelek, 2009). Children engaging with activities through play will use mathematical understandings and ideas as part of that play (van Oers 2016). Although English (2016) considered children from preschool (ages beyond the focus of this paper), her points are relevant here, particularly her description of children's willingness to engage with mathematics as a "natural eagerness" (p. 1081). She proposed that it needs to be recognised that young children are able to engage with mathematical ideas and that their mathematical understandings need to be revealed and built on. Meaney (2016) reiterated the idea that young children can engage in mathematical ideas, but also argued that the focus on the theories of Piaget and Vygotsky to emphasise thinking (over children's actions with their bodies) encourages a deficit model. She suggests that adults need to be wary of focusing only on children's language, rather they should also consider children's actions, as "emphasising the importance of language in learning has tended to position toddlers, with limited language fluency, as insufficient human beings" (Meaney, 2016, p. 23).

Bates, Latham, and Kim (2013, p. 1) state that "powerful emotions surrounding the application of math skills" can impact on an individual's willingness to engage with mathematical ideas, whether formally in a classroom situation or informally in everyday life. Adults' own experiences with mathematics, and the anxieties that may be associated with it,

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could prevent them from noticing the mathematics that is evident in young children's lives and the excitement with which young children engage with that mathematics (Korelek, 2009). Lee and Ginsburg (2009) stated that adults might not engage young children with mathematics due to their own fear of mathematics and "their own unfortunate encounters (and subsequent low feelings of competence) with mathematics" (p. 38, parenthesis included in original). Korelek (2009) stated that adults need to "notice, value, and build on children's excitement as they explore mathematics from the earliest age" (p. 10), and that they may need to sideline their past experiences regarding mathematics to enable them to do so.

Young Children Engaging with Mathematics

Young children engage with mathematics to make sense of the world. Bjorklund (2018), in her discussion of her earlier research from 2007 and 2008, described young children (toddlers) mathematising or making sense of the situation through the use of mathematics – that is, "sense making of mathematical relationships" (p. 41). Toddlers are able to engage in mathematical thinking during play. Garvis and Nislev (2017) identify opportunities for mathematical play to engage children's curiosity, show mathematics as a social activity, and make mathematics relevant in the child's everyday life and that of their family. They also indicate that the enjoyment children experience when engaged in play has a positive impact on children's mathematical learning.

Geist (2009) described a range of activities that infants and toddlers might engage in that involves mathematics. These activities include dumping blocks out of a bucket and moving the blocks of one colour to a separate pile, playing musical instruments, placing containers of different sizes inside each other, slicing fruit for snacks, crawling through larger objects such as boxes, and using water or sand to fill and empty containers. All the aforementioned activities would be familiar to educators and parents, but the mathematical opportunities may not be as well understood. Geist (2009) indicated possible mathematical thinking that the young children may have with whilst engaged with these activities, including classification, quantity, representational thought, identification and comparison of attributes, ordering and sequencing, and spatial relationships.

Johansson, Lange, Meaney, Riesbeck, and Wernberg (2016) analysed eight hours of video taken at a Swedish preschool. They used Bishop's (1988) six mathematical categories as a frame to analyse what was evident in the video. They also described the activities as either including mathematics in an incidental way (instrumental) or focusing on mathematics (pedagogical). If just the activities described as instrumental are considered, it was evident that all six mathematical categories were present in the videos, including some instances where more than one mathematical category was evident in one activity. The range of Bishop's mathematical categories that were identified and the instances where more than one of the mathematical categories was present, could indicate that children willingly engaged in mathematical ideas without the targeted input of an adult.

Meaney (2016) focused on toddlers' engagement with locating and proposed that the supposition that language is needed to engage with mathematical ideas is inaccurate. Videos of two children demonstrated problem solving involving locating, specifically, climbing on a bench and over a play frame. Each of these activities involved the child physically moving their whole body in relation to the objects and both children were able to successfully negotiate their movements with the objects. Meaney (2016) concluded that the toddlers were able to "learn about locating themselves in space" (p. 23) using actions to lead to learning.

Palmer, Henriksson, and Hussein (2016) outlined the mathematical ideas evident in the routine of changing nappies. A range of mathematical ideas were present in the observations,

including counting, weight, volume, spatial orientation, time, size, and proportion. However, Palmer et al. (2016) found that there was great variety in the terms used for mathematical ideas that were observed by the educator - not all educators recognised the range of mathematics evident in this frequent activity.

Lenses for Identifying Young Children's Mathematics

Phillipson, Sullivan, and Gervasoni (2017) provided neurological and biological research to supplement the work of educational theorists Piaget and Vygotsky to draw a link between the impact of learning and learning opportunities during children's early years and progress in later years. This, they propose, highlights the importance of the role of families in early childhood learning and provides emphasis on how "families (and educators) can engage with children in ways that will promote early mathematical development (p. 8). Three 'lenses' are described below to demonstrate how the mathematics in the activities young children engage with can be made visible to adults. They each describe a way to focus on mathematics but differ in both the focus and how the focus can be used by the adult to make visible and accessible the mathematics young children engage with. To illustrate the use of these lenses, examples are included from the literature used above to describe how young children can engage with mathematics in their early years.

The activity - Bishop (1988)

Bishop (1988) described six fundamental mathematical activities as universal, "in that they appear to be carried out by every cultural group … and are also necessary and sufficient for the development of mathematical knowledge" (p. 182). The six mathematical activities are counting (for example, numbers), measuring (such as comparisons and units), locating (incorporating position and orientation), designing (for example, properties of shapes and objects), playing (incorporating rules, procedures, and processes), and explaining (such as classifications, generalisations, and explanations). Bishop (1988) stated that the inclusion of more than one of these activities would provide greater significance due to their interaction. He proposed that these six mathematical activities could be used as a structural framework to enable engagement with mathematical ideas. With young children, a consideration of the activities they engage in as part of their everyday actions, and a deconstructing of these, could enable the mathematics involved to become visible.

Johansson et al. (2016) used Bishop's (1988) fundamental mathematical activities in their analysis of the mathematics children can engage with and which "forms the children's experiences and interests" (p. 28). All six of Bishop's mathematical activities were evident, and the authors stated that there were instances where more than one of Bishop's mathematical activities could be seen to be present. Also of interest is that Bishop's mathematical activities were present in those instances described by the authors as instrumental, where "mathematics was incidental" (p. 29) and not the main focus.

The mathematical components - Greenberg (2012)

Greenberg (2012) used five mathematics components to identify and discuss mathematical opportunities and involvement. The mathematics components are number and operations (such as quantities and counting); shapes and spatial relationships (identifying objects, shapes, and positioning); measurement (incorporating qualities); patterns, relationships, and change (including recognising and creating repetitions); and collecting and organising data (such as collection and analysis of information). She proposed that the

use of these components would increase the awareness of mathematics in everyday routines, enabling the identification of mathematics in everyday environments, both to discuss and to extend mathematical engagement. It could be that with young children, the identification of these components in their everyday happenings may enable the recognition of the mathematics involved.

Geist (2009) described eight examples of activities infants and toddlers might engage with and identified the mathematical components that could be evident. Although published before Greenberg (2012), the process used by Geist reflects how Greenberg's focus on components could be used to identify the mathematical thinking young children engage with. Geist emphasised the importance of educators *actively observing* what infants and toddlers do to enable the identification of mathematical components.

The language - Platas (2017)

Platas (2017) focused on the vocabulary as a way of identifying the opportunities for developing mathematical ideas. She recognises that mathematical understandings have been developed through both informal and more targeted supportive environments that provide opportunities that encourage "sustained engagement and positive interactions" (p. 34). Platas (2017) described teacher maths talk, which considers the vocabulary used and the questions that could be asked to develop mathematical ideas during interactions in the learning environment. Although the focus of the paper was on teacher maths talk and the aim of the paper was to encourage children's use and understanding of mathematical words and language, the recognition of language that is mathematical in what children say and in describing what children are doing may assist in the identification of mathematics young children are engaged with. This wider consideration of the language young children and adults use to engage with and describe what children are doing reflects the of the fact that young children do not always need to use language to be engaged with mathematics (Johansson, Lange, Meaney, Riesbeck, & Wernberg, 2014).

Palmer et al. (2016) recorded language involved in the everyday routine event of a nappy change at a Swedish preschool. The mathematical content of the utterances on the recordings was analysed and allocated to mathematical categories. What was noted by the authors was that mathematical ideas were not always verbalised by the educators. However, what was not noted is whether the educator chose not to verbalise mathematical ideas or whether the educator could not see the mathematical ideas. It may be that the mathematical ideas were not noticed by the educator (Cohrssen & Tayler, 2016; Korelek, 2009) and therefore not commented on or the educator did not take the opportunity to verbalise the mathematical ideas (Platas, 2017).

What is argued for?

This paper is not arguing for a formal academic curriculum with a focus on mathematical instruction (Elkind, 2012) to develop mathematics, but for a heightened awareness to create the opportunities for young children to revel in their "natural eagerness" (English, 2016, p. 1081) and participating adults being able to identify the mathematical ideas that may be present in everyday events (Geist 2009). However, this requires adults who are able to recognise and encourage the mathematics concepts that children are using. The lenses in the previous section can provide support for adults in identifying the mathematical ideas young children engage with in their everyday, spontaneous, and curiosity-driven lives.

Before educators, caregivers, families, and the general community can engage young children with mathematics, opportunities for adults to re-think or re-vision how they see and engage with mathematics needs to occur. Lee and Ginsburg (2009) identified nine misconceptions that early childhood teachers and preservice early childhood teachers held regarding young children's engagement with mathematics. One of these, that "young children are not ready for mathematics education" (p. 38), illustrates adults may not consider focusing on mathematics with young children. The eight other misconceptions that were identified would serve to limit the mathematics that children engage with or how they might engage with mathematics. The beliefs and attitudes, and potential misconceptions regarding the capacity for young children to engage with mathematics, has the potential to impact on what early childhood educators do regarding young children and mathematics (Sayers, 2013). In a similar way, parents and caregivers may not want to engage young children in activities involving mathematics (Koralek, 2009) or to recognise the mathematics involved in what young children do in their everyday lives (Palmer, et al. 2016).

The first step for adults not involved in early childhood education, would be the realisation that young children do engage with mathematical activities and mathematical thinking and that these activities and thinking are wide and varied. The use of any of the three lenses provided earlier in this paper (Bishop, 1988; Greenburg, 2012; Platas, 2017) would enable mathematics to be made visible to the adults engaged with young children. Bishop (1988) highlights how the engagement with the six universal activities is sustained and focused, reflecting the eagerness described by English (2016). Greenberg (2012) explained how mathematics is evident in everyday activities with infants and toddlers and that they are capable of using mathematical ideas and concepts to make sense of their world. She outlined that adults need to become aware that everyday activities involve mathematics and that the language used is highly mathematical. Likewise, Platas (2017) provided a range of mathematical vocabulary and questions which could be used in everyday contexts and activities. These would assist an adult to recognise and engage with the mathematical thinking of young children, reflecting what Meaney (2016) argued for - a "focus more on what young children can do and how this might provide insights" (p. 5).

Thornton (2018) describes the importance of mathematics in early childhood as "fostering intellectual curiosity" (p. 276) and providing opportunities for children to experience positive emotional responses such as delight and joy. Enabling adults to *see* the mathematics that young children engage with could open opportunities for adults to extend this engagement through conversations and questions. Geist (2009) provides many examples of descriptions, statements, and questions that can be used with young children to "encourage the natural mathematical interests" (p. 39). Likewise, Platas (2017) provides a range of questions that can encourage children to think mathematically within the context of their everyday activities. Although some of these questions are overtly mathematical, the contexts in which they can be used would are closely linked to the children's everyday lives, enabling them to "describe our world mathematically" (p. 34).

There is the risk that once adults realise that young children can engage in mathematical thinking, they will push for more formal educational experiences for young children. Elkind (2012) warns of the risk of moving the focus of early childhood education to formal academic instruction. He proposes that a focus on academic instruction might emphasis rote-based or formal educational curricula, which could value knowing instead of understanding. This could also lead to a move away from children initiating their activities due to intrinsic curiosity to educators using (or imposing) activities with "very little challenge, interest, or novelty" (p. 86). These experiences might then build to reflect those that disengaged the

educator might have developed from their learning of mathematics, such as their perceived low confidence (Lee & Ginsburg, 2009) or their fear of mathematics (Bates et al., 2013), repeating history and condemning further generations to mathematical disengagement.

Lembrér and Meaney (2014) highlight the risks of schoolification, particularly in terms of whether the child is viewed as lacking mathematical knowledge, instead of bringing mathematical understandings with them. This could be a more negative interpretation of Kinnear and Whittmann's (2018) "mathematical potential (p. 20) – although they have potential, young children also have current mathematical understandings. Lembrér and Meaney (2014) also warn of the risk of changes in the mathematical activities an educator may create, based on schoolified curricula and expectations. Lembrér (2015) continued this point, proposing that schoolification could lead to mathematical focused activities selected for young children (potentially those such as described by Elkind), rather than having the opportunity engage with mathematics through play (Garvis & Nislev, 2017) and natural curiosity (Thornton, 2018).

Conclusion

Early childhood is an age when initial engagement with mathematics occurs. Young children's engagement with and making sense of the world necessitates the use of and development of mathematical thinking (Bjorklund 2018). Play experiences also involves mathematical thinking (Garvis & Nislev, 2017). Unfortunately, adults do not always see the mathematics involved or recognise the capacity of young children to willingly engage with mathematical ideas.

Early childhood educators struggle to recognise mathematics in the activities of young children. This may be due to their personal mathematical experiences which in turn, can impact on their capacity to identify mathematical opportunities in activities children engage with (Anders & Rossbach, 2015). However, early childhood teacher education courses do challenge and change early childhood preservice teachers' beliefs and attitudes towards mathematics and teaching and learning mathematics for young children (Cohrssen & Tayler, 2016). Kinnear, Lai, and Muir (2018) highlight the importance of cohesion between the learning environment and the mathematical understanding and knowledge held by the educator. This has moved from acknowledging that children can engage with mathematics to a consideration of the "ethical obligations to provide mathematically purposeful environments where young children can make sense of mathematics and develop their mathematical thinking (Kinnear, Lai, & Muir, 2018, p. 2), though these should heed the advice of Elkind (2012) and Lembrér and Meaney (2014).

The engagement with families is also being embraced, with Phillipson et al. recognising the contribution of the family to the child's development of mathematical understandings. As with educators, adults in families need to be able to identify and recognise mathematics in their everyday lives and contexts. Families need to be provided with support to enable them to identify the mathematical understandings evident in their children's everyday experiences (Greenberg, 2012). Bjorklund (2018), Geist (2009), Meaney (2016), Palmer et al. (2016), and Garvis and Nislev (2017) all described the mathematics that may be evident in everyday activities that families might experience with young children. These are all opportunities waiting to happen that will enable children to reach their "mathematical potential" (Kinnear & Whittmann, 2018, p. 20) with their "natural eagerness" (English, 2016, p. 1081).

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Verification and Validation: What do we mean?

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Verification and validation are essential aspects of mathematics and beyond in STEM, but these constructs are not consistently defined in research nor in curricula documents. In this theoretical paper, we argue that verification and validation are largely characterized as binary judgments by teachers and researchers about what students do. We then present empirical examples of student work to show this view does not account for students' thinking as they resolve problems. We conclude that in order to foster learners who are confident and capable in STEM fields, it is necessary to revisit how verifying and validating activities are conceptualised and developed across years of schooling.

International calls continue for a greater focus on STEM education and an increased emphasis on mathematics to meet social and economic challenges into the future (English, 2016). Advocates of STEM agendas have championed "well-developed curricula that concentrate on twenty-first century skills including inquiry processes, problem-solving, [and] critical thinking" as well as content knowledge (English, 2016, p. 3). Recent trends have called for studies of how to leverage student achievement in one area to support similar gains in others, especially where subjects are naturally integrated (e.g., use of simulation apps and computer coding in conjunction with mathematical modelling to gain understanding of a real-world problem such as scheduling of in-patient transport in hospitals). As mathematics needs to play more of a foundational role accessing key concepts and providing investigative tools for interdisciplinary problems (Marginson et al., 2013), many curriculum authorities have responded by advocating an increase in mathematical modelling (e.g., National Governers Association Center for Best Practice & Council of Chief State School Officers [NGACBP&CCSO], 2010). Indeed, Sokolowski (2015) has confirmed that mathematical modelling activities generate positive learning effects when compared to other teaching methods in any mathematical content domain. As modelling both relies on, and fosters, many critical thinking skills identified as 21st Century Skills desired for daily life (English & Gainsburg, 2016), this is not surprising.

Yet, classroom modelling does not approximate professional mathematical modelling with regard to verifying a model or validating the modelling. This capability is fundamental to all STEM disciplines as mathematical models are developed based on aspects of the real world that modellers come to understand are valued by their clients. This involves modellers using their prior knowledge of the real world that impinges on the problem being modelled, researching the context of the situation they need to model, as well as mathematical knowledge when formulating a model and when verifying and validating the model(s) constructed or applied. In contrast, classroom modelling is usually developed based, at least partially, on pedagogical concerns. As recommendations for both curriculum and teachers shift towards eliciting and building on students' ways of reasoning (NCTM, 2015), expectations on students and teachers are evolving especially with regards to how students are to verify and validate their models.

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In this paper, we will examine existing research, curriculum documents, and student activity to argue that there is inconsistency in the meanings of verification and validation in both research and educational literature that prevents development of robust and rigorous criteria for studying these important constructs and developing them fully in classrooms. In particular we argue that curriculum documents reflect the inconsistency found in research literature and therefore may not support teachers in developing students' verification and validation skills. Using examples of student work, we then illustrate various aspects of verification and validation that may be overlooked with simplistic or broad definitions. Our contention is that for the field to move forward, it is necessary to stop asking questions that can be answered dichotomously, such as: *Did the student validate the model?* Instead, we propose asking questions that enable documenting students' thinking and metacognition such as: *How did the student validate the model?*

Verification and Validation in Curriculum Documents

In order to ascertain what messages about verification and validation are conveyed in extant curriculum documents, a selection of these were analysed. In Australia, there is a national curriculum, but states are responsible for education. In Victoria, the current curriculum is presented in the *Victorian Curriculum Mathematics: F-10A* (VCAA, n.d.) and the *Victoria Certificate of Education: Mathematics* (Grades 11-12) (VCAA, 2015). Similarly, in the United States of America (USA), a guiding national document, *The Common Core State Standards: Mathematics* (NGACBP&CCSO, 2010), exists but local jurisdictions are responsible for the curriculum. Texas, for example, has its own competence-based standards, the *Texas Essential Knowledge and Skills* (TEKS) (SBOE, n.d.). *TEKS* address all grades although in Grades 9-12, students select to study mathematics with a particular content focus (e.g., Algebra I or Pre-calculus).

Recognising that important aspects of mathematics should be included in curricular documents in order to be subsequently valued and fostered in the classroom, Kim and Kasmer (2006) analysed 35 USA state standards documents (Grades 0-9) to determine whether they supported key aspects of reasoning. Reasoning for verification was the researchers' primary focus. A list of keywords related to reasoning for verification were used to analyse grade-level expectations expressed in the documents. The analysis found that verification was expected mainly in upper grades. Particular tools were expected to be used to verify results (e.g., calculators). Other instances of expected verification were to verify predictions, conclusions, solutions, and mathematical relationships and ideas.

Drawing on the methodology of Kim and Kasmer (2006), we selected the terms, valid/validate and verify, and additionally considered terms that might be indicative of verification and validation. The key terms selected for our analysis were: compare, check, dimensional analysis, draw conclusions, estimate, ideal, justify/ justification, limitations, predict, reasonableness/ reasoning, reflect, special/extreme case and valid, validity, and verify. Where terms have multiple meanings, only those where the meanings could be seen to be supporting verification and validation were included (i.e., comparing attributes or geometric reflections was excluded). All authors coded at least one document and then these were cross-checked and collated by the third author.

Table 1 shows at what grade levels the key terms were identified in the *TEKS* (SBOE, n.d.). Terms that did not appear were excluded. Across the *TEKS*, the focus is on validation and verification of mathematical results. Students are expected to determine the reasonableness of solutions and select appropriate models, but not to develop, construct, or adapt models. They are expected to *justify*, *compare*, and *assess reasonableness*.

Table 1
Grades and Subjects where Key Terms Identified in Texas Curricula

Key Term	K	1	2	3	4	5	6	7	8	A1	A2	G	P	M	Q	IS	D	S	AR
Compare					✓		✓	✓	✓	✓		✓		✓	✓		✓	✓	✓
Draw conclusions	✓	✓	✓	✓			✓	✓	✓					✓					
Estimate				✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Justify	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Limitations												✓	✓		✓				
Predict	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Reasonableness	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Valid/validate												✓		✓	✓				
Verify									✓			✓							✓

Note. A1 Algebra I, A2 Algebra II, G Geometry, P Pre-calculus, M Mathematical models, Q Advanced quantitative reasoning, IS Independent study, D Discrete mathematics, S Statistics, AR Algebraic reasoning.

In Victoria (see Table 2) students are expected to *assess reasonableness* of estimates, answers, and results. In upper secondary, they verify results and solutions and test the validity of conclusions, arguments, and models. There are many examples related to use of technology such as to "relate the results from a particular technology application to the nature of a particular mathematical task (investigative, problem solving or modelling) and verify these results" (VCAA, 2015, p. 36). What is missing is a viable description of *how* students are supposed to make these judgments and against what standards.

Table 2
Grades and Subjects where Key Terms Identified in Victorian Curricula

Key Term	F	1	2	3	4	5	6	7	8	9	10	F	G	M	S	FM	MM	SM
Compare	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓			✓		
Check								✓		✓	✓	✓				✓		
Draw conclusions											✓			✓		✓	✓	✓
Estimate	✓			✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Justify		✓		✓		✓		✓			✓							
Limitations											✓[a]	✓	✓	✓				
Predict											✓[a]		✓			✓		
Reasonableness				✓		✓	✓	✓				✓	✓	✓	✓	✓	✓	✓
Valid / validate											✓			✓	✓	✓	✓	✓
Verify						✓			✓				✓	✓	✓	✓	✓	✓

Note. F Foundation mathematics, G General mathematics, M Mathematics methods, S Specialist mathematics (Yr 11), FM Further mathematics, MM Mathematical methods, SM Specialist mathematics (Yr 12), [a]10A.

Verification and Validation in Research Literature

In this section we briefly review a selection of research studies where verification and validation are a focus and then consider what could be the sources of conflicting findings.

When a class of Victorian Year 6 students was introduced to mathematical modelling, there was little transfer of problem-solving techniques from other classroom mathematics

experiences to modelling, particularly checking (Brown & Stillman, 2017). The checking that occurred included questioning the validity of statements, judging reasonableness and logic of answers and using empirical testing to show correctness of a proposed solution. There were thus only the rudiments of verification and validation shown. In another primary school study, but in Japan, Kawakami (2017) reported that model validation triggered students' combining of the models they constructed in their internal modelling world with the external model constructed in the real world through data generation and collection. In a secondary classroom context in Texas, assumptions made implicitly or explicitly in formulating a model based on student interpretations of problem contexts were reported as feeding forward into verification and validation activity (Czocher & Moss, 2017). Czocher (2016) found that students engaged in validating throughout their modelling activity, but that it was not solely focused on a prediction or result. Indeed, Czocher (2013) showed previously that there is important overlap between validation, verification, and metacognition. Stillman (2000) reported Year 11 students using prior knowledge of real world task contexts in verification activities by enhancing decision making and as a means of checking progress or judging the reasonableness of interim or final results. These studies show validation and verification occurs in classrooms but encompasses more than checking correctness of computation. In contrast, Blum (2015), drawing on several German studies, noted absence of validating in students' solutions despite educational standards for mathematics requiring validating of mathematical results and checking, comparing and evaluating mathematical models with respect to the real situation. Over 80% of Year 6-11 students in a study by Ludwig and Reit (2013) did not validate the solution to better adjust their symbolic model to the given situation.

What gives rise to such conflicting results? Firstly, a partial answer could come from previous work. Stillman, Brown and Galbraith (2010) indicate that student modellers in even lower secondary can overcome low intensity blockages to their progress whilst modelling by harnessing metacognitive activity including reflection on actions that allows rectifying errors. However, blockages of high intensity occur where the modellers resist accommodating new contradictory information resulting in cognitive dissonance (Festinger, 1957). Task solvers' approaches to resolving cognitive conflicts in order to maintain their cognitive structure could be the key to whether or not verification and validation activities are manifest in the classroom and thus seen by researchers. *Pseudo-learners,* according to Raychaudhuri (2013), successively stockpile items of knowledge almost linearly making connections primarily from the context where the knowledge was taught. They do not recognize cognitive conflict as lack of connection means questions of conflict do not occur. However, they will compartmentalize the conflicting pieces if pointed out to them; so, the conflict will cause no perturbation to their cognitive structure and there will be no evidence of either verification or validation.

Secondly, caution is needed in interpreting research results as there is no consensus in the use of terms such as verifying and validating with validating often being defined explicitly as only a last step and verifying of the model mathematically being implicitly understood to occur (e.g., Blum, 2015). Still others see verification as multi-faceted and likely to occur when any interim results are derived, or decisions taken that impinge on the models produced (Stillman, 2000). Indeed, Czocher (2013) problematized the complex role played by validation in modelling as it accounts for ascertaining both the model-situation fit and verifying that its analysis was conducted correctly. Validation includes ensuring that the model is based on assumptions that represent real world problem constraints, and must therefore include interim checks, whether or not they lead to model revisions.

Empirical Examples

In this section we present empirical evidence that whether verification and validation occur cannot be effectively treated as a dichotomous decision. We note that each example is drawn from research data. We argue that thinking of verification and validation as "checks" on interim or final results limits conceptualizing how these critical skills should be fostered. In each case, we problematize the questions: *Is the model valid?* and *Has the student validated?*

Erin was calculating the daily cost of a food stall franchise at an international multi-venue event, as part of her modelling of a cost plan for the organisers. An internet search resulted in her choosing a total estimated cost per stall for the duration of the event. She calculated the daily cost using a 7-day week as $3600 per stall. Whilst recording further information about a venue, she suddenly recalled it operated for only four days. She inferred her daily cost estimate was incorrect and recalculated it as $6200. As she recorded the new result, she expressed her doubts of its correctness, so repeated the computation but obtained the same result. Having checked her intuitive doubt, she reluctantly accepted the result but sat for a moment, thinking it through again. This is an example of metacognitive awareness where a metacognitive experience (Flavell, 1979), an intuitive feeling that the result of the calculation was too large triggered the cognitive task of verifying the result. Without a benchmark to judge correctness in the real situation her feeling was intuitive. Intuitions can be a source of productive ideas, but they can hinder thinking and reasoning (Fischbein, 1987) so verification is necessary. On the surface this appears to exemplify merely checking by redoing a computation, but more is happening here for Erin. She verified her result, checking that the number made sense, a paragon of the objectives set by the curriculum. Thus, there must be more to verifying a prediction than checking it against real data or validating a model than checking whether it makes sense.

Ari and Tony were working on how long it takes an average family to fill a wheelie bin that holds up to 48 kg of rubbish. Through an internet search they established an average Australian family produces 153.85 kg of waste weekly. Ari divided 153.85 by 48 on his calculator and stated: "It takes 3.2 days to fill a full bin." Other group members challenged this, prompting him to repeat his computation and hold up his calculator to show them the result of 3.2. When Tony suggested 153.85 was 100%, Ari worked forward from 48 mentally estimating that 3.2 by 48 gave 100% of 153.85. Tony was still unconvinced encouraging Ari to express his thinking to dispel this puzzlement. Tony argued that 7 days was to little time to make over 150 if 48 took 3 days. Ari calculated 48×3.2, insisting: "To fill a full bin, it takes 3.2 days and gets to that [showing Tony his calculator] 153". Tony tried to introduce a cognitive conflict for him by saying if it was 48kg for 3 days then it takes 3 times 48 for a week, which is 9 days not 7. Ari remained adamant: "but I just worked it out then". Tony countered with: "There's not 9 days in a week." Others in the group supported Tony but Ari was convinced his calculation was correct as he had verified it by repeating it and doing the reverse by estimation and with a technological tool. He persisted with his way of thinking. As trying to facilitate cognitive conflict for Ari had not worked, Tony used a direct approach suggesting using $153.85 \div 7$ which he agreed was the waste per day and then $48 \div 21.97$ giving 2.2 days to fill the bin. Ari then conceded.

According to the relevant curriculum, students in the context of "mental, written, and technology assisted forms of computation" are to "routinely use estimation to validate … their answers" (VCAA, n.d., p. 65). This was of no help to Ari as he verified his calculations several times including using estimation. There is far more to be considered in real situations. Firstly, we can ask: Would researchers say Ari verified his model? Certainly, he checked the prediction was correct, but this did not lead to finding flaws or revising his model. Secondly, we can ask

about Tony's role in the milieu. Tony experienced cognitive conflict when presented with Ari's answer and was able to identify its source. He then tried, unsuccessfully, to provoke cognitive conflict for Ari.

On another modelling task, Mance sought an expression for the quantity of buffering agent in a fish tank as a function of time, *t*. To do so, he needed to create a differential equation that would model the rate of change of quantity of buffering agent in the tank as a buffering solution entered and well-mixed solution left. The problem statement gave the concentration, *C*, of the buffering solution as $1 - e^{-t/60}$ g/L and the solution entering at a rate of 5 L/min. He assumed rates of liquid entering and leaving were equal and obtained $\frac{dc}{dt} = 5(1 - e^{-t/60})$ g/min. He then proceeded to validate that his model represented the real-world situation saying: "If you just multiply those two together, you'll have 5 times the buffering strength entering and that'd give you g/min. It's asking for how much buffering is in it at any point in time. If you were to plug in a time for that you'd be multiplying for a minute rate. So I think, the strength of the buffering solution. Yeah, that'd be right. So, I think it's *C* is equal to 5 times that. Because if you plug in time you're gonna get an answer in grams and that's what you want."

Mance's equation was incorrect. It did not account for liquid leaving the tank nor for change in concentration of buffering agent in the tank. However, Mance validated his model checking it satisfied the question posed. A more appropriate way to view Mance's work is to ask not *whether* he validated, but *what* aspect of the modelling process he validated, *how* he did so, and the sources of cognitive conflict that led him to validate. He used dimensional analysis to check his set up of the model and then examined whether he thought the model would lead to an answer. His prior academic experiences taught him to doubt his models until he checked the units but as they gave him the correct unit, he inferred incorrectly that his modelling was correct.

Discussion

Through analysis of curriculum documents, research literature, and empirical examples we have identified inconsistencies in how verification and validation are treated. The first inconsistency is conceptual and pertains to what object is verified or validated. In curriculum documents and in some research literature, validation is conceptualized as a check carried out at problem end. Indeed, some conceptualize validation as possible *only if* and after the student obtains a result (Ludwig & Reit, 2013). In our empirical examples, only Tony and Ari verified a final result. Erin verified an interim result and Mance validated the representativeness of his differential equation prior to solving it. The complexity of mathematical modelling presents many opportunities for errors or different ways of thinking about a problem and many opportunities for verification and validation.

The second inconsistency is methodological. It is the grain size of validation and verification. Grain size determines what is observable and what is to be observed. Concerns captured by the questions: *Is the model valid? Has the student verified or validated?* correspond to a coarse grain size. As methodological or pedagogical questions, they are intended to observe the student's final product and ascertain whether or not the answer is correct. They are evaluative questions with dichotomous answers. Coarse grain size analysis is unclear whether the student verified or validated in our empirical examples because not all results are correct but in all cases the student engaged in verifying activity in attempts to ascertain whether the model was adequate. Further, Ari and Mance convinced themselves that their models were adequate. In Erin's work we observe her verifying an interim result by redoing her computations. In contrast, concerns captured by the questions: *What is being verified or validated and how?* have a fine grain size. In Tony and Ari's discussion, we observe debate

over which result is correct. Mance used dimensional analysis to monitor his ongoing work, not because he suspected something amiss, but because he was confirming that "all is well" (Goos, 2002, p. 286). Thus, fine grain questions are both more descriptive and more revealing of student thinking allowing for teacher intervention during modelling rather than waiting until the end.

The third inconsistency is theoretical. Viewing validation and verification as dichotomous judgments that have a normatively correct answer ignores psychological and experiential aspects of the student as a rational actor. That is, because the dichotomous view emphasizes the end product of modelling over the model construction process, the locus of control for determining whether a model is adequate is external to the student. The brief analyses of the empirical examples and the review of related literature above suggest that multiple theoretical constructs are construed as impetus, means, and consequence of validating and verification activity. At minimum, an adequate theory of verification and validation should include constructs such as cognitive conflict, intuition, reflection, metacognition, and account for students' prior experiential and academic knowledge.

Shifting the view of verification and validation from a dichotomous judgment about a process or action to a richer description of students' ongoing activity means encouraging learners and teachers to attend to more than success in resolving the modelling problem. It encourages attending to activities and skills that support learning how to carry out validating activities. These attendant skills would then replace the narrow product-oriented definitions of verification and validation currently found in curriculum documents. We do not suggest attending to the processes of modelling and verifying and validating activities *in place of* students obtaining correct answers. We advocate emphasizing these activities to foster learners who respond to cognitive conflict with restructuring of knowledge and changing their approaches rather than learners who resolve cognitive conflict via compartmentalization (Raychaudhuri, 2013). Finally, a descriptive rather than dichotomous view of verifying and validating activity begs the question: *How can verification and validating activities be provoked?* In the product-oriented dichotomous view the only sites for verification and validation occur as a result is reached and at problem end. Teacher moves might include asking, "Does the (final) answer make sense?" or pointing out an error in reasoning or judgment. As in the Ari and Tony example, just pointing to an error is often not sufficient to incur cognitive conflict necessary to provoke validating. A descriptive, fine grained view of verification and validating activities allows teachers to respond to students' modelling activities not as though there is a single, high-stakes act of metacognitive decision making at the end but rather throughout the process.

Conclusion

We have shown that verification and validation are in fact more complex and nuanced activities than reflected by curriculum documents. Specifically, we have shown it is possible for students to engage in validating activity without arriving at a correct model or answer because it is possible to have any combination of correct or incorrect assumptions, models or results. Thus, if the only proficiency required is that students check their obtained results against some correct answer at the end of their resolution of the problem, teachers and researchers miss students' natural ways of reasoning and how to build on them. As a research community we can shift away from asking dichotomous, evaluative questions towards richer questions that ground discussions of pedagogy in student reasoning such as: *What is the student validating and how?* and *How can validation be provoked in this moment*? In line with these conclusions, a richer view of verification and validation is necessary in order to align teaching and

assessment with the cognitive and metacognitive activities that support skills needed by successful STEM students.

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Different worlds: Looking deeply at context in the sustainability of PD for collaborative problem-solving mathematics

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In this paper, I look at multiple layers of context to explore the *figured world* of mathematics teaching in primary schools in Chile. I collected data from 15 teachers during 18 months after their participation in professional development for collaborative problem-solving mathematics. I found constraints to impact on the teachers' implementation of new pedagogy at personal, local, institutional, national, and global levels. Whilst these constraints acted directly to constrain practice, they also acted to produce the imagined realities for teachers of mathematics in their particular world.

As I approached the school, I saw immaculately dressed children dropped off by their parents and greeted by staff with a kiss. The security guard let me in through the imposing gates and I entered a concrete building that surrounded a very small courtyard. Children of all year levels were playing in this small space and I could see no grass anywhere. It was dark and freezing. Entering the classroom, I saw paint peeling from concrete walls and long rows of desks. Soon 45 year-one students took their places, quietly sitting and attending to their teacher. They were extremely well behaved and appeared happy to chant and skip count and later to choral their answers to a mathematics exercise in unison. I was definitely in another country, was I in another time?

Research reports typically include a section titled "context" in which the local and institutional contexts are briefly described. Such brevity does not do justice to the hugely complex and multi-layered aspect of the worlds in which we teach and learn mathematics. Most often, MERGA participants come from educational contexts that are resource-rich and may have considerably less experience in schools that appear on first glance deprived and that display many other cultural differences embedded in day-to-day practices.

My experiences living, working and researching in a context very unlike my own motivated this paper. In 2015, I left New Zealand to take up a post-doc in Santiago, Chile and soon after, I began a research project looking at teachers' identities and experiences after their participation in a programme of professional development (PD) in collaborative, problem-solving mathematics. On my first visit to a mathematics classroom in a mid-low income area of Santiago, I was struck by the vast differences between this world and my own memories of teaching primary school in New Zealand. These differences were evident in both the environment and the typical teaching-learning contract. It soon became apparent to me that the changes expected of teachers through the PD were far greater than I had assumed they would be when I designed my research. This had implications for data collection and analysis of my own project, as well as the overall success and sustainability of the PD programme. Only by close attention to context would I be able to make sense of my findings.

In this paper, I wish to explore context deeply. I will use Holland and colleagues' concept of figured worlds (Holland, Skinner, Lachicotte, & Cain, 1998) to describe and analyse primary mathematics teaching in Chile through the various layers of context – from the

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personal and local layer of classroom and community through to institutional, national and global contexts. This world I will compare and contrast with the figured worlds of: mathematics teaching in New Zealand, and of reform-oriented discourse. My aim is to answer the question: *How do the various layers of context influence success and sustainability in PD for collaborative problem solving in mathematics?*

Theoretical Framework

I draw upon Holland and colleagues' (1998) concept of figured worlds. A figured world is "a social and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (p. 52). In this case, the figured world at play is the world of teaching mathematics at the primary school level in Chile. The key 'characters' in this figured world are teachers and students, but also school principals, colleagues, parents and caregivers. The most relevant 'act' to this research project is the teaching and learning of mathematics, and the interpretations of this act as developed historically, socially organized and reproduced by stylized interactions between teachers and learners.

Wider political and social contexts, and the more local and personal contexts, are instrumental in the formation and structuring of the storylines within a figured world. Firstly, storylines are a central narrative or a taken-for-granted unfolding of particular activities, such as teaching. Secondly, in the world of learning or teaching mathematics, context may refer to the political context of education policy, the social world of class systems, racial and gender relations, and local contexts of individual schools and school networks. The specific context helps generate the storylines drawn upon in people's understandings and constructions of their figured world. The typical storylines created by actors in the world are key to analysis; they tell us much about the figured world and provide a "backdrop for interpretation" (Holland et al. 1998, p. 54) of performances and activities; the meanings of everyday events are figured against these storylines.

The Situated Nature of PD

Within the broader literature on PD there is certainly attention paid to context or the "situated nature" of PD (Avalos, 2011). Teacher PD is a complex process, thus there is a need to examine "the interacting links and influences of the history and traditions of groups of teachers, the educational needs of their student populations, the expectations of their educational systems, teachers' working conditions and [their] opportunities to learn" (Avalos, 2011, p. 10). Within mathematics education, research on PD is a growing field (Skott, van Zoest, & Gellert, 2013; Sztajn, Borko, & Smith, 2017), yet in a recent review of PD research in the NCTM compendium, little attention is paid to issues of context, rather focus is on impact, sustainability and PD comparisons (Sztajn et al., 2017).

Some authors within our discipline have found context to be an important factor in success and sustainability of PD programmes. Gresalfi and Cobb (2011) examine both the institutional context and the context for PD to understand the distinct identities that teachers construct through their participation. They argue that these identities, formed in context, are central for teachers becoming motivated to improve their classroom practice. Similarly, Battey and Franke (2008) used the notion of identity and examined how teachers participated in both the settings of PD and the classroom in order to understand the relationships between these. They note that even high-quality PD does not always translate to changed practices. I offer a final example, based in Chile. The research involved revisiting a teacher in the year

following her PD. Whilst this teacher had initially taken on board the PD and integrated it into her classroom programme, when the researchers returned to visit the following year they found she had abandoned the methodology and was engaged in another, contrasting, PD (Gellert, Espinoza, & Barbé, 2013). These authors suggest the wider context of policy and fast-paced reform worked to limit her formation of professional identity. In general, the literature regarding PD which attend to teacher identity, and understand identity using a sociological frame, tend to consider context as highly important in the production of identity and correspondingly the success of the PD. However, these studies do not always look at context in wider terms, examining the multiple layers of context, and thus limits a more complete understanding of the success and sustainability of PD programmes.

A PD Programme in Chile

One professional development (PD) program currently operating in Chile provides an 8-month workshop promoting non-routine, collaborative problem solving in the mathematics class. The pedagogy is more student-centered than typically seen in Chilean classrooms, which is characterised by the teacher dominating and controlling classroom talk (Gellert et al., 2013; Preiss, 2010). The programme promotes students working in groups to solve non-routine mathematics problems and then discussing their results during plenary sessions. The PD follows principles of 'reform' mathematics and shares characteristics with other programs internationally (e.g. Koellner, Jacobs, & Borko, 2011). In 2015, this project provided PD to 140 Chilean teachers from grades 1 to 8. Currently the programme is undergoing extensive scaling-up in a variety of locations along the country.

The PD promoted substantial pedagogical change and ultimately aimed to generate a radical and permanent shift in participants' mathematics teaching. Such a change is not easy (Guskey, 2002) and teachers may experience "identity conflicts" as they expose their work to scrutiny and are required to "modify proven practices" (Avalos & de Los Rios, 2013). Such difficulties are evident in the research evaluating effectiveness of PD (e.g. Desimone, 2009) and other research analysing teacher identity change after PD, such as described above. Post-course evaluations generated positive responses; teachers reported changed attitudes and strong support for the new pedagogy. Participants demonstrated significant changes in their beliefs to be more aligned with 'reform' methods (Cerda et al., 2017). However, it is important to uncover whether these teachers continued developing their practice over subsequent years, whether they incorporated new pedagogies more permanently into their regular practices and identity enactments, and what difficulties they may have faced without the continued PD support.

Methods

I invited all teachers who completed the PD project in 2015 to participate in this follow-up study, and 15 teachers volunteered. The participating teachers all taught in either municipal (public) schools or private-subsidized (voucher) schools. They taught populations of mid to low socio-economic status in urban or rural schools.

During 2016, I engaged with teachers in a series of four 'email interviews' to elicit narratives about their experiences with mathematics teaching and in particular of enacting the changed pedagogy of the PD. Each email contained approximately five questions to stimulate reflective responses and typically included a question asking specifically about their teaching of problem solving; for example: "Have you taught problem solving this year? Please tell me how it went?" (Email 1), "Can you tell me about your most successful

problem-solving activity this year?" (Email 4). It was hoped the reflective emails would generate narratives through which teachers could construct their own understandings of their experience (Sparks-Langer & Colton, 1991), and also that using emails would allow for greater reflection, giving teachers the time and space to consider their responses, in a way that a regular research interview does not (James, 2015). Some teachers asked to respond to the email questions via telephone or using the *whatsapp* cell-phone application; in these cases, I emailed teachers the questions one week beforehand to allow a similar reflection time.

Other data collected included classroom observations, undertaken primarily to better understand the specific context and therefore aid email data analysis. I also used these observations to generate further personalized questions for each teacher. The final phase of the data collection was an interview and additional observation with ten of the teachers, those who wished to continue with the project, and this took place in May to June of 2017. Darinka Radovic, a Spanish-speaking colleague from Chile accompanied me to these interviews and assisted in subsequent analysis. Questions here focused more specifically on identity enactments and teacher change, and I used the interviews as an opportunity to clarify the themes uncovered in the previous phases of the research.

All interviews were transcribed, and I coded firstly according to broad research objectives. These were: Evidence of change or learning related to PD, constraints and affordances in mathematics teaching including problem solving, and teacher identity performances. These broad topics produced the initial codes, then within each topic all data was re-coded inductively. Following the methods of Braun and Clarke (2012), the codes were collapsed and combined to form themes. At this point, I used the concept of figured worlds to interpret the themes. In picturing the figured world of primary mathematics teaching and learning in Chile I drew upon my own experiences of primary mathematics teaching in New Zealand, which comprises a markedly different figured world.

Contextual constraints

Four main themes regarding constraints and affordances emerged in the teachers' discourse: students, curriculum, time, and resources. Whilst each of these were described as a constraint in their teaching of mathematics via collaborative problem solving, at other times teachers considered these same aspects to be affordances. This suggests that these aspects are not constraints in themselves, rather it depends upon the specific context and individual views of the teachers and is related to their identities as teachers of mathematics (see Darragh & Radovic, submitted). For the purpose of this paper, I will focus on how these operated as constraints and look at the influence of context on each constraint.

Comments about students dominated teachers' discourse and, in the majority, teachers spoke about their students as being deficient in some way: they were seen as lacking, academically or intellectually, or lacking group work skills (see also Darragh & Valoyes-Chavez, submitted). Often comments related the problems to the students' backgrounds - for example pairing a low SES situation with academic issues:

> Their socio-cultural level is low, they do not have good reading habits, they are disrespectful and intolerant of each other […] some fight, twelve children have untreated ADHD, they are rude and have no interest in learning. (Marcela)

The constraint purportedly caused by students emerged very early in the data set. Teachers frequently mentioned students as being one of the difficulties they felt they would

face in the teaching of problem solving and as being a limitation that they face in teaching more generally.

Although the recently updated national curriculum emphasises problem solving (MINEDUC, 2012), teachers spoke more of the enacted curriculum (Remillard, 2005) as developed by the management team in their particular school. In the case of teachers working in voucher schools, these were run by organisations which develop their own curricula (albeit based on the national document) to develop teaching plans and in some cases textbooks. At some grade levels, the teachers must follow highly scripted lesson plans that follow a strict sequence of lessons. Other comments pertained to the curricular constraint generated by national exam called SIMCE[1]:

> The challenge that comes to me this year is that my class should take the SIMCE and I would like to have the skills to get a good score. (Maob)

The constraints on teaching generated by standardised testing are well documented in the literature (Morgan, Tsatsaroni, & Lerman, 2002; Pausigere & Graven, 2013; Walls, 2008). Within Chile, these tests are high-stakes for the teachers and schools rather than for the students. Students do not ever learn their test scores but those of the schools are published, schools earn bonuses for high marks and this monetary reward is passed on to teachers. Teachers also spoke about their evaluation being connected to test results, despite purporting to be based on the dimensions of quality teaching.

> There is a contradiction between what the 'framework for good teaching' asks of us, that is, written criteria and ideal descriptors in 4 different areas on which our job performance is evaluated, because when the time comes to say if you're good or bad teacher it is the results of the SIMCE standardized tests which weigh in (María).

Thus, for those teachers who taught a SIMCE year class (annually in 4th, 8th, and biannually in 6th grade for mathematics), there were significant pressures to cover the large quantity of content associated with the test, and to teach problem solving in the style of SIMCE, rather than the 'non-routine' version of the PD.

In addition to time needed to prepare for SIMCE tests, teachers referred to the large amount of content they needed to teach within a limited period. Teachers also felt they needed more time outside of the classroom to plan, to reflect and to meet with colleagues.

> The conditions [in Chile] are not the best, put in relation to the quantity of class hours and hours to plan – it is the worst. We have many hours in front of the children, without sufficient time to prepare, revise and correct material. I think it is a great debt, in my school I count on more time for planning (Rosario).

The teachers in public schools often brought up lack of resources as a constraint, in particular:

> In the public schools, in which I work, they don't provide all the necessary supplies to do our job (paper, photocopying, pens, amongst others). They ask for teaching materials, technology and such is the bureaucracy that they arrive years later, do not arrive, or arrive in insufficient quantities for an entire course. There is no autonomy to manage resources according to the realities of each school (María).

The constraints described above were not the case for all teachers at all times. In fact, within each of the themes the teachers also spoke of affordances. The students were viewed as surprisingly capable, some schools did allocate time specifically for problem solving in the curriculum and the amount of resources available depended on the wealth of the school.

[1] SIMCE is an acronym standing for *System of Measurement for the Quality of Education* in Spanish.

In 2015 after a seven-week strike, public school teachers gained a guaranteed 30% of their working hours set aside for non-contact time. These teachers spoke of the changes as an affordance on their ability to plan for and enact new teaching pedagogies.

However, these contextual constraints were not only evident across a range of teachers, they were also evident as key discourses even in their absence, for example, Javiera's comment suggests sufficient resourcing is certainly not something automatically assumed:

> Thanks to God, this school can count on good conditions. When we need any material, the school obtains it. We have various concrete teaching materials with which the children can experience and better visualize geometry. The rooms are good – not too big, they have everything to create a good lesson. (Javiera)

Similarly, the surprise that teachers expressed of their students actually being able to do collaborative problem solving suggests their expectations were actually the opposite. In this manner, even when talking of an affordance to the teaching of mathematics and problem solving, the teachers at times reinforced the notion of these aspects as being constraints.

Storylines in figured worlds

We may gain an understanding of the figured world of teaching primary mathematics in Chile by close attention to the storylines evident in this data. Here I have space to discuss two: a *storyline of deprivation*, and the *neoliberal storyline*.

The storyline of deprivation is evident across a number of the constraints mentioned by the teachers. Schools are deprived of resources, teachers are deprived of time, and students are lacking, both academically and financially, they are deprived of 'quality' backgrounds. This particular storyline is produced by the local context of lower socio-economic status populations. Here, the personal and local contextual layer of students and their family background interacts with the wider political context in which Chile has the greatest gap between rich and poor in the developed world (Bellei & Cabalin, 2013). This disparity is reflected in educational inequality and visible in the large number of schools servicing communities in lower socio-economic levels, a demographic pertaining to the majority of teachers in this study. When working in such a context teachers tend to draw from, and contribute to, these storylines of deprivation (see also Healy & Powell, 2012).

A second storyline evident in the data is a more familiar story worldwide; it is the neoliberal story of testing and performance evaluations. This is a global storyline of performance evaluations of teachers, measured through their students' test results, and influencing curricula. This layer of context operates at the global level, but specifically in Chile, the market-based educational system (Bellei & Cabalin, 2013) was adopted early and completely. Schools are privatised, giving rise to the impact of context at an institutional level - the local interpretation of the curriculum can supersede the official curriculum. Nation-wide policies can be seen as a key influence particularly in the use of nation-wide standardised testing, SIMCE exams. Arguably, the market model of education may also indirectly contribute to the deprivation storyline for those working in schools that lose funding as vouchers follow the students to other schools.

Following Holland and colleagues (1998), figured worlds are a "realm of interpretation" in which characters, such as students, are recognised in particular ways, significance is assigned to certain acts, such as problem solving teaching, and certain outcomes are valued over others, such as SIMCE test results and performance evaluations. The data here demonstrates how students are recognised as lacking and problematic, problem-solving teaching is seen as constrained, and results in SIMCE tests, which ask questions that are shallow versions of problem solving, are valued over rich mathematical tasks. The neoliberal

and deprived storylines are produced in various layers of context and these storylines work against the success and sustainability of the PD.

Contrasting worlds

Reform discourse provides an alternative figured world for mathematics learning and teaching. In this world, the act of non-routine problem solving is valued and strongly endorsed in research literature. The PD provided to the teachers in this study drew from this world, aligning the strategies with tried and true approaches that were developed elsewhere in the world, in countries that shared some contextual aspects yet differed significantly in other ways. A limitation of the PD is that it may not have adequately adapted the project to the world of primary mathematics teaching in Chile; there did not appear to be consideration of typical storylines of deprivation and neoliberal performative education in the delivery of PD.

Secondly, I wish to mention the contrast in the figured worlds of mathematics teaching and learning in New Zealand and Chile. I designed a research project based on assumptions derived from my own origins from a very different figured world. This world did not have lack of space in the classroom; it was well resourced, with students and teachers accustomed to autonomous group work and no nation-wide exam system. I imagined a world where change would be rapid because it was not so very different. The contrasting reality required an adjustment to data analysis. Rather than investigating how teacher identity might change in response to new pedagogies, I was able instead to look at the role of context in the formation and enactment of teacher identity and surmise the crucial role of the culturally produced identity in the success or failure of PD (Darragh & Radovic, submitted).

By way of conclusion, I suggest that the various layers of context influence success and sustainability of PD. It does so firstly by generating real constraints on the act of teaching and learning and secondly by producing storylines which limit the possibilities of change. Specifically, the figured world of primary mathematics teaching in Chile generates storylines that teachers may draw upon and contribute to in citing their difficulties in the enactment of new pedagogies. An attention to all the layers of context is thus crucial in understanding teacher change and working towards success and sustainability of PD.

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Fluency with number facts – Responding to the Australian Curriculum: Mathematics

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This paper reports on results of a targeted and extended (two school terms) program for building Grade 3 and Grade 4 students' facility with number facts and application for mental computation. As part of a larger project, this paper reports results from two schools. Results indicated strong gains in number fact recall and mental computation for both cohorts at both schools. The similar gains in outcomes at both schools suggests the power of a targeted and extended program to build basic fact fluency for mental computation.

Introduction

Teaching the Australian Curriculum: Mathematics (ACARA, 2017a) is to address the development of the four proficiencies of understanding, fluency, problem solving, and reasoning. Of most importance to this project is the term fluency. The focus of the research reported here was on building Year 3 and Year 4 students' fluency with the basic facts of addition (and subtraction) and multiplication (and division) respectively.

Fluency is a term often misunderstood. Fluency is about skills in "choosing appropriate procedures, carrying out procedures flexibly, accurately, efficiently and appropriately" (ACARA, 2017b). Fluent number fact knowledge spans far beyond conventional automatic mastery practices involving rote learning (Baroody, 2006). Students demonstrate fluency when they "calculate answers efficiently, when they recognize robust ways of answering questions, when they choose appropriate methods and approximations…" (ibid). Having the ability to automatically recall facts fluently does not mean students will have strong conceptual understanding (Hurst & Hurrell, 2016).

In the Australian Curriculum: Mathematics (AC:M), the development of number fact knowledge is an explicit feature in Year 3 and Year 4. In Year 3, the *content descriptor* is expressed as follows:

- Recall addition facts for single-digit numbers and related subtraction facts to develop increasingly efficient mental strategies for computation (ACMNA055)
- Recall multiplication facts of two, three, five and ten and related division facts

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(ACMNA056)

The associated aspect of the achievement standard is stated as follows: "By the end of Year 3, students … recall addition and multiplication facts for single-digit numbers."

For Year 4, the content descriptor is as follows:

- Recall multiplication facts up to 10 x 10 and related division facts (ACMNA075)

The associated aspect of the achievement standard is stated as follows: "By the end of Year 4, students … recall multiplication facts to 10 x 10 and related division facts."

Number fact recall is a component of the AC:M, yet the pathway to attainment of these achievement standards is not prescribed. The term 'recall' may also serve to impact approaches to developing number facts. Does recall mean automatic/instant recall of a number fact, or the capacity to use strategies to achieve a correct calculation? How long does a teacher wait for a student to 'recall' a number fact before she/he determines that the student has attained the achievement standard?

Conceptual Framework

It can be argued that the purpose of number fact recall is to assist mental computation and estimation, one of the most-used mathematical skills of adults in their daily lives (Northcote & McIntosh; 1999). It has been long-established that automatic recall of basic facts is a prerequisite for mental computation facility (Sowder, 1992).

Research has highlighted the value of instructional programs that emphasise strategic and flexible thinking and provision of opportunities for students to explore, discuss and justify their strategies and solutions (e.g., Blote, Klein, & Beishuizen, 2000; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). Young children bring to school sophisticated number knowledge and strategic thinking around number, and there is general consensus that rich learning environments, where students are provided with opportunities to explore number combinations and arrangements, assist students to derive their own strategies for basic fact combinations (e.g., Baroody, 1985; Fuson, 1992) and the development of number sense (Wright, 1996). However, some students do not develop efficient strategies for basic facts, and predominantly rely on inefficient counting strategies for mental computation (McIntosh & Dole, 2000; Mercer & Miller, 1992; Ruthven, 1998; Steinberg, 1985). Explicit strategy instruction, modelling, discussion, questioning, feedback and guided and independent practice has shown pleasing results, including for early years learners as well as students identified as exhibiting mathematics learning difficulties (e.g., Bryant, Bryant, Gerston, Scanmacca, & Chavez, 2008; Gerston, Jordan, & Flojo, 2005). There is also an extensive body of research that has shown that explicit teaching of strategies for particular groups of basic facts has facilitated fact recall and application in problem solving (e.g., Mercer & Miller, 1992; Rightsel & Thornton, 1985; Steinberg, 1985; Thornton & Smith, 1988).

The conceptual framework for this study was based on the premise that it is important for teachers to spend time in promoting thinking skills and strategies for basic facts with the goal of automaticity of basic facts. If learners are armed with automatic recall of basic facts as well as a wide range of thinking strategies, they are in a strong position to develop fluency in mental computation.

The research question of interest in this project was: What is the impact of an enriched program of number study targeting basic number fact recall upon mental computation fluency for children in (a) Year 3 associated with addition and subtraction; and (b) Year 4 associated with multiplication and division?

The Study

A key element of our design was to align the project goals with the needs of participating teachers. We recognized the importance of providing teachers with authentic, practice-based learning opportunities drawn from research into basic number fact development, opportunities to experience these investigations as learners themselves, and opportunities to share their ideas and experiences with colleagues, including the challenges encountered and their insights into the process. We drew on the Loucks-Horsley, Love, Stiles, Mundry and Hewson (2009) research and development framework as the methodology to our study.

As part of a larger study, this paper reports on results from two schools (A and B). School A is a large (approximate enrolment of 950) primary school (Prep-6) located in an outer-urban community of low to middle income (ICSEA 985). All six Year 3 classes (140 students) and five Year 4 classes (134 students) participated in the study. School B is a mid-range (approximate enrolment of 750) primary school (Prep-6) located in a coastal community of middle to high income (ICSEA 1073). All four Year 3 classes (87 students) and five Year 4 classes (129 students) participated in the study.

Instrument

A purpose-designed pen and paper Number Fact Quiz was developed by the researchers. It consisted of 60 items organised into three sets. Sets 1 and 2 consisted of 25 items each that targeted number facts for either addition and subtraction (Grade 3) or multiplication or division (Grade 4). Each set was organized according to perceived complexity (i.e., items in Set 1 were perceived to be less complex than items in Set 2). Set 3 consisted of 10 items requiring mental computation (that is, at least one of the numbers was greater than 20).

Procedure

At the beginning of the school year, the pre-test was administered to each class of students. To ensure a level of consistency, the lead teachers at each school implemented the test with each grade level at their respective schools. They read each item to the students and allowed 6-8 seconds for the basic number fact items (Sets 1 and 2) and 10 seconds for the mental computation items (Set 3). Project teachers attended a half-day professional development session where strategies and teaching approaches for developing number fact recall and mental computation was presented. The place of mental computation and basic fact recall within the Australian Curriculum: Mathematics was also revisited. Over a 20-week period (two school terms), classroom teachers prioritised number fact lessons for at least 15 minutes per day. Project leaders conducted school and classroom visits and undertook ad hoc classroom observations during each school term. The post-test occurred at the beginning of the last school term.

Results

The pre and post-test results for Grade 3 at both Schools A and B for each set of items on the quiz are presented in Figures 1 – 3. The pre and post-test results for Grade 4 at both Schools A and B for each set of items on the quiz are presented in Figures 4 – 6. What is most notable from all figures is the substantial gain in performance between the pre and post-test. In both schools, students' performance is 100% or very close to for particular number fact items, particularly for those items in Set 1.

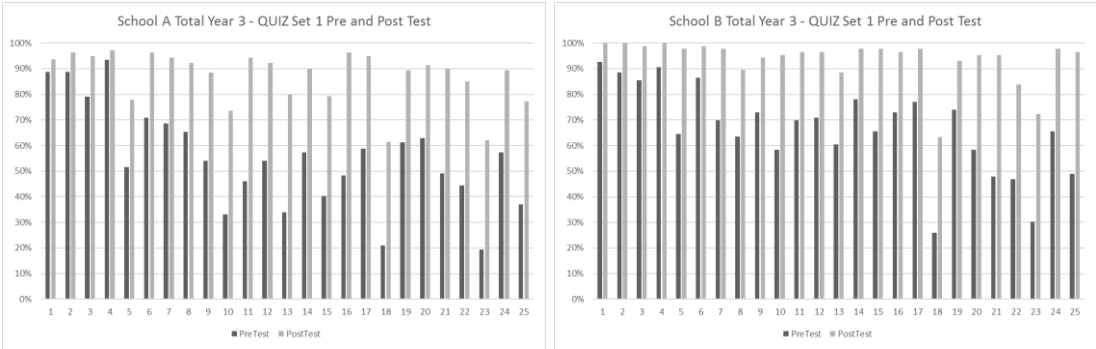

Figure 1. Set 1 pre and posttest scores for Year 3 School A (*n* = 140) and School B (*n*= 87) respectively

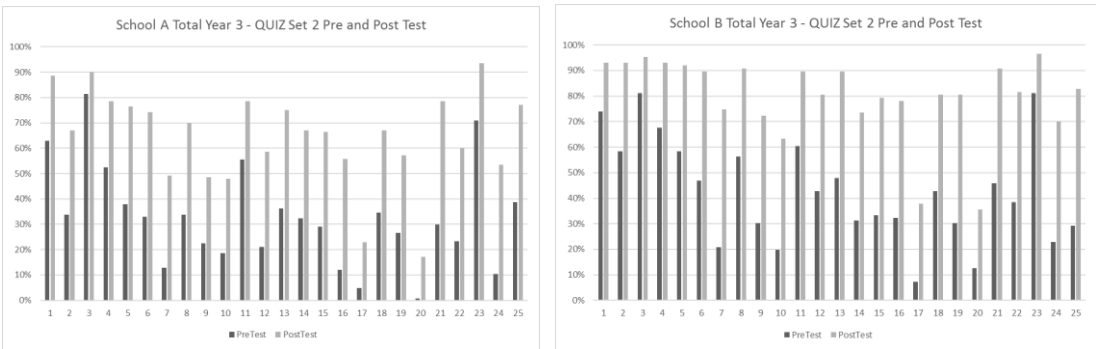

Figure 2. Set 2 pre and posttest scores for Year 3 School A (*n* = 140) and School B (*n* = 87) respectively

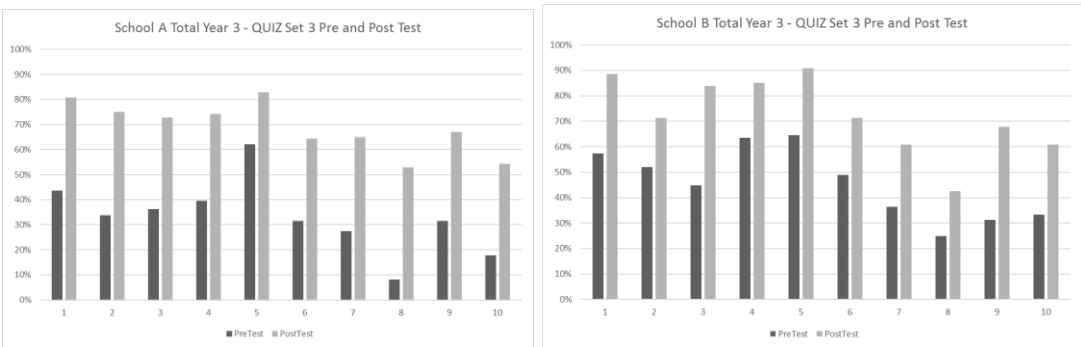

Figure 3. Set 3 pre and posttest scores for Year 3 School A (*n* = 140) and School B (*n* = 87) respectively

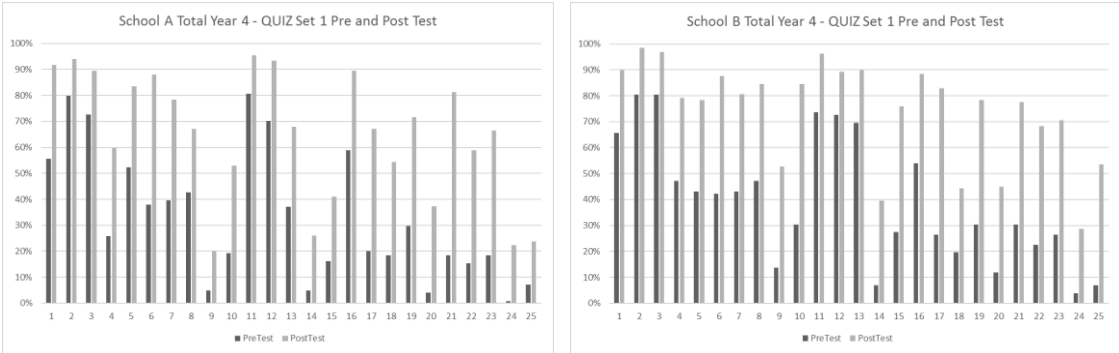

Figure 4. Set 1 pre and posttest scores for Year 4 School A (n = 134) and School B (n = 129) respectively

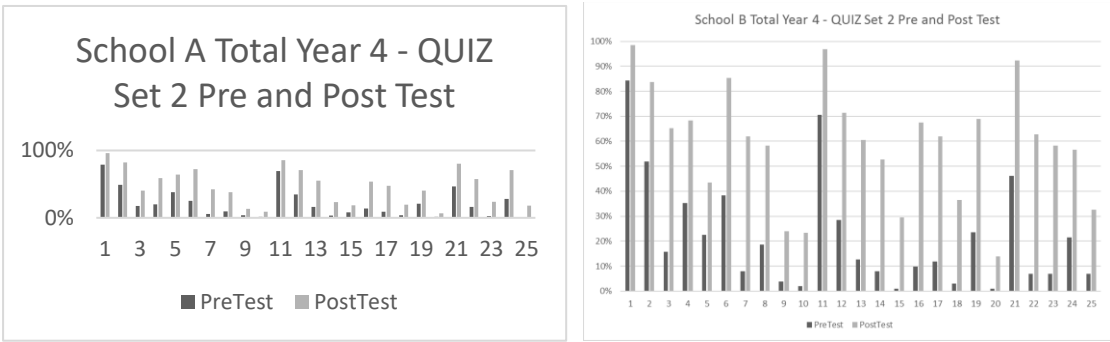

Figure 5. Set 2 pre and posttest scores for Year 4 School A (n = 134) and School B (n = 129) respectively

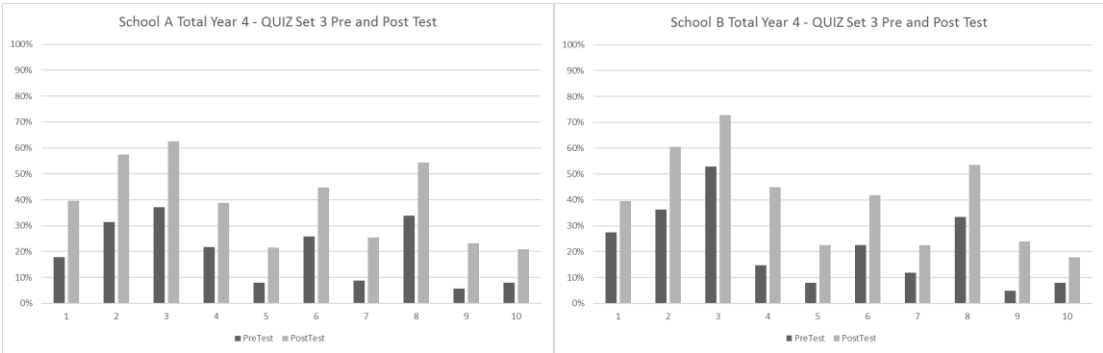

Figure 6. Set 3 pre and posttest scores for Year 4 School A (n = 134) and School B (n = 129) respectively

The overall mean score for each section of the quiz for School A and School B by year level is presented in Table 1. From Table 1, it can be seen that each Year level in both schools had considerable gains in results at the pre-test mark compared to the post-test. The year level cohorts from both schools started at similar but slightly different baselines and the overall performance on each set of quiz items was greater for School B than School A.

However, the percentage increase gains for School A and School B in each year level cohort is similar.

Table 1
School A and B pre and post-test mean scores for each section of the quiz

	School A		School B	
	Pre-test	Post-tests	Pre-test	Post-test
Grade 3				
Set 1 (/25)	14 (56%)	22 (88%)	17 (68%)	24 (96%)
Set 2 (/25)	8 (32%)	16 (64%)	11 (44%)	20 (80%)
Set 3 (/10)	3 (30%)	8 (80%)	5 (50%)	7 (70%)
Grade 4				
Set 1 (/25)	8 (32%)	16 (64%)	10 (40%)	19 (76%)
Set 2 (/25)	5 (20%)	12 (48%)	6 (24%)	15 (60%)
Set 3 (/10)	2 (20%)	4 (40%)	2 (20%)	4 (40%)

Discussion

The results presented in Table 1 show the overall gains from the pre-test to the post-test for both cohorts in both schools. The data indicate that the Grade 3 students returned much stronger results on the pre-test that their Grade 4 counterparts. The Grade 3 test items all focused on addition and subtraction whilst the Grade 4 test was only multiplication and division. What is noticeable are the strong gains on the post-test for the Grade 4 students indicating the value of a targeted focus on multiplication and division number facts. Also noticeable is the similarity of gains for each school particularly in relation to Set 3.

Set 1 and Set 2 items on each test for each cohort indicate a different level of difficulty, as student performance was relatively better for Set 1 items than Set 2 items. This was purposely built into the design of the instrument, and results suggest that some types of number facts are easier for students than others. For the Grade 3 test, Set 1 included counting-on and doubles facts and Set 2 was associated with the ‘make a 10’ strategy. For the Grade 4 test, Set 1 included multiplication and division with zero, one, doubles (multiply by 2), double doubles (multiply by 4) and multiply by 3. Set 2 included items associated with multiplying by 10, 5, 9 and square numbers. Set 3 items were not presented as number facts and required students to apply their number fact knowledge to items requiring mental computation. Grade 3 students performed better at applying their addition and subtraction number fact knowledge than Grade 4 students in applying their multiplication and division number fact knowledge. It is argued that the Grade 3 students in this study demonstrated much greater fluency in addition and subtraction mental computation than their Grade 4 counterparts did with multiplication and division mental computation.

Teachers reported that the grouping of facts by strategy provided them with valuable diagnostic data upon which they could focus their teaching. As can be seen from Figures 1-6, there are marked dips in performance associated with particular items. For example, the majority of students in both Grade 3 cohorts experienced difficulty with item 13 (□ +7 = 9), item 18 (□ - 2 = 4), and item 23 (complete the triad: 8, ?, 2). All of these number facts were classified in the ‘counting on 2’ category and were in Set 1 of the quiz. Teachers reported their surprise that such ‘easy’ facts caused difficulty to their students, but realised that the

symbolic representation of the items on the test would have most likely contributed to student performance. Varying the symbolic representation of number facts became a teaching point in the classrooms. From the Grade 4 data, noticeable difficult items were items 9, 14 and 24 in Set 2, all of which are associated with multiplication by 4. Items 20 and 25 (also in Set 2) also caused difficulty, and these items are both associated with multiplication by 3. Teachers reported that analysing the data and identifying patterns of difficulty provided them with greater focus for their teaching. The results of this study suggest the value of targeting groups of basic facts to facilitate recall and application in problem solving, as suggested in the literature (e.g., Mercer & Miller, 1992; Rightsel & Thornton, 1985; Steinberg, 1985; Thornton & Smith, 1988).

During this project, the teachers committed 10-15 minutes per day to number facts. There was no prescribed approach, but guidelines were provided. There were three common classroom practices employed that included focused teaching, practice and whole class number talks. Focused teaching comprised visual and hands-on activities that exposed students to multiple representations of facts/number combinations to develop understanding of the connections within fact families. Practice Time (3 x 10 minutes per week) included warm ups, games, partner quizzes. Students engaged in paired and group card and dice games and other activities designed by their teachers as well as students. Whole class Number Talks occurred once a week (approximately 15 minutes) with the teacher providing a question and requiring students to share their thinking with the class. Such questions might include discussing strategies for mentally computing 98+45, for example. Results from this study, in all sets of the post-test for both cohorts in both schools, indicate the value of providing opportunities to explore, discuss and justify their strategies and solutions, as suggested in the literature (e.g., Blote, Klein, & Beishuizen, 2000; Gravemeijer, Cobb, Bowers, & Whitenack, 2000).

Conclusion and Implications

This study aimed to explore the potential of an enriched program of number study on mental computation fluency in Grade 3 and Grade 4. The results of this study indicate the value of a targeted focus on number facts throughout the year. Results also indicate the developing flexibility of students in carrying out procedures for mental computation given the strong gains in performance on the pre and post-tests. The data provides some measure of students' capacity to apply basic fact knowledge for mental computation and arguably some measure of their fluency. Further research, with a more refined instrument, would assist in exploring students' fluency with number facts and application in problem solving. The results do indicate the capacity of Grade 3 with addition and subtraction but indicate that that Grade 4 students have further distance to travel to consolidate multiplication and division facts. These results suggest a need for a continued and targeted focus on multiplication and division facts beyond Grade 4.

Of importance to the teachers in this project was how they provided students with opportunity to learn the stated content descriptors within the AC:M associated with recall of addition and subtraction facts (ACMNA055 and ACMNA056) and recall of multiplication and division facts (ACMNA075) and to report on students' attainment of aspects of the associated achievement standard for Grade 3 and Grade 4 respectively. On the pre and post-test quiz, students were provided with 6-8 seconds to respond to the number facts items, which means that students had time to apply strategies for number facts if they could not recall them instantaneously. Whilst this project did not specifically seek students' and teachers' interpretation of the meaning of the term 'recall', it is reasonable to suggest that

the emphasis on the provision of an enriched program of number fact study emphasized the importance of building students' capacity to use strategies to achieve a correct calculation rather than emphasizing automatic/instant recall of a number fact. Further research will ensure that a more meaningful interpretation of the term 'recall' in the AC:M will be explored.

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Pilot Study on the Impact of In Situ Spaced Professional Learning on Teachers' Mathematical Knowledge of Multiplicative Thinking

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This paper reports on a pilot study that incorporated an alternative professional learning model that was school-based and focused an identified area of need: multiplicative thinking. The shift to multiplicative thinking can be challenging for both students and teachers due to its multifaceted nature. The study involved the delivery of six modules of learning related to multiplicative structures, pedagogical approaches to learning and subsequent between session activities to staff in 14 participating primary schools. Our findings suggest such a model of professional learning that includes enactment and reflection supports change in teachers' pedagogical content knowledge.

In the current political climate, there is increased pressure on teachers to improve student-learning outcomes in mathematics education. Within Australia and New Zealand major initiatives such as, 'Count Me In Too' in New South Wales, the 'Victorian Early Numeracy Research Project', and the 'New Zealand Numeracy Development Project' were implemented to improve the professional capabilities of teachers and subsequently raise student achievement in mathematics (Bobis et al., 2005). Key components of these initiatives included the development of learning frameworks for teachers designed to identify student learning and inform planning; professional learning programs for teachers; the use of one-to-one interview assessment tools; and the appointment of consultants or numeracy coaches to support teachers in their planning and teaching. The aim of these projects was to link professional learning to students' learning and classroom practice through off-site professional learning with the support of a numeracy leader, or external mentor (Bobis et al., 2005).

This paper presents evidence of an alternative professional learning model that was offered in situ (based in individual schools); required a whole school commitment; and targeted at a specific area of need, namely multiplicative thinking. We argue that such a model that involves teachers as learners in a learning community, and directly relates to classroom practice, has the potential to impact on teachers' mathematics content knowledge (MCK), pedagogical content knowledge (PCK), and subsequent student learning.

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Theoretical Background

The research literature drawn on to inform this study included teacher professional learning models and the importance of multiplicative thinking, and the difficulties of teaching and learning this content.

Recent studies highlight the need to situate professional learning for teachers in realistic contexts as part of the on-going work in schools, in contrast to one-off models of professional development (Bruce, Esmonde, Ross, Dookie, & Beatty, 2010). Teachers are seen as learners and schools as learning communities (Clarke & Hollingworth, 2002). Bruce et al., (2010) support Clarke and Hollingworth's notion of professional learning being embedded in classroom experiences and practices within the school context, and argue that such professional learning is characterised as occurring in sustained and iterative cycles of planning, practice and reflecting. Furthermore, Desimone (2009) suggested that professional learning should be over an extended period of time, because significant change in teacher practice, and subsequently student learning can take up to five years. Timperley, Wilson, Barrar and Fung (2007) suggested that, "professional development that led to sustained better practice, had a focus on developing teachers' pedagogical content knowledge in sufficient depth to form the basis of principled decisions about practice" (p. xivi). Cobb, Wood and Yachel (1990) suggested an effective motivator for change could be to create "cognitive conflict" in the teachers' minds, by challenging their approach prior to them attempting to modify their classroom practice.

Clarke & Hollingsworth (2002) developed an Interconnected Model of Teacher Professional Growth that elaborated on a linear professional learning model proposed earlier by Guskey (1986). This model indicates a shift in emphasis in relation to professional learning and teacher change, from perceiving teachers as passive participants in a deficit model, to seeing change as a complex process that involves learning and growth. Within such a model teachers are considered "active learners in shaping their professional learning through reflective participation in professional learning programs and in their practice" (Clarke & Hollingsworth, 2002, p. 948).

The Interconnected Model of Teacher Professional Growth highlights four domains (external, personal, practices, and consequences) within a change environment. Each of the domains is connected so that change in one domain leads to changes in other domains through processes of 'enactment' and 'reflection'. Enaction is the process of interpreting and acting on a set of beliefs and pedagogy. In other words enaction is putting new ideas or new beliefs into practice. Reflection works with enaction to ensure that the implemented action is actively and carefully considered over time. The model focuses on two reflective practices: reflecting on the changes in teacher beliefs; and reflecting on the implementation of the new knowledge or new pedagogy. It also emphasises the change environment, and the impact the environment in which the teachers' work, has on teacher change.

Others (e.g., Clarke, Clarke, & Roche, 2011; Sowder, 2007) emphasised the importance of understanding how students think about and learn mathematics. Sowder purported that students provided an interpretive lens that "helps teachers to think about their students, the mathematics they are learning, the tasks that are appropriate for the learning of that mathematics, and the questions that need to be asked to lead to better understanding" (p. 164). Clarke, et al. (2011) argued that the use of task-based one-to-one interviews builds teacher expertise through "enhancing their understanding of individual and group understanding of mathematics" (p. 901) and thus building teachers' PCK and their MCK related to particular content matter.

Multiplicative Thinking

A recurring theme in the literature is that multiplicative thinking is central to students' mathematical understanding is the basis of proportional reasoning, and a necessary prerequisite for understanding algebra, ratio, rate, scale, and interpreting statistical and probability situations (e.g., Hurst & Hurrell, 2014). Some researchers argue that the difficulties associated with students' lack of proportional reasoning are related to their limited experiences of different multiplicative situations (e.g., Greer, 1988) or to their reliance on additive thinking when multiplicative thinking is required (e.g., Hurst & Hurrell). Sullivan, Clarke, Cheeseman, and Mulligan (2001) argued that teachers' reluctance to engage students in problems that gradually remove physical prompts and encourage students to form mental images of multiplicative situations is possibly why students do not make the transition to abstracting. Greer (1988) suggested three ways to overcome a reliance on additive thinking: first to include more complex number combinations in word problems so that the appropriate operation cannot be intuitively grasped; second to provide multi-step word problems, rather than single operation word problems; and third to experience the different multiplicative situations (Equal Groups, Rate, Multiplicative Comparison, and Rectangular Area/Array). Given the complexity and importance associated with developing multiplicative thinking, teachers need to have a sound mathematics content knowledge and pedagogical content knowledge for developing multiplicative thinking in their students.

Informed by the research literature this pilot study was situated within the teachers' own school and was directly related to their practice. Multiplicative thinking was identified by the teachers as a current concern, and that they were struggling to move students from additive to multiplicative thinking. The professional learning (PL) was spaced across three terms, and focused on developing teacher content knowledge relating to multiplicative thinking and pedagogical practices. Part of the PL required participants to conduct one-to-one interviews with students to assist the teachers to understand how their students are developing multiplicative thinking- thus building the teachers' PCK (Clarke et al., 2011).

In this paper we address the research questions: What is the impact of an in situ, spaced, professional learning on teachers' pedagogical content knowledge for developing multiplicative thinking in their students? What challenges do teachers experience when planning for and teaching multiplication and division?

Methodology

The following study is a pilot study of professional learning involving 14 schools out of a possible 57 Catholic primary schools in a New South Wales Catholic Education System. The results of this study will inform a larger scale study to include more system primary schools over a period of four years (2016 -2019).

The approach taken in this pilot study involved five 90-minute professional learning modules, delivered to each school across three school terms. The focus of the PL was to increase teachers' pedagogical content knowledge relating to multiplicative thinking and how students develop multiplicative thinking. The PL targeted Stage 2 (Year 3 & Year 4) teachers. However, each school had identified in their School Action Plans a high proportion of other students at their schools who were still reliant on additive strategies for multiplication and division. The pilot schools wanted the PL to be delivered to all teachers from Foundation to Year 6. The PL included between session activities that required participants to administer a multiplicative thinking interview with a sample of students from each grade and for teachers to trial tasks with their whole class.

Professional learning structure. The research team, led by a university academic, developed a series of modules with Teaching Educators (TEs) from a New South Wales Diocese. All the PL modules followed the same structure and included:

1. a professional reading about the multiplicative structure;
2. opportunities in Modules 2-5 to analyse student data (student work samples) after the tasks had been completed in the mathematics lesson;
3. opportunities for teachers to reflect on these student work samples - the observations of multiplicative thinking in student responses were recorded in teacher reflective journals;
4. opportunities for teachers to solve a series of learning tasks focused on each multiplicative structure that they would then plan to teach to their students after each module; and,
5. teaching the tasks from each module as a between module activity.

Professional learning modules. Module 1 was an overview of multiplicative structures and introduced teachers to a new multiplicative thinking interview; Module 2 examined the use of arrays as a multiplicative structure; Module 3 examined the multiplicative structure 'times as many'; Module 4 examined the multiplicative structure of allocation and rate; and, Module 5 involved an analysis of interview data and teacher reflection of student learning. Within each module there were challenging tasks related to the content and ways to adapt and extend tasks. Throughout each module important ideas about learning mathematics with understanding (exploring, reasoning, questioning, justifying and reflecting) were included. The elements of a lesson structure (problem solving, sharing solution methods and discussing the effectiveness of solution methods) and planning lessons were also considered in each module. Each module also included a resource pack for teachers.

Seven TEs facilitated the PL at participating schools they were aligned to across three terms (Term 2-4) as part of each schools' regular after school mathematics PL. Some TEs facilitated the PL in two or more schools. Pre- and post-surveys were completed by the teachers and leaders within the first and last modules. The facilitators' materials consisted of:

- a multiplicative thinking interview given to a sample of three students from each grade at two points in time (pre- and post- the teacher professional learning modules);
- a teacher survey of mathematical content knowledge about multiplicative thinking and how students learn to think multiplicatively in problems situations at two points in time (pre- and post- the professional learning);
- four professional readings for teachers about each multiplicative structure;
- a series of tasks focused on each multiplicative structure; and,
- work samples of students' solutions to the mathematics tasks taught after each module.

Participants. The participants for this pilot study included all classroom teachers lead teachers, Assistant Principals and Principals at each participating school (N=230). The school principals, teachers and students involved in the pilot study agreed to be part of the research. The Diocese Catholic Education Office gave permission for the research to be undertaken and University ethics approval was received.

Data collection. The data reported in this paper is from two of the nine open response questions in the teacher pre- and post-surveys. These questions were designed to gain insights into teachers' understanding of how students develop the ability to think

multiplicatively in problem situations and the perceived challenges they face when planning and teaching multiplication and division. Two open response questions reported here include:

- How do you believe students develop multiplicative thinking?
- What are the main challenges you experience when planning and teaching multiplication and division? Why do you think this?

Data analysis. The teachers' responses to open response items were entered into a spreadsheet, coded then categorised through the analysis of data using a grounded theory approach (Strauss & Corbin, 1990). If a teacher shared multiple ideas or themes, each was coded as a separate response. The first two authors independently coded the teachers' responses using open coding to identify key themes. In collaboration, these authors conducted a further cycle of coding to derive 10 agreed categories. These ten categories were further refined to create eight categories with the input of the other authors. The frequency of responses for each category was collated and patterns identified across pre/post data.

Results and Discussion

This section presents the results relating to the open response questions, including teachers' perceptions of how students develop multiplicative thinking, and teachers' main challenges experienced when planning and teaching multiplication and division. There were less responses for the question related to challenges associated with planning and teaching as some of the participants were not classroom teachers.

Table 1 shows the pre- and post-responses related to the eight categories developed from the analysis of the data. The categories relating to multiplicative data (1-5) are presented in order of the percentages of responses in the pre survey data, then the general categories.

Table 1
Percentage of Responses Relating to How Students Develop Multiplicative Thinking

Category	Pre (n=244)	Post (n=236)
Multiplicative categories		
1. Using arrays, partial arrays and visualising the structure	11	28
2. Moving from additive moving to multiplicative thinking	8	2
3. Use of multiplicative language and recognising the relationship between multiplication and division	7	8
4. Being challenged to use more efficient strategies	4	15
5. Experiencing multiplicative structures	0	17
General categories		
6. Materials and representations moving to abstract thinking	30	11
7. Engaging in real life problems and open tasks	27	12
8. Teacher demonstration and practice	13	7

Prior to the PL approximately 70% of responses related to general pedagogical approaches to mathematics, compared to 30% post the Pl. The percentage of responses relating to multiplicative thinking (70%) was more than double that prior to the PL (30%), which appears to suggest that the professional learning program had a positive impact on how they considered students develop multiplicative thinking. The TEs indicated that they emphasised the use of materials and representations, moving to abstract thinking and real-

life problems and open tasks in their work with schools, so it is not surprising to see the higher proportion of responses for these two categories. More responses focused on the need for students experiencing different multiplicative structures, being challenged to use more efficient strategies and the use of arrays, after the professional learning.

The data related to the second survey question is presented in Tables 2 and 3. The teachers' responses relating to planning are presented in Table 2. The categories are ordered according to the responses for the pre-survey data related to multiplicative thinking first and then general pedagogical issues.

Table 2
Percentage of Responses (Pre/Post) Relating to Challenges when Planning

Category	Pre (n=217)	Post (n=196)
Multiplicative thinking issues		
1. Incorporate multiplicative language in planning and teaching	6	10
2. Provide experiences of both aspects of division	5	7
3. Provide experiences of the different multiplicative structures	2	12
4. Moving students from using additive to multiplicative strategies	2	8
General pedagogical issues		
5. Catering for diversity: Planning open-ended task	47	32
6. Writing enabling and extending prompts	1	8
7. Using relevant real life contexts and problems that challenge students	25	16
8. Moving students from materials and representations to abstract thinking	6	4
9. Making links between assessment data and syllabus expectations	4	2
10. Time to plan as a team	3	1

The main challenges teachers had post the PL were still predominantly related to catering for diversity, and using relevant real-life problems that challenge students. Planning open-ended tasks and problems relating to real life context is challenging for teachers because generating such task requires sound understanding of the key ideas underpinning the mathematics content and knowledge of their students' conceptual understanding. Although the PL engaged teachers in these aspects they are still areas of concern for teachers.

While their knowledge of the components of multiplicative thinking has increased, so has the challenge of incorporating them into their planning as they are still assimilating some of this new learning. As Desimone (2009) indicated, change takes time, so ongoing support from Lead Teacher and TEs would be encouraged. These results also indicate that aspects relating to planning need to be incorporated into the full implementation of this project.

Table 3
Percentage of Responses (Pre/Post) Relating to Challenges when Teaching

Category	Pre (n=217)	Post (n=196)
Multiplicative thinking issues		
1. Encouraging students to use efficient methods to solve problems	14	30

2. Encouraging students to make links between multiplication and division	10	25
3. Encouraging the use of multiplicative language	12	25
General pedagogical issues		
4. Finding the balance between explicitly teaching strategies and allowing students to generate their own.	19	4
5. Predicting challenges or possible questions students might have and knowing how to respond in the moment	11	6
6. Allowing students enough time to struggle before intervening	10	5
7. Parental pressure to teach the "times tables" and long division	10	2
8. Lacking knowledge to teach this content well	14	3

As reported in Table 3 the responses prior to the professional learning suggest more of a tension between getting the balance right between explicit teaching and student generated strategies, knowing when to hold back from telling, and how to respond in the moment. While these aspects were still present in the post-survey responses they were not the main challenges. The majority of post-survey responses (80%) related to challenges associated with teaching related to multiplicative thinking compared with 36% of responses before the professional learning. In particular, encouraging students to use efficient methods to solve problems, make links between multiplication and division, and to use multiplicative language, were aspects of their practice they found challenging. Having greater awareness of the importance of each of these components in assisting students to move from additive to multiplicative thinking is encouraging.

The post professional learning survey data relating to challenges in planning and teaching for multiplicative thinking highlight a need to place greater emphasis on task selection, catering for diversity, links between multiplication and division, and multiplicative language around planning and teaching in the revised professional learning program.

Concluding Comments

This PL was informed by earlier research (Clarke & Hollingsworth, 2002: Clarke et al., 2011; Timperley et al., 2007) in that it was design around an identified area of need; teachers were active participants in the learning; and involved enactment, reflection and between session tasks. Enacting new mathematical practices and reflecting on the impact on student learning has an impact on change in teachers' understanding of how students learn. As one Year 3 teacher (Sophie) reflected in her diary:

> The language of times-as-many was challenging for students initially but once they had more experience with tasks like this, I saw a shift in the strategies they used and they were using multiplicative language and making connections between multiplication and division.

Noticing such a shift illustrates Sophie's own growth in understanding how students develop multiplicative thinking.

In relation to the research questions, the findings suggest that providing in situ targeted professional learning across a year that includes teachers conducting one-to-one task based interviews has potential to improve teachers PCK and CK with respect to multiplication and division. Although not reported here, several teachers indicated that conducting the one-to-one interviews provided a greater understanding of the complexity associated with how students develop multiplicative thinking, highlighting the value of including interviews as part of the professional learning. The findings indicated areas to explore in more depth in the full implementation of this PL model. These include making the links between multiplication and division more explicit, ways to differentiate learning, task selection, and

planning. We conclude that the teachers appreciated the opportunity to engage in PL in their school, related to an identified area of need, and to build collective understanding and work collaboratively within and across year levels.

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Opening classroom practice to challenge: The role of trust in mathematics teachers' collaborative inquiry involving co-teaching.

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When working together to enact new and challenging pedagogies, sharing classroom practice is a key resource to inform teachers' inquiry conversations. Understanding the role of trust in collaborative inquiry represents an important tension when teachers are sharing aspects of their work to interrogate and improve their practice. The study used a design-based methodology to explore the affordances of teachers' collaborative inquiry for teacher learning. Expanding the inquiry activity to include co-teaching created productive conditions to promote trust and support challenging conversations and thus had the potential to support teachers to transform mathematics teaching|learning.

Expanding what teachers know and can do is an important mechanism to promote increased student success in mathematics classrooms (Alton-Lee, 2012). There is a long-recognised need for improvements in the mathematical experiences and outcomes of many of our students (OECD, 2016). A significant influence on the educational success of students is the quality of the teaching they experience (Hattie, 2009) and what teachers know and believe about mathematics fundamentally influences their teaching (Adler & Ball, 2009). Teachers inquiring into their practice in collaboration with their colleagues is increasingly seen as a productive approach to strengthening classroom practice (Owen, 2015). Opening teachers' classroom practice can provide teachers with access to an expanded repertoire of practice ideas and classroom events. Where teachers are working together to enact new and challenging pedagogies, the sharing of classroom practice for instance through classroom observations, the sharing of classroom video, and classrooms with more than one teacher, is a key resource to inform inquiry conversations.

In the context of transforming their pedagogical practice, teachers need opportunities to articulate and evaluate their knowledge of mathematical content (Bobis, Higgins, Cavanagh, & Roche, 2012). In particular, opportunities to discuss new learning support teachers to shift from thinking about ideas to thinking about how they might be applied to enhance teaching (McPhan, Pegg, & Horarik, 2008). Hunter (2007) suggests that teachers' discussions play an important role in transforming teachers' beliefs and attitudes, promoting reflection on habitual practice, and creating opportunities for changed practice. Kazemi and Franke (2004) found that individual and shared experiences mediated what ideas teachers' made public within their collaborative activity and this shaped the nature and direction of the group's work. However teachers' interactions can both support and constrain possibilities for their professional learning (Little, 2003). For instance, norms of collegiality where validation, rather than critical reflection, is the focus can shut down critical questions and thus constrain teacher learning (Allen, 2013). Thus, to be productive for teachers' ongoing learning, professional conversations need to surface and challenge problematic aspects of teachers' practice as well as affirming effective practice.

Respectful relationships are central to teachers' collaborative activity. Respect among participants and a commitment to shared decisions are key factors promoting teachers' open

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engagement in conversations about their practice (Robinson, Hohepa, & Lloyd, 2009). Making classroom practice public involves risk for teachers and requires trust. Accordingly, trust is often seen as a prerequisite to teachers being willing to share their teaching with others, and where trust is compromised this can act as a barrier to teachers working together productively. Teachers are reluctant to expose weaknesses in their practice where there is a fear of negative consequences and where teachers are reluctant to take perceived risks, change can be constrained (Le Fevre, 2014). Understanding the role of trust in both affording and constraining change represents an important tension when teachers are sharing aspects of their work in order to interrogate and improve their practice.

The research approach

A sociocultural perspective was taken with the aim of appreciating the multiple, socially and culturally constructed realities of participants' experiences (Schoen, 2011) and the study drew on appreciative, authentic and participatory approaches. The research is grounded in assumptions concerned with equity, caring and social justice, valuing strength and difference as foundations for growth and learning, and privileging community over individual goals. It used a design-based methodology to explore the affordances of teachers' collaborative inquiry for teacher learning in the context of primary mathematics teaching|learning. Design-based research supports the "learning conditions which current theory promotes as productive but which may not be commonly practised, nor completely understood" (Design-based Research Collective, 2003, p. 5).

The study involved working in an urban New Zealand primary school over a 6-month period with three teachers referred to here as Pat, Casey and Kris, to design and implement an approach to collaborative teacher inquiry with a focus on strengthening mathematics teaching|learning. The project was explicitly focused on the generation of practice-based pedagogical knowledge and aimed to document the processes involved in knowledge production. Regular group meetings were held to develop ways for the teachers to share their mathematics teaching. The primary aim was to design a flexible and adaptive approach to teachers' collaborative inquiry, including resources to support its enactment. Between meetings, the teachers engaged in agreed activities in relation to mathematics teaching|learning including video-recording mathematics lessons and later co-teaching lessons in pairs. The negotiated shared inquiry focus for the teachers' practice was developing their use of "talk moves" (Chapin, O'Connor, & Anderson, 2009) as a pedagogical approach aimed at strengthening their target students' mathematical language and supporting them to engage in mathematical discourse.

The group of teachers met three-weekly on seven occasions usually for an hour or more at the end of the school day. The meetings included reflective conversations about classroom events, the sharing of classroom video, discussions of research-informed articles, and planning for future activities. I participated in and audio-recorded group meetings and observed a mathematics lesson in each classroom followed by a semi-structured interview with each of the teachers at the beginning and end of the study. I transcribed the interviews and group meetings verbatim and listened to the audio recordings repeatedly as the transcripts were analysed. The transcripts were coded thematically using an open-ended approach (Creswell, 2014) to identify patterns that emerged from data. A cultural-historical activity theory (CHAT) framework (Engeström, 2009) provided a conceptual tool to identify elements of the activity of teachers' collaborative inquiry, including contradictions that arose and actions taken to resolve them. As such, it was particularly important to note and account

for data that departed from dominant patterns (Braun & Clarke, 2006). This paper draws on data primarily from the group meetings and final teacher interviews.

Findings

The findings trace the teachers' collaborative inquiry approach through two distinct stages of design, although in practice there was considerable overlap in the process of transformation from the initial to the final design stage.

Initial Design Stage: Sharing Video

At the beginning of the study, the group designed a collaborative inquiry approach that broadly paralleled that of video clubs (van Es & Sherin, 2008) whereby the teachers video-recorded mathematics lessons in their classrooms, reviewed the recordings and self-selected an excerpt for the group to reflect on at a subsequent meeting. For teachers, making videos of their practice public and available for others to scrutinise represented a risk because

> [you] just don't want that perception … of people thinking that you're weak in teaching [Kris]

Sharing practice, particularly aspects of their practice that were identified as problematic, required teachers to trust their colleagues particularly as some teachers recalled negative past experiences of having their teaching observed by others. As part of a performance appraisal process, for instance, judgements about the quality of a teacher's practice had carried the risk of punitive action in some cases. The teachers felt that when they were open about challenges in practice, they could expect different responses from school leaders than they would from a colleague, for instance Kris suggested:

> if somebody talks critically or honestly about [their concerns for] a child, if it's senior management … it becomes a big deal whereas if it's colleague to colleague with no title attached … now there's *two* heads together to unpack why is that child stumbling

Hi-lighting an apparent contradiction, Kris later went on to suggest that school leaders should trust teachers to engage in robust professional conversations:

> if it is a true professional conversation that there's gotta be that trust there. If you trust that group to be having those conversations [then] actually more impact might be had because it's not going to be reported back on, it's not going to be judged against

In relation to sharing classroom video, two of the teachers recalled previous experiences of viewing video excerpts in a professional development context where the purpose was unclear, and the critique was overly negative and personal. Nevertheless, the teachers believed that reflecting on classroom video with colleagues had the potential to be instructive and accordingly two teachers volunteered to share excerpts from their classrooms. Contrary to the perceived risks associated with sharing video, the teachers' initial experiences focused on celebration rather than critique and this appeared to promote future sharing. As Kris commented:

> we didn't kind of say these are the positives these are the negatives but … she had this safe group that really *acknowledged* what she was doing in her classroom and really *celebrated* the mathematical learning that was going on; *that's* what I think made the difference. Once that initial hurdle was done then we were inundated with them weren't we, and that's that pride

In one case Casey, who had initially declined to even watch the video I had recorded in her classroom, later showed video excerpts of her teaching to the wider teaching staff as part of a literacy-focused staff meeting she was leading. Kris wondered

whether that would've happened if she hadn't shown us videos [as part of the study]

Sharing video was voluntary and excerpts were usually chosen with an explicit learning purpose whereby statements like, "I'm showing you this because …" became routine. This appeared to support teachers to take the risk of exposing weaknesses in their practice and positive experiences of sharing video in this context appeared to promote relationships increasingly characterised by trust. However, despite the explicit aim of improving teacher practice, opportunities for teacher learning were largely limited to the teacher who was sharing the particular video excerpt. For instance, during the sharing of video from Pat's classroom, other teachers did not pick up and engage in a discussion of the mathematics or Pat's practice and Pat's recount of this event was left mostly unexamined by the group as a whole. In this case, although the teachers had access to representations of Pat's practice, including through video and descriptions of classroom events, the learning opportunity appeared to be mostly limited to providing a forum for Pat to reflect on her practice. Expanding the inquiry approach to include co-teaching afforded enhanced opportunities for teacher learning for the larger group through active participation in the co-construction of practice, and this is discussed in the following section.

Final Design Stage: Co-teaching Mathematics Lessons

In keeping with a design-based study, the teachers' collaborative inquiry approach was continuously revised throughout. A feature of the design to emerge at the end of the study was a co-teaching arrangement whereby pairs of teachers planned, taught and reflected on mathematics lessons together. The co-teaching approach aligned with Murphy and Scantlebury's (2010) description of "two or more teachers teaching together, sharing responsibility for meeting the learning needs of students and, at the same time, learning from each other" (p. 1). Where and with whom the teachers co-taught varied on each occasion so that they taught together in their own and each other's classrooms as depicted in figure 1 below.

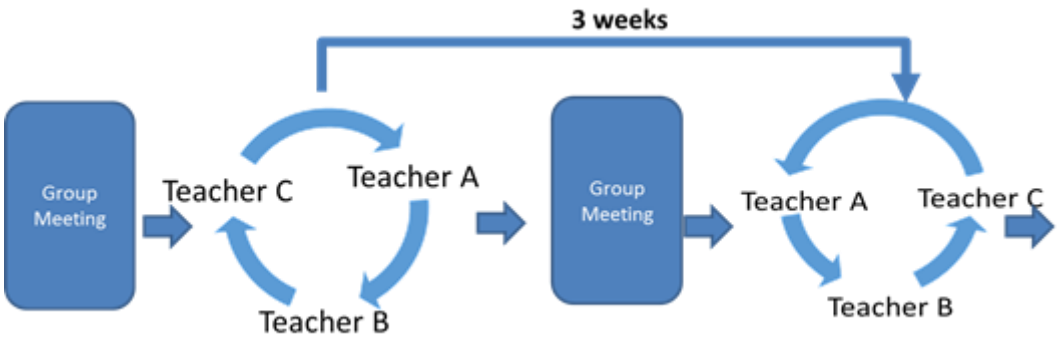

Figure 1: The co-teaching inquiry cycle

The teachers had suggested that some co-teaching arrangements might be problematic where they had previously experienced challenges in their relationship with a co-teaching partner. Contrary to the teachers' expectations, co-teaching provided a context for strengthening the trust within collegial relationships, particularly for co-teaching pairs where there had been some initial reluctance to work together. Reflecting on a co-teaching episode

involving two teachers who had previously experienced challenges in their professional relationships one of the teachers commented:

> that willingness to open up from that particular person to say I need help that was actually really powerful. That's what collaboration is

At the outset of the study, group members explicitly positioned themselves as learners and equals through the process of negotiating of their group *kawa*, or protocol for working together. Pat suggested that this orientation towards learning in the teachers' shared work supported productive co-teaching relationships:

> if people … all come in to say that I'm going to learn something from [co-teaching] then they probably won't have that much of an issue

Pat saw that the teachers' common learning goals supported the development of trust amongst them because they understood why someone was doing something:

> the advantages that we have is that we had already learnt about our talk moves and we know the purpose of having think [time]… it comes down to trust … if someone says we're gonna do some think time now then we have to trust that that's the right time to do it

Nevertheless, where relationships had been challenging in the past teachers needed their colleague's actions to reflect the learning stance they were articulating:

> people were quite clear at the beginning about roles and why they were there, but it also takes time for what's said to be actioned … words sound great to other people, but that trust has to be earnt [Kris]

The shared experience and a sense of shared accountability for the learning of a group of students supported the emergence of trust between co-teaching pairs. As Kris suggested:

> there's that accountability … even though it was my class I was accountable when you came into my room like you were accountable. It was almost like you're my mate and I didn't wanna let you down by leaving you hanging but you didn't wanna let me down by not buying into what the learning conversation was

Elaborating on the idea of teachers protecting one another's esteem, or *mana* as the teachers described it, Kris commented:

> that whole kind of concept of I've got your back … its not even I've got your back, it's that it's okay not to have it right all the time and if you haven't got it right I'm not gonna shoot you down

Comparing co-teaching to formal observations of teaching such as for appraisal, Kris suggested:

> co-teaching *I* think was that shared experience that actually you didn't do it but neither did I so that *shared* responsibility when we did talk moves when we looked at that there was that *real* the *sharing* of what was going on in the classrooms

In this way, teaching together appeared to provide support for teachers to take risks and try new pedagogical practices where the responsibility for "getting it wrong" didn't lie with any one teacher individually. Co-teaching diffused the risk of teaching challenges being exposed as weaknesses because the focus was on the students' learning rather than on an individual teachers' practice. In contrast to individually-taught lessons, reflective conversations about co-taught lessons increasingly involved teachers raising challenges in relation to classroom events. For instance, reflecting on a video excerpt from a lesson co-taught by Pat and Casey, Kris questioned the impact on the students' opportunities for think time where there were two teachers in the classroom. She had noticed that one teacher tended to fill the space left by the other when two teachers were co-instructing. She framed her question to soften the challenge she was making to Pat and Casey's practice:

> I wondered … what impact that had on the lesson for the learners … that's a snapshot of the lesson so it probably wasn't all like that but it's something that I felt too

As Pat and Casey responded to her, Kris was affirming and empathetic, and assured them that she wasn't judging them:

> *Pat:* sometimes you can't do everything perfectly
>
> *Kris:* I'm not critiquing that or anything
>
> *Casey:* you're just asking us what we think [GM#6]

Kris later remarked that she would not have raised such a challenge with just any group of colleagues:

> it's not only the questions we ask ourselves as the team but also the questions we ask each other to develop them further … like the think time or whatever that there's some people I wouldn't have asked that to because of the trust issue whereas I could ask it here and know that it wasn't a personal thing that it was accepted as a constructive question to promote thinking

Exchanges involving the teachers challenging and justifying aspects of practice became more frequent. They recognised that their co-teaching experiences and the conversations they were having were influenced by the shared understandings that the group was developing through their regular reflective conversations at group meetings. As Casey said,

> we've got a lot of pre-knowledge we're bringing already

The teachers had previously seen difficult working relationships as an unavoidable product of incompatible personalities whereas in contrast co-teaching appeared to create opportunities to reimagine and build increasingly productive relationships centred on their shared accountability for teacher and student learning.

Discussion

In the early stages of the project, teachers were reluctant to collaborate with a colleague where they perceived a lack of trust, however the sharing of classroom video and the experience of co-teaching together afforded opportunities for trust to be developed amongst members of the group. Furthermore, the influence of trust appeared to be iterative whereby increasingly trustful relationships promoted increased levels sharing of teachers' practice, and this in turn supported the kinds of robust, learning-oriented conversations that could both promote shifts in practice and strengthen trust. This is an important finding as it hi-lights how avoiding working with particular colleagues due to a perceived incompatibility and associated lack of trust can be self-fulfilling and constrain opportunities for developing productive professional relationships.

In traditional teaching arrangements where teachers are individually responsible for the learning of a group of students, a tension can emerge whereby exposing classroom challenges can direct attention to the quality of the individual teacher's practice and thus make the teacher vulnerable to the risk of negative critique and punitive action. The teachers in this study were increasingly willing to open their practice to the scrutiny of others through the sharing of classroom video and so reflecting together on episodes of classroom teaching opened opportunities for teacher learning. The opportunity to examine one's practice within a community in which relationships are characterised by professional trust supported the professional learning of the teacher whose practice was being examined. Nevertheless, there were limitations to the extent to which teachers could access the practice of others and thus the conceptual resources available (Horn et al., 2016), and conversations about individual

teachers' lessons tended to be characterised by affirmations of practice and challenging one another's practice was avoided.

In contrast, co-teaching – the act of jointly engaging in the teaching task – served to focus teachers' attention on the shared goal of student learning and thus away from their individual practice, perhaps removing a potentially competitive structure which might compromise the relationship between two teachers. In particular, the teachers explicitly identifying as learners and equals appeared to support the group to engage with one another in ways that interrupted previous patterns of participation. In CHAT terms, the teachers' actions were increasingly directed towards a common object and this supported their growing sense of the collective. The shifting of attention from their feelings about one another and their focus on an individual teacher's practice, to a shared and perhaps more neutral focus on the children's learning, redefined what constituted successful collaboration. The teachers started to see that challenging problematic practice served to promote thinking and support learning. Consistent with Roth and Tobin (2002) is the finding that co-teaching produced expanded resources for teachers with which to support both the learning of their students and their own learning. Achievements and challenges in the teachers' shared work were collectively realised outcomes thereby teachers experienced working together as both promoting success and providing support, which in turn promoted increasingly positive feelings about working together.

Within the co-teaching inquiry activity, the development of trust within the group was emergent and contingent on both teachers' actions. This involved the teachers taking risks and responding to the risk-taking actions of others within their shared activity. Furthermore, teachers' actions towards attending to and upholding one another's mana supported the building of trust that then opened space for teachers to engage in increasingly robust, learning-focused conversations. The teachers' engagement in and reflection on jointly constructed practice, that of a co-taught lesson, appeared to represent highly productive conditions for promoting the risk-taking and challenge necessary for teachers to transform mathematics teaching|learning.

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Senior Secondary Probability Curriculum: What has changed?

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In this paper, the probability content in senior secondary mathematics Victorian curriculum between 1978 and 2016 was classified and compared, by content, context, procedural complexity, the SOLO thinking frameworks and use of technology. While probability continues to form an important and increasing component of the curriculum, it has moved from the popular general mathematics to the specialised subjects. The senior secondary probability curriculum as described by the textbooks, provided students with a variety of levels of complexity and thinking in both 1978 and 2016, although low complexity questions were more common in 2016.

The mathematical knowledge and skills of Victorian students has faced scrutiny because of reported declining performance in the international studies TIMSS and PISA (Mullis & Martin, 2015; Thomson, De Bortoli, & Underwood, 2016). The number of students studying advanced mathematics is decreasing (Kennedy, Lyons, & Quinn, 2014). Much debate has occurred around why this has transpired. As part of this discussion, this paper describes the changes to the senior secondary mathematics curriculum, in particular the content and complexity of the probability component of the curriculum.

As an experienced teacher of senior secondary mathematics, I am interested in the recent changes to the curriculum and spectacle if this is related to the perceived decline of mathematical knowledge of Australian students in popular discourse. As a contribution to investigating this issue, this paper provides a comparison of the changes to one element of the curriculum. The topic of probability was chosen as it has recently gained a greater focus within the senior secondary mathematics curriculum (VCAA, 2015). Yet probability is often overlooked and is not even mentioned in the TIMSS Advanced study (Mullis & Martin, 2015).

In this paper, the 1978 curriculum was compared to the 2016 newly modifed Austalian curriculum. The comparison year of 1978 was chosen due to the availability of the ACER report, *Changes in Secondary School Mathematics in Australia, 1964-1978* (Rosier, 1980). The second comparison year of 2016, was when the Australian Mathematics curriculum was updated in an attempt to align Australian curriculum across all states and territories, and incorporate recommendations from international research (ACARA, 2012). The changes to the senior secondary mathematics curriculum included an increase in the focus on the topic of probability, as described by the formal written curriculum (VCAA, 2015). The textbooks were used as an indicator of this intended curriculum, following the example of TIMSS (Lokan & Greenwood, 2001).

Research questions

The research questions considered in this paper are:

1. How has the topic of probability within the Victorian senior secondary mathematics curriculum changed over the years 1978-2016?
2. Which methods have been found to be most appropriate for classifying

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mathematical questions?

Background

This section describes the structure of the mathematical subjects in the comparison years of 1978 and 2016. Mathematics was non-compulsory in senior secondary schools in both years.

Each mathematics subject in both years was described in its own textbook, which many teachers used as a pseudo- curriculum (Lokan & Greenwood, 2001).

In 1978, senior mathematics subjects in Victoria were General Mathematics, Pure Mathematics and Applied Mathematics. These subjects contributed to the HSC (High School Certificate) (Rosier, 1980). General Mathematics was the most popular mathematics and was taken by any student wanting to continue to university. General Mathematics and Pure Mathematics could have been taken alone, while Pure Mathematics was also a co-requisite for the specialised Applied Mathematics. Pure and Applied mathematics lead to tertiary studies where mathematics was an integral part of the discipline. Four figure mathematical *tables* (Kaye & Laby, 1977) were supplied for all examinations, which included probability distribution *tables* and miscellaneous formulae.

In 2016, the senior mathematics subjects in Victoria were Further Mathematics, Mathematical Methods and Specialist Mathematics, which contribute to the VCE (Victorian Certificate of Education) (VCAA, 2015). Further Mathematics was designed for general preparation for employment or further study and was the most popular subject. Mathematical Methods could be taken alone, with Further Mathematics or was a co-requisite for Specialist Mathematics, which was designed for further study in engineering or pure mathematics. CAS (Computer Algebra System) calculators and bound reference books were assumed for some examinations, but not all.

Methodology

Mathematics curriculum has been investigated from various perspectives, including content, context, procedural complexity, levels of thinking and use of technology. These perspectives were supported by the frameworks used in PISA, TIMSS, and the SOLO taxonomy (Structure of Observed Learning Outcomes) (Biggs & Collis, 1991).

The content of the senior mathematical subjects was described, focusing on the probability topic and highlighting the recent curriculum changes.

The context surrounding the mathematical questions were classified as these influence engagement according to the international study of PISA (Thomson et al., 2016). A focus on personal contexts which students could relate to, such as food or costs of phone plans, were considered more engaging than abstract contexts.

Procedural complexity as described in TIMSS (Hiebert, 2003) was based on the number of decisions: with low complexity problems described as involving less than four decisions; moderate complexity involving four or more decisions and possibly containing one sub-problem; and high complexity requiring more than four decisions and two or more sub-problems.

Levels of thinking is one method frequently used for classifying student responses. The formal curriculum guidelines for both comparison years refer to thinking and reasoning as being significant (Rosier, 1980; VCAA, 2015). When considering levels of thinking, the SOLO taxonomy has been found to be a valuable method for classifying student responses, especially in probabilistic thinking (Mooney, Langrall, & Hertel, 2014). SOLO has also been

used to design assessment questions (e.g. Collis & Romberg, 1992). The aim of the current study was to classify mathematical questions using the SOLO framework and to describe the anticipated level of thinking expected for students' responses to these questions.

SOLO classifies responses and questions with unistructural/multistructural/relational (UMR) cycles within a neo-Piaget's framework (Biggs & Collis, 1991). Classification of questions according to the SOLO level of thinking required was determined through the following considerations. It is anticipated that Year 12 students would be at Piaget's formal mode of thinking. A question was classified as requiring unistructural thinking if it needed one formula or concept, including definitions. A question was deemed as requiring multistructural thinking if it involved one formula but used in different ways; for example, using the formula backwards, or if it involved two concepts. Relational thinking involved using several formula or concepts and relating them together.

The current use of technology including CAS calculators has been well documented (e.g. Geiger, Forgasz, Tan, Calder, & Hill, 2012) so will not be discussed in detail here, however calculator use has certainly changed over the years and therefore warrants a mention.

Method

This paper concentrates on the intended mathematical curriculum as described by the most popular textbooks of 1978 and 2016. Textbooks have been used as the pseudo- intended curriculum at the senior secondary level (Lokan & Greenwood, 2001; Son & Diletti, 2017). Each question in the review section of the textbooks was categorised by:

- Content: probability topics
- Context: abstract, occupational, personal
- Procedural complexity: low, moderate, high
- Expected SOLO level required to solve the question: prestructural, unistructural, multistructural, relational, and extended abstract
- Technology: CAS calculator or *tables* required.

Example of analysis method.

The method of analysis used in this study can be described though the use of two examples. The first example is from the multiple-choice section of the 2016 Specialist Mathematics exam 2, where use of a CAS calculator and reference book is assumed.

> A random sample of 100 bananas from a given area has a mean mass of 210 grams and a standard deviation of 16 grams. Assuming the standard deviation obtained from the sample is a sufficiently accurate estimate of the population standard deviation, an approximate 95% confident interval for the mean mass of bananas produced in the locality is given by **B.** (206.9, 213.1) (VCAA, 2016, p. 9)

This question involved using the formula from the formula sheet supplied, with little interpretation, with the fact that z = 1.96 needed for a 95% confidence interval being the only information to remember or obtain from the reference book. $\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$ $\bar{x} = 210, s = 16, n = 100$ are taken from the question. This was a question with low procedural complexity as only one formula was used, and less than four decisions were involved. Students needed only one concept of theory to solve this question so would only demonstrate thinking at the SOLO level of unistructural. This question used material which was new to the subject in 2016 and was not in any subject in 1978. Although a calculator was needed, a CAS calculator would have been unnecessary. This question had a personal context of food, as did several of the Specialist Mathematics questions.

The second question to demonstrate and clarify the analysis method is from the 1978 Applied Mathematics textbook review section.

> If X is a normally distributed variate with mean 4, and the probability of X exceeding the value 8.34 is 0.015, calculate (i) the standard deviation of X, (ii) Pr ($X > 1 \mid X < 4$). (Fitzpatrick & Galbraith, 1974, p. 103)

The solution was found by using the cumulative normal distribution *tables* to find the solution to Pr (Z > z) = (1 - 0.015), where z = 2.17. The formula $z = \frac{8.34 - 4}{\sigma}$ was then used to find standard deviation, $\sigma = 2$.

The second part of the question requires the conditional probability formula, $\Pr(X > 1 \mid X < 4) = \frac{\Pr(1 < X < 4)}{\Pr(X < 4)}$ which was found on the supplied formula sheet. All values were transformed into the standard normal curve and the cumulative distribution *table* was used again. This question involved more than four decisions and two sub-problems and consequently was seen as a high procedural complexity. Using the SOLO structure, this question would require students to be at a relational level of understanding, as the student would need to use several concepts and relate them together. This question would have been easier to complete with a CAS calculator, to save converting to the standard normal curve. This question was taken from the textbook, selected from a past exam, and involved an abstract context.

Results

Probability questions from the popular senior secondary mathematics textbooks used in Victoria in 1978 and 2016 were initially classified by content and subsequently classified by context, procedural complexity, level of thinking and use of technology.

Description of the probability content

The percentage of mathematics students who studied the different mathematics subjects is interesting. The percentage of mathematics students who studied General Mathematics in 1978 and Further Mathematics in 2016 was very similar at 64-66%. The percentage of mathematics students who studied Pure Mathematics in 1978 and Mathematical Methods in 2016 are also similar at 36-34%. The percentage of mathematics students who studied Applied Mathematics in 1978 at 33% was much higher than the 9% who studied Specialist Mathematics in 2016. This decrease in the number of students studying advanced mathematics has been an ongoing area of concern (Kennedy et al., 2014).

In 1978 most students completed either General Mathematics or Pure Mathematics, with the majority of Pure Mathematics students also studying Applied Mathematics. The topic of probability made up a quarter of the core content in General Mathematics, but was not part of Pure Mathematics. Probability was included in Applied Mathematics, but this topic was internally assessed and not in the external examinations. The probability content in General Mathematics and Applied Mathematics overlaps, with discrete and continuous probability components in both subjects, including probability using calculus integration. Both subjects focus on four distribution types, where the main decisions involved the use of formula from the formula booklet or *tables*. This structure insured nearly all students who studied mathematic, studied topics of probability.

In 2016 most students studied Further Mathematics and/or Mathematics Methods. Some of the Mathematical Methods students also studied Specialist Mathematics. Probability

content is not included in the Further Mathematics subject except for a brief reference to the normal distribution. Mathematical Methods includes discrete and continuous probability including calculus integration. Two probability distribution types were studied. In 2016 probability was increased in the Mathematical Methods and Specialist Mathematics subjects, to include statistical inference, sampling, and confidence intervals. This was to bring the subjects in line with the Australian Curriculum and was also recommended by international research (ACARA, 2012).

The topic of probability forms a large component of several mathematics subjects in both 1978 and 2016. The probability topic has moved from the popular General Mathematics subject to the less popular specialised subjects. The study of particular probability distributions in 1978 was decreased and the link between calculus integration and probability increased. The interpretation of probability problems increased by including inference and confidence levels in 2016.

Analysis of the probability questions

Each of the probability review questions from the popular textbooks in 1978 and 2016 were solved and classified according to context, procedural complexity, level of thinking and use of technology. A summary of the results is provided in Table 1. The 1978 review questions comprised short answer questions and extended response while the 2016 questions also included multiple-choice questions. The number of review questions in the Mathematical Methods textbook in 2016 was much higher than the other subjects, due to the inclusion of a large number of multiple-choice questions. The 1978 Pure Mathematics and 2016 Further Mathematics subjects were not included in Table 1 due to the lack of probability content.

Context remained a mixture of abstract, occupational, and personal. There were a large number of abstract questions without a real-life context, especially in the multiple-choice Mathematical Methods questions. Questions about broken machines, phones, and laptops were popular.

Procedural complexity was fairly consistent over the four subjects, except for the large number of low procedurally complex questions in the 2016 Mathematical Methods multiple-choice questions. The highest number of high procedurally complex questions is in the 2016 Specialist Mathematics subject. Questions usually explicitly stated which distributions to use, which decreased the decisions required and hence the complexity.

In 1978 a book of *tables* was used instead of the CAS calculators of 2016. Ten of the 1978 review questions would have been considerably easier if a CAS calculator had been available, for calculations using distributions or integrals. The number of questions requiring either *tables* or CAS calculators more than doubled in 2016. All the 2016 probability questions could have been solved with *tables* or a scientific calculator.

Table 1.
Analysis of Probability Review in Senior Secondary Mathematics, 1978 and 2016

Criteria		1978 review General Mathematics (Fitzpatrick, 1974)	1978 review Applied Mathematics (Fitzpatrick & Galbraith, 1974)	2016 review Mathematical Methods (Evans, Lipson, Jones, & Greenwood, 2016)	2016 review Specialist Mathematics (Evans, Cracknell, Astruc, & Jones, 2016)
Number of questions		28	31	71	30
Context	Abstract	10	5	40	17
	Occupation	12	15	26	9
	Personal	6	11	5	5
Procedural complexity	Low	11	13	52	8
	Moderate	7	9	14	11
	High	10	9	5	11
SOLO	Prestructural	0	2	0	0
	Unistructural	10	11	41	10
	Multistructural	11	6	21	8
	Relational	7	12	9	12
CAS Calculator or *tables*		5	7	22	10

Discussion and Conclusion

The data reported above provide some insights into the intended curriculum in 1978 and 2016 as indicated by the review sections of the most popular textbooks. The topic of probability has moved from the popular 1978 General Mathematics to the more specialised 2016 Mathematical Methods. This was disappointing due to the need for an increase in the probabilistic thinking needed in the general population (Batanero, Chernoff, Engel, Lee, & Sánchez, 2016).

There was a large increase in the number of review questions in the 2016 textbook, and there continued to be many probability questions which did not require *tables* or CAS calculators. There was a mixture of abstract, occupational and personal contexts for the questions, with many more questions of an abstract nature in the 2016. In 1978, there were 15 abstract questions, while in 2016 there were 57 abstract questions. Abstract questions were considered to be less engaging than real-life contexts (Vincent & Stacey, 2008).

While there continued to be a mixture of questions of varying complexity in all the mathematics subjects, the 2016 Mathematical Methods textbook included considerably more questions of low level complexity. This disparity could be exaggerated by the structure of the textbooks. In 1978 the review chapters were divided into *Sets* of questions of mixed complexity. In 2016 the review sections of the textbooks started with the low complexity questions and finished with questions involving high-level complexity. This could result in teachers and students not reaching the high-level questions at the end of the review chapter.

The review chapters themselves may not be a good indication of the textbooks as a whole. The complete textbooks need scrutiny, including the demonstration questions.

Low complexity questions tended to be unistructural (using one concept), while highly complex questions tended to be relational (relating several concepts together).

Multiple choice questions tend to be of lower complexity and unistructural, although this was inconsistent. In 1978 there were no multi-choice questions and 24 low procedural level complexity question while in 2016 there were 60 low level questions, most of them of the multiple-choice style. A few infrequent multiple-choice questions did involve the understanding of many concepts and contained complex worded problems.

Classifying the questions became more difficult in the extended response questions, as they were scaffolded with many sub-questions, making them of a high procedural complexity. Each part of the question might be a single concept or might be related to each other. Both years had similar numbers of high level complexity questions, with 19 in 1978 and 16 in 2016.

The SOLO taxonomy, the structure of observed learning outcomes, was designed to classify student responses (Biggs & Collis, 1991). It has been used here to classify the questions as demonstrated by Collis and Romberg (1992). This assumed the students' previous experiences were known, as a question solved for the first time requires a higher level of thinking than a well-practiced question (Chick, 1998). Some of the highly complex questions in the review section of the textbook were very similar to the examples from the textbook, which would make them less difficult for the students. Classifying questions by complexity or thinking is also difficult as mathematical questions can be answered in different ways.

Analysis of the intended curriculum or textbooks is scarce in senior secondary mathematics and even more infrequent in the area of probability (Son & Diletti, 2017). Further investigation is needed in how the teachers and students use the textbooks and other curriculum resources. Students are not expected to learn from textbooks on their own in Australia (Lokan & Greenwood, 2001). The teacher's implementation of the curriculum is influential. The wider curriculum including teachers and students' perceptions will be encompassed in a larger study of the senior secondary mathematics curriculum, where the intended, implemented and attained probability curriculum will be investigated through thinking frameworks.

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A Comparison of STEM and non-STEM Teachers' Confidence in and Attitudes towards Numeracy

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This study reports initial results of a larger study examining teacher practices in numeracy in the secondary sector. Survey data examined STEM teachers' confidence in mathematics topics and attitudes towards numeracy in everyday life, and the responses were compared to teachers trained in non-STEM areas. The data indicate that teachers' specialist backgrounds influence certain aspects of their numeracy. Similarities between groups highlight the contribution made by all teachers to secondary school numeracy education.

Increasing demands on Australian teachers to equip students with the knowledge, skills and dispositions to participate as effective citizens have been led by government (Ministerial Council on Education Employment Training and Youth Affairs [MCEETYA], 2008), and is driven by the Australian Curriculum (AC) and its embedded General Capabilities (Australian Curriculum Assessment and Reporting Authority, 2013). The AC and General Capabilities propose a path for education that delivers content knowledge within traditional learning areas such as Science and English, while the General Capabilities sit alongside these traditional content areas and highlight additional skills necessary for students to contribute to a global society. Teachers of all grade levels and subject areas are responsible for developing students' General Capabilities, which include such things as literacy, critical and creative thinking, and ethical understanding. Numeracy is also considered a General Capability with the aim that students:

> … develop the knowledge and skills to use mathematics confidently across other learning areas at school and in their lives more broadly. Numeracy involves students in recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully (Australian Curriculum Assessment and Reporting Authority, 2013, p. 31)

To support the implementation of the AC and General Capabilities in schools, teachers in Australia are required to provide evidence against the Professional Standards for Teachers (Australian Institute for Teaching and School Leadership [AITSL], 2011) as a part of mandatory teacher registration processes. The Professional Standards seek to articulate quality teaching practices and describe expectations for teachers' professional knowledge, practice and engagement. Pertinent to this study, two of the Standards relate to numeracy Capability. The first concerns the expectation that teachers, "apply knowledge and understanding of effective teaching strategies to support students' literacy and numeracy achievement" (p. 11), and the second emphasises teachers' capacity to interpret student assessment data to inform and evaluate learning and practice. Thus, teachers must have strong professional numeracy skills for the purposes of managing their professional administrative work, as well as for effective teaching practice in support of student numeracy outcomes.

Teachers' capacities to embed numeracy skills and model numerate behaviour is paramount to supporting students who are required to meet numeracy standards for both

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credentialing purposes and as future citizens of a global society. Teachers' numeracy capacity has been the subject of international comparison in studies such as The Programme for the International Assessment of Adult Competencies (PIAAC) and the Adult Literacy Life skills Survey. Golsteyn, Vermeulen, and de Wolf (2016) compared the literacy and numeracy skills of primary and secondary school teachers relative to other survey respondents and found that teachers score on average higher on literacy and numeracy tests than the country average. At the lower end of the distribution, the lowest scoring teachers significantly outperform the lowest scoring other respondents. The highest performing secondary teachers are comparable to other respondents and the primary teachers are only slightly outperformed than the best other respondents. Although Australian data is limited due to differences in local collection methods there are no reasons to expect that the local data would be different from the international data.

Research into teaching numeracy in Australia has predominantly examined the practices of teachers with a strong mathematical background, or those who teach mathematics or numeracy, particularly in the primary or middle school sectors (e.g., Beswick, Watson, & Brown, 2006). Relatively few studies have explored the practices of secondary teachers, although perceptible differences between primary and secondary teachers in relation to aptitude and confidence in teaching mathematics and numeracy have been identified (Forgasz, Leder, & Hall, 2017; Watson, Beswick, Caney, & Skalicky, 2006).

The professional numeracy demands of teachers relate to the mathematical methods and analytical skills necessary for ensuring high quality work in a professional capacity (Steen, 1990) and is influenced by context (Beswick, 2008). The practices and professional learning requirements of experienced teachers is an emerging field of study. With the value of equipping teachers with the requisite knowledge of mathematical processes and procedures that underpin numeracy in classroom learning is beginning to be examined (Callingham, Beswick, & Ferme, 2015; Ferme, 2015).

For teachers in secondary schools, professional numeracy skills must support the administrative requirements of their work, as well as providing a basis of numerate practices within the specific context of their subject area. Thus, the scope of professional numeracy demands of teachers are as contextually broad as the subjects they teach, include the core requirement of strong mathematical confidence and attitudes, and rely on strong pedagogical knowledge specific to numeracy. The numeracy demands of the profession beyond what is taught to students are recognised by pre-service and experienced teachers (Forgasz et al., 2017), but many teachers lack confidence in planning, teaching, assessing and creating appropriate learning tasks for the development of numeracy in their students (Goos, Dole, & Geiger, 2012).

STEM education (Science, Technology, Engineering and Mathematics) is increasingly acknowledged as critical to addressing the rapid growth of professions that rely on scientific and technical services (Council of Australian Governments, 2015). STEM subject areas are inherently connected by the way they apply mathematical ideas to solve problems, and integrate naturally across disciplines within the real world (Johnson, 2012). It follows therefore, that the numeracy demand of STEM subjects is higher than that of other disciplines. Nevertheless, it is important to recognise that all learning areas have distinctive numeracy demands in relation to the type of mathematical knowledge required by students in order to demonstrate successful learning (Goos, Geiger, & Dole, 2010). Teacher confidence and attitudes in numeracy, therefore, are critical to student outcomes irrespective of subject.

The Study

The study described here used a quantitative survey design to answer the research question: To what extent does STEM specialisation effect teachers' confidence in mathematics topics and attitudes towards numeracy?

This paper reports on two sections of a survey which was part of a mixed methodology study about secondary teachers' practices in and understanding of numeracy.

Participants. Forty-seven teachers from eight regional and metropolitan government secondary schools (Grades 7 to 12) in NSW were participants. Their tertiary qualifications reflected the diversity in educational pathways leading to formal accreditation as secondary teachers, with representatives from a range of undergraduate and graduate-level teacher education pathways. The participant profiles represented similar specialist teaching area qualifications, age and gender demographics when compared to recent teacher workforce data (Commonwealth of Australia, 2014). Figure 1 shows the participants' specialist areas, and the prevalence of teachers with more than one specialist area. For example, of the six English teachers who completed the survey, three also had specialist qualification in Humanities, and one also had qualification in Languages.

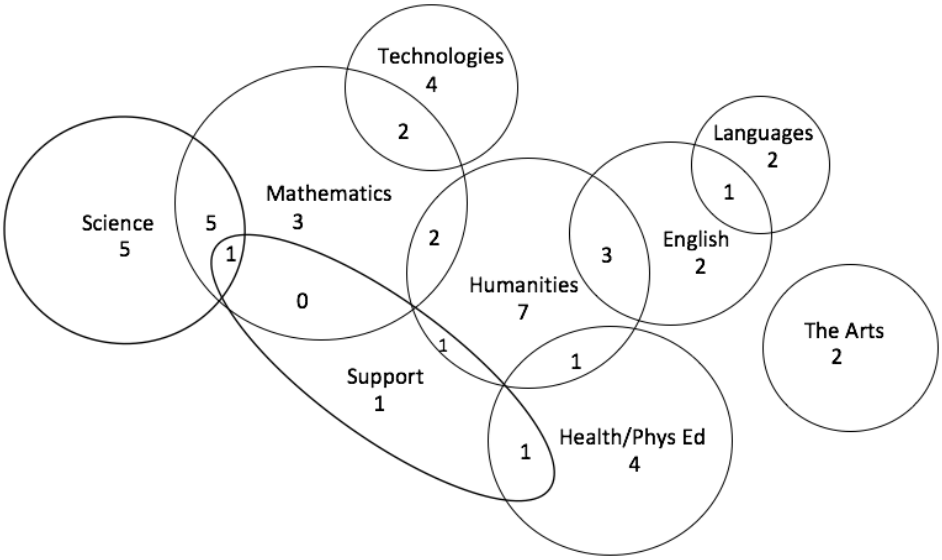

Figure 1. Representation of Participants' specialist areas.

Instrument. The survey comprised six confidence items and eleven attitude items, and were adapted from instruments used in previous studies examining teacher confidence and beliefs (Beswick et al., 2006; Watson, 2001). The confidence items were modified to focus on mathematical topics to reflect the Australian Curriculum: Mathematics strands, and the attitude items were based on Beswick's (2006) instrument and examined participants' attitude towards numeracy in everyday life. This set of items refer to numeracy rather than the original quantitative literacy to reflect the local preference for these often interchangeably used terms. Teachers responded to all items on 5-point likert scales from 1 (Strongly Disagree) to 5 (Strongly Agree).

Procedure. The teachers completed the approximately 15-minute survey via an online commercial platform. There was no time limit. Results from participants were grouped according to their teaching specialisation. Table 1 shows the numbers of teachers with STEM and non-STEM specialisations and the numbers whose highest level of mathematics studied

was secondary or tertiary. That participants with STEM specialisations were much more likely to have studied mathematics at the tertiary level reflects the significance of mathematics in STEM (Johnson, 2012).

Results and Discussion

Participants' highest level of attainment in formal mathematics was recorded. The context in which it was studied was also noted, i.e. whether it was undertaken as part of a secondary school leavers certificate, a tertiary mathematics subject (e.g. Engineering Mathematics), led to a qualification in a mathematics-dependent field such as Physics, or was part of a postgraduate degree. Note that only one participant did not undertake senior secondary mathematics (Grades 11 and 12) at all. It is not known whether participants considered statistics to be a mathematics subject in this context, although the example provided in the survey item ("Engineering Mathematics") was intended to indicate to participants the item's focus on mathematics rather than statistics.

Table 1
Highest mathematical attainment of participating teachers

Teaching Specialisation	Mathematics Attainment	
	Tertiary	Secondary
Science, Technologies, Mathematics	17	5
Other (The Arts, Humanities and Social Sciences, Languages, English, Physical and Health Education)	4	21

The means and standard deviations were calculated for each survey item for each of the groups. A two-tailed *t*-test comparison of scores was also calculated. Missing data were not included in the calculations. Pooled results for the mean and standard deviation for each set of items were calculated. These results appear in Tables 2 and 3 respectively.

Table 2
Means and standard deviations of teachers' responses to Confidence items

Item	STEM teachers (n=21) Mean	St. dev	Other teachers (n=26) Mean	St. dev	*t* (2 d.p.)	*p*-value
1. Estimating and calculating with whole numbers	4.95	0.21	4.32	0.90	3.22	0.0024**
2. Recognising and using patterns and relationships	4.73	0.46	4.20	0.87	2.56	0.0139*
3. Using fractions, decimals, percentages, ratios and rates	4.91	0.29	4.12	0.73	4.76	0.00002**
4. Using spatial reasoning	4.59	0.59	3.79	1.06	3.11	0.0032**
5. Interpreting statistical information	4.68	0.57	3.96	0.68	3.93	0.0003**
6. Using measurement	4.91	0.29	4.28	0.74	3.74	0.0005**
Pooled Results	4.80	0.30	4.08	0.69	4.24	0.0001**

Note: * indicates significance at $p < 0.05$ and ** indicates significance at $p < 0.01$

That there were statistically significant differences between the groups for all items reflects the overall higher confidence in mathematics of STEM-trained teachers compared to teachers trained in other areas, analogous to the evidence of differences between primary and secondary teachers (Watson et al., 2006).

At least 84% of all participants indicated Agree or Strongly Agree to each of the six confidence items. With respect to the mathematics strands identified, both groups of teachers were most confident in *Estimating and Calculating with Whole Numbers* and least confident in *Using spatial reasoning*. The latter item's lower confidence level may reflect teachers' lack of knowledge of what constitutes spatial reasoning due to unfamiliarity with mathematical terms (Ferme, 2015) or the narrow conception of numeracy many teachers have (Callingham et al., 2015). The only non-response recorded for confidence items was for this item.

Previous research on teacher perceptions has identified that teachers tend to focus on numerically-based aspects of numeracy (Callingham et al., 2015) which is reflected above in the results for Items 1, 3 and 6. Ninety six percent of participants indicated either Agree or Strongly Agree for Item 6. The disparity between groups for Items 3 and 6 may be because they pertained to proportional reasoning. The essential nature of proportional reasoning in developing higher-level mathematical ideas, as well as applications in other subjects, and the difficulties students have with developing proportional reasoning has been well researched (e.g., Hilton, Hilton, Dole, & Goos, 2016). Many teachers have the same conceptual difficulties as students (Sowder et al., 1998). Given measurement's strong basis in proportional reasoning (Lesh, Post, & Behr, 1988), these findings support previous research in that teachers' confidence in proportional reasoning may impact upon other mathematical topics (Beswick et al., 2006; Sowder et al., 1998).

Table 3
Means and standard deviations of teachers' responses to Attitude items

	STEM teachers		Other teachers			
Item	Mean (n=21)	St. dev	Mean (n=26)	St. dev	*t*	*p*-value (2-tailed)
A. I need to be numerate to be an intelligent consumer	4.77	0.43	4.48	1.00	1.27	0.2116
B. I am confident that I could work out how many times I would need to tile my bathroom	4.82	0.85	4.32	0.80	2.06	0.0449*
C. I often perform calculations in my head	4.86	0.35	4.04	0.98	3.74	0.0005**
D. Understanding fractions, decimals and percentages is becoming increasingly important in our society	4.27	0.70	3.76	1.09	1.89	0.0658
E. Numeracy is just as necessary for citizenship as literacy	4.59	0.59	4.28	0.84	1.45	0.1552
F. I have difficulty identifying mathematical patterns in everyday situations	*1.63*	*1.26*	*2.48*	*1.34*	-2.09	0.0428*

G. Proportional reasoning is needed to understand claims made in the media	4.10	0.83	3.50	0.93	2.25	0.0299*
H. Given the price per square metre, I could estimate how much carpet I would need for my lounge room	4.80	0.89	4.38	0.82	1.64	0.108781
I. Mathematical ideas are not always communicated well in newspapers and the media	4.32	0.84	3.80	1.04	1.86	0.069106
J. I often use mathematics to make decisions and choices in everyday life	4.41	0.67	4.12	0.88	1.25	0.216087
K. I can easily extract information from tables, plans and graphs	4.91	0.29	4.44	0.82	2.54	0.0147*
Pooled Results	4.57	0.35	4.07	0.46	4.07	0.0002**

Notes: Item F was reverse scored for the purposes of calculating the pooled result. * indicates significance at $p < 0.05$ and ** indicates significance at $p < 0.01$.

The eleven attitudes items show an overall similar difference between the two groups of participants as for the confidence items, with higher means and lower standard deviations in the STEM group. There were statistically significant differences for five of the items (B, C, F, G and K). Once again, the data indicates that appreciation of proportional reasoning (Item G) was different between STEM-trained teachers and others, however an item on media communications (Item I) that often involve proportional concepts such as percent and risk showed no significant difference. Differences for Item G may again be linked to limited of knowledge of the term among non-STEM teachers (Ferme, 2015) or teacher's conceptual difficulties (Sowder et al., 1998).

The remaining four items for which there were statistically significant differences (B, C, F and K) could arguably be linked by the requisite mathematical confidence required to perform these tasks. Item H is similar to Item B but in this case the difference between groups was not significant. Steen (1990) observed that, "unless the mathematics studied in school is understood with confidence … it will not be used in any situation where the results really matter" (p. 7). In the context of secondary schools and teachers' capacity to embed numeracy within their subject area, these data suggest that teachers other than STEM teachers may not be able to fully exploit the opportunities the curriculum offers to support students' numeracy outcomes, including in terms of demonstrating a positive attitude towards much of numeracy.

Much of teachers' other numeracy-dependent professional activity involves interpreting data (Item K). The STEM group with a high mean and a very small standard deviation. For the non-STEM group, the mean was quite high but so was the standard deviation, hence the significant difference. By contrast, the confidence item (Item 5 – interpreting statistical information) that dealt with a related area of mathematics was the second lowest-scoring item for both groups. In this case, the separation of the aptitude-focused *interpreting statistical information* in the confidence set from the application-focused *extracting information from tables, plans and graphs* in the attitude set may account for the disparity between them, echoing previous research around the influence of context on teachers' beliefs

(Beswick, 2008) and confidence around interpreting statistics in the media (Watson, 2001). However, as graphs, diagrams, tables, maps and plans are commonly used in many learning areas (Goos et al., 2010) participants would reasonably have a strong familiarity with these concepts and skills as part of their classroom teaching, reinforced by overall high numeracy performance compared to other professions (Golsteyn et al., 2016).

The role that numeracy plays in everyday life is encapsulated in items A, D, E, G, I and J. That there were no significant differences between the two groups is possibly an outcome of the overall mathematics attainment level of all participants, in that 83% of participants had studied an advanced-level mathematics course or above in senior secondary school, exposing them to the range of high-level mathematics knowledge and skills necessary for the modern age (Steen, 1990) and demonstrating the high level of education all teachers have (Golsteyn et al., 2016). Items A and J had the narrowest differences between means, suggesting that teachers' specialist backgrounds have little influence on attitudes towards the important role that numeracy has on their own lives or that of their students. It appears that there is little difference in teachers' disposition or "willingness and confidence to engage with tasks, independently and in collaboration with others, and apply mathematical knowledge flexibly and adaptively" (Goos et al., 2010, p. 212) when accounting for specialist background.

Conclusion

The data reported here are consistent with previous research identifying differences in teacher confidence dependent on their specialist teaching area (Beswick et al., 2006; Forgasz et al., 2017) and identifies that secondary STEM teachers in particular have a greater confidence and tend to have more positive attitudes when compared to other secondary teaching areas. While the interdependence of mathematical concepts and skills within STEM subjects are widely known (Johnson, 2012) attitudes towards numeracy amongst all teachers was positive overall and numeracy is recognised by both groups as playing an important role in everyday life.

The difference in confidence in and attitude towards numeracy between STEM and other teacher groups may be rooted in the same conceptual difficulties that students experience (Sowder et al., 1998). Studies have indicated that sustained programs of professional learning that provide opportunities for teachers who lack confidence in mathematics to experience success are beneficial (Beswick, 2008; Forgasz et al., 2017; Watson et al., 2006).

Teachers generally are numerate individuals and supporting non-STEM specialist teachers to improve confidence in the mathematical foundations of numeracy would have multiple benefits, particularly when effort is made to reconceptualise knowledge (Sowder et al., 1998). Assisting non-STEM teachers to become more confident in mathematical knowledge may also better highlight to them the opportunities present in curriculum and professional contexts to develop more positive numeracy outcomes for their students.

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The explicitness of teaching in guided inquiry

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Guided inquiry pedagogy is gaining recognition for promoting deep learning as students connect concepts, understandings and meanings to defend and justify their mathematical ideas. Research describes how it promotes the development of deep understandings, yet the approach can seem at odds with explicit teaching pedagogies that show potential for a rapid rise in mathematical achievement in solving simple, routine tasks. Additional pressure from timed, standardised tests can contribute to confusion about choice when teachers consider implementing pedagogies with which they are less familiar. This paper illustrates what explicit teaching looks like in inquiry as year five students explore angles in polygons.

Planned inquiry teaching and learning experiences in mathematics, as teacher resources, are becoming more available as classroom teachers seek to know more about the approach to try with their own students. Presentations amongst mathematics education researchers at MERGA, sharing research on inquiry pedagogy, raise further interest in the pedagogy and institutions such as The Australian Academy of Science have invested time and money through the ReSolve project, to draft, trial and publish a large number of mathematical inquiries for classrooms around Australia. As classroom teachers conduct trial inquiries in their classroom for the first time, surprise about the levels of difficulty or challenge for students and an unsure feeling about how to facilitate such high levels of intellectual quality are expected reactions by teachers. There has been an emphasis placed on explicit classroom pedagogy to raise the academic standards of students quickly and many teachers have become comfortable using this approach. Explicit teaching, in a sense, does take place in guided inquiry yet practically, it is not quite clear what this entails. An experienced inquiry teacher will recognise teachable moments as a moment of struggle and engineer a way forward to support learners to recognise the significance of knowing and understanding such a new concept. This paper will illustrate teacher engineering in inquiry pedagogy and how it is explicit, as students learn and make conjectures about measurement and geometry concepts.

The students in the Year 5 classroom highlighted in this paper needed to know how to use a protractor to measure angles (ACARA, 2017). Their classroom teacher engineered an inquiry that would provide opportunities to repeatedly create angles and measure those using protractors. Illustrated below are some of the teachable moments in this inquiry that reflect a sense of explicit teaching, including the teacher recognising teachable moments, their students' perturbations, an illustration of how the teacher moved forward, and the students' reactions/learning. This paper adds to an extensive body of work exploring inquiry pedagogy in primary classrooms, building on iterative phases of Design Research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The teacher-researchers and authors of this paper were interested in capturing the 'explicitness' of teaching in inquiry pedagogy to illustrate to other teachers new to this approach, what explicit teaching entails in the inquiry *How can I accurately predict the sum of the internal angles of any polygon?*

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Literature

Although this paper explores the teaching of mathematics through inquiry pedagogy, the authors are focused on recognising how teachers navigate with students through their moments of unsureity, towards sense making about mathematics. Guided inquiry presents an investigation approach to teaching and learning mathematics through four phases: Discover, Devise, Develop and Defend. (Makar, 2012). The authors of this paper argue that guided inquiry pedagogy provides multiple opportunities for explicit teaching within and between these phases, often identified as a Checkpoint in the inquiry process. The Checkpoint helps teachers to recognise difficulties (and successes) students are having and to evaluate whether explicit teaching is required. Explicit teaching may be required in guided inquiry due to a number of reasons. The exploration may have become too large (e.g. students may have collected a large number of data and are unable to see relationships between the data), or a roadblock to a new topic presents and exploration is not necessary to move learners forward (e.g. students need to know how to use a protractor to measure angles. This coming to know process for students has previously been translated by one of the authors as a process whereby learners traverse a complicated series of emergent and concomitant potentialities, engineered by the teacher (Fry, 2016). In guided inquiry, the teacher encourages emerging complex connections students make as potentialities and includes these as problems to solve, concomitant to the inquiry question. Valuing potentialities and including them as pathways to explore in guided inquiry contributes to developing a learning community in the classroom and there is much research reporting on such a classroom culture (Fielding-Wells, Dole & Makar, 2014; Goos, 2004; Makar, Bakker & Ben-Zvi, 2015). The teacher's role to challenge and scaffold teaching and learning in inquiry mathematics classrooms has been explored in terms of providing mathematical evidence, scaffolding reasoned discourse and in creating socially productive classrooms (Anthony & Walshaw, 2009; Hunter, 2012; Hunter, Hunter, Jorgenson & Choy, 2016; O'Brien, Makar, Fielding-Wells & Hillman, 2015). This paper builds on this research to illustrate how one year five teacher makes teaching and learning about angles in polygons explicit, with her students.

Teachers have been surprised about the levels of difficulty or challenge for students when implementing inquiry pedagogy in their mathematics classroom. Student mathematical learning in inquiry has often exceeded teacher expectations and student confidence is enabled when they achieve success in overcoming such mathematical challenge (Hunter, et. al, 2016). In inquiry, the intellectual quality of lessons can improve significantly over time with specific gains in higher order thinking and the problematising of knowledge (Makar, 2016). Guided inquiry presents contextualised investigations that include ambiguity to open pathways for solving the problem and to open up ways to answer the question with students making decisions about how they navigate the problem-solving process (Makar, 2012). Since the early to mid-2000s, Australian teachers have heard the effectiveness of explicit instruction for particular students (Rowe, 2006; Hattie, 2008; Melony, 2015). On the other hand, Hunter (2012) provided evidence that when a teacher taught lessons procedurally, student disengagement increased. It was reported that explicit teaching also limits students' opportunities to exercise conceptual agency (Anthony, 2013). In a problem-solving sense, explicit teaching approaches such as direct instruction strive to minimise misinterpretations by presenting carefully planned problems to suit guided practice of the process being taught (Hattie, 2008). The challenge for teachers new to inquiry is to navigate ambiguity in the mathematics classroom which is less apparent in an explicit teaching approach, to facilitate

high levels of intellectual quality and success in mathematical learning without explicitly stating how to work things out.

Method

Although non-interventional, the illustrations of teaching in inquiry analysed in this paper aim to contribute to a growing body of knowledge of inquiry pedagogy to teach mathematics. As the classroom teachers and authors of this paper become interested in teacher-led research themselves, and an interest generally in understanding how to engineer meaningful learning experiences through inquiry, the type of research begins to reflect a more participationist focus (Sfard, 2005). Part of a larger study to understand teaching and learning mathematics through inquiry, design research methodology allows the authors to build on previous iterations of study to understand the learning ecology, contributing to subsequent phases of testing and revision (Cobb, et al., 2003). The research presented here is qualitative in nature with analysis based on grounded theory methodology (Corbin & Strauss, 2008), through qualitative content analysis (Flick, 2009).

Context and participants

The year five classroom depicted in this paper was situated in a large Metropolitan school in South East Queensland. The class was an even mix of boys and girls of different backgrounds including some with EAL/D and Special Education needs. This inquiry took place in the second semester when norms around the classroom culture of inquiry had been established. The classroom teacher had participated in a longitudinal study investigating inquiry teaching in the classroom and was conducting her own research into classroom practice. She had been teaching with guided inquiry for a number of years and was becoming more comfortable with the teaching approach. The inquiry question asked students *How can we accurately predict the sum of the internal angles of any polygon?* Students already had language associated with naming polygons including being able to identify different types of triangles.

Data collection

Part of a larger study, the first three lessons of the inquiry were filmed as the class explored learning in the Discover, Devise and Develop phases. The focus on explicit teaching was not an intention of the inquiry and the teacher in the video was not aware of this research focus at the time. The videos were viewed and analysed by both researchers independently, to firstly gain a general sense of when explicit teaching of a concept took place, similar to the process of open coding (Flick, 2009). These instances were compared and categorised so that an agreed understanding could be made between the researchers about the elements constituting explicit teaching in the inquiry context. The authors used the process of axial coding to further analyse the relations between categories and to interrogate the data further for patterns related to explicit teaching in guided inquiry.

Initial viewings emphasised the identification of key mathematical concepts to do with measurement of angle that were explored in this sequence of lessons (Table 1).

Table 1
Key mathematical concepts to do with measuring angles

Student pre-understandings, difficulties and errors
A polygon where all the sides are the same length, is a regular polygon
When comparing two similar polygons where one has been enlarged, the larger shape would have larger internal angles
The sum of the internal angles of different scalene triangles will differ because scalene triangles have sides of different lengths
Inaccurate measuring: The importance of double checking measurements and the issues of not closing the corners of a polygon when tracing a shape

These identifiable moments within the lessons required further analysis to help characterise the associated teachable moments. Closer analysis would focus on how the teacher engineered a way forward for the learner in terms of explicit teaching.

Results

The students were devising their plans for finding out how to accurately predict the sum of the internal angles for any polygon. It had become obvious that some students were moving to the next phase of the inquiry and had started collecting data to answer the question. Some students traced around pattern blocks as an easy way to generate shapes to measure. Other students used a ruler to draw polygons with a particular number of sides.

Regular and irregular polygons

The teacher stops the class at a Checkpoint within the Devise phase for students to share their progress with others. One keen student, Nicholas, offers to share his plan and based on his measurements of two hexagons, states that he doesn't think there is a way to predict the sum of the internal angles of a polygon. Two of his hexagons have the exact same angles and a third has a totally different sum of the internal angles.

> Nicholas: "My first hexagon which is the… perfectly… even one, was…"

The teacher sees Nicholas is struggling to think of the term to describe the hexagon with sides the same length and interjects:

> "Do you know the word for that, that hexagon where all the sides are exactly the same? It's a regular hexagon. If it's regular, then all its sides and all its angles are the same. So, if that's regular, what do we call (Nicholas') hexagon that has still got six sides…"

This is a quick intervention by the teacher and before she finishes Nicholas responds with the word 'irregular' and continues to use these words to explain his plan. In this instance the teacher explicitly provided the students with the language of regular polygons, to support him to complete his statement. This counts as a Checkpoint in guided inquiry as all students are focused on the speaker and hear the language lesson in the context of the inquiry.

An enlarged shape has larger angles than its original shape

Later in the lesson but part of the same Checkpoint, Annabel shares her groups' plan with the class. A key idea they share is that they hope to predict the angles of four different

shapes, then measure them to generate evidence about the accuracy of their predictions. The teacher asks the class to comment and Ramon shares his suggestion of focusing on one shape such as a pentagon, and to draw another larger pentagon to measure and compare the angles. Another student, Rick, builds on this idea and makes a conjecture that the sum of the internal angles of the smaller shape would be less because it is a smaller shape. The idea that an enlarged shape will have greater angles than the original similar shape is incorrect:

> Teacher: "And you don't know until you have tested it. You don't know if you will come up with that at all."

At this point the teacher praises all the students for thinking about the problem systematically and sharing reasonable ideas. She highlights the importance of students listening to the ideas shared and not always agreeing with each other's plans. She hopes that the students will explore this line of investigation to test the conjecture made.

Irregular polygons: Scalene triangles

The following day and just before the class moves into the Develop phase of the inquiry, the teacher reviews the idea of measuring regular and irregular polygons that Nicholas proffered the previous day. This conversation takes place before students begin to put their devised plans into effect. Students then begin to collect evidence to answer the inquiry question and record measurements and calculations about the angles they are measuring in their scrapbooks. Building on the idea of measuring the internal angles of regular polygons to compare to the internal angles of irregular polygons, Nicholas turns his focus to triangles only. He discards the idea of testing scalene triangles as they are 'always different' yet is unable to explain how or if this property will change the sum of the internal angles of a scalene triangle and the teacher encourages him to test this. The students put their plans into action and soon after the teacher approaches Nicholas to check his progress. He quickly shares how the two scalene triangles he has measured both have a different result for the sum of the angles. The class had previously shared that they had heard of the sum of the internal angles in triangles always adding to 180 degrees and Nicholas has pursued the idea of using irregular polygons, such as scalene triangles, to test.

> Teacher: "Can I just check with a protractor and test your theory?"

She asks him to identify one of the scalene triangles he has measured. As she measures the internal angles, Nicholas eagerly looks at what she is doing. The teacher talks about this process using a think–aloud strategy, to make explicit the process she is modelling. For instance, she adds a dot on the page at the end of the straight edge of the protractor and states out loud how this can help her measure the angle. The teacher points to the protractor and Nicholas moves closer to take over and read the measurement. The teacher keeps checking to read the measurement until the student recognises that the angle is 103 degrees. He had previously measured this as 105 degrees and the teacher recognises this as a minor discrepancy. She identifies a different angle as a 'tiny, tiny little' angle to measure next and highlights to the student that he has recorded a measurement of 145 degrees.

> Teacher: "If you know this is a tiny little angle, what angle is it?"

Nicholas quickly explains his mistake in measuring the external angle and answers the teacher's question.

> Teacher: "So if it's acute, can it be 145 degrees?"

She highlights the error and acknowledges how this has happened. She continues to measure the remaining angle in the scalene triangle with Nicholas to check his previous measurements. He recalculates the sum of the internal angles to discover a different total.

> Nicholas: "Wow! 183 and 182!"

The teacher encourages him to try to create and measure another scalene triangle to test but he has become interested in checking his measurements on the previous triangle.

Variation due to inaccurate measuring

Towards the end of the lesson, the teacher creates a Checkpoint to bring the class back together. She asks students to share what they have discovered and Nicholas comments that the sum of the internal angles in *some* scalene triangles are the same; after all, two that he measured 'had' 183 degrees and the other one of them 'had' 178 degrees. The teacher asks the class to comment while she records the measurements on the board for all students to see. Quite a few other students in the class had focused on measuring the internal angles of triangles and the teacher asks the class to consider their measurements. She refines the inquiry question to *How can we accurately predict the sum of the internal angles of a triangle?* to guide the Checkpoint discussion.

It is at this point that the teacher describes seeing 180 degrees in a number of books as she had travelled around the classroom and asks Africa, one of those students, to share their results. Africa explains that they had measured three triangles and the sum of the internal angles for all three triangles was 180 degrees. Other students confirm their efforts and the teacher records six measurements of 180 degrees on the board to reflect the calculations different students had made. She returns to the idea that there was also a total recorded of 183 degrees, 182 degrees and 178 degrees. Annabel makes a claim that most people got 180 degrees, yet this does not account for the other measurements. Students make possible suggestions such as adding up the measurements incorrectly, measuring different kinds of triangles (scalene and isosceles) and not measuring the angles correctly. The teacher pauses on this point, noting how the other measurements are close to 180 degrees and that these results may are only a couple of degrees away from 180 degrees.

> Teacher: "These are really close. 178 is really close and 183. So, if you've made an error of 1 or 2 degrees then that will make a slight variation. So, can you 'answer the question' and can you prove using evidence? When you know something – think you've come up with an answer that's correct – you need to prove this to us. Show the class how you answered it."

The Checkpoint highlights how inaccurate measuring could cause variation in the data students collect and this becomes the focus for the remainder of the lesson. The class considers how to overcome this issue and it becomes clear that every measurement will need to be checked by another student to confirm it is accurate.

The students continue to measure to gather evidence and the teacher uses this time to travel around the classroom to gain feedback about how what each group is doing. After a short while the teacher pauses the class again. She has noticed that when some students have measured the internal angles of polygons using pattern shapes they have traced in their books, the corners are not precise and are difficult to measure. Tracing shapes had resulted in some polygons with rounded corners.

> Teacher: "When you are tracing shapes (models on the board) this is what is happening on the corner. And if you measure from there the angle is completely different to if you measure on the corner. Where the 2 lines cross is where to measure the angle from so this might be the difference in

measuring angles but if you are trying to be accurate then this could be significant. If you're out by one degree by every angle in an octagon, then what will you be out by? 8 degrees! So be careful."

The students continue to measure and check the angles they have measured, in pairs.

Discussion

Illustrations from two lessons have been presented, spanning the Discover, Devise and Develop phases of the inquiry *How can we accurately predict the sum of the internal angles of any polygon?* Four key mathematical ideas were identified and are listed in Table 1. Often inquiry questions include ambiguity in the pathways students can take to solve a problem and we use the term 'presented' here in the discussion to highlight that these concepts arose within the inquiry although the teaching intent, as guided by the Curriculum, was on using a protractor to measure angles. The nature of the inquiry question does lend itself to exploring Geometry understandings outlined by the Curriculum and this illustrates the richness of the task. The interactions are used to highlight how the teacher makes mathematical teaching and learning explicit through guided inquiry.

In the first instance, there is a need for teaching the definitions of regular and irregular polygons, presented by a student's need to communicate his ideas using this vocabulary. There is an opportunity for the teacher to "jump in" to assist students with developing their understandings of these concepts at a point in time when the students need to know this. The vocabulary and related mathematical understandings can assist in moving the inquiry forward. Comparing the internal angles of regular and irregular polygons that are similar then becomes the next investigation focus for many of the students. When one student presents the idea that an enlarged shape has larger angles than its original shape, the teacher acknowledges this as a worthwhile endeavour to pursue. In this instance, the teacher has not intervened to correct the student. She sees value in spending time exploring this idea as a way to deepen connections between mathematical concepts of transforming shapes through enlargement. One student (Nicholas) decides to focus on irregular polygons (scalene triangles) and when he finds that the sum of the internal angles of two different scalene triangles is not the same, the teacher joins him in his investigation into knowing. This is a second example of the teacher placing explicit emphasis on the need to follow a line of investigation: do enlarged shapes have greater angles than smaller, similar shapes and do all scalene triangles have internal angles that sum to 180 degrees. Placing value on the students' solutions makes it explicit to the class that challenging these ideas will contribute to a collective understanding, as a community of learners, about shapes and measuring angles. Finally, the fourth illustration presents the teacher using Checkpoints that focus on the students' efforts and the issue of variation in angles measured due to inaccurate measuring. The teacher connects measurement errors made, directly to the concept of evidence which needs to be reliable. Although she doesn't use the term reliable the emphasis she places on accuracy of measurements makes explicit the importance of reducing variation. This provides further purpose for students to continue to use a protractor to measure angles.

Knowing when to 'jump in', when to pursue students' incomplete ideas or partial understandings, and when to make connections between mathematical concepts seems to be skills this teacher uses to make mathematical learning explicit through guided inquiry. Rather than an emphasis on explicit teaching, the explicitness in guided inquiry focuses on explicit learning. Explicit teaching and guided inquiry are very different pedagogies when taken at face value, yet the importance of the teacher making learning explicit exists in both. It will be useful to explore how teachers engineer the explicitness of learning mathematics

in other guided inquiries to inform classroom teachers generally about how experienced inquiry teachers make learning explicit.

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Generating Ideas for Numeracy Tasks across the Curriculum

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The purpose of this article is to provide insight into how teachers identify initial ideas for the design of numeracy tasks. A design-based research approach was employed utilising classroom observations, video-stimulated recall techniques and semi-structured teacher interviews. Data collection and analysis were informed by a rich model of numeracy as well as generic principals of task design synthesised from relevant literature. Data analysis indicated that there were at least two approaches to generating ideas for numeracy tasks both of which were compatible with the principles of task design employed in this study.

The term numeracy is one of a number of terms used internationally (e.g., Australia, Canada, New Zealand, South Africa and the UK) for the capability to make effective and critical use of mathematics in personal, academic, workplace, and civic life (Geiger, Forgasz, & Goos, 2015). Although numeracy is often associated with the mastery of basic arithmetic skills, it is now understood that being numerate, in an increasingly globalised world characterised by rapid technological and economic change, must involve the capability to use mathematics to exercise critical judgement and to explore and bring to resolution real world problems (Steen, 2001).

All Australian students are expected to build the capability to apply mathematics to solve real world problems through engagement with Australian Curriculum (ACARA, 2017) that specifies numeracy as one of the general capabilities to be developed in all subjects, not just in mathematics. However, the lack of advice to teachers on how to design effective numeracy tasks, from within the Australian Curriculum, educational jurisdictions or other sources, threatens the successful implementation of this goal.

Burkhart and Swan (2013) argue for the importance of task design in improving the teaching of mathematics leading to enhanced student learning outcomes. While acknowledging the role quality tasks to effective teaching practice, Schoenfeld (2009) argues that task design principles are rarely made explicit and so it is difficult to for others, including teachers, to adopt effective approaches to task creation and adaptation. This situation is confounded by a lack of insight, due to limited research, into how teachers generate ideas that can serve as the basis for the design of mathematical activities, especially those that promote students' numeracy capability.

The purpose of this paper is to report on an aspect of a three-year longitudinal study that aimed to generate new understandings about how teachers design and implement effective numeracy tasks. The aspect attended to here is the processes teachers employed to generate ideas for numeracy tasks. In doing so the following research question will be addressed:

What processes do teachers utilise when generating ideas in initial stages of designing numeracy tasks?

In responding to this question, the following will be described and discussed: underpinnings of task design; methodological approach; two illustrative classroom vignettes; and findings and opportunities for further research.

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Principles of Task Design as Lenses for Introspection

Two complementary theoretical perspectives were used to guide the initial stages of the larger study from which data for this paper is drawn: (1) the 21st Century Model of Numeracy (Goos, Geiger, & Dole, 2014); and (2) generic principles of task design. These perspectives were employed as guidelines for teachers' attempts to design numeracy tasks and as a means of structuring teachers' introspection on the processes they employed to think of initial ideas for tasks. These perspectives are outlined below.

The 21st Century Numeracy Model (Goos, Geiger, & Dole, 2014) has been validated via a series of research projects as a basis for: auditing curriculum documents for numeracy opportunities; structuring teachers' design of numeracy tasks; planning for the implementation of numeracy tasks; and examining teachers' learning trajectories in relation to effective numeracy practice (e.g., Geiger, Forgasz, & Goos, 2015; Goos, Dole, & Geiger, 2011). There are four dimensions central to the model – contexts, mathematical knowledge, tools, and dispositions that are embedded in a critical orientation to the use of mathematics. A critical orientation has been established as the dimension central to sustaining students' interest in and persistence with an activity; providing the reason for activation of the other model dimensions and requiring the use of inquiry approaches when seeking solutions (Geiger, Forgasz, & Goos, 2015). These dimensions are summarised in Table 1.

Table 1
Dimensions of 21st Century Numeracy Model

Contexts	The use of mathematics to act in and on the world, thus in a range of real world situations both within schools and beyond school settings.
Mathematical Knowledge	Concepts and skills; problem solving strategies; estimation capacities.
Dispositions	Confidence and willingness to use mathematical approaches to engage with life-related tasks; preparedness to make flexible and adaptive use of mathematical knowledge.
Tools	Use of material (e.g., models, measuring instruments), representational (e.g., symbol systems, graphs, maps, tables) and digital (e.g., computers, applications, internet) tools to mediate and shape thinking.
Critical Orientation	Use of mathematical information and activity to: make decisions and judgements; form opinions; add support to arguments; challenge an argument or position.

Generic principles for effective tasks design were synthesised from relevant research literature. A brief outline of the literature underpinning these principles is presented below and summarised in Table 2. More detailed descriptions of the generation of these principles have been reported elsewhere (e.g., Geiger, 2016).

The fit to circumstance of tasks with local conditions and constraints is important for effective implementation as most tasks are developed for specific curriculum and school contexts (Kieran, Doorman, & Ohtani, 2013). Such circumstances also include considerations such as the pedagogies adopted to implement tasks and teaching resources available within a school.

Challenge is important for students if real learning is to take place. Most guidelines for improving learning outcomes stress the need for teachers to extend students' thinking by posing extended, realistic, and open-ended problems (e.g., City, Elmore, Fiarman, & Teitel, 2009). By posing such challenging tasks, teachers provide opportunity for students to take risks, to justify their thinking and to work with other students (Sullivan, 2011). Challenge also includes opportunities for students to make decisions and judgments and so exercise and develop their capacities to use mathematics critically (Goos, Geiger, & Dole, 2014).

While it is important for students to engage with learning experiences that offer challenge, tasks must also appear to be achievable, that is, challenging yet accessible (e.g., Sullivan, Clarke, & Clarke, 2013). Further, for tasks to be accessible they must be transparent; that is, it is clear what students are expected to do, and there must be points of entry where every student can begin an activity (Burkhart & Swan, 2013).

Table 2
Generic Principles for Effective Task Design

Fit to circumstance	Accommodating curriculum requirements and other affordances or constraints within a school setting, for example, teaching materials available within a particular school.
Challenge	Extending students' thinking by including elements of challenge in tasks provides opportunity for reasoning, risk taking, and the justification decisions.
Challenging yet accessible	Tasks must feel achievable to all students regardless of their prior history of achievement.
Complementary pedagogies	The pedagogical approach must match the demands and instructional intention of the task.
Transparency	In order for students to engage fully with tasks, activities must not only be accessible but also transparent in relation to expected outcomes – there is clarity around what is required of students to achieve success.
Opportunity to make decisions and judgements	The opportunity to make decisions and judgements introduces a critical demand into a task and provides purpose for students to engage with an activity.

Research Design

Methodological Approach

A design-based research approach was employed with the teacher professional learning component based on a framework devised by Loucks-Horsley, Love, Stiles, Mundry and Hewson (2003) that situates effective professional learning within teachers' own school-based contexts. The selection of this approach was appropriate as the study involved iterative cycles of intervention and improvement utilised to enhance teaching practice while working in school classrooms – known to be complex and contextually rich (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003).

Research was carried out in three iterative phases; each consisting of three interventions in the form of teacher/research workshops between which researchers conducted school

visits where teacher designed numeracy tasks were implemented. Teacher/researcher workshops were initially based on input from the researcher about the nature of numeracy, the generic principles of task design, and emersion activities in the form of exemplar numeracy task. Over time, these workshops evolved to opportunities for the cooperative development (researcher/teachers) of new principles of design and implementation. During researcher visits, numeracy tasks designed by teachers were trialled with lessons video-recorded. Video-stimulated recall interviews with teachers were conducted as soon as convenient after an observation in order to discuss critical events or phases during a lesson. Additionally, pre- and post semi-structured interviews were carried out in association with each classroom observation. Interview questions was structured around the elements of the 21st Century Model of Numeracy and the generic principles of task design. A particular focus of these interviews was on how teachers had generated the ideas for the tasks they implemented. Data was synthesised into case studies of teachers and students that was used to identify changes over time. This paper is based on data drawn from classroom observations and teacher interviews.

Participants

Five teachers from both Queensland and Victorian schools were recruited for Phase 1 of the project with an additional five teachers, one from each of the Phase 1 schools, agreeing to participate in Phases 2 and 3. Phase 1 teachers were purposively selected (Burns, 2000), firstly for their capability to design rich numeracy learning tasks, established through previous collaboration in numeracy-based researchers projects, and secondly for representation across learning areas. The quality of tasks developed through previous collaborations is evidenced by publication in peer reviewed research articles (e.g., Geiger, Goos, & Dole, 2013). Phase 2 teachers were recruited by Phase 1 teachers from their own schools, providing mutual support for each other's contributions as the project progressed. Phase 1 and 2 teachers represented a range of learning areas and sectors of schooling – Secondary English (1), History (1) Mathematics (1), Music (1), Science (1) Technology and Design (1), and Early Childhood/ Primary teaching (4). Because of space limitations, this paper reports on the accounts of two teachers; the first an early childhood teacher and the second a secondary teacher of English. These teachers were chosen because they reported different approaches to generating ideas for numeracy tasks.

Two Approaches to Generating Ideas for Numeracy Tasks

In this section, two vignettes are presented by way of illustration of how two teachers adopt different approaches to generating initial ideas as the basis for numeracy tasks embedded in learning areas other than mathematics.

Vignette 1 – Looking Through the curriculum

Olive is an early childhood teacher working in a school within the Catholic education sector. The school was located in a satellite city 45 km from a state capital. The observation on which this vignette is based took place during Phase 1 (Year 1) of the project. Olive's school had recently acquired an adjacent block of land on which they were planning to build additional classrooms as part of a school expansion. While plans for new buildings were underway, Olive had obtained permission to involve her students in creating a *prayer garden* in the backyard of this property. As part of this initiative, she had developed a series of tasks

for her group of preparatory (prep) students (typically 5-6 years of age) as part of an activity rotation that integrated geography, religious studies and mathematics.

In one of these tasks, students were asked to determine if a long rectangular bench-seat could be moved to a different position within the garden without moving the seat itself as this was too heavy for young students. No formal measuring tools (e.g., tape measures, rulers) could be utilised as the students had not yet learned about formal units of measure. Instead, 30 cm by 30 cm square tiles were provided as measuring tools for informal units.

After the teacher had explained the task, students discussed among themselves how they could go about the activity. After this discussion, students used the tiles to determine a measure of the bench seat length by placing the tiles end-to-end across the top of the seat. Once the length of the bench seat had been covered, the tiles were gathered up and moved by the students to the proposed new site for the seat. Once in position at one end of the designated space, students laid out the tiles end-to-end on the ground until they reached the other end of the area in question. Left over tiles were then piled up at the end of the space. When asked, students concluded that there was not enough space to move the bench seat to the proposed space and that another place would need to be found.

After the lesson, Olive was asked how she had thought of the task. She replied by saying her starting point was reflecting on the objectives of relevant curriculum documents she needed to address at that time of the year (O – Olive; I – Investigator).

> O: Yeah, I guess I always like to look at my curriculum; I know that we have to meet the needs, obviously, of the curriculum and make sure, yeah, I was looking, yeah, through and I saw that this idea of spaces and how we can change a space to meet a new purpose.
>
> I: So, you had the idea of a prayer garden, but then you were also looking through your geography thinking well I'm going to do that next.
>
> O: Yes, as well, yeah.
>
> I: And you put the two together.
>
> O: It's a hands-on approach and the kids can visually see it as well, and they can be a part of it. I think that's more meaningful to them as well. It's very easy to talk about it and show photos, but unless they're actually experiencing it, I think, yeah.
>
> I: How did you bring the maths into it then? Was that a forced thing or did that just come to you?
>
> O: No, I think it did just kind of came. I think it's knowing your curriculum and knowing what needs you need to meet. I think that's the starting point.
>
> I: And knowing lots of bits of curriculum at the same time.
>
> O: Yes, absolutely, yeah.

At the same time, she also considered what resources or aspects of the environment could be utilised and eventually brought curriculum and the resources offered by the built environment together.

> O: I just think it's probably more engaging for students if they can be a part of that environment and get their hands dirty and now they're talking about ripping down fences and taking out poles and weeding.
>
> I: Are you conscious of just seeing things like that, and just thinking what to do with it?
>
> O: Yeah, it's an opportunity. I think you've got to look at things and think it's an opportunity.

Olive believed it was her familiarity with curriculum documents that allowed her to pick out relevant strands from both geography, religious education and mathematics and bring these together through the opportunity made available by the purchase of the new property.

When asked if this was typical of the way she developed numeracy tasks, she replied that she had worked hard to be thoroughly familiar with the curriculum requirements for any year level she was teaching and that she used this as a lens when looking to create new activities. While maintaining a focus on the curriculum, Olive also took advantage of a local resource in the form of a new aspect of the built environment.

Vignette 2 – Archiving Ideas

Richard is an English teacher in a government secondary school situated in a regional centre. He was observed during Phase 3 (Year 3) of the project conducting a Year 9 lesson in which students were required to write a letter to a new pen pal who lived on Horn Island – located off the northern Australian coastline in the Torres Strait. As preparation for this assignment, Richard had asked his students to research a number of aspects related to the island including:

- population
- land area
- number of schools
- distance from their home to Horn Island
- means of transport and travelling time from their home to Horn Island
- frequency of transport to and from the island

Richard had asked students to gather this information in order to help them understand the life circumstances of their new pen pals and so provide them with starting points for their first letter. While gather this information, students were asked to identify the advantages and disadvantages of living on Horn Island. Students worked enthusiastically on this task through the lesson, regularly expressing surprise at what they found, for example, the population of Horn Island is only 539 people, a small fraction of the population of the country town in which students lived. The lesson concluded before the task was completed with Richard telling students it would be continued the next day.

When asked how he had come up with the idea for the lesson, Richard replied that he was always on the lookout for opportunities to promote students' understanding of the challenges faced by others in the world (R – Richard).

> R: So, the background to the lesson was that for sometime I've wanted to have my students communicate with people from indigenous backgrounds. So that's just a general goal I had. And by pure fluke my son just happened to get a job on Horn Island flying teachers around. So, I thought that opportunity was staring me in the face to ask my son if he would mention to a teacher the idea of the students at Mt Erin writing to their students. And he did that and so I decided this would be a terrific project for the kids to be involved in where they would explore the differences lifestyles between Torres Straight Islanders and people from around this area.

In the above except, Richard also indicates that he was searching for ideas that could promote broader educational goals than those specific to curriculum documents. In this case it was his son's engagement with the people of Horn Island that served as the basis for an idea which he 'parked' until he could use it in his classroom at some future date. He made the connection with this archived idea and the English unit on writing to a pen pal when looking forward through his teaching program for the semester (R – Richard; I - Investigator).

> I: How did you think of the idea?
>
> R - By firstly starting with the decision to look for a way of connecting students with other students with an indigenous background. So, I started with... that was my goal.

I: Okay so was that independent of curriculum?

R: Independent of curriculum.

I: So, you thought sometime during the year you want to do this?

R - Yeah as soon as possible… so I ended up thinking "I'm going to have this as a goal" so from that goal I was going to look for a way of addressing it

I - Okay and then, so you're working the way you often have that you get an idea and you park it until you see some part of the curriculum coming up where it will fit. Have I got that right?

R - Yes

The task Richard developed from the idea allowed him to bring into the classroom an aspect of teaching he valued – an understanding of others through awareness of their life circumstances. At the same time, while this was the driver for identifying an initial idea for a task, Richard shaped the resulting activity to satisfy specific curriculum requirements as well as the global aims of the subject he was teaching – letter writing and the promotion of empathy for others.

Discussion and Conclusion

In identifying ideas to be used as the basis for numeracy activities embedded in learning areas other than mathematics, Olive and Richard used two quite different approaches.

Olive relied on a deep knowledge of curriculum across learning areas to generate ideas. In this approach she looked to bring forth ideas by making connections between the specific curriculum objectives of different learning areas. In the vignette reported here, she found a connection between geography, religious education and mathematics. These aspects of curriculum were brought together in fitting to circumstance her teaching with an opportunity provided by the built environment. The problem she posed for students, provided developmentally sensitive challenge yet was accessible through the tools she provided which allowed students to bring the critical question embedded in the task to resolution. She worked with students as a whole group and one-on-one to ensure the demands of the task were transparent while employing an investigative pedagogy consistent with the problems solving focus of the activity. Students were provided the opportunity to make a decision about whether the bench-seat could be moved to another place in the garden.

By contrast, Richard did not use the curriculum as a lens through which to look out into the world for a teaching idea but rather as a framework or overarching plan to which he could attach teaching ideas he had already identified as having potential at some future time. In his approach curriculum requirements were viewed as a way of facilitating a broader educational purpose rather than providing an initial direction for task development. This meant that he fit to circumstance the idea he had previously identified to the requirements of curriculum. He challenged students to make use of mathematics-based evidence to argue for the advantages and disadvantages of living on Horn Island. In developing such arguments, students were required to make judgements and to form opinions. Because the task required students to connect the life circumstances of a pen pal on Horn Island to that of their own community and the demands of the task related to mathematical knowledge were limited, they found the task both assessable and transparent.

The above analysis indicates that while Olive and Richard took different approaches to identifying initial ideas for development into tasks, the activities they developed were consistent with the generic principles of task design they had been encouraged to use when designing tasks. While this paper reports on only two cases, they provide tentative evidence

that initial ideas for numeracy tasks across the curriculum can be generated in different ways yet can still be shaped to fit the broad specifications of the principles of generic task design. In the cases reported here, task design was initiated in two different ways – drawing intensively on curriculum objectives or upon broader values and beliefs.

These findings indicate that further research is required to establish what other approaches can be utilised by teachers to generate numeracy tasks for implementation across the curriculum. This research could also seek to establish if teachers' approaches generating ideas are invariant or adaptable in relation to the circumstances of their practice.

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The practice of using NAPLAN numeracy test results: A review of the literature

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Considered by many as a source of valuable data and a potential to improve mathematics education, a significant amount of studies have been conducted on The National Assessment Program – Literacy and Numeracy (NAPLAN) test results. This study is a systematic literature review of 86 peer-reviewed journal articles published between 2008 and 2017 to investigate how NAPLAN numeracy test results were used in those studies. Findings showed NAPLAN results were used primarily to map student progress and identify strengths and weaknesses in teaching.

The National Assessment Program – Literacy and Numeracy (NAPLAN) is an annual assessment for of Australian students in Years 3, 5, 7 and 9, undertaken since 2008. This standardised test assesses students' reading, writing, language (spelling, grammar, and punctuation) and numeracy administered by the Australian Curriculum, Assessment and Reporting Authority (ACARA). The Federal Government pushes such assessment to achieve public accountability, demonstrate transparency, and maintain public confidence in the standards of the education system throughout Australia (Klenowski & Wyatt-Smith, 2012).

The NAPLAN test plays a key role in establishing and raising standards of learning (e.g., Hardy, 2014; Polesel, Dulfer & Turnbull, 2012). It is assumed that NAPLAN test results create opportunities for thoughtful dialogue and discussion to improve teaching and learning practices. In this regard, ACARA (2017) identified various areas to use NAPLAN test results for teachers, parents, schools and government bodies. For example, NAPLAN test results could be used to help teachers to challenge higher performers and identify students needing support. For parents, the NAPLAN test results supply individual student level reports to enable parents to see their child's progress over the course of their schooling. It also provides each school aggregated data to identify strengths and weaknesses within their teaching programs (Polesel, Dulfer & Turnbull, 2012). According to ACARA (2017), the NAPLAN test results can be used to;

- Challenge higher performers and identify students needing support
- Map student progress, identify strengths and weaknesses in teaching and set goals.
- Discuss progress with teachers and compare performance against national peers.
- Support good teaching and learning, and school improvement.

The present study used a systematic literature review of 86 peer-reviewed journal articles which focused on NAPLAN numeracy test results. The four purposes of the NAPLAN test results were categorised for the systematic literature review. The study investigates how NAPLAN numeracy test results were used in the reviewed journal articles in relation to the four purposes of NAPLAN results listed by ACARA (2017). The study focuses only on the numeracy test results. As a result, the study is guided by two research questions. Firstly, which purposes for the NAPLAN results are focused on in studies? Secondly, what are the gaps in using NAPLAN numeracy test results? The contribution of this study lies in the procedures used to review the articles in particular to mathematics education research, the use of

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 322-329. Auckland: MERGA.

mathematics test results to inform practice in mathematics education and identify gaps to inform future mathematics education studies.

Background

Since 2008, the NAPLAN test results are available and reported as a mean scale score compared to the national minimum standard (such as the skills and understandings students can demonstrate at their particular year of schooling, in a specific subject area or domain) (ACARA, 2017). The report is also available to be selected by gender, indigenous status, language background, geolocation, parental occupation and parental education at each year level and for each domain (reading, writing, spelling, grammar and punctuation, and numeracy) of the test (ACARA, 2017).

Results from NAPLAN test have a number of potential uses. They can be used to monitor the performance of the education system, inform classroom practice, ensure that students have met required educational standards and encourage teacher and schools for their students' performance (ACARA, 2017; Rosenkvist, 2010). The NAPLAN test results can also provide schools with data to analyse and sense trends occurring in schools that can inform planning and policy decisions (Perso, 2009). According to ACARA (2017), the NAPLAN results can be used for four purposes.

Firstly, the NAPLAN test results can be used to challenge higher performers and identify students needing support (ACARA, 2017). In this regard, there is a considerable body of research literature (e.g., Nichols and Berliner 2007; Stobart 2008; Taubman 2009; Darling-Hammond 2010) cited in (Lingard & Sellar, 2013, p.634) demonstrating the effects of standardised testing results to inform teachers' pedagogical practices and improve students learning outcomes. Therefore, individual NAPLAN results can support teachers to plan for individual student improvement (Perso, 2009).

Secondly, the NAPLAN test results provide useful information to map student progress, identify strengths and weaknesses in teaching and set future goals (ACARA, 2017). The NAPLAN test results are available in aggregated forms at the national and school level. As a result, schools can gain detailed information from NAPLAN test results about how they are performing and identify strengths and weaknesses which may lead to further attention and interventions. To identify strengths and weakness of students numeracy competency for future intervention, Hardy (2014) urged that the validity of the NAPLAN test is considered (measures of students' actual learning of mathematics) and to ensure well-understood measures of students' achievement.

Thirdly, the NAPLAN test results are good source of information to discuss students' progress and compare their performance against national peers (ACARA, 2017). This comparison and the reported outcomes of the test enable the Australian public to develop a general national perspective on student achievement and, more specifically, an understanding of how their child and schools are performing in relation to the national standards (ACARA, 2017). Such data are assumed to develop confidence in Australians that education resources are allocated to ensure that all students achieve meaningful learning during their time at school (Guenther, 2013).

Finally, at the system level, the NAPLAN test results provide education ministers with information about the success of their policies and resourcing in priority curriculum areas (ACARA, 2017). It also provides ministers with the capacity to monitor the success of policies aimed at improving the achievement of different student groups, such as indigenous students. Such data provide an additional suite of information, thus enhancing the capacity for evidence-based decision making about policy, resourcing and systemic practices at the system level (Klenowski & Wyatt-Smith, 2012).

In addition to the four purposes, the NAPLAN test results have become a powerful tool to describe education systems, assess teaching quality and determine school funding formulae (Guenther, 2013).

A significant number of studies have been conducted on NAPLAN and used NAPLAN results in their reports for various purposes. Hardy (2014) used NAPLAN numeracy results as useful data for grouping students to help improve their numeracy capabilities, and as a stimulus for teacher professional development. Burrows, Goldman, Olson, Byrne, and Coventry (2017) used NAPLAN test results to show the impact of increased consumption of sugar-sweetened beverages on numeracy test scores and suggested strategies to improve the students' numeracy competency.

This study used a systematic literature review of peer-reviewed journal articles published between 2008 and 2017 which focused on NAPLAN numeracy test results. The study investigated how NAPLAN numeracy test results were used in these studies in relation to the four purposes of NAPLAN numeracy test results.

Method

This study used a systematic literature review. The search was conducted in five scientific databases (i.e., Education Resources Information Center (ERIC), Web of Science, Scopus, Science Direct, and Academic Search Complete). The general search terms for all databases included the Boolean operators 'AND' and the wildcard (*) function. To limit the scope of the study, the review was limited to peer-reviewed articles published between 2008 and 2017, and full-text availability was required. In addition, studies focusing either only NAPLAN numeracy or both of NAPLAN literacy and numeracy were used in the review. When the studies were on both numeracy and literacy aspect, only the numeracy aspect was considered for further review.

Similar to the suggestion by Cronin, Ryan, and Coughlan (2008), this study followed the steps shown in Figure 1 to conduct the systematic review. The review passed through 5 steps from identifying the search term to categorising search results according to their focus.

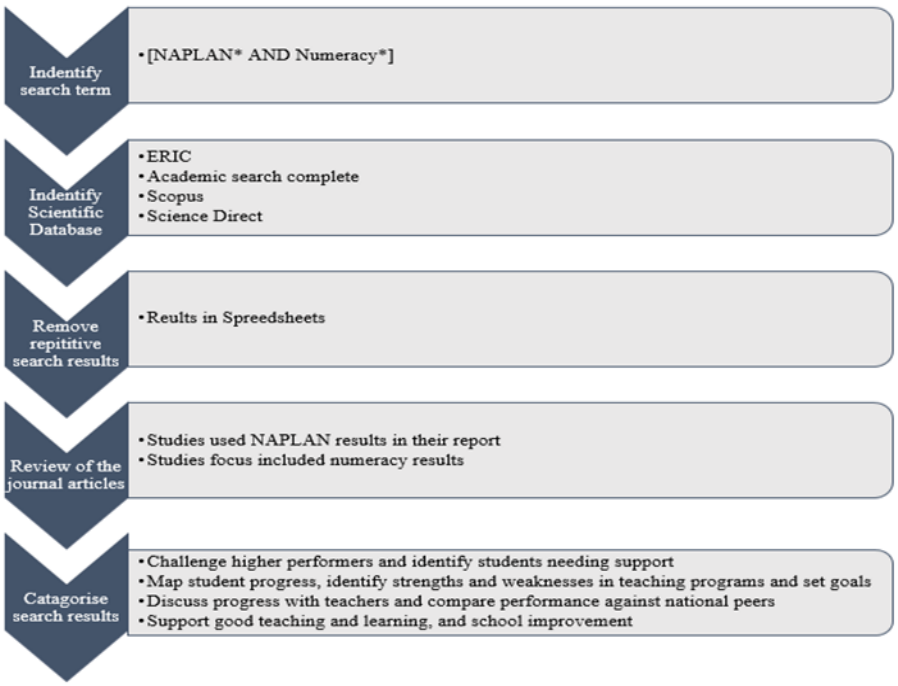

Figure 1. Database search and review process.

The initial search yielded 284 studies. However, further screening (availability of the full report, relevant to NAPLAN test results and removing repetitive search results) resulted in 160 journal articles.

Additional screening (studies used NAPLAN results in their report and studies focusing on NAPLAN numeracy test results) revealed 86 peer review journal articles. The search result across each search engine and publication year is shown in Table 1. All the studies used the quantitative NAPLAN numeracy test results (mean scores) in their reports.

In the end, these journal articles were reviewed and categorised according to the four purpose criteria (ACARA, 2017) mentioned in the background section.

Table 1

Relevant Search Results for Review

	Year published									
Search Source	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**Total**
Scopus	0	0	1	2	5	5	5	8	8	34
Science Direct	0	0	0	2	0	5	7	7	12	33
ERIC	4	1	1	2	3	0	1	0	0	12
Academic Search complete	0	0	0	0	0	0	0	6	0	6
Web of Science	0	0	0	0	0	0	0	0	1	1
Total	**4**	**1**	**2**	**6**	**8**	**10**	**13**	**21**	**21**	**86**

In order to get a more comprehensive understanding, articles were further categorised according to year level (Years 3, 7, 9 or only primary, secondary or both or no year level focus) and type of school (government or non-government [independent, Catholic] or both or not mentioned). A spreadsheet was used to document, extract information about the categories and analyse the data.

Results and Discussion

As shown in Table 2, seven studies (8.1%) used NAPLAN numeracy test results in their report to identify higher performers and students needing support. For example, Perso (2009) analysed NAPLAN test items and indicated that students need to be taught how to deal with the literacy demands of a task if students are to become numerate.

Table 2

Purpose and Year Level

	Year published									
Purpose	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**Total**
Map student progress, identify strengths and weaknesses in teaching and set goals	0	0	1	0	0	3	4	4	13	25 (29.1%)
Support good teaching and learning, and school improvement	0	0	0	1	1	1	5	5	3	16 (18.6%)
Discuss progress with teachers and compare performance against national peers	0	0	1	2	1	2	2	5	2	15 (17.4%)
Challenge higher performers and identify students needing support	2	0	0	0	0	1	0	2	2	7 (8.1%)
Others	2	1	0	3	6	3	2	5	1	23 (26.7%)
Total	**4**	**1**	**2**	**6**	**8**	**10**	**13**	**21**	**21**	**86**

A large number of studies (N = 25 (29.1%) used NAPLAN test results in their report to map student progress, identify strengths and weaknesses in teaching and set goals. Using multilevel modelling to account for within-school variables, Chua, Khan, Humphry, and Hassell (2017) analysed NAPLAN test results to estimate the effect of the national partnerships on student performance. The results indicated that on average male students performed higher in the numeracy test. Vetter, O'connor, O'Dwyer, and Orr (2015) argued the importance of fitness for general numeracy competency. Similarly, Burrows et al. (2017) showed that increased consumption of sugar-sweetened beverages was associated with significantly lower test scores in numeracy. To identify students needing support for their numeracy competency, Brew, Toelle, Webb, Almqvist, and Marks (2014) used NAPLAN test results to investigate the effect of omega-3 fatty acid supplementation on subsequent numeracy performance in children. All these studies used NAPLAN test results to identify the possible reason for students' weak achievement in numeracy with little suggestion on the possible strategies to improve their numeracy competency.

The search results revealed sixteen studies (18.6%) focusing their report on supporting good teaching, learning, and school improvement. Hardy (2015, p.335)'s research showed schools that dominated with high numeracy results indicated good teaching practice and demonstrated the schools' focus on the students as 'valued capitals.' Similarly, Polesel, Dulfer, and Turnbull (2012) demonstrated that NAPLAN test results are used in schools as a source of information to plan intervention and engaging in curriculum development practices to improve teaching and learning in the school.

Fifteen (17.4%) studies focused on NAPLAN results to map student progress and compare performance against national peers. Ford (2013) analysed the inequality of achievement between indigenous and non-indigenous students in the States and Territories, with particular reference to New South Wales and the Northern Territory. In relation to comparing students against national peers, Marks (2016) used NAPLAN results to show the relative effects of socio-economic, demographic, non-cognitive and cognitive influences on student achievement in Australia. Others used NAPLAN results to compare national peers and suggest improved strategies for teachers in relation to mobile learning (Males, Bate, & Macnish, 2017), disability (Teather & Hillman, 2017), class size (Watson, Handal, & Maher, 2016) and NAPLAN scores.

There were a large number of studies (N = 23 [26.7%]), in the search result, which used NAPLAN numeracy test results in their report, with a different focus than those listed by ACARA (2017). For example, a study by Quinnell and Carter (2013) draws the reader's attention to the large variety of symbols, abbreviations, and conventions used in the NAPLAN numeracy tests. Norton (2009) provided a critique of the Year 9 NAPLAN numeracy test and how results might inform teaching mathematics. Norton (2010) used Year 9 NAPLAN numeracy test results to examine pre-service teachers' mathematics content knowledge. Rogers, Barblett, and Robinson (2016) investigated the impact of NAPLAN numeracy tests on student, parent and teacher emotional distress in independent schools. This result showed that the NAPLAN test results were not limited to the four purposes listed by ACARA (2017).

The review also considered the trends of these studies across publication year as shown in Figure 2. Studies published toward 2017 have an increased focus on supporting good teaching and learning through analysing test results.

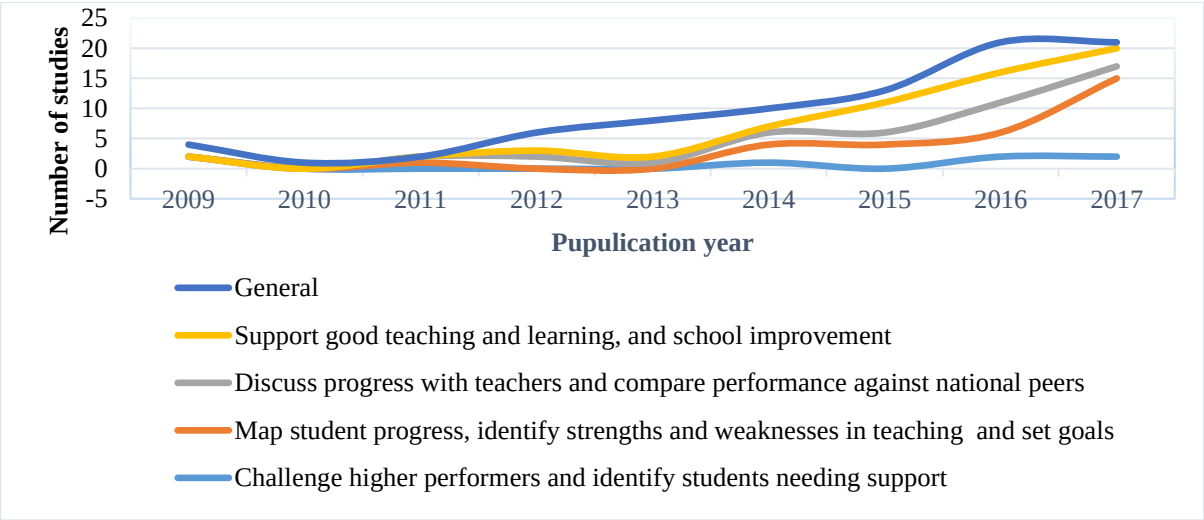

Figure 2. Trends of using NAPLAN results from 2009 to 2017.

As shown in Table 3, studies conducted at primary schools (N = 15), focused on their report to map student progress, identify strengths and weaknesses in teaching. Whereas, a significant number of studies were conducted combining both primary (Years 3 and 5) and secondary (Years 7 and 9) school NAPLAN numeracy test results (N = 37) and their primary focus was diverse (such as mathematics teachers perception on NAPLAN testing).

Table 3

Trends in the Studies Across Year Levels

Purpose	Year level							
	3	**5**	**7**	**9**	**Primary**	**Secondary**	**Both**	**Total**
Map student progress, identify strengths and weaknesses in teaching and set goals	5	2	2	1	4	4	8	25
Support good teaching and learning, and school improvement	3	1	0	1	4	3	4	16
Discuss progress with teachers and compare performance against national peers	2	0	0	0	1	3	9	15
Challenge higher performers and identify students needing support	1	1	0	0	4	0	1	7
Others	1	0	0	0	2	4	15	22
Total	**12**	**4**	**2**	**2**	**15**	**14**	**37**	**86**

The search result showed that most of these studies were conducted with a combination of government and non –government schools (N = 40[46.5%]). Twenty-five studies (29%) were conducted in government schools, and eight were conducted in non – government schools. The remaining studies (N = 13[15.1%]) didn't explicitly mention the type of schools included in their report.

Conclusion

The purpose of this review was to investigate how NAPLAN numeracy test results were used in the 86 peer-reviewed journal articles in relation to the four purposes of NAPLAN

results listed by ACARA (2017). From the review results, NAPLAN numeracy test results were used for various purposes in the reviewed studies. However, as this systematic review showed, a large number of studies used NAPLAN test results to map student progress, identify strengths and weaknesses in teaching. Hardy (2014) showed the importance of using NAPLAN test results as a source of evidence for grouping students to help improve their numeracy capabilities. Au (2013) examined the impact of childhood obesity on academic performance and identified children requiring support to improve their numeracy competency. Smith et al. (2014) grouped students as breakfast skippers and non-breakfast skippers impacting their NAPLAN scores on which breakfast skippers scored lower NAPLAN scores in numeracy. These studies identified possible reasons for students' performance, with limited recommendations for relevant interventions to improve students numeracy competency.

Interestingly, a significant number of studies used NAPLAN numeracy test results different from the four purposes provided by ACARA (2017). For example, Tayler et al. (2016) showed the importance of NAPLAN scores to provide evidence on how best to invest in Early Childhood Education and Care. Males, Bate and Macnish (2017) studied the impact of mobile learning on students NAPLAN scores. These studies show that the NAPLAN numeracy test results can be analysed for a wide variety of purposes, not limited to the four purposes provided by ACARA (2017). In addition, these studies (Males, Bate & Macnish, 2017; Tayler et al., 2016), used well-thought-out, valid research to evidence the impact of various interventions, (e.g., using technologies) on students' performance in numeracy.

A limited number of studies used the NAPLAN numeracy results to challenge higher performers and identify students needing support. It was the expectation of this study that majority of the reviewed articles would focus on this purpose. Polesel, Dulfer & Turnbull (2012) warn that the focus on NAPLAN test results has shifted the culture of some schools to 'teach to the test' and identify students needing support prior to sitting NAPLAN. . It is the contention of this study that studies identifying students needing support will support schools to plan interventions for students who are in most need with the intention of increasing success in their numeracy. In this regards, Hardy (2015) suggested that identifying high or low NAPLAN tests scores of students is an essential element in the provision of support and design interventions.

The authors recommend the following directions for future research on NAPLAN numeracy test results. First, even though a large number of studies used NAPLAN test results to map student progress, identify strengths and weaknesses in teaching, none of these questioned the validity of the NAPLAN tests. Future studies focusing on the validity of the NAPLAN tests are fundamental to use the results for multiple purposes. In this regard, Hardy (2014) recommended the importance of valid tests to inform future planning and intervention. Secondly, as the review results elicited, a limited number of studies used NAPLAN numeracy test results to identify students' weakness and suggest possible strategies to improve the result. Future studies could focus on identifying students' weakness in a specific mathematics branch such as algebra, measurement, geometry, probability or statistics. This could support teachers, schools, and policymakers to plan relevant interventions at national, school and individual student level. Finally, future studies should focus on assisting teachers with data analysis of NAPLAN numeracy results (such as item analysis, using item analysis report) to evaluate students' performance and plan their teaching and learning programs.

This study is limited to a review of peer-reviewed journal articles and a few search engines. Broader inclusion of publications (such as conference papers, and books) and a range of search engines could provide a more compressive picture about the use of NAPLAN numeracy test results. However, this study could be used as a starting point for similar studies.

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Making Mathematics Accessible for All: A Classroom Inquiry

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Patterns of participation in mathematics are often affected by power and status structures in the classroom. This case study focuses on two 10-year-old students who have achievement and status and power issues in mathematics, within a class of predominantly Maori and Pasifika students from low socio-economic backgrounds. The findings illustrate the impact teachers have on the opportunities to participate available to students in the mathematics classroom, through practices which explicitly address status issues. Importantly, they show that unless teachers intervene to address inequities and promote participation the status quo of diverse students underachieving in mathematics remains.

Mathematics is a social endeavour and therefore a key component of mathematical inquiry learning is active participation of students in the classroom community. There is considerable research evidence which emphasises the positive relationship between student participation and their achievement in mathematics (e.g., Anthony & Walshaw, 2007; Barnes, 2005; Ing et al., 2015). However, for some students accessing the mathematical discourse holds its own challenges, and therefore examination of the nature of patterns of participation to ensure equitable access to mathematical learning is a critical issue in classrooms (Barnes, 2003; Barnes, 2005; Bennett, 2014; Civil, 2014; Lack, Swars, & Meyers, 2014; Rubel, 2017).

One aspect which needs consideration are the power and status structures which exist in the classroom and which shape participation in both positive and negative ways (Civil, 2014). The focus of this paper is on two students who currently have achievement challenges in mathematics. The aim is to examine each student's participatory practices, evaluate their learning opportunities, and investigate the factors that promote or inhibit opportunities to learn. These include, for example, rich tasks that build mathematical understandings, participation in group work, and opportunities to communicate mathematical thinking.

The specific research questions explored in this paper are:

- What factors in the learning environment inhibit or enhance student's participation in a mathematical inquiry community?
- What actions can teachers take to proactively and consistently promote equitable patterns of participation in the classroom?

Literature Review

In the past two decades attention has focused increasingly on the role of participation as a way to gain equitable outcomes in mathematics. One particular focus has been placed on the role power relations and status hold, in who gets to participate in positive ways in mathematics classrooms. Civil (2014) links equity, power and status relationships in mathematics classrooms to participation. These power and status structures, present within wider society, play out in many different ways in classrooms often linked to race, ethnicity, gender, socioeconomic status, and even personal popularity. They not only influence who

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gets to participate in mathematical discussions but also how contributions are valued. Different contributions may be ranked according to an expectation of competence tied to an allotted status. Lack and colleagues (2014) illustrated the ways in which status perceptions influence student interactions. In their study, they found that high status students dominated classroom discussions, working on the assumption that their contributions were most valid.

The social construction of race and white spaces is another significant element of equity discussions in mathematics education (Rubel, 2017). Rubel argues that race is pivotal in perpetuating societal inequities and widening gaps in mathematics opportunities and outcomes. She advocates teaching mathematics for social justice through teachers explicitly exploring the dominant role of "whiteness" in the mathematics classroom and developing equity-directed instructional practices (p.66). For example, through culturally relevant pedagogy teachers connect mathematics instruction to students' cultural practices, out of school experiences, and real lives, rather than just reflecting and valuing the typically white, middle-class cultural practices of schools (Rubel, 2017).

Many teachers assume that students are able to work collaboratively in small groups. However, many studies (e.g., Barnes, 2003, 2005; Hunter, 2007) show the negative impact on some students when teachers do not specifically attend to how students participate in the discourse. For example, Barnes (2003) illustrated the power relationships which evolved when she identified the social positions that the students either took or had assigned to them by others. She describes the position of "outsider" – students who are frequently ignored and treated as though they do not have the same rights as others to contribute to discussions. For these students being positioned as "outsiders", resulted in them having little power, and few opportunities to make productive contributions. This showed that the approval of an idea had less to do with its usefulness or correctness than with who proposed or who supported it.

Teachers hold an important role in ensuring high-quality, equitable participation for all students in the mathematical discourse. Bennett (2014) suggests a number of pedagogical strategies teachers can use to build strong classroom culture based on equitable access for all to participate. Creating classrooms that focus on reasoning, deep conceptual understanding, and communication of mathematical thinking are key factors because student understanding of what mathematics is, and their self-perception as a learner of mathematics, powerfully influences their participation (Ministry of Education, 2009) and their identity as a mathematical learner. But also, the classroom norms need to convey an expectation of active participation from all students. Civil (2014) identifies cultural responsiveness as also important in the mathematics classroom. She argues that when this is considered students are encouraged to participate, contribute and have opportunities to learn. However, teachers also need to monitor and actively work to resolve status and positioning as students interact in large and small groups. Civil (2014) provides one strategy to address status through assigning competence. Barnes (2005) explains that through drawing attention to and praising good ideas or solutions as an example of assigning competence the teacher is able to gradually increase recognition from the class that all students have something of value to contribute. Other factors of importance include the use of heterogeneous grouping structures and rich tasks which reflect students' experiences and knowledge, and require multiple abilities to solve (Civil, 2014; Lack et al., 2014). These give every student opportunities to participate and illustrate that there are different ways to be "smart" in maths, which challenge student perceptions of what it means to be "good" at mathematics (Civil, 2014).

Teachers also need to interrogate their own beliefs about status and their role in facilitating the talk. Importantly teachers need to ensure that they do not fold back to traditional teacher-driven discussion, labelled by Barnes as a "teacher helping pupil storyline" (2003, p.2). The teacher's role is to contribute to the discussion through probing student thinking, asking and supporting students to ask clarifying questions or helping guide students in their reasoning (Bennett, 2014). This positions teachers as the pedagogical experts, rather than the intellectual authority, within the classroom (Bennett, 2014).

The theoretical framework underpinning this study is a socio-constructive view of learning mathematics. Learning is viewed as a fundamentally social activity, where students make sense of mathematics through active participation (Perry, Geoghegan, Owens, & Howe, 1995). Using this framework supports focusing on the mathematical learning students construct through participating in the classroom community and provides insight into the role of the teacher in classrooms where students collaboratively create mathematical understandings (Perry et al., 1995).

Methodology

This small-scale study was conducted at an urban, New Zealand, primary school with predominantly Maori and Pasifika students from low socio-economic home environments. The study was conducted over four weeks and is based on a series of three classroom observations. A qualitative, case study approach was used to gather data to answer the research questions. The data collected were recorded observations of classroom lessons, field notes, and interviews involving a series of open ended questions with the two students. Analysis of the data consisted of determining themes based on evidence of participation in the classroom, and teacher actions which facilitated or precluded this.

Two Year 8 students were selected for the study, based on their teacher Sarah identifying them as struggling to achieve in mathematics. The first student, Huia, is of Maori descent and the second student, Meilani, is of Cook Islands descent. Both girls are actively involved in their cultures, particularly through kapa haka, drumming and dance. Huia and Meilani are puzzles of practice for Sarah because both students are achieving at or above national standards in all other subjects but are below national standards in maths.

Results and Discussion

Perceptions of Mathematics and Doers and Users of Mathematics

On interview, both Huia and Meilani stated that they did not like mathematics.

> Huia: Maths is just about adding numbers in different ways. I don't like maths. It's hard learning new strategies.
>
> Meilani: Maths is ok but I don't really like it. I like doing times tables but I'd be better at maths if I knew more divided bys. You need to practise things like times tables and know how to work things out.

Clearly, their explanations for not liking mathematics centred on their perception of mathematics as facts and strategies to be learnt rather than making connections to the concepts and relationships within mathematics. Some of their antipathy towards mathematics could be attributed to the problems used in the observed classroom which appeared to focus on procedures and correct answers, rather than exploring and understanding mathematical concepts and relationships. For example:

> There are 72 ice creams in the freezer. Two eighths of them are eaten by Huia and her friends. Five eighths are eaten by Meilani and her friends. How many are left over?

While the teacher has included Meilani and Huia's names in the problem, it does not take the cultural diversity of them into consideration. This resulted in Huia when asked how it felt to be Maori in maths stating:

> It's English maths about English things.

Huia provides evidence of the teacher's implicit positioning of her own cultural values and practices as the norm. To connect mathematics to students' realities teachers need to learn about their student's heritage, home languages, interests, everyday activities, and out of school lives, and develop strategies to effectively teach students who have cultural backgrounds and experiences which differ from their own (Rubel, 2017). The tasks students engage with not only determine the mathematics they learn, but how they come to think about, develop, use, and make sense of mathematics (Anthony & Walshaw, 2009). Situating mathematics tasks in students' cultural contexts empowers them to participate through considering mathematics as part of their own identities and lives (Anthony & Walshaw, Rubel.

How students also see themselves as learners in the mathematics classroom and how this affects their status and positioning is influenced by how students are grouped. The teacher stated that she used flexible grouping, but in reality, she split the class into those working at Level 2 - 3 of the Curriculum, and those working at Level 3 – 4. Her justification was so that "children with similar abilities could work together". This meant that students were positioned and taught according to the teacher's perception of their mathematical ability. For example, as students worked on a problem about whose family had eaten more pizza; either the family who ate 6/8 of their pizza or the family who ate 7/10 of their pizza. The teacher acted in different ways as she engaged with the different groups. When she went to Meilani's group she explicitly intervened and asked leading questions which allowed them no opportunities to explore solutions or contribute a range of ideas. For example, she asked:

> What would be easier to compare than eighths and tenths, how could you find something the pizzas have in common?

Through such actions the cognitive demands were lowered and those students the teacher considered were lower in ability were provided with less opportunity to participate in higher order thinking.

The students knew they were placed in lower ability groups and this also affected their attitudes to mathematics. For example, Huia stated:

> Other people know and other people understand maths but I don't. You feel like their ideas are better than yours.

Her statement illustrates her awareness of who the "smart" students were and where they stood in the classroom hierarchy in terms of who got to talk and whose contributions were valued in both group and class discussions. Huia saw herself as an "other".

While small group discussions can provide opportunities for students to extend their thinking, poor communication within groups limits participation and engagement with the task. Observations provided evidence that most groups in the class used cumulative talk, where everyone uncritically accepted and agreed with what other people said, rather than doing what Mercer (2008) describes as necessary in constructive mathematical discourse where ideas are challenged in the process of constructing knowledge. The students also need to actively participate in meaningful discourse through engaging in mathematical practices

(Bennett, 2014). In all three observations both Meilani and Huia's groups drew diagrams showing fractions of a whole, but the representations were not connected to any reasoning. Meilani and Huia appeared uncertain about how to participate in such practices as justifying, arguing, and generalising. However, they were given no support or scaffolds to learn these skills.

Questioning

Teacher questioning to support students to engage and participate in mathematical discourse also acts as important scaffolds for students to access deep and rich reasoning. Teachers frequently ask students open-ended questions after they have solved a problem, for example, "how you solved the problem?", and can draw out an initial student explanation. However, teachers find it more difficult to follow up on student ideas and ask questions that support students to participate in making their thinking explicit or understanding other students' strategies (Franke et al., 2009).

The observations illustrated a pattern the teacher took when groups explained how they solved the problem. The teacher would intervene and ask questions such as:

> Are you following that? Do you agree with that?

However, when receiving a yes or no response from the students she did not press for clarity or expect that the students would justify their reasoning. Thus, the discussions held little evidence of the students collaboratively constructing mathematical ideas or developing new perspectives and understandings. This also limited Huia and Meilani's access to broaden understandings and make rich connections across different student's reasoning.

Status and Positioning of Learners

On interview Meilani and Huia both stated that people in their class treated each other with respect "sometimes", and they both stated that the "popular people" bullied others. These statements illustrate the way they perceived themselves in contrast to others and affected their participation and contributions to the mathematical discourse.

When asked "Who's good at maths in your class?" Meilani and Huia both named the same students.

> Huia: The teacher thinks they're good at maths too. They're usually chosen to explain their ideas and they can talk about their ideas.

Clearly, the students perceived as holding low status were generally expected to be less competent. This allowed them to take a passive role where they did not need to question or contribute unless responding to the teacher's closed questions. Observations showed that Huia's body language consistently conveyed her own perception of her low status or "otherness". She usually sat looking at the ground and seemed like she wanted to disappear. This was further reinforced by the teacher who in one classroom observation initially asked Huia to explain her thinking. However, when she felt Huia was taking too long she took the paper and said:

> Sorry Huia we're running out of time so maybe we'll come back to this. Lei can you explain how you solved the problem please?

Lei, one of the students considered "good at maths", then showed how to multiply the two denominators to find a common denominator. Children watch and interpret teacher's

actions to see what they value. The teacher's actions in this instance conveyed a clear message that Huia's input was of less value than that of others in the group.

Barnes (2003) describes some students as “attention-avoiders”. During the observations Meilani appeared interested and often took up a position as a helper, for example, writing everyone’s names or asking questions, but not a position where she would influence others through sharing her own thinking or reasoning about a problem. During one observation the teacher asked Praise, a member of Meilani’s group with high status, to explain. While Praise was talking Meilani was speaking quietly to Chontel who was sitting next to her.

> 8 represents 2/8 and ⅝ with one group left over. You could draw ⅞ ‘cos ⅜ plus 4/8 is ⅞.

While it was clear that Meilani was engaging with the reasoning she was reluctant to share her thinking with the larger group. A role the teacher needed to do in this instance would be to publically notice and respond to Meilani’s whispered comments. Through such means the teacher could give Meilani more confidence in her voice and position her as someone competent in mathematics with strengths, abilities, and valid ideas to contribute.

Conclusion and Implications

Meilani and Huia both face barriers to participating in mathematics and this had resulted in them having a negative disposition towards mathematics, a sense of “otherness” and passive participation which had resulted in lowered achievement. As Barnes (2005) illustrated students who participate less, learn less.

The teacher role is significant in creating patterns of participation in the mathematics classroom. To promote equal status interactions and participation amongst students requires teacher intervention (Barnes, 2005; Civil, 2014). Unless teachers intervene to equalise rates of participation, “‘the rich get richer,’ and the gap in academic achievement widens” (Cohen, Lotan, Scarloss, & Arellano, 1999 as cited in Civil, 2014, p.7). As Anthony and Walshaw (2009) explain the teacher’s actions directly affects what is happening and for who. In the case of Meilani and Huia unless there is effective teacher intervention, in line with Lack and colleagues (2014) propose, these two students are positioned in ways that cause them to have less opportunities to participate and therefore achieve.

One action which is important given that mathematics is a social endeavour would be the need for the teacher needs to address status issues. The social construction of mathematical learning takes time and patience, however, creating a strong culture of participation is imperative for developing all students as capable and confident mathematicians with a deep understanding of mathematics (Bennett, 2014; Lack et al., 2014). Bennett (2014) argues that key actions include active support for collaboration, and the building of a caring, inclusive and a respectful learning community. To give Meilani and Huia the best opportunities to participate the use of ability grouping needs to be addressed. As Civil (2014) proposes grouping students needs to focus on their different strengths and ways of thinking within heterogeneous groups where groupworthy problems tasks that incorporate students’ cultural identities are used which draw on multiple competencies. This supports students to have multiple ways to learn from each other (Anthony & Walshaw, 2009) and also would support Meilani and Huia to connect school mathematics with the mathematics they use in other parts of their lives.

In such a setting the culture of participation will promote productive mathematical discourse, and Sarah can make this meaningful and rich by expecting and supporting all students to participate in mathematical practices and communicate their explanations, justification, and argumentation (Bennett, 2014).

The implication we need to consider is the reality that there is far more to participation issues than students being shy or reluctant to share their thinking. Participation in a mathematical inquiry community is about how the teacher establishes and maintains the classroom as a safe and equitable learning environment where every student develops a positive mathematical disposition, can see the value of mathematics in their work, actively participates in learning mathematics, and believes they can succeed.

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Zooming-in on Decimals

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This study investigated how six Year Four (9-10 years old) students interacted with a dynamic (zoomable) digital number line to demonstrate knowledge and understandings of decimal fractions. Results from a task-based interview indicated that the zoomable number line is able to assist students in developing conceptual understandings of decimal density, place value and relative size. The app proved to be a powerful mathematical representation of decimal fractions because of its dynamic affordances that allowed the students to 'see' concepts that would otherwise be 'unseeable' when using a traditional static number line.

Decimal fractions are considered to be of great significance in the primary mathematics curriculum due to their application and use in everyday life. However, decimal fractions and the related areas of ratio and proportion, are recognised as the most challenging and complex areas of mathematics for young children to learn (Lamon, 2001). Such complexities arise as students are exposed to fractions in numerous symbolic representations, including ratios, common fractions, decimals, and percentages, without deep understanding and knowledge of how use each notation correctly (Bobis, 2011). Decimal fractions notation can be difficult for children to comprehend, as part of the relationship, the denominator, is hidden (Steinle & Stacey, 2004). The relative size of numbers written in decimal notation is instead expressed through place value, and many young students struggle to extend their whole-number place-value knowledge to the comprehension of decimal fractions.

One of the key concepts required for understanding of decimal fractions is *decimal density*. Decimal density refers to the continuity property of rational numbers, whereby, between any two decimals there are an infinite number of other decimals (Widjaja, Stacey & Steinle, 2008). Many young children have extensive difficulties in recognising the density of decimals, as it is void in their whole-number knowledge (Widjaja et al., 2008). The discreteness feature of whole numbers is incongruous for understanding the density of decimals (Widjaja et al., 2008).

If students do not have a strong conceptual understanding of decimals they are more susceptible to developing a set of misconceptions associated with the topic area. Misconceptions are predictable, systematic mistakes that arise from an incorrect interpretation; broadly they are a mistaken way of thinking and understanding about a mathematical concept (Steinle, 2004). Young students' misconceptions about the meaning of decimal number notation has been well documented in the works of Steinle and Stacey (e.g. Steinle & Stacey, 2004; Steinle, 2004, Steinle & Stacey, 2001). However, less evidence has been accumulated on how to avoid and address these misconceptions. How then can teachers support children's comprehension of key concepts such as place value and decimal density when decimal fractions are introduced?

An important feature of mathematics education is working with appropriate mathematical representations to assist students in understanding the abstract concepts that are central to learning mathematics, yet there is uncertainty around the best representation for decimal fractions. Number lines have been suggested as a powerful representation of decimal fractions as they show where a decimal 'fits' between two-whole numbers

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(Thomson & Walker, 1996), which relates to decimal density. However, the mathematical meanings of the structure and multiple uses of number lines, can also be problematic for children (Teppo & van den Heuvel-Panhuizen, 2014; Way & Harding, 2017). The affordances of dynamic digital representations may offer a better learning alternative than traditional static number lines (Moyer-Packenham et. al., 2016; Tucker, Moyer-Packenham, Shumway& Gordon, 2016), but little research has been conducted about this particular digital tool. It is very difficult to display hundredths, and thousandths on a paper-based number line, however the sophisticated functions of the interactive web-app make this easily achievable. For example, users are able to stretch the number line and zoom-in to see how tenths 'fit' between two consecutive whole numbers and zoom-in again to see how hundredths are located between a pair of tenths.

The focus of this study was on exploring how children's interactions with the zoomable number line might support development their conceptual understanding of decimal fractions. The research question addressed by this paper asks; How do Year 4 students interact with dynamic digital representations, namely a 'zoomable number line', to demonstrate knowledge and understandings of decimal fractions?

Methodology

The theoretical position of this study is grounded in the work of Piaget (1978), who first introduced the task-based interview to uncover children's thinking about posed problems and believed that "true understanding takes place when the student makes discoveries for themselves" (Assad, 2015, p.17). A major tenet of constructivism suggests that learning occurs when students are actively involved in a process of meaning and knowledge construction, mediated through existing knowledge, as opposed to passively receiving information (Simon, 1995, p.116). This study recognised students' existing knowledge of decimal fractions and anticipated that individual students would respond to their explorations of the digital number line in different ways.

Task-Based Interviews

Gerald Goldin's (1993, 2003) research links constructivist theory with structured clinical interviews as a means of understanding conceptual knowledge, higher-level problem-solving processes, and children's internal constructions of mathematical meanings. Goldin (2003) outlines two key functions of task-based interviews, firstly, to observe the mathematical behaviour of children, often in an exploratory problem-solving context, and secondly, to draw inferences from observations to form conclusions about the problem solver's possible meanings, knowledge structures, cognitive processes, and whether these change through the course of the interview. Therefore, the tasks must be sufficiently open-ended to allow each student to respond in their own preferred way, and designed to maximise dialogue between the researcher and student (Hunting, 1997). The researcher uses probe questions to encourage further explanation, for example 'Can you tell me what you are thinking?', but remains neutral regarding 'correctness' of responses. The purpose of the interview is neither to teach nor to test knowledge, but rather to explore the student's ways of thinking and interacting with the mathematical representation.

Participants

The participants of the task-based interview were from a class of Year Four students (9-10 years old) in a suburban school in NSW. This year level was chosen because, according

to the syllabus, in Year 3 decimal notation (tenths) has been introduced, and Year 4 students are working towards being able to "model, compare and represent decimals of up to two decimal places", and being able to "place decimals of up to two decimal places on a number line" (BOSTES, 2012, p.142). The class teacher was asked to recommend six children who would be comfortable in talking one-to-one with the researcher. Although the children were not intended to be representative of the class, the teacher was asked to ensure that the children spanned a range of general mathematical achievement levels, in case there were differences in the ways in which low and high achievers interacted with the digital number line. Six children were considered sufficient to reveal a variety of student responses, while not exceeding the imposed time constraints of an undergraduate Honours project.

Tasks

A set of five tasks were designed, with increasing levels of challenge, to encourage the students to interact conceptually with decimal notation and the positioning of decimals on a number line (see Table 1).

Table 1:
Interview tasks

Task 1 – Base knowledge Show the student card '4.2' What is this number? What is this? (interviewer gestures to decimal point) What does it mean?	Task 3 – Locating a decimal number Locate and describe its location on the digital number line. 0.3 0.12 0.9 1.55 Extension: *4.915* *9.819* How did you locate the decimal fraction on the number line? How did you know what numbers the decimal was between?
Task 2 – Introduction to app The participant freely investigates the app by manipulating the toggles to zoom in and out, and scroll along the number line. What do those lines represent? What does that decimal number mean? What do you notice about the 'zoomable number line' app? Are there any features of the app that are confusing or difficult to use? What are some interesting or useful features of the app?	Task 4 – Comparing numbers Which is larger? (Predict then check) 0.4 or 4.5 0.86 or 1.3 0.3 or 0.4 1.85 or 1.84 3.71 or 3.76 Extension: 3.92 or 3.481 4.08 or 4.8 What strategy did you use to predict which decimal was larger? How do you know it is larger?
Task 5 – Contextualised problem solving *Five gymnasts are entered into a competition. Four of the gymnasts have performed their routines. Their scores, out of ten, were 9.8, 9.75, 9.79, and 9.76. What score must the last gymnast get in order to win the competition?*	

The interview began with a conversation about decimal fractions and an exploration of the functionality of the app. Following this, students located a sequence of one- and two-decimal-place numbers on the digital number line. They then compared a series of pairs of decimal numbers by identifying which was larger. The fifth task was presented as a contextualised word problem requiring the comparison of four decimals.

Data Collection and Analysis

Video-recording the interview was critical to collecting quality data as it was necessary to capture what was said and done by the child simultaneously, including gestures, expressions and emotional responses. In addition, a screen-recording tool from the application *QuickTime Player* was used to document manipulative actions that students made on the 'zoomable number line' web-app (Pierce, 2017), accessed on an iPad.

Data analysis involved repeated viewings of the video/audio recordings and screen-captures, to document the key features of each child's responses, with particular attention to manipulation of the digital number line, the mathematics concepts demonstrated, and the relationship between the two. These significant sayings and doings were then analysed for similarities and differences across the students' profiles, using a thematic analysis approach (Braun & Clarke, 2006).

Findings

The analysis revealed four major themes: decimal density, whole-number thinking, place value and relative size of decimals, and problem-solving. The four themes are presented in order of their predominance across the six students.

Decimal density

While completing the tasks, most students attended to the idea of decimal fractions existing between whole numbers, and smaller decimals existing between those decimal fractions, that is, the concept of decimal density. Through their actions and dialogue, they explained the zoom function of the digital number line as, "… a closer view to show you how far apart the numbers are and what decimals are in between" (Student 1).

For example, when locating the decimal 1.55, Student 5 explained "one-point-five-five is in the middle of one-point-five-six and one-point-five-four and also in the middle of one and two." By using the zoomable number line the student was able to demonstrate his developing understanding of number density as he recognised how wholes can be continually divided into smaller parts by describing one-point-five-five being in the middle of 1 and 2 as well as 1.56 and 1.54. The zoomable number line allowed Student 5 to exhibit his understanding of number density as he used the zoomed-in screen display (as seen in Figures 1 & 2) by gesturing to it when he named multiple locations of 1.55 on the number line.

Two students extended this idea as they commented on the continuity property of decimals whereby between any two decimals there are an infinite number of other decimals. Student 4 uncovered how, between any two distinct numbers, there are an infinite number of decimal fractions as he continually zooming in around the number 1. He was interested in discovering how many place values could be revealed in a decimal. The function of zooming-in to reveal increasing place values fascinated him. Similarly, when Student 6 was freely investigating the zoomable number line app he described his actions aloud: "It's zooming in… and now it's showing the decimals… and now it's showing the hundredths… and now it's showing the thousandths… and now it's showing the tens of thousandths."

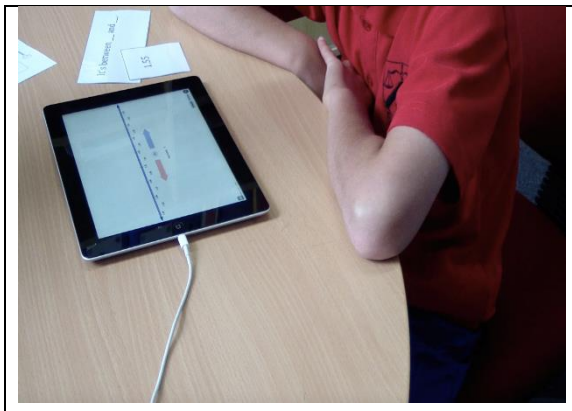

Figure 1. Screenshot of Student 5's interview showing zoomed in screen display of the web-app to complete a decimal location task.

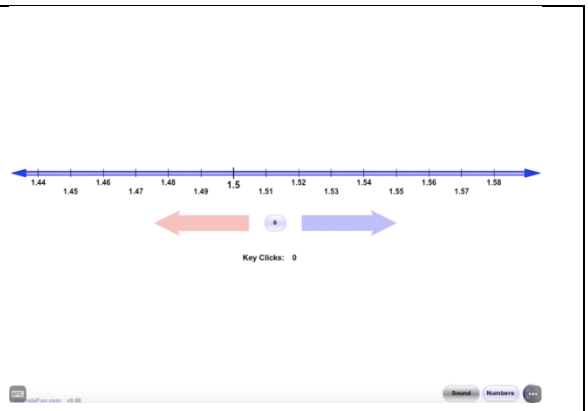

Figure 2. Screenshot of the 'Zoomable Number Line' web-app that was recorded using QuickTime Player whilst Student 5 was manipulating it.

Shifting 'whole number thinking'

Four out of the six students interviewed in this study displayed conceptual confusion based on their treatment of decimals as whole numbers. These students also displayed difficulty coordinating the number of parts and the size of the parts, as they had not developed the decimal-fraction link. The decimal-fraction link refers to the idea that decimals, like fractions, allow us to describe parts of a unit quantity (Steinle & Stacey, 2001). This was evident in the reading of decimals as whole numbers, for example, "one-point-fifty-five". However, during the interview, the students' conceptual understanding of decimals shifted from whole-number knowledge to being able to recognise and apply the decimal-fraction link through their interaction with the zoomable number line. For example, Student 3 was asked to locate the decimal 0.12 on the zoomable number line and describe its location by stating "it's between _ and _". Student 3 zoomed into 0.9 without revealing the hundredths and stated "zero-point-twelve is between zero-point-eleven and zero-point-thirteen". When prompted by the interviewer he zoomed into show hundredths and was then able to locate 0.12 by "scrolling back". The next question asked the student to locate 0.55 and describe its location on the number line. For this task, Student 3 instantly zoomed-in to reveal hundredths around the 1 and was then able to accurately locate 1.55.

When answering the same question, Student 1 went to locate 0.12 as a decimal after 0.9, after zooming in to reveal hundredths and seeing the numbers 0.88, 0.89, 0.9, and 0.91 on the number line he started scrolling left to locate the decimal after 0.1. The interviewer asked the student why he had self-corrected his working the student responded, "...because I read it wrong" the interviewer asked, "when did you realise that?" the student answered, "when I zoomed in". When asked to locate 0.12, Student 2 similarly to Student 1 went to locate it after 0.9, he realised the error when he zoomed in to reveal hundredths.

Place value and relative size of decimals

As described above, a common misconception held by children is thinking that the numerals to the right of the decimal point are another whole number. Student 3's persistent whole-number thinking led him to believe, in the context of comparing decimals, that "longer is larger". The scrolling function of the digital number line enabled the student to correctly compare the relative sizes of sets of decimals, which assisted him in discovering

the concept of place value in decimals. For example, Student 3 was asked to predict whether 3.92 or 3.481 is larger, the student answered "three-point-four-hundred-and-eighty-one" reasoning that "four hundred and eighty-one is larger than ninety-two". The student was prompted to use the zoomable number line to check his answer. Student 3 accurately located 3.92 on the zoomable number line. He then zoomed in further to reveal thousandths and when probed he noted "I have to scroll back to find three-point-four-hundred-and-eighty-one". The interviewer asked the student again which of the two was larger and why, to which he responded "three-point-ninety-two because it's further up on the chart [number line]". At the end of the task-based interview the student was asked whether the app assisted him in answering a particular question, he replied "yes, the last question because I thought that the four-hundred-and-one decimal would be larger than the ninety-two decimal but it wasn't".

Problem-solving, explaining and justifying

Students with a stronger understanding of decimal notion still struggled with the abstractness of decimal notion, but the digital number line provided a tool for making predictions, testing them and then questioning their findings to 'see' the concept from various perspectives. In particular, Students 5 and 6 engaged with the dynamic nature of the zoomable number line to assist them in confirming and verbally justifying their solutions for the open-ended problem-solving task.

For example, after reading the *Task 5* problem-solving question aloud, Student 5 answered, "they must get an addition of zero-point-zero-six to win". When asked to explain his working the student reasoned, "because the lowest score is nine-point-seven-five and the highest is nine-point-eight and if you added zero-point-zero-six then it will be one over the next one and they will win." The student was asked to prove his answer using the zoomable number line he said, "the lowest score is at this point [gestures to 9.75] meanwhile the next one is one-point over [gesture to 9.76] and the next one is one-point under the highest [gestures to 9.79] and the highest is here [gestures of 9.8]." (See Figure 3)

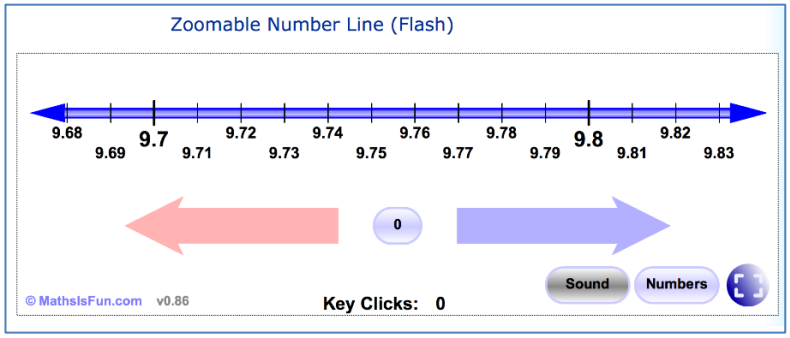

Figure 3. Screen shot of Student 5's number line during justification of problem solution. (Pierce, 2017 at *Maths is Fun*)

The interviewer asked, "so what score could the competitor get to win?", student answered "6 more [gestures from 9.75 up 6-hundredths to 9.81] because then they will manage to get just one over the second-place at nine-point-eight-one." The interviewer then asked "did the zoomable number line help you answer the questions?" the student replied, "Yes, seeing the numbers on the number line helped me answer the question, I could see exactly where each number was and zoom in to see specifically where it was to make it easier."

Discussion and Conclusion

As anticipated, the ways in which each of the six students used the digital number line varied across the tasks, with a strong link between their approaches and existing understandings of base-10 numeration and decimal fractions. However, regardless of their prior knowledge, all children showed progress in their thinking during the 20-minute interview, particularly regarding decimal density, whole-number thinking and recognising the role of place value in communicating the relative size of decimal fractions. The students' interactions with the zoomable number line, at times, generated productive cognitive conflict, as the dynamic representation of decimal fractions challenged their existing understandings and misconceptions. Outside of the research-interview situation, these would have been ideal 'teaching moments'.

The number line is one of the few models that is functional for discussing the density of decimals. However, Steinle and Stacey (2004) have commented on the limitations of a static number line as they note that some students view it as a discrete representation of numbers at isolated points. The digital functions of the zoomable number line allowed the students to uncover decimal concepts that a static number line is not able to readily illustrate. The ability to move forwards and backwards along the number line enables students to see the continuous sequence of decimals, which assists them gaining a sense of relative magnitude and of addition and subtraction (Teppo & van den Heuvel- Panhuizen, 2014), and this was readily facilitated by the digital number line.

Pierce, Steinle, Stacey, & Widjaja (2008) argue for the need for fundamental reorganisation of whole-number prior knowledge in understanding the density of decimals. The dynamic affordances of the digital number line appeared to prompt the required reasoning in students, as they zoomed-in to reveal decimal fractions between two whole numbers, and zoomed-in further to see smaller and smaller parts appearing.

While keeping in mind the inferential limitations of a small-scale study, the findings clearly indicate the potential of dynamic, digital number lines for use as effective learning and teaching tools to develop conceptual understanding of decimal fractions, particularly regarding number density, comprehending place-value notation and the relative size of decimal fractions. All students, with minimal prompting from the researcher, engaged in self-directed exploration of decimals, and self-correction of reasoning as a direct result of their interactions with the zoomable number line. As the digital number line is capable of illustrating concepts that cannot be effectively represented on a traditional static number line, further research into how children perceive decimals with this digital tool could inform effective teaching practices for the prevention and correction of common misconceptions about decimal fractions.

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Unexpected Outcomes of a Family Mathematics Story-Time Program

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Research suggests mathematical stories can support mathematical learning. We discuss an unexpected outcome caregivers of reception year learners (Gr R age 5-6yrs), participating in a mathematics story-time program, shared in interviews. The program, implemented with Gr R learners' caregivers in two South African schools, explained and demonstrated a dialogic approach to reading mathematical stories. While data indicated success in the intended specific mathematics outcomes (e.g. numeral and number word recognition; finding one more/less), we focus here on a broader learning outcome all parents shared. That is, children changed their way of being a learner participant in the family/community.

Background and Context

South African learners tend to perform poorly on both international and regional mathematics and literacy comparative measures (see Graven, 2014). Furthermore, the performance gaps according to socio-economic status are among the most extreme (Reddy et al., 2015). Redressing inequality in education has been a priority since South Africa's first democratic elections in 1994 and education is seen as a vehicle for redressing persistent economic inequality. The South African Numeracy Chair (SANC) at Rhodes University (the incumbent is the first author), is mandated to work at the research and development interface to address the challenges of primary mathematics learning in so called previously disadvantaged communities. Much research points to early childhood learning opportunities being particularly influential in setting the educational learning trajectories of learners and particularly important for closing educational gaps between the rich and the poor (see Atweh et al., 2014). Increasingly South African research is calling for early intervention, especially in mathematics, as by Grade 4 the majority of Grade 4 learners are already considered to be two grades behind grade level expectations, (Spaull and Kotze, 2015). In 2016 the SANC introduced the Early Number Fun (ENF) program that brought over 40 Grade R teachers, and district and provincial departmental officials, together on a monthly basis. In the program participants engaged with SANC project researchers on selected research-informed resources that were considered potentially useful in supporting Grade R (reception year age 5-6yrs) student mathematics learning. Participants provided feedback on their in-class use of resources and adaptations were made accordingly (resources are available on www.ru.ac.za/sanc/enf).

Building on emerging local research that points to the benefits of using a narrative approach in the teaching of mathematics to primary learners (Roberts, 2016; Takane, Tshesane & Askew, 2017) the first author designed a series of mathematics stories for Grade R teacher use with learners. While the ENF focus was on mathematics learning, it was considered important to find ways to blend this work with literacy that is a key part of teachers' daily work. Mathematics story-books provided a powerful resource for blending these two key learning areas. Following highly positive feedback on the use of mathematics storybooks and their related resources from teachers it was decided to extend the use of these

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books and activities to parents. The first author's experience of running family mathematics events across several local schools, communities and after-care centres, indicated strong parent willingness to support their children's learning but also that there was a need for provision of age appropriate resources and guidance in the use of these for strengthening mathematical learning, reasoning and communication.

Thus, the mathematics story-time program was introduced to caregivers of Grade R learners, that involved three one-hour sessions (spaced out weekly or monthly) in which four story books were provided along with related resources (e.g., paper finger puppets, laminated numeral and number word cards, dice, cards). To date the program has only been run in two South African schools and plans are underway to run the program with a school in Australia mid 2018. Research into the process of implementing, and the effects of, the program is ongoing and will feed into subsequent iterations of the program. In this paper, we draw predominantly on data to illustrate the impact of the program on families. It is not the intent here to provide a deep, or theoretical analysis of these data. Simply, our intent here is to establish the impact of the program from the data collected.

Framing Assumptions

All SANC programs are broadly informed by a Vygotskian perspective of learning where learning is considered socially mediated and historically and culturally situated (1930). For Vygotsky language is the critical socio-cultural tool that enables development of concepts and meaning making, first socially "between people (interpsychological)" then individually "inside the child (intrapsychological)" (Vygotsky 1930, p. 48). Thus, an assumption of the program was that for learners to develop rich mathematical understanding they needed to engage actively with mathematical ideas in social situations. The dialogic reading approach (see Whitehurst et al., 1988), with its interactive technique where the reader prompts children to discuss ideas and answer questions related to the stories read to them, coheres well with these assumptions. Encouraging learners to pretend read the stories, using the same dialogic approach with others, provided further opportunity for children to talk about mathematical and other ideas in the stories. Furthermore, a constructivist approach informed the development of the storybook program and resources were informed by literature around learner progression (e.g. Clements and Sarama, 2009) and what young learners can be expected to know and do mathematically by this age (5-6yrs). This aligns with the South African curriculum content and its progression for Grade R learners and is similar to the Australian curriculum.

The Family Mathematics Story-Time Program

The aim of the program is to involve parents in engaging with their children about mathematical ideas in the reading of picture book mathematics stories to strengthen their children's number sense, mathematical language and love of reading and enacting stories with mathematically-engaging content. In each session the first author modelled dialogic reading to parents of each story with 3-5 children and a child would then pretend read the story back. Thereafter demonstrations of children engaging with mathematical ideas using the accompanying resources (e.g. the puppets, the numeral and number word cards, the flash cards of 'more' and 'less') were done. In particular attention was focused on questions to ask children as one engaged with the resources. Parents were provided three sessions in which this dialogic approach to reading the stories was explained and demonstrated. Parents were encouraged to engage with learners about what was happening mathematically (and

otherwise) in the story; to encourage prediction of what would happen next and to have children similarly 'pretend' read the story.

The first three books each tell a basic number story of five or ten characters moving from one place to another, one or two at a time, and focuses on the changing quantities as this movement happens. For example, the first story begins with 5 monkeys in a small tree, no monkeys in a big tree and a moaning monkey who provides the stimulus for jumping to the big tree. See Graven and Coles (2017) discussion of teacher and learner engagement with the story in two classroom contexts. Below is an example of one page of the story:

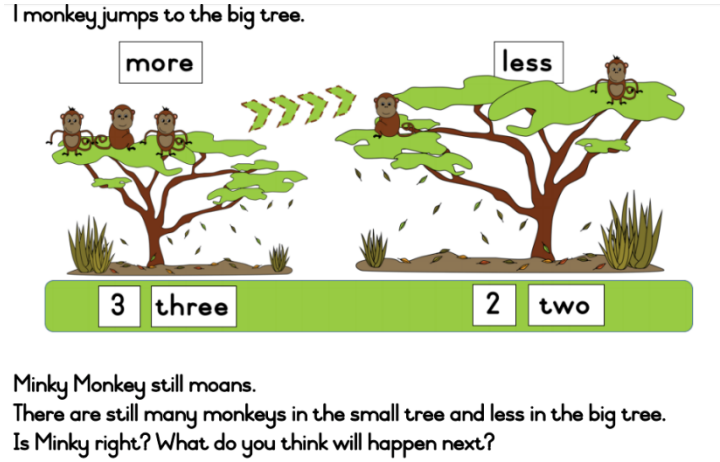

Figure 1: Page 4 of the 'Monkeys in the tree' book (SANCP, 2016)

In the demonstrated reading children were asked questions throughout like 'Okay so if another monkey jumped then how many will there be in each tree when I turn the page? Where are there more monkeys? Where are 'less' monkeys? How many monkeys are there altogether? The second book is similar with 5 children under a small umbrella at the start of the story. The third book (vastly adapted from its first version used in ENF) involved 10 frogs on a small lily pad and unfolds with the frogs jumping in pairs to a large Lily pad. A story-board page was provided at the end of each of the first three books which had only the pictures of the two trees/umbrellas/lily pads etc. Children were encouraged to use this page, with the finger puppets and flash cards (numerals, number words and the words 'more' 'less'), to retell the stories by moving the puppets from picture to picture. The final story involved two children collecting firewood for their 'Gogo' (granny) and carrying the sticks on their heads (as can be seen locally). The children in the story, Busi and Thabo, sometimes pick up one or more sticks and the sticks sometime fall. This story is not patterned in the same way as the other three stories but similar questions are asked such as 'how many sticks altogether?' 'now who will have more sticks?' and 'how many more sticks does Busi have?' While dice and cards were also provided and basic games demonstrated at the end of each session to families these were not a main focus of activities and instead were used to reinforce concepts, language and skills developed in the stories (numeral recognition; how many altogether? where are more? etc.).

In sessions the likely progressions of children in the various intended mathematical skills and concepts were made explicit to caregivers so that they could assist in mediating children's progression. The intended skills that were built into the stories and demonstrated in the dialogic reading were: number and numeral recognitions (first to five then to ten);

counting objects and collections of objects accurately up to ten; subitising up to 6 (aided by dice); working out what one/two more/less of a quantity is; working with the comparative language of more and less and later saying how many more (noting the difference) in each collection. So for example for it was noted that children may move from counting the characters or objects in stories and dice by: sometimes making errors with one to one counting (e.g., touching one object more than once in a count or touching it once and counting more than one number); they should later count correctly with one to one correspondence; they should eventually subitise (know instantly there are three monkeys or 4 dots on a dice without counting); once subitising they can count on from how many in one place to get the total number in both places or on both dice.

Research Sample and Data Gathering

The interpretive research, drawing on aspects of design experiment research (Cobb, Confrey, Di Sessa, Lehrer and Schauble, 2003) that is highly interventionist in nature used qualitative data gathering methods aimed at understanding both caregiver and learner experiences as a result of their program participation. The primary data-gathering instrument was interviews. Two caregiver interviews were conducted - one at the start and another at the end of the three sessions. These interviews were a one-on-one interview and were audio recorded. All interviews were transcribed. Learner interviews were task-based where children were asked to: pretend read the monkey story, retell the story (using various props), and play with two dice – in each of these activities learners were asked a range of questions that provided data as to learner progress in relation to the various intended mathematical learning outcomes. Video recordings were made of each of the sessions. These have not been transcribed as their primary purpose was to enable revisiting what was communicated and discussed in sessions.

The data used in this paper are derived from the transcribed post-session interviews with the caregivers who participated in the mathematics story time program at one of the two schools it has been implemented in to date (as described below). The interview questions included for example: Can you tell us about your and your child's experiences of reading these story books together? Could you please tell us about your and your child's experiences of these activities? Has the way in which your child interacts with the stories changed at all after reading the books a few times? Explain. Do you think there have been any social/emotional advantages to reading these number stories with your child? Explain.

Unexpected Outcome

Across the post session interviews in both schools there were multiple comments from caregivers about the ways children engaged with others in their extended families and in their local communities had changed. While a key aim of the program was to encourage primary caregivers to spend time engaging with children around mathematical ideas we had not expected the extent to which mathematical engagement spread, initiated by the children, to other members of the family and community. We chose to focus on this rather unexpected outcome for this paper. While a similar story could be told drawing on the data of caregiver interviews in the other school, we have chosen this school because the second author attended the final session at this school and observed several of the post-session parent and learner interviews.

Both schools serve predominantly learners who under apartheid were classified non-white. In both classes the medium of instruction is English even while the vast majority of learners speak either isi-Xhosa (the language of indigenous people in the Eastern Cape area)

or Afrikaans (a language of Dutch origin spoken extensively by those of mixed race classified under apartheid as 'Coloured'). The school we focus on here draws learners from predominantly poor, working-class or unemployed backgrounds.

The parents/caregivers of nine children in the Grade R class of the chosen school participated in three sessions run weekly at the school in the evenings from 5:30 to 6:30 pm. All nine parents or caregivers participated in interviews at the start and end of the sessions. In some cases different caregivers came to the different sessions (such as a different parent, an aunt or a grandmother). The interviews were conducted on a one to one basis by the first author and a research assistant who attended all sessions and assisted with the distribution of resources and video recording the sessions. The second author attended and assisted with the recording of five of the interviews following her attendance of the third session in this school. All names used below are pseudonyms. Responses to interview questions were coded according to themes. The theme we share here of increased and child-led mathematical engagement emerged in all nine sets of caregiver interviews.

Findings

In all interviews comments were made about how the children increasingly initiated mathematical learning activities, even beyond with their caregiver/s who attended the sessions and beyond the stories. Below we provide examples of utterances from all nine interviews of this changing engagement with different family/community members. We begin with exemplar comments of changing engagement with parents, then with siblings or cousins and finally with others in their community such as neighbours and friends. Thereafter we provide a fuller vignette of one child's changing engagement and way of being to illuminate this in a richer more holistic way. In sharing the data we have ensured that we have provided at least one quote from each of the nine caregivers interviewed.

Examples of utterances about changing ways of being with parents

> Jaya's mother: "She would be shy because she doesn't always read to her father. But now with this project she always reads to him."

> Zandi's father: "She was not interested in reading, but now she asks what we are reading, and then tomorrow she wants to act what you were reading yesterday for her… I don't have much time, only on weekends. I sit with them and we take all the books, and from Saturday to Sunday we go through all the stories… I have learned to also teach the young ones because I used to say to them I didn't have time."

> Cal's father: "Like I said to my wife, for us reading the book, and to help him read as well, it is eye-opening for us as well to let him start at a young age. It's like a new seed for a tree to come out, shall we say."

> Leanne's mother explained that she was more confident reading to her daughter and that now it is "routine for them every night…They will come with the book, even though I think, 'I want to watch this movie' I would rather sacrifice that."

> Saide's mother: "I don't want to lie. I didn't even remember taking a book before my baby went to bed and read it. But after I got those books, when I come home at 5 o'clock, 'mommy can we read a story?' I don't get bored, I read the story and he also reads the story for himself, and when we go to bed again we read the story."

Examples of utterances about changing ways of being with siblings/cousins/aunts

> Nathi's mother: "He is telling the big sisters the stories…He used to fight with other people, but now it has brought them together with the games and the stories…"

Jaya's mother: "We did it [reading and re-enacting] together, me and the eldest and Jaya. Even the baby was sitting there. Always enjoying [inaudible] that one…because now in this project we have time to read together as a family."

Lee-Ann's mother: "she explains to her brother how the books work…So she is telling him 'Listen here, do you know that six and four is ten?' with that attitude. But I am glad. I am not going to interfere because she is in the learning process. Then she says, 'let me show you how I got ten.'"

Jean's mother: "She asks you to help her with the sticks and the cards, and the monkeys. It's very exciting for her… She likes to play on her own, but she also asks me to play with her, and my sister also."

Examples of utterances about changing ways of being with neighbours/friends

Eli's granny: "He goes to his friends. Like yesterday he showed them the book, and they asked questions, and then Eli told them what the story was about."

Sade's mother: "In class she was very quiet I think, but because she has those books now, whenever we go to other people's houses she will tell them about [them and] these interesting games, and the numbers and then she is the teacher. Which she really is."

Leanne's mother: "Yeah because in the class she is very quiet, I think, but because she has these books now, whenever we go to other people's houses, she will tell them about these interesting games, and the numbers, and then she is the teacher."

Elsa's changing way of being and engaging with others as told by her Aunt

In order to provide a more holistic picture we now provide a fuller story of one child (Elsa) as told by her aunt. Elsa's aunt looks after her in the afternoons as her mother works in a city about 160 km away. Elsa's aunt attended the sessions in place of Elsa's mother who was only able to attend one session. We selected this interview because it captures multiple changing way of engaging with others in learning activities that were noted across interviews. While not all interviews tell the story of such oppositional change from 'quiet and shy' to 'talkative and open', as is shared below, they all indicated increased engagement with others around mathematical ideas mathematical engagement and all provided instances of their children initiating or taking the lead in such engagement.

Elsa's aunt explained how Elsa had become someone who asked lots of questions and initiated regular engagement with her and her mother about mathematical ideas:

She asks lots of questions…She says 'Mummy please can you come and sit here I want to ask you something. Tell me how many plates are in the cupboard…' and then when she comes home from school she says, 'Can you please read the book? Can we please do the cards and dice?' She wants to choose what she wants to do…I read to her then she mentions how she reads to me. 'I am the mom you are the child'.

Later she explained that this engagement 'even' extended to her father. Comments about engagement with fathers came up in several interviews across both schools as can be seen in some of the examples above. Elsa's aunt commented that "Even with her father. 'Daddy how many wheels on the car?' and he says 'It's four. You can count. If I take one away, how many are there?'" Elsa's aunt further explained how Elsa would get her to act out the story pretending spoons were monkeys and cups were the trees and would use blankets for umbrellas. They would then move spoons around and Elsa would ask questions such as 'how many are under each umbrella?' She explained how Elsa asked for a pen because she wanted to write the numbers (1-5). She then explained how Elsa engaged with "her big cousin in Grade 7" and argued with him mathematically:

> The big cousin in Grade 7 now. He says (to her) 'What is this nonsense you are doing? I don't understand.' I say, 'Come Barry, come and sit here.' And she (Elsa) argues with him when he says 'it is five more'. And she says, 'no it is two more'.

This relates to a page in the "Busi, Thabo Sticks and the fire" storybook where on Busi has 5 sticks on her head and Thabo has 3 sticks on his head. So while Barry is right that Busi has 5 sticks and this is more than Thabo's - Elsa is arguing that Busi only has 3 more than Thabo and not 5 more. This distinction was made explicit to parents in the sessions. In South Africa it is common in national assessments for students throughout primary grades to answer questions of 'how many more' by simply stating the quantity that is more as Elsa's cousin is reported to have done. Elsa's Aunt went on to explain how much Elsa was enjoying engaging with mathematics ideas and 'helping' others with these ideas:

> She wants to do so much and you can see she is enjoying it. She is experiencing more about numbers. She tells her brother she will help him with maths. She says 'Come and sit, you are also going to get clever'.

Elsa's Aunt further stated that Elsa was talking and 'opening up' to the family more:

> She is not actually my daughter but my sister's daughter. My sister works in (city). She was quiet before but now she is talking. Her mother was here for the last session, but she could not make it today because she was working a bit late…. because of the things she does now she is opening up to us. She speaks more now, then she asks all of us to tell her something.

She added how Elsa was now more willing to play with other children:

> She did not want to play with the neighbours before, but now she wants to go out and play with the other children, but we don't want her to go because it's not safe in the street. So when the children ask if she can play, we tell them they must play in the yard. She calls them to come play… When they play she goes and fetches paper and the pencils… she wants to read the book about the monkeys to those children. They are all in Grade R. She says 'come I am going to read to you'…. She never had friends calling, but now she is more interested in teaching them how to count…. She was a shy girl but now she has more friends, talks and does stuff.

'Playing teacher' or 'playing school' with the resources given with other children was noted in several other interviews. At the end of the interview Elsa's aunt re-iterated: "Our child was very quiet, but now when you came, its like you took her out of that quiet corner. She opens up to everyone."

Concluding Remarks

All caregivers interviewed indicated a willingness to attend future programs as they reported that they had gained so much from participating in the workshops. In all the interviews an increase in children's confidence to initiate engagement with others about mathematical ideas is indicated. Indeed, confidence to engage mathematically is likely to increase the more one has opportunities to speak the language of mathematics. From the interview data it seems that the provision of these storybook resources and the demonstration of a dialogic reading approach provided opportunities for engaging mathematical ideas that went beyond the intended caregiver-child reading scenario. Data showed that children regularly played with the resources and books both by themselves and with other family members and other children. Perhaps a key enabler in this was that the resources, while handed to parents, were to be given to their children providing for a sense of ownership of them. Leanne's mother explained that she saw this as important in enabling the independence and agency her daughter had developed in guiding her own learning. She explained "She doesn't have to depend on the other person. 'Come and play with me'. She can do it on her own and can be in charge of the games. It could be a factor why she likes it so much." Later

she added “But the thing is, because they are her books, she wants to be the main actor in reading the books. So she want to tell you, she want to show you how it works… she wants to be in charge”. In concluding the interview Leanne’s mom explained the learning opportunity enabled by this home based versus classroom-based story time program as:

> ‘What is nice about it, is, if ever they have difficulties in understanding in these sessions they are shy and they won’t ask questions. But in their own space at home, they are more confident. I think kids learn more when things are informal and not as formal as in the classroom.

What we need to do from the work reported here is to frame this within a model of parental engagement and learning for the children. It is clear from the data presented here that the project has built the intellectual capital of both parents and learners both in terms of literacy and mathematics. We now need to undertake a deeper analysis of the data to build this theoretical case. We are also keen to see how the same project is realised in the Australian context and to test the viability of a theoretical model across the two diverse contexts.

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How do Students Create Algorithms? Exploring a Group's Attempt to Maximise Happiness

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This paper addresses the need for empirical research on the processes by which students create algorithms. I analyse the collaborative work of three high-school students on a contextualised graph theory task, in which they created an algorithm for maximising the happiness score of a seating arrangement. The group found an optimal arrangement but created an algorithm that did not fully account for this arrangement. The group's written algorithm reflected only the properties of their optimal arrangement that they explicitly noticed after creating the arrangement. And, these explicitly-noticed properties aligned with the group's predominant contextual considerations.

Discrete mathematics – "the math of our time" Dossey (1991, p. 1) – is ever-growing in prominence due to its significance in computer science and the many real-world applications of its sub-branches (e.g., probability, logic, combinatorics, and cryptography). Accordingly, mathematics education research has sought to promote the teaching and learning of discrete mathematics (Hart & Sandefur, 2017; Kenney & Hirsch, 1991).

At the heart of discrete mathematics lies *the algorithmic approach* (Kenney & Hirsch, 1991) which entails solving a problem by devising and analysing an algorithm that constructs a solution. The algorithmic approach distinguishes discrete mathematics from traditional mathematics such as algebra, calculus, and topology (Kenney & Hirsch, 1991). Thus, developing students' competence with the algorithmic approach is a central theme in discrete mathematics education research (Morrow & Kenney, 1998; Hart & Sandefur, 2017).

Past research aimed at promoting students' competence in the algorithmic approach has largely focused on: 1) explicating the processes by which experts create algorithms (e.g., Weintrop et al., 2016); and 2) designing tasks (henceforth referred to as *algorithmic tasks*) that require students to create their own algorithms (e.g., see Morrow & Kenney, 1998 for numerous examples). The overarching goal of these two lines of research is more or less to identify *how the experts operate*, and then create tasks in which students are expected to operate in similar ways. While this past research has been useful in advancing our understanding of how experts create algorithms, it has lacked attention to the processes by which students create their algorithms.

The study reported in this paper is a first attempt to address the foregoing gap in the literature, by exploring the process through which a group of three students create an algorithm in a contextualised graph theory task. The aims of this study are to: 1) explore the idiosyncratic interpretations that students employ for the various requests of the task (e.g., the request to find a solution for the case at hand; the request to create an algorithm that finds a solution); 2) explore how these interpretations interact; and 3) discuss how the eventual algorithm reflects (or does not reflect) these interpretations.

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Background Literature

The need to provide students with more opportunities to engage with non-routine algorithmic tasks, and to equip them with the tools to create their own algorithms has been well-acknowledged in the literature (Hart & Sandefur, 2017). Two predominant lines of research have emerged in response to this need. The first line of research (e.g., Weintrop et al., 2016) has revealed common characteristics (e.g., recursion and induction) underlying the processes by which experts create algorithms. These processes are then construed as the processes that students *should* engage with when they work on algorithmic tasks. Furthermore, characteristics of the experts' algorithms, such as efficiency and generalisability, are taken to be the measures of quality for assessing students' algorithms. While this line of research is useful in providing ideals to which students could aspire, it has been acknowledged that it is unreasonable to evaluate students' final algorithms with respect to expert-qualities, without understanding the processes by which students' algorithms emerged (Hart & Sandefur, 2017). The second line of research (see Morrow & Kenney, 1998 for numerous examples) complements the first by designing algorithmic tasks and discussing the potential processes that students could engage in when working on these tasks. Despite the usefulness of this line of research, the hypothetical student activity that guides the design of these environments is potentially limiting, due to disparities between what students are expected to do, and what students actually do on these tasks (e.g., see Cai et al., 1998).

In a rare empirical study that analysed how students create algorithms, Cai et al. (1998) distinguished between *a procedure* and *an algorithm*. The former refers to the entire process by which a solution is found, while the latter refers to communicating the process in a succinct way which utilises recurring steps and patterns in the process. For instance, consider the problem: *how many days are there between March 24th and April 21st of any given year*? A procedure involves, say, counting every day between the two dates, and coming up with 28 days. An algorithm involves, say, noticing the recurrence of seven days between these two dates, and thus getting to 28 days by way of 7 x 4. Cai et al. observed that the students easily constructed a procedure but struggled with creating an algorithm. The students regurgitated their procedure when asked for an algorithm, and there was no evidence that the students noticed any recurring patterns in their procedure. Studies such as Cai et al. (1998) reveal nuances of the algorithmic approach (or the process of creating algorithms) that perhaps experts take for granted, but with which students might struggle. The study reported in this paper can be situated within research such as Cai et al. (1998) that seeks to expose, by way of empirical data on student activity, nuances of the algorithmic approach, and propose ways for helping students cope with these nuances.

Theoretical Framework: Considerations of Aptness

In the context of tasks asking participants to pose mathematical problems, Kontorovich (2016) discussed the construct of *considerations of aptness* "to capture uncertainties and doubts of a poser together with the meaning that is eventually attributed to the vague terms [stated in the task instructions]" (p. 246). The construct comprises five types of considerations (see Kontorovich, 2016), but for my study I focus on one of them: *considerations of aptness to the task*, which "is concerned with the poser's attempt to satisfy explicit requests of the given stimulus" (p. 246). The use of considerations of aptness in this study is motivated by the aim to explore the idiosyncratic interpretations that students employ for the various requests of the task.

I use *considerations of aptness to the task* to refer broadly to the students' attempts to fulfil the multiple requests of the task. The students' interpretations of these requests can be inferred from their idiosyncratic attempts to satisfy the explicit requests. One thing to note, however, is that the algorithmic task (used in this study) contains multiple explicit requests such as: finding an optimal solution for a given case; creating a method for finding an optimal solution for the general case; and framing the method as a letter. Thus, there will be different types of considerations of aptness to the task, depending on which particular explicit request the students are addressing.

Research Questions

The overall aim of the study (to reiterate) is to explore the process by which a group of students create an algorithm in a contextualised graph theory task. More specifically, through the lens of considerations of aptness to the task, the study explores the following questions: 1) What considerations of aptness to the task does the group employ? 2) How do these considerations of aptness interact? 3) How are these considerations of aptness reflected (or not reflected) in their final algorithm?

Method

The data for this study is taken from the collaborative work of three students who at the time of data collection, were in a Year 12 mathematics (calculus) class at a high school in New Zealand. All three students knew each other well and were recruited as part of a larger research project that explores students' engagement with discrete mathematics through contextualised tasks (Yoon, Chin, Griffith Moala, & Choy, 2017). The group worked on *The Birthday Seating Task* (Davies, Chin, Griffith Moala, & Yoon, 2016; adapted from https://xkcd.com/173), incorporating the theme of optimisation which is one of the central themes in all areas and contexts of discrete mathematics (see Hart & Sandefur, 2017). The fifty-minute session took place outside of class time and was video-recorded. The group worked in the presence of an interviewer who answered clarification questions but avoided providing mathematical hints.

The task begins with some warm up questions that familiarise students with weighted graphs (networks) in the context of different relationships. For example, in the graph below (Figure 1), the nodes represent people, and the number on an edge between two nodes represents a "happiness score" for the corresponding people's relationship. After completing the warm-up questions, the students are given the following scenario:

> Michael is turning 15 and has decided to invite his friends to the movies this weekend. He creates a Facebook event and invites his best friends. 7 friends have confirmed that they will attend. Michael decides to make a seating plan beforehand as the cinemas will only provide them with one row of seats, and he knows some of his friends don't get along. He draws a graph that represents the relationships between the seven confirmed friends [Figure 1] and shows the happiness scores between each pair of people [NB: if there's no edge between a pair of nodes (e.g., A & E or D & F) then you can assume that their happiness score is 0] (Davies et al., 2016, p. 9).

The instructions for the task are (stated below). After reading the instructions, the group the group were told to work together and that they must agree on everything that they include in their algorithm.

> Create an algorithm (method) that Michael can use to find the best seating arrangement (i.e., the one with the highest total happiness score) for his friendship graph. Remember all of Michael's friends must sit in one row at the cinema. Write a letter to Michael in which you:1) State the best seating arrangement; 2) Explain your method for choosing the best seating arrangement, and how/why it is

guaranteed to give you the best seating arrangement; 3) Describe how Michael can adapt your method to choose a seating arrangement if more of his friends (other than the 7 given in the graph) show up unexpectedly at the cinema. Remember that some of the unexpected friends might not get along with. some of the 7 confirmed friends (Davies et al., 2016, p. 10).

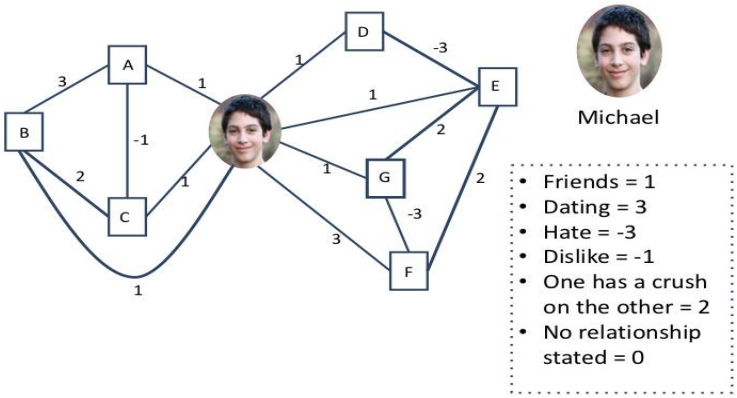

Figure 1. Michael's friendship graph.

To analyse the data, I first read the annotated transcript of the group's work and identified which requests of the task (e.g., the request to create an algorithm; the request to find the best seating arrangement) they were addressing and divided the transcript into excerpts corresponding to these different requests. I then compared these excerpts and sorted them into four basic categories that I interpreted as the primary considerations of aptness to the task that the group employed. These were considerations of aptness to: (i) the best seating arrangement for the given graph; (ii) a method for finding the best seating arrangement for the given graph; (iii) the real-world context of the task (i.e., a group of people with particular relationships going to the movies); and (iv) the unexpected friends. I then explored the interactions among these four considerations of aptness throughout the group's work and examined how the considerations were reflected (or not reflected) in the final algorithm. In the next section, I give an *account-of* (Mason, 2002) the group's work, summarising *what happened* in the entire session. Then, I describe two themes that emerged from the analysis. Throughout the session, the students and interviewer switched between "algorithm" and "method", and I preserve both when describing and analysing their work.

A Summary of the Group's Work

The group begins by agreeing that, "Michael should be in the middle, because it's his party." They then discuss how "the two people that hate each other the most" should be seated at the ends of the row, "that way they are farthest away from each other" and they don't "spoil things". Then, Sia notices that Michael is the only person with whom *D* has a "positive relationship." They remark that perhaps *D* should sit next to Michael, because that is "where *D* would feel most comfortable [and] if you don't put *D* next to Michael, he's just not going to have any fun, because he hates *E*, and no one else knows him." Then, each student creates a seating arrangement. Sia and Para both create (separately) *G-E-F-M-D-A-B-C* which has a score of 13, and Heti creates: *A-B-C-M-F-E-G-D* which also has a score of 13 (note, the decisions they make while they are creating these arrangements are not evident in the data). They examine the arrangements and say, "they both don't have any negatives". Para then asks, "how about the ones who show up unexpectedly?" Sia responds, "we'll just

assume that they get along with the people at the ends of the seating row". Para adds, "yeah, it's kind of rude to show up unexpectedly right?" Heti says, "Yeah, why couldn't they just confirm that they will come? It's so inconvenient!" Para continues, "OK, so just put them on the edges." Then, Para says, "It would be so much better if we just gave all of them tickets, and then they choose where they want to sit." Heti nods and Sia responds, "Nah, because if you think about it that'll be awkward if everyone chooses, because it's not just a party, it's the movies. So, what if like *E* comes and sits next to *F*, and then *B* also comes and sits next to *F*? Then Michael can't sit next to *F*, but *F* and Michael is a three, and *B* and *M* is only 1." Para argues, "But if they like each other then they will sit together." Sia responds, "Yeah but that won't always happen if everyone gets to choose where they sit."

Para reads the task instructions aloud, and says, "OK so we need to explain our method." Heti starts writing, then asks, "which one [of their two arrangements above] should we give?" Sia says, "this one (*G-E-F-M-D-A-B-C*) right? Because we want *D* next to Michael. 'cause that's where *D* will feel most comfortable." They further endorse their seating arrangement by saying that it "has no negative relationships", and every person in the arrangement sits next to at least one person with whom s/he has a positive relationship, "so everyone has a good time". Heti then asks, "so our method was…?". Para responds, "our two goals was [*sic*], keep the negatives ones away from each other, and keep *D* next to Michael." Both Sia and Heti nod in agreement. Heti finishes writing the letter [see Figure 2]. Sia then looks at the task instructions again, and says "Oh man it's that thing again, we have to like think about a different situation, and show that our method still works". Heti and Para both say, "We've already done that. We've said just assume they get along with *G* and *C*." Sia says: "Oh yes, that's right. Cool."

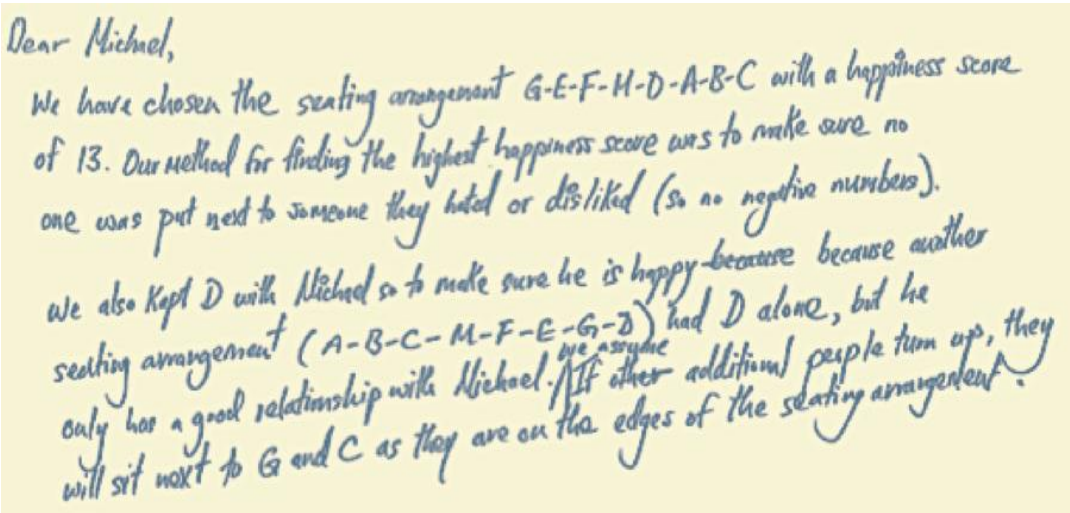
Dear Michael,
We have chosen the seating arrangement G-E-F-M-D-A-B-C with a happiness score of 13. Our method for finding the highest happiness score was to make sure no one was put next to someone they hated or disliked (so no negative numbers).
We also kept D with Michael so to make sure he is happy ~~because~~ because another seating arrangement (A-B-C-M-F-E-G-D) had D alone, but he only has a good relationship with Michael. We assume If other additional people turn up, they will sit next to G and C as they are on the edges of the seating arrangement.

Figure 2. The group's letter containing their algorithm.

Findings

Two main themes emerged from my analysis of the students' considerations of aptness. I describe these two themes in turn.

The predominance of contextual considerations

From the outset, the group's considerations of aptness to the real-world context is

evident. For instance, they say that Michael should sit in "the middle because it is his party." And, "the two people who hate each other the most [should sit] on the ends of the row" so that they don't "spoil things". Furthermore, after noticing that person *D* had a positive relationship with only Michael, they decided to seat *D* next to Michael because "that is where *D* would feel most comfortable" and "otherwise he just won't have any fun." These considerations to the real-world context seem to influence the group's considerations of aptness to the best seating arrangement. Though the two seating arrangements they created (*A-B-C-M-F-E-G-D* and *G-E-F-M-D-A-B-C*) were equivalent in terms of *total happiness score*, the group endorsed the latter because: *D* sits next to Michael ("where he [*D*] feels most comfortable"); and every person in the arrangement sits next to a person with whom s/he has a positive relationship ("so everyone has a good time"). Competition between considerations of aptness to the real-world context and considerations of aptness to a method for finding the maximal seating arrangement is noticeable when the group address the issue of the unexpected friends. Para remarked that "it would so much easier if we just give them tickets and they choose where to sit", but Sia countered with a scenario exemplifying how Para's suggestion might not yield the highest happiness score. Ultimately, the group's considerations of aptness to the real-world context impact their considerations to the unexpected friends, as evidenced by their remarks: "seat them on the ends of the row" because "it's kind of rude to show up expectedly" and "it's so inconvenient".

The algorithm reflects only explicitly-noticed properties of the optimal arrangement

The group's letter contains a seating arrangement, *G-E-F-M-D-A-B-C*, which they endorsed as the best, and a method for finding the best seating arrangement. I infer strictly from their letter that the group's method comprises three rules: 1) avoiding negative relationships; 2) keeping D next to Michael; 3) placing the unexpected friends on the edges of the optimal arrangement. These three rules align with the group's predominant contextual considerations. For instance, "avoiding negatives" aligned with *everyone having a good time*, and "keeping D next to Michael" aligned with *making D feel comfortable*. These three rules (particularly the first two), however, do not fully account for the group's optimal arrangement: That is, creating an arrangement for the given graph using only these two rules, would not necessarily yield the group's optimal arrangement. For example, the two arrangements *C-B-A-M-D-G-E-F* and *B-E-G-C-F-M-D-A* (among others), both of which have lesser happiness scores (11 and 6 respectively) can be obtained via these two rules. Evidently, the group's optimal arrangement has particular properties that distinguish it from these two arrangements. One such distinguishing property is: *each person sits next to a person with whom s/he has the highest relationship* (*or the next highest if the highest one is taken*). The group's final algorithm does not account for such distinctions.

What may have led to the emergence of the group's written algorithm (i.e., in particular, one which does not fully account for their optimal arrangement)? To answer this question, I note that the group's algorithm was written after they created the optimal seating arrangement. Further, the three rules in group's algorithm can be traced to explicit remarks made after the creation of the optimal seating arrangement. For example, *avoiding negatives* can be traced to the remark that the optimal arrangement had no negative edges. Also, *keeping D next to Michael* and placing the *unexpected the friends on the edges* can both be traced to the group's aforementioned considerations of aptness to the real-world context. In contrast, the group did not make any explicit remarks regarding, for instance, how each person in the optimal seating arrangement sits next to a person with whom s/he has the highest relationship (or the next best if the highest person has already been taken).

It thus seems that the final algorithm reflects only those properties of the optimal solution that the group explicitly noticed after the creating the arrangement. And, as mentioned above, these explicitly-noticed properties were ones that aligned with their predominant contextual considerations. This might suggest that the group considered the aptness of the optimal arrangement primarily with respect to the real-world context.

Discussion

The two main findings of this paper were: 1) the group's final algorithm consisted of three rules that reflected only the properties of the optimal arrangement that the group explicitly noticed (after creating it); and 2) the explicitly-noticed properties aligned with the group's predominant considerations of aptness to the real-world context. These two findings help explain how the group's algorithm (in particular, the two rules of "avoiding negatives" and "keeping *D* with Michael") did not fully account for their optimal arrangement. For instance, one distinguishing property of the optimal arrangement that the group's algorithm did not account for was: *each person sits next to a person with whom s/he has the highest relationship (or the next highest if the highest one is taken*). I infer that although the actual rules that guided the group's decisions while they were creating the arrangement were not explicit in the data, discrepancies exist between the actual rules used and the rules communicated in the algorithm. From this inference, I claim that the unaccounted-for distinguishing property emerged via a rule that was not expressed in the final algorithm. Such a rule could be that of "maximising locally", which stipulates that the next person to be seated is one who has the highest relationship with the most recently seated person, ignoring any effects of future choices.

The omission of a rule such as "maximising locally" from the group's final algorithm may indicate discrepancies between *how the students created the optimal solution* (*arrangement*) and *the students' report on how they created the solution*. The latter might involve reflecting on the solution found and re-creating the former. These discrepancies seem related to Cai et al.'s (1998) distinction between *procedure* and *algorithm*. To recall, *procedure* refers to the entire process by which a solution is found, while *algorithm* refers to communicating this process in a succinct way that utilises recurring steps and patterns in the process. The students in my study, I claim, took the step from procedure to algorithm, as evidenced by the properties of their solution that they noticed (e.g., "no negative relationships") and the manifestation of these properties in their algorithm. However, there properties of the solution that the group did not explicitly notice, but which likely emerged from rules that the group actually used to create the optimal arrangement. The absence of explicit remarks pertaining to, for example "maximising locally" might suggest that the group used such a rule subconsciously, while they were creating their optimal solution. Cai et al. (1998) argued that the transition from procedure to algorithm requires students to understand these rules at a conscious level. I hypothesise that a possible prerequisite for *understanding a rule at a conscious level* is explicitly noticing a property of the solution that closely corresponds to the rule. For example, the rule of "avoiding negatives" corresponded to the "no negative relationships" property that the group explicitly noticed. In contrast, the rule of "maximising locally" could not be traced to an explicitly-noticed property of the solution. Furthermore, the alignment of these explicitly-noticed properties with the group's predominant contextual considerations of aptness suggests the significance of that to which the aptness of the solution and the final algorithm are considered. That is, the sorts of properties explicitly noticed and ultimately reflected in the final algorithm might be influenced by those aspects of the task that the students deem particularly important to

address.

Discrepancies between *how the students created the solution* and *the students' report on how they created the solution* can be construed as a challenge that students might face when they engage with algorithmic tasks. As such, what can be done to help students externalise more faithfully the rules (which at times are used subconsciously) that actually govern how they create their solution? Two lines of suggestions come to mind. First, the question can be approached from a task design perspective (e.g., Watson & Ohtani, 2015) that focuses on developing questions in the task that would help elicit these subconscious rules by, for instance, directing their attention to particular aspects of the solution they have created. Alternatively, the question can be approached from a metacognitive perspective (Schoenfeld, 1985) which focuses on developing students' awareness of the rules they are using, while they are using it to create a solution. Exploring these alternative (complementary) approaches in the context of students engaging with algorithmic tasks warrants further research.

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Disciplinary Literacy in the Mathematics Classroom

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Nationally and internationally, teachers are being held increasingly accountable for student achievement, particularly in light of high stakes literacy and numeracy tests. Policies have been implemented that are designed to improve educational outcomes through raising student literacy levels across all school years in all subject areas. This has resulted in all teachers being seen as teachers of literacy. Research around the teaching of literacy in mathematics supports the view that focusing on the language of mathematics will assist students to move from the concrete to the more abstract understandings required in the older year levels (Schleppergrell, 2007). However, this can be challenging for teachers who might be subject, but not language, specialists. In this paper, we report on a case study that investigated literacy teaching practices in a Year 7 mathematics classroom and specifically, the practices around teaching mathematical report- writing and the conditions that might have enabled or constrained them. Findings suggest that while teaching the general writing required in mathematics might be part of teaching practice, if practices are to change, school leaders need to provide both time and money to enable teachers to develop their knowledge of specific disciplinary writing practices.

Introduction

Learning mathematics encompasses learning the language of mathematics. In general, the significance of literacy and its connection to student achievement has been established with Wise (2009) stating that "literacy is, in reality, the cornerstone of student achievement, for any student in any grade" (p. 46). There have been concerns over the literacy levels of adolescent students, given the decline in the performances nationally in National Assessment Program - Literacy and Numeracy (NAPLAN) and internationally [Programme for International Student Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS)]. However, as Wise (2009) has emphasised, the impact of poor performance is "not just on the individuals, but on the national economic condition and the strength and stability of our society" (p.46). It is for these reasons that literacy has been flagged as an essential twenty-first century skill and is reflected in policies and curriculum reforms at the national Melbourne Declaration of Educational Goals for schooling (MCEETYA, 2008) and the international level (for example, No Child Left Behind, 2001). On a national level literacy is embedded in all Australian subject area curriculum documents.

In the field of mathematics, research has shown how the nature of the mathematics classroom has also changed (Thompson & Rubenstein, 2014) so that the literacy of mathematics is increasingly important. Research has focussed on the language required to succeed, especially in the higher levels of schooling (Schleppergrell, 2007) and strategies have been suggested to incorporate mathematical literacy into the classroom (Hillman, 2014; Thompson & Rubenstein, 2014). However, this can be difficult for teachers, especially at the middle and senior school phases of schooling, with teachers being content area specialists but often less overtly familiar with the language of their subject (Gillis, 2014; Moje, 2008). Moreover, a renewed emphasis on writing in the disciplines (van Drie, Van Boxtel, &

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 361-368. Auckland: MERGA.

Braaksma, 2014), is also showing that content area writing has resulted in more generic literacy skills being taught, rather than those writing skills specifically required in the disciplines (Fang & Schleppergrell, 2010). Bazerman (1988) also showed how reasoning and rhetoric was different in each discipline. Thus, discipline-based literacy teaching practices are needed that are less general in nature and instead, focus on the subject specific requirements for each task.

Disciplinary Literacy

Over the last few decades there have been a variety of ways of describing how to teach literacy across the curriculum. These range from content area reading, content area writing, to writing across the curriculum. In Australia, the term "curriculum literacies" (Cumming & Wyatt-Smith, 2001) has been used, though increasingly, the term "disciplinary literacy/ literacies" is being employed (Moje, 2008; Shanahan & Shanahan, 2008). Current research into disciplinary literacy has shown how each discipline understands and uses language differently (Fang & Schleppergrell, 2010; Schleppergrell, 2007). Therefore, teachers need to engage students in the specific writing practices required to compose the required texts of the subject or field (Shanahan, 2015; Wyatt-Smith & Cumming, 2001). However, Klein, Boscolo, Gelati, and Kirkpatrick (2014) argued that while students may be taught discipline specific writing strategies in some subjects (e.g., History), it is more likely that they are taught only to write a more general text rather than one that is specific to the particular discipline. Fang and Schleppergrell (2010) argued similarly that content area writing has resulted in more generic literacy skills being taught, rather than those writing skills specifically required in each discipline.

Furthermore, education has grown increasingly complex and the nature of tasks students encounter daily in the classroom requires a flexible repertoire of language skills (Cumming & Wyatt-Smith, 2001). Adolescent students, in particular, encounter a wide range of tasks and topics and are expected to master a range of written genres, sometimes from one lesson to the next. Assessment tasks often favour the written mode and so writing to demonstrate learning requires students to write for specific audiences and purposes. They also need to use specific generic structures. In mathematics, such tasks often take the form of reports. These reports need to follow specific rules in terms of structure, grammar and language choice (Michigan Department of Education, 2012).

The Literacy of Mathematics

Mathematical learning is a complex process, and as an integral part of students' mathematics education, they need to be able to write effectively to communicate their findings to a range of audiences (Michigan Department of Education, 2012). Thompson and Rubenstein (2014) suggested that mathematical literacy requires "the ability to connect and translate … mathematical modes of communication" (p. 105). Hillman (2014) argues that students need to be introduced to "reading, thinking, speaking and writing" (p. 399). Teachers can use any number of strategies to teach writing, however, there is often little explicit teaching of the specific writing required to succeed (Kibler, 2011). Research has also shown that literacy instruction in mathematics often focuses primarily on vocabulary and word meanings (Wilson, McNaughton & Zhu, 2017). As all teachers are teachers of the specific literacy of their disciplines, and with the focus on improved student outcomes, there is an increasing need to incorporate the teaching of literacy and writing into daily education practices. In Australia, this teaching will be informed by the national curriculum documents

but as Kitson (2015) has shown, these documents may not be particularly supportive. In this paper there is a focus on the mathematics teaching practices of one teacher to examine what she taught about report writing, and what enabled and constrained her teaching practices. The theory of practice architectures (Kemmis & Grootenboer, 2008; Kemmis, Wilkinson, Edwards-Groves, Hardy, Grootenboer, & Bristol, 2014) is used as the analytical framework.

The Theory of Practice Architectures and Mathematics Education

A practice perspective on mathematics education has been outlined previously (see Grootenboer & Edwards-Groves, 2013; 2014), but briefly the theory of practice architectures conceptualises practices, which are comprised of characteristic "sayings, doings and relatings", as being enabled and constrained by conditions and arrangements (i.e., practice architectures) in any given site (Kemmis, et al., 2014). For example, the practice of teaching fractions uses characteristic *sayings* like 'denominator' and 'equal parts'; *doings* such as 'completing exercises'; and, *relatings* including the students relationship to the teacher and their peers. These are enabled and constrained by *cultural-discursive* arrangements like the shared understandings of mathematical language; *material-economic* arrangements such as the teaching space and resources; and, *social-political* arrangements including the school rules and students' emotional relationships to fractions from their previous experiences. There are two relevant implications of this theory here: (1) that practices, including mathematics education practices, are realised, and need to be understood, as site-based; and, (2) to develop practices there needs to be a concurrent development of the practice architectures that enable and constrain the practices.

The Study

The data reported here is part of a larger study conducted in 2016 that examined the teaching of literacy across the curriculum in the middle years at an independent school in South-East Queensland, Australia. The study was ontological in nature and employed qualitative methods to examine and explore teaching practices at this specific site. The study identified literacy teaching practices across several subject areas and examined relationships between these practices and the practice architectures that enabled and constrained them.

Data Collection

The participant group in the larger study was comprised of middle school teachers across various school subjects including mathematics. Data were gathered via classroom observations, in-depth semi-structured interviews, and document analysis. The school had recently embarked on a period of teacher professional learning that encouraged mentoring and coaching which included classroom observations and personal reflections on their teaching journey. Therefore, the participants all mentioned they were comfortable with the researcher's presence in the classroom. For the purposes of this paper, the findings related to the literacy teaching practices of the mathematics teacher (Diane – a pseudonym) are the focus. Diane is an experienced teacher having taught for over 25 years. She also was Co-Head of Mathematics at the time of the study, taking responsibility for the Year 7 to 9 mathematics program. Data collection was negotiated with the teacher and took place at the start of and towards the end of term one, and one lesson in term three, in 2016. The lessons were audio-recorded and detailed field notes taken. Diane nominated the lessons that were observed but she mentioned to the researcher that she had not specifically changed any of her practices because of the study. Diane also participated in two semi-structured interviews,

and some informal discussions that occurred after each lesson had been observed. The interviews were recorded and lasted around 30 to 45 minutes each. She also participated in a group interview with all participating teachers at the conclusion of the study. The group interview lasted around 45 minutes. During the individual and group interviews, key aspects of practice were discussed.

Data analysis

Data were analysed using several steps consistent with the theory of practice architectures. Initially, the transcripts of the classroom observations were analysed to identify general themes that emerged. After the initial identification of themes, transcripts were re-read and analysed specifically for the practices related to the literacies of mathematics and then, using the lens of practice theory (Kemmis et al., 2014), data were re-analysed for what they revealed about the specific "sayings", "doings" and "relatings" of these practices. Finally, in order to establish what enabled or constrained these practices, the data were examined to clarify the particular conditions and arrangements - the cultural-discursive, material-economic and social-political condition that existed in the site. Collectively, the analysis generated findings related to literacy teaching practices in reading, writing, speaking, listening and viewing of texts in mathematics. However, for the purposes of this paper, only the writing practices, and specifically, how Diane taught report writing, will be discussed.

Findings and Discussion

During the research period, Diane was observed initiating students into practices that saw writing as a product to demonstrate learning. In particular, Diane taught the report writing practices prior to an examination. The purpose of the lesson was to review the structure and contents of a mathematics report. Discussions with Diane revealed this was the only lesson that would be devoted to teaching students about report writing that year. During this lesson, Diane utilised scaffolding and modelling strategies to teach the students the particular writing required. In this case, the mathematics assessment required a very specific report format to be used. Diane's teaching practices around the structure can be analysed using her "sayings" "doings" and "relatings".

Practices of Teaching Mathematical Report Writing

Diane used scaffolding language (sayings) to teach the students about the particular structure the mathematics report required. Her opening question was related to the contents of a mathematics report; "who can think what are some of the things in a maths report?" She scaffolded the students further by asking a clarifying question; "Who can think what goes first?" This language demonstrated that writing a mathematics report requires sequential thinking, as indicated by the adjective "first". Diane also led the students through the order required in the contents of a mathematics report utilising time markers such as; "and then your name and then you'd put, …". Diane stressed the importance of following this order by using repetition to reinforce this practice, and further reinforced this practice by writing the list on the whiteboard (field notes, 17/8/16).

Diane also used suggestive language to prompt students to think further about the requirements using words such as; "And maybe a photo …, you can do a screenshot ", thus teaching the students that it was possible to add images to the report. The practice of writing a report also included aspects related to the vocabulary of the structure as evident in words

such as “title page, table of contents, reference list”, and some associated synonyms; “some people just call it Contents”, as well as the purpose of some of the headings; “So that I can actually find – or whoever is reading the report can find what they’re looking for”. Thus, while Diane was teaching the students about the necessary structure, she was also teaching secondary mathematical literacy practices such as the acceptability of using synonyms (e.g., “I don’t mind if you vary that a little bit, I just want you to basically do this”), the purposes behind the content and also the necessity of using templates while writing the report: “You have to set it up first. You can have a template”. Alongside this, she taught incidental vocabulary and structure that was not an accepted part of this particular report; “Not an Index that goes at the back – you don’t need one for a report; Yes, a bibliography is when you include all the things you look at, the whole research job. This is only the ones you used”, and also other unacceptable practices (e.g., “You don’t have to print your report off – it’s on Haiku”).

Diane also taught practices associated with specific report structure – the introduction and conclusion explaining that both introductory and concluding paragraphs need to contain certain points and be of a certain length; “Introduction, I would think you would probably have five to ten lines. And what is an introduction? What’s it need? Conclusion. And what is a conclusion?” Here, Diane was using the question and answer teaching strategy to encourage students to consider what they already knew about writing introductions and conclusions and to relate this prior knowledge to the current task. However, what was evident in her *sayings* was a distinct lack of specifically mathematical language. Hence, while this pedagogical practice addresses an important part of a well-rounded mathematics education, it does not ‘sound’ particularly mathematical.

During the teaching of the relevant practices related to writing a mathematics report, the teacher and students were involved in specific actions (*doings*), and the writing of the report also involved specific actions. The scaffolding provided by Diane helped the students to be *apprenticed* (Hillman, 2014) into the required practices. In this case, we see how Diane helped the students develop some of the specific requirements for this report – the structure, some of the vocabulary, some of the thinking involved – using scaffolding, ‘question and answer’, and repetition of key points in particular.

As part of the apprenticeship approach (Hillman, 2014), students were invited to participate actively in their learning but to be guided through this learning. During this lesson, Diane related to the students as the guide or mentor, and as the authority in the classroom (*relatings*). She controlled the activities for most of the time, asking questions and encouraging students to consider what they already might know about writing a report, its structure and associated vocabulary. In an interview with Diane, she affirmed this relationship describing herself; “I would like to think I was a warm demander” (interview 14/3/2016*)*. Again, these doings and relatings do not appear to be particularly ‘mathematical’, and yet they are an integral part of the literacy of mathematics.

The Practice Architectures of Teaching Mathematical Report Writing

According to Kemmis et al. (2014) practices do not exist in isolation, but they are enabled and constrained by the specific practice architectures in the site. The data presentation now turns to a description of the specific site arrangements and conditions and a discussion of how they enabled and constrained Diane’s teaching practices. While there is much that can be discussed about the site-specific cultural-discursive, material-economic and social-political arrangements, only those relevant to Diane’s teaching of report writing will be discussed here. In particular, Diane’s practice of teaching report writing was enabled

and constrained by the cultural-discursive arrangements that existed at the site. The cultural-discursive arrangements included an assessment sheet of eight pages that contained language related to both the assessment and the mathematics. The language of assessment used here mentioned "Understanding and Fluency" and "Problem solving and reasoning" as two criteria for marking, and later these criteria were detailed using words related to writing such as "description, use of appropriate language, clear explanation". This shows how the mathematical processes were assessed using literacy skills and knowledge. The inclusion of these criteria likely enabled Diane's teaching as she was able to teach the students about some of the language of the report genre. However, there was an assumption that students shared her understanding of these terms as she did not explain them further.

Further enabling and constraining conditions included the mathematical language in other words such as "facts", "procedures", "investigate", "evaluate" and "justify". While these words were part of key marking criteria, and key to effective communication in mathematics (Hillman, 2014), they were not taught as part of the writing lesson observed. Thompson and Rubenstein (2014) indicated that in order for students to achieve success in mathematics, they must be able to "write in ways that expose their reasoning" (p. 105). Hence, it is likely that whilst the assessment sheet provided some of the cultural-discursive arrangements that enabled the explicit teaching of the structure, other arrangements might have constrained her explicit teaching of the more abstract mathematical language. The assessment sheet contained three pages scaffolding the structure of the report but these three pages only mentioned the use of "correct formal, impersonal language". Research has demonstrated how useful the explicit teaching of mathematical language is (Shanahan & Shanahan, 2008; 2012), but also that many subject teachers do not have this literacy knowledge.

Diane admitted in an interview (14/3/2016) that she lacked professional learning in literacy and suggested the lack of time available for professional learning for topics other than those related to mathematical content or other specific school initiatives (for example, using technology) might have played a role. Thus, the material-economic arrangements at the site, leading to a lack of professional learning in literacy, might have constrained Diane's teaching of these important aspects of mathematical learning.

The social-political arrangements at the school favour a traditional, hierarchical structure. Teachers are expected to be both authority and expert. This was evident in an observation of a staff meeting at the start of the year (20/1/2016) where the Deputy Principal spent an hour reminding teachers of the discipline code in the school, using words such as 'follow the framework" without inviting feedback, and reflected also in seating arrangements – staff were seated in rows in a classroom whilst the Deputy stood at the front. Diane utilised a similar traditional transmissive pedagogy that reinforced her role as the authority in the classroom. Her authority was also evident in her language choices: "Go get what you need (to one student). Keep still (to another student). No, it has to be done through Turn It In". The tone of command in modal verbs such as "Go, Keep, Has to" indicated her role as authority in the classroom. There was another arrangement that possibly enabled Diane's teaching. The class had been streamed and this was the "top" class in Mathematics. This arrangement might also have enabled Diane to teach the report structure required to demonstrate learning quite easily, unhindered by other potential constraints such as behaviour management that was observed in other classes during this study.

Conclusion and Implications

The writing practices evident in this lesson showed Diane teaching students about the particular structure associated with writing the mathematics report to demonstrate their learning. Diane taught certain aspects of the report genre to the students, more specifically scaffolding and modelling the structure of the report. She also focused on some vocabulary, supporting similar research findings (Wilson, McNaughton & Zhu, 2017). Nevertheless, Diane is a mathematics teacher and not a literacy teacher, and secondary school teachers are usually content experts rather than literacy experts. While she scaffolded and modelled aspects of writing the report for the students, she did not appear to focus on the other necessary, but more abstract, language features. Diane taught some related vocabulary but mathematical language is much "more complex than just studying vocabulary" (Thompson & Rubenstein, 2014). She also needed to focus on aspects such as analysis and reasoning. As Hillman (2014) states, "mathematically literate students are able to analyse, reason and communicate ideas effectively" (p. 401).

We have shown how the lack of professional development in the literacy of mathematics might have constrained Diane's teaching. Hillman (2014) has argued the need for more professional learning for subject teachers to enable them to identify the specifics of the literacy of mathematics. Researchers such as Moje (2008), and Fang and Schleppergrell, (2010), have offered a range of pedagogical approaches and strategies that can be employed in order to identify and then teach students about the underpinning literacies of a subject, and Fang and Schleppergrell (2010) argue for teachers to make writing practices more visible and allow students to critique them. This suggests that while Diane's teaching practices demonstrated a commitment to teaching aspects of literacy, there is much more that can be done. What is clear from this particular case is that if practices are to change at the individual level, then changes need to occur in the allied practice architectures. Thus, it is not sufficient simply to mandate practice change through policy reform, and specifically here it seems that professional learning is needed to provide time and space for teachers to become more knowledgeable and skilful in the specific subject literacies of mathematics. School leaders must recognise this need for teachers to be both content experts and to develop their knowledge of specific disciplinary literacy to enable them to initiate students into the required complex disciplinary writing practices. School leaders need to provide time and money for professional learning in literacy. In this way, we might see the improved learning outcomes required. This is an important time to develop teachers' knowledge and dedicated teachers such as Diane have commenced the journey.

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Maintaining Productive Patterns of Teacher-Student Interactions in Mathematics Classrooms

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In preparing to study teacher-student interactions in mathematics classrooms in Iran, this paper as a literature review considers relevant mathematics education literature. I explore which aspects of classroom environments orient researchers to judge patterns of classroom interactions as productive. I examine patterns of classroom interactions that were empirically linked to student understanding. This paper highlights the importance of productive patterns of teacher-student interactions in promoting student learning, examining authoritative and dialogic teaching as two opposing approaches.

Introduction

Researchers, who adopt a sociocultural perspective on learning (see, for example, Goos, 2004), view classroom interactions as the means that facilitate student learning. The ways in which the teacher and the students interact with each other matter and give rise to potentially very different classroom mathematical practices and thus shape student learning. According to Lave and Wenger (1991), learning is seen as a social event in which teachers and students give meaning to the classroom interactions, where student participation is the focus. Similarly, Loef Frank, Kazcmi, and Balley (2007) stated that the ways teachers and students interact with each other in the social context of mathematics classroom are important to student learning. In addition, Moschkovich (1999) noted that students are expected to engage in classroom interactions and develop mathematical thinking, such as making reasons, describing conjectures, and clarifying ideas.

Considering the Iranian context, teacher-student interactions have been characterised by teachers doing most of the talking in a mathematics classroom and leaving little space for student-to-student talk. In such context, classroom interactions are mainly built on close questions teachers give students to work through from their textbooks, rules teachers give students to remember, abstract calculation and procedures teachers explain to students, and the correct answer teachers emphasise. In this way, classroom interaction has been limited to questions teachers raise, short responses students give, and the definite responses teachers provide the class with. The pattern of teacher-student interactions the Iranian students have experienced discouraged student mathematical understanding and worked against mathematical thinking because students are not encouraged to participate in classroom discussions, explain their ideas, provide evidence, and make argumentation. In other words, classroom interaction is all about listening to teacher talk and remembering rules. The fact is that in Iranian high school settings teachers are prone to follow textbooks, to be loyal to the course program (Sepasi, 2000), and students have already remained passive recipients of knowledge (Kamyab, 2004).

In this paper, I adopt the sociocultural perspective on learning and consider two questions as I review mathematics education literature related to patterns of teacher-student

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interactions. First, what can we learn from research on a particular pattern of teacher-student interactions? For instance, does a particular pattern of classroom interaction support student conceptual understanding, and, if so, how? Second, what do teachers do to productively support classroom interactions? Finding answers to these questions from research studies is important for my future study because I want to examine the patterns of teacher-student interactions in Iranian mathematics classrooms. I aim to understand how similar or different patterns of classroom interactions in the Iranian classrooms are from those identified by research literature.

The goal of this paper is to review research studies that investigated patterns of teacher-student interactions in mathematics classroom (e.g., Mehan, 1979), and to examine the studies (e.g., Moschkovich, 1999) that offer findings from an empirical study they made about instructional strategies teachers use to support productive patterns of teacher-student interactions.

It is worth examining patterns of teacher-student interactions through this overview of literature due to two reasons. First, this overview made me aware of particular patterns of teacher-student interactions used in the Western context that might be similarly used in the Iranian context. Second, as a result of reviewing these patterns of classroom interaction, I came to understand effective strategies that Western teachers used to promote productive patterns of classroom interaction. In my future study, I will consider whether and how Iranian teachers use the similar or different strategies to develop productive patterns of classroom interaction.

Learning Theory

The view of learning from sociocultural perspective is consistent with the Vygotskian perspective, which emphasises the role of classroom interaction in student learning (Vygotsky, 1978). To Vygotsky, learning involves the sharing of meanings as the learner interacts with more competent peers. Vygotsky (1978) made his idea of learning clear through the term "Zone of Proximal Development" (ZPD). He defined the ZPD as "the distance between the actual development level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers" (p. 86).

The idea is that learners learn best as they engage in working with more skilled persons, and it is through such joint engagement that learners acquire new concepts. I can infer that the function of ZPD is to involve the learners' collaboration when they solve the problem. The aim of the collaborative endeavor is to enable the learners to solve problems that cannot be solved in the absence of competent peers. This aim requires the teacher to adopt and support a particular pattern of classroom interaction that encourages the learners to make their thought explicit as they participate in sharing ideas, making reasons, and providing a conclusion. If we draw on Vygotsky's notion of ZPD claiming that learning is dependent on the learners' interaction with more competent peers, then, we need to identify particular forms of classroom interactions that are more likely to facilitate the student learning through collaborating with skilled persons.

Similarly, Lave (1988) and Wenger (1998) conceptualised learning as being inherently social. They viewed learning as a matter of participating in the classroom practice as the learners interact and learn together. To Lave and Wenger, the concept of practice indicates doing in a social context that gives meaning to what the learners do.

There has been growing interest in several studies that have shed light on teacher-student interactions from sociocultural perspective and the way in which student learning is

enhanced in a whole-class discussion (Goos, 2014; Razfar, 2012; Stein, Engle, Smith, & Hughes, 2008; Ball, Goffney, & Bass, 2005; Civil & Planas, 2004; Engle & Conant, 2002).

For the purpose of this paper, I want to focus on productive patterns of classroom interaction that can help students to develop a better conceptual understanding. The productive pattern of teacher-student interaction has been defined as a "purposeful talk on a mathematics subject" (Pirie & Schwerzenberger, 1988, p.460). Guiding the development of the purposeful talk requires teachers to take a significant role in a mathematical classroom ensuring all students actively engage in it. Kazemi (1998) characterised productive pattern as a form of classroom interaction in which the development of student's mathematical thinking had been supported. That is, we will consider a mathematical classroom productive if student outcome resulting from teacher-student interaction enables them to make reasoning that justifies procedures rather than statements of the procedures themselves, and to make relations among multiple strategies (Kazemi, 1998).

Patterns of Teacher-Student Interactions in the Mathematics Classroom

Overview of studies in which patterns of teacher-student interactions are examined can enable me to infer whether and how patterns of interaction in the Iranian classrooms look more like this or that pattern explored in research studies. In addition, this can help me identify the messages that teachers and students might form in my future study, where a different or similar pattern of interaction may emerge with respect to the Iranian classroom.

Mehan (1979) analysed a particular pattern of classroom interaction that minimised the opportunities for students to participate fully in classroom discussions. He found that the predominant pattern of turn-taking often involves teacher's evaluation who normally initiates the sequence by questions requiring mathematical fact recall. The turn sequences in this pattern have three parts including teacher Initiation (I), student Response (R), and teacher Evaluation (E).

Within this pattern of turn-taking (IRE), students must develop particular strategies to perform well (Mehan, 1979) because the pattern requires the students to respond to the teacher's initiation not only with the correct content but also with the correct communicative conventions; otherwise, the student's response may be disregarded by the teacher (Mehan, 1979). In addition, students need to know when and how to respond, and what kinds of questions teachers are asking when they initiate the sequence. This pattern has been criticised from different points of view. For example, the IRE disadvantages the students from cultures where this form of classroom interaction is uncommon (Tharp & Gallimore, 1988); and it provides little opportunity for students to verbalise their thought or comment on their peers' ideas (Wood, 1992). In contrast, Wells (1993) noted that evaluation is the most typical function of the third move.

According to Scott, Mortimer, and Aguiar (2006), the IRE pattern requires the teacher to take *authoritative* interaction approach, presenting a certain point of view, leading students through a question and answer routine, and creating a close pattern of interaction. Through authoritative approach, the teacher does most of the talking, allowing little space for students to express their ideas. Scott, Mortimer, and Aguiar (2006) added that more than one voice may be heard in authoritative interaction but there is no exploration of various points of view.

Some researchers have assessed another pattern of classroom interactions that is built on the Initiation, Response, and Follow-up (IRF). This pattern begins with either a student or the teacher posing a question or initiating a topic. Then, the initiator takes the response to

move the conversation forward. Next, the conversation continues for as long as the teacher and students wish to talk about the topic (Anneberg Learner Media Organization, 2004).

While some researchers have mentioned that the IRF promotes student understanding, others noted that it can constrain student learning. For example, some researchers noted that the IRF reflects a narrow mode of knowledge transmission and cannot facilitate student understanding (Orsolini & Pontecorvo, 1992; Tharp & Gallimore, 1988). The idea is that such a dominant pattern of teacher-student interactions, in which the third-turn follow-ups is on the part of the teacher, is basically teacher-oriented and can be problematic in providing students with opportunities to develop communicative competence (Nassaji & Wells, 2000).

In contrast, some researchers (see, for example, Mercer, 2000) noted that using follow-up turn leads in a stronger communicative base. Christie (2002) mentioned that rejecting the pattern in which follow-up is supported delivers a message about a tendency to neglect looking at the nature of meanings in constructing ideas, roles of teachers and students at the time of constructing shared meanings, and the importance of such pattern in enabling students to develop a better understanding.

Using the IRF pattern in the classroom, the teacher adopts a *dialogic* interaction approach in teaching, making a sequence of prompts, using probing questions to engage students in an ongoing discussion, and creating an open pattern of teacher-student interaction. The dialogic approach allows the student to come to a new idea through the process of exploring and talking. Throughout the lessons, students have a degree of agency to provide hypotheses, explain ideas, pose questions, and provide reasons.

In dialogic approach, there is an attempt to work on student's views in a way that the teacher might adopt an approach that allows for comparing and contrasting ideas. The teacher, then, tries to make explicit how those ideas are relevant to one another. In this way, students are provided with opportunities to connect their ideas to each other. The teacher evaluates these ideas so as to prompt others to engage in discussions and offer different interpretations of the event. Here, an important point is that the direction of progress of whole-class interactions is impacted not only by the teacher but also by students' contributions.

According to Aguiar and Mortimer (2003), the *transition* between authoritative and dialogic approach supports meaningful learning as different teaching objectives are addressed. The extent of transition between authoritative and dialogic approaches might be related to the teachers' perspective of teaching and learning. The point is that teacher decision of the transition between dialogic and authoritative approach must lead to student conceptual understanding.

Consequence of Teacher-Student Interaction Approach to Student Learning

The consequence of authoritative and dialogic interaction patterns to student learning is tied to their merits and demerits. The merit of authoritative approach depends on the purpose it is used to serve on particular occasions. For instance, mathematical practices involve certain goals, such as solving a word problem, focusing on mathematical content, acquiring technical terms, or understanding mathematics textbooks within an overall lesson sequence. Drawing on this, the teacher uses certain strategies, supporting students to bring various ways of talking. During a classroom practice such as acquiring technical terms, the initial discussions may follow the authoritative pattern of interaction, but the further conversational turns may not follow this pattern when the instructional goal is to focus on mathematical content. The former objective allows the teacher to elicit the correct response and save time to uncover more mathematical content.

Despite this merit, the authoritative approach is not free from instructional shortcomings. The first shortcoming of this approach is that it does not support the conceptual goal of constructing knowledge (Wells & Mejia-Arauz, 2006). This might be because of the fact that this approach doesn't fully support student-to-student talk and students are provided with little opportunities to express their ideas and make reasoning.

The second shortcoming is that authoritative approach does not encourage student's justification of their ideas and hence limits student's effort to develop argumentation. From the sociocultural perspective, if learning is viewed as opposed to constructing arguments, then the value of mathematical reasoning is devalued. In addition, some researchers argued that if students are not encouraged to think on their own, they do not develop complex thinking skills in mathematics learning (Cobb, Bowers, 1999; Voigt, 1995).

In contrast to authoritative approach, the dialogic approach has potential to engage students in more interactive patterns of interactions, supporting students to think about their solutions, evaluate ideas, justify explanations, and provide reasoning. In the one hand, such pattern of interaction enjoys the merit of developing mathematical thinking as students participate in whole-class discussions. On the other hand, developing mathematical thinking skills may take students a great deal of time during the school semester. That is, covering instructional goals in a classroom where the teacher takes dialogic approach might not be achievable during the intended time. I can infer that to achieve instructional goals teachers need to constantly shift between dialogic and authoritative approaches so that they can save time, cover more mathematical content, and develop student learning.

In addition to constantly shift between the approaches of classroom interactions to support student learning, there are productive instructional strategies that can help develop student conceptual understanding. Through the review of literature, I focus on the strategies that had been widely and rarely documented in research studies because it can ensure me that the instructional strategies the Iranian teachers use are more likely to fall between such extreme ranges.

Productive Instructional Strategies

I reviewed research studies in which researchers had examined instructional strategies that productively supported patterns of teacher-student interactions. That is, the strategies supported classroom interaction in ways that lead in student understanding.

For instance, in Moschkovich's study (1999), the teacher used particular instructional strategies (that are widely-documented in similar research studies, see, for example, Goos, 2004; Engle & Conant, 2002, Kazemi, 1998) including mathematical linguistics clue, revoicing, comparison, and contrasting, to encourage student-to-student talk and help students to reflect upon each other's contributions. The teacher also kept the discussion ongoing by asking students for making hypotheses, prediction, justification, clarification, argumentation, and summarising. Moschkovich concluded that these strategies can provide students with opportunities to actively participate in discussions, develop conceptual understanding, and uncover the mathematical content.

Loef Franke, Kazemi, and Battey (2007) examined different effective strategies that teachers can use to support student understanding. In their study, out of several certain strategies (e.g., revoicing, aligning students on the basis of their ideas, highlighting positions, pointing out an implicit aspect of student explanation) proven to be productive in supporting teacher-student interactions, I choose filtering approach that is rarely addressed in similar studies I reviewed. It is the way that helps the students to make details explicit in their explanation. Using this approach, the teacher can focus students' attention on a certain idea.

In this approach, students are encouraged to share their solutions and assess one another's ideas. It is followed by the teacher's filtering, choosing which idea worthwhile pursuing with the whole class. Through this structured way that teachers use to interact with the whole class, students are given opportunities to value and discuss solutions, producing more meaningful and productive discussions. In addition, this approach allows the students to self-evaluate their solution and reproduce more valuable ideas. In this way, they can develop sophisticated thinking skills because they need to figure out what the weakness of their solution is and how they can come to a better solution.

Conclusion

First, I provide a view of classroom interactions from researchers' point of view who adopt sociocultural perspective on learning. I then characterise the ways teachers and students interact in the Iranian context where classroom interactions is dominated by the teacher talk. Next, I draw on the work of Vygotsky, describing the term ZPD and its connection to teacher-student interactions. After that, I review the features of productive patterns of classroom interaction, describing that student outcome enables them to make reasoning. Evidence from research studies exist to support the need for promoting productive classroom interaction, guiding students to make sense of mathematical content. Further, in order to make clear the difference between more productive and less productive classroom interactions, I provide a dichotomy between "IRE" and "IRF" patterns of teacher-student interactions following authoritative and dialogic approaches. Finally, I draw on some researchers' work (e.g., Moschkovich, 1999; Loef Frank, Kazcmi, & Balley, 2007) who examined instructional strategies that have the potential to create conditions to develop productive patterns of teacher-student interactions. I review the strategies in order to understand how similar or different these instructional strategies are with what I may observe in the Iranian context, and whether the strategies can lead to a productive pattern of teacher-student interactions in Iran.

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The Distinction Between Mathematics and Spatial Reasoning in Assessment: Do STEM Educators and Cognitive Psychologists Agree?

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Mathematics in Australia specifies spatial reasoning as a general capability within the curriculum. However, psychological research to date limits spatial assessment to psychometric tests leaving little room for a well-defined spatial curriculum. Although there are clear relationships between mathematics and spatial thinking, the independence in the measurement of the two constructs in research literature is rarely explored. In the present study, professionals in the fields of STEM Education and Cognitive Psychology evaluated mathematics and spatial assessment items. The results show evidence for a distinction between the two constructs in the content of the items, however with a caveat that thoughtful selection of assessment items is crucial to ensure independence in the measures.

Throughout the world there is growing advocacy for developing spatial thinking within school curriculums (Mulligan, 2015; National Research Council, 2016; Newcombe, 2017). Although some countries remain grounded in traditional computational mathematics, others, such as Australia are moving towards a wider view of mathematical competency. The definition of Numeracy in the Australian curriculum addresses the need to develop students' mathematical skills that can be applied to the real world (ACARA, n.d.). This includes specific reference to spatial reasoning. The focus on numeracy in the Australian agenda has resulted in a shift in assessment content (Logan & Lowrie, 2017). Although the nature of assessment is designed as a barometer for educational outcomes, there is little doubt that these outcomes influence classroom practice, thus driving the assessment-curriculum cycle (Doig, 2006).

Mathematics and Spatial Assessment

Within standardised mathematics assessment in Australia the balance between traditional mathematics and spatial content is changing (Lowrie & Diezmann, 2009). In mathematics problem solving, spatial processes are advantageous for assessment success (Lowrie, Logan & Ramful, 2016a). Likewise, for items that appear fundamentally spatial, there is evidence to suggest some numerical processing may be required (Maybury & Do, 2003). Mix and Cheng (2012) proposed that research needs to identify "psychological distinction" (p. 205) between mathematical and spatial processing in order to distinguish between the constructs. There is scant literature on the classification of mathematics tasks by processing requirements, instead mathematical categories tend to be defined by content strands within curriculums (Mix & Cheng, 2012). While different countries support the teaching of different techniques and heuristics for assessment with varying degrees of success (Lowrie, Logan & Ramful, 2016b), spatial processes have been found to support the development of mathematical proficiency, particularly when encountering novel tasks (Lowrie & Kay, 2001). Although neither spatial reasoning nor mathematics can be thought of as unitary constructs (Mix & Cheng, 2012) the enduring relationship cited within the

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literature begs the question, are they distinct proficiencies when it comes to academic assessment or do they exist on a continuum?

The nature of the components that comprise spatial reasoning have been explored in multiple ways but are often grounded within factor analytic studies based on multiple measures (Carroll, 1993). One theoretical distinction within spatial reasoning is the idea that spatial reasoning can be defined in terms of mental transformations of an object compared with transformations of one's own viewpoint (Sorby, 1999). The differences between these two types of spatial processing may be the result of different task demands or in individual strategy preferences (Hegarty & Waller, 2005). Despite the categorisation of processing differences, the definitions are still bound within psychological measures. Spatial reasoning has a reputation for its lack of theoretical underpinnings or frameworks, often being restricted to narrow definitions aligned to the psychological tests used to measure the constructs (Hegarty & Waller, 2005). Despite the increasing acceptance that spatial reasoning is embedded within many aspects of numeracy, there is still limited research establishing how to incorporate traditional measures of spatial thinking into mathematics curriculums (Mulligan, 2015).

Therefore, the question remains, if space and number are inherently linked and there is a push towards spatial content within school curriculums and assessment, how do we separate traditional psychological measures of spatial thinking from practical curriculum-based assessment?

Aim of the Present Study

The relationship between mathematics and spatial reasoning has been explored extensively through correlational and longitudinal studies (see Mix & Cheng, 2012 for a review; Casey et al., 2015). Researchers are confident that the development of strong spatial skills equips students in STEM fields and mathematics in particular (Mulligan, 2015). Whether this relationship is due to shared underlying processes or problem-solving strategies is still under debate (Mix & Cheng, 2012). Nonetheless, research has rarely questioned the independence of the two constructs as output measures, even for geometry items that are inherently spatial (Clements & Battista, 1992). Validated measures in research carry an assumption of content validity as they have undergone rigorous testing and peer-review (Peter & Churchill, 1986). However, in the case of spatial reasoning and mathematics where there is theoretical overlap and the lines distinguishing the two skills are blurred, this assumption of content validity can no longer be guaranteed.

Peter and Churchill (1986) define two characteristics of valid measures, 1) the measures do not understate the intended constructs and 2) they do not assess extraneous characteristics. It would seem from their conclusions that there is a midpoint in which measures should sit and that the only evaluation of content validity at our disposal is subjective judgement. In research exploring the mathematics-spatial relationship this position may be hard to find (particularly with reference to point 2), yet researchers rely on the content validity of the separate measures in declaring construct independence. Therefore, the aim of the present study was to explore the content validity of mathematics and spatial measures through the ratings of discipline professionals in order to validate the separation of numeracy and spatial assessments. Secondary to this, we aim to examine the psychological distinction between the two constructs within assessment.

Method

Participants

A participation request was sent out to the authors' network within the fields of mathematics education, educational psychology, cognitive psychology, pre-service teacher education and school teachers. Eighty-four responses were collected. Thirteen participants were removed from the analysis as they completed less than 50% of the survey. This left a total of 71 participants from four countries, see Table 1 below for geographical breakdown. Based on the supplied information, participants were categorised as STEM Education professionals (N = 58) or Cognitive Psychology professionals (N = 13).

Table 1
Participant Geographical Demographics

Country	Australia	U.S.A.	U.K.	New Zealand	El Salvador	Saudi Arabia	Unknown	Total
N	51	14	2	1	1	1	1	71

Measure and Procedure

Participants completed an online survey via an anonymous Qualtrics link. Demographic information was collected and then participants were asked to rate 38 items on a continuum of purely mathematics (score = 0) to purely spatial (score = 100). Participants moved a horizontal slider to rate each item, there was no numerical input required however data was recorded as the numerical equivalent of slider placement. The items rated were from the Spatial Reasoning Instrument (SRI; Ramful, Lowrie & Logan, 2017) and a set of numeracy items developed for a larger project examining mathematics and spatial reasoning in primary school students. These items were designed to reflect NAPLAN items (ACARA, n.d.) and covered geometry and measurement and number and algebra content strands. The items were randomly presented, and no information was given as to whether they were from the numeracy or spatial assessment.

Results and Discussion

For all items except two, there were no differences (using t-tests) between STEM Education professionals and Cognitive Psychology professionals in the ratings of individual items (p >.05). Two spatial items requiring reflection across a diagonal line (see Figure 1 for an example) were rated significantly higher (i.e., closer to purely spatial) by Cognitive Psychologists (*Means* = 93.33 and 91.47) than STEM Educators (*Means* = 69.95 and 76.00), $t(46.89^2) = 23.39$, $p < .001$, $d = 1.28$ and $t(48.71^1) = 3.38$, $p = .001$, $d = .97$. Cognitive Psychologists on average rated these items as more spatial, while STEM Educators placed them closer to a mixture of the two constructs due to the mathematical conventions (i.e., reflection, diagonal) in the question.

Most respondents were Australian (71.8%; see Table 1) and as a result of unequal sample sizes in other countries of origin the sample was classified as Australian and non-Australian for comparison purposes. For all items except one there were no significant differences (based on t-tests) in ratings based on country of origin ($p < .05$). One numeracy item (see

[2] Adjusted values used due to violation of Levene's test for Equality

Figure 2) was rated lower on the scale (i.e., closer to purely mathematics) by Australian professionals (M = 18.38) than non-Australian (M = 38.31) respondents, $t(51) = 3.51$, p = .001, d = 0.99. This may be a result of the role of the number-line in spatial literature (Edmonds-Wathen, 2012). This item may be considered a vertical number-line.

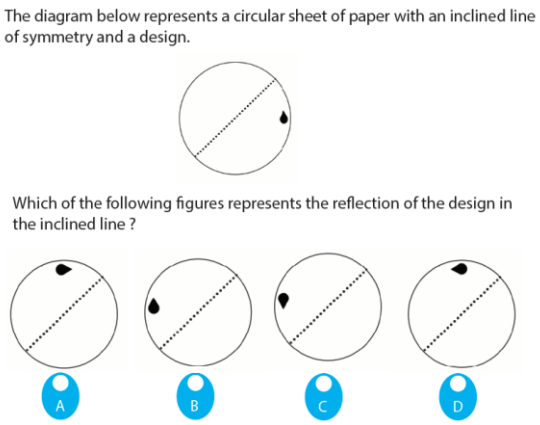

Figure 1. Spatial item (SRI; Ramful et al., 2017) rated differently by STEM Educators and Cognitive Psychologists

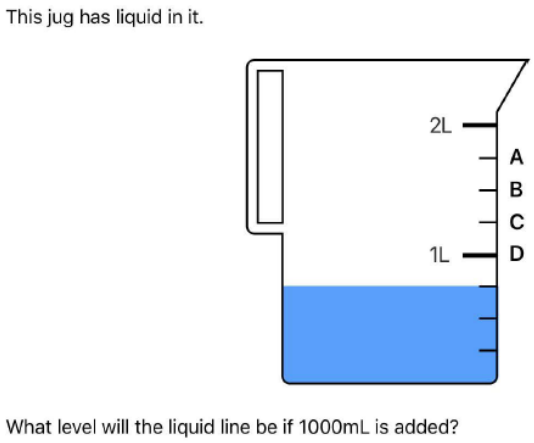

Figure 2. Numeracy item rated differently by Australian and non-Australian professionals

Factor Analysis

An exploratory factor analysis was performed because it was unclear how many factors may underlie the assessed items. A Principal Components extraction method was used and the scree plot of Eigenvalues was examined to establish the number of factors for consideration. The point at which the scree slope flattened suggested that there were three factors present in the dataset. A forced orthogonal rotation did not produce component correlations greater than .32 (r = .017), therefore no rotation was performed (Tabachnick & Fidell, 2001). The resulting factor loadings are presented in Table 2. Factor loadings of 0.32 or greater are commonly regarded as adequate for establishing the existence of a factor (Tabachnick & Fidell, 2001). Items that load greater than 0.32 on more than one factor are cross-loaded. Cross-loaded items are identified in Table 2, these items were attributed to the dominant factor when calculating factor scores (see Table 3). The three factors accounted for 60.32% of the variance in the model.

Table 2
Factor Loadings >0.32 Based on a Principal Components Analysis with a Promax Rotation of 38 Items from the Item Assessment Survey (N = 75)

Item Source	Description	1	2	3
SRI	Rotation of a figure	.90		
SRI	Rotation of a figure	.89		
Numeracy	Rotation of an object in Euclidean plane	.84		
SRI	Rotation of a figure	.82		
SRI	Directions on a map	.80		

SRI	Alternative perspective of a figure	.79		
SRI	Identifying the irregular shape of a net	.77		
SRI	Map rotation	.77		
Numeracy	Direction on a map	.75		.51
SRI	Identifying the irregular shape of a net	.75		
SRI	Rotation of a bike image	.75		
SRI	Rotation of a pentagon	.74		
SRI	Alternative view of a single block	.74		
SRI	Paper fold	.73		
SRI	Determining position from alternative perspective	.72		
SRI	Rotation of a dog image	.71		
SRI	Directions on a map	.71		
SRI	Alternative view of a set of blocks	.71		
SRI	Alternative view of a set of blocks	.69		
SRI	Identifying a slice of a 3-D shape.	.68		
SRI	Identifying the irregular shape of a net	.66		
SRI	Map navigation	.65		
Numeracy	Net of Cube	.63		
SRI	Lines of symmetry	.54		
SRI	Determining order from alternate perspective	.52		.49
SRI	Reflection across a diameter** - reflected object only	.43	.33	
SRI	Reflection across a diagonal** – reference object	.42		
SRI	Perspective taking	.36		.75
SRI	Perspective taking	.32		.71
Numeracy	Bar Chart		.83	
Numeracy	Fractions		.81	
Numeracy	Calculation on a vertical number line*		.76	
Numeracy	Number patterns		.76	
Numeracy	Number patterns		.70	
Numeracy	Number line		.67	.45
Numeracy	Bar Chart		.66	
Numeracy	Rotating fuel gauge		.57	
Numeracy	Rotation of an object in degrees		.40	

Note. * denotes difference by Nationality, ** denotes difference by profession

The tests were chosen to assess two separate constructs, numeracy and spatial reasoning. The exploratory factor analysis, however, revealed three underlying factors. The pattern that

emerged was in line with the literature on spatial thinking that the constructs while related do not load on a single factor (Carroll, 1993; Sorby, 1999). Factor one is made up of 27 items that link to spatial transformations such as mental rotation and spatial visualisation (Sorby, 1999), for example rotation of a dog image or identifying a slice of a 3-D shape. These items included numeracy items that could be solved using spatial transformations alone (such as identifying one side of the net of a cube). The nine items in factor two are numeracy items that involve a degree of computation and understanding of numerical conventions. It is noteworthy that the cross loaded item is a reflection of a diagram with no reference object in the answer options. This is one of the items previously discussed that differed in rating based on assessor profession. Although intended as a spatial reasoning item, STEM Education professionals rated this item as more mathematical than Cognitive Psychologists due to the inclusion of numeracy concepts such as reflections. However, this cross-loaded item lacked a reference object in the answer options (compared with a similar reflection item with a reference object, see Table 2). Why these two items did not both cross-load on the two factors is unclear but may be influenced by the presence (or lack of) a reference object in the answer options. Factor three contained five items that cross-loaded on the other two factors (only two of these items were dominant) but could be characterised as requiring a viewpoint change, as opposed to a mental transformation. Of note is that not all perspective change items fell into this factor, this could be attributed to the ease of doing a mental transformation for some spatial items rather than a perspective change (Hegarty & Waller, 2005). The cross-loading of the number-line item and with this secondary spatial factor is interesting. Research has shown strong links between representations of number and space in Australian populations (Edmonds-Wathen, 2012), particularly when displayed on a horizontal left to right path. The vertical number-line item assessed did not produce the same relationship, however as previously noted was rated differently according to country of origin. It is noteworthy that the spatial factor aligned to the number-line item was the orientation factor, perhaps owing to the egocentric spatial features of location and direction embedded within the number-line (Edmonds-Wathen, 2012). Average factor scores are presented in Table 3.

Table 3
Average Rating Scores for Each Factor

	Spatial Transformations (27 items)	Numeracy (9 items)	Perspective Taking (2 items)
Rating Average (S.D.)	81.53 (12.07)	28.91 (19.11)	87.58 (15.06)

The loading of items onto three factors rather than two supports the notion that on a content validity level, neither spatial reasoning nor mathematics are unitary constructs or completely independent as literature suggests (Mix & Cheng, 2012). The cross-loading of particular items also support Mix and Cheng's (2012) conclusion that numeracy and spatial reasoning assessments on the surface are not entirely distinct measures. The scale on which the items were measured had 0 as purely mathematical and 100 as purely spatial. The ratings presented in Table 3 lie towards the ends of the continuum as would be expected from separate measures of mathematics and spatial reasoning. However, the scores were not at the extreme ends of the continuum, suggesting that despite the surface delineation of the ratings of the two constructs, there are common components in mathematics and spatial reasoning assessment. Although separate factors, spatial transformations and perspective taking factors

were both within the spatial end of the continuum, and significantly different in their ratings, $t(57) = .247$, $p = .02$, $d = .33$. Given there are only two items in the perspective taking factor, there is a limit to the conclusions which may be drawn. Regardless, it might be the case that items in this factor are further removed from mathematics due to the ego-centric transformations as opposed to the object-centric transformations more closely aligned to mathematical problem-solving (Kozhevnikov & Hegarty, 2001).

Conclusions and Recommendations

If Australian mathematics education research is to reflect the Australian curriculum it is important to explore the overlap between the psychological construct of spatial reasoning and the embedded curriculum content, as reflected by curriculum assessment.

The results of this study suggest that despite the apparent similarity of some numeracy (based on traditional Australian NAPLAN items) and spatial reasoning items (Ramful et al., 2017), there was a distinction between the two as rated by STEM Education and Cognitive Psychology professionals. As with Maybury and Do (2013), it appeared that some of the spatial items required a degree of mathematical content knowledge and some of the numeracy items were dependent on spatial processing. However, overall the two constructs were independent. The present work relied on ratings from professionals across STEM Education and Cognitive Psychology and therefore the conclusions drawn reflect on the content validity of the items and not the processing demands of the items per se. Given the reliance on content validity in defining measures for comparison (Peters & Churchill, 1986) and the close relationship of the two constructs (Mix & Cheng, 2012) it is important to separate the content of assessment items to draw conclusions about the nature of the mathematics-spatial relationship. Future research may address the connection between the processing of the two assessment types in addition to the content validity.

Although the sample was heavily Australian where the curriculum explicitly identifies the role of spatial reasoning in numeracy, there were few differences across countries where the role of spatial reasoning in the curriculum is less explicit (e.g., the United States; National Research Council, 2006). The present items assessed where drawn from the Australian curriculum despite a large volume of longitudinal work originating in the United States (e.g., Casey et al., 2015). There are interesting cross-cultural opportunities to explore the mathematics-spatial distinction across curriculums in future.

The results of the study shed light on some of the defining characteristics that distinguish mathematical from spatial assessment. Regardless of country of origin or industry specialisation, mathematical conventions appear to mark the delineation between mathematics and spatial reasoning. Within spatial reasoning there is a further divide between mental transformations and perspective changes. While curriculum achievement measures and psychometric tests of spatial reasoning continue to be used as the foundation of relational research, it is important to distinguish the line between the two. Validating their independence enables researchers to move towards a better understanding of the underlying relationships between mathematics and spatial thinking and how they can be developed to support one another.

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What do Culturally Diverse Students in New Zealand Value Most for their Mathematics Learning?

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Classrooms around the world are becoming increasingly diverse. A major challenge faced by educators is achieving equity for all mathematics learners. To achieve equity, educators need to acknowledge and cater for this increasing diversity which includes attending to values of their students. Drawing on survey responses and individual interviews, this paper explores the mathematics education values espoused by 227 middle school students in New Zealand, including Pākehā/European, East Asian, Māori and Pāsifika learners. Results from this study provide insight into what is valued in the mathematics classroom and may assist teachers to develop pedagogy and classroom culture which aligns with students' values.

New Zealand, like many countries, has an increasingly diverse student population. The latest Ministry of Education (MoE) reports (2016) show that 51% of all New Zealand school students identify as Pākehā/European, 24% as Māori, 10% as Pāsifika, 11% as Asian, and 2% as Middle Eastern/Latin American/African. The term Pākehā refers to people of non-Māori heritage, typically of White European descent. Māori are the indigenous cultural group of New Zealand. Pāsifika describes a multi-ethnic group of individuals who were born in New Zealand, or who migrated from the Pacific Islands. This group identifies themselves with the islands and/or cultures of Cook Islands, Samoa, Tonga, Tokelau, Niue, Fiji, Tuvalu, and the Solomon Islands.

Currently, New Zealand has one of the largest ethnically driven achievement gaps across all developed countries, with Māori and Pāsifika students much more likely to underachieve comparable to all other ethnicities (Education Assessment Research Unit and New Zealand Council for Educational Research, 2015). To disrupt these trends and achieve equity for students from all cultures, educators must attend to diversity by enacting culturally responsive teaching practices (Averill, Te Maro, Easton, Rimoni, & Smith, 2015). Several research studies (e.g., Civil & R. Hunter, 2015; J. Hunter et al., 2016) show equitable education outcomes can be achieved when we attend to cultural values in the classroom. However, there appears to be minimal research specifically exploring the types of values espoused by mathematics learners from different cultures, particularly for students from indigenous or marginalised cultures.

Values are a relatively recent field in mathematics education research. Values have been conceived as a dimension of our affective system (the other affective dimension being our beliefs, attitudes and emotions); as a motivational trait; volitional in nature; or as a sociocultural construct (Seah, 2016). For the purpose of this study, values will be defined as "the deep affective qualities which education aims to foster through the school subject of mathematics" (Bishop, 1996, p. 169) and are reflected by the "convictions which an individual has internalised as being the things of importance and worth" (Seah & Andersson, 2015, p. 169).

Bishop (1996) proposed three types of values found in mathematics education: (a) moral or general education values (e.g., *fairness, respect*); (b) mathematics values which relate to mathematics as a discpline (e.g. mathematics is about *control*); and (c) mathematics education values which relate to learning and pedagogy (e.g., valuing *group work*). The present study will

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focus on the third subtype, mathematics education values, as these values contribute to effective mathematics learning because they reflect the personal and cultural learning preferences of both teachers and students. Specifically, this paper will explore the following research question: what are the most important mathematics education values espoused by a cohort of culturally diverse learners in New Zealand?

International Literature Exploring Mathematics Education Values

There appears to be a scarcity of literature specifically exploring the types of mathematics education values espoused by students from different cultures, with the majority of research focusing on East Asian countries. After reviewing the East Asian values literature (Law, Wong, & Lee, 2011; Lim, 2015; Zhang et al., 2016; Zhang & Seah, 2015) the most consistently reported mathematics values included values internal to the student, for example *achievement, effort and practice*; as well as external values relating to the teacher or pedagogy, for example *teacher led activities, teacher creativity, explanations, strictness, board work*, and instruction involving *multiple methods*. For example, Zhang and colleagues (2016) found that students across China valued *achievement* as the most important aspect of their mathematics learning.

Outside of Asia, a small number of research studies have explored the mathematics education values espoused by Swedish (Österling & Andersson, 2013; Österling, Grundén, & Andersson, 2015) and Australian (Seah & Peng, 2012) students. These studies reported several internal mathematics education values, such as *achievement, memorisation/recall, effort/practice*, and *concentration*; as well as external values relating to the teacher and pedagogy, for example, *personalised help* from the teacher, *clarification* and *explanations, hints, worked examples, strictness, a quiet/relaxing classroom* and *problem solving*.

To date, there are no known research studies specifically exploring the types of mathematics education values espoused by students in New Zealand, however several studies provide some insight into valuing in New Zealand classrooms. For example, Anthony (2013) explored students' values towards a "good" mathematics teacher and student, comparing values from a high socio-economic and a low socio-economic school. Anthony reported that students from both schools valued a teacher who *cared* about his/her students, and who provided *clear explanations*. However, in terms of a "good" student, students from the high socio-economic school (and predominantly Asian and Pākehā/European ethnicities) expressed a high proportion of internal and independent learning values (e.g., *effort* and *practice, knowing multiple mathematical strategies*). In contrast, students from the low socio-economic school (and predominantly Pāsifika and Māori) espoused a greater proportion of collaborative values (e.g., *sharing, mathematical communication, respect*) compared to the other ethnic groups. Likewise, other studies (e.g., Hunter & Anthony, 2011; Sharma, Young-Loveridge, Taylor, & Hāwera, 2011) investigating Māori and Pāsifika perspectives on learning experiences in the mathematics classroom affirmed these students endorsed collaborative values in the mathematics classroom, such as *respect, positive relationships*, *reciprocity* and *group-work*.

However, it can be challenging comparing mathematics education values across the literature due to inconsistent methodologies and differences in the interpretation of values. For example, the values generated through interviews (e.g., Lee & Seah, 2015) may be very different to the values generated through questionnaires (e.g., Österling et al., 2015). Thus, to make valid cultural comparisons in mathematics education values, consistent methodologies are required.

Methodology

This paper reports on the mathematics education values espoused by a cohort of middle school students (Years 7 and 8) in New Zealand. The participants included 227 students from four urban schools, who identified as one of four ethnic backgrounds, including 30 Pākehā/European, 25 East Asian, 41 Māori and 131 Pāsifika students. The majority of the Pākehā/European and Asian students were from higher socio-economic home backgrounds and attended a high decile school. In this school, the teachers loosely followed the New Zealand Numeracy Project (MoE, 2009) and grouped students based on ability/achievement of specific goals. These Pākehā/European and Asian students usually worked independently, or within small groups. All the Pāsifika and Māori students attended one of three lower decile schools and came from low-socioeconomic home environments. These schools had been involved in an ongoing professional development and research project entitled Developing Mathematical Inquiry Communities (DMIC) (Hunter, Hunter, & Bills, in press). The DMIC project focuses on the development of culturally responsive teaching and pedagogical practices. Students in the DMIC project typically work in heterogeneous groups to engage in inquiry based mathematical activities.

The study followed a mixed methods approach and focused on the use of student voice to guide the interpretation of values. All students completed a survey where they ranked 12 mathematics education values in order of their importance: *utility, effort/practice, flexibility* with strategies, *accuracy, teacher explanations, mathematical clarity and understanding, family, peer collaboration/group-work, peer collaboration/communication, persistence, respect* and *belonging*. These values were drawn from policy documents and research literature (e.g., Clarkson, Bishop, FitzSimons, & Seah, 2000; MoE, 2011, 2013). As children often find it difficult to articulate their own values (Clarkson et al., 2000), each value was incorporated into a value statement, for example the statement "if I can't solve a difficult maths problem I need to keep working at it" indicated the mathematics education value of *persistence*. Following the survey, all students were interviewed and asked why they had selected their three most important values.

To determine the degree of importance for each value, the proportion of students who ranked each of the twelve values in their top three values was analysed. All interview data was wholly transcribed and guided by a grounded theory approach where codes, categories and themes were developed from the data itself. For example, a student response explaining why they ranked the value *utility* as most important was: "most jobs probably involve maths so it will be important for it to be in my life" was coded into the node of future education/jobs.

Findings and Discussion

In Table One the four highest ranked mathematics education values for each of the four ethnic groups are identified. The following section explores student explanations for the top two values (bolded in Table 1) for each ethnic group to understand the importance of these values for the students.

Utility

The survey statement "*It is important for maths to be useful in real life or my future"* was used as a value indicator for the mathematics education value of *utility*. Students from all cultural groups expressed that *utility* was important for their mathematics learning, with around half (n=111/227) of all students ranking this value in their top three values.

Table 1
Summary of the Most Important Mathematics Education Values Across Cultural Group

Cultural Group (*n*)	**Highest Ranked Values**	**Frequency (*n*)**
Pākehā/European (30)	**Utility**	**57% (17)**
	Teacher explanations	**53% (16)**
	Clarity & understanding	43% (13)
	Flexibility	27% (8)
East Asian (25)	**Utility**	**64% (16)**
	Clarity & understanding	**52% (13)**
	Effort/practice	32% (8)
	Persistence	28% (7)
Māori (41)	**Utility**	**39% (16)**
	Peer collaboration/group-work	**39% (16)**
	Family	32% (13)
	Respect	29% (12)
Pāsifika (131)	**Utility**	**42% (55)**
	Peer collaboration/group-work	**37% (49)**
	Family	24% (31)
	Effort/practice	24% (31)

Note. The frequency column indicates the percentage and number of students in each ethnic group who ranked the value in their top three mathematics education values.

During follow up interviews, more than half the students (*n*=54/104) ranking *utility* in their top three expressed that proficiency in mathematics was important for their future education or career goals. This included students equating proficiency in mathematics with future employability: *Because if your maths skills aren't the best you might have a bit of trouble finding a job* (Asian male), or that engaging with mathematics allowed access to a specific occupation (e.g., teacher, banker, accountant). Other students identified that mathematics was important for their future education: *You can learn lots of maths for your future so that you might go to university* (Māori female). Alternatively, students (*n*=39/104) perceived utility as important for everyday and practical activities (e.g., shopping, building, money): *In my dad's job he needs to use it every day and we're doing renovations at our house and using maths to calculate area* (Pākehā/European male) and: *My Mum said if I never learn maths properly I won't know how to pay bills or count money* (Pāsifika male). Interestingly, these last two statements suggest that utilitarian values can be influenced by the values expressed by parents and also highlight the impact of out of school experiences.

Valuing *utility* means that these students desired mathematics which was either practical, or relevant to their own lives or the world around them. Students desired mathematics which had a purpose, that related to everyday activities, or which impacted upon their future success. Given the growing technological and digital economy (Ministry of Business Innovation and Employment 2017), there is a strong message in New Zealand that mathematics is important for future employment and economic advancement. Previously, Young-Loveridge and

colleagues (2006) found that an overwhelming majority of students held beliefs about the importance of mathematics for the future, suggesting that utilitarian mathematics values are reflective of societal values and not necessarily distinguishable by cultural differences.

Utilitarian mathematics education values have been reported elsewhere in the international values literature. For example, Österling and Andersson (2013) reported that Swedish middle school students highly valued "connecting mathematics to real life" (p.22). Similarly, Barkatsas and Seah (2015) found the favourite mathematical tasks reported by students across Australia and China involved real life scenarios. The strong *utility* values held by students both in the current study and other published research reaffirms the need for mathematics teachers to provide authentic learning experiences with opportunities for students to apply concepts and skills to real life scenarios. This is particularly important for marginalised students where linking school mathematics with the real world (of the students) improves mathematical engagement (Averill et al., 2015).

Teacher Explanations

The mathematics education value of *teacher explanations* was explored through the statement "*My maths teacher needs to explain it to me properly so I understand".* This value was important for many of the Pākehā/European students, with fifty three percent (*n*=16/30) of these students ranking *teacher explanations* within their top three values.

The reason provided by all Pākehā/European students (*n*=16/16) for the choice of this value indicated they perceived that clear and accurate explanations from their teacher were necessary for their own mathematical progression and/or achievement. For example, one student stated: *A teacher is supposed to teach you what you're doing and if I don't understand it then I probably won't be able to figure out that question*, whereas another student responded: *Because otherwise I get stuck and then I just don't do it.* One student associated teacher clarity with her own performance in tests: *If my teacher doesn't explain it properly then I won't learn that and when I'm in a test I can't remember it so in the test I don't really do that well.*

The responses of these Pākehā/European students indicate that they often look to their teacher as the mathematical authority in their classroom, therefore, it was important to them that their teacher possessed relevant and expert mathematical knowledge. Values relating to *teacher explanations* have been reported in earlier research. For example, Seah and Peng (2012), as well as Österling and her colleagues (2015) reported that Swedish and Australian students valued *teacher explanations*, which included systematic and detailed teacher instructions. These Swedish and Australian students expressed the view that their peers often made mistakes which the students found confusing, therefore valuing the accuracy of their teachers' explanations. Other research from New Zealand also demonstrates that students value *teacher explanations* as a component of effective mathematics learning (Anthony, 2013). For example, when Anthony explored what students valued for "good" mathematics teaching and learning, she found students from a predominantly Pākehā/European dominated school valued a *helpful teacher* who provided effective *explanations and clarity*.

Mathematical Clarity and Understanding

The statement *"It is important that maths is clear and makes sense to me"* was used as a value indicator for the mathematical education value of *mathematical clarity and understanding*. Fifty two percent of the Asian students (*n*=13/25) ranked this value in their first three values.

The majority of the Asian students (*n*=10/13) who ranked this value in their top three responded that mathematical *clarity and understanding* was important because it facilitated

their mathematical progression and achievement. For example, one student explained: *If you don't know what it means you don't know how to answer it, you need to understand and it needs to be more clearer*. Two students also equated mathematical clarity with accuracy, for example a student explained: *Because if maths isn't clear it will be wrong*.

For these Asian students, valuing mathematical *clarity and understanding* meant that these students felt discontent when their mathematics did not make sense to them. The importance of this value for the Asian (and Pakeha/European students) may be a reflection of mastery culture in their classroom. Earlier research from New Zealand (Meissel & Rubie-Davies, 2016) found that the majority of Pākehā/European and Asian mathematics students expressed strong mastery orientation goals. It appears the value of *clarity and understanding* has not appeared elsewhere in the values literature. In other research studies (e.g., Österling & Andersson, 2013; Seah & Peng, 2012; Zhang & Seah, 2015), students from Sweden, Australia and Asia expressed values relating to the source of their mathematical clarity and understanding (e.g., *worked examples*, *teacher explanations*).

Peer Collaboration/Group-Work

To investigate the mathematics education value of *peer collaboration/group-work*, the statement *"I learn more in maths by working with other children"* was used. Thirty-nine percent (n=16/41) of Māori students and thirty seven percent (n=49/131) of Pāsifika students ranked this statement within their top three values.

During follow up interviews it appeared that there was link between this mathematics education value and a key Māori and Pāsifika cultural value of *reciprocity* (MoE, 2011, 2013). Students viewed *peer collaboration/group-work* as providing reciprocal learning opportunities: *Because you can help other children and that would help you* (Māori male). Many Māori (n=12/16) and Pāsifika (n=34/49) students spoke of the benefits of working together, sharing new strategies and gaining new knowledge from their peers. For example, a Māori student responded: *Because we have more than one idea when we're working with someone,* also a Pāsifika student stated: *So you can get different strategies*. Alternatively, a smaller group of Māori (n=3/16) and Pāsifika (n=8/49) students discussed the opportunities they had themselves to help others, for example one student explained: *So when they are stuck I can help them* (Pāsifika female).

Collaboration is a core collectivist cultural value for Māori and Pāsifika people (MoE, 2011, 2013). In this study, the Māori and Pāsifika students valued *group work* because sharing ideas and strategies helped the students to progress and improve their own, and their peers' mathematics. Earlier research by Sharma and her colleagues (2011) found that Pāsifika students recognised the benefit of collaborative mathematical learning both for building their own mathematical understanding and progressing their peers' mathematical understanding. Similarly, Anthony (2013) found that Pāsifika students valued a social arrangement in the classroom, which suited their collaborative ways of learning. This contrasts research from Asia (e.g., Law et al., 2011; Zhang et al., 2016) which demonstrated an absence of collaborative mathematics values from East Asian students and research from Sweden (Seah & Peng, 2012) that highlighted Swedish middle school students valuing *independent working* due to their perception that listening to their peers' conflicting strategies was confusing rather than helpful.

Conclusion and Implications

Understanding what students' value in the mathematics classroom is important and has implications for effective and culturally responsive mathematics instruction. In the New

Zealand context, it appears that there has been limited research exploring students' self-reported mathematics values. The findings from the current study provide a useful starting point for developing an evidence base in New Zealand, particularly for marginalised students, as well as contributing to the international literature relating to the role of values and valuing in mathematics education.

In the current study, students from all ethnicities valued the *utility* of mathematics. The commonality of this value across international research studies (e.g., Barkatas & Seah, 2015; Lee & Seah, 2015; Lim, 2015; Österling & Andersson, 2013) suggests *utilitarian* values are influenced by societal and educational values. In terms of other important values, the Pākehā/European students espoused teacher focused and independent learning values (i.e., *teacher explanations*, *clarity and understanding*, *effort/practice*). Conversely, the Māori and Pāsifika students espoused collectivist mathematics education values (i.e., *peer collaboration/group-work*, *family*, *respect*). Opposing cultural values may, in part, explain these differences in the students' mathematics education values. For example, research studies (e.g., Hofstede, Hoftstede & Minkov, 2010) show New Zealand (typically European) people hold individualist cultural values, whereas Māori and Pāsifika people endorse collectivist cultural values (MoE, 2011, 2013). An alternative explanation for these differences in mathematics education values may be linked to differences in classroom experience. The Māori and Pāsifika students were in classrooms involved in the DMIC project which promotes collaborative values and classroom norms. It would be valuable to investigate further the distinction between mathematics education values inculcated through cultural values, or the result of classroom experiences/pedagogy.

Acknowledging and harnessing values in the mathematics classroom has important implications for culturally responsive and effective mathematics teaching and learning. As Seah (2016) writes "how do we go about facilitating students' appropriate valuing such that it helps them to study mathematics more effectively? The first step would be to have a good idea of what is currently being valued by students" (p. 4). By recognising what is valued (or not valued) in the mathematics classroom, teachers can develop classroom culture or pedagogy which aligns with the students' values, or explicitly address inappropriate values, or values which may contradict the classroom norms and pedagogy.

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Readiness to Teach Secondary Mathematics: A Study of Pre-Service Mathematics Teachers' Self-Perceptions

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This study evaluated pre-service teachers' (PSMTs) perceptions of their own readiness to teach secondary mathematics. The study was conducted at an Australian university across two campuses, in different states. Specifically, PSMTs' perceptions of their preparedness were explored in terms of mathematical content knowledge, pedagogical content knowledge, and mathematical knowledge for teaching. Findings indicate that while the majority of participants feel that they have the requisite content knowledge to confidently teach Lower School secondary mathematics, further training is required to develop their content and pedagogical knowledge, especially for upper secondary mathematics.

Scholars have argued that the pre-service, secondary mathematics teachers (PSMTs) must possess a substantial level of both mathematical content and pedagogical knowledge (Ball, 2008; Schoenfeld & Kilpatrick, 2008; Norton, 2010). Similarly, the professional experience (or practicum) is also considered essential for pre-service teachers' training, and instrumental in developing their overall teaching craft, perspective and philosophy (Cox et al., 2013). This study builds on a previous study, which explored pre-service teachers who were completing a Graduate Diploma of Secondary Education (GDE) and their self-perceptions of readiness to teach secondary mathematics (Hine, 2015). In this study, we extended the scope of this previous work by evaluating students enrolled in a GDE, Master of Secondary Teaching (MTeach), and Bachelor of Secondary Education (BEd) programs across two university campuses, situated in different states in Australia. This study used semi-structured qualitative interviews to support survey-generated data.

Research Aims and Significance

This research project has two specific aims. The first aim is to investigate the self-perceptions of PSMTs as they prepare to teach mathematics for the first time. The second aim is to explore how these PSMTs understand and perceive their 'readiness' to undertake this role, by analysing their self-perceptions against key themes presented in the theoretical framework. The significance of this research lies in the assumption that current tertiary education courses adequately prepare students for a secondary mathematics teaching role, and that research into this area can strengthen future efforts in preparing PSMTs.

Theoretical Framework

Three interrelated themes form the theoretical framework for this research, namely: Mathematical Content Knowledge (MCK), Mathematical Pedagogical Knowledge (MPK), and the domains of Mathematical Knowledge for Teaching (MKT). These themes are now explored within the context of preparing PSMTs for the teaching profession.

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Mathematical Content Knowledge (MCK)

There is a substantive literature base to support the claim that knowledge of mathematical content is central to its teaching (Norton, 2010). Ma (1999) contended that such knowledge is concerned with the depth, breadth, connectedness, and thoroughness of mathematical concepts and theory. Additionally, Schoenfeld and Kilpatrick (2008) asserted that proficient mathematics teachers possess a broad and deep knowledge of the mathematics taught at school level, as well as knowing multiple methods of representation and how ideas develop from conceptual understanding. Empirical studies have suggested strongly that the knowledge of mathematics teachers positively affects student achievement (Baumert et al., 2010; Campbell et al., 2014). In this paper, MCK is defined as knowledge related to or underlying the secondary school mathematics content assessed at Years 7-12.

Pedagogical Content Knowledge (PCK)

Following extensive research on the relationship between teachers' mathematical content knowledge and their ability to teach, there is clear and growing evidence to support a positive association on this relationship (Ball et al., 2005; Ma, 1999). Scholars have suggested that teachers require a development of PCK, which has been described as an intersection of subject knowledge and pedagogical knowledge (Delaney et al., 2008). For this study, PCK can be understood as knowing a variety of ways to present mathematical content and assisting students in deepening their understanding of mathematics (Ma). More recently, the profound knowledge of mathematics and methods of representing it to students has been described as MKT (Delaney et al.).

Domains of Mathematical Knowledge for Teaching (MKT)

In light of Shulman's proposal that teaching knowledge is a complex, multi-dimensional construct (1999), Ball et al. (2008) analysed extensively the work of mathematics teachers and hypothesised a conceptual framework for MKT. As represented in Table 1, this framework comprises two overarching domains, Subject Matter Knowledge and PCK, each of which are comprised of three sub-domains. Subject Matter Knowledge comprises the sub-domains: Common Content Knowledge (CCK), Specialised Content Knowledge (SCK), and Horizon Content Knowledge (HCK). PCK consists of the sub-domains Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC).

Table 1
Domains of MKT. Adapted from Ball et al. (2008, p. 403)

Subject Matter Knowledge	Pedagogical Content Knowledge
Common Content Knowledge (CCK)	Knowledge of Content and Students (KCS)
Specialised Content Knowledge (SCK)	Knowledge of Content and Teaching (KCT)
Horizon Content Knowledge (HCK)	Knowledge of Content and Curriculum (KCC)

Methodology

Methods

This study was interpretive in nature and used qualitative research methods to collect and analyse data about how PSMTs perceived their readiness to teach secondary mathematics. For this investigation, the researchers developed and used two online, qualitative surveys and semi-structured qualitative interviews to collect data from participants. Participants were asked to respond to a ten-item survey prior to commencing a teaching practicum experience. Immediately following the teaching practicum experience, the participants were asked to respond once more to the same survey. Then, both researchers invited all participants to participate in a semi-structured interview. In this manner, the researcher was able to determine the extent to which any of the participants' self-perceptions of readiness had changed following their experience in the classroom. The survey items and interview questions are included within this section.

Research Participants

The entire student cohort enrolled in courses for secondary mathematics pedagogy at the one Australian university was invited to participate in the research. Specifically, of the 53 students enrolled in these courses across both campuses, 20 elected to participate in the pre-practicum survey, 14 in the post-practicum survey, and six participated in the interview. The demographic details of the survey and interview participants are included in Table 2. PSMTs in the GDE and MTeach completed a two-week practicum. Across the four-year degree, BEd students complete eight mathematics content courses and a mathematics pedagogy course, or six mathematics content courses and two mathematics pedagogy courses, and undertake four practicum experiences totalling 32 weeks in schools.

Table 2
Summary of Participants' Demographic Data

	Gender	Age	Degree	Major	Specialisation
Pre-Practicum Survey Participants [n=20]	13 Female	17-25 = 14	GDE = 8	Math = 11	Math = 9
	7 Male	26-35 = 4	MTeach = 2	Science = 5	Science = 7
		36-45 = 2	BEd = 10	Other = 4	Other = 4
Post-Practicum Survey Participants [n=14]	7 Female	17-25 = 9	GDE = 7	Math = 7	Math = 7
	7 Male	26-35 = 5	MTeach = 1	Science = 3	Science = 5
			BEd = 6	Other = 4	Other = 2
Interview Participants [n=6]	5 Female 1 Male	17-25 = 6	GDE = 3 BEd = 3	Math = 5 Science = 1	Other = 5 Math = 1

Survey and Interview Items

Nine items comprised the pre-practicum and post-practicum surveys of this research. Survey items 1-4 were for participants to indicate specific background information regarding their age, gender, and prior tertiary studies. Survey items 5-9 directly assisted the researchers

in pursuing the specific aims of the research. These items required participants to adopt a critically reflective stance towards their perceived readiness (before & after the practicum) in teaching secondary mathematics. The interview schedule was comprised of survey items 5-9. The research participants had been furnished with the terms *MCK* and *MPK* in their mathematics pedagogy courses.

5. Describe your readiness to teach secondary mathematics students in terms of the mathematical content knowledge and skills you currently possess.
6. In what area(s) of mathematical content knowledge do you feel you require further training?
7. Describe your readiness to teach secondary mathematics students in terms of the mathematical pedagogical knowledge and skills you currently possess.
8. In what area(s) of mathematical pedagogical knowledge do you feel you require further training?
9. Overall, describe your readiness to teach mathematics to secondary students.

Data Analysis Process

The researchers analysed qualitative data collected from the pre-practicum and post-practicum surveys (items 5 - 10) and interviews according to a framework offered by Miles and Huberman (1994) which comprises the three components: data reduction, data display, and drawing and verifying conclusions. Within each of these components the researchers executed the following operations: coding, memoing, and developing propositions. Codes developed by the researchers were attached to gathered data via pre-practicum surveys, post-practicum surveys, and interviews, and were selected from those data based on their meaning. In particular, the codes were developed according to the domains of MKT (Ball et al., 2008) which are delineated in Table 1.

Findings

The key findings of this research have been generated exclusively by participant responses from the surveys and interviews. Overall, PSMTs responses suggested a self-perceived degree of readiness within the themes of MCK and MPK. These findings have been summarised in tabulated and discursive formats, and in alignment with the six domains of MKT. Findings from post-practicum interviews are also included.

Mathematical Content Knowledge - Readiness

Nearly all of the PSMTs stated they felt ready to teach mathematics before the practicum experience (17 of 20). One participant (who had CCK, SCK and HCK) stated

> I feel confident to teach the content of secondary mathematics. I have recently completed mathematics content units which I did not find difficult. I feel I have a good conceptual understanding of the different mathematical concepts I will be required to teach and feel confident that I will easily be able to 'brush up' on any topics (if need be) before I am required to teach them.

Following the practicum, all PSMTs (14 of 14) declared they were ready to teach in terms of their MCK. Specifically, all participants stated that they had CCK, and many of these expressed feeling confident in teaching Lower School classes (i.e. Years 7-10) only. Herein one participant (who had CCK) described

> [I feel] Good overall although there were some topics in Year 11 and Year 12 classes that I had not seen for a long time. I think that I'll need to take the time to learn this content properly and master it.

> Things like matrices, some parts of vectors, proofs and pieces of calculus. I'm ready overall, and really ready for Lower School classes.

The reported self-perceptions of PSMTs' readiness in MCK are displayed in Table 3.

Table 3
Mathematical Content Knowledge – Perceived Readiness

Pre-Practicum	Relative Frequency	Post-Practicum	Relative Frequency
I Feel Prepared	17 of 20	I Feel Prepared	14 of 14
I Have CCK	19 of 20	I Have CCK	14 of 14
I Have SCK	7 of 20	I Have SCK	3 of 14
I Have HCK	1 of 20	I Have HCK	0 of 14

Mathematical Content Knowledge – Further Training Needed

Before the practicum, all PSMTs were able to identify an aspect of their MCK that they required further training (see Table 4). In particular, most PSMTs identified these aspects as HCK (20) and SCK (17). One participant (who needed SCK & HCK) reflected

> I need to consolidate my content knowledge especially for the advanced classes. Year 8 content knowledge I'm fine, it's probably everything for Year 9 and Year 10 advanced classes that I need to practise. Things like algebra, probability trig[onometry], indices and especially the harder examples.

Similar to pre-practicum responses, the PSMTs continued to focus on HCK and SCK as areas for further training post-practicum. For example, one PSMT (who needs HCK) stated "I feel as though I only need further training with Extension content as I have never taught an Extension class, and only had the opportunity to observe one". Another PSMT (who needed SCK & HCK) listed various curriculum topics: "I will need to refresh the higher skills of calculus, trig[onometric] relationships, geometry, matrices, and linear algebra".

Table 4
Mathematical Content Knowledge – Further Training Needed

Pre-Practicum	Relative Frequency	Post-Practicum	Relative Frequency
I Need HCK	20 of 20	I Need HCK	13 of 14
I Need SCK	17 of 20	I Need SCK	11 of 14
I Need CCK	1 of 20	I Need CCK	5 of 14
I Need None	0 of 20	I Need None	1 of 14

Mathematical Pedagogical Knowledge - Readiness

A majority of PSMTs (17 of 20) claimed they felt ready to teach in terms of their MPK, particularly with regards to KCS. From those who expressed that they felt prepared, one participant (who has KCS) stated

> Coming from a high school education where it was…based off the 'chalk and talk' style of teaching, I felt I did not have as much knowledge on different pedagogical skills and knowledge that can be used to engage students in mathematics. Coming to university…taught me there are many different ways mathematics should be taught to students…I feel much more ready after doing some units.

After the practicum experience, 13 of 14 PSMTs expressed feeling ready, and particularly in terms of their KCS. One participant (who had KCS) stated

> I'm pretty happy with my teaching so far. I felt I was learning new things each week with my classes, like how to break down concepts so that the younger school students can understand better. My mentor was really helpful in showing me how to make a lesson engaging for younger students, like splitting up the activities, getting students involved, and checking work.

A summary of PSMTs' self-perceptions of readiness in MPK in presented in Table 5.

Table 5
Mathematical Pedagogical Knowledge – Perceived Readiness

Pre-Practicum	Relative Frequency	Post-Practicum	Relative Frequency
I Feel Prepared	17 of 20	I Feel Prepared	13 of 14
I Have KCS	17 of 20	I Have KCS	13 of 14
I Have KCT	2 of 20	I Have KCT	1 of 14
I Have KCC	0 of 20	I Have KCC	0 of 14

Mathematical Pedagogical Knowledge - Further Training Needed

Prior to the practicum experience, 17 PSMTs identified a need for further MPK training. Moreover, a majority of these expressed they required KCS, KCT or KCC (or any combination of these domains). One PSMT (who needs KCS, KCT & KCC) wished to become more proficient in "Diversifying the teaching of the content. If it is explained one way and students do not understand, how do you change your thought process to adapt and meet their requirements?" Following the practicum, all participants nominated something to work on, pedagogically speaking. One PSMT (who needs KCC & KCT) stated

> I think that learning how to be more creative with lessons so it's not the same kind of lesson each time. I did try to avoid this so the students wouldn't get too bored but planning huge and exciting lessons takes so much time! Finding new or different ways to help students connect their knowledge to new ideas would also be helpful.

A summary of PSMTs' response for further MPK training is presented in Table 6.

Table 6
Mathematical Pedagogical Knowledge – Further Training Needed

Pre-Practicum	Relative Frequency	Post-Practicum	Relative Frequency
I Need KCC	17 of 20	I Need KCC	14 of 14
I Need KCT	17 of 20	I Need KCT	14 of 14
I Need KCS	11 of 20	I Need KCS	1 of 14

I Am Unsure	2 of 20	I Need None	0 of 14

Overall Readiness to Teach Secondary Mathematics

Nearly all PSMTs stated that they felt ready to teach secondary mathematics prior to the practicum experience (18 of 20). Such assertions of readiness were conditional, however; over half of those PSMTs stated they needed to develop elements of their MCK, MPK or both of these knowledge domains. For instance, one PSMT (needing SCK & HCK) qualified her self-perception of readiness with "Lower secondary I feel 90% confident. Upper secondary I do not feel confident at all, maybe 40% at that. I could learn the content the night before the lesson. I am aware that this is not good going into prac[ticum]". Following the practicum, an overwhelming proportion averred feeling prepared to teach (13 of 14). Again, all of these responses were qualified with an expressed need for PSMTs to develop professionally in MCK and MPK domains. While one participant expressed how he was "Itching to get started", another (who needed SCK & HCK) stated "Overall, I feel as though I am quite ready to teach in secondary schools. There are definitely a few gaps [in my content knowledge] but nothing that I don't think won't be sorted out after a year or two of teaching in my own classroom". Approximately half of the pre- and post-practicum cohorts reported feeling ready to teach Lower School classes but conceded that elements of their MCK and MPK for Upper School courses required improvement (see Table 7 for a summary of participant responses).

Table 7
Overall Readiness to Teach Secondary Mathematics

Pre-Practicum	Relative Frequency	Post-Practicum	Relative Frequency
I Feel Prepared	18 of 20	I Feel Prepared	13 of 14
I Need KCT	14 of 20	I Need SCK	7 of 14
I Need KCC	13 of 20	I Need HCK	7 of 14
I Need HCK	12 of 20	I Need KCC	3 of 14
I Need SCK	11 of 20	I Need KCS	2 of 14

Discussion and Conclusion

Findings in this study supports previous findings and revealed that PSMTs are generally confident in their ability to teach lower secondary school mathematics (Years 7-10). However, many are still working towards developing the SCK and HCK required to teach upper secondary mathematics, especially Specialist/Extension courses (Tables 3 & 4). While the majority claimed to be ready to teach secondary mathematics to varying degrees prior to the practicum (17 of 20), all 14 participants in the post-practicum survey indicated that they possess the requisite MCK to teach lower secondary mathematics. Therefore, these claims suggest that participation in the pedagogical unit of study or the practicum positively influenced PSMTs self-perception of readiness with regards to having the necessary content knowledge, at least for teaching Years 7-10. There was also a shift in students' self-perceptions regarding the level of content knowledge that they possess post-practicum.

Before the practicum 7 of 20 students indicated they had sufficient SCK and 1 of 20 had HCK. Following the practicum only 3 of 14 students indicated that they have SCK and none for HCK (Table 3). This change is surprising, as many participants have completed up to second-year tertiary or higher levels of mathematics, which surpass even the highest level of secondary mathematics. A potential explanation for this self-perceived lack of SCK and HCK post-practicum could be that PSMTs have not yet mastered these mathematical skills and content knowledge, and therefore do not feel confident teaching it.

It is also possible that PSMTs' lack of confidence to teach upper secondary mathematics is compounded by their self-perceived MPK. Indeed, 17 of 20 students claimed to possess the requisite MPK to teach Years 7-10 prior to the practicum, which increased to 13 of 14 in the post-practicum survey (Table 5). All 14 participants in the post-practicum survey indicated that they lack the MPK to effectively teach Years 11 and 12, especially the Specialist/Extension courses (Table 6). Some common explanations offered by interviewees for this self-perceived deficiency included limited exposure to upper secondary classes during their practicum, an expressed need to develop MCK, or not seeing a direct link between university level mathematics and what is covered in the upper secondary mathematics syllabus. Overall, this study revealed that PSMTs perceive themselves to be ready to teach Years 7-10; however, more support is required for the development of their MPK and mastery of MCK to teach upper secondary mathematics.

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Free-Response Tasks in Primary Mathematics: A Window on Students' Thinking

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We administered specially-designed, free-response mathematics tasks to primary students (*N* = 583, ages five to 12 years old). Our focus was on whether (i) the children's responses could be reliably assessed, and (ii) the responses could provide insights into children's mathematical thinking. We used a contemporary comparative judgement technique, interviews with four teachers, and analysed a sample of six responses to make inferences about the students' mathematical thinking. We found that the sampled responses' scores, interviewees' comments and qualitative features of the sampled responses led to consistent insights on the children's mathematical thinking. We argue that free-response tasks should supplement traditional assessments in primary mathematics.

Assessment tasks are a ubiquitous method for understanding students' thinking in primary mathematics. In New Zealand common assessment tools include Progressive Assessment Tests (PATs) which are standardised multiple-choice item tests. However, a common criticism of such tests is the predominance of short, closed items that emphasise the recall and application of isolated facts and procedures (Berube, 2004). Such tests risk promoting a narrow and arguably distorted view of students' mathematical thinking (NCETM, 2009). In comparison, consider tasks designed to assess writing ability. These might include short items such as spelling or grammar, but also free-response items that ask students to produce a piece of descriptive writing. This latter is invaluable to primary teachers as a window onto students' thinking in the context of writing (Brindle, Graham, Harris & Hebert, 2016), yet free-response tasks in mathematics assessment are rare.

We investigated the use of specially-designed, free-response tasks in primary mathematics classrooms. The tasks comprised a short prompt (see Table 1) followed by a blank page for the student's response, analogous to a writing task in language lessons. A traditional barrier to the use of such assessments in mathematics is that they are difficult to score in a meaningful and reliable manner (Laming, 1994). We applied an emerging assessment technique, based on comparative judgement (Pollitt, 2012), that has been reported to overcome this barrier in secondary school and university mathematics (Bisson, Gilmore, Inglis, & Jones, 2016; Jones & Alcock, 2014; Jones & Inglis, 2015). The comparative judgement technique was used to address our first research question: *Can primary children's free-response mathematics work be reliably assessed?*

A reported advantage of free-response mathematics tasks, at the secondary level, is that they can provide a window onto children's mathematical thinking (Jones & Karadeniz, 2016). This hypothesis formed our second research question: *What insights into primary children's mathematical thinking do the assessed responses provide?* We explored this research question using two methods. First, primary teachers involved in assessing the children's responses were interviewed to investigate what features they focussed on when making assessment decisions. Second, we sampled six responses, two low, two medium and two high-scoring, and analysed their features. Our analysis investigated the consistency

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between the scores of the six sampled responses, the teachers' comments during the interview, and qualitative analysis of the responses themselves.

Methodology

Context and Participants

The assessment tasks were administered to students (*N* = 583, ages five to 12 years old) in a low socio-economic primary school. Following this, twenty teachers participated in a staff meeting where they undertook the process of comparative judgement (described later in the paper). Four teachers agreed to be interviewed after the staff meeting to investigate their perceptions of the assessment process. They included a Year 2–3 teacher, a Year 6 teacher, and two Year 7–8 teachers.

The Tasks and Administration

Fourteen tasks (see Table 1 for examples) were designed by the researchers and selected by members of the school's senior leadership team as relevant to the areas of focus in mathematics lessons during the previous term. Students in Years 0–3 completed three tasks each and students in Years 4–8 completed four tasks each. Over a week, students were provided with 10–15 minutes to complete one task each day individually. Teachers wrote the responses for the children in Year 0–2 when required.

Table 1
Example assessment task prompts and year levels

ID	Task prompt	Years
1	Write and draw everything you know about addition and subtraction.	0 – 4
2	Write one or more tricky word problems for a friend involving multiplication or division. Show how you would solve them.	3 – 8
3	Write and draw everything you know about the operations (addition, subtraction, multiplication, division).	4 – 8
4	This is a graph of favourite fruit of one class of children. [pictogram of fruit shown here] What statements can you make about the children's favourite fruit?	0 – 2
5	These graphs show boys and girls favourite winter sports. What statements can we make about the girls and boys favourite winter sports? [two bar graphs shown]	3 – 4
6	This graph shows how many hours people in two different classes watched television and did their homework over the week. Think about things such as the mean, mode, median and range. [two dot plots shown here] What statements can we make about the two different classes of children and how much time they spend watching television and doing homework?	5 – 8

Assessing the Tasks

Free-response mathematics tasks do not lend themselves to scoring. This is partly because rubrics assume a set of pre-defined response types (Jones & Inglis, 2015), whereas the tasks used here were designed to generate a wide range of responses without anticipating

the responses in advance. Moreover, even if a rubric could be designed, the marking is likely to be unreliable; that is assessors would judge the extent to which a given response matches the rubric inconsistently (Murphy, 1992). Therefore an alternative assessment method was required and we used comparative judgement (Pollitt, 2012).

Comparative judgement requires no rubric or scoring and instead assessors are presented with two responses and are simply asked which student has demonstrated the 'better' mathematical thinking. Many such binary decisions are collated from a pool of assessors, and are used to estimate a parameter for each script using the Bradley-Terry model (Bradley & Terry, 1952). The parameters are then scaled and can be treated as scores for the test responses (Jones & Alcock, 2014). Recent research has demonstrated that using comparative judgement to assess free-response mathematics tasks produces reliable and valid outcomes that are robust across a diversity of learners from school students to undergraduates (e.g. Bisson et al., 2016). A feature of comparative judgement is that assessors can compare responses to the different tasks and still make consistent and valid judgements about which student is 'better'. This feature is important here due to the different tasks administered to students, as exemplified in Table 1.

Due to space constraints we do not detail or make the case for using comparative judgement here. Readers are referred to technical explanations on the use of comparative judgement for assessment (e.g. Bramley, 2007; Pollitt, 2012).

Assessment Outcomes

The 1912 test responses were anonymised, scanned and uploaded to a comparative judgement website (nomoremarking.com) for assessment. Twenty teachers from the school and 10 other individuals (with a teaching background who now work within professional learning and development) were recruited to comparatively judge the responses, and they completed between 72 and 1400 judgements each, resulting in a total of 12888 judgements. For each pairwise judgement the teachers were instructed to select the 'better response'. The binary decision data was statistically modelled (Pollitt, 2012) to produce a parameter estimate of the 'quality' of each response. The parameter estimates were then scaled to produce a set of scores ($\mu \approx 50$, $\sigma \approx 15$).

The reliability of the assessment outcomes were investigated using standard techniques (Bramley, 2007; Pollitt, 2012). First we calculated the Scale Separation Reliability (Bramley, 2007), a measure of the consistency of teachers' judging considered analogous to Cronbach's alpha for traditional scoring procedures, and found that this was satisfactory, $SSR = 0.83$. We then calculated a misfit statistic for each judge and each test response to investigate whether any individual teachers had judged anomalously or any responses been judged inconsistently by different teachers. Following the standard practice of considering any misfit statistic greater than two standard deviations above the mean misfit statistic to be misfitting (Pollitt, 2012), we found that none of teachers, and just 59 test responses (0.3%) were misfitting. Taken together these measures suggest the assessment outcomes were reliable.

Data Collection and Analysis: Interviews

Four self-selected teachers were interviewed individually for between 20–25 minutes. The interviews were audio-recorded, wholly transcribed and analysed using grounded theory (Corbin & Strauss, 2007) to identify themes. We present our findings with respect to how the teachers made their judgements, and how they interpreted student understanding of two areas prominent in the data: numeracy and statistical literacy.

Results

Students' Thinking: Teachers' Observations

Three of the teachers began their interviews by reflecting on the open-endedness of the tasks. They described how this resulted in a wide variety of responses while also allowing different entry points and levels on which the tasks were answered. One teacher noted that the format of the task meant students "can show a lot more of what they know [compared to traditional tasks]".

Analysis suggested there were three key themes in relation to the criteria that teachers used to make pairwise judgement decisions. First, all four interviewed teachers stated that they preferred student responses that were mathematically sound rather than a social response; for example one teacher said "some of the responses were very personal, and so you knew that, that was not a mathematical answer". Second, three of the teachers' expressed a preference for responses that used examples, illustrations, and explanations to provide evidence of student thinking. For example, one teacher said "looking at the evidence, how they prove it". Third, three of the teachers said they preferred student responses that were meaningful in terms of the task prompt. For example, one said "some type of link that linked to the question". Another teacher said that when judging the statistical literacy tasks she was "looking for a statement that did actually relate to the data … rather than just writing out a whole bunch of numbers".

Teacher Judgements of Responses to Number Questions

Our analysis of the interviews revealed that the teachers commented frequently on the arithmetical component of student responses. Three teachers said that responses to the numeracy tasks often demonstrated an understanding of the properties of operations. In particular, three of the teachers noted that students commonly identified the relationship between repeated addition and multiplication and two described how students identified the inverse relationship of addition/subtraction and multiplication/division.

These teachers also observed that students had good recall of basic facts and were confident with addition. However, some students appeared to misinterpret the prompts and "just wrote down hundreds of plus problems, but they couldn't actually say, you know, which words show that it's addition". Similarly, two teachers noted that in many responses the students "don't really know how to write a division or a subtraction word problem and solve it that well". The teachers also commented that most students appeared to equate "tricky" with large numbers and then wrote problems that they could not solve.

Teacher Judgements of Responses to Statistical Literacy Questions

Our analysis also revealed that the interviewed teachers commonly referred to responses to statistical literacy tasks. Three of the teachers noted that many students were able to interpret simple graphs such as pictograms and bar graphs and make simple statements from these. They also noted that many students struggled with the more complex graphs and having to reconcile two graphs. For example, one teacher said that "certain types of graphs and data displays were consistently bad". All four teachers said that the students tended to make general observations about the data but in many cases had difficulty constructing specific statements. For example, one teacher observed there was "[only] a handful that actually were able to make a good statement saying that this graph is showing". Three

teachers also noted common written student phrases of the type “oh that one’s the most, that one’s the least on a graph” when attempting to make a statement from the graphs.

Two teachers (both of Year 7–8) noted that some of the students appeared to know what the mean, median, and mode were but did not use these in a meaningful way to convey information about the set of data. For example, one teacher said “it was irrelevant to actually what the question was asking them”. One of these teachers reflected that “they [students] don’t often think that deeply about it or understand what statistics is for”.

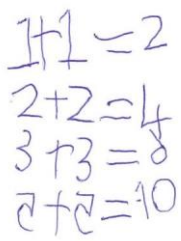
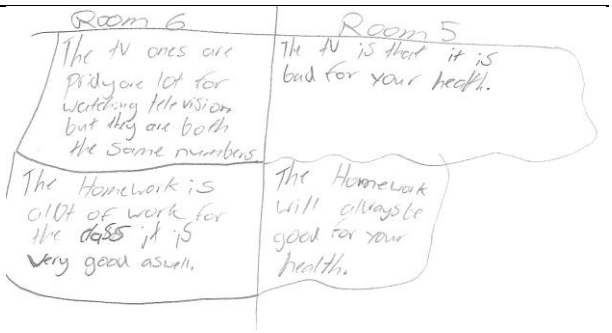
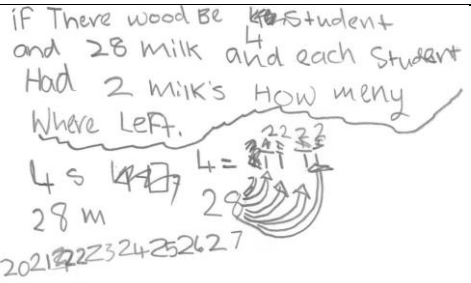
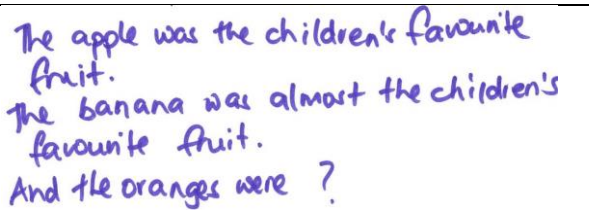
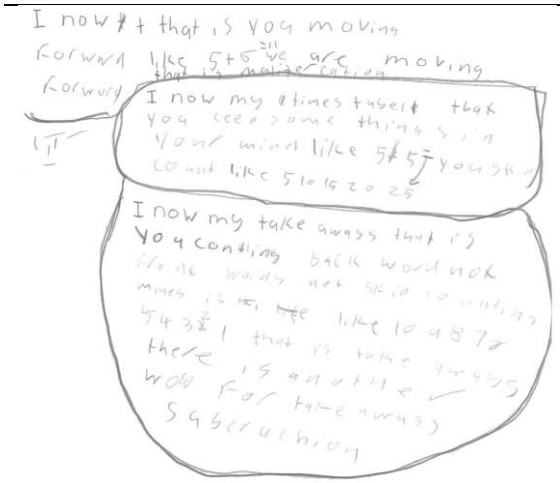
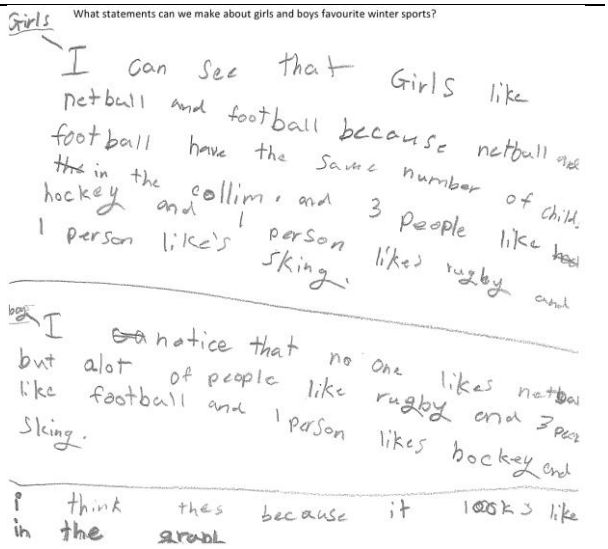
Numeracy tasks	*Statistical literacy tasks*
	
Task 1, 25th percentile, Year 2	Task 6, 25th percentile, Year 6
	
Task 2, 50th percentile, Year 5	Task 4, 50th percentile, Year 2
	
Task 3, 75th percentile, Year 5	Task 5, 75th percentile, Year 4

Figure 1: Sampled responses to the tasks in Table 1.

Students' Thinking: The Tasks

To gain further insights into students' mathematical thinking we sampled six responses. In light of the interview analysis just reported we sampled responses to five tasks from across the age ranges that focussed on numeracy (Tasks 1 to 3 in Table 1) and statistical literacy (Tasks 4 to 6). For each task type (numeracy/statistical literacy) we sampled the three responses closest to the 25th, 50th and 75th percentiles of the assessment scores. The sampled responses are shown in Figure 1.

Numeracy Tasks

Figure 1 displays increasing sophistication from the 25th to the 75th percentile in the responses to the numeracy questions. The response at the 25th percentile shows four addition equations, with no written explanations, drawings or any focus on subtraction as requested in the task prompt (Table 1). In terms of the general themes of the interviewed teachers preferring the use of illustrations and links to the task this response was lacking. The additions are all of the form $x + x = 2x$, consistent with young children's informal reported strategy of doubling (Ter Heege, 1985). The responses at the 50th and 75th percentiles are also partial responses to the task prompt: only one example is given in the response to Task 2, and multiplication and division is omitted from the response to Task 3. However they show increasing sophistication: the Task 2 response implicitly involves multiplication or division, makes use of context (although the use of context was not mentioned by the interviewed teachers), and diagrams; the Task 3 response contains a written metaphor of movement for addition and subtraction, and this metaphor implies the inverse nature of operations which the teachers cited as influencing their judgements. Although the response does not overtly describe multiplication or division, repeated addition and subtraction are in evidence. Therefore the increasing sophistication of the numeracy tasks sampled show good consistency with our analysis of the teacher interviews. However, none of the sampled tasks showed evidence of students using arithmetical examples that were too complicated for them to calculate.

Statistical Literacy Tasks

Similar to the numeracy tasks, Figure 1 suggests increasing sophistication in the student responses to the statistical literacy prompts from the 25th percentile to the 75th percentile. The response at the 25th percentile shows a social response to the prompt, comparable to what Watson and Callingham (2005) classified as an idiosyncratic response in their statistical literacy construct. In this type of response, personal beliefs and experiences dominate, as illustrated by the student's statements "The TV is that it is bad for your health" and "The homework will always be good for your health". The student response aligns with the teachers' observations of student difficulties in interpreting and understanding the more complex graphs. The Task 4 and Task 5 responses both provide statements that are related to the prompt and task. This aligns with the teachers' preference for responses linked to the question. They both show similarities to the category of informal responses on Watson and Callingham's construct with interpretations of basic one-step graphs provided but little justification for these "The apple was the children's favourite fruit". However, while the Task 4 response provided two simple statements, the Task 5 response made a range of statements about the data and included some basic data reading from the graph. The responses also reflected the teachers' preference for evidence that students could interpret simple graphs and make statements about these. However, none of the sampled responses

showed evidence for the teachers' observations that some students misuse and appear to misunderstand averages.

Discussion

Mathematics assessments are often criticised for privileging a narrow, fragmented view of children's mathematical thinking. We addressed this criticism by designing free-response tasks and administering them to primary students of various ages.

To inform the first research question we applied a comparative judgement technique to assess the responses. The outcomes were found to be reliable, and this finding is consistent with the use of free-response tasks with older groups of students (Bisson et al., 2016; Jones & Alcock, 2014; Jones & Inglis, 2015; Jones & Karadeniz, 2016). Moreover, a comparison of the scores and qualitative analysis of six sampled responses, triangulated with the comments of four interviewed teachers, supported the validity of the assessment method. Specifically, the highest scoring responses showed more mathematical sophistication and better reflected the teachers' comments on what they valued when assessing than the medium scoring responses; and, likewise, the medium scoring responses compared to the low scoring responses.

To inform the second research question, we interviewed a sample of four teachers after they had completed their comparative judgements, and undertook a qualitative analysis of six sampled responses. For both types of task – numeracy and statistical literacy – there was consistency between the scores, the qualitative features of the sampled responses, and the interviewed teachers' comments: higher scoring tasks better addressed the task question, were mathematical rather than social, and made use of examples and illustrations. For the numeracy tasks the scores and qualitative features were consistent with teachers' preference for sophisticated use of arithmetic operations, including evidence of knowledge of the reversibility of operations. For the statistical literacy tasks the scores and features were consistent with teachers' preference for meaningful written interpretations of graphical representations that were accurate and mathematical (rather than social).

Taken together, our findings provide support for the use of free-response mathematics tasks with primary students both for summative assessment, if scored using a reliable holistic assessment technique such as comparative judgement, and for providing teachers with a window onto children's mathematical thinking.

Limitations

We are aware of three limitations with the present study. First, no prior mathematics achievement data were available due to ethical constraints on data collection, and so we could not further validate the assessment outcomes. Second, only four teachers were interviewed and six responses sampled, threatening generalisation. Third, while free-response tasks have been reported to be well-suited to assessing conceptual knowledge (Bisson et al., 2016), they are less appropriate to providing insights about procedural knowledge. Therefore free-responses methods should be combined with traditional assessment approaches to provide a fuller picture of children's mathematical thinking (Jones & Inglis, 2015).

Conclusion

Traditional assessments have the advantage that they can be reliably scored, but provide only a narrow window onto children's mathematical thinking. Free-response tasks have the

potential to provide richer insights, but historically have been extremely difficult to score reliably. Using comparative judgement we have provided evidence that this barrier can now be overcome. We recommend teachers and researchers consider using free-response tasks to supplement traditional assessments, thereby providing reliable and valid insights onto the mathematical knowledge of primary students.

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Using a Contextual Pāsifika Patterning Task to Support Generalisation

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Pāsifika cultures have a rich background of mathematics including a strong emphasis on patterns used within craft design (Finau & Stillman, 1995). However, there have been limited studies which have investigated the use of contextual Pāsifika patterns in mathematics classrooms. The aim of this study was to explore how contextual Pāsifika patterning tasks can potentially support young children to develop their understanding of growing patterns. Ten lessons using Pāsifika and Māori patterns were undertaken with 27 Year 2 students (6-year-old). In this paper, analysis of one of the lessons is used to examine how a contextual task assisted these young students to generalise growing patterns.

Mathematical achievement of culturally diverse students is a challenge in many countries. Teaching in ways responsive to the cultures of our students is an important step in enhancing equity of access to mathematical achievement and enacting educational policy (e.g., Ministry of Education, 2012). Within New Zealand, similar to other countries, there is a changing student population that is increasingly culturally diverse. This includes a large number of Pāsifika students, a heterogeneous group from a range of Pacific Island nations and including both those born in New Zealand who identify themselves with the Pacific Islands and those who have migrated from the Pacific Islands (Coxon, Anae, Mara, Wendt-Samu, & Finau, 2002). In New Zealand schooling, Pāsifika students' results are characterised by under-achievement when compared to students of other ethnicities (University of Otago & NZCER, 2014). Deficit theorising is frequently used by educators to explain this under-achievement with Pāsifika cultures being positioned as mathematically deficient (Hunter & Hunter, 2018; Turner, Rubie-Davis, & Webber, 2015). However, Pāsifika cultures have a rich background of mathematics including a strong emphasis on patterns used within craft design (Finau & Stillman, 1995). This paper investigates the use of contextual Pāsifika patterning tasks to support young children to develop their understanding of growing patterns.

Research Literature

Over the past decades, early algebra has been the focus of both research studies and curriculum reform with calls for a greater emphasis on the teaching and learning of algebra in primary classrooms (Blanton et al., 2018; Ministry of Education (MoE), 2007). Both patterning activities and functions offer an opportunity to integrate early algebraic reasoning into the existing mathematics curriculum. Evidence from research studies highlights that young learners can engage in early algebraic reasoning and generalise from patterning tasks (Blanton et al., 2018), and this supports students development of deeper understanding of mathematical structures (Warren & Cooper, 2008). Early algebraic thinking comprises of three key components: (1) generalising mathematical relationships and structure; (2) representing generalised relationships in diverse ways; and (3) reasoning with generalised relationships (Blanton, et al., 2018; Kaput, 2008; Warren & Cooper, 2008). For the purpose

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of this study we will be focusing on generalising mathematical relationships and structures. It is argued that before students can generalise relationships, they must be able to identify the underlying mathematical structure (Warren & Cooper, 2008). However, it is often the way in which the pattern structure is represented, in conjunction with teacher instruction, which inhibits primary school students to successful access mathematical structures.

For the purpose of this study we are considering the structure of growing patterns. Linear growing patterns are characterised by the relationship between elements which increase or decrease by a constant difference. Students are often introduced to growing patterns through visual images of mathematical shapes (e.g., squares, circles) as a series of stages adding in one direction of the pattern (e.g., adding on another line of squares or a layer of circles) as evident on www.nzmaths.co.nz. Less common in curriculum documents and teaching materials are the introduction of growing patterns as instances of growing in multiple directions (e.g., in the shape of a cross growing in four directions).

The majority of studies have considered how students see the structure of growing patterns and how they form generalisations with geometric growing patterns often not embedded or connected to students' culture. The aim of this study is to explore how contextual Pāsifika patterning tasks can potentially support young children to develop their understanding of growing patterns. In particular: (1) How do young culturally diverse students see the structure of Pāsifika patterns? and, (2) How do young culturally diverse students generalise the structure of Pāsifika patterns?

Theoretical Frameworks

There are two theoretical frameworks that underpin this study; first, a framework for generalisations in early algebraic thinking as identified by Radford (2010); and, that of culturally responsive pedagogies with a particular focus on Pāsifika values and culture.

Generalising mathematical concepts must go beyond just the act of noticing (Radford, 2010). For all elements of a pattern sequence, students must develop the capacity to see the underlying structure and articulate this algebraically (Radford, 2010). Underpinning this assertion is Radford's (2010) three 'layers of generality': factual, contextual and symbolic generalisations. Factual generality is an elementary level of generalisation where students engage heavily in gestures, words and perceptual activities often attending to particular instances of the pattern rather than general elements across the pattern. In developing a contextual generalisation, students will often refer to the "the next figure which supposes a privileged viewpoint from where the sequence is supposedly seen" (Radford, 2010, p.52). Finally, the symbolic level requires students to replace words with symbols such as letters to express the generality of the rule. The majority of studies in this area utilise tasks from a Western context with few drawing on students' cultural backgrounds.

Despite mathematics being positioned as a value and culture free subject area (Presmeg, 2007), researchers (e.g., Bishop, 1991; D'Ambrosio, 1985, Tate, 1995) have shown that mathematics is a cultural product. The perspective taken within this paper is that the teaching and learning of mathematics cannot be decontextualised from the learner as this is wholly cultural and closely tied to the cultural identity of the learner. Similar to Tate's (1995) argument related to African American students in the USA, we contend that failing to provide Pāsifika students with tasks and learning experiences that are centred on their traditions, experiences, and culture is the major reason for inequity in mathematics education in New Zealand. To develop culturally responsive mathematics classrooms, educators need to ensure that tasks are set within the known and lived, social and cultural reality of the students. Acknowledging that students bring their own cultural ontology (ways of being,

knowing and doing) and discourse to the classroom provides an opportunity for students of diverse cultural backgrounds to make more meaningful connections to mathematics (Miller, Warren, & Armour, 2018). Many Pāsifika learners have a rich environment of patterns from cultural activities and artefacts. For example, cultural activities such as the Samoan sasa (slap-dance) draw strongly on patterns as does Cook Island drumming and drum dances. An emphasis on patterns is also evident in cultural artefacts ranging from Tongan and Samoan ngatu/siapo/tapa (a form of bark-cloth) to Cook Island tivaevae (quilts). We contend that these contexts provide a means of drawing upon the mathematics already evident in Pāsifika culture to develop young learners' early algebraic reasoning.

Research Design

This research reports on one aspect of a larger study which focused on the use of authentic patterns from Pāsifika and Māori culture to develop young culturally diverse students' understanding of functional patterns. It was conducted with one classroom of Year Two students in a low socio-economic, high poverty, urban school in New Zealand. Twenty-nine students (aged 6 years old) participated in the study including 17 male and 12 female students. The students were predominantly of Pāsifika descent (n = 24), with three students from an indigenous New Zealand Māori background, and two students from South East Asia. The teacher in this classroom was an experienced teacher who had been involved in an ongoing professional development and research project entitled *Developing Mathematical Inquiry Communities* (for more information see Hunter & Hunter, 2018).

Drawing on the design of a classroom teaching experiment (Steffe & Thompson, 2000), students participated in ten 30-minute lessons exploring and developing their understandings of functional growing pattern generalisation. The students were taught in small groups with between 12 – 14 students involved in each lesson. Each lesson involved a similar structure with the launch of the task, paired work, a large group discussion and a teacher facilitated connection to a generalised rule. Students in this classroom had previously engaged with tasks involving repeating patterns but growing patterns were unfamiliar as this is not a curriculum expectation until Year Four (MoE, 2007). The focus of this research paper is two lessons focused on a pattern from a Cook Island tivaevae.

The video footage of the lessons was wholly transcribed and analysed to identify themes. To manage these documents a coding system was utilised to determine how to examine, cluster, and integrate the emerging themes (Creswell, 2008). Researchers coded the data at each phase with respect to early algebraic thinking, teacher actions, Pāsifika values, and student actions and met to discuss their themes and recode any data.

Findings and Discussion

The findings draw on the analysis from one lesson to provide an exemplar of how an authentic cultural pattern can be used to develop early algebraic reasoning. This includes an examination of the task structure and launch, teacher actions and student responses.

Task Structure and Launch: Engaging in Pāsifika Culture

Tivaevae is a traditional form of Cook Island quilting which involves groups of women designing, cutting, and embroidering these quilts. These are usually only given as gifts on special occasions such as weddings or significant birthdays. Designs are often based on plants and flowers and frequently incorporate forms of growing patterns. The task was

designed in collaboration with the teacher using a photograph of part of a pattern of a tivaevae. The focus was on the number of leaves on the pattern.

Figure 1. Cook Island Tivaevae task.

The teacher began the lesson by acknowledging the cultural knowledge of her young students. After showing the class a photo of a tivaevae, she positioned a child of Cook Island heritage as an expert to share her knowledge of tivaevae with the other students:

> Mereana (excitedly): My Mama, she makes that, she makes heaps, my whole family does.
>
> Teacher: Does she? Wow, you can help me then. I am so pleased that you came and sat over here Mereana because I knew that you would know about this.
>
> Mereana: It's a Cook Island… (pause) tivaevae
>
> Teacher: Yes it's Cook Island, and they are beautiful, aren't they?
>
> Mereana: We use it for weddings and birthdays
>
> Teacher: Listen to Mereana, Mereana is going to tell us, who makes them?
>
> Mereana: My Mama and the girls in our family.

In this example, the teachers' actions supported the students to begin making a meaningful connection to mathematics in relation to their cultural context.

The teacher then began to orient the students to the structure of the pattern: *let's just get really clear where the leaves are, Sima, so we are all going to do it together because otherwise we will all be talking about different designs*. In doing so, she consistently drew student attention to the constant four in the middle of the pattern.

Student Approaches to the Pattern: Seeing the Pattern in Multiple Ways

Initially, the students attempted to draw or count to find the number of leaves for the pattern positions. For example, Sebastian and Cruz began by counting the twelve leaves for position one (four leaves in the centre (4) and two leaves up each of the four stems (8)) and for position two counted another eight leaves up to 20 leaves. After the third position, they noted the regularity of the increase by eight. They used this to continue to count by visualisation for the successive positions. For example, when they came to position four which was not pictured, Sebastian and Cruz then counted from 28 to 36.

At this point, noticing that many of the students were either drawing or counting, the teacher stepped in to press the students beyond counting:

> Teacher: What I noticed is that lots of people were busy drawing that picture and you were doing lots and lots of counting. But sometimes when you count really big numbers and draw lots and lots of leaves, what happens to our counting?
>
> Tiare: You lose the count.

This teaching moment was key to shift student attention to more explicitly noticing the structure of the pattern rather than using a count all or count on technique. The teacher finished the first lesson by selecting a pair of students to share the pattern that they noticed:

> Mereana: (indicating on the picture to outside circle of leaves) Whenever you add the leaves you add eight.
>
> Teacher: So what did you notice? Say it again
>
> Mereana: You can, if the number gets higher, you just add eight leaves
>
> Teacher: So who understands what she is talking about? What does she mean, when it gets higher we keep adding eight?

This is the first instance of a generalisation articulated by a student. Mereana is seeing the structure of the pattern as adding eight leaves as the numbers get higher. This demonstrates that she is coordinating two variables in the pattern: (i) the structure of the eight leaves; and, (ii) the position of the pattern (if the number gets higher). The teacher then pressed Mereana to link her explanation back to the structure of the pattern. When this occurs Mereana begins to talk about the pattern as individual instances.

> Teacher: Can you show on that picture behind you?
>
> Mereana: (Draws circle with finger around first eight leaves outside) There is eight on the first one, and then eight on the second one (draws imaginary circle around the next layer of eight).

In this instance, Mereana has formed a factual generalisation (Radford, 2010). She is articulating the pattern as instances and using gesture to support her reasoning. This aligns with past research that indicates that the use of gesture for young students as they articulate generalisations appear to be a key stage in their development of algebraic thinking (Miller, 2015; Radford, 2010). It is unclear whether she sees the pattern more generally or as only series of instances, however it is clear that she has been able to begin to notice and articulate the two variables of the pattern with only a few explicit teaching moments which facilitated the students to attend to the variables. Research highlights that young students often refer to only one variable of the pattern and do not attend to the co-variational relationship of the two variables. In addition, it appears that there is more success if the variables are embedded in the one structure so the students cannot ignore it (e.g., kangaroo tails and ears) rather than separated (e.g., geometric shapes and words or number cards under the pattern indicating the position) (Miller, 2015). In this case it appears the tivaevae pattern provided an opportunity for these young students to begin to see co-variation.

Lesson Two: Three alternative approaches

The structure of the tivaevae pattern allowed the students to see it growing in multiple ways. At the beginning of the next lesson, the teacher first facilitated student awareness that the pattern could be seen in multiple ways: *different people see it growing in different ways, have a little think first.* She then provided the students with time to talk in pairs to further develop their ideas of how the pattern of leaves was growing.

The teacher noticed that students viewed the pattern as growing in three distinct ways. She carefully selected specific students to share the alternative ways that they saw the pattern growing. The teacher began by reminding students of the constant in the middle: *just before we say about the outside can someone remind us how many leaves there are on the inside? What about in that middle bit?* Following this, she asked a pair to share:

> Asher: The pattern is, they're putting two each on the outside.
>
> Teacher: Two what? Asher? Two what?

Asher: Ah, leaves.

Teacher: Good what are they putting on the outside each time Aurora?

Aurora: Two leaves.

Teacher: Two leaves, just two or two on every?

Aurora: Side.

Teacher: Two on every side.

As the students shared, the teacher introduced simplified representations of how the students saw the structure of the pattern.

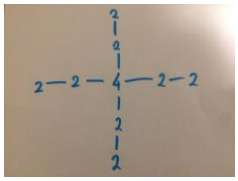

Figure 2: Simplified representation of pattern structure.

In this moment, not only is the teacher having students attend to the structure but she is also displaying the visual contractions of the pattern that students are articulating. In this instance, there has been a transfer from a cultural context into a mathematical context. Following this the teacher then asked student to share an alternative solution, similar to the example from the previous lesson:

Teacher: Ok so now Sima and Seini, and I think you guys all did something else similar I want you to talk about what you did, what did you notice?

Sima: Every time you add on leaves you add eight.

Teacher: So can I just check how many were in the middle on your one?

Seini: Four.

Teacher: It's still four in the middle, you said, every time we add eight. Turn and tell your buddy why they're saying they'll always add eight.

While the students did not articulate the constant variable in the pattern of four (the centre leaves) as an additional variable in the pattern, the teacher was drawing their attention to it as a way for them to consider their generalisations. Following this, the teacher again introduced a simplified representation to show how the students saw the structure of the pattern: *So, the first one's around it, can you see that? So it's like this, the patterns you've got your four in the middle and then it goes eight and then eight, and then eight, can you see like that, it's growing like that.* (draws diagram – see Figure 3)

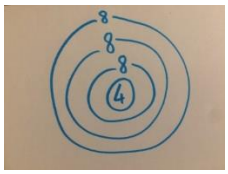

Figure 3: Simplified representation of pattern structure.

In contrast to the above figure, other students saw the pattern growing as the pattern number eight times around the stalk. That is, they saw the leaves growing up one stalk rather than in circles around the centre of the pattern.

Ngaire: There were three leaves on each stalk.

Teacher: On each stalk or on each side of each stalk?

Ngaire: Each side.

Teacher: So, there were three leaves over here, and how many leaves on this side?

Ngaire: Three.

Teacher: Have a think, have a look at that one and think about how that one works? Cos Ngaire has seen it a different way. What would come down on this stalk?

Ngaire: Three.

Teacher: Three where?

Ngaire: Three on each.

Teacher: Three here and three here (writes a three on each side of the stalk for each stalk).

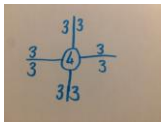

Figure 4: Simplified representation of pattern structure.

Importantly, following the sharing of the solution strategies, the teacher provided an opportunity for other students to access the ways in which their peers visualised the pattern growth. For example, using the description generated by Asher and Aurora, the teacher asked the students to describe what the third position would look like and what the seventh position would look like:

Tiare: (referring to one branch) Two times three.

Teacher: We could use our two times tables couldn't we. So there's two and two and two. So you could keep going out, how many twos would you go out for the seven?

Sebastian: Seven.

It was at this point that the students shifted their thinking from seeing the pattern as additive to multiplicative. For young students' it is often challenging to see the multiplicative structure of a growing pattern without teachers making this explicit for students (Warren & Cooper, 2008; Miller, 2015). In addition, Tiare has provided a factual generalisation (Radford, 2010) where she is referring to one instance in the pattern, however the key point here is that students are now being to see the multiplicative structures of the pattern.

Conclusion

This study begins to add to new knowledge about the use of culturally relevant tasks being used to develop early algebraic thinking for students. Acknowledging that the students brought their own cultural knowledge to the classroom provided an opportunity for these culturally diverse students to make more meaningful connections to the mathematics presented in the lesson. It is evident that contextual Pāsifika patterning tasks, such as a tivaevae pattern, can support young children to develop their understanding of growing

patterns and begin to articulate generalisations. There were opportunities for: (i) students identifying multiple structures of the pattern, which were both additive and multiplicative; (ii) being to identify and articulate covariational relationships; and, (iii) form both factual and contextual generalisations. There were clear shifts in both student thinking and teaching actions across the two lessons. Students moved from count all strategies, to identifying additive thinking to then multiplicative thinking through considering the structure of the pattern. The teacher actions of mirroring student thinking and using mathematical diagrams supported students as they made generalisations and further contractions of the pattern structure.

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Preparing for the Final Examination in Abstract Algebra: Student Perspectives and Modus Operandi

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This study focuses on the undergraduate mathematics students' perceptions and applied techniques for the preparation for their final examination in Abstract Algebra. The results of this study suggest that the revision for the final examination involves, firstly, the review of the lecture notes, followed by the solution of the coursework together with the use of model solutions and the solution of the past papers. The order of the last two activities varies. An often-occurring revision technique involves, instead of a linear succession of the aforementioned activities, a 3-dimensional spiral approach towards revision, with the three activities interchanging until the students who apply it feel that they have achieved adequate object-level and metalevel learning.

Many studies have reported on undergraduate mathematics students' difficulty with Abstract Algebra (Ioannou, 2012). It "is the first course in which students must go beyond 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes" (Dubinsky et al., 1994, p268). A typical first Abstract Algebra course requires deep understanding of the abstract notions involved, as well as the application of techniques in the preparation of coursework and final examination. An important element that causes students' difficulty with Abstract Algebra is its 'abstract' nature (Hazzan, 1999). The deductive way of teaching Abstract Algebra is unfamiliar to students and, in order to achieve mastery of the subject, it is necessary to "think selectively about its entities, paying attention to those aspects consistent with the context and ignoring those that are irrelevant" (Barbeau, 1995, p140). In addition, Gueudet (2008) suggests that many pedagogical issues emerging in undergraduate Mathematics Education are based on the transition from secondary to tertiary Mathematics, which can still occur in their second year. In fact, student difficulties in Abstract Algebra may be an indication of problematic transition, mainly due to the particular nature of this course (Ioannou, 2012). The aim of this study is to investigate the student perspectives and applied study skills for the preparation for the final examination, an essential part of their learning process and assessment. For the purposes of this study, I will use the Commognitive Theoretical Framework (CTF) (Sfard, 2008).

Literature Review

Research in the learning of Abstract Algebra (Theory of groups and rings) is relatively scarce compared to other university Mathematics fields. Even more limited is the commognitive analysis of conceptual and learning issues (Nardi et al., 2014). The first reports on the learning of Abstract Algebra appeared in the early 1990's. Several studies, following mostly a constructivist approach, and within the Piagetian tradition of studying the cognitive processes, examined students' cognitive development and analysed the emerging difficulties in the process of learning certain group-theoretic notions.

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Furthermore, the construction of the newly introduced abstract algebraic notions is often an arduous task for novice students and causes serious difficulties in the transition from the informal secondary education mathematics to the formalism of undergraduate mathematics (Nardi, 2000). Students' difficulty with the construction of these concepts is partly grounded on historical and epistemological factors: "the problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (expected to understand) the concepts today" (Robert & Schwarzenberger, 1991).

Nowadays, the presentation of the 'fundamental concepts' of Group Theory, namely group, subgroup, coset, quotient group, etc. is "historically decontextualized" (Nardi, 2000, p169), since historically the fundamental concepts of Group Theory were permutation and symmetry. Moreover, this chasm of ontological and historical development proves to be of significant importance in the learning of Abstract Algebra for novice students. From a more participationist perspective, CTF can prove an appropriate and valuable tool in our understanding of the learning of Abstract Algebra due both to its ontological characteristics, as well as its epistemological tenets.

Research suggests that students' understanding of the notion of group proves often primitive at the beginning, predominantly based on their notion of a set. Students often have the tendency to consider group as a 'special set', ignoring the role of binary operation. Iannone and Nardi (2002) suggest that this conceptualisation of group has two implications: the students' occasional disregard for checking associativity and their neglect of the inner structure of a group. An often-occurring confusion amongst novice students is related to the order of the group G and the order of its element g. This is partly based on student inexperience, their problematic perception of the symbolisation used and of the group operation. The use of semantic abbreviations and symbolisation can be particularly problematic at the beginning of their study. Nardi (2000) suggests that there are both linguistic and conceptual interpretations of students' difficulty with the notion of order of an element of the group. The role of symbolisation is particularly important in the learning of Abstract Algebra, and problematic conception of the symbols used probably causes confusion in other instances. In addition, an important means for coping with the level of abstraction in the context of Abstract Algebra is the use of visual images. In fact, their use plays a significant role, since they serve as a meaning-bestowing tool (Ioannou & Nardi, 2009a).

Theoretical Framework

CTF is a coherent and rigorous theory for thinking about thinking, grounded in classical Discourse Analysis. It involves a number of different constructs such as metaphor, thinking, communication, and commognition, as a result of the link between interpersonal communication and cognitive processes (Sfard, 2008). In mathematical discourse, objects are discursive constructs and form part of the discourse. Mathematics is an autopoietic system of discourse, i.e. "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p129). Moreover, CTF defines discursive characteristics of mathematics as the word use, visual mediators, narratives, and routines with their associated metarules, namely the how and the when of the routine. In addition, it involves the various objects of mathematical discourse such as the signifiers, realisation trees, realisations, primary objects and discursive objects. It also involves the constructs of object-level and metadiscursive level (or metalevel) rules. Thinking "is an individualised version of (interpersonal) communicating" (Sfard, 2008, p81). Contrary to the acquisitionist approaches,

participationists' ontological tenets propose to consider thinking as an act (not necessarily interpersonal) of communication, rather than a step primary to communication (Nardi et al. 2014).

Mathematical discourse involves certain objects of different categories and characteristics. Primary object (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p169). Simple discursive objects (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization. Compound discursive objects (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary." In the context of this study, groups are an example of compound d-objects. The (discursive) object signified by S in a given discourse is defined as "the realization tree of S within this discourse." (Sfard, 2008, p166). The realization tree is a "hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations and so forth" (Sfard, 2008, p300).

Sfard (2008) describes two distinct categories of learning, namely the object-level and the metalevel discourse learning. "Object-level learning […] expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse" (Sfard, 2008, p253). In addition, "metalevel learning, which involves changes in metarules of the discourse […] is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008, p254).

Methodology

This study is part of a larger research project, which conducted a close examination of Year 2 mathematics students' conceptual difficulties and the emerging learning and communicational aspects in their first encounter with Abstract Algebra. The module was taught in a research-intensive mathematics department in the United Kingdom, in the spring semester of a recent academic year. This module was mandatory for Year 2 mathematics undergraduate students, and a total of 78 students attended it. The module was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a doctorate student in the mathematics department. The module assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of the semester.

The gathered data includes the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 members of staff's interviews (5 members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave 3 interviews each), student

coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, the collected data of the 13 volunteers has been scrutinised. Naturally all sources of data have been appropriately analysed, and the conclusions of the data analysis have been triangulated.

Finally, all emerging ethical issues during the data collection and analysis, namely, issues of power, equal opportunities for participation, right to withdraw, procedures of complain, confidentiality, anonymity, participant consent, sensitive issues in interviews, etc., have been addressed accordingly.

Data Analysis

Data analysis suggests that for the revision for the final examination, twelve of the thirteen students (12/13) study the lecture notes, solve the coursework using the model solutions given by the lecturer and solve a various number of past papers (Student O is the exception). Preparation for the final examination, and the final examination per se, is the final stage in the students' learning process and at this stage students are invited to resolve any commognitive conflicts[3]. As the following excerpt suggests, usually, the first step for revising is the *study of the lecture notes*. Students' approaches vary, but their predominant aim is to go through the definitions and theorems, both to improve their object-level learning but also to memorise the ones that will possibly be asked to state. In addition, five of the thirteen students (5/13) students produce their own revision notes, which help them to improve their object-level learning and assist them in memorizing easier.

> I normally write out my notes, a lot... Hmm, yeah like I make revision notes, and I do revision cards. And I normally just sit and rewrite out the definitions a million times and the theorems a million times, and just like – do the revision cards and get people to test me and I'll write them down, and then I'll work through past papers and all the problem sheets. Student A

Seven of the thirteen (7/13) students study their lecture notes without producing revision notes. This is usually the first step for their revision. Studying the lecture notes for the final exam requires a different, all-inclusive, approach from the preparation of the coursework. Studying the lecture notes for the exam is a 'renewed task' leading to improved learning of the theory. As the following excerpt suggests, having a holistic picture of the entire theory, and consequently having already, up to a certain extent, created realizations of the involved d-objects and realization trees, makes the task of revision and objectification a different experience.

> Usually, like the coursework… we start from the lecture notes…and usually I am trying to understand everything… not like when we prepare a coursework. For the coursework we do not have much time so we are going for the exercises… I believe that if you do not understand something, then you cannot understand what it follows as well… In the past, I used to make my own notes, but since it was time consuming, I decided to stop that… I study the notes and I highlight the important things… Something that I need to see again… I study only from the notes… Student B

The above excerpt is a representative example of all thirteen students' awareness regarding the different approach that should follow for the examination revision. Student B expresses her desire to change her study approach and wishes to improve her learning. She identifies that solving a mathematical task without studying the related narratives and routines is a faulty approach. For her, studying the lecture notes as part of the final revision is a task that has to be faced anew. Experience has led her to prioritise efficiency in her

[3] Commognitive conflict is defined as a "situation that arises when communication occurs across incommensurable discourses". (Sfard, 2008, p 296)

study skills and approaches, as well as the awareness of the demands of examination revision.

The next step in the twelve of the thirteen (12/13) students' revision is usually the *solution of coursework and past exam papers*. There are two distinct categories of students regarding which task is undertaken first: six of the thirteen (6/13) students are studying the coursework first and six of the thirteen (6/13) students start with past exam papers. Studying the coursework first, together with the given model solution, is an important step in the learning process. As the following excerpt suggests, this revision approach allows students to have the chance to exactly locate their weakness and improve their object-level learning of the definitions of certain d-objects. Consequently, this process will allow them to successfully cope with the level of abstraction, improve the structure of the realization trees of these d-objects and objectify them, something that it will permit them to enhance their metalevel learning.

> Um, probably with the questions that we've been given, and with the solutions, I'm hoping to like – help teach myself how to do it… and then I learn by doing past exam papers, mainly, […] I tend to do like quite a few years back, like do all of them, and once I've done them, go back, and like the questions that I […] wasn't able to do before, I try and do it again, cos I've hoped that I've taught myself. Student C

Student C considers working with the coursework and the model solution as a means to 'teach herself' the *how* and *when* of the routines involved. It is a chance to correct and/or improve her object-level and metalevel learning, application of metarules and solving techniques, and consequently overcome any knowledge gaps resulted in the learning of this new mathematical discourse. Using the solutions, the particular students will be able to observe the metalevel rules of Abstract Algebra in practice and learn how they should be applied. For these students, model solutions are apparently an indispensable tool that can be used in order to resolve any preexisting commognitive conflicts and improve the realization trees. These students will possibly have the chance to realize not only the metadiscursive level rules, but it will also allow them to understand how they should approach a mathematical task in general, namely, specifying the routine prompts[4], applying the decided course of action, and successfully completing the task.

Another benefit from working with the model solutions while revising the coursework exercises is the improvement of self-confidence. Although only Student D expressed so overtly this perception, it is important to be highlighted, since other students have implied it as well. When this task is completed, he then works with the past papers.

> I don't generally look at the exam papers… only slightly towards the end – only because they can freak you out if you – I like spending a few days building up your confidence just reading through lectures notes and that sort of thing – examples of the course sheets I like looking through them for a while then go... […] You need confidence. If I have confidence I am quite good. I can actually generally breeze through even if I don't actually know the answers entirely. Student D

Another revision technique is by studying the past papers first and then the coursework. Student E is planning to start by working with the past papers, identifying the demands of the examination as well as his weaknesses.

> Um, well I'll definitely be looking at past papers. […] Then go to lecturers and just get feedback on what I've done, and then they'll help me say like oh no don't do this, or yeah, you're doing all right in this bit. So any kind of gaps in my knowledge hopefully they'll – help fill in. […] I kind of look at what would come up on the exam […] have a little look at lecture notes, maybe a few problem

[4] Elements of situations whose presence increases the possibility of the routine's performance.

> sheets, then maybe get and attempt another one, with a bit more knowledge. Actually I do one, that I kind of do with my lecture notes open really, then try – as I'm getting a little bit better, try and do it without the lecture notes, cos obviously that's gonna be what's happening in the exam. Student E

This technique allows the students to identify the difficulties they will possibly face in their examination, identify the expected types of questions they will probably need to solve and therefore adjust their revision in order to overcome the new demands and revise the appropriate mathematical routines. Student E is the only student that is overtly willing to ask assistance from the lecturer in solving past papers.

Regarding the overall process of revision for the final examination, five of the thirteen (5/13) students have clearly stated the interchange of the three activities, namely studying the lecture notes, reworking the coursework and solving the past papers. This approach can be described as a *3-dimensional spiral approach towards revising*. Students that follow this revision approach, work with the lecture notes, coursework and past papers in an interchangeable way until: first, they have overcome any commognitive conflicts caused by the nature of Abstract Algebra, and second, they have improved their metalevel learning. In each spiral cycle of revision their level of comprehension improves.

Unlike Student E, Student F does not require any assistance from the lecturer but instead he is marking his solutions of the past papers by himself.

> And then start past exam papers, and get the solution and see what they're looking for in the exam questions… So do a few of that, and do a proper exam conditions, and… […] Mark it – no – I done it first, mark it, and then look at the marks scheme, yeah, mark myself and see how much I get? And then, after a few days, redo the paper again, to see how much I improve, or which area I still don't understand or something. Student F

Student F's approach towards revision has some very useful and interesting elements, such as the solution of past papers under exam conditions, or the repetitive, spiral like, approach already encountered in Student E's case. His active approach to revision is manifold with repetitive cycles that possibly allow him to better objectify the material and enrich his experience. Relying solely on his marking, though, without asking for any external control might jeopardise his learning. Student F's examination results (50%) do not show that his revision scheme has led to the expected outcome.

The weakest students, Students G and H, expressed a similar perspective regarding what makes a good examiner, according to which good examiner is the one whose papers are the same every year: *Last year we had a very good lecturer… his papers were exactly the same every year, but with different numbers…* Student G. This statement suggests that these students adopt a 'utilitarian' perspective of learning Mathematics at the university level. This statement, as well as their performance, suggests a difficulty in the transition from secondary education towards university Mathematics education and its demands. It indicates that their mathematical thinking is, according to Sierpinska (2000), only practical, based on *prototypical* examples that they need to see in the past papers, and not *theoretical*.

Finally, Student O is the only student who does not revise by using the three elements of revision, namely the revisit of lecture notes, the solution of coursework making use of the model solutions, and the solution of past papers. He rather uses only the first two. In particular, Student O makes use of books, in parallel with the lecture notes and coursework, and on many occasions places special emphasis on the way he reads the books and the relaxed pace of his reading.

> It's a matter of revising what you have done and revising your coursework answers, going through books just being relaxed. I am sure most other people would have a different approach would be do past questions, but I prefer to be more relaxed more laid back about it. […] I don't want to go through

> it at top speed, just go through it normally. Hopefully it will sink in. But then I read it and then close the book and try to reproduce what they have... Student O

Student O's perception is quite distinct, indicating his effort to, not only, approach revision in a superficial way and get a good mark, but rather as a chance to improve his object-level and metalevel learning in the discourse of Abstract Algebra. He considers exams as an opportunity to widen his object-level and metalevel skills, overcome any possible commognitive conflicts that occurred in the coursework and hopefully achieve endogenous discursive expansion. The last is encapsulated in his phrase "sink in".

The above excerpt indicates maturity in his way of reading a mathematical text. Student O has realised that reading a mathematical text is fruitful only if the pace of reading is not fast, but rather compatible with the difficulty of the test and the speed in which an individual grasps the various aspects of the discourse. His approach is overall mature, indicating successful transition towards university Mathematics and its norms.

Conclusion

The above analysis suggests that the revision for the final examination is a 'renewed contract' for mathematical learning, during which students need to develop and/or apply certain techniques. They are invited to revisit what they have been taught, localise the conceptual gaps and overcome the remaining misconceptions. The majority of students adopt a similar approach towards revision for the final examination. Usually the revision process initiates by revisiting the lecture notes. The predominant aim is to engage again, after having acquired more experience, with the various mathematical narratives, namely definitions, theorems, lemmas and proofs, both to improve their object-level learning and memorise the ones that are most likely to appear in the examination paper. The second step of revision is either to the study of the coursework questions in parallel to the given model solutions or attempt to solve past papers. Regarding the solution of coursework using the model solutions, the discussion above indicates that for many students it is an important step in their learning process. Students have the opportunity to compare their solutions with the model solutions and precisely localise their errors. This will enable them to resolve any misconceptions related to these errors, by improving their object-level learning regarding the involved d-objects and will also help them to resolve problems with the governing metalevel rules and, more generally, with proof production. This process requires autodidactical skills (self-teaching) that will enable them to teach themselves, among other things, the *how* and the *when* of the involved routines, and to correct and/or improve their learning and solving techniques. Regarding the solution of past papers, many students at this stage try to specifically identify the definitions, theorems and proofs that are likely to be included in the examination paper, to pinpoint possible mathematical tasks that they may be asked to prove or solve, to extend their experience by solving the past papers as such, and, moreover, to have an opportunity to apply their solving skills, knowledge and understanding to a variety of tasks. The revision process is often nonlinear, and students use the three elements interchangeably until they feel that they have achieved adequate object-level and metalevel learning.

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The Role of Executive Function and Visual-Spatial Working Memory in the Development of Mathematics Skill

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This paper explores what we can learn from research that early cognitive processes support the development of children's mathematics skills. The role of two cognitive processes in working memory in the development of early mathematics was investigated: executive functions (EF) and visual-spatial (VS) ability. Children's mathematical skills were considered in relation to EF and VS. Recent research suggests that EF skills, which include monitoring and manipulating information in mind (working memory), suppressing distracting information and unwanted responses (inhibition) and flexible thinking (shifting), play a critical role in the development of mathematics performance. The number development also can be explained solely in terms of domain-general processes such as VS.

Most children at primary level face various problems in mathematical skills. An early step should be taken to identify the problem of Mathematics learning and this can help in providing appropriate treatment for children in the early stages. The numeracy and mathematical concepts of Mathematical concepts are an important and clear component of a very close relationship in Mathematics achievement, but there are other cognitive factors that play a significant role in contributing to the development of better Mathematics achievement (Cragg and Gilmore, 2014). Two interrelated cognitive processes in particular are critical for children's mathematics achievement: executive functions (EF) and visual-spatial (VS) skills (Bull and Lee, 2014; Cragg and Gilmore, 2014).

Among the factors that influence the problems of students' mathematical skills are cognitive factors namely executive function skills that control cognitive processes in thinking and behavior (Toll, Van der Ven, Kroesbergen and Van Luit, 2011). Furthermore, cognitive skills in maintaining and manipulating information for working memory systems are at a critical level for students in solving Mathematical problems (Raghubar, Barnes and Hecht, 2010). These cognitive skills are needed to monitor and control cognitive processes throughout complex cognitive tasks in solving problems (Miyake, Friedman, Emerson, Witzki, Howerter and Wager, 2000). Executive function skills include set-shifting ability, working memory and inhibition. These three skills are interconnected with each other and are the key executive functions in designing and solving problems (Garon, Bryson and Smith, 2008; Miyake et al., 2000).

The Role of EF in Mathematics Skill

The role of EF in the development of early numeracy is further investigated in relation to other cognitive variables that are important in this mathematics development. There was also empirical evidence suggesting that EF ability was already involved in learning early numeracy (Gathercole and Pickering, 2000) and children's VS skills also play a unique role in the development of mathematics (Meyer, Salimpoor, Wu, Geary and Menon, 2010).

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The EF is a skill in controlling the attention that involved in solving problems with various strategies, controlling responses, storing information in mind, preventing other distractions, compromising in a situation, reflecting on past experiences and planning in the future (Zelazo, Blair and Willoughby, 2016). EF is very important in the development of mathematics because those children with a weak memory in executive functions have difficulty in mastering mathematical skills, difficult to remember and understand instruction throughout the classroom activities (Raghubar, Barnes and Hecht, 2010).

EF is divided into three sub components. There are working memory, shifting and inhibition skills (Miyake et al., 2000). Working memory (WM) refers to the ability to store and manipulate information simultaneously. According to the influential working memory model proposed by Baddeley (1986, 2000), WM consists of (1) a central executive that coordinates incoming and outgoing information (i.e., an attentional control system), two slave systems; (2) the phonological loop for storing verbal information (verbal short-term memory) and the (3) VS sketchpad responsible for storing respectively VS information on a short-termed basis (VS short-term memory); and (4) an episodic buffer responsible for temporarily storing multimodal information from various sources (Miyake et al., 2000). According to Diamond (2013) WM is involved in storing and retaining information in mind as well as how individuals manipulate and capable of making mental changes to such information. For example working memory is required for individuals to remember information early and can link the information in the future. This skill is very important in academic development especially when performing mental calculations mentally where children can combine new information in solving mathematics problems.

According to Yeniad, Malda, Mesman, van IJzendoorn and Pieper (2013), cognitive flexibility skill (set-shifting ability) is needed in a use of various strategies to solve the mathematical problems. Shifting may be relevant for mathematical achievement, for example when switching between arithmetic operations is required. This skill refers to the individual's ability to change the mental set to something new according to the needs of the situation. The first step in cognitive flexibility (shifting) is to produce a mental representation that refers to the various strategies used in problem solving (Garon, Bryson and Smith, 2008). According to Diamond (2013) this skill is important to change the thinking approach in solving mathematics problem or to translate mathematical sentences according to their own understanding. An example is how children’s perception is dealing with situations in solving mathematics problems with various ways and the individual's ability to think more than one way. Inhibition is the ability to eliminate unnecessary actions and disruptive processes (Diamond, 2013; Zelazo, Blair and Willoughby, 2016) and address the high response responses in learning (Zelazo et al., 2016). Inhibition may be relevant for math when results of intermediate solving steps need to be suppressed and used in a next step, in order to arrive at the correct answer.

EF contributes to the achievement of Mathematics (Raghubar et al., 2010; Toll et al., 2011) especially in the knowledge of numbers sense (Blair and Razza, 2007; Geary, 2010). For example, children with low inhibitory control that is one component of EF may be less likely to evaluate and switch mathematical problem-solving strategies when they prove ineffective (Bull and Scerif, 2001). EF seems to be particularly important for word problems, which require students to build and manipulate models of the problems in their heads (Fuchs, Geary, Compton, Fuchs, Hamlett, Seethaler, 2010b). According to Swanson (2006), mathematical problem solving is complex because it involves multiple phases, a mental representation strategy and produces planning when information is stored in memory. Thus, these EF skills are essential to be applied to children throughout the mathematical learning

process so that they are easy to do the calculation process by applying the strategies in mathematical problem solving.

The Role of Visual-Spatial in Mathematics Skill

WM is a temporary storage in the working memory model by Baddeley (1986) that has a processing system consisting of central executives, verbal working memory and visual-spatial working memory (VSWM). Verbal working memory stores sound or word structure information while VSWM maintains spatial memory and visual information and manipulates mental images. This working memory system controls the process of storing information while engaging in complex cognitive activities involving aspects of VS and language (Miyake and Shah, 1999). The VSWM is involved the manipulation of spatial information. For example, when visualizing the side of a three-dimensional object, remembering and reproducing a spatial arrangement or organizing numbers on a page to manipulate mathematically (Baddeley, 1996).

VS refers to the individual's ability to imagine an object and ability to create and manipulate the image mentally (Lerner, 2003). This skill should be mastered by students so that they can see and describe something more deeply and can relate the concept of representation in solving the given problem. Arithmetic problem solving requires VS skill to translate into a particular representational form and it requires a strategy (Imbo and LeFevre, 2010). According to Metcalfe, Ashkenazi, Rosenberg-Lee and Menon (2013) VS skill is the best contributor to mathematical problem solving. It means that when the individuals have high skill in VS, they might also be able to solve the mathematics problems very well. According to van Garderen (2006), the VS ability leads to more abstract thinking and can be used by students to solve mathematics problems more easily. The student's understanding in mathematics is more clearly by develop a visual representation of diagrammatic or symbolic representations that can represent situations in mathematical problems. Mastery in these skills can indirectly motivate students to design strategies for mathematical solutions. When associating a number to any quantity, VS skills are needed for visual representation in mathematical problem solving. Hence, the number patterns will be easier to remember when the VS skills are developed (Hoon, 2002; Krajewski and Schneider, 2009).

EFs and VS Skills in Mathematics Learning

EF and VS skills are cognitive abilities that are highly related (Miyake, Friedman, Rettinger, Shah and Hegarty, 2001). VS tasks require a complex multistep of mental manipulations that considerably engage EF to encode and analyze the problem by manipulating spatial information, store VS information temporarily, plan a strategy for solving the problem, monitor performance, shift attention (cognitive flexibility) and adapt the strategies during performance (Miyake et al., 2001). According to work with adults, experts posit that EF is needed to a greater extent in VS tasks than in phonological tasks, which may be more easily automated (Miyake et al., 2001).

EFs and VS skills are specific associations emerging for subdomains of mathematics, such as problem solving (Cragg and Gilmore, 2014). Children use EF to set goals and arrange their behaviors in a wide range of tasks, for example, to recall and apply strategies to answer math questions successfully, to switch between operations and notations, to store and retrieve the necessary parts to solve a complex multistep problem, and to inhibit or suppress any inappropriate strategies (Blair, Ursache, Greenberg, Vernon-Feagans and the Family

Life Project Investigators, 2015). For example, mathematics problem solving requires EF skills, which refer to the ability that can recall and manage information in different ways, switching attention and using working memory to follow directions (Geary, 2013). Moreover, VS skills contribute to children's mathematics performance through their ability to use adaptive strategies to solve arithmetic problems (Geary and Burlingham-Dubree, 1989) and interpret numerical information spatially (Gunderson, Ramirez, Beilock and Levine, 2012). VS skills also may be related to the understanding of part/whole relationship and units involving a dimensional geometric, which is implicated in problem-solving tasks (Verdine, Irwin, Golinkoff and Hirsh-Pasek, 2014).

EF Deficits in Children with Mathematics Difficulties

EF deficits have often been identified as potential targets of early intervention efforts designed to help young children experiencing mathematics difficulties (Bull and Scerif, 2001; Toll et al., 2011). Children with executive functioning deficits more frequently fail to complete multi-step instructions by their teachers and to finish complex tasks (Gathercole, Lamont and Alloway, 2006). For example, working memory and cognitive flexibility deficits have been reported to constrain children's counting abilities, fact retrieval (Geary, Hoard, Nugent, & Bailey, 2012), and mathematics problem solving (Andersson, 2008). This may occur due to the children struggling with storing, manipulating symbolic information and shifting between several strategies or following multistep solution procedures (Toll et al., 2011).

WM is considered important for mathematical performance because information from long-term memory must be stored and manipulated during mathematical problem solving (Andersson, 2008). Deficits in WM ability can disrupt the representation and articulation of numbers during the counting process which lead to secondary deficits in numerical processes (Zamarian, Visani, Delazer, Seppi, Mair, Diem and Benke, 2006). Deficits in WM are more strongly predictive of learning difficulties in mathematics skill during early childhood. Thus, early intervention designed to prevent learning difficulties may be more effective.

Experimental Studies

Milisavljevic and Petrovic (2008) study was conducted on students (8-16 years) of 124 people to study EF in schools for students with intellectual problems in learning. The aspect that is seen is a strategy used by students with intellectual problems with students who have no intellectual problems in solving a given mathematical problem. Unstable EF development has been identified. It is found that the problematic student fails to solve mathematical problems when given large numbers and it shows low and weak EF levels. The solution path requires regular formula planning for the given questions and will involve the cognitive process of the individual. Hence, the weak performance of the test is influenced by low working memory development, which is the bridge for EF (Cabeza and Nyberg, 2003).

Crocker, Riley and Mattson (2015) conducted a study on the relationship between mathematics and aspects of WM and VS memory of children that exposed to the effects of alcohol consumption by pregnant mothers. The two groups involved were control groups and treatment groups of 56 children. They are evaluated in mathematical achievement in terms of VS memory by using neuroscience technology tools called CANTAB and used "Spatial Recognition memory" and "Pattern Recognition Memory" tests. This finding suggests that VS skills are diminishing in mathematics achievement for children who exposed to alcohol consumption by pregnant mothers. Furthermore, the impact of alcohol

on the child caused difficulty in mathematics achievement such as the weakening process in numerical skills and might interrupted the cognitive processes that focused in EF ability. This incredible disorder or illness usually occurs in the parietal part of the brain (parietal lobe) especially when it involves a calculation in mathematics function. The brain parietal part affects the mathematical function in children's cognitive development (Cantlon, Davis, Libertus, Kahane, Brannon and Pelphrey, 2011).

Geertsen, Thomas, Larsen, Dahn, Andersen, Krause-Jensen, Korup, Nielse, Wienecke, Ritz, Krustrup and Lundbye-Jensen (2016) have studied the relationship between motor skills with cognitive function domain measurements as well as academic achievement of mathematics in early childhood adolescents. A total of 423 children (9 years) were selected in this study. They used the Cambridge Neuropsychological Test Automated Battery (CANTAB) technology to measure the cognitive function domain in EF. Spatial Working Memory (SWM) test was used to evaluate how far the individual can maintained and manipulated spatial information and used strategy efficiency in mathematics problem solving. Academic achievement was tested on their ability to solve mathematics problems with 50 questions of mathematical problems (addition, rejection, multiplication, geometry and probability). The data clearly shows that the EF domain assessed by CANTAB had a significant positive relationship with the level of mathematics achievement. The students who achieved the high level scores in mathematical problem solving had a strong ability for spatial skills in manipulating information. It shows that cognitive processes can help individual to decide and control the motor in the learning process (Geertsen et al., 2016).

Implication and Conclusion

There is now increasing evidence of a strong relationship between EF and VS skills, in particular of cognitive processes in working memory and children's mathematics achievement. Specifically, EF and VS skills may be two cognitive skills that are theoretically important for children's early success in mathematics in elementary school. Some children may have trouble with processing numerical information and performing numerical tasks accurately because they may have problems in more general cognitive skills (EF and VS) rather than problems with understanding the information (Kolkman, Koresbergen and Leseman, 2013). This paper also provides empirical support for the evaluations of school based, multi-component interventions designed to address the early of learning difficulties through the EF deficits, particularly in working memory. Teachers need to provide frequent opportunities for children to engage with spatial activities in children's school curricula, such as using a diagram to create a block structure in which they can practice their one-to-one correspondence as well as their spatial skills (e.g., mental rotation). Moreover, others have suggested that instruction designed to foster a deeper understanding of mathematical concepts and to build connection between numbers, words, and ideas may consequently be a better way to enhance EF processes (Clements, 2013).

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Improvising in the Primary Mathematics Classroom

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If classrooms are dynamic, then mathematics teachers will need to improvise mid-lesson from time to time. Teachers' capacity to improvise is usually analysed via a cognitivist lens. This study contrasts a cognitivist analysis of a primary teacher of mathematics with an ecological analysis. The ecological approach was able to develop a more detailed characterisation of teacher improvisation where attention to, and manipulation of, environmental entities supported improvisation. This characterisation of improvisation is posited to have potential in developing novice teachers' capacity to 'think on their feet' in mathematics lessons.

Teachers' capacity to improvise mid-lesson has been posited to be a key element of teacher expertise which facilitates student learning in mathematics (Hatano & Inagaki, 1986). While there are theoretical frames which are argued to be able to model this capacity to improvise mid-lesson (Schoenfeld, 2011), a conceptualisation of teacher improvisation that can enable the structured development of this capacity is lacking (Grossman et al., 2009). This report compares two theoretical frameworks – Schoenfeld's (2011) model of teacher in-the-moment decision making and a framework developed from ecological psychology (Gibson, 1979) – which both seek to enable structured analysis of teachers' mid-lesson improvisation. Data collected as part of a larger study of mathematics teachers' in-the-moment practice is used to contrast each account of teacher improvisation. The data were collected using Head-mounted, Video-cued recall interviews (HMV interviews), a data collection method which was developed in firefighting (Omodei, McLennan, & Wearing, 2005), and deployed in a Foundation-level (first year of primary school) mathematics class in a primary school in Melbourne.

Literature Review

This report is the result of a lucky event that occurred during data collection. The larger study that this report draws from (Jazby, 2016) investigates how mathematics teachers adapt mid-lesson by combining both video and interview data collection techniques. It was envisaged that during the head-mounted camera data collection phase of the study, participants knew that a researcher was coming to their school to attach a camera to their head, so it would be unlikely that improvisation would occur. Luckily, given the researcher's interest in how teachers think on their feet, one of the study participants (Hannah) found that her meticulously planned lesson was not going to work, and she needed to improvise. Hannah had provided 5 and 6 year-old students with collections of 30-40 icy pole sticks. She had expected the children to group the sticks into bundles of ten to be able to count the collections of sticks effectively. Instead, the children were able to count the sticks by ones accurately and quickly, and Hannah felt that she had to improvise if she was going to be able to meet her goal for the lesson, which was to explore grouping strategies and place value with the students. The data captured from this lesson provides a rare opportunity to analyse an improvised sequence of teaching that occurred spontaneously on a day when cameras

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happened to be rolling. Given the rarity of such occurrences (Lipshitz, Klein, Orasanu, & Salas, 2001), this event provides a means to contrast and compare different theoretical models of teacher improvisation, so that the way in which each model frames the phenomena can be contrasted.

The research question which guides this study is: what characterisation of teacher improvisation in a mathematics lesson emerges when Hannah's episode of improvisation is analysed using different theoretical lenses? Two lenses are used to provide contrasting characterisations; a cognitivist lens, and an ecological lens. For the sake of brevity, one cognitivist model – Schoenfeld's (2011) theory of teacher in-the-moment decision making – is presented as being representative of cognitivist approaches. While Schoenfeld is not the only researcher to present a model of mathematics teachers' in-the-moment behaviour, his decision making model provides an example of an information processing account of teacher improvisation which is particular to mathematics education, and is well established in the field. Schoenfeld (2011) argued that teachers engage in a process of goal prioritisation which leads to the selection of particular courses of action mid-lesson. When decision points are reached mid-lesson, teacher actions can be modelled by assuming that the teacher engages in a calculation of subjective expected values relating to the options that are available to the teacher. In the situation faced by Hannah – realising that her lesson plan was not going to work several minutes into the lesson – Schoenfeld's (2011) model would characterise her behaviour as being driven by a change in goal prioritisation. A new goal, developing a new course of action for example, would be prioritised by Hannah, and she would start to engage resources such as her knowledge to think of alternative courses of action. As alternatives are arrived at, a calculation of subjective expected values can be used to select the best course of action. Schoenfeld (2011) acknowledges that this particular process (running through each option in your mind and working out the perceived costs and benefits of each course of action) cannot be performed at a conscious level in time-pressured decision making, but he claimed that this calculation enables accurate modelling of what decisions teachers are likely to make. Schoenfeld's model (2011) is characteristic of an information processing account of in-the-moment cognition (Jazby, 2016; Lipshitz et al., 2001). The main drivers of improvisation in this kind of account are internal mental processes and mental entities.

The alternate lens considered in this study is derived from ecological psychology (Gibson, 1979). Jazby (2016) developed an ecological account of mathematics teacher noticing which draws on ecological models of in-the-moment behaviour developed in the research area of Human Factors (Kirlik, 1995). In an ecological model, a person's cognition is viewed as occurring within an environment, and how a person interacts with their environment – largely through perceptual interaction – is considered as drivers of behaviour with less emphasis on internal mental processes and entities. While entities such as knowledge are not seen to be completely irrelevant to skilled task performance (Kirlik, 1995), less weight is accorded to mental entities and processes when behaviour is viewed through an ecological lens. Instead, a skilled performer is argued to have developed perceptual routines (ways of moving through a task environment and deploying their attention) which enable them to put themselves in the right place at the right time to perceive environmental entities that can provide guidance for behaviour (Gibson, 1979; Jazby, 2016). In the situation faced by Hannah, the realisation that her lesson plan would not work could result in her engaging a perceptual routine (a way of moving around the classroom and deploying her attention) which would enable her to find environmental structures that could help her know what to do next. As she perceived environmental structures (such as student movements, utterances and the mathematical representations students were making) she

would perceive these entities as being meaningful. The way a student piled counters, for example, could be seen by a teacher as indicating that student is engaging a particular type of mathematical thinking.

Kirlik (1998) pointed out that skilled performers not only are able to identify meaningful environmental structures mid-performance, they are also able to manipulate their environment in ways which create or increase the likelihood that such structures will exist. If you want to know what a student is thinking about a counting task, for example, you could direct a student to count objects and watch what they do. This would count as a manipulation of environmental structure according to Kirlik's ecological model. Unlike the characterisation produced by an information-processing model, teachers are not passively waiting for environmental structures to emerge mid-lesson in an ecological model; they actively manipulate the classroom so that particular environmental structures emerge. As a performer attends to meaningful environmental structures, they perceive an affordance structure (Kirlik, 1995) which provides a sense of what is possible within the current environment. Gibson (1979) argued that perception of an affordance is the perception of what can be done. Perception of a chair, for example, carries with it perception of 'sit-ability' with very little deliberative cognitive processing required (Gibson, 1979). Kirlik (1995) argued that as a skilled performer attends to multiple meaningful environmental structures simultaneously, they develop an awareness of an affordance structure rather than perceiving individual affordances and constraints. In Hannah's situation, Hannah is theorised to be attending to multiple meaningful environmental structures as she improvises. This will give her an awareness of what she can and cannot do mid-lesson, as her attention to her environment will enable her to 'pick up' information that can guide her behaviour more than relying on an internal cognitive process that relies on memory and information processing (Gibson, 1979).

Table 1
Primary Drivers of Teacher Improvisation in Cognitivist and Ecological Models

	Driver	Description	Relevant research
Cognitivist account	Teacher knowledge	Often described in terms of PCK, CK and PK – what teachers know about the content and teaching	Schoenfeld (2011)
	Teacher goals	What a teacher wants to do; these change as the lesson progresses	Schoenfeld (2011)
	Prioritisation of goals	A mental process which selects which goals a dominant at a particular time	Schoenfeld (2011)
	Calculation of subjective expected values	A mental calculation regarding the perceived benefit/cost of a particular course of action	Schoenfeld (2011)
Ecological account	Perceptual routines	The way in which a teacher moves and deploys attention mid lesson	Jazby (2016)
	Meaningful environmental structures	The specific structures within the classroom that a teacher sees as meaning something relevant to teaching	Jazby (2016), Kirlik (1995)
	Manipulation of environmental structures	Teacher actions which create new sources of environmental structure	Jazby (2016), Kirlik (1998)
	Perception of an affordance structure	What a teacher perceives as being possible within current classroom conditions	Jazby (2016)

When data relating to improvisation is analysed from a cognitivist or an ecological perspective, particular drivers of improvisation are posited to exist. Table 1 provides a summary of key drivers of teacher improvisation in Schoenfeld's (2011) model and contrasts them with the drivers posited by researchers who employ an ecological lens. Each lens posits a different ontology of entities which drive behaviour – a cognitivist lens directs researchers to look for mental processes and entities, while an ecological lens directs researchers to analyse perceptual behaviour and environmental structure. In order to ascertain how each theoretical lens characterises Hannah's improvisation, the same set of data are analysed, but different drivers of behaviour are used to guide coding of the data. Particularly in the post-lesson interview stage of data collection, a cognitivist lens has been used to code evidence of cognitive process or epistemic claims that can be used to develop a model of Hannah's knowledge and cognition during the period of improvisation. The ecological approach leads researchers to code perceptual interaction and the environmental structures Hannah attends to. By contrasting these two approaches to analysing the same data set, it is hoped that two characterisations of Hannah's improvisation can be developed and contrasted.

Method

Head-mounted, Video-cued recall interview method (HMV interview method) has developed from research concerned with investigating in-the-moment decision making in fields such as firefighting in the 1990s (Omodei et al., 2005). As part of a larger study (Jazby, 2016), three teachers were asked to wear a head mounted camera as they taught a mathematics lesson they had planned themselves to their regular class. Approximately 10 minutes after the lesson, teachers reviewed the head-mounted footage and provided a commentary on what they had been thinking, feeling, and attending to following a free-recall protocol developed by Omodei et al. (2005). The aim of the method is to use the head-mounted video footage to prompt 're-experiencing' of the activity of research interest, so that the detail of recall data collected is more detailed and less narrativised than data collected without use of a head-mounted camera (Omodei et al., 2005).

This study analyses data collected from Hannah's classroom. Hannah, unexpectedly, found that her lesson plan was not working mid-lesson and felt that she had to improvise. Hannah was in her 6th year of full-time employment as a primary teacher and her students were Foundation students (first year of school – approximately 5-6 years old). Hannah first noticed that her lesson plan was not working during a phase of between-desk instruction. This phase of the lesson occurred after an introduction phase of the lesson and prior to a summing up phase at the end of the lesson. In order to analyse Hannah's improvisation, data collected from the approximately 15-minute period of between-desk instruction was prepared.

During this period, three sources of data were collected: stimulated recall data collected post lesson, video data collected from the head-mounted camera, and video data collected from a fixed position camera. Data that could provide evidence of Hannah's cognitive processes – including the knowledge activated mid lesson, her goals, how she prioritises goals and her calculation of subjective expected values may be present in the recall data collected. The researcher asked Hannah to describe such processes as the free-recall interview began, and a few prompts such as, "what were you thinking?" and "what was your goal here?" were used if Hannah paused the video and singled out a segment of the lesson as being significant.

Coordination of head-mounted and fixed-position camera data enabled perceptual routines to be identified. The fixed position camera captured how Hannah moved around the

classroom while the head-mounted camera captured what was in Hannah's field of view as she moved her head. Camera data could not be used to ascertain what she was attending to. Recall data provided evidence of this, as Hannah made statements such as, "I was looking at what these two were doing" while pointing to a pair of students when reviewing the head-mounted footage. When Hannah made statements that provided evidence of what she had been attending to, the head-mounted video data provided information relating to the way in which the environment was structured when Hannah was attending to it. Hence, the video data was not used to infer what Hannah was attending to, as the post-lesson interview provided Hannah's recall of her what drew her attention mid-lesson, but by coordinating the recall and video data the structures within the environment that Hannah recalled attending to could be identified. This enabled meaningful environmental structures to be identified, and as Hannah took action, changes in these structures could also be seen in the video data.

While no particular prompts were provided regarding perception of an affordance structure, Hannah made comments regarding 'seeing' what she could or could not do during the lesson. Description of meaningful environmental structures were frequently followed by comments relating to what could and could not be done in relation to those structures. For example, Hannah recalled, "I saw them counting by ones and then I looked over there and I saw them bundling and I thought, of course the ones are going to be quicker because you're just going 1, 2, 3, 4, 5, 6, 7 and the tens are going to be longer because they actually have to stop and bundle them … so it [the lesson plan] wasn't going to work; I'm going to have to change the lesson". When viewed in terms of an affordance structure, perception of the students' counting strategies via attention to the meaningful structure of the students moving icy pole sticks also carries information about what can or cannot be done if the lesson goals are to be accomplished. Hence, these particular data have been taken as evidence that Hannah perceived a constraint in relation in continuing with the lesson plan, and this perceived constraint, which guided her subsequent behaviour, was perceived via attention to particular elements of environmental structure with negligible recourse to deliberative cognitive processing. She also perceived affordances from attention to meaningful environmental structures. When asked why she stopped the class at a particular point in the lesson, she recalled that, "I saw that they'd counted by ones but then I could hear these two boys saying something about bundling … so I wanted to use them as an example and get them to show that they'd changed their mind". The improvised class discussion that took place after Hannah perceived this opportunity for action involved asking the boys questions about their strategy and why they'd changed their minds. This was also coded as evidence of an affordance structure.

Results

Direct Evidence of Cognitivist Drivers of Mathematics Teacher Improvisation

Hannah made 5 epistemic claims during this period of improvisation. Three of these claims related to knowledge of particular students, one claim was a general claim regarding how children learn, and the last claim related to mathematics. None of the epistemic claims made directly relate to how Hannah developed an unplanned task while concurrently engaged in teaching a lesson. Hence, there are not enough directly stated epistemic claims made which would enable an account of Hannah's improvisation to be constructed.

Hannah did not recall her teaching goals without prompting during the post-lesson interview. Her stated goal was "to teach the kids about grouping into 10s". When asked whether her goal changed during the lesson, she stated, "no". As she stated that she only had

one goal during the lesson, there is also no evidence of goal prioritisation evident in the available data. There is also no evidence of calculation of subjective expected values in Hannah's recall data.

Direct Evidence of Ecological Drivers of Mathematics Teacher Improvisation

In terms of perceptual routines, at the beginning of the between-desk instruction phase of the lesson, Hannah stood at the front of the classroom and made small head movements left and right. She recalled attending to what the students were doing with icy pole sticks at this point. She claims that the realisation that the planned activity would not work occurred during this perceptual routine. She then began to walk between desks and switched her attention between student faces and the mathematical representations that students were constructing on their desks. She then directed students to move with her between desks, looking at the mathematical representations that had been created, before directing the students to stand on one side of a table while she led a discussion at three different tables centred around the representation that had been created. In this phase, she was able to keep most students' faces within her field of view. She then returned to her between-desk routine, before returning to the front of the class. When she returned to the front of the class, she split her attention between the area where the students were working, and a table which had unused manipulatives on it.

Primarily, Hannah attended to environmental structures which related to students' mathematical activity. In the recorded lesson, this meant icy pole sticks. Figure 1 shows two different ways in which students structured icy pole sticks while counting. When Hannah pointed to the image on the left during the recall interview, she recalled that this pair of students "got it" – referring to grouping by tens, which was the mathematical focus of the lesson. When reviewing the image on the right, Hannah recalled that she took this structure to mean that the students were counting by ones mid-lesson. Hence, these are examples of environmental structures which Hannah perceived as meaningful mid-lesson. Hannah's comments and gestures during post-lesson interview provides evidence of the environmental structures she was attending to mid-lesson. The images captured from the head-mounted camera, like those presented in Figure 1, provide evidence of how the environment was structured, while the language Hannah used to describe each structure provides evidence of the meaning perceived.

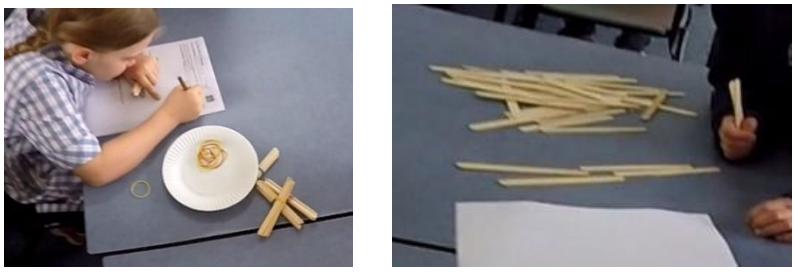

Figure 1. Student use of manipulatives which conveyed student thinking.

Hannah manipulated the task environment before the lesson began. As the previous literacy lesson ended, she added mathematics manipulatives (icy pole sticks) and worksheets to each table. As Hannah moved between desks, she gave students directions which changed the environmental structures which were present. At one point she asked all students to draw

the strategy that they had used when counting sticks. When asked why she did this, she said, "well, I know this isn't going to, you know, help them. But I'm hoping that I'll get something from this". She then engages in a between-desk perceptual routine which gives her perceptual access to the drawings that students are creating. Some of these representations were perceived as meaningful and related to student thinking (e.g. "I could see how they were making uneven groups").

As Hannah recalled what she could see or what she noticed during the lesson during the post-lesson HMV interview, she frequently described how perception of a meaningful environmental structure carried a sense of what she could or could not do. During the 15 minute segment of the lesson analysed, 17 instances could be identified where Hannah recalled what she attended to, and this could be coupled with a statement which described what she perceived she could or could not do in the lesson.

Discussion

Each theoretical lens enabled a different approach to analyse of this single data set. Contrasting the two lenses, a cognitivist approach led to limited direct evidence of the drivers of improvisation in this episode. Hannah could not recall what she was thinking or the knowledge she accessed in-the-moment in the post-lesson interview in a high degree of detail. Of course, this does not mean that Hannah was not thinking or relying on knowledge, but that even when HMV interviews – a research method which is a well regard form of Cognitive Task Analysis (Omodei et al., 2005) – are used to gather data, limited evidence of these internal mental processes can be identified. König et al. (2014) found that mathematics teacher noticing required teacher knowledge that was organised into complex mental schema which were activated with automaticity. Perhaps teacher improvisation also requires the use of mental processes that are so automatic, little direct evidence of these process can be gathered by researchers.

In contrast, applying an ecological lens to Hannah's improvisation enabled the identification of drivers of improvisation to be directly identified in the HMV data in much more detail. The way she deployed attention mid-lesson via perceptual routines could be mapped from coordination of data sources. Her perceptual routines put her in the right place at the right time to see meaningful environmental structures. These structures were described by her in the interview data and were also captured in the camera footage. She also took action mid-lesson to create environmental structures, and attention to these structures gave her a sense of what she could and could not do (an affordance structure) mid-lesson and her recall of events provides some evidence regarding what she saw as being possible during the period of improvisation. This creates a characterisation of her improvisation as being driven by providing instructions and materials to children that would increase the likelihood that meaningful environmental structures would be created during the lesson. She employed perceptual routines which increase the likelihood that she would be able to see meaningful environmental structures if they arose. When she saw environmental structures which meant 'counting by ones' on most students' tables, this gave her a sense that she was constrained in following her lesson plan (a perception of the affordance structure of the class at that point). In order to ascertain a new course of action, rather than engaging in a mental process of searching her mind for an alternative, she engaged in perceptual behaviour: she moved around the class, manipulating environmental structures, "hoping that I'll [Hannah] will get something from this". Though an ecological lens she got a sense of what she could and could not do in the environment, and her subsequent unplanned activity that was deployed on the fly, was arrived at by active perceptual behaviour which attended to what was happening in

the moment more than by engaging in any calculated or cognitively demanding internal mental analysis.

Conclusion

This study demonstrates the potential utility of an ecological framework by contrasting it with the more commonly employed approach of analysing teacher improvisation using a cognitivist framework. Teachers' perceptual interaction with a classroom can be captured and analysed using data collection techniques such as HMV interviews. Teachers' internal cognitive processing – particularly processes which are likely to be automatic – are more difficult to capture and analyse. Because ecological entities such as perceptual routines and meaningful environmental structures can be identified directly in the presented data set, a more detailed characterisation of teacher improvisation can be developed. This characterisation identifies teacher behaviours that could be directly taught to novice teachers. These behaviours could enhance novices' capacity to know where to look for information to guide their teaching mid-lesson, how to create conditions which will increase the likelihood that there will be entities which are worth attending to in the classroom, and how to think on their feet when go awry. This would address the issues raised by Grossman et al. (2009) who argued that teacher education lacks structures which render these elements of classroom practice teachable to novices. The potential utility of an ecological framework provides justification for further ecological research of mathematics teachers' in-the-moment practice.

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Speaking Spatially: Implications for Remote Indigenous Learners

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As part of a much larger study where spatial reasoning is the focus, this paper draws on the language aspects of this strand of the curriculum. The quarantined part of the project discussed in this presentation is based in remote Indigenous schools. We draw on the challenges of the concept of symmetry and where the language of instruction (and mathematics) is a foreign language. We pose questions of the nuanced, and often complex, language of spatial reasoning and the impact this has on the performance of Indigenous learners when applied to the enacted practices in school mathematics. We conclude by raising concerns and directions of the subsequent phases of the project.

The Importance of Studying Space

The spatial component of the mathematics curriculum is somewhat different from the other content areas of mathematics. Most notable – unlike number, measurement, chance and data – the space strand does not have a heavy reliance of number. While there are aspects of the space strand, such as angle, that do incorporate the use of number, by and large, this strand focuses on a different way of thinking and is very rich in terms of the spatial vocabulary that is required. It is this language component of the space strand that is the focus of this paper. We draw on one example from our larger project where we were exploring the notion of symmetry and reflection.

There is considerable pressure on teachers to be accountable and to ensure coverage of state-mandated curriculum. This is an international phenomenon, but with an overcrowded curriculum, there is often an emphasis on the study of number with the consequence that the space strand is a poor cousin in the study of mathematics. Newcombe and Frick (2010) have argued that the study of space is an important component in the development of mathematical skills. There is a strong acknowledgement that the spatial skills developed in the early years correlate with spatial reasoning in later years. For example, it has been found (Levine, Ratliff, Huttenlocher, & Cannon, 2012) that when children between 2 and 4 years engage in puzzle play, there is a strong prediction on performance on non-linguistic spatial transformations at the age of 4.5 years. This suggests that early experiences are formative in the generation of spatial reasoning, and language as will be discussed in a subsequent section.

But the study of space has been expanded with new forms of learning made possible in environments not incorporated in conventional paper and pencil work. In their innovative study using robotics, Khan, Francis and Davis (2015) have proposed that the study of space also opens up new forms of knowing through unification of the physical context, biodynamics of the body moving through space; coordination of sensorimotor skills, along with cognitive processing of events and actions. This multiplicity of learning events, they contend, makes for a different set of competencies that are made possible through the other strands of the mathematics curriculum. The space strand opens up new and different possibilities for learning mathematics. We take this importance of space as fundamental to

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this project: “Equity and Spatial Reasoning: Reducing the mathematics achievement gap in gender and social disadvantage”.

Language of Space

There is some debate as to whether language influences performance in spatial reasoning but what appears in the literature is the importance of building a strong spatial language in order to be able to compete successfully on spatial tasks (Dumitru, Joergensen, Cruickshank, & Altmann, 2013). The impact of language on spatial performance was highlighted in a comprehensive study of non-hearing learners whose spatial language was restricted (Gentner, Özyürek, Gürcanli, & Goldin-Meadow, 2013). When students were asked to perform on non-linguistic spatial tasks, it was found that there was a strong link between students’ poor performance and their lack of spatial language. Similarly, other studies have shown the importance of working with families to build spatial language (Polinksy, Perez, Grehl, & McCrink, 2017) and the positive impact these strategies have on students’ subsequent performance on spatial tasks.

The importance of learning, hearing, using the language of space is a critical aspect of spatial understandings and performance. Using relational spatial language (such as top, middle) can enhance performance on tasks where students have had to find hidden cards (Loewenstein & Gentner, 2005). In a study of early years students working with block assembly (Verdine et al., 2014), it was found that students relied on language skills in order to build various assemblies with the blocks. Such studies suggest the importance of having a strong spatial language in order to perform well on spatial tasks.

The importance of parental talk with early years students cannot be underestimated. Pruden, Levine and Huttenlocher (2011) observed (and measured) parental spatial talk with their children. It was found that the level of parental language predicted children’s spatial language as well as their performance on spatial tasks. Miller, Vlach and Simmering (2016) reinforce this view, but extend it to argue that the language relevant to the task was a greater predictor of spatial skills; it surpassed factors such as demographics and language per se. They suggest that the quality of the spatial language needs to be considered rather than just the quantity of spatial terms used. This finding has significance in terms of the research that there is a correlation between the social background of students and their performance in spatial tasks. Many authors noted the importance of language skills in relation to successful completion of the tasks, but also reported that there were differences in the students’ language skills that related to their socio-economic status (SES) backgrounds with low SES families reporting that they used less spatial language than their middle-SES peers (Verdine et al., 2014).

Implications for Equity

It has been recognised that educationally-disadvantaged students are likely to have difficulty navigating the various forms and components of spatial and graphic representations (Heinze, Star, & Verschaffel, 2009) —with such challenges heightened for female students (Hegarty & Waller, 2005). More specifically, students from low SES backgrounds have been found to be more at risk in this strand of study than their middle SES peers (Verdine et al., 2014). At the same time, it is acknowledged that teachers are challenged by the prospect of including spatial understandings in their teaching (Stylianou, 2010) and this can relate to their feelings of anxiety in teaching the content (Gunderson, Ramirez, Beilock, & Levine, 2013).

Equity and Indigenous Learners

When considering the Australian context and the intent of our project for the Indigenous cohort in the larger study, we are cognizant of a number of factors to take into account. We need to acknowledge the cultural context/s within which we work. Here, the notion of space and place are quite unique and different from that represented in the standard school curriculum. Second, we need to recognize that the students in remote contexts speak a home language (or many languages) that are not the language of instruction. As such, the ways of speaking spatially may be very different from that of the school register. The vocabularies of the home and school may be quite different so consideration needs to be made of this difference.

First, consider the cultural and geographical context. In the Australian context there was a wide range of work undertaken in the 1980s that was predominantly ethnographic in orientation. The seminal work of Watson drew attention to Indigenous ways of knowing. In particular Watson-Verran and Chambers (1989) documented the ways that Yolgnu people mapped their land according to historical and cultural events rather than the protocols used in standard mapping taught in schools. They worked extensively to document the intimate connection between land and mathematics among the Yolgnu people. Their corpus of work highlights different ways of thinking and working mathematically, spatially and culturally that needs consideration in these contexts. In work with Warlpiri people, Harris (1991) found that the people tended to use compass points more often than relative terms such as left or right, not only in reference to land, but also in relation to the personal including the body. The findings of Verran-Watson (also known as Watson, Verran, & Watson-Verran) are not unique to the Australian context and have been recorded in other Indigenous ways of knowing (Tsai & Lo, 2013).

Second, in more contemporary work undertaken by Edmonds-Wathen (2011, 2012) where the author explores the spatial language and ways of thinking spatially of the Iwaidja people, the importance of the relationship between language and space for Indigenous learners, and their teachers is highlighted. In her work with a range of remote Indigenous communities across Australia, (Jorgensen (Zevenbergen), 2016, 2017) has documented the impact of Indigenous languages on learning and the importance of language strategies used by teachers to support learning for Indigenous learners for whom the language of instruction (Standard Australian English) is different from the home language(s) that the students bring to school.

Third, we need to acknowledge that the language of the teacher, and hence instruction, may be quite different from that of the learners. In remote contexts, most of the teachers are new graduates or recent graduates, often new to remote teaching and often with little to no experience working with Indigenous learners and community. Collectively, this requires some work to be undertaken with teachers as well as students.

Space within the Australian Curriculum

Within the Australian educational context, there is now a common curriculum that provides teachers and systems across the nation with a framework for mathematics. It is broken down to year levels and content areas (strands, general capabilities, cross curriculum priorities). For the purposes of this paper, we draw on the strand component and focus on the "geometry" component and draw on the content descriptions in order to identify the content and language demands on teachers and learners. The spatial strand co-exists within measurement. The content in Space is broken into two main areas – "shape" and "location

and transformation". In organizing the curriculum, there are explicit learning outcomes that are then broken into elaborations that provide teachers with more explicit descriptions of learning. For example, at the year 3 level (Australian Curriculum Reporting and Assessment Authority, 2017) under "location and transformation" there is a general statement of content:

> Identify symmetry in the environment

The elaborations for this outcome were listed as:

> Identifying symmetry in Aboriginal rock carvings or art
> Identifying symmetry in the natural and built environment

> In the year 4 curriculum statements, students expected to be able to "Create symmetrical patterns, pictures and shapes with and without digital technologies" (ACARA, 2017).

Table 1
Content of the Geometry Strand within the Australian Curriculum: Mathematics

Shape	Prism, cylinder, cone, cube, sphere, net, skeleton, cross section
	2d shapes and 3D objects – corners, faces, edges
	Properties of shapes, objects
	Symmetry, angles as measures of 'turn'
	Regular and irregular shapes
	Classifying angles, measuring angles, comparing angles
	Connecting shapes with nets
	Construction of prims and pyramids
	Angles on a straight line, angles at a point, vertically opposite angles
	Perspectives of 3D objects and combinations of objects
	Classifying shapes via their properties
	Angle sums of triangles and quadrilaterals, co interior angles and transversals
	Parallel lines
Location and transformation	Everyday language of location and direction – turns, directions, distance
	Giving directions
	Transformations – slides, turns, flips
	Grid maps
	Scales and legends on basic maps
	Symmetrical patterns
	Describing routes using landmarks and directional language
	Rotational symmetries,
	Enlargement transformations
	Cartesian coordinates
	Describing translations, reflections rotations on Cartesian plane

Teachers across Australia, regardless of context are expected to teach to this framework. The conservative ideology behind the reform has been to suggest that remote Indigenous

students should be exposed to this framework so that ALL Australians have the same opportunity to learn and hence the same opportunity to achieve. Such an approach, while fundamentally flawed in its assumption that all learners start from the same position, has been mandated for all teachers and systems. In examining the elaborations across the geometry component, the content has been noted in Table 1. The content, and by implication the complexity of the inherent language can be inferred. It is beyond the scope of this paper to provide a comprehensive account the linguistic demands of this sub-strand of the curriculum, but suffice to say at this point that it is extensive.

A Case of "Symmetry"

From the larger project, we sought to identify students' understanding of symmetry and reflection. This is part of the Year 3 experiences in the Australian curriculum. As our targeted year levels were Years 3 and 5 it could be assumed that the topic would have been covered in the school context. The task we used was to have a series of photographs of children undertaking activities such as ballet, football, and dancing including one of an Aboriginal boy doing a dance (see Figure 1. for the actual stimulus pictures used). The task was the students to mirror the image, and then to draw it. We found the latter component difficult and time consuming so we have opted to delete this component. The task involved a series of children in various activities. The stimulus pictures were carefully selected so that body parts would be in different positions – e.g. left arm up, right arm horizontal –so that we could assess whether or not students were able to demonstrate their understanding of reflection on a line of symmetry. These pictures were selected to incorporate inclusiveness across the full project where we are looking at urban/rural, high SES and low SES and Indigenous/immigrant students. So, we sought to have stimulus pictures that would embrace the experiences of the students, including genders. Of interest is that the examples in Figure 1. were the only pictures that the remote students selected to draw.

Figure 1. Symmetry Stimulus Pictures.

Findings

Working with three cohorts of students across two remote communities where there were 3-4 students in any one group, we show them the stimulus pictures (n=10). Initially, the approach of the project was to have the students see the images as reflections around a line of symmetry. Working with the cohorts of students, it was unclear the understanding of the concept or language of "symmetry". This meant that the students were initially unable to

complete the task. Unlike other cohorts of students (urban, rural) once the term "symmetry" was used, they were able to engage with the task, with minimal intervention or teaching.

Working with the Indigenous students, the concept of symmetry was modelled using mirrors to show the reflection of the images and how they looked different and the reversal of arms and legs. Students struggled with this and the mirrors became a distraction from the task – mostly with the students looking at themselves! When the task was altered so the image was placed in front of the student (either on a desk or the ground) and students were asked to mirror the image or to reflect the image, students were able to do the task. By copying the image in a bodily manner, it appeared that the task was more accessible for the students.

When asked to explain their body position, the students tended to rely on gesture to indicate why, for example, they had a right arm up or the left foot raised. They were more likely to give a flick of the head and/or point to a part of the picture and then show their body position. The gestural explanation/justification appeared to be a preferred mode of explanation. For this part of the project, we were only in the community for a brief period so the time need to build trust and rapport was limited but we were able to have the students elicit non-verbal responses to this task that demonstrated their understandings of the task. It was clear from their body language and mirroring of the stimulus pictures that they were able to mirror the body positions. This suggests to us that the students were cognizant of the concept but the (school) language of symmetry needed to be developed.

Implications

While this project is in its trial year where we are working on tasks/activities to identify spatial reasoning of our targeted cohorts, the initial work in two remote Indigenous communities has highlighted a number of considerations for subsequent phases. At this point, we are unclear as to the role of school language and spatial reasoning when working with remote Indigenous students whose home language is different from the language of instruction. We will spend more time in community in 2018 to build greater familiarity, trust and rapport with the students and families so that we may be better able to access positive interactions with the students. We will also attempt to build tasks that are culturally more responsive to the communities and their activities so as to engage the learners in the tasks. Our attempt in the work cited here (the incorporation of an Indigenous dancer) was a positive step. We note that most of the girls picked the ballerina which has very little connection to this context and so we are curious as to how much 'culturally responsive' activity is needed to better understanding Indigenous ways of knowing around spatial reasoning.

One key consideration for the project is timeliness. The three tasks we undertook were very time intense. The task cited here took more than 30 mins to implement with small groups hence our decision to omit the drawing task from future work. The video and photographic recording of students – for example as they posed in the front of each photograph – was a very positive process as it was quick and easy to implement without stopping the flow of the activities.

Language of Symmetry

What occurred in these testing contexts was that the symmetry task was the most difficult of the three tasks undertaken. In this task, it was clear that students could show symmetry as a point of reflection, but what was problematic was the language of symmetry. Using the term initially was met with blank faces so strategies were need to scaffold the students in

order for them to access the task. While terms such as “mirror” and “reflection”, and the use of a mirror, did not scaffold the students, it was more apparent that modelling served a better scaffolding technique than the use of language per se.

Using one picture as a catalyst or prompt, students were asked to look at the stimulus picture and look at a particular body part – e.g. the arm of the right side of the picture with a point to the right. Usually two of these prompts was sufficient for the students to crack the code of the game (Zevenbergen, 2000) and engage successfully in the tasks. Through this scaffolding, language did not hinder success per se as students were able to grasp and demonstrate their understanding of the task. What is unclear at this point in time is whether the students have been able to mathematise (in a school sense) the concepts embedded in the task or were they simply copying the actions and engaging in a non-mathematical game. The discussion that followed gave us some indication that they had understood the mathematics of the task which is encouraging, even in the absence of a formal mathematical register.

To date, what we have learned is that we may need to be more open to ways of working spatially – and linguistically – for remote Indigenous learners. The formal language may not be part of their mathematical habitus at this point in time. Teachers may require support to assist them to develop this formal language using strategies identified in the Remote Numeracy Project (Jorgensen, 2017). The Remote Numeracy project preceded the current project and documented the strategies teachers and schools used to support numeracy learning. One of the key learnings from this earlier project was around the scaffolding of language and as such, the outcomes of that project could assist our current students to access and use the formal discourse of school mathematics, particularly in the area of spatial reasoning.

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Effectiveness of Applying Conceptual Change Approaches in Challenging Mathematics Tasks for Low-Performing Students

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This article reports on the effectiveness of an intervention using conceptual change approaches within challenging tasks, on the mathematics gains for low-performing year 3-6 students in in six primary schools. Quantitative data from PAT-Maths testing for each year showed a consistently large effect size of 0.7 compared to expected gain data from DECD. All six experimental groups caught up with DECD expectations within one year. Over the two years, students from years 3-5 gained an additional 27 months of mathematics learning over the expectations and students from years 4-6 gained 29 months, indicating the potential of the approach for closing educational gaps for low-performing students.

In recent decades, the assessment revolution (Broadfoot and Black, 2004) has contributed to a prioritising of quantitative data for accountability and predictability, and subsequently, an emphasis on testing. As the pressure on teachers to produce high results has increased (Gaffney & Faragher, 2014), concerns have been raised in Australian schooling regarding effective intervention strategies for closing the educational gap for low-performing and at-risk students (Ewing, 2011; Masters, 2009).

One strategy often recommended to close educational gaps for low-performing students is direct instruction (Ewing, 2011; Farkota, 2003; Masters, 2009). Direct instruction is relatively simple to implement (Ewing, 2011), however critics argue that an instructivist approach leads to overemphasis on memorisation of procedures rather than conceptual understanding (Cooney, 2001). Conversely, grappling with challenging tasks supports conceptual thinking and making connections in mathematics, thus producing greater overall learning gains (Boaler & Staples, 2008; Stein & Lane, 1996). However, research literature regarding the use of challenging tasks with low-performing students indicates a strong tendency for teachers to reduce the cognitive demand in tasks when they perceive that students may struggle (Archambault et al., 2012; Stein & Lane, 1996).

Conceptual change programs, which are gaining momentum for their high impact on student learning in science classrooms (Hattie, 2015), may provide a way for teachers to address student difficulties within challenging mathematics tasks without reducing the cognitive load (Kennedy, 2015a). While research literature regarding the application of conceptual change approaches in mathematics classrooms is limited (Swan, 2001), research by Kennedy (2015a) connected lesson structures developed for conceptual change in science (Erilymaz, 2002) with highly-regarded lesson structures for challenging tasks in mathematics (Lapan et al., 2006) to develop a model for using conceptual change questioning when students struggle in challenging tasks.

The current study used quantitative data gathered on standardised tests over a two-year period to explore the effectiveness of an intervention based on Kennedy's (2015a) model. The research explored the improvement in learning and the effect of the intervention at "closing the educational gap" (Ewing, 2011, p.66) for low-performing year 3-6 students in six South Australian primary schools.

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Literature review

The move towards greater accountability in recent decades (Gaffney & Faragher, 2014) has led to trialling of intervention strategies to improve learning gains (Farkota, 2003) and "close the educational gaps" (Ewing, 2011, p.66) for low-performing students. In a review of numeracy and literacy achievement across Australian schools, Masters (2009) found that by year five a gap of 2.5 years was observable between the lowest 20% and highest 20% of students and increased with each subsequent year of schooling. This observation led Masters to conclude that, "Australian students who slip behind in their literacy and numeracy learning during their primary years often never catch up." (Masters, 2009, p.vii).

Reports on Australian schooling recommend direct instruction as an effective intervention for low-performing students (Ewing, 2011; Farkota, 2003; Masters, 2009). With an effect size (Cohen's *d*, Cohen, 1988) of 0.60 in standardised testing (Hattie, 2015), direct instruction is an appealing strategy for school leaders who feel pressured to improve student results in such testing (Masters, 2009), thus setting the benchmark by which other interventions may be judged (Ewing, 2011).

In contrast, while research on challenging tasks has found large overall learning gains for students (Boaler & Staples, 2008; Stein & Lane, 1996), the type of evidence gathered does not easily align with the quantitative data produced by standardised testing. Australian research into challenging tasks has tended to focus on student engagement in questions that are authentic and complex rather than simple or standardised, such as those that require making connections between concepts, devising solution strategies and exploring multiple pathways to solutions (Sullivan et al., 2006; Sullivan et al., 2013; Sullivan et al., 2014). This presents a significant dilemma for schools and teachers who wish to trial challenging tasks yet are increasingly judged on improved student achievement from standardised testing (Gaffney & Faragher, 2014; Masters, 2009).

Concerns have also been raised regarding how teachers use challenging tasks with low-performing students. Researchers have observed that teachers have difficulty identifying potential barriers to learning prior to engaging in challenging tasks (Mousley et al., 2007), may be unsure how to address student misconceptions and alternative conceptions that are uncovered (Son & Kim, 2015), and tend to reduce the cognitive demand in tasks when they perceive students may struggle (Archambault et al., 2012; Stein & Lane, 1996). Creating a class culture of persistence (Sullivan et al., 2013), teacher questioning in response to correct or incorrect answers (Boaler & Staples, 2008; Swan, 2001), and the use of enabling prompts to maintain the cognitive load (Sullivan et al., 2006), have been identified as key components for supporting students who struggle in challenging tasks.

Conceptual change programs may provide a way of addressing the concerns raised by researchers with using challenging tasks for low-performing students (Kennedy, 2015a), while still producing the quantitative data valued within the current climate of the assessment revolution. Research on the impact of conceptual change programs in science classrooms have consistently produced large effect sizes of 1.16 (Hattie, 2015), indicating significant potential for closing educational gaps. When applying a similar approach to mathematics, Kennedy (2015a) found that incorporating conceptual change questioning into challenging mathematics tasks encouraged students to alter their existing conceptions.

Conceptual change approaches centre around the idea of motivating learners to change their own ideas or beliefs (Mayer, 2008; Posner et al., 1982; Resnick, 1983). They typically involve using a challenging situation or problem paired with discrepant events and questioning (Erilymaz, 2002; Swan, 2001) to create cognitive conflict (Posner et al., 1982; Resnick, 1983) whereby a learner recognizes an anomaly in his or her own thinking and then actively constructs

a new model that explains the observable facts (Mayer, 2008). In an analysis of 20 incidents of conceptual change within challenging mathematics lessons, Kennedy (2015a) identified the following common phases:

Launch: Engaging learners with a challenging mathematical problem and identifying alternative conceptions through their initial responses.

Exploration: Experimenting with learners' own ideas to solve the problem. When alternative conceptions were identified, the teacher juxtaposed discrepant events with questioning, until cognitive conflict was observed.

Accommodation: Cognitive conflict was increased with questioning and discrepant events until learners recognised that their existing ideas were anomalous or inadequate. At this point learners were observed to change their own minds and develop a new idea or conception that better explained the observable facts.

Resolution: Learners solved the initial challenging problem using their new idea.

Generalisation: Learners applied the new conception to solve a challenging problem.

The current study explores the effectiveness of an intervention based on Kennedy's (2015a) model for conceptual change in challenging tasks, on the learning gains for low-performing students. In acknowledging the current emphasis in Australian schools on improving the gains for low-performing students on standardised testing (Masters, 2009), data presented in this paper are drawn from year 3-6 classrooms from six primary schools over a two-year period using the Progressive Achievement Tests in Mathematics (PAT-M) developed by the Australian Council for Educational Research (ACER).

This paper is underpinned by several assumptions. The first assumption is that almost all students are capable of succeeding at mathematics to high levels (Askew et al., 1997). Second is the assumption that effective teaching can be defined by learning gains for students (Askew et al., 1997), and that in the end, schools are judged by increases in student achievement (Gaffney & Faragher, 2014). A third assumption is that for learning to be effective, students need to think deeply, connect ideas and be challenged (Boaler & Staples, 2008). The final assumption is that when new information conflicts with learners' existing conceptual understanding, it is either rejected outright or accommodated by changing the underlying conceptions (Posner et al., 1982; Resnick, 1983).

Method

Context

Leaders from six public primary schools in regional South Australia requested a professional learning project to improve learning gains for their students. Three of the schools were medium sized (pop. 100-300), three were small (pop. 60-100) and all were located within 50km of Adelaide. Leaders cited concerns about their students' performance on PAT-M and NAPLAN testing. They noted that student data from their first year of PAT-M testing (2015) were lower than expected, and that their numeracy data from 2015 NAPLAN testing showed low cohort gain for years 3-5 (see *Results* for more detail).

Participants

All year 3-6 teachers and students from the primary schools participated in the study. For the purpose of this report, data are limited to the learning gains made by the lowest 20% of students by pre-test scale score, in line with the approach taken in Masters' (2009) report. Each of the primary schools participated in annual online PAT-M testing from September-October

in 2015, 2016 and 2017 according to protocols set by South Australia's Department for Education and Child Development (DECD). All tests were marked externally by ACER and the data were returned to each school for further analysis. At the school level, data were examined to identify all students who participated in any two consecutive tests. These data points were deidentified, then grouped by year level for 2015-2016 and 2016-2017, forming six groups for further analysis (see *Table 1*).

Professional learning model

Helping teachers to reconsider and change their own beliefs is considered an important element in effective professional learning (Beswick, 2008). Within this project a conceptual change lens was applied to professional learning, for its potential to encourage teachers to reconsider and alter their own ideas or beliefs (Mayer, 2008) thereby influencing their practices (Hawley & Valli, 2000). Throughout eight days of live professional learning, leaders and key teachers from each school both raised and engaged with problems they had experienced with classroom teaching exposing their beliefs regarding the nature of mathematics, the nature of learning, the nature of problem-solving and the nature of teaching (see Askew et. al, 1997). Discrepant events such as observing modelled lessons using conceptual change, predicting and then reflecting on student responses, and trying out the teaching approach with students were used to help participants to reconsider and change their own beliefs, before they trained their remaining teachers. All teachers were provided with support from their leadership, two days of live professional learning with the researcher, an 18-hour sequence of webinars, and a set of adaptable lesson plans created by the researcher (Kennedy, 2015b) as a starting point for implementing conceptual change approaches. In the first year of the study the prepared lessons were used for 1-2 lessons per week by teachers, however, in the second year the teachers moved towards creating their own lessons using a similar structure.

Data analysis

PAT-M testing was selected for this study as it provided objective and norm-referenced information on students' level of achievement, their skills, and understanding of mathematics (Lindsey, Stephanou, Urbach, & Sadler, 2005), and as data gathered from PAT-M testing formed the basis for Australian research on the impact of direct instruction (Farkota, 2003). In the current study, published standards of expected achievement (SEA) developed by DECD (DECD, 2016) were substituted for control groups due to ethical and practical considerations, with effect size calculated using Cohen's *d* (Cohen, 1988).

To help quantify the impact of the intervention on closing the gaps for low-performing students, effect sizes for all experimental groups were also calculated in terms of months of additional mathematics gain over that which was expected in the SEA. As advance in achievement tends to change with students' year levels (Lee et al., 2012) the time-indexed approach to effect size for mathematics interventions developed by Lee et al. (2012) was applied to calculate months of additional gain. As an additional control, cohort gain data obtained in NAPLAN for the two years immediately prior to the study (2013-2015) were compared with data obtained throughout the study (2015-2017).

Results

Data are presented below to examine the effect of the intervention on the PAT-Maths growth in mean scale score for each experimental group in comparison to the DECD SEA. For each experimental group, data from paired t-Tests are presented to demonstrate statistical

significance, along with the means, standard deviations, and effect sizes. *Table 1* presents pre- and post-test comparison data for each experimental group, control data from the SEA and the effect size of the experiment. Standard deviation for each experimental group was calculated using pooled data across the pre- and post-test results. All experimental groups made statistically significant gains, with P figures at or below 0.001.

Table 1
Characteristics of Experimental Groups and PAT-M Data, by Year Level

	Year 3-4		Year 4-5		Year 5-6	
	2015-2016	2016-2017	2015-2016	2016-2017	2015-2016	2016-2017
n	22	21	17	21	20	16
Experimental group gain (pre-test and post-test means)	14.7 (96.8-111.6)	12.9 (98.1-111.0)	6.9 (108.1-115.0)	7.6 (104.4-111.9)	9.0 (113.9-122.9)	10.7 (111.1-121.8)
DECD SEA expected gain[a]	9 (101-110)		2 (110-112)		8 (112-120)	
Standard deviation	6.37	5.29	4.85	7.03	4.99	5.58
t Stat	-8.21	-8.46	-3.47	-4.15	-6.15	-7.70
Effect size (Cohen's *d*)	0.90	0.74	1.01	0.79	0.21	0.49
Gain in months	14	10	16	13	5	13

[a] DECD Standard of Expected Achievement Gain Data (DECD, 2016)

The first year of the project produced a large mean effect size of 0.70 across all experimental groups. This was consistent in 2016-2017 (d = 0.67). Data pooled across the two years of the project showed an additional gain of 27 months for years 3-5 students over the SEA, and an additional gain of 29 months for years 4-6 students. *Figure 1* illustrates the gains in shown in *Table 1* for each experimental group, compared with expected gains. In five out of the six experimental groups the pre-test means were lower than SEA recommendations. In all cases, the post-test means for the experimental groups were the equal to or higher than the SEA and growth exceeded the SEA.

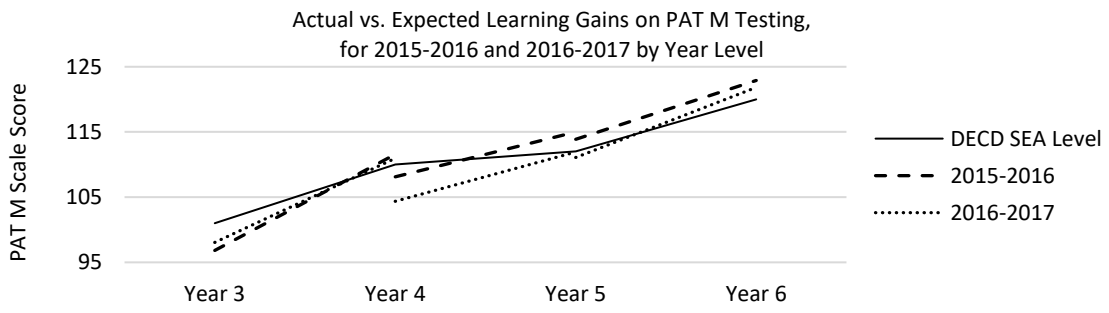

Figure 1. Learning gains by experimental group vs. expected learning gains from DECD SEA in scale score for PAT-M testing, 2015-2016 and 2016-2017.

To check the validity of the high effect size in PAT-M data, additional control data from NAPLAN testing for the two-year period prior to the study were compared with data obtained during the study. *Table 2* shows the comparative NAPLAN cohort gain data for the research cohort with the cohort gain for all South Australian schools. Due to the small size of three of the schools, some cohort gain data were not available for comparison. Each available school's cohort gain was calculated by subtracting the mean scale score of the pre-test from the post-test for the stated time-period, then compared with the same calculation made using publicly available data on all South Australian schools from the NAP website. *Table 2* below shows that the percentage of schools in the study meeting or exceeding SA cohort gain for years 3-5 rose from 60% to 83% over the period of the study. This improvement in control data from NAPLAN supports the validity of the PAT-M data.

Table 2
Comparing year 3-5 Cohort gain data in NAPLAN prior to and during the study

Percentage of schools that met the 2013-2015 SA gain[a]	3 out of 5 schools (60%)
Percentage of schools that met the 2015-2017 gain[a]	5 out of 6 schools (83%)

[a]Cohort gain calculations were made using data from the MySchool and NAP websites.

Discussion

The research reported in this article explored the effect of an intervention using Kennedy's (2015a) model for integrating conceptual change within challenging problems on the mathematics learning gains made by low-performing students in six SA primary schools. The primary finding was that the student achievement improved significantly above the expected levels on standardised testing, thus closing the educational gap faced by low-performing students.

The intervention reported on in this study produced a high effect size of 0.7 over and above the annual gain expected for PAT-M provided in the DECD SEA, for the lowest 20% of students. This was consistent across both years of data collection. As teachers are under pressure to improve student results on standardised testing (Gaffney & Faragher, 2014) PAT-M data analysis was selected for this study to enable comparison with existing research. The effect size of the intervention reported on in this study exceeded the benchmark of 0.60 set by direct instruction (Ewing, 2011; Hattie, 2015).

This study sought to quantify the effectiveness of the intervention on closing the educational gap experienced by low-performing students. Masters (2009) reported that the educational gap between the lowest and highest 20% of year five students was 2.5 years. Data from this study indicate that the lowest 20% of students from years 3-5 gained an additional 27 months over the expected growth, and the lowest 20% of students from years 4-6 gained an additional 29 months over the expected growth (see Lee et al., 2012). Over the course of the study all experimental groups were observed to catch up with the recommended SEA for PAT-M within 12 months, with the growth for all groups significantly exceeding the recommendations. These findings indicate significant potential for conceptual change interventions based on Kennedy's (2015a) model, in closing educational gaps for low-performing students.

Limitations

This study has several limitations. In terms of participants, this study contains data from a relatively small number of students, all from small to medium primary schools in regional South

Australian schools. A larger study, with more varied schools, is recommended to confirm the findings. A second limitation is that baseline data for PAT-M gain were not available for this study, so control data from both the DECD SEA and from NAPLAN were used to compensate and to check the reliability of the growth. It would be useful to repeat the study with a larger group of students and teachers for whom baseline PAT-M gain data are available to confirm the findings. A final limitation is that the researcher used her own resources for part of this study as other research and resources on conceptual change approaches in mathematics are limited. To manage this potential conflict of interest, all data were collected, analysed and reported using external sources from ACER (PAT-M), the Australian Curriculum and Assessment Authority (NAPLAN) and DECD (SEA). All data are available on request. Future reports derived from the current study are expected to report on the impact of the professional learning model on teacher beliefs and practices, as well as examining the effectiveness of interventions based on Kennedy's (2015a) model on the learning gains made by high-performing students.

Conclusion

The results of this study indicate that standardised mathematics testing results for low-performing students from years 3-6 improved when an intervention based on Kennedy's (2015a) model for conceptual change within challenging tasks was implemented for 1-2 lessons per week over a two-year period. A large and statistically significant effect of 0.7 was observed across each experimental group, for each year of the project. Over the course of the study, all experimental groups caught up with recommendations provided by DECD for PAT-M, with low-performing students from years 3-5 gaining an additional 27 months over the expected growth, and low-performing students from years 4-6 gaining an additional 29 months over the expected growth. This suggests that Kennedy's (2015a) model may provide a viable intervention for improving the learning gains and addressing the education gaps experienced by low-performing students.

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Issues of Equity in a Mathematical Inquiry Classroom

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Issues of equity exist within classrooms just as they exist more broadly within society in general. This paper looks at issues of equity affecting access to learning for girls working on challenging mathematical tasks in collaborative groups with boys. It examines measures that teachers can take to mitigate inequity and foster a culture within their classroom that supports equitable learning outcomes for all students. Aspects of Engle, Langer-Osuna, and de Royston's *Influence Framework* (2014), are used to examine issues affecting access to the conversational floor, spatial orientation of group members, and the extent to which participants respond to the ideas of the female member of a collaborative group involved in mathematical inquiry. Video stimulated reflective dialog (VSRD), was employed to elicit responses and impressions from the female student regarding issues of equity that arise during group interactions. This study highlights the dynamic nature of status within heterogeneous groups, and evaluates the effectiveness of various teacher interventions in relation to equity within a mathematics classroom.

The *Developing Mathematical Inquiry Communities* (DMIC) formative intervention project facilitates teachers to develop culturally responsive and collaborative environments that effectively engage diverse learners in mathematical reasoning, leading to increasingly equitable achievement outcomes (Alton-Lee, Hunter, Sinnema, & Pulegatoa-Diggins, 2011). High value is placed on cultural diversity and the inclusion of all students in the class, with learning contexts that are drawn from the rich cultural milieu of students' lives. In such a setting, increased onus is placed on students, both in terms of the co-construction of mathematical learning, as well as the management of difficult social dynamics within groups, such as sharing their reasoning with others, agreeing, disagreeing, listening to the ideas of other group members, taking turns and collaborating towards a shared solution path. Research studies (e.g., Banks et al., 2001; Boaler, 2006; Brown & Redmond, 2016; Engle, Langer-Osuna, & McKinney de Royston, 2014; Esmonde, 2009; Franke et al., 2015, Layva, 2017) show that the complex nature of interactions within heterogeneous collaborative groups can be difficult to navigate, leading to some students being marginalised or isolated within their group, inhibiting equity of access to learning. For girls, these challenges can at times be even greater as they are faced with issues around unfair distribution of talk time, social and physical dominance of boys, and the reluctance of some boys to value or engage with the ideas of girls (Esmonde, 2009; Langer-Osuna, 2017; Leyva, 2017; Radovic, Black, Salas, & Williams, 2017).

This study will examine issues affecting equity for girls working in collaborative groups with boys and ways in which teachers can promote equity within a mathematical inquiry community, to ensure that all students have access to learning. For the purpose of clarity, this paper defines equity as "a fair distribution of opportunities to learn" (Esmonde, 2009, p.1010).

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Review of Literature

While in New Zealand, statistics point to girls achieving higher levels of academic attainment in mathematics than boys (Ministry of Education, 2016), narratives of male dominance in mathematics, and the idea that mathematics is a masculine domain still persist (Leyva, 2017; Radovic, Black, Salas, & Williams, 2017; Louie, 2017). The practice of ability grouping which is widespread in New Zealand primary schools (Anthony, Hunter, & Hunter, 2016), reinforces this masculinisation of mathematics through the implied or explicit values that this grouping practice promotes (Leyva, 2017). A learning environment that places high value on completing tasks quickly, without communication, as well as student awareness that they are artificially stratified into groups according to their perceived ability, fosters a culture of competition and exclusion within ability grouped classrooms which may be better suited to boys (Radovic et al., 2017; Louie, 2017).

Louie (2017) argues that ability grouping is a mechanism for cementing socially constructed hierarchies around gender, status, language, authority and ethnicity. Such social hierarchies exist within a classroom, just as they exist in society in general, partially because the underlying values of education systems are themselves socially constructed to reflect the values of the dominant culture (Louie, 2017). Students' attitudes and behaviours towards their peers demonstrate these socially constructed ideas and beliefs about masculinity and femininity most acutely with seven to eight year olds (Skelton et al., 2009), who constitute the focus of this study.

While the *DMIC* approach deliberately addresses inequity of learning outcomes for diverse students (Anthony et al., 2016), issues of equity still affect access to learning for some students as they work together in heterogeneous groups. This is largely because students can experience difficulties as they collaborate, communicate, share ideas and take turns with their peers (Anthony et al., 2016; Boaler, 2006). In such situations, they are required to exhibit a set of prosocial skills that enable equitable interactions within the group so that mathematical reasoning can take place (Anthony et al., 2016). However, these skills may not yet be adequately developed (Boaler, 2006). Although Walshaw (as cited in Esmonde, 2009), suggests that girls may be better suited to cooperative learning tasks, such as those found in a *DMIC* classroom, groups are not usually comprised exclusively of students of the same sex, and girls can become marginalised in a group even if there are more girls than boys (Esmonde, 2009).

Langer-Osuna (2017) suggests that access to the conversational floor can be restricted because of a "hubris penalty" (p.242), affecting who is allowed to exhibit social or intellectual authority within a group, and that girls frequently have to display more competence than boys in order for their ideas to be listened to. Girls can be prevented from participating because they are positioned by boys as low status within the group (Leyva, 2017). This restricted access to having ideas heard has implications for students working in dialogic groups, where learning is co-constructed. If, as Cohen (1998) argues, students who "talk more, learn more" (p. 19), then this points to an imbalance of equitable access to learning based on gender.

In order to effectively mitigate gender inequities in a mathematics classroom, teachers must attend to non-mathematical factors such as the equal sharing of intellectual authority, allowing equitable access to the conversational floor, the inclusion of all students, and the positioning of diverse learners as experts, in order to ensure equitable access to learning (Langer-Osuna, 2017; Brown & Redmond, 2016; Mueller, Yankelwitz, & Maher, 2011; Esmonde, 2009). To do this, teachers must be knowledgeable about students' cultural and family background, as well as their interests (Banks et al., 2001; Cohen, 1998). Teachers

need to design tasks that are engaging, relevant and connected to students' experiences (Bills & Hunter, 2015). This knowledge of students' lives becomes increasingly important for teachers wanting to ensure equity for girls who are experiencing difficulty in being listened to when working in a group. By actively positioning girls as experts, through both effective task design and specific teacher talk, teachers can publicly assign competence to individuals, calling on their cultural or experiential background to reposition them as intellectually authoritative (Cohen, 1998).

Increased equity can most effectively be accomplished by creating a classroom culture where diversity, student voice, and equity are explicitly valued above competitiveness, speed, conformity and exclusivity (Boaler, 2006). By reframing classrooms as places where equity is highly valued, teachers can promote a growing awareness for students that prosocial attributes, such as listening to others (Franke et al., 2015), showing fairness, being empathetic and compassionate, taking turns and valuing the ideas of others have positive implications not only for effective participation in a collaborative mathematics group, but also for future life as a member of a diverse society (Boaler, 2006; Louie, 2017).

Methodology

This study was conducted with students from a culturally diverse, lower socio-economic, state integrated Catholic primary school in a main centre in a suburban area. The students involved came from Niuean, Māori, Latin American and Indian cultural backgrounds. Three students were selected for this case study, one girl and two boys. Although the interactions of three students are analysed, the experiences and impressions of the female student, in terms of equity, are the main focus of this case study, with the aim of answering two key questions: What factors affect equity for girls working in collaborative mixed ability groups? And, how can teacher interventions promote equity in a mathematical inquiry community?

The study required a student to reflect on interactions within their dialogic, heterogeneous group while they worked on a culturally connected, challenging mathematical task. Video stimulated reflective dialogue (VSRD) was used (Hargraves et al., 2003; cited in Pratt, 2006), so that students could have uninterrupted access to their group work as it took place. The small group work was 25 minutes long. Following the session, the female participant was interviewed to gather reflections and impressions about aspects of the lesson video while watching it back with the interviewer.

Three components of Engle, Langer-Osuna, and de Royston's *Influence Framework* (2014) form the basis of the analytical approach used in this case study, with the aim of building an understanding of girls' experiences in terms of equity, when working in mixed gender collaborative groups:

How much access does the participant have to the conversational floor within the group?

To what degree do other students within the group take up or respond to the ideas of the participant?

How is the participant spatially oriented in relation to other students within the group?

The lesson video and the VSRD student interview responses are analysed separately in this paper. Firstly, the lesson videos are analysed in relation to the three components of the *Influence Framework*. To do this, group interactions are organised into four episodes, and recorded as numbered proposal negotiation units to present a narrative of group interactions (Engle et al., 2014). Secondly, VSRD is used to elicit responses from the female participant. Interview questions relate to the three components of the *Influence Framework*.

Findings and Discussion

Episode One

The problem (Figure 1) had been introduced by the teacher and the group had time to think about the problem individually before they talked with the group.

Rosary Beads Task October is the month of the Rosary. Room 8 is planning to celebrate the month of the Rosary by making a set of Rosary beads for each student. We have to figure out how much string we will need, and how many beads...We will be working in groups of three. How much string will we need for three students to make their Rosaries? How many beads will we need for three students to make their Rosaries?

Figure 1. Rosary bead mathematics task.

The initial episode shows how Luci's idea was assessed by Dev and Jordan and how she was positioned as a lower status member of the group:

1 Luci:	We could measure our necks like this...
2	...to find how long the string should be
3	[puts the measuring tape around her neck and makes the shape and size of a Rosary]
4 Dev:	But we're not going to wear it
5 Jordan:	Yeah we are
6 Dev:	No we're not
7 Luci:	Yes we are
8 Dev:	No we're not
9 Luci:	Okay
10	Wait, I thought we were going to wear them
11 Jordan:	[picks up beads]
12	We need to do this
13	[looks for different coloured beads]
14 Luci:	[Keeps working with the measuring tape to find the length of string needed to make the Rosary but does not talk about her idea with the boys]

The idea put forward by Luci was of merit and could have led to further discussion among the group. However, Luci's idea was immediately challenged by Dev. Langer-Osuna (2016) suggests that when girls give directions to male peers, they are frequently rejected as inappropriate, adding to feelings of marginalisation within a group. Luci initially defended her idea, but then publicly agreed with Dev, demonstrating an acceptance that Dev had higher status and more intellectual authority than she did. Luci continued to work with the measuring tape to find the measurement that the problem asked for, but she did not talk about what she was doing with the others. Although Luci was part of a social group, her mathematical practice became private because her idea was devalued by her peers.

For the following 5 minutes Luci spoke for a total of 16 seconds and was given the pen by Jordan, so that she could record his ideas. While her spatial orientation remained consistently open to the group throughout this episode, the boys turned away from Luci, to face each other. When asked about this episode Luci responded:

> Jordan told me to write his ideas, but he didn't ask about mine... I felt weird because I wanted to have my ideas heard. They're not answering the question. When I talked they cut me off… I felt angry and disappointed… Jordan is in charge because he decided to be in charge.

The interaction positioned Jordan as the most authoritative and Luci as the least authoritative, coerced into a subordinate role despite the merit of her initial idea. For Luci, having her idea rejected led to her withdrawal from the conversational floor. She was assigned a role that did not require her to share her thoughts and her equitable access to learning was denied by both Dev and Jordan because her ideas were not valued, and she was positioned as low status.

Episode Two

Following nearly two minutes of Luci's silence, and in the absence of a solution path, a teacher intervention was used in an attempt to reposition Luci as a "local authority" (Engle et al., as cited in Langer-Osuna, 2017, p.239) in the context of the mathematical task. Prior knowledge of the student's family life enabled the teacher to call on Luci's knowledge and experience of using a measuring tape with her nana:

15 Teacher:	Luci, have you seen anyone using a measuring tape before?
16 Jordan:	I have
17Luci:	Yes. My um…
18 Jordan:	I have. My mum...
19 Teacher:	That's good but hold on please Jordan, you are interrupting Luci
20 Luci:	My nana makes clothes so she uses it
21 Teacher:	Luci has seen her nana use a measuring tape, so she might be a good person to listen to when you're working together on this problem
22 Jordan:	[Gets up and moves away from the group to get a tissue...
23	Comes back and sits down with the group]
24 Luci:	We could measure around your neck to see how long each Rosary will be
25 Jordan:	[blows his nose while Luci is talking]
26 Dev:	[Gets up and leaves the group...
27	...walks outside the room]
28	I just need to go toilet
29 Jordan:	Okay
30	[Gets up to put his tissue in the bin...]
31 Luci:	It's about 74 centimetres...
32 Jordan:	[Returns and sits down]

Apparent in this episode is how spatial positioning reflects attitudes about intellectual authority (Engle et al., 2014). While Luci's idea was logical and presented in a clear way with the use of appropriate materials to support her reasoning, both boys were unable or unwilling to engage with her idea either verbally or spatially. By proposal negotiation unit 30, Luci was explaining her reasoning, but neither of the boys were present to listen to her. Therefore, while the teacher repositioning of Luci as a local authority did give her increased access to the conversational floor, the boys quickly denied her that access by removing themselves spatially.

Luci explained what she thought about this episode:

> He's being rude to me but I keep talking anyway. I felt annoyed because he's not even listening to me. It's disrespectful. I listened to him but he didn't listen to me.

In terms of equitable access to learning, in this episode all of the students were prevented from meaningfully engaging with mathematical reasoning, because there was a lack of the necessary prosocial skills required in order for equitable group-work to take place (Anthony et al., 2016).

Episode Three

This episode outlines a teacher directed group conversation where students were asked to reflect on, and share their thoughts about the importance of fairness and respect when working collaboratively with others:

33 Teacher:	Luci, how do you feel about the way your group are working together on this problem?
34 Luci:	I feel angry… because I listen to their ideas and they don't do that for me
35 Teacher:	What do you think about that Dev and Jordan?
36 Jordan:	I think it's bad because it's not nice and you should listen
37 Dev:	I think I'm sorry Luci
38 Luci:	It's okay
39 Teacher:	Is it fair to not listen to other people in your group?
40 Luci:	No
41 Dev:	It's not fair because we need to share
42 Jordan:	Like share your idea
43 Teacher:	So how can we show respect to each other when we are working together?
44 Jordan:	Be fair to them
45 Luci:	Listen to their ideas

During the teacher intervention, students articulated their awareness of the importance of showing respect and listening to other peoples' ideas. The aim of this episode was to redistribute status within the group, in order to best ensure equitable access to learning for all students. By directly and explicitly raising the issue of fairness, the teacher facilitated a group conversation where ensuring equity became a focus for the group as they continued to work on the task, leading to greater access to learning for all group members (Langer-Osuna, 2016).

Episode Four

The following episode demonstrates how an explicit discussion about prosocial skills and fairness led by the teacher facilitated the group to begin to work in a more effective manner and re-positioned the students more equitably in terms of their status within the group:

46 Luci:	If we all have measuring tapes we can all measure how much string we will need
47 Dev:	Okay [picks up a measuring tape]
48 Jordan:	[Picks up a measuring tape and begins measuring around his neck]
49	My one is almost up to 100 centimetres
50 Dev:	Mine is 84 centimetres
51 Luci:	Mine is 74 centimetres
52 Jordan:	Is this R.E or maths?
53 Luci:	So yours is 100 centimetres?
54 Jordan:	Yeah
55 Luci:	[Records on sheet of paper]
56 Dev:	And mine is 84 centimetres
57 Luci:	[Records on sheet of paper]
58	And my one is 74 centimetres
59	[Records on sheet of paper]
60	Okay, so we need to add this all up
61 Jordan:	Okay, I think it's going to be in the two hundreds

This episode demonstrates a more equitable interaction between group members. Luci repeats her initial idea, but this time the boys listen and respond. They follow Luci's direction and engage in a mathematical discussion towards a shared solution path. In this example, the

teacher intervention was not about positioning Luci as a local authority, but more explicitly about equity. Luci reflected:

> It was more fun because we did it together. We listened to each other.

Student interactions from episode four show that they could take turns, share ideas and discuss a solution path. Spatial equity is another feature of this episode (Esmonde, 2009). Students stayed together, facing each other as they used the measuring tapes, and Luci recorded measurements on a sheet of paper placed in the middle of the group. The conversational floor was shared between the three students, and all mathematical contributions were valued verbally by other group members. The ideas of all participants contributed to the success of the group as a whole.

Conclusions and Implications

The four episodes outlined in this case study provide insights into issues affecting equity for girls participating in collaborative problem solving. They also demonstrate that teacher interventions can influence the ability of a mixed gender group of students to work effectively and equitably towards a shared solution path. While the repositioning of Luci as a local authority did eventually have an impact on the success of the group, it was only after the teacher had explicitly raised issues of equity, drawing out what the students already knew about the importance of listening, taking turns and showing respect to others, that the group was able to work productively and engage in shared mathematical reasoning.

Episodes one and two show that the boys were initially unable, or unwilling to engage with Luci's idea, regardless of its merit. Rather than participating in a mathematical dialogue about why they did not think her idea was worthy of attention, they instead denied her access to the conversational floor by either marginalising her spatially, or by discounting her suggestion without mathematical justification. Skelton et al. (2009) suggests that seven to eight-year-olds draw particularly heavily on societal gender stereotypes when they interact with each other. While it is difficult to measure the validity of this position within the scope of this case study, it may be fair to suggest that aspects of the dominant culture's gender narrative exist within any classroom, and that student interactions may be affected by these subtle discourses (Leyva, 2017). Episodes three and four demonstrate the impact that attending to social equity can have in engaging students more effectively in mathematical reasoning, and ensuring that all students can access learning opportunities.

There are a number of implications for teachers in mathematics classrooms where learning takes place in collaborative, dialogic groups. Firstly, because students are required to work collaboratively in an environment where risk taking and the sharing of mathematical reasoning occurs, the classroom culture needs to place high value on equity, and respect for the contributions of all students. To do this, teachers need to explicitly attend to matters of equity, both as part of the setting up of class norms, as well as when issues arise within collaborative groups. Students' attitudes and ideas about masculinity, femininity and gender should be openly discussed and challenged in relation to equity. Similarly to the DMIC formative intervention's encouragement of task design contexts that empower students from diverse cultural backgrounds (Bills & Hunter, 2015), mathematical problem contexts employed by teachers should serve to foster positive ways for students to relate to one another, regardless of gender, with the overall goal of creating a classroom culture where diversity in all forms is a focus for celebration rather than an area of prejudice and misunderstanding.

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What Can We Learn about Students' Mathematical Understanding from their Writing?

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Growing from the commognitive framework, this study is concerned with objectification – a special way of talking about mathematical objects, which is key to one's concept formation. The study explores how objectification can be manifested in the discourse on square roots that unfolds in writing. The data comes from a class in a foundation programme where 11 students worked on a specially-designed assignment after intensive engagement with roots. The findings point to the task-dependency of students' written talk about square roots, a struggle to coordinate words and symbols into coherent narratives, and an avoidance of verbal formulations. Theoretical and practical implications are drawn.

Introduction

As mathematics educators, we seem unanimous in our calls for learning mathematics with understanding and for developing deep, strong and well-connected mathematical knowledge among all our students. Yet, more often than not, we produce and absorb stories about students' struggles to reach the desired levels of understanding and knowledge quality. While coming from different countries, classrooms, and even decades, it is rather remarkable how similar such stories can be. Their robustness in space and time evidences the complexity of our educational enterprise; a complexity that requires not only innovative technological and pedagogical approaches for coping with it but also new theoretical lenses for making sense of it. Indeed, what does it mean to "learn mathematics with understanding" and how does "deep, strong and well-connected mathematical knowledge" look like when metaphors are put aside? I propose that answering these questions requires epistemologically solid theories with operational definitions that allow researchers to communicate effectively and be accountable for recommendations that they offer to practitioners.

The discursive framework of Sfard (2008) may be considered as an instance of a theory with the above-mentioned characteristics since it has been acknowledged for providing a comprehensive system of insightful conceptualizations of mathematics and its learning (e.g., Güçler, 2014; Nachlieli & Tabach, 2015; Shinno, 2018). For example, in relation to what is colloquially called "concept understanding", the framework introduces the notion of *objectification* – a special way of talking about mathematical objects as living outside of a human discourse, just as their material congeners. Indeed, mathematically competent discursants might not be aware of how similar their communication about mathematical intangibles and perceptually accessible things-in-the-world sound (as an exercise, notice the structural and syntactical similarities between the sentences "an absolute value of x is the square root of x squared" and "Rexy is the dog of the neighbours"). Such objectified talk, however, is atypical to newcomers who encounter a mathematical object for the first time (Sfard, 2008).

Through multiple examples of small children using numbers as part of their talk, Sfard (2008) illustrates how objectification can be used for analysing the development of their numerical discourse. Yet, it is less clear how this kind of analysis can be conducted in the case of mature students who engage with more advanced mathematics and how objectification can manifest itself when one's discourse unfolds in writing.

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The study reported in this paper explores this overarching interest in the case of a class of students in a foundation programme who engaged with square roots. Two complementary reasons yielded the decision to focus on this topic. First, in Kontorovich (2018a), I position roots as a cross-curricular concept that students encounter several times in their mathematics education journey, but the definitions (and consequent properties) of roots can change radically from one encounter to another (e.g., real roots in arithmetic and algebra, root functions in calculus, complex roots). Hence, different cohorts of school teachers and university lecturers might benefit from an evidence-based picture of students' engagement with this rarely explored concept (see Shinno, 2018 for an exception). Second, as it probably is in most cases where an algebraic entity is under consideration, symbolism is ingrained in the discourse on square roots. By untangling the relation between symbolic and verbal counterparts of students' discourse, this study offers a theoretical refinement of the notion of objectification as it emerges from a written communicational medium.

Commognitive Framework in a Nutshell

In her framework, Sfard (2008) positions mathematics as a collectively maintained discourse, which is in a constant flux because its participants differ in their aims, thinking, and commitment. A participation in the discourse requires an individual to communicate with others and with oneself, with the latter being defined as *thinking*. Hence, the neologism *commognition* is often used in regard to the framework as a combination of "communication" and "cognition".

Unlike colloquial discourses that often revolve around material entities, the commognitive framework posits that mathematical objects are discursive – i.e., they come into being through humans' words, symbols, narratives, and routines. As it has been mentioned in Introduction, the discursive nature of mathematics is easy to miss due to the objectified ways competent discursants communicate about its objects. Specifically, the ubiquitous mechanisms of *reification* and *alienation* make the objects sound as capable of a mind-independent existence: "Reification is the act of replacing sentences about processes and actions with propositions about states and objects" (Sfard, 2008, p. 44). Operatively speaking, reification is manifested in one's usage of mathematical words as nouns rather than as adjectives and verbs (e.g., "the square root of 9 gives us 3" is a reification of "after extracting the square root from 9 we got 3[5]"). Alienation erases the human agency from a narrative, which results in impersonal sentences (e.g., "the square root of 9 is 3"). To clarify, Sfard (2008) argues that objectification is an unavoidable feature of human discourses, which allows them to grow in communicative and practical effectiveness. Indeed, notice how reification and alienation compress the sentences in the examples into the concise $\sqrt{9}=3$, which can be handled now as an object. In this way, the discursive changes that someone goes through when communicating about mathematical objects are interpreted by commognitive analysts as tangible evidence of this someone's *learning*.

The notion of mathematical object has been used extensively until now but it has not been operationalized yet. The commognitive operationalization uses the notion of *signifier* – a perceptually accessible entity that can be realized into another signifier, which in turn, can be realized into another signifier, and so on, forming a *realization tree*. Thus, a *discursive*

[5] As surprising as it may sound, literate discourses on square roots vary in different countries (Kontorovich, 2018b). According to the Israeli school curriculum, in the field of reals, a number b is a square root of a if $b^2=a$. The expression "*the* square root" and '$\sqrt{}$'-symbol are used to refer to non-negative roots only. In this way, the symbolic statement $\sqrt{9}=3$ is correct when it represents the function $f(x)=\sqrt{x}$ at $x=9$ and when it is considered as an operation between numbers.

mathematical object signified by *S* is a personal construct consisting of the realization tree of *S* within a particular discourse. Let us consider an example: a student was assigned with a question "Give the square root of x^2" and she wrote "$\sqrt{x^2}$". From the commognitive standpoint, the production of the answer involved at least three signifiers that were successively realized by the student: the written "square root", the phonetical "skweə ruːt" and the symbolic "$\sqrt{x^2}$". Our dissatisfaction with the student's answer may be captured in terms of the "length" and "richness" of her realization tree – two criteria that attest to the quality of her discourse. Another quality criterion might pertain to the situatedness of her realization trees – i.e., how stable they are when the same signifiers are mentioned in situations involving different interlocutors and interactions.

Research Aim

The previous section shows that the notion of objectification can be analytically powerful in cases where a teacher or researcher has access to students' oral talk. Yet, in many classrooms, written communication is the accepted medium for students to demonstrate their mathematical proficiency and for a teacher to provide a constructive feedback. Then, the aim of this study is to characterize how reification, alienation, and objectification can manifest themselves in students' written narratives; specifically, through the use of symbols and words.

Method

The data for this study comes from a class of 11 students, who at the time of data collection, were enrolled in a foundation programme affiliated with a large technological university in Israel. The participants were eighteen- and nineteen-year-olds who finished school with the minimal mathematical requirements of the national educational system. The participants can be considered as typical students of a foundation programme since it is intended for those, whose school achievements are not sufficient for getting accepted to the universities and faculties of their choice. The programme provides its students with an opportunity to apply for academic studies based on their achievements in intense foundation courses in mathematics, physics, and English. As part of the mathematics course, ten teaching hours are dedicated to roots, where students engage in solving hundreds of questions on the topic. The data was collected nearly three weeks after the topic was covered.

I collected the data with an assignment consisting of 17 tasks. In the first 15, the students were asked to extract square roots from numbers and parameters. Due to a special choice of numerical values and algebraic structures (see Figures 1a and 2a for examples) the computational complexity of these tasks was substantially lower than the ones that the students encountered as part of their programme studies beforehand. The last two tasks were concerned with the radical symbol and the definition of a square root. These two tasks, as well as the request to explain their symbolic manipulations in writing, might have been less familiar to the students. I distributed the assignment in one of the mathematics lessons and asked the students to work on it individually. While their work was not time-limited, all students submitted their assignments in 25 minutes.

The principles of commognitive research were employed for analysing students' responses (Sfard, 2008). Specifically, I systematically contrasted students' use of symbols and words and traced the changes that their narratives underwent throughout the assignment. This analysis resulted in categories that were interpreted with the theoretical framework and led to Findings. The excerpts that I use to illustrate the findings in the next section are

translations of the responses that the students provided. In translation, I aimed at preserving the idiosyncratic structure of students' narratives even when it came at a cost of violating the rules of English grammar.

Findings

Due to space limitation, I structure this section around two findings that emerged from the analysis of students' assignments: the struggles with objectification and words, and the task-dependency of objectification. Excerpts from the assignments of Anna and Betty (pseudonyms) are used to illustrate the findings.

Struggles with Objectification and Words

Figure 1a illustrates a square-rooting routine that was identified in Anna's responses to the tasks in which square roots were extracted from squared numbers and parameters (i.e. $\sqrt{\blacksquare^2}$): she started with converting the radical symbol to the power of half, followed with reducing the powers to 1, and concluded with the number or parameter that has been squared initially. In this way, the length of Anna's symbolic strings was more or less the same in all the tasks. Accordingly, while she showed robustness when simplifying '$\sqrt{x^2}$' into 'x', Anna's systematic adherence to compound symbolic chains indicates that there is no immediate link between the two signifiers in her realization tree and she needs a multi-step procedure for converting one symbol into another. In contrast, when roots were extracted from square numbers (e.g., $\sqrt{169}$), Anna provided immediate answers without capturing any procedure in a form of a written text.

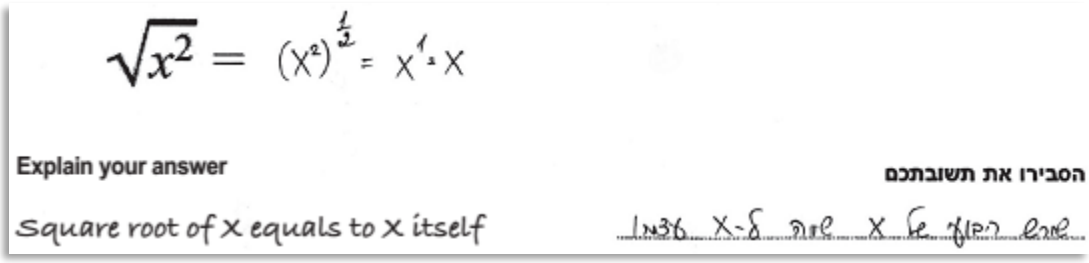

Figure 2a. Excerpt from Anna's assignment and its translation

Let us attend now to Anna's narratives that involve words for exploring whether they reflect her reliance on a multi-step realization procedure described above (see Figures 1a, 1b, and 1c for examples). When contrasted with the narratives of her classmates, two features become notable in Anna's responses: First, her usage of hybrid signifiers that combine words with symbols (i.e. "square root of x", "the expression $\sqrt{a}$" and "Expression $\sqrt{}$"). Second, while some of the students wrote that "roots do" or "are" something (see Figure 2a as an example), Anna uses the verb "equal", which is characteristic to the symbolic medium. Accordingly, I claim that her narratives are verbalizations of procedures, in which some symbols are converted into other symbols. Indeed, the narrative in Figure 1a is a summary of the simplification process that she carried out. In Figure 1b, Anna converted the verbally expressed relation between a and b into symbols, and then converted it once again into a narrative where some symbolic signifiers were copy-pasted and others were translated into words. A similar instance is evident in Figure 1c, where the radical symbol is described as turning into power. In this way, in none of the 17 tasks in the assignment, Anna exhibited an

objectified talk, in which square roots are treated as extra-discursive objects, when words and symbols are tangible means for capturing the mathematical intangibles. Instead, her narratives revolved around root-related symbols as they were the objects themselves.

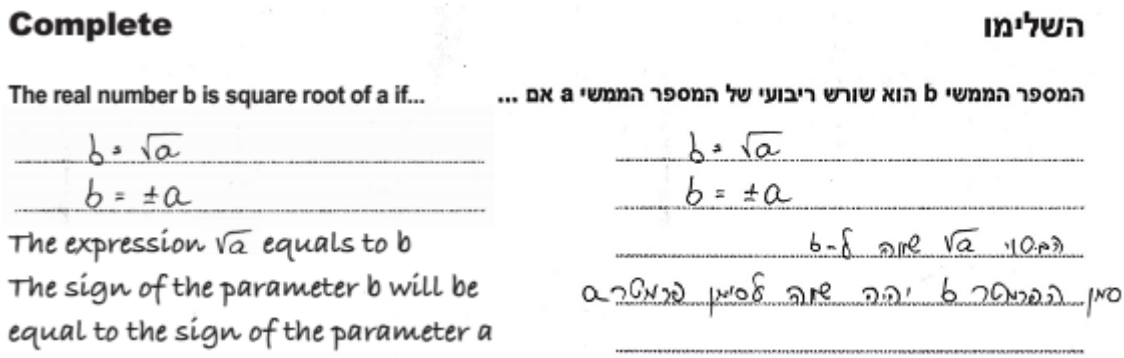

Figure 3b. Excerpt from Anna's assignment and its translation

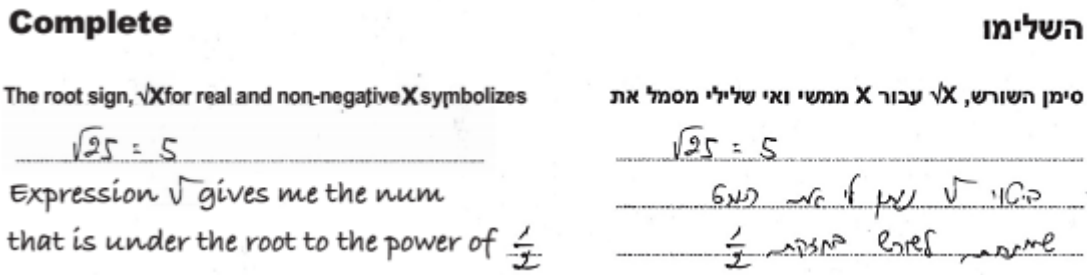

Figure 4c. Excerpt from Anna's assignment and its translation

While my interpretation of Anna's verbalizations may sound critical, her attempts to verbalize deserve appreciation. Indeed, the responses of six students to the assignment were purely symbolic. Their explanations to the simplification tasks consisted of elaborated symbolic strings that started with the assigned prompts and ended with symbols that the students provided as final answers. Furthermore, the students did not provide any response to the last two tasks, which were the most "wordy" tasks in the assignment. This preference to symbolism and avoidance of words might be interpreted as an indication of a struggle to mathematize through written texts.

Another aspect of the struggle pertains to mismatches between students' symbolic and verbal narratives. In Figure 1a, for example, Anna's narrative is concerned with x, while her symbolic string is prompted by x^2. In Figure 1b, the two symbolic sentences seem to contradict. Anna's verbal narrative clarifies that she realizes "$b=\pm a$" into "equal signs" rather than "equal values".

Task-dependency of Objectification

I switch now to three excerpts from Betty's assignment. Figure 2a and 2b illustrate that in the first simplification tasks she provided immediate answers. This suggests that Betty's realizations for the assigned prompts were automated. Her explanatory narratives, in turn, slightly differed in their degree of objectification. In Figure 2a, we witness her copying the assigned prompt, crossing off the power and the radical symbol, and providing a verbal narrative using "the root operation". In the narrative, 'root' has the status of an adjective, which might suggest a not fully reified usage of the word. Alternatively, it may be suggested that Betty's narrative compresses the process that she went through when engaging in the

task of explaining her answer rather than in the *task of providing* it in the first place. Indeed, in Figure 2b, no signs of a process are evident in Betty's symbolic writing, which aligns with her using the signifiers "the root" and "roots" as nouns. Similar traits were found in five additional responses that Betty provided.

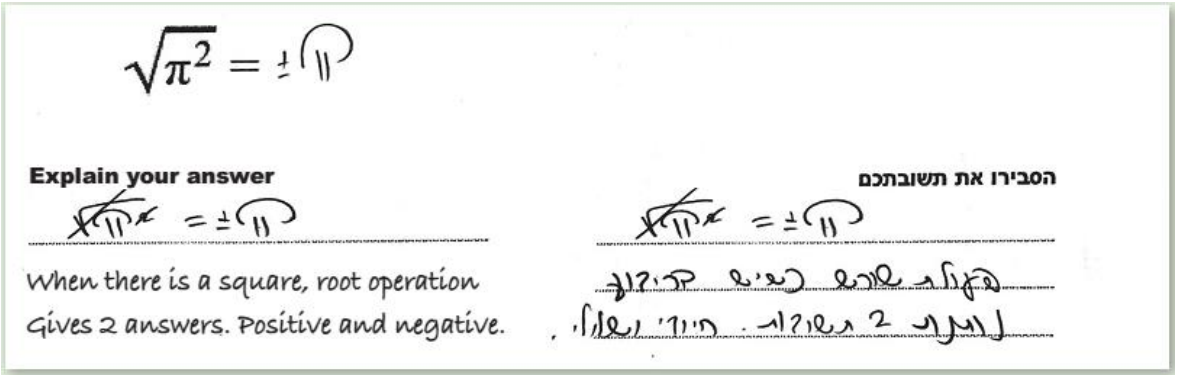

Figure 2a. Excerpt from Betty's assignment and its translation

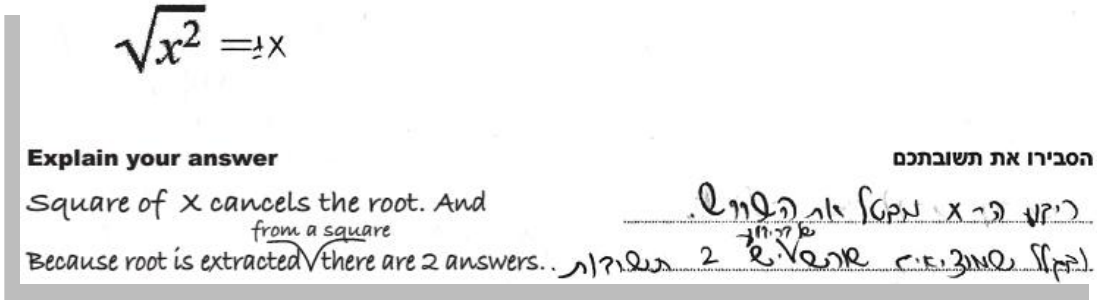

Figure 2b. Excerpt from Betty's assignment and its translation

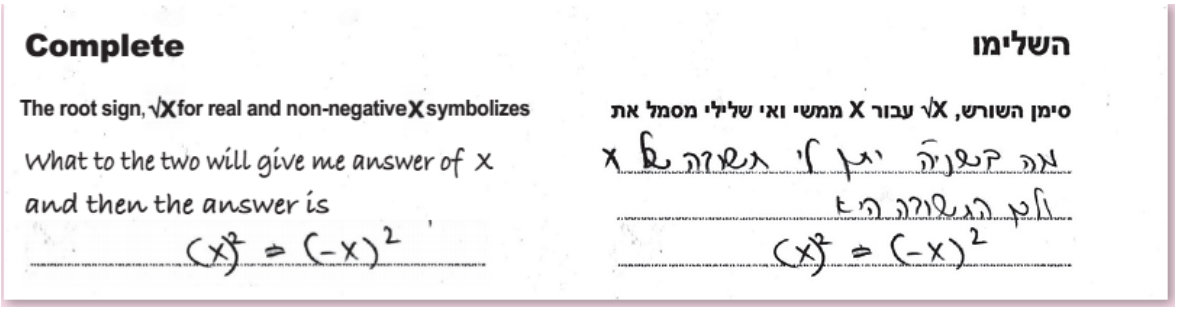

Figure 2c. Excerpt from Betty's assignment and its translation

A decrease in the objectification degree is evident in Betty's response to the task on the radical symbol (see Figure 2c). In regard to alienation, she resorts to a personal sentence for the first time in the assignment. Notably, Anna went through a similar collapse in alienation in the same task (see Figure 1c again). A collapse in reification is manifested in two ways: First, Betty uses the pronoun "what", which releases her from realizing "$\sqrt{x}$" into a more specific signifier (compare to Anna, whose realization was "the number" with special properties). Second, Betty's reference to a future action (i.e. "will give") indicates a processual formulation. Yet, while Anna's narrative captures the first step in the process – a transition from the root symbol to the power of half – Betty's narrative is focused on the final result of the action.

Despite the mismatch between the verbal and symbolic components of her narrative (i.e. "answer of x" versus "$(x)^2=(-x)^2$"), Betty succeeded to complete the assigned sentence and

referred to squaring – one of the defining attributes of square roots. This cannot be said about six other students, who attempted the task but were mostly concerned with the signs of x and with providing numerical examples. These responses can be interpreted as a mismatch between the task the students were assigned and the one they completed.

Summary and Discussion

In the past decade, we notice a substantial growth in the body of research that uses the commognitive framework as a lens for scrutinizing mathematics learning and teaching (e.g., Güçler, 2014; Nachlieli & Tabach, 2015; Shinno, 2018). Yet, this research rarely pays attention to the notion of objectification that has been described as key to concept formation in mathematics (Sfard, 2008). The study at hand addresses this gap by exploring how objectification can manifest itself when mature students engage in a discourse on square roots through writing.

One finding of this study pertains to task-dependency of objectification. Through a fine-grained analysis of one student's assignment (Betty), I showed that her narratives in different tasks differed in degrees of objectification. This finding resonates with the literature on mathematics lecturers and teachers. For instance, Güçler (2014) shows excerpts from a calculus classroom where a lecturer wrote objectified sentences on the blackboard but when communicated orally, they were often about processes instead of objects. This discursive move can be explained with different tasks that the lecturer was engaged in: recording "a mathematical truth" versus explaining it to the class. On the same point, Nachlieli and Tabach (2015) propose that teachers intentionally conflate their talk on objects and people in order to reduce the distance between mathematics and students. When considered together, the studies of Güçler (2014), Nachlieli and Tabach (2015), and the one reported in this paper might suggest exploring one's objectification range that she demonstrates in different tasks rather than ascribing her to a single objectification degree.

On the one hand, the suggestion might seem contradictory to Sfard (2008), who wrote that "once the project of objectification is completed, its results seem irreversible. This is why the adults seem incapable of seeing as different the things that the children cannot see as the same" (p. 141). On the other hand, Sfard's observation is a generalization of her findings from research on children's numerical discourse – probably the first mathematical discourse that children encounter in their lives. Thus, future research may be interested in exploring whether "the project of objectification" occurs differently in discourses that revolve around different mathematical objects. Indeed, it seems obvious that the discourses differ in their affordances to verbalize and symbolize, which should be reflected in how discursive objects are individualized by the learners. The language also shapes the discursive affordances, and then, this research venue might turn to be especially fruitful if teams of researchers will explore how the "same" mathematical objects are learned and taught in different countries.

Another finding of this study is concerned with collapses in objectification that were particularly evident in the tasks asking students to communicate the radical symbol and the definition of a square root. Mathematics Education discipline has accumulated a considerable body of evidence pointing to students' struggles to operate with definitions and notation (e.g., Güçler, 2014) when the lion's share of studies can be ascribed to the cognitive paradigm (e.g., Tall & Vinner, 1981). The commognitive standpoint, in turn, allows proposing that the source of the struggle is not exclusively in students "not knowing how definitions and notation work" but in a broader difficulty to discuss symbolically-signifiable objects. The participants in this study struggled to produce narratives that combine symbols

and words into coherent sentences; in multiple cases, the students either wrote about symbols as they were the objects (see the assignment of Anna) or avoided the usage of words. The preference to symbolism may be interpreted as a manifestation of compressed talk where compound verbal sentences turn into laconic symbolic strings; the same compression that Sfard positions as "irreversible". Yet, there are tasks for which symbolism becomes insufficient and verbalism is required.

Another important aspect that deserves systematic scrutiny pertains to instances where "reteaching" is required because students individualized incorrect rules of a mathematical discourse. In this study, for instance, the students were divided between whether $\sqrt{x^2}$ equals x or $\pm x$, but no one suggested $|x|$ as a correct answer. So, is an objectified talk a curse or a blessing in cases where the whole discourse needs to be revised? In the same way, if the path to objectification goes through reification and alienation, how does a dis-objectification occur? While these questions might seem purely theoretical, they have high practical value. In large heterogeneous classes, a teacher can never be fully sure whether the mathematics that she teaches is new to the students or whether it distorts something well-familiar to them. Accordingly, thorough investigations of these questions may lead to research-based pedagogies that teachers can use in their daily practice.

One practical suggestion that can be made based on this study is that students should be provided with sufficient opportunities to develop different aspects of their mathematical discourse. These opportunities are inseparable from both the tasks that a teacher brings to the classroom and the classroom rules of communication that often remain tacit. For assessing these tasks and rules, the teacher may self-reflect through questions such as: "Is it sufficient for my students to specialize in symbolic manipulations for succeeding in a particular topic?"; "Are the symbolism and verbalism complementary sides of the same coin in my classroom or does one just recap the other?"; "Who usually verbalizes both orally and in writing in my classroom?". I believe that answering these questions may equip teachers with practical insights on how the mathematical discourse of their students can be improved.

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Designing Data Collection Instruments to Research Engagement in Mathematics

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Engagement is a multifaceted concept that has attracted recent attention by researchers both in Australia and internationally (Attard, 2012; Chan, Baker, Slee, & Williamson, 2015). For many years mathematics education has been seen as boring and dull, and students have disengaged from a relatively early age in learning and participating in mathematics (Grootenboer & Marshman, 2016). Therefore, there seems to be an imperative for research and action into this issue, as low levels of engagement among school students can put them at risk of decreased participation and, ultimately, low levels of academic achievement (Fredrick et al, 2004). It is evident that there are three types of engagement consistent across the literature; emotional, behavioural, and cognitive. However, it appears that there are no existing data collection instruments that specifically focus on capturing student engagement in mathematics. Thus, some tools have been designed that derive specifically from the theoretical framework on engagement with the aim of being theoretically robust, conceptually relevant, and practically manageable.

The concept of engagement in mathematics has become a growing concern for researchers in Australia and internationally in recent years (Attard. 2012; Chan, Baker, Slee, & Williamson, 2015). Improving engagement is believed to be the means of ameliorating low levels of academic achievement and high levels of student boredom (Fredricks, et al, 2004). There is an imperative for research in this field as low levels of engagement can result in low achievement and participation. Consequently, this has the potential to affect our country's perennial shortage of mathematically literate citizens (Attard, 2011). There are also historical, economic, and practical reasons for the budding research interest in school engagement (Fredricks, Blumenfeld, & Paris, 2004). Many students now view mathematics as dull and inaccessible, and so disengage (Grootenboer & Marshman, 2017). Observations such as these are especially alarming as the new and rapidly changing international economy requires workers with mathematical knowledge in order to synthesise and evaluate new information, problem solve, and think critically (Fredricks, Blumenfeld, & Paris, 2004). Even though school is mandatory, students need to be engaged in their education if they are to succeed at school and thrive thereafter. For example, most higher education courses require specific levels of mathematics upon entry. Students who are disengaged from mathematics throughout their schooling, not only face a limited choice of courses available to them but additionally they limit their capacity to grasp the mathematical perspective present in everyday life experiences (Sullivan, Mousley, & Zevenbergen, 2005). McPhan, Moroney, Pegg, Cooksey, and Lynch (2008) claim that central to increasing participation rates in mathematics are teaching strategies in the early years that engage students in investigative learning. Students who are engaged in mathematics in the early years are more likely to learn, find a sense of satisfaction from the experience, and therefore progress to higher-level mathematics courses (Marks, 2000).

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Engagement in Learning Mathematics

Attard (2011; 2012), a key Australian researcher into student engagement in mathematics, suggests that effective mathematical engagement occurs when a student is enjoying the subject, can easily see the relevance that their work has to their own lives and future, and can make meaningful mathematical connections between the classroom and outside the school environment. Also highlighted, is the significance of choice and creativity in the mathematical learning context, and the suggestion that, if students are engaged in activities that encourage creativity and that provide opportunities to make decisions about their learning, their engagement will increase. Attard (2012) also incorporates the notion of "fun" in her studies, stating that "most of the 'good' [fun] lessons discussed by the students were those that include physical activity, active learning situations involving concrete materials, and/or games" (p. 11). Additionally, Fägerstam and Blom (2013) state; "The pupils in this study all described positive experiences regarding the outdoor lesson… all of them spontaneously uttered remarks such as 'it was fun'" (p. 68). Similarly, Brunsell, Fleming, Opitz, Ford, and Ebrary (2014) found that "joyful" learning was significantly connected to better learning. Fägerstam and Samuelson (2014) also report that outdoor learning provided students with a more enjoyable approach to education. It is likely that an individual's sense of enjoyment or feeling like an activity is fun can have direct, positive impacts on their school learning experiences.

Motivation concepts are suggested to have significant relevance, and are often synonymous, where the conceptualisation of engagement is discussed. Students' motivation to complete tasks dramatically increases when games are included in mathematics (Attard, 2011; 2012). Additionally, when students can make links between the mathematics they are learning and "real" life their engagement significantly increases (Ajmal, 2013; Attard 2012). Thus, is it critical that students are able to make links between what they are learning, their knowledge, and both inside and outside classroom experiences (Opitz & Ford, 2014).

The literature frequently suggests that outdoor learning is an effective pedagogical approach to increase student engagement and it is often suggested that students perform significantly better in outdoor activities (Fägerstam & Samuelson, 2014; Haji, Abdullah, Maizora, & Yumiati, 2017; Young & Marroquin, 2008). Attard's (2012) research suggests "the incorporation of tasks that mirrored life-like situation appears to have been a strong factor in engaging students in mathematics tasks, as were the tasks that required the students to take the mathematics out of the classroom and into the school playground" (p. 11). Waite (2011) reports on the benefits of outdoor learning stating, "Another very important aspect of our findings was the levels of involvement of children in planning and use of outdoors. This seemed to ensure a greater sense of ownership, more engagement and higher levels of usage…Enjoyment and engagement of the whole child was common across all the case studies" (p. 78). Young and Marroquin (2008) also report on the effectiveness of lessons taken outside on the playground, "Reluctant students were more apt to engage in the activities and volunteer to explain their thinking or justify their answers" (p. 282). As observed, the word 'engage' and its derived forms are not uncommonly used in literature discussing outdoor learning.

Previous Studies into Engagement in Mathematics

Reflected in the research literature, the multifaceted nature of engagement has been commonly defined around three dimensions; emotional, cognitive, and behavioural (Fredricks, Blumenfeld, & Paris, 2004), and this is discussed in the next section. For

example, Skilling (2014), using Fredricks, Blumenfeld, and Paris' (2004) framework, investigated engagement in a mathematical educational setting. She found that students who were emotionally engaged tended to demonstrate interest and enjoyment, and cognitively engaged students demonstrated effective planning, and they managed and regulated their learning. Also, students who are behaviourally engaged actively participated, persisted, and asked questions.

Also, in an Australian context, Attard (2011; 2012) comes from an educational background rather than a psychology background and has devoted a significant measure of research to the different domains of engagement. Using the work of Fredricks et al (2004) she has developed literature on the practicalities of the theories of engagement.

Theoretical Framework

Given the literature on engagement and the previous studies outlined above, the work of Fredricks, Blumenfeld, and Paris (2004) is seen as seminal and provides a sound and robust theoretical framework for investigating engagement in mathematics education. This will now be outlined and discussed

Types of Engagement

Fredricks, Blumenfeld, and Paris (2004) suggest that emotional engagement "encompasses positive and negative reactions to teachers, classmates, academics, and school" (p. 60). They explain that there is a direct correlation between how students react to these school experiences and their willingness to do work. However, Attard (2011; 2012) does not define this particular domain of engagement as other researchers have and instead labels it as belonging within the affective domain. She centralised her analyses not around the internal state (emotions) of students' engagement, but rather around students' experiences with school and their associated affective responses. Grootenboer and Marshman (2017) also attempted to comprehensively characterise the affective domain through the interrelated dimensions of beliefs, values, attitudes, and emotions. Often seen to be related, emotional engagement and motivation can be seen as synonymous but in reality they are very distinct concepts. Motivation encompasses internal (emotional), private and unobservable aspects while engagement is the manifestation of these qualities that are observable on the outer (Skilling, 2014). However, research in the field of motivation is often critical if the deeper influencing factors of students' engagement is mathematics are to be understood holistically (Skilling 2014).

Behavioural engagement refers to an individual's active participation and involvement in academic and social activities (Attard, 2011; 2012). Finn, Pannozzo, and Voelkl (1995) emphasise that inherent to the construct of behavioural engagement is the concept of participation, which is a crucial component in achieving positive academic outcomes. Behavioural engagement is most commonly conceptualised in three interrelated ways. The first entails how an individual adheres to classroom norms and follows the rules (Finn et al., 1995). The second aspect of behavioural engagement is concerned with students' actions such as their "efforts, persistence, concentration, attention, asking questions, and contributing to class discussions" (Fredricks, Blumenfeld, & Paris, 2004). Often the term "effort" is seen to be problematic as it is included in definitions of both behavioural and cognitive engagement and distinctions are not always made clear (Fredricks, Blumenfeld, & Paris, 2004). The third dimension involves an individual's participation in school activities such as sports or leadership roles (Finn 1989; Finn, Pannozzo, & Voelkl, 1995).

Attard (2011; 2012) defines cognitive engagement as an individual's investment, acknowledgment of the value of learning, and willingness to go above and beyond the minimum requirements. Cognitive engagement relates to the desire for hard work, persistence in problem solving, and endurance in the face of failure (Attard 2011; 2012; Connell & Wellborn, 1991). Fredricks, Blumenfeld, and Paris (2004) explore cognitive engagement from two different perspectives where one encompasses the psychological investment in learning (Newmann, Wehlage, & Wisconsin, 1995) while the other targets cognition and emphasises strategic learning (Zimmerman, 1990). The literature concerned with cognitive engagement as a psychological investment defines it as an individual's direct efforts towards learning and mastering the knowledge, skills and crafts associated with academic work (Fredricks et al, 2004). Again, the word effort is indistinctly used across definitions of engagement as it parallels those found in research on behavioural engagement. Psychological definitions of cognitive engagement also strongly resemble definitions found in the motivation literature. Students who are intrinsically motivated demonstrate persistence in the face of hardship and experience a sense of satisfaction when given challenging tasks (Brophy, 1987). The alternative definition of cognitive engagement emphasises students demonstrating highly strategic learning qualities. Often described as being self-regulated, strategic students complete tasks by using metacognitive strategies to arrange and assess their cognition (Zimmerman, 1990). They are efficient in suppressing distractions, as well as maintaining and regulating their efforts to sustain their cognitive engagement (Fredricks, Blumenfeld, & Paris, 2004). They explain that both definitions of cognitive engagement are valuable and that neither alone can adequately deal with the qualitative aspects of engagement.

The conceptualisations of emotional/affective engagement, behavioural engagement, and cognitive engagement incorporate a wide variety of constructs. It is critical to acknowledge that these engagement factors are not isolated processes occurring within the individual, but rather they are dynamically interrelated and a shift in one can dramatically influence the others.

Developing Research Tools and Approaches

As observed, the word 'engage' and its derived forms are not widely used in literature discussing mathematics learning, albeit that it is seen as important in practice. Yet it is seldom seen that a distinctive definition of engagement is integrated and the word is often cryptically used. The majority of the literature seems unsuccessful in making conceptual links between the different types of engagement and the effectiveness of outdoor learning and therefore there are no available instruments that would be suitable for such a study.

It appears that there are no existing instruments that specifically focus on capturing student engagement in mathematics learning. Therefore, some tools have been developed that derive specifically from the theoretical work of Fredricks, Blumenfeld, and Paris, (2004) that was outlined and discussed above. The aim was to have data collection tools/approaches that were theoretically robust, conceptually relevant, and practically manageable. The three data collection tools/approaches relate specifically to the three dimensions of the theoretical framework as is outlined in Table 1 below.

Table 1
Data collection

Dimension of Engagement	Data Collection
Emotional engagement	A survey that the students will complete at the conclusion of each lesson.
Behavioural engagement	Observations of students participating in the lessons using an observation framework.
Cognitive engagement	Student work samples will be collected in each lesson

The data collection tools/approach are designed to be used together around the observation of a particular mathematics lesson. Specifically, the data re behavioural engagement is collected by observing the students behaviour in the lesson; the data re emotional engagement is collected through a student survey immediately at the end of the lesson; and, the data re cognitive engagement is collected by analysing student work samples that were generated in the lesson.

Emotional Engagement

To identify levels of emotional engagement, the survey (see Figure 1 below) was designed based on the work of Fredricks, Blumenfeld and Paris (2004), but also drawing on the findings of Skilling (2014) and Attard (2011; 2012). Fredricks, Blumenfeld, and Paris, (2004) defined emotional engagement as showing interest in a task, and they defined "interest" as displaying "enjoyment of the activity" (p. 63). Similarly, Skilling (2014) suggests that students who are emotionally engaged "demonstrate interest and enjoyment" in mathematical tasks (p. 589). To this end, the first two items in the survey focussed on enjoyment and interest. Furthermore, although not the sole focus of mathematics education, Attard (2011) identified that the element of "fun" can play a significant role in student engagement, and she suggests that "the element of fun was identified as an element of "good" mathematics lesson" (p. 371). Therefore, the third item asks students about the fun in the lesson. Finally, the last item encompassed these features of engagement by asking it the student 'would like to do that lesson again'.

As would be evident, the survey would be given to the students after the observed lesson and is designed to be quickly and easily completed by primary school students.

Emotional engagement Survey

	Strongly agree	Agree	Uncertain	Disagree	Strongly Disagree
I enjoyed that lesson					
I thought the lesson was interesting					
I had a lot of fun during that lesson					
I would like to do that lesson again					

Do you have any other comments about the lesson?

Figure 1: Emotional Engagement Survey.

Behavioural Engagement

To understand and establish levels of behavioural engagement, the behavioural engagement observation checklist (see Figure 2 below) was designed, again building on the work of Fredricks, Blumenfeld and Paris (2004) and also using Skilling (2014). Fredricks suggests that behavioural engagement "includes behaviours such as effort, persistence, concentration, attention, asking questions, and contributing to class discussion" (p. 62). Similarly, Skilling suggests that students who are behaviourally engaged "actively participate, persist, and ask questions" (p. 589). These features and defining qualities were used to develop the instrument as can be seen in the left column. The intervals in which the data will be observed and recorded on the behavioural engagement observation checklist will be dependent on the nature of the lesson but will most likely be every 3 to 5 minutes, or in accordance with the phases of the lesson.

A key feature of the observation checklist was that it had to be able to be completed 'in the moment'. This necessitated that the data collected was simple and manageable. Clearly this means that the data collected will not necessarily have as much detail as would be desired but was seen as feasible and adequate given that the data will be collected through observations of mathematics classes.

Behavioural Engagement Observation Checklist - Example

Key
A = All
M = Most
H = Half
S = Some
N = None

Are the students...	5 min	10 mins	15 mins	20 mins	25 mins	30 mins	35 mins	40 mins	45 mins	Notes
	(INSERT SPECIFIC LESSON PHASES HERE)									
actively participating?										
asking questions?										
contributing to class discussion?										
persisting with tasks?										
displaying strong levels of concentration ?										

Additional Notes:

Figure 2: Behavioural Engagement Observation Checklist.

Cognitive Engagement

Third, Attard (2011), and Connell and Wellborn (1991), suggest that students who are cognitively engaged show a desire for hard work, persistence in problem solving, and endurance in the face of failure. The literature concerned with cognitive engagement define it as an individual's direct effects towards learning and mastering the knowledge, skills and crafts associated with academic work (Fredricks et al., 2004)

While the actual nature of the student work samples will be largely dependent on the specific topic of the lesson and the activities involved, in general the following features will be looked for: evidence of mathematical conceptual development /learning vis-à-vis the focus of the lesson; evidence of higher-order thinking related to the topic of the lesson; and, evidence of persistence and sustained effort in the activities of the lesson;

These are less definitive than the processes and tools for the first two dimensions of engagement, but this is inevitable given the unknown variations in primary school mathematics lesson. Also, as was noted previously, the conceptualisations of cognitive engagement are more diverse than behavioural and emotional engagement, and the definitions are varied and tend to include aspects that are also seen as part of the other two dimensions. It is envisaged that after these criteria are trialled with some empirical data, then it will be possible to refine them further.

Limitations and Discussion

There are clearly a number of limitations with the data collection tools and processes outlined in this paper, and a number of these have already been noted. The emotional engagement survey specifically relies on students being able to know and report their emotional responses after the lesson. This might be further complicated if the students are younger, or if there is some time gap between the lesson and the administration of the instrument. Also, it seems likely that students emotional response will be largely related to their later experiences of the lesson (i.e., they may not recall as clearly their emotional responses from the initial activities of the lesson). Finally, it is highly relied on that students will be able to make distinctions between the words 'enjoyed', 'interesting' and 'fun'. Apart from these practical concerns, it is also possible that aspects of emotional engagement are not captured by the survey items, and this will be monitored through the initial data analysis.

The behavioural engagement checklist is limited because it relies on the researcher observing and evaluating 'all' behavioural features present in the moment. Also, the recorded data is quite crude as in how many students are displaying the noted behaviours, but it is expected that they would give a perspective on the general engagement of the class in response to the lesson. Of course, if the lessons were video-recorded, or if there were more observers, then the complexity and volume of data and analysis would be improved.

Finally, the nature of the cognitive engagement is less clear and lacking precise definition. This is largely due to the lack of theoretical and conceptual clarity in the literature. However, given that the student work samples can be analysed after the event without the pressure of time, it is expected that thoughtful consideration will be possible.

While the tools and processes outlined in this paper are in the early stages of development, they are designed to be grounded in a robust theoretical framework. In this way, it is hoped that they will be useful to understand engagement in mathematics education.

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The Five Question Approach: Disrupting the Linear Approach to Mathematics Teaching

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Student disengagement is influenced by the degree of success that is experienced in the mathematics classroom. In turn, success is often determined by the depth of understanding that students gain during predetermined time frames. This paper reports on the Five Question Approach to teaching mathematics which provides teachers with greater flexibility in content delivery, pacing and consolidation of content. This qualitative case study draws on data collected in three Australian secondary classrooms. Findings indicate that the Five Question Approach led to increased student engagement, academic improvement and a significant decrease in examination anxiety.

Introduction

Student engagement with mathematics in the middle years (Years 5 – 8) is a significant issue and has been the focus of considerable research. The level of students' engagement typically declines from the end of primary through to the first two years of secondary schooling (Martin, Way, Bobis, & Anderson, 2015; Plenty & Heubeck, 2013). Arguably such disengagement could have contributed to a decline in the performance of Australian students in mathematics in Programme for International Student Assessment, PISA, testing from 2000 to 2012 (Masters 2016). One aspect that may influence student engagement is the 'traditional' mathematics teaching approach. In this approach students follow step-by-step teacher led procedures, set out in worked examples, and complete copious, repetitive questions with a focus on finding one correct answer.

All schools in NSW are required to have a scope and sequence document outlining the order and duration of syllabus topics for each year. The linear, fixed time approach to the teaching of mathematics may also be related to a lack of success for some students. Many teachers follow the scope and sequence document prescriptively, moving from one topic to the next according to the time allocated and irrespective of student competence. The Five Question Approach (FQA) may provide a solution to this problem. This paper introduces the FQA and provides details of its application in two Australian secondary schools. While the FQA has been used in classrooms over many years, no empirical evidence is available on its efficacy for improving student learning and engagement.

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Literature Review

Traditional approach to teaching

In the typical Australian classroom, mathematics content is taught according to a scope and sequence document that provides a plan of the order of topics and the duration of time spent on each topic across the school year. Each school develops its scope and sequence document from the syllabus documents and the class teacher follows this plan in their teaching program. The teaching and learning structure is therefore linear with a fixed time allocated for each topic and little linking between topic areas (Rhorer, 2015). In the week preceding the half yearly or yearly examinations there is usually a period of dense and rushed revision where all topics areas are revisited. This intensive revision can lead to a high level of examination stress (Plenty & Heubeck, 2013).

In most mathematics textbooks questions on each topic are grouped together and students generally work on sets of similar questions before moving on to the next topic. This massed, as opposed to spaced practice promotes 'overlearning', which is the belief that the longer the practice directly following the new learning, the longer the material will be retained (Rohrer & Taylor, 2007). An alternative approach, 'mixed review', which is based on a combination of mixing questions on multiple topic areas (interleaving), and then spacing the questions over time, has been shown to result in a significant improvement in test results (Rohrer, 2015, Rohrer, 2009). The spacing and mixing of questions using repetition with variation, where questions are similar but different, may allow students to deepen their understanding of the concepts over time (Kapler, Weston, & Wiseheart, 2015, Handa 2012).

According to Steffe (2010), Anthony and Walshaw (2009); and Anderson, White, and Sullivan (2005), mathematical learning takes place when students are engaged in the solution of problem style tasks involving open ended, investigative style questions where various solution paths are discussed. Ideally these tasks need to be engaging and specifically selected by the teacher to achieve the desired educational goals. This contrasts with the traditional teaching approach in which the students follow step-by-step teacher led procedures where students are provided with examples, and repetitive, similar exercises are completed in class and usually finished for homework. Often when marking the exercise, the answer is given without any discussion and there is no concluding summary of the concepts from the lesson. It is this traditional method of teaching that is the most common practice currently used in Australia and this is generally considered to have contributed to a decrease in engagement (Sullivan 2011, Stacey 2010, Ball, Sleep, Boerst, & Bass 2009).

Student engagement

Engagement is associated with the depth to which the students relate to their classroom work and can be conceptualised as including three dimensions (Attard, 2014). Behavioural, cognitive and affective dimensions are manifested in active participation, valuing of learning and the willingness to be involved (Attard, 2014). Engagement with mathematics is an issue in Australia and internationally resulting in a general decline in participation in mathematics at higher levels and a decline in performance internationally over the past 20 years (Kennedy, Lyons, & Quinn 2014, Plenty, & Heubeck, 2013). While there are many factors that influence student engagement in the classroom one aspect contributing to the decrease in engagement may be a result of the restricted time allocated by the linear scope and sequence which influences the pace of lessons (Ollerton 2009).

FQA approach

The Five Question Approach (FQA) involves the presentation of five questions from different content areas at the start of every lesson. Students are required to complete the questions in order, from one to five, to ensure that they do not omit the questions that they are less confident in or do not want to do. While the first four questions focus on procedural fluency, the fifth question is conceptual, often open-ended, and enables students to deepen their knowledge. Students who have completed the first four questions quickly spend more time and delve more deeply into the concepts involved. The FQA provides opportunities for students, over a longer variable time period, to consolidate procedural knowledge and develop conceptual knowledge of content areas that they may not have previously understood due to the fixed amount of time allocated to that topic. The FQA may address the shortcomings of the linear fixed-time approach by allowing teachers to take small steps towards a change in practice without completely changing their entire approach to teaching.

This research aimed to provide some direction for improvement in the teaching and learning of mathematics and mathematics pedagogy in general. The central research questions were:

1. What influence does the FQA have on perceived and actual academic performance of students in mathematics?
2. What influence does the FQA have on student engagement in mathematics?

Methodology

To evaluate the efficacy of the FQA in practice both qualitative and quantitative data were collected from two schools over the course of one school year. Quantitative academic achievement data was collected from the half year and end of year examinations held at both schools. This data was used to answer part of the first research question. The data on student engagement and the perceptions of their academic improvement were collected through student focus group discussions, teacher interviews and classroom observations. Participants were provided with pseudonyms to protect their privacy. This qualitative data was collected at the start of the year, and at the end of Terms 1, 2 and 4.

Contexts

Data was collected from three classes across two Catholic secondary schools. Each year group, Years 7 to 10, had approximately 180 students spread across seven classes. Both schools graded their mathematics classes in all years, based on the end of year examinations, and followed a linear scope and sequence document. All four teachers (one class was taught by two teachers in a job share) commented on the pressure they felt in having to complete the scope and sequence in the required time, describing it as very content heavy and stating that in previous years some topics were omitted due to time constraints.

Two classes were selected from the first school, one Year 9 and one Year 10, and were both the fifth graded classes out of seven. Participants were trained mathematics teachers with more than 10 years of teaching experience. The school's 2015 National Assessment Program – Literacy and Numeracy, NAPLAN, year 9 numeracy results were below average when compared to similar schools. One Year 8 class was selected from the second school, which was the second highest graded class out of seven. The teachers worked in a job share arrangement. Both teachers were trained mathematics teachers but with less than five years of teaching experience. The school's 2015 NAPLAN year 9 numeracy results were close to those of similar schools.

Data

For each participating class the teacher(s) selected a sample of six students to participate in focus group discussions. The sample was stratified by selecting female and male students across a broad range of academic performance levels within each class. Based on the teachers' knowledge of the students the selected student's engagement levels in year 8 ranged from disengaged to highly engaged while the year 9 and 10 classes ranged from disengaged to a moderate level of engagement. The qualitative data on engagement and perceived academic performance was collected from classroom observations, teacher interviews and focus group discussions all of which occurred four times, at the start and end of term 1 and the end of terms 2 and 4. Each source of data was analysed for emerging themes prior to a cross-case analysis to seek new or common themes (Flick 2009).

Findings and Discussion

Time to learn was the strongest theme. The students discussed how under the traditional structure there was not enough time to understand each topic. According to Gina, a Year 9 student, "Last year I didn't understand most of the topics. We have a week or two to finish a topic but really you need more time." The students made it clear that they required additional time to understand the material. The FQA provided the opportunity for the teacher to focus on areas of concern until the students developed their understanding. With the amount of time per lesson spent on the FQA varying from ten minutes to most of the lesson there was concern about completing the scope and sequence. However, by year's end all classes had completed the scope and sequence within the required time frame, albeit by following a different, non-linear route using the FQA.

Teacher Perceptions

Prior to using the FQA each of the four teachers taught in a similar manner and all described themselves as 'traditional' teachers. The first classroom observations clearly indicated that the teachers followed the same traditional structure as previously described. The lessons did not revise any previous material and the teachers continued with the next aspect of the current topic, whether or not the students were ready, so that the topic would be completed within the set time.

This practice appeared to change with the teachers using the FQA to focus on the areas that were causing the students' difficulties. For example, the changes to Jenny's teaching centred around her organisation and thinking. She intended to focus on students' understanding of the concept rather than spending the allotted time and moving on to the next topic whether or not the students understood: "…and I am guided by how we're not all there yet, let's keep going. Let's get this, we will get this and then we will move on" (Jenny, Year 9 teacher). By teaching that way she felt that the students achieved success and that they felt positive about themselves and became more engaged. Jenny found the FQA gave her much greater freedom in the classroom as the time restrictions were removed: "…if it goes somewhere you just let it because that's okay …you just pick up in the next lesson… I've really enjoyed getting that understanding" (Jenny, Year 9 teacher).

Student Perceptions

The students stated that they liked the topic areas that they understood and in which they performed well. Ella, a year 8 student, didn't like or understand the algebra topic when

studied in the previous year. She commented that she did not have sufficient time to develop an understanding of algebra as the teacher moved too quickly through that topic area. By the end of the year her level of cognitive and affective engagement had changed from low to high. Other students made similar comments about the fast pace of the lesson and insufficient time to learn, reflecting findings from Ediger (2012). When thinking back to the previous year Allie said: "we do algebra at the start for a week and wouldn't understand it and then we'd have a test at the end of the term and we didn't know it because we only did it at the start" (Year 9 student). The other students in the focus group agreed and felt that the repetition of the FQA allowed better understanding and built their confidence.

Rather than having a topic started and finished over two or three weeks the students preferred to continually revisit the topic area which gave them time to understand and they retained that knowledge for a longer time period aligning with the findings from research by Rohrer (2015), Dunlosky, Rawson, Marsh, Nathan, & Willingham (2013); and Bird (2011). The focus group students preferred and enjoyed the flexibility and the variety of the FQA. Brenda represented the opinions of the group and she said "…if it's just the same question all the way down, you get lost and think about other things. But with FQA you must focus, it makes you think a lot harder and engage in the lesson" (Year 8 student), indicating an increase in cognitive and affective engagement. The teachers agreed that the students enjoyed the challenge and variety of the different topics of the FQA which enabled them to deeply investigate the concepts and the subsequent discussion promoted even greater understanding.

Engagement

All teachers raised the point that, in general, engagement was higher in mathematics classes that contained more capable mathematics students. The students commented that they enjoyed the topics in which they were confident which aligns with the teacher statement on engagement. The first Year 8 classroom observation showed a compliant class but there was little evidence of deep engagement. The students appeared to be able to easily answer the questions given but displayed little enthusiasm and the activities did not have any cognitive challenge. Over the course of the year the questions increased in cognitive challenge and the level of engagement increased. The initial Year 9 and 10 classroom observations showed classes that were not engaged with significant class time was used to manage behaviour and the attention level of the students. Over the course of the year the level of engagement in those classes increased and the need for classroom management decreased. Harriette, the year 10 teacher, felt the students were more engaged and confident due to the extra time spent on problem areas in the FQA. She noticed that the students' level of engagement with the FQA was much greater than during the traditional part of the lesson and felt that this was due to the repetitive nature of the work they were required to complete.

All the participating teachers used tests to ascertain the students' level of understanding but the time restrictions prior to implementing the FQA prevented any remediation. The teachers stated that the analysis of the results was supposed to influence teaching to remedy the issues identified. "We're supposed to do assessment for learning, but time restrictions make it impossible – you do the task and you go, this is what mark you got and move on" (Karen, Year 8 teacher). In this case the tests could have contributed to the students' lack of engagement because of a lack of remediation caused by time limits.

The FQA provided an opportunity to address the problem areas indicated in the test. The teacher included poorly answered test questions in the FQA which allowed the students opportunities to develop their conceptual understanding that they lacked. Although the

students all stated that the FQA took time from the textbook theory work, Brenda made this comment that provides some evidence that the combination of cognitive, operative and affective engagement results in deep engagement (Attard 2014):

> I think the five questions is better than theory because we're attempting questions, and you remember sitting there and working out things, and attempting them. And I think that the best lessons are if you remember working it out in your head, and if you get it wrong, sort of talking about it and discussing it is really good (Year 8 student).

Academic Improvements

The focus group students commented that the FQA with its revision, repetition with variation, consolidation and interleaving had helped in the reduction of examination stress. The students had developed an understanding of concepts over time which provided the opportunity for them to develop a more complicated map or network in their memory rather than a simple linear progression (Ollerton, 2009). Allie said: "…when you have an exam it's like, oh I know how to do this because like I do it in five questions, but you don't actually realise that in class". Gina went further when she said "… everyone in the year should (do the FQA), because it really does help when you get to exams, you actually have a better understanding of what you have to do and how to do it" (Year 9 student).

All three classes showed improvement in their half yearly and yearly examination results. The marks were used to rank the students based on their academic performance in that examination. The rankings were used to show the relative movement of the students from the sample classes compared with other students in the year group. As the content in the examination was directly related to the content of the five questions this analysis gave an indication of the influence of the FQA. The rank for each student was compared with the initial rank from the previous year that was used to form the class. These rankings were compared, and the data displayed on a scatter plot.

The scatterplots show the results for the current class comparing the final examination ranking with the final examination ranking from the previous year. The straight line indicates equal rankings so points above the line show an improvement in ranking. There was greater improvement in the Year 9 and 10 classes when compared with the Year 8 class, but this was because there was significantly greater room for improvement. In Year 8 there were nine students from the 8M2 class that were placed above 20th ranking. As a result, students that were nine places or less below the line may have improved or performed at the same level. Therefore, twenty of the students improved and eight did not when compared against their Year 7 performance.

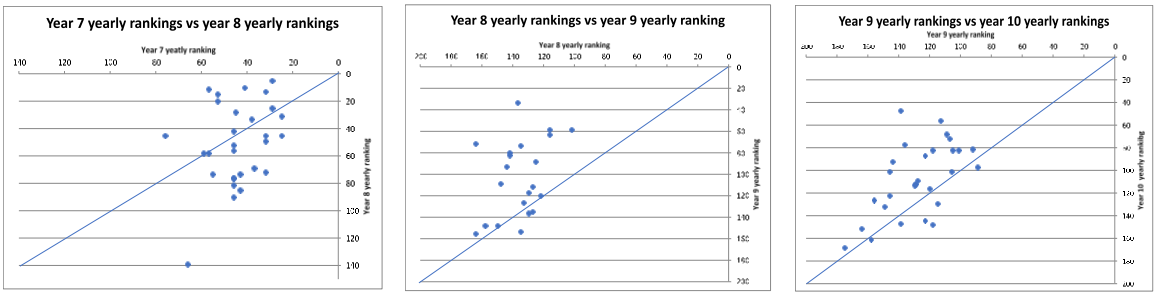

Figure 1: Comparison of yearly examination rankings

Importantly, all the students felt that they were better at answering questions due to the repeated practice over time and would be more likely to attempt questions because of the

FQA. The teachers said that the student enjoyment of mathematics class and as a result their behavioural engagement, was enhanced by the FQA. The positive attitude through success in the FQA increased students' perceptions that their academic performance was improving and as the results showed, academic performance did improve for most of the students.

Conclusion

The FQA positively influenced the level of engagement and academic performance in all three classes. The students and teachers found the repetition of questions over time provided the opportunity for students to develop understanding and confidence. The increased confidence helped to reduce examination anxiety for many of the students and as a result their examination performance improved. The greater discussion around mathematical concepts, and explanation of multiple solution pathways, showed an increase in engagement. All teachers and students indicated that they would like to continue with the FQA in the next year with one school adopting the practice across the entire faculty.

The success of the FQA in improving engagement and academic performance through multiple exposure to content areas over extended time periods has implications for all 'traditional' teachers. The emphasis on textbook exercises and worksheets that replicate the worked example are not as effective as spreading the questions over time. Strict adhesion to the time allocated in the scope and sequence may not allow all students to develop sufficient understanding and consequently the lack of understanding and success is disengagement. The study showed that the 'traditional' classroom structure did not provide many opportunities for students to deeply engage with the mathematical content and concepts. The FQA provided multiple opportunities for understanding and hence success from meaningful conversations about mathematical concepts which was borne out through the study as engagement increased in all classes.

This study had some limitations which should be recognised. Firstly, only secondary schools were used in the study and both schools were secondary Catholic schools. Both schools graded their classes and as a result the students had a strong focus on their examination performance as that determined their class allocation. It would be difficult to predict the results of the FQA in schools where students are grouped heterogeneously. Such a context might provide challenges for the teacher in determining the optimal set of five questions for each class. The design of the five questions requires teachers to have a strong pedagogical content knowledge and as a result, out of field teachers may find the writing of the questions difficult.

This research has the potential to be expanded by involving more school systems, both secondary and primary schools, teachers, and students, in particular, non-denominational schools and schools using a mixed ability approach to mathematics classes. The collection of data from two schools and three different year levels provided some insight into the value of the FQA. Extension of the study into more schools may provide a deeper understanding of the potential of the FQA to alleviate some of the issues involved with a 'traditional' teaching environment with fixed time linear approach to teaching mathematics.

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Insights into a Year 4 Student's Spatial Reasoning and Conceptual Knowledge of Rectangular Prisms

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While spatial reasoning skills have been found to predict mathematical achievement, little is known about how primary students' conceptual understanding of three-dimensional objects develops. This paper reports insights into a Year 4 student's spatial reasoning when constructing and describing the properties of rectangular prisms, using Froebel's Gifts, in an interview. Categories of the van Hiele framework were used to analyse student responses in conjunction with a specially designed analysis tool. The findings highlight the benefit of using a one-to-one interview to shed light on one student's spatial reasoning and conceptual knowledge of rectangular prisms.

Knowledge of geometrical shapes and solids is regarded as a core element of geometry and mathematics education in primary schools. However, Clements and Sarama (2011) suggest that geometry and spatial reasoning are areas of mathematics that are often ignored, or receive little attention in the early years' classrooms. Within Australian contexts there is little research relating to the use of geometric construction materials such as Froebel's Gifts (German designed geometric building blocks). By gaining insights into students' conceptual knowledge of three-dimensional objects, including rectangular prisms, we can extend our understanding of how they develop spatial reasoning and visualisation skills. Informed by literature relating to primary students' spatial reasoning and knowledge of three-dimensional objects, the rationale of this Australian pilot study was to provide new insights into students' spatial reasoning and their knowledge of prisms, using a German task-based interview.

Geometric reasoning is introduced into the Australian Mathematics Curriculum (ACM) at Year 3 when students develop the skills to make models of three-dimensional objects and can describe key features (ACARA, 2017). This implies that students need to generate the properties of three-dimensional objects via constructions. In doing so, they must consider the relationship between the different objects such as a cube is a special rectangular prism (Sinclair & Bruce, 2015; Lehrer & Curtis, 2000).

In a recent review of mathematics education in the early years MacDonald, Goff, Docket, and Perry (2016) concluded that, "there continues to be a dearth of Australasian research in the area of geometry (and space) in the early years" (p. 176). Sinclair and Bruce (2015) stressed the importance of extending geometry teaching at the primary school level shifting from an emphasis on how to name and sort shapes by properties, to learning experiences that develop children's spatial reasoning through active meaning making. Dindyal (2015) suggested that when it comes to constructing "children should be given experience with: free construction materials (clay, plasticine, ropes, boxes), geometric construction materials (lego, pattern blocks, meccano, tangrams), constructing with paper (paper folding, paper cut-

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outs), constructing on paper (drawings of shapes, patterns)" (p. 523). Also, student learning becomes richer when learning occurs through intentional guided play, and when students are encouraged to participate in geometric reasoning.

Lehrer and Curtis (2000) outlined a lesson with Grade 3 students when they experimented with finding the five "perfect" solids (Platonic solids). The students were not told the "rule" or properties of perfect solids but created solids with polydrons and conjectured about the rules depending on whether the teacher confirmed their newly created solid was "perfect" or not. The students used the language of edges, faces, vertices, congruence and angles to compare the solids and determine the rule. Similarly, Ambrose and Kenehan (2009) reported on a teaching experiment in which 8- and 9-year olds built and described polyhedra over several lessons. The results indicated that the students improved in their geometric reasoning and "began to identify, enumerate and notice relationships between component parts of polyhedral" (p. 158).

There are also connections between spatial reasoning and spatial visualisation that contribute to students' ability to decode three dimensional designs and effectively construct models of these. Previous studies (e.g., Battista & Clements, 1996; Reinhold, 2007) highlighted the important role of spatial visualisation, including the "ability to comprehend [and apply] imaginary movements in three-dimensional space or manipulate objects in imagination" (Pittalis, Mousoulides, & Christou, 2007, p. 1073). For example, Battista and Clements (1996) reported on students who failed in coordinating two different orthogonal views for the construction of a rectangular prism (e.g., a 4 x 5 x 3 prism presented as a diagram), when counting the number of single cubes. A lack of coordination was evident whenever the students were unable "to recognize how they [the orthogonal views of a prism] should be placed in proper position relative to each other" (p. 267). This prevented some students from forming "one integrated mental image of the objects" (p. 272). Their initial conception of rectangular prisms was characterised as having an uncoordinated set of faces. Students who were more successful at enumerating the number of cubes to make the figure were more likely to reconstruct the image as layers, and multiply one layer by the number of layers, or add iterations of layers (sometimes by skip counting). Those who struggled to structure the image in layers appeared to use structuring that was "local rather than global" (p. 275), that is, the students considered small sets of cubes in a side, row or column, making the structuring and counting of cubes very complex. This *local* structuring is similar to Reinhold's (2007) finding that students focus on isolated features and visualised "in bits" rather than visualise the entire structure.

Gutiérrez, Jaime, and Fortuny (1991) described the van Hiele levels (1986) in terms of three-dimensional objects, which aimed to ascertain a students' level of geometric accuracy by measuring the degree to which students' thinking adhered to a level or levels. The following condensed version of the specific three-dimensional descriptors for the van Hiele Levels 1 to 3 (Gutiérrez et al., 1991, p. 242) informed the data analysis in our study.

> Level I (Recognition): Students consider three-dimensional objects as a whole and recognise and name solids (prisms, cones, pyramids, etc.). They distinguish a given solid from others on a visual basis using reasoning of the type "it looks like..." They do not explicitly consider the components or properties based in order to identify or name a solid.
>
> Level 2 (Analysis): Students identify the components of solids (faces, edges, etc.), and describe properties of three-dimensional objects in an informal way. They are unable to logically relate properties to each other, nor logically classify solids or families of solids. Students are able to discover properties of the solids by experimentation.

Level 3 (Informal deduction): Students are able to logically classify families of solids (e.g., classes of prisms or rounded solids, regular polyhedra). Definitions are meaningful for students, and they are able to handle equivalent definitions for the same concept.

When teaching geometry, it is important for teachers to be aware of a possible trajectory that shows the growth of students' geometric understanding (e.g., van Hiele, 1986) as well as being able to select rich tasks that can extend student learning. A study by Moss, Hawes, Naqvi, and Caswell (2015), provided the kind of professional learning that helped teachers to focus on rich tasks and to look for student understanding within their actions of constructing three-dimensional objects.

As part of ongoing research exploring Australian primary students' interactions with Froebel's Gifts, we interviewed 23 students. For the purpose of examining one case study as an example of the rich insights gained through this research, we present one Year 4 student's interview data. The following research question underpins this study:
What insights can be gained from a Year 4 student's construction of rectangular prisms, regarding his spatial reasoning and conceptual knowledge of prisms?

Method

This exploratory case study is of one student (Mark) who was part of a pilot project that investigated students' spatial reasoning and conceptual knowledge of cubes and cuboids (rectangular prisms). The project involved Year 3 and 4 students (aged 8-10) from two Australian primary schools (N=23). Students were asked to respond to a one-to-one geometric reasoning interview, designed by German authors (Reinhold & Wöller, 2016), using Froebel's Gifts. The Australian teachers reported that wooden blocks were not used as part of the school's curriculum and it was unlikely that students had experienced tasks of this nature. Mark (pseudonym) was selected because his responses and reasoning after constructing rectangular prisms were influenced by a "learnt definition" of the properties of prisms. His responses were also representative of those of other students.

The interview was initially used with German students to assess Year 3 and 4 students' geometric knowledge and spatial reasoning. As the term cuboid (used in the original interview) is not used in the Australian curriculum it was substituted for rectangular prism. The following questions were asked in the interview:

1. Close your eyes and imagine a rectangular prism. Describe what you see.
2. Please finish this sentence," A rectangular prism is…."
3. I want you to build a rectangular prism using these blocks (2cm cubes).
4. How do you know this is a rectangular prism?
5. Can you build a different rectangular prism?
6. How do you know this is a rectangular prism?
7. I want you to build a rectangular prism using these blocks (small rectangular prisms). Please explain what you are doing.
8. How do you know this is a rectangular prism?
9. Can you build a different rectangular prism?
10. How do you know this is a rectangular prism?
11. Compare this cube (one 2cm cube) with this rectangular prism (small rectangular prism). How are they the same and different?
12. Is this construction (4 cubes together in single layer) is a rectangular prism? Why? Why not?

Froebel's Gift Sets 3 (congruent cubes) and 4 (congruent rectangular prisms) were used in the interview. Reinhold, Downton, Livy, and Wöller (2017) provide further explanations of the Gift Sets including their geometrical features and pedagogical intentions underpinning the use of these blocks in primary mathematics education.

Data collection and analysis

The Australian authors conducted the one-on-one interviews with each student. One provided the Froebel block sets and asked the interview questions, while the other filmed the student's constructions and took field notes. Each interview took approximately 30 minutes to complete. Grounded theory was used to develop guidelines for detecting student interview responses. Each author independently used open coding while viewing the videos, identifying key themes related to each student's response. In collaboration, the authors conducted a further cycle of refined coding (Table 1) to reach agreement on interpretations and categories. The first column of our data analysis tool shows three categories and corresponding questions from the interview.

Table 1.
Categories and Codes Used for Analysing Student Constructions of Rectangular Prisms.

Category (and interview question numbers)	Codes
Explanation, argument and justification. (Questions: 1, 2, 4, 6, 8, 10, 11, 12)	1.1No response
	1.2 Incorrect response
	1.3 Using informal language
	1.4 Informal language and enumerating
	1.5 Name properties and use correct mathematical terms and enumerate
	1.6 Reference to 2D and 3D relationship
	1.7 Other responses
Construction processes using cubes/ prisms (Questions: 3 and 5 when using cubes)	2.1 No Construction
	2.2 Single cubes/prisms randomly placed
	2.3 Single cubes/prisms randomly placed leading to rectangles
	2.4 Single cubes/prisms forming rows or columns
	2.5 Multiple cubes/prisms forming rows or columns.
(Questions: 7 and 9 when using rectangular prisms)	2.6. Orientating the prisms when building layers or walls
	2.7 Starting with cubes/prisms then "stretching"
	2.8 Other
Completed product	3.1 No response
	3.2 Incorrect response (incomplete rectangular prism)
	3.3 Uses one piece (small rectangular prism)
	3.4 One layer (e.g., using 2 cubes)
	3.5 Two layers or more with four rectangular and two square faces
	3.6 Six rectangular faces

The following results report Mark's responses to questions 2 to 10 and 12. Questions 3 to 6 refer to his construction of rectangular prisms using small cubes and questions 7 to 10 refer to his construction of rectangular prisms using small rectangular prisms.

Results and Discussion

The results include the interview questions and reference to the codes in Table 1. The discussion of results makes links to the theoretical framework (van Hiele's levels) presented in the review of literature.

When Mark was asked to complete Question 2, "A rectangular prism is…." he said, "It is longer and bigger than a square and has eight corners and six sides." We coded this response as 1.4 Informal language and enumerating (Table 1) because Mark did not use correct mathematical language for the properties of his rectangular prisms such as faces and vertices or the geometric names of the faces.

Constructing a Rectangular Prism with Cubes

For Question 3, Mark was given a set of blocks (cubes) and asked, "Build a rectangular prism." First, he used six blocks to make a layer 2 (length) by 3 (width) by 1 (height) (Figure 1a), then he added a second layer making a 2 by 3 by 2 (Figure 1b). He rotated his construction (Figures 1c and 1d) and his completed product includes two square faces and four rectangular faces (Figure 1e). We coded Mark's construction process as 2.4 Single cubes (one by one) forming rows, and 2.6 as he rotated the construction part way through to position the square face directly in front of him. The completed product was coded as 3.5 Two layers or more with four rectangular and two square faces.

Figure 1. Mark's sequence when correctly constructing a rectangular prism with cubes (a, b, c, d, e).

For Question 4 when asked to explain, "How do you know this is a rectangular prism?" Mark said, "It has 6 sides, 8 corners, it is larger and more stretched out than a cube." We coded this response as 1.4 (Informal language and enumerating) rather than 1.5 (Name properties and use correct mathematical terms) because he referred to faces as sides and vertices as corners.

For Question 5 when asked to construct a different rectangular prism using these cubes he constructed a 3 by 4 by 1 prism, then added a second layer (Figure 2). The second prism was wider and longer. Rather than placing cubes one by one as he had done for Question 3, he placed three blocks down row by row when making the base and this was coded as 2.5 Multiple cubes forming rows or columns.

Figure 2: Marks two responses when making rectangular prisms with cubes.

The final product in Figure 2 (3 by 4 by 2) was coded as 3.6 Six rectangular faces. This response was the first time he had made a prism with no square faces.

When asked to justify (Question 6) why the response in Figure 2 was a rectangular prism and what was the same or different, Mark replied, "They both have 6 sides and 8 corners. They are different because one is larger than the other." Note he did not make reference to the shape of the faces. This response was also coded as 1.4.

He was then prompted to notice the faces of each construction. Mark identified the faces of the larger prism (in front, Figure 2) as rectangles and said, “The other has rectangles and squares [faces]” (see, Figure 1e).

Next, he was asked if they were both rectangular prism and stated, “No not exactly... because this is not a square [pointing to the face of the larger prism].”

Both constructions (Figures 1e and 2) were correct, although Mark was not convinced. Mark’s explanation suggests he had an incomplete or prototypical view of prisms. His perception seemed to be that all rectangular prisms have square faces at the ends and he had difficulty recognising and naming a construction with six rectangular faces as a prism.

Constructing a Rectangular Prism with Rectangular Prisms

For Questions 7 and 8 Mark was asked to construct a rectangular prism using rectangular blocks and to justify. First, he used nine blocks to construct a rectangular prism 3 by 1 by 3 (see, Figure 3 a). He then took blocks away leaving a 2 by 1 by 2 (Figure 3b) and said, “This is not a rectangular prism because this [face] is not a square,” [pointing to the end of the prism - Figure 3b]. Mark then removed two blocks and said, “This is a rectangle prism because it has six sides and eight corners, this is a square and this is a rectangle prism,” [pointing to the faces Figure 3c].

Figure 3. Mark’s sequences when making a rectangular prism with rectangular prisms (a, b, c).

When asked why Figure 3b was not a rectangular prism Mark said, “Well if you add these two up it is not a square [and pointed to the front face of Figure 2b which is not a square] but it doesn’t have to be, but I prefer it to be.” This construction was coded as 2.5 Multiple prisms forming rows or columns. The completed product was coded as 3.5 Two layers with four rectangular and two square faces. Mark’s justification as to why Figure 3c was a rectangular prism suggests his explanation is 1.4 because he is still relying on informal language to describe the properties of prisms.

The dialogue of this section of the interview reflects a shift in Mark’s thinking relating to the properties of a rectangular prism. He confirms that a prism does not require square ends but that he “prefers it to be.” As the interview progresses he starts to further articulate aspects of geometric reasoning when constructing and justifying his responses.

For Question 9, “Can you build a different rectangular prism?” Mark rotated the blocks placing them sideways rather than flat and made a 3 by 1 by 2 prism (Figure 4a).

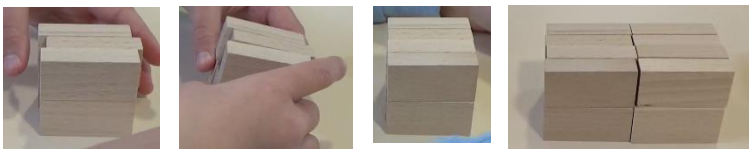

Figure 4. Mark’s second construction using rectangular prisms (a, b, c, d).

Mark tilted the blocks (Figure 4b), suggesting he was attending to the shape of the front face (rectangle) rather than the other faces. Although he had made a prism, Mark chose to continue constructing adding another column (Figure 4c), then extended the width making a 4 by 2 by 2 prism (Figure 4d). This construction was coded as 2.5 multiple prisms and 2.6

orienting the prism when building layers. The completed product was coded as 3.5 as it had two square faces and four rectangular faces.

For Question 10 he was asked, "How do you know if this is a rectangular prism?" Mark said, "It is longer and bigger than a square," highlighting his incomplete understanding of the properties of a prism. He named the properties of the prism using informal language and was coded 1.4.

Finally, Mark was asked to clarify if the construction was a rectangular prism. Mark again demonstrated uncertainty as illustrated in the following dialogue:

> This could be a square but it is not a cube. Yes, we could call it a rectangular prism [then he changed his mind] … it was not because the sides are the same [pointing to the four sides] … this side is not larger than the other [focusing on the property of length]. It is a shape but it might be an irregular shape and I cannot name it.

While Mark demonstrated some understanding of prisms (e.g., he constructed [and possibly visualised them] in layers; understood the attributes of a square though not its inclusiveness with rectangles; and that prisms have 8 corners and 6 sides [faces]), he incorrectly believed that all rectangular prisms have two square faces and that those square faces should be orientated at the front and back of the prism. Mark's spatial reasoning suggests he is operating at van Hiele Level 2 because he described the properties of a prism using informal language and was unable to logically relate the properties of his constructions to one another.

In summary, this study provided an in-depth analysis of Mark's spatial reasoning and lack of conceptual knowledge when constructing prisms. Our findings contribute to the literature by highlighting the importance of students constructing prisms and deriving the properties of prisms through experimentation.

Conclusion and Implications

A critical analysis of a Year 4 student's responses to a one-on-one interview provided insights into Mark's conceptual knowledge of the properties of a rectangular prism. Having to justify and explain his construction assisted with a shift in his understanding of the properties of a rectangular prism, rather than relying on learnt knowledge that a rectangular prism has 6 faces and 8 corners. We also acknowledge only one student was reported here, however the results and discussion revealed the importance of questioning and probing student thinking to elucidate their knowledge about rectangular prisms.

Importantly, Mark's ability to draw on spatial and mathematical language to identify and describe the features of and relationship between two- and three-dimensional shapes (square, rectangle, vertices) limited his capacity to engage in more sophisticated, reasoned responses to our questions. Mark focused on isolated bits of information rather than being able to visualise the entire structure of the constructions, suggesting an inability to simultaneously integrate multiple aspects of visual or cognitive information, which concurs with earlier findings (Battista & Clements, 1996; Reinhold, 2007).

Our findings suggest that construction of three-dimensional objects supports students' development of geometric reasoning. As highlighted within the ACM (ACARA, 2017) students need to generate the properties of three-dimensional objects via construction. In doing so they are able to logically classify families of solids and work toward achieving Level 3 of the van Hiele framework. As previous research indicated students' geometric reasoning improves when they have opportunities to construct solids with materials; use spatial language to describe the properties of prisms; and begin to notice relationships between the properties of prisms (Ambrose & Kenehan, 2009; Lehrer & Curtis 2000).

Implications for teachers include engaging students in investigations that involve generating the properties of three-dimensional objects via construction with a range of materials. Second, that teachers engage students in exploring non-prototypical forms and to look for logical connections and relationships between objects. Third, conducting an interview could also provide teachers with an opportunity to notice how students explain and justify their constructions, and identify their stages of geometric thought. As Moss et al. (2015) found, professional learning helped teachers to notice student understanding within students' actions of constructing three-dimensional objects.

Further exploration of our research within an Australian context would extend the work of earlier studies (Reinhold & Wöller, 2016; Sinclair & Bruce, 2015). Ongoing in-depth analysis of individual students' geometrical concepts of rectangular prisms will also aim to strengthen these findings and connections with the van Hiele levels. In addition, the data analysis tool used to analyse and code student interview responses adds to the literature and could be useful for teachers when assessing students' understanding of three-dimensional objects. In subsequent papers, we will report further on the use of these categories and codes after analysing the other 22 (and future) student data.

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Proposed Structural Refinements to the Interconnected Model of Teacher Professional Growth

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This paper proposes structural modification of the Clarke and Hollingsworth (2002) Interconnected Model of Teacher Professional Growth (IMTPG) based on results from doctoral research that studied the changes in mathematical knowledge and beliefs of two Year 5/6 teachers as they implemented a four-week, innovative curriculum unit. These inclusions into the current model expand its analytical, interrogatory and predictive functions. This in turn increases its relevance for those implementing professional learning at the school level, enabling them greater insight into the aspects of a teacher's world that require more support, and possibly more challenge.

The process of teacher change is an important element in the overall success of professional development programs, yet one that many such initiatives fail to consider (Guskey, 2002; Justi & Van Driel, 2005). Research on the features of professional development that promote teacher change (Clarke, 1997; Elmore, 2002; Sowder, 2007) suggest learning occurs gradually and iteratively, and that the success of interventions depends on the individual and context. This variability has led to calls for a research emphasis away from *whether* a program is effective or not, to a focus on *how* programs work in particular settings to promote teacher learning, and what the learning pathways of teachers with different knowledge, beliefs and pedagogical practices look like (Goldsmith, Doerr, & Lewis, 2013; Simon & Tzur). The product of this shift presents 'naturalistic' generalisation opportunities (Stake, 1995) for schools designing and implementing similar professional learning strategies.

The purpose of this research was to explore the impact of teaching an innovative mathematics curriculum on teachers' knowledge and beliefs about mathematics and mathematics teaching and learning. As innovative tasks are conceptually demanding, such immersion experiences have the potential to build mathematical knowledge while teaching. The Interconnected Model of Teacher Professional Growth (IMPTG) (Clarke & Hollingsworth, 2002; see Fig. 1) provided a lens to describe and interpret meaning constructed by teachers in this change environment. The epistemology of constructionism was chosen to explore teacher's constructed realities as they engaged with their world. A theoretical perspective of intepretivism was employed as this study aimed to draw an in-depth understanding of the interpretations each teacher made within the school context. Case study was the adopted methodology as it complemented the study's epistemology and theoretical perspective, facilitating an understanding of the complexity of the phenomenon under study (Merriam, 1998).

The IMTPG sets out possibilities for change in one or more of four domains: external, personal, of practice, and of consequence. It represents a move away from previous linear modelling of teacher change (Guskey, 1986), suggesting there are multiple and cyclic growth pathways. It aligns with an increasingly popular view of change as professional growth, acknowledging teachers as active and reflective learners, socially situated within a learning

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environment (Clarke & Hollingsworth, 2002). The model was employed to describe interaction between domains, the mediating processes of enactment (putting a new idea, belief or practice into action) and reflection, and the resultant (professional) growth. The term reflection was viewed in the same vein as Dewey (1910), that is, "active, persistent and careful consideration" (p. 6). Clarke and Hollingsworth (2002) suggested change may occur in one domain of a teacher's world but this may not lead to change in other domains. A 'change sequence' is represented when two or more of the identified domains are connected by reflective or enactive links. This may be momentary experimentation, not necessarily sustained. A 'growth network' on the other hand represents considered and long-lasting change in cognition and/or behaviour.

The External Domain represents information sources or stimuli outside a teachers' day-to-day context. The other three domains can be described as belonging to the teacher in the context of their daily work, that is: teachers' professional experimentation, teachers' knowledge and beliefs, and teachers' salient outcomes. In this research, the main source of stimulus in the External Domain was teaching the *Some of the Parts* unit (Britannica, 2006). In the personal domain, the Mathematical Knowledge for Teaching model (MKT) (Hill, Ball & Schilling, 2008) was used to label the kinds of knowledge present in the practice and reflections of the participating teachers while implementing the innovative curriculum unit.

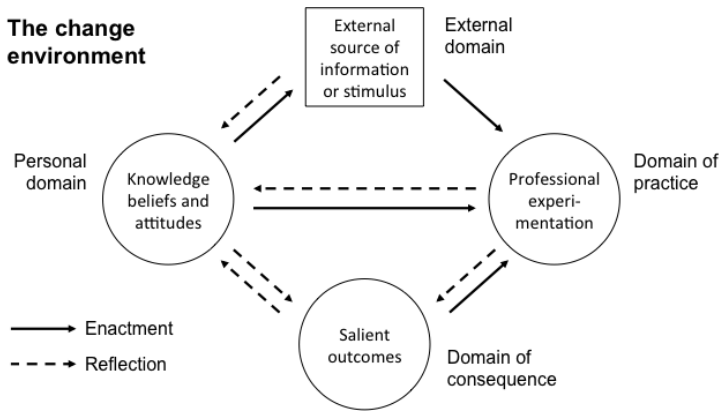

Figure 1. The Interconnected Model of Teacher Professional Growth (Clarke & Hollingsworth, 2002).

Many researchers have used the IMPTG to identify pathways of change as a result of professional learning (Lebak, 2015; Wongsopawiro, Zwart, & van Driel, 2017; Zwart, Wubbels, Bergen, & Bolhuis, 2007). Most of these studies involved science and mathematics high school teachers.

Characteristics of the IMPTG model have been both narrowed and reinterpreted by researchers to aid analysis. Lebak (2015) used the IMPTG to model factors mediating change in beliefs and practice of a high school science teacher in a year-long video-supported reflection process. Recognising the complexity involved in studying the relationship between beliefs and enacted practice, analysis using the IMPTG model involved only the attribute of beliefs in the Personal Domain. Changes in the participating teacher's knowledge of his students' capabilities were evident in the discussion but did not form part of the modelling; they were viewed only in relation to the teachers' beliefs about what students could do. Justi and van Driel (2006) reinterpreted Clarke and Hollingsworth's (2002) definitions of change and growth in their analysis of beginning science teacher's knowledge of scientific models and modelling. Due to the time constraints of the research period, they

used the criterion of complexity to identify the difference between "superficial change" (p. 443) and growth rather than how long the change was maintained.

Zwart et al. (2007) adjusted the IMPTG model in four ways. Two of these were definitional, and two structural. The researchers broadened the concept of reflection to suit their understanding of the term to incorporate the intention of such a process, and reconsidered the Person Domain as an integrated whole rather than the individual attributes of knowledge, beliefs and attitudes. In a more pragmatic vein, Zwart et al. divided the External Domain to reflect the specific professional learning introduced into the research (reciprocal peer coaching, and other generally available sources of information), and delineated the preparation and implementation of lessons in the Domain of Practice. Wongsopawiro et al. (2017) adapted the External Domain and Domain of Practice in a similar way. Unlike Zwart et al., however, they highlighted four PCK elements in the Personal Domain to better reflect change in PCK.

Reflection is acknowledged as an important internal process when studying the progression of teacher change (Darling-Hammond, 2005; Mewborn, 1999; Sowder, 2007). Research using the IMPTG has identified patterns in change that suggest focused and structured reflection on students' learning was an important catalyst for PCK development (Wongsopawiro et al., 2017) and that reflection occurred more often than enactment as the first mediating process for change (Zwart et al., 2007). Considering its pivotal influence on changes in teacher knowledge and as a stimulus to engage teachers in the change process, greater insight into the influence of reflection on teacher cognition and behaviour would be beneficial. The current IMPTG structure has some limitations in this regard. While successfully representing change between domains it fails to demonstrate change within a domain. Considering the emphasis in reform-oriented teacher professional learning on developing teacher knowledge and beliefs, such adaptation to the model may be assistive.

Methodology

This case study research involved an experienced male and female teacher at a single New South Wales government primary school site. Mark taught Year 5 and Debbi taught Year 6 (pseudonyms). Both volunteered to be participants in the research. Each teacher implemented the four-week fractions unit called *Some of the Parts* (Brittanica, 2006). Based on the principles of Realistic Mathematics Education (RME; see, e.g., Streefland, 1991), this unit progressively introduced 'models of' realistic contexts to provide 'models for' student thinking, including fraction bars, double number lines and the ratio table. The teacher support guide provided details about the underlying mathematics of each section within the unit, how to implement the tasks and the associated modelling, outlined possible student responses and ways to assess student understanding. During the unit, students were encouraged to conjecture, explain and justify their reasoning, thereby promoting conceptual understanding through "reinvention" of mathematics (Gravemeijer, 1999). These features led to the unit being considered innovative.

To gain a broad understanding of how each teacher designed and implemented lessons in their normal routine, both Mark and Debbi were observed for three pre-intervention lessons. The Copur-Gencturk (2015) classroom observation record assisted in recording these aspects of each lesson, as well as the mathematical discourse and sense-making promoted, and the classroom culture. In this pre-intervention phase, both Mark and Debbi completed two questionnaires developed as part of the Teacher Education and Development Study in Mathematics (TEDS-M) (Tatto, et al., 2008). The first questionnaire assessed their mathematical and pedagogical content knowledge and the second questionnaire their level

of agreement with statements relating to teachers' beliefs about learning mathematics, mathematics achievement, and the nature of mathematics. Background forms generated data about personal mathematics history and professional development experiences.

The innovative curriculum trail period was four weeks, with four lessons planned for each week. Meetings were held with each teacher in the term preceding implementation to assist with the familiarisation of the unit's structure, the underlying philosophy of RME and the models promoted. Both Mark and Debbi were given their own copy of the teachers' guide and student response book to read over the two-week holiday break. In the first week of the following term, time was put aside for each teacher to clarify any questions before the trial started. Semi-structured interviews designed to prompt the teacher to reflect on how the lesson went, how it matched their intentions, moments they felt were significant, decision-making junctures, what they had learned, and what they believed the students had learned were held after lessons in all phases of the research. An extended reflective interview was held at the end of the research period.

Data Analysis

Making sense of the data in this study drew upon both direct interpretation and categorical aggregation, attaching meaning to small collections of impressions within a single episode, then searching for patterns and relationships (Stake, 1995). Constant comparison began with initial observations, incidents at interviews, notes made through direct observation of lessons or documents. These incidents were compared with others both within and across sets of data (Merriam, 1998). Consistent refinement addressed the large quantity of data generated and allowed questions arising to be asked *during* the field experience as well as after its completion. The search for patterns and themes in this research started with open coding. After locating themes and assigning initial codes, data were condensed and organised into preliminary categories via axial coding.

Results and Discussion

Following brief background on the teachers, most of this section focuses on the specific modelling of growth and change of each teacher and how an adapted version of the IMTPG assisted with this.

Both Mark and Debbi had taught for around 40 years; both said they loved teaching mathematics. Mark had only taught either Year 5 or 6 apart from three years in lower primary grades. He had tutored mathematics to Year 12 level to "keep his mind active" and thought there was great benefit in spending time doing complex mathematics. Mark felt there was "nothing really that kids can throw at me that I can't answer off the bat" (31.7.15), indicating how confident he felt in responding to problems and explaining the underlying mathematics while in the act of teaching. Mark's belief about his role of instructor and explainer in the classroom was tightly held; a position that was well understood by his students. Debbi had taught her current grade for more than 10 years. On her background survey she reported to be highly confident both in teaching mathematics in general, and in catering for the needs of higher achieving students. Debbi indicated she would like to be a specialist mathematics teacher in the primary setting in the future. Interestingly, both Mark and Debbi scored noticeably higher on their PCK percentile score than their MCK score on the TEDS-M assessment. A third of the difficulties Debbi encountered in this assessment related to extended reasoning, suggesting that explanations of mathematical ideas may be challenging for her.

Mark and Debbi had very different responses to the innovative curriculum trial. Mark adopted a growth perspective in relation to his knowledge and beliefs about the teaching of fractions. Implementing *Some of the Parts* tasks challenged Mark's knowledge of teaching difficult concepts like equivalence, and his long-standing beliefs about the most effective way to do this. The changes Mark observed in the engagement of his students and their ability to solve complex fraction problems challenged his knowledge of how students interacted with content. At times Mark found the tasks and ensuing changes in his classroom confronting but he trusted the expertise of the unit writers and was willing to persist. Debbi was concerned about the applicability of the unit from the beginning of the research trial. She felt it was not current or challenging enough for her higher achieving students, demonstrating a lack of trust in its progressive modelling approach. Debbi used only some of the realistic tasks and the associated models. Her often stated concerns that the tasks did not challenge her 'top students' resulted in disparate presentation of lessons. Such prioritising of one group of students over the rest of her class resulted in limited interaction with the underlying RME philosophy and most of the models presented. Growth evident in Debbi's knowledge of how to use ratio tables to promote understanding of proportional reasoning was also driven by her focus on the extension of higher achieving students.

Modelling Mark and Debbi's pathways of change using the IMPTG assisted analysis of the categories of knowledge and beliefs affected by the introduction of innovative curriculum into the classroom setting. In this modelling it became evident that the results of the research could be better represented. Participating teachers had made changes within the Personal Domain, in particular an internal relationship between teacher's knowledge, beliefs and attitudes. In some instances there were changes in both knowledge and beliefs, in others there were changes in knowledge but no reported changes in beliefs. One teacher changed attitudes as a result of changes in knowledge and beliefs; the other teacher's early attitudes towards the innovative unit dominated the change process. The mediating process for each of these options was reflection. While facilitating analysis in one domain or from one domain to another, the current structure of the IMPTG has not anticipated modelling of change *within* in a domain. If the IMPTG is considered a model of possibilities, this anticipatory status suggests that it can also function to represent lack of change or growth. An interesting opportunity, then, is created to compare why change may have occurred in one domain and not another. An expansion of the structure of the Personal Domain might allow greater representation of such an occurrence. As this research aimed to look closely at changes in the knowledge and beliefs of teachers implementing innovative curriculum, such structural inclusions have potential to be helpful in the data analysis phase.

Representation of Changes in a Refined Structure

In this section I outline two examples from the data generated in this study that demonstrate change within an IMPTG domain, one for Mark (Fig. 2) and one for Debbi (Fig. 3). These examples contribute to my argument for refining the structure of the IMPTG. Codes reflecting the current IMPTG domains are used in the following examples, that is: ED (External Domain), DoP (Domain of Practice), and DC (Domain of Consequence). Codes for the adapted version appear as PD-K (Personal Domain - Knowledge), PD-B (Personal Domain – Beliefs) and PD-A (Personal Domain- Attitudes). The mediating processes are also outlined.

Domain Relationships	Mediating processes	Description

ED to DoP	Enactment	Using the innovative curriculum introduced Mark to models he had not used to teach fractions including fraction bars, the ratio table, and double number lines.
DP to DC	Reflection	As he used the models, Mark reflected on how much he was able to hear students' understanding and the reasoning they were able to demonstrate.
DC to DoP	Enactment	Mark followed the curriculum's presentation of such models with fidelity.
DP to PD-K	Reflection	Using a range of models to represent big ideas like equivalence caused Mark to reflect on the knowledge he had and was gaining in relation to teaching fractions.
PD-K to PD-B	Reflection	In Mark's reflections about instructional advantages of using such models (knowledge of content and students, knowledge of content and teaching), he stated his new belief: that this was a better way to teach fractions than he had used previously in his career.
PD-B to PD-K	Reflection	This change in beliefs about teaching fractions through hands-on activities and models promoted reflection on Mark's new knowledge of curriculum options and his intention to use them again.
PD-K to DC	Reflection	Changes in Mark's knowledge and beliefs about using models to teach fractions encouraged further reflection on the salience of such knowledge, and the benefits for his students.

Figure 2. Mark's IMPTG pathway of change in relation to using models to teach fractions.

The reflection that occurred from Mark's change in knowledge about the instructional advantages of using the RME modelling to change in belief that this was a better way *for him* was an important result in this research. It demonstrated that reflection on new knowledge had changed his beliefs, not action or further reflection on aspects outside his Personal Domain.

In the case of Debbi, change within the Personal Domain is reflected by intensification of what is known (knowledge of curriculum), and what is believed (what 'lazy teaching' looks like). The reflection while immediate was not fleeting; it had been considered for a long time and was being activated by an external stimulus (*Some of the Parts*).

This example represents 'lack of change,' and reinforcement of current thoughts and beliefs. Cognition is activated but knowledge, beliefs and/or attitudes are not changed in this process. This mapping extends the current possibilities for which the IMPTG can be engaged. Such resistance is not unusual in the process of change, but a vehicle needs to be present to map it. Without the proposed refinement in the structure of the Personal Domain shown in Figure 4, mapping of Debbi's initial reaction to the unit would have been restricted, and it is this initial reaction that affected much of the enactment that followed.

In making these structural refinements I present my assumptions about the terms knowledge, beliefs and attitudes. Thompson (1992) distinguished between knowledge and beliefs in two ways: conviction and consensuality. Beliefs can be held with different levels

Domain Relationships	Mediating processes	Description
ED to PD-K	Reflection	Debbi was introduced to the *Some of the Parts* unit as part of her orientation to the innovative curriculum trial. Looking through the unit, Debbi reflected on her personal knowledge of curriculum "units like this" (11.6.15) in England.
PD-K to PD-B	Reflection	Recalling this knowledge, and looking further through the unit caused Debbi to reflect on her beliefs about the best way to teach mathematics; her belief intensified the longer she looked at the teachers' manual and students' workbook.

PD-B to PD-A	Reflection	Debbi believed the *Some of the Parts* unit would not be effective in extending her higher achieving students, leading to rejection of many of the activities in the unit even before they had been trialled. Debbi voiced immediate concerns about its appropriateness for her students, describing such units as "lazy teaching" (11.6.15). Voicing this belief changed Debbi's initially positive attitude to being part of a research trial.
PD-A to DoP	Enactment	This early negative attitude affected the degree to which Debbi was prepared to trial the curriculum in her classroom and her willingness to experiment with the models presented. There was little reflection from the DoP back to the PD as this negative attitude was so pervasive.

Figure 3. Debbi's IMPTG change pathway in relation to using the *Some of the Parts* unit.

of conviction but knowledge is not. Knowledge is consensual (there is general agreement about ways to judge its validity) whereas beliefs are subjectively held understandings thought to be true. Attitudes are less cognitive than beliefs and felt more intensely. They are considered harder to change than beliefs (Phillip, 2007).

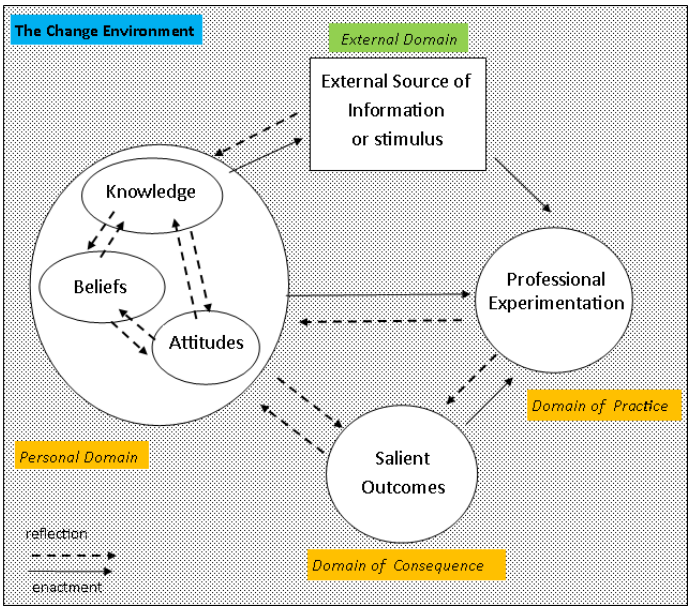

Figure 4. Proposed refinement of IMTPG model's Personal Domain.

Conclusion

Empirical identification of patterns in teacher growth are needed to identify how professional learning initiatives like innovative curriculum work in particular contexts. While there were only two teachers involved in this study, the proposed structural refinement of the IMPTG expanded the extent to which data generated in this study could be analysed. Mapping change and 'lack of change', and the associated pathways may provide professional learning providers with information about the aspects of a teacher's Personal Domain that require challenge and support. I acknowledge of course that further examples are needed to assess the validity of my proposed enhancements of the model.

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Developing a Rubric for Assessing Mathematical Reasoning: A Design-Based Research Study in Primary Classrooms

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Despite mathematical reasoning being a proficiency included in mathematics curricula around the world, research has found that primary teachers struggle to understand, teach, and assess mathematical reasoning. A detailed rubric involving the three reasoning actions of analysing, generalising and justifying at five proficiency levels was refined according to feedback from teachers. At different stages of the study, teachers used the rubric to assess their students' reasoning and provided feedback about its usefulness.

In the Australian Curriculum: Mathematics (Australian Curriculum and Assessment Authority [ACARA], 2017), reasoning is explicitly stated as a proficiency to be developed in students and is defined as being the '… capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising'. Despite its emphasis in many curricula around the world, research on teachers' knowledge and understanding of reasoning indicates that many teachers need support in enacting and assessing many aspects of this proficiency (Blanton and Kaput, 2005; Clarke, Clarke & Sullivan, 2012). This need has resulted in calls for more opportunities for teachers to learn about students' mathematical reasoning and its development (Francisco & Maher, 2011). Our previous research found that teachers developed their knowledge and understanding of reasoning through demonstration lessons and teaching it themselves (Loong, Vale, Herbert, Bragg & Widjaja, 2017). In addition, teachers need to know how students' reasoning can be assessed formatively. This assessment allows teachers to monitor students' reasoning proficiency and further develop it through regular planning of tasks that elicit a variety of reasoning actions, other than a commonly used action like explaining (Clarke, et al., 2012). A rubric for assessing mathematical reasoning will help teachers be aware of the reasoning actions and formatively assess the reasoning articulated and displayed by students. Pegg, Gutiérrez and Huerta (1998) noted that a method of assessment may not fit the specific requirements of teachers for various reasons, for example, it may be too time consuming, require an understanding of the topic or nature of learners' responses not accessible to the teacher, or may not be appropriate for a school context. For this reason, we chose to use a design-based research methodology (Wang & Hannafin, 2005) where teacher participants helped refined the rubric we designed to a level that teachers find useful. This paper reports on our experience in developing a formative assessment rubric for reasoning for reSolve, a national project funded by the Australian Federal Department of Education and Training

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managed by Australian Association of Mathematics Teachers (AAMT) and Australian Academy of Sciences.

Background

Reasoning Frameworks

Jeannotte and Kieran (2017) identified two categories of reasoning definitions in curriculum statements: the "structural aspect of mathematical reasoning (p. 7)" and the "process aspect of mathematical reasoning" (p.9). It is the "process aspect of reasoning" that is included in the Australian Curriculum. The search for similarities and differences, and processes related to validating are the two main categories of reasoning, where comparing and contrasting objects leads to forming conjectures and generalising. Ellis (2007) identified three levels of comparing and contrasting when analysing Year 7 student responses to growing patterns tasks: 1) relating, 2) searching and 3) extending. Lannin, Ellis and Elliot (2011) reorganised these categories and combined conjecturing and generalising to nominate four essential understandings of generalising: 1) developing statements; 2) identifying commonality and extending beyond original cases; 3) recognising a domain for which the generalisation holds; and 4) "clarifying the meaning of terms, symbols and representations" (p. 12). Validating enables students to convince others that a conjecture or generalisation is justified (Carpenter, Franke & Levi, 2003). Explanations are not sufficient to be convincing. Carpenter et al. (2003) identified three classes of justification to describe the ways in which primary students justify and argue: "appeal to authority; justification by example; and generalisable arguments" (p. 87). The authors tested these categories and levels of reasoning when investigating Year 3-4 students' reasoning when working on a commonality problem. Table 1 displays the levels of reasoning for the three reasoning actions identified in the students' reasoning.

Table 1
Reasoning Actions and Levels: 'What else belongs?' (Vale et al., 2017)

Reasoning Actions	Reasoning Levels
Comparing and contrasting	Noticing (seeing) similarities or relations
	Noticing commonalities and differences
	Searching for commonalities
Generalising	Forming conjectures about common properties
	Extending a common property through further examples
	Generalising properties
Justifying	No justification
	Appealing to authority or others
	Explaining a common property using an example or counter property
	Verifying that the common property holds for each member of the group
	Extending generalisation using logical argument

Assessing Reasoning

The literature reports three types of assessment, namely, assessment for learning (AfL) where information collected from assessments are used to modify teaching and learning (ARG, 2002; William, 2011); assessment of learning (AoL) where achievements are summarised for the purpose of recording and reporting to relevant parties (Harlen, 2007);

assessment as learning (AaL) where the student monitors what they are learning and uses that feedback to make adjustments, adaptations and major changes in what they understand (Earl, 2003). AfL and AaL are formative types of assessment whereas AoL is summative in nature. Formative assessment enables the teacher to systematically gather evidence and provide feedback about learning while instruction is underway. We concur with Pegg and colleagues (1998), on the benefits of "forms of assessment which allow for the interpretation of learners' responses within a framework of cognitive growth… allows teachers to see where their learners are on some developmental ladder and, at the same time, provide advice on possible pathways for future teaching endeavours." (p.4).

The Use of Rubrics for Assessment

Rubrics have been designed and researched for their efficacy in promoting thinking and learning as well as making the assessment criteria required transparent to students (Panadero & Jonsson, 2013). Instructional rubrics have also been found to be useful for teachers. Andrade (2000) for example advocates their use as they help teachers teach, make assessing student work quick and efficient, and help teachers justify to parents and others the grades that they assign to students. Using a rubric allows the teacher to infer the gap between the students' current learning and desired instructional goals, identifying students' emerging understanding or skills so that they can build on these by modifying instruction to facilitate growth.

Rubrics and models that are available for assessing reasoning include those for geometry such as the van Hiele's levels of geometric reasoning (van Hiele, 1986) and the SOLO taxonomy (Pegg et al., 1998). However, these are mostly frameworks that detail the progression of reasoning for a particular content domain rather than for mathematical reasoning in any content domain. Given teachers' need for assessment tools that are pragmatic and usable in schools that tracks cognitive growth in students, our research adopted a design-based methodology to work with teachers to create this rubric.

Methodology

Design-based research is a systematic but flexible methodology that aims to improve educational practices through iterative cycles of design development, implementation, and analysis in collaboration with practitioners (Wang & Hannafin, 2005). Wang and Hannafin identify five basic characteristics of design-based research: pragmatic, grounded, interactive, iterative and flexible, and integrative and contextual. These characteristics are evident in our design process as it seeks to solve a practical on-going issue of how to assess the reasoning proficiency. It is grounded in the theoretical frameworks of teaching and learning of mathematical reasoning and real-world implementations of it. Our design is one where teachers interact with the research team to iteratively refine the rubric. Our research focused on creating a rubric that enables teachers to use it in the everyday context of their teaching. It also aimed to provide teachers with sample lessons that elicit reasoning and examples of teachers' use of the rubric.

Participants

The participants included the six members of the research team, 32 teachers from four primary schools in Victoria, Australia, and a critical friend expert from AAMT/AAS who provided constructive feedback on the rubric in the penultimate stage of project. Two

teachers from each year level across Years 3-6 from each of the schools trialled the rubric twice in their grade, each time with the most recent version of the rubric.

Methods

There were four stages in our design-based research project. In Stage 1, the Mathematical Reasoning Research Group (MaRRG) developed an initial reasoning assessment rubric for trial in schools. In Stage 2, the research team provided professional development (PD) on mathematical reasoning to each of the four participating schools to develop teachers' awareness of mathematical reasoning is, and ways to elicit it in their classroom. A one-hour whole school PD workshop on assessing mathematical reasoning was conducted by two researchers at each school prior to teachers trialling the assessing mathematical reasoning rubric. We provided PD because it is crucial in assisting teachers to come to an understanding of the nature of mathematical reasoning (Loong et al., 2017) before attempting to assess it. Researchers met with participating teachers to discuss trialling the reasoning tasks and rubric, and how to use the teacher observation schedule. Pairs of teachers in the same year level selected one of the tasks provided by the research team to teach and observed each other teaching the same lesson. Two researchers observed each lesson. The observing teacher and researchers used the observation sheet to record evidence of student mathematical reasoning. The teaching pair together with the two researchers then engaged in a post-lesson discussion lasting between 30 minutes to one hour. The focus of the post lesson discussion was on teachers assessing student reasoning that they observed, students' work samples and the teacher observation schedule using the rubric. Feedback from the post lesson discussion at the first school led to a modification of the rubric.

In Stage 3, the other three schools trialled Version 2 of the Assessing Reasoning Rubric using one of the tasks provided by the research team. Feedback for further modification occurred successively as each school attempted to use the rubric to assess their children's reasoning. For a second round of trials at each school the researchers then provided links to useful resources for locating tasks with a reasoning focus to support teachers in their development of a follow up lesson intended to include opportunities for students to reason and for teachers to assess. The intention was for them to trial the revised rubric to assess children's reasoning in these tasks. In the second round of Stage 3, classroom teaching and learning of the reasoning task and post lesson discussions with teachers were video-taped to provide data for exemplar materials for the assessment of mathematical reasoning. In Stage 4, the rubric was further revised using feedback from teachers in Stage 3 and the *reSolve* critical friend. The final simplified version was presented to teachers at a mathematics education conference and feedback was gathered using field notes.

Results and Discussion

Stage 1

Our previous research (Vale et al. 2017) provided an initial framework (Table 1) for designing an assessment rubric for mathematical reasoning. However, this framework was based on a particular type of generalisation problem, involving forming conjectures about a common property and was therefore not necessarily appropriate for other generalisation and justification tasks. We investigated other frameworks for other types of reasoning tasks: Lannin et al. (2011) for early algebra problems and Carpenter et al. (2003) for justification and proof tasks. We also consulted NRICH (1997-2018) where five steps were identified in

the progression of reasoning proficiency for tasks involving the testing of conjectures. These were:

Step one: Describing: simply tells what they did.

Step two: Explaining: offers some reasons for what they did…

Step three: Convincing: confident that their chain of reasoning is right…

Step four: Justifying: a correct logical argument… uses words such as 'because', 'therefore', 'and so', 'that leads to' ...

Step five: Proving: a watertight argument… (https://nrich.maths.org/11336)

Synthesising our findings, we decided upon three key reasoning actions to be included in the initial rubric: 'analysing', 'forming conjectures and generalising' and 'justifying and logical argument.' We found that each of the reasoning verbs included in the definition of reasoning in the Australian curriculum aligned with one of these key reasoning actions. We thought about attempting to identify reasoning outcomes for each year level but the research does not provide evidence for this. Students at a young age are capable of providing a watertight argument relative to their knowledge of content and use of materials and symbols. Conversely, without the opportunity to develop reasoning proficiency students in later years may not have developed the proficiency to notice, generalise and justify. As well, students may display different developmental levels across the three reasoning actions. Consequently, we decided upon five levels in the reasoning learning trajectory for use in the rubric: 'not evident', 'beginning', 'developing', 'consolidating' and 'extending'. Intentionally, the levels were not aligned to school year levels. Figure 1 provides the descriptors in the rubric for the "developing" level for each reasoning action (space does not allow exhibition of all levels for this version).

Stage 2

As a result of feedback from teachers in the post lesson discussions during the first round of the teaching and learning of reasoning tasks at two schools, the following modifications were made. Changes to the rubric included:

Formatting the rubric to fit on a single A4 page.
Providing a space below to include teacher's comment "Evidence of reasoning"
Highlighting/Bolding keywords in the rubric
Reducing "wordiness" of rubric
Consistency of tense and wording in bullet points

Further issues that arose from Stage 2 included the limitations in assessing student reasoning solely on the use of a work sample. Many teachers commented on the ways students often expressed their reasoning verbally and through gesture. They suggested ways teachers could capture this evidence to complement the work sample to provide a more accurate assessment of students' reasoning actions/capabilities. Modifications were made to the rubric to include space for teachers to include evidence of students' gestures and verbal explanations, as listed above and teachers used a revised version in Round 2 of the trials.

	Analysing	Forming Conjectures and Generalising	Justifying and Logical argument
Developing	• Notices a common numerical or spatial property. • Sorts and classifies cases according to a common property. • Orders cases to show what is the same or stays the same and what is different or changes. • Recalls, repeats and extends patterns using numerical structure or spatial structure. • Describes the case or pattern by labelling the category or sequence.	• Communicates a rule about a ***property*** using words, diagrams or number sentences. • Communicate a rule about a ***pattern*** using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition. • Explain the meaning of the rule using one example.	• Checks the truth of statements using materials and informal methods. • Uses known facts to verify that the statement, common property, or rule for a pattern holds for each case. • Uses a counter example to refute a claim. • Starting statements in a logical argument is correct and accepted by the classroom • Detecting and correcting errors and inconsistencies using materials, diagrams and informal written methods.

Figure 1. Excerpt from Assessing Mathematical Reasoning Rubric - Version 1.

Stage 3

The first two schools and teachers were observed again using tasks that they had found themselves and schools 3 and 4 were observed twice using tasks provided by the team and those selected by the teachers. All teachers in this stage provided feedback on Version 2 of the rubric. They made positive comments in relation to the modifications made and the overall "user-friendliness" of the rubric, for example, bolding of words, one-page format, reduced wordiness.

> Well the first improvement is it's all on one page… it's much more user friendly when it's all on one page. (School A, Year 5/6 teacher, Round 2)

> I like the use of bold. (School A, Year 5/6 teacher, Round 2)

Overall it is a simpler and less wordy rubric yet still provides support to teachers with less well-developed understanding of reasoning who may require more complete explanations and guidance how to move forward in their planning for reasoning.

> … I quite like it because it's making me learn what they should be doing, I'm thinking maybe I should be encouraging them to verify the truth of what they're saying more … (School B, Year 5 teacher, Round 2,)

> … if we can use it as not only to inform us about our students but for where to next. (School A, Year 5/6 teacher, Round 2)

However, feedback from the teachers at these schools also indicated that for teachers with a good understanding of mathematical reasoning, the rubric was still too wordy. Although they have not come across a reasoning rubric, they preferred a simple version. For example:

> It has to be simplified there are too many aspects … it is too time consuming … (School B, Year 4 teacher, Round 2)

However, for teachers still coming to terms with the nature of mathematical reasoning a detailed rubric was useful and provided much guidance and language to use in reports.

> But the statements would be good for us as well to put in reports and stuff, I haven't done as much for maths reasoning because I've always thought it's more of a high school thing and I haven't really

> thought about it being in primary, but now I'm realising it can…I could use that language to help write reports and stuff of what they need to do next (School B Year 5 Round 2)

There did appear to be some ongoing confusion between formative and summative assessment. Some wanted year level reasoning outcome statements. Teachers tended to view positively assessment that has a summative slant. The following teacher put it this way:

> You could almost have it as a summative assessment cos rather than having not evident, beginning, developing change it to language of F, 1, 1.5, 2, 2.5. We have it in our brains, makes it more practical like the Vic Curric. … (School B, Year 4 teacher, Round 1)

Stage 4

Our critical friend provided pertinent insights into the clarity of the rubric and possible improvements to it. This feedback together with the feedback from teachers in Stage 3 enabled us to make further changes to the rubric condensing it to a page with simple dot points for teachers to use as well as record observations and feedback. Figure 2 is the simplified version. We simplified the headings for the three key reasoning actions: analysing, generalising and justifying. This final version was presented to 35 teachers at a mathematics education conference and feedback from the teachers was positive.

Assessing Mathematical Reasoning Rubric

Student Name: **Reasoning Task:** **Date:**

Observation of student's reasoning:

	Analysing	Generalising	Justifying
Not Evident	• **Does not notice** common property or pattern.	• **Does not communicate** a common property or rule (conjecture).	• **Does not justify**.
Beginning	• **Recalls** random known facts or **attempts to sort** examples or **repeats** patterns.	• **Attempts to communicate** a common property or rule (conjecture) for the pattern.	• **Describes** what they did and **recognises** what is correct or incorrect. • **Argument** is not coherent or does not include all steps.
Developing	• **Notices** a common property, or **sorts and orders** cases, or **repeats and extends** patterns. • **Describes** the property or pattern.	• **Generalises: communicates** a rule (conjecture) using mathematical terms and records other cases or examples.	• **Attempts to verify** by testing cases and **detects and corrects** errors or inconsistencies. • Starting statements in a **logical argument** are correct.
Consolidating	• **Systematically searches** for examples, extends pattern or analyses structure to form a conjecture. • Makes **predictions** about other cases.	• **Generalises: communicates** a rule (conjecture) using mathematical symbols and **explains what** the rule means or **explains how** the rule works using examples.	• **Verifies** truth of statements by confirming all cases or **refutes** a claim by using a counter example. • Uses a correct **logical argument.**
Extending	• **Notices and explores** relationships between properties.	• **Generalises** cases, patterns or properties using mathematical symbols and **applies** the rule. • **Compares** different expressions for the same pattern or property to show equivalence.	• Uses a **watertight logical argument**. • **Verifies** that the generalisation holds for *all* cases using logical argument.

Comments (feedback, reasoning prompts for further development):

Figure 2. A simplified Assessing Mathematical Reasoning Rubric.

Conclusion

This paper described a design-based research study that drew upon the expertise and experience of teachers to refine the assessment of reasoning rubric. Teachers' feedback was valuable in refining the rubric to provide sufficient detail for teachers to understand what mathematical reasoning is and what to look for when assessing children's reasoning. Whilst

teachers who are confident in their knowledge and understanding of mathematical reasoning felt that a more useful rubric would be a summative rubric, the detail in the rubric was helpful for teachers who needed further development in the area of developing reasoning in students and in reporting student progress. Setting up the rubric with developmental stages and descriptions for each reasoning action provided insights into each of the three reasoning actions as individual learning trajectories. Teachers will be provided with both the detailed and simplified versions of the rubric. How pragmatic and successful the rubric is in meeting the needs of primary teachers remains to be seen. Follow up research might reveal if this is a useful tool for teachers.

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Variations on a Theme: Pre-service Mathematics Teacher Reflections Using an Affect-based Critical Moment Protocol

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Examining critical incidents in the mathematics classroom is a useful way for pre-service teachers to understand the experience of teaching. This paper examines the development and trialling of variations of a novel affect-based critical moment protocol that enables pre-service mathematics teachers to reflect on their teaching performance. The emotions experienced in these moments were examined using self and group reflection, considering the thoughts and actions occurring immediately prior to, or during those moments. The four case studies presented report on trialled variations of this reflection process in a range of programs and delivery modes in four regional Australian universities.

Teacher performance is generally assessed by observation of classroom practice. In initial teacher education (ITE) programs, such observations are traditionally made by a mentor classroom teacher during practicum. There is a multitude of processes and literature around the practicum experience and an expansion in the range of strategies designed to improve teaching performance, including group or team reflection strategies (Woolcott et al., 2017). Studies of critical incidents (events) and/or critical moments, which include reflective pedagogical analysis and interpretation, provide an important way of determining the aspects of a lesson to examine and improve teaching performance (Tripp, 2011; Yeigh et al., 2016).

Critical incidents and critical moments, often treated or identified separately, have become important, in developing the reflective practices of teachers and in ITE (Griffin, 2003). Studies focussing on critical moments have included emotion and/or affect of pre-service teachers (PSTs) (Yeigh et al., 2016), and detailed measures of teachers' emotional states at particular times in a classroom (Tobin & Richie, 2012), but both critical incidents and critical moments are largely determined through observation of teaching as it is related to student learning. The lack of reliability and validity of findings from observational studies is widely reported (Madigan & Ryan, 2011) and critical incident or critical moment protocols do not escape this uncertainty. They are necessarily judgmental and, in pre-service practicum their effectiveness may be constrained by subjective differences in evaluations of both competence and confidence by an observer (Huntly, 2011).

Practicum can be an emotional experience and the emotional content of critical incidents or moments requires careful consideration. The critical moment protocol discussed in this article deals with self-determined emotion or affect as an important discussion point for

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reflective self-evaluation of teaching performance. The innovative protocol presented here was developed in a project investigating how to increase the perceived teaching competence and confidence of mathematics and science PSTs. The protocol builds on teaching performance being linked to how an individual feels at particular moments during teaching (Ritchie et al., 2014). The protocol includes affect-based critical moment analysis as part of the PST learning experience, sometimes, but not always, referring back to incidents or thinking processes immediately prior.

Background

Critical incident analysis in the classroom began in the late 1980s, and was used as a focus for reflective practice, for example in the case of a behavioural incident (e.g., spitting in class, see Tripp, 2011). Meyer and Land (2005) extended the conceptualisation of critical incidents to include those when threshold concepts are understood, that is, when previously inaccessible ideas are understood which lead to "significant shifts in perception of a subject" (p. 373). This concept extension embraces 'teachable moments', when either a misconception has been observed or a desire to know has been sparked in the students providing a teaching opportunity (Patahuddin & Lowrie, 2015) to develop deeper conceptual understanding (Griffin & Ward, 2015). Woods (2012) proposed that, as a critical incident is unplanned and unanticipated there is a higher potential for impact on the affective/emotional state. Recent developments have broadened the conceptualisation as critical moments are not distinguished primarily on the basis of an event or incident and can be used to exam learning in a range of different contexts (Woolcott et al., 2017). In ITE, for example, PSTs can use critical moment analysis to improve their classroom performance while on practicum.

This report uses the term 'affect-based critical moments' to distinguish lesson performance foci from the event-based critical incidents discussed above. The protocol was developed as a part of a broader project, *It's part of my life: Engaging university and community to enhance science and mathematics education* (IPOML), undertaken across the Regional Universities Network (RUN) (Woolcott et al., 2017). The project developed around a model derived from teacher education processes related to a collaboration nexus, previously described for Australian contexts (Gahan et al., 2011). The Enhancement-Lesson-Reflection (ELR) process (Figure 1a) was designed and trialled to develop the perceived competence and confidence of PSTs to engage with and inspire classroom science and mathematics learners. The ELR process shows PSTs how to use science and mathematics to solve problems in their local region through collaborations with university and community experts. Ideally, the ELR process is iterated (Figure 1b) and 1, 2, 3 and 5 cycle iterations have been documented.

Initially the critical moment protocol was used in face-to-face reflection sessions, but this article aims to examine trialled variations of this reflection process adapted to a range of programs and delivery modes across the RUN universities. This protocol is supported in most cases by peer observers and university educators through engagement in collaborative group reflection following self-reflection on selected positive and negative critical moments. The reflection has an added dimension in being structured around lesson study.

The critical moment protocol

Each teaching PST was asked to identify, from the video of their lesson, a number of critical moments, representing an important (positive or negative) emotional feeling or

experience. In initial trials, PSTs identified two critical moments, generally less than two minutes in duration, for each third of the lesson and recorded their perception of the start and end time for each moment. Later trials used variations adapted to their teaching contexts depending on whether PSTs were: attending a face-to-face session at a university campus with university educators and mathematicians; teaching classroom students, large school groups at a university campus or peers in a university class; and/or, undertaking a group reflection either face-to-face or online (Table 1). An emotion diary (Ritchie et al., 2014, Yeigh et al., 2016), such as seen in Figure 1c, was used by both the teaching and observing PSTs to explore experienced or observed emotions from the lesson. An individual may feel different emotions at different times while teaching mathematics/science and this protocol allows PSTs to consider and reflect on the emotions they associated with each critical moment.

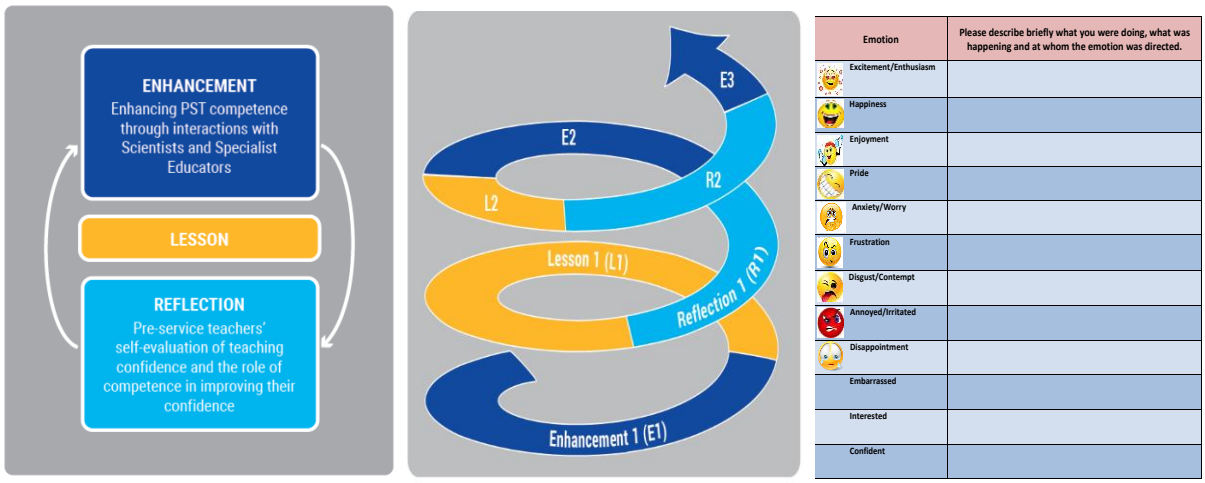

Figure 1.a. The Enhancement-Lesson-Reflection (ELR) process. b. The iterative path of the ELR process. c. The initial emotion diary with emotions; Excitement/Enthusiasm, Happiness, Enjoyment, Pride, Anxiety/Worry, Frustration, Disgust/Contempt, Annoyed/Irritated, Disappointed, Embarrassment, Interested and Confident

The teaching PST then engaged in a reflection, ideally as a group discussion guided by university educators, concerning what the teaching PST was doing or thinking just prior to and during each critical moment. The reflection discussion is structured around the following questions:

1. What happened that made you see this as a critical moment? What were you doing or thinking just before this moment?
2. What was the main emotion you felt at the time? How strongly was it felt?
3. (If reflecting with others) What did others think about your emotion? Was it the same as your view?
4. What would you do if you had an opportunity to recreate that moment in future lessons?

This served to focus how a PST could utilise or maintain positive, or perhaps change negative, emotions during future lessons. This paper examines the critical moments being selected by the PSTs and their reflections to determine: *How engaging with variations of the critical moment protocol can facilitate change in PSTs' confidence and competence when teaching mathematics;* and, *How protocol variations can be successfully adapted to a range of ITE contexts.*

The Study

A mixed methods approach involving four embedded case studies is reported. Cases were purposefully selected based on their focus on mathematics teaching, but present a diversity of characteristics with respect to mode of delivery, program variation (including number of iterations of the process), geographic distribution, size of PST and school student cohorts involved, and availability of mathematics experts. The variation between cases provided an opportunity to compare the effect of the protocol and better understand its deployment across this diversity and the associated variation in protocol implementation. The case studies and participants are summarised in Table 1.

Table 1
Reflection process variations reported in this study in Cases 1 through 4

Trial delivery context	**Reflection participants**	**Reflection process variation**
Case 1: Reflection on campus, on lessons delivered at a school (Year 6)	Four PSTs from a BEd-Primary and three university educators	Self and group reflection face-to-face with university educators, ELR with 3 iterations
Case 2: Reflection on campus within university tutorials, on presentation of conference posters by two PSTS	A class (40 students) of one-year Grad Dip of Education (Primary) and one university educator	Class reflection face-to-face with university educator, assignment submission including a written self-reflection, ELR with 2 iterations
Case 3: Reflection during a series of community events every 3 weeks at a university campus (25-40 Year 9 and 10 students from 15 schools), PST delivered lesson face-to-face and online as part of ELR	Five secondary PSTs and up to six university educators	Face-to-face reflection with PSTs outside of event at university campus, ELR with 5 iterations
Case 4: ELR online (asynchronous and synchronous), instruction embedded into the university Learning Management System for an assessment task built around the reflection	A class of PSTs (secondary mathematics) and one university educator	Self-reflection assignment supported by a video of the lesson and PST recollection of experiences during teaching, 1 iteration

Data collection and participants

Data from the four separate cases outlined in Table 1 were collected from June 2013 to December 2016. In each case the entire ELR process involved either a group of up to six PSTs or a larger tutorial group, up to three university educators, and/or up to two university mathematicians, who collaborated to help develop mathematics lessons, poster presentations or workshops delivered by the PSTs. In Case 4, PSTs worked in partnership with community mathematics experts. Reflections were assisted by the educators, either in face-to-face, blended or online (synchronous and asynchronous) learning environments, and/or assessed as an assignment within the university education curriculum.

Data collected included video transcripts of the reflection sessions, as well as transcribed or written responses of participants in semi-structured interviews, recorded after each lesson, as well as prior to or after a reflection session. The semi-structured interviews allowed for flexible exploration of emerging themes as interviews unfolded. Data were also available in the written responses made on forms included in the protocol resourcing.

Data analysis

Data were first coded and scored using constant comparative analysis and then coded to nodes using the qualitative data analysis software NVivo Version 10 (QSR International). The nodes were then cross-coded with categories of meaning significant to emergent themes. The researchers responsible for implementation in each case provided an overall summary of each case that was also used to identify key themes, identify process changes over time and to clarify similarities and differences in implementation.

Results and discussion

This section discusses how, for each case study, PST confidence and perceived competence was supported using the affective based critical moment protocol.

The effect of common protocol components

All PSTs reported increased confidence and competence following collaboration and/or self-reflection—no negative feedback was reported. Interestingly, PSTs reported increased confidence when others did not notice their negative emotional states, since this meant that any lack of confidence felt by them was not noticed by observers (and, therefore, not something the PST should worry about). One PST went further in noting:

> … the fact that our peers don't really notice when a student teacher has a negative critical moment can be flipped. I have been able to spot that 'glazed' look on students' faces when they don't understand a concept being taught, which is particularly useful when they do not vocalise their lack of understanding. (PST)

Another typical comment, supported by Tripp (2011), was that no one had previously asked the PSTs how they felt during their teaching, even though it was a very emotional time. This comment echoes the stress of the practicum experience, a stress that perhaps observing teachers and university educators do not always acknowledge:

> The opportunity to have a peer observation and reflection program in a real classroom setting allowed us to allay our fears and stresses with our peers and university educators as observing mentors. That our feelings were acknowledged helped to build our confidence. On prac, the mentor never actually asked 'How do you feel?' (PST)

The protocol allowed PSTs to play a stronger role than when observer reports are used for reflection, as PSTs facilitate their own self-judgement and self-determination. The focus on short self-identified elements was the subject of a comment in Case 2—the PST identified a negative moment as after students misunderstood a teaching instruction, or positive moment as when students presenting in classroom feedback were able to demonstrate to the class an understanding of the intent of the curriculum point:

> The reflection process provides a very effective structure that encourages reflecting carefully on smaller sections of their lesson rather than retelling the whole experience which can often not lead to significant improvements. By looking at small sections of the lesson, this method is non-threatening and appears to lead to much more specific discussion and then improvements. (Educator)

The commonality of process was evident in comments from the workshops in Case 3, where PSTs conducted mathematical modelling enrichment classes (Axelsen et al., 2017) with school students, but with separate enhancement and reflection sessions. An emerging theme was the support that such a collaborative reflection provided—support lacking in practicum contexts, but common to all variations reported here:

> Compared to going on pracs, in this program there is much more of a review process and you can have that enhancement and for me that was really important. For me it confronted me and made me change my teaching track and that was because I had the opportunity to sit back and look at how I had gone. Because in prac there is none of that; there is no review—it is more about ticking the boxes. (PST)

While reflection variations that included iterations provided feedback mechanisms that led to improved confidence and competence, PSTs using the single iteration seen in the assignment of Case 4 appear to have gained some benefit also:

> The reflection process has shown me just how much I can learn from videoing myself teaching a lesson, watching it and reflecting on it. Whilst I was aware of the benefits of reflection as determined by lecturers and texts, I had not yet actively engaged in 'watching myself teach'. This process allowed me to critically reflect upon everything from my body language to the way I explained difficult concepts and thus learn how to better myself as a teacher. In the future, I will aim to consistently reflect upon my teaching, videoing lessons and gaining feedback from students when possible, so that I can continue to learn from my mistakes and recreate my successes. (PST)

PSTs believed the protocol assisted them develop their ongoing mathematics teaching:

> In terms of lesson plan reflections, it is easy to see that although other pre-service students may have not noticed anything wrong, in some cases a mentor will have noticed such things. Importantly, the ability to take constructive criticism is easier when a student teacher already has an idea of what might have not been working. This is a kind of preparedness for critical feedback. Again on my 2nd practicum, where I at least feel a lot more is expected of me and my mentor has certainly expected a lot, the experience of emotion diaries (the video reflection as a whole) has allowed me to cope better with this experience. (PST)

Most importantly, the protocol enabled PSTs to also consider emotion, including those of their students, as an indicator of teaching effectiveness. For example, in response to the question, "What aspects of the trial did you find most interesting?" posed after a lesson, one PST said:

> Reflecting on emotions rather than practice, assessing student emotions instead of performance based assessment, assessing key emotional moments that others are possibly unaware of (PST)

The effect of protocol variations

The ELR protocol as initially trialled in Case 1 (Figures 1a and 1b) became 'variations on a theme', as the ELR process and the critical moment reflection protocol, was adapted in other ITE programs. Case 1 may be seen as an ideal way to improve confidence and competence in PSTs in practicum, although it would be time and resource consuming unless it was adapted more directly to on-site utilisation with enhancement and reflection sitting alongside lesson delivery (see Woolcott et al., 2017).

In Case 2, the effect was similar, but in an adaptation to a university tutorial class, rather than a school class. The enhancement occurred with mathematicians in lectures and the educator modelled the critical emotion reflection process using two critical moments from a previous lecture—one positive, one negative—and invited the pre-service teachers to reflect on them with her. After poster presentations, two PSTs taught lessons to their peers based

on their posters. PSTs then identified critical moments from the video recording, followed by a group rating of these critical moments and group reflection and discussion.

> The experience changed their way of thinking. It opened their eyes that maths isn't so prescriptive. They complained that the emphasis was on thinking—so it pushed them. It changed their mindset. (Educator)

Case 3 illustrates the ELR reflection protocol in a workshop context, a context that may easily be adapted to professional learning experiences (Woolcott et al., 2017). The educator facilitated the reflection process where PSTs, an educator and mathematician reflected after a workshop and then discussed the plan for the next workshop. The lesson and reflections session was repeated every three weeks for five iterations. All PSTs taught a lesson. The iterative process helped reduce PSTs' nervousness with regard to teaching mathematical modelling with its open-ended nature of inquiry:

> I had been pretty nervous dealing with the unknowns but coming in today on the back of the other sessions, it's not going to be that difficult. (PST)

> The Enhancement–Lesson–Reflection process taught me a lot about trying to get the kids more engaged. It helped me to focus on the maths side of things rather than focusing (too much) on the modelling process. It taught me to try to get the kids engaged without giving them too much. It taught me to make it (the problem) real life with lots of variables and to facilitate rather than telling them (the students) how to do it. It really taught me to change the way I teach. (PST)

In contrast, Case 4 successfully adapted the critical moment protocol to teaching PSTs studying online. The reflection phase required self-reflection on three critical moments for short lesson segments taught to peers (family and friends in most cases). The self-reflection was completed as an assignment supported by viewing a video of the lesson. Students who fully engaged in this task reported that it was demanding, but beneficial:

> The final assessment task was very challenging, however provided an excellent opportunity for reflection and skill development. (PST)

Conclusion

Findings from these variations support the use of the critical moment protocol in examining the impact of emotions on thinking and behaviour in the PSTs' classroom teaching experience. PSTs learnt how to identify and analyse their teaching-related affective states in order to assess their own emotions and to understand the relationship between emotional literacy and effective pedagogy (Woolcott et al., 2017; Yeigh et al., 2016). Many PSTs valued reflecting with a video of their lesson. The educator emphasised the importance of using critical moments, *small sections of the lesson*, rather than the whole lesson, since this led to more focussed discussions in the reflection sessions so that PSTs could identify specific aspects for improving their teaching. PSTs also acknowledged that the collaborative nature of the ELR sessions improved their confidence and competence.

The study emphasises the importance of reflective analysis of affect for teacher performance, and thus links the affective protocol to increasing competence through increased pedagogical confidence. The critical moment protocol appears to be a method that can be embraced easily by PSTs, and potentially by teachers to support a reflective process for improving teaching and learning, and which appears to be relatively non-judgmental, being essentially a self-regulatory process.

Ensuring those who facilitate reflection and affective analysis are skilled in guiding PSTs in the exploration of their experiences, while also considering the perspectives of peers and pedagogical mentors, appears essential. Overall, these findings indicate that it is important

for PSTs to better connect emotional awareness to their teaching goals, as an aspect of pedagogical confidence and competence. These findings appear to be consistent across all the variations of the protocol supporting its adaptability, although additional research and trialling is needed to determine long-term effectiveness of the process.

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The Beliefs about Mathematics, its Teaching and Learning of those Involved in Secondary Mathematics Pre-Service Teacher Education

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Secondary mathematics pre-service teachers often have different experiences of mathematics and its teaching and learning during their initial teacher education. This paper documents the beliefs about mathematics, its teaching, and its learning, of mathematicians and mathematics educators who teach secondary mathematics pre-service teachers. The beliefs of the surveyed sample of eighty-two academics and differences between groups were characterised using descriptive statistics and one-way comparisons between groups ANOVA. Generally, respondents had a Problem-solving view of mathematics and those with education backgrounds were more in agreement with that method of teaching.

Within secondary pre-service mathematics teaching programs, mathematicians typically teach the mathematical content courses whilst mathematics educators teach the mathematics pedagogy courses. However, the practice of mathematics in university is often different from mathematics teaching and learning practices in secondary schools. The preliminary research reported in this paper is ultimately concerned with how pre-service teachers reconcile the different pespectives of mathematics that might be communicated to them by mathematicians and mathematics educators. The research began by surveying mathematicians and mathematics educators who teach secondary mathematics pre-service teachers about their beliefs about mathematics and mathematics teaching and learning.

Theoretical Background

Australian Curriculum: Mathematics (ACM) identifies mathematics as an inquiry discipline requiring a problem-solving pedagogy. The problem-solving proficiency strand states, "Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively" (ACARA, n.d.). Teachers' beliefs strongly influence their classroom practices (McLeod, 1992; Mosvold & Fauskanger, 2014), which in turn influence their students' beliefs about mathematics and their ability to learn it. Beswick's (2005, 2012) study of twenty-five secondary mathematics teachers showed that individual teachers can hold differing beliefs about mathematics in schools, and as a discipline, and can bring these conflicting beliefs about mathematics learning to the classroom, thus influencing the classroom environment.

Ernest's (1989) study of mathematical beliefs describes three different conceptions of mathematics: Problem-solving, Platonist, and Instrumentalist views. In the Problem-solving view mathematics is a dynamic field of human invention where the teacher is a facilitator helping students become confident in posing and solving problems. The Platonist view sees mathematics as a structured, unchanging body of knowledge where the teacher is an explainer helping students towards conceptual understanding. The Instrumentalist view takes mathematics to be a collection of procedures, facts and skills with the teacher an instructor supporting students to master skills and procedures.

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A limited number of studies has investigated university mathematicians' beliefs about their teaching and their impact on student beliefs and practice. Carlson and Bloom (2005) studied how mathematicians solve problems and their associated emotional perspectives. Dreyfus and Eisenberg (1986) investigated the aesthetic value of mathematics and recommended that teaching include the "aha" of problem solving and that "considerations of two or more solution paths could bring practical benefits, developing a familiarity with different solution methods and deeper conceptual understanding" (p. 9). However, Burton's (1999) study of 70 mathematicians found they believed students needed to learn mathematics before they could begin mathematising, which Burton (1999) described as "objective mathematics they, as teachers, thrust towards reluctant learners" (p. 20).

The preparation of secondary mathematics teachers is shared between discipline experts (mathematicians) and education experts (mathematics educators), who teach separate courses in initial teacher education (ITE) programs that may unintentionally communicate different visions of mathematics. Therefore this paper addresses the following research questions: (1) What are the beliefs about mathematics and mathematics teaching and learning espoused by mathematicians and mathematics educators who teach pre-service teachers? (2) Are there differences in the beliefs of these two groups?

Research Design and Data Collection Methods

A survey was developed in two sections using a five-point Likert scale to elicit responses (strongly disagree to strongly agree). Twenty six items from Beswick's (2005) survey of teacher beliefs were used in the first section, *Beliefs about mathematics, its teaching and its learning*. The second section comprised seven items developed from Ernest's (1989) three conceptions of mathematics.

An invitation to complete the survey online was sent to Australian mathematicians, statisticians, and mathematics educators involved in ITE programs via the Mathematics Education Research Group of Australasia, Australian Mathematics Society, and the Heads of School/Faculty of Education and Mathematics or Science at all Australian universities for forwarding to the relevant staff. There were 82 (from 120) respondents who completed all items in the survey. They represented 35 different Australian universities and five international universities, while three where retired and three looking for work. Forty-nine (60%) were male, 33 (40%) were female, and the median age was 46. Sixty respondents (73%) taught mathematics content subjects only, eight (10%) taught mathematics pedagogy only and 14 (17%) taught both discipline and pedagogy. There was a wide range of qualifications amongst the respondents which included: PhD in mathematics 44 (54%); PhD in education 12 (15%); PhD in mathematics and a Graduate Diploma in Education (GDE) 11 (13%); and no PhD 15 (18%). Descriptive statistics and one-way between groups ANOVA with Bonferroni post-hoc tests were used to analyse data using SPSS.

Analysis and Discussion of Responses for the Whole Group

The results are organised around participants' conceptions of mathematics, beliefs about learning mathematics, and beliefs about teaching mathematics.

Conceptions of Mathematics

The survey items 1-7 in Table 1 are linked to Ernest's (1989) three conceptions of mathematics: Problem-solving (items 1, 2, 7); Platonist (items 3, 4); and Instrumental (items 5, 6). Considering first the Problem-solving view of mathematics: all respondents identified

mathematics as a "continually expanding field of human inquiry" (1) and most 71 (87%) agreed (or strongly agreed) with it "remaining open for revision," (2) and "interrelated and sharing methods of inquiry with other areas of knowledge" (7). Interestingly, 57 (70%) disagreed with the Platonist view that mathematics was a "static but unified body of knowledge," (3) whilst they were reasonably evenly spread in their views as to whether mathematics was "discovered not created" (4) (29 or 35% agreed, 29 or 35% undecided, 24 or 30% disagreed). No one agreed with the Instrumental view that mathematics is an "unrelated collection of facts, rules and skills" (5) but 8 (10%) agreed that mathematics is "entirely distinct from other fields" (6) and is "computation" (item 20 from Table 2). These responses indicate that respondents generally have a Problem-solving view of mathematics as a discipline with some aspects of Platonist views.

Table 1
Survey Responses about Conceptions of Mathematics

	Item Number	D	U	A
1	Mathematics is a continually expanding field of human inquiry.	0	0	82
2	Mathematics is not a finished product, and its results remain open to revision.	6	5	71
3	Mathematics is a static but unified body of knowledge, consisting of interconnecting structures and truths.	57	13	12
4	Mathematics is discovered, not created.	24	29	29
5	Mathematics is a useful but unrelated collection of facts, rules and skills.	78	4	0
6	Mathematics is entirely distinct from other disciplines.	68	6	8
7	Mathematics and other areas of knowledge are interrelated or partly integrated, sharing concepts and methods of inquiry.	3	8	71

Note: D = Strongly disagree or Disagree; U = Undecided; A = Strongly Agree or Agree

Beliefs about Learning Mathematics

Beswick (2005) identified items 1-3 and 5-8 in Table 2 as identifying a Problem-solving view of learning mathematics. Most academics (at least 90%) agreed with items 1-3, 5 and 6. However, they were less in agreement about being "fascinated with how students think" (7) (10, 12% undecided; 67, 82% agreed) and "providing interesting problems to be investigated in small groups" (8) (23, 28% undecided; 55, 67% agreed). The Platonist view of mathematics learning differs from the Instrumental view as in the Platonist view the learner is actively constructing their knowledge wheras in the Instrumental view the learner is passively receiving the knowledge. Items 16-17 were difficult to distinguish between the Platonist and Instrumental views: just under half the academics agreed with this Platonist/Instrumental view while about one quarter disagreed. Thirty-eight (46%) agreed mathematics "should be presented in the correct sequence" (18), a more Platonist view, while 21 (26%) disagreed. Items 19 and 21 represented an Instrumental view. Thirty-four (41%) agreed the best way to learn was an "expository style" (19) and 25 (30%) disagreed. However for item 21 only 2 (2%) believed that "telling students the answer was an efficient way of facilitating mathematics learning" and 61 (74%) disagreed. This indicates a general belief that mathematics learning is best achieved with problem solving but the content needs to be structured so content and skills are related.

Table 2
Survey Responses about Beliefs about Mathematics and Beliefs about Learning and Teaching Mathematics (Beswick, 2005).

	Item Number	D	U	A	
1	A vital task for the teacher is motivating students to solve their own mathematical problems.	4	1	77	PSL
2	Ignoring the mathematical ideas that students generate themselves can seriously limit their learning.	4	4	74	PSL
3	It is important for students to be given opportunities to reflect on and evaluate their own mathematical understanding.	2	1	79	PSL
4	It is important for teachers to understand the structured way in which mathematics concepts and skills relate to each other.	1	2	79	PL
5	Effective mathematics teachers enjoy learning and "doing" mathematics themselves.	0	5	77	PSL
6	Knowing how to solve a mathematics problem is as important as getting the correct solution.	1	1	80	PSL
7	Teachers of mathematics should be fascinated with how students think and intrigued by alternative ideas.	5	10	67	PSL
8	Providing students with interesting problems to investigate in small groups is an effective way to teach mathematics.	4	23	55	PSL
9	Mathematics is a beautiful, creative and useful human endeavour that is both a way of knowing and a way of thinking.	0	3	79	
10	Allowing a student to struggle with a mathematical problem, even a little tension, can be necessary for learning to occur.	1	3	78	PSPT
11	Students always benefit by discussing their solutions to mathematical problems with each other.	7	18	57	PSPT
12	Persistent questioning has a significant effect on students' mathematical learning.	5	22	55	PSPT
13	Justifying the mathematical statements that a person makes is an extremely important part of mathematics.	1	2	79	PSPT
14	As a result of my experience in mathematics classes, I have developed an attitude of inquiry.	8	20	54	PSPT
15	Teachers can create, for all students, a non-threatening environment for learning mathematics.	5	13	64	PSPT
16	It is the teacher's responsibility to provide students with clear and concise solution methods for mathematical problems.	19	24	39	PIL
17	There is an established amount of mathematical content that should be covered at each grade level.	20	22	40	PIL
18	It is important that mathematics content be presented to students in the correct sequence.	21	23	38	PIL
19	Mathematical material is best presented in an expository style: demonstrating, explaining and describing concepts and skills.	25	23	34	IL

20	Mathematics is computation.	68	6	8	
21	Telling the students the answer is an efficient way of facilitating their mathematics learning.	61	19	2	IL
22	I would feel uncomfortable if a student suggested a solution to a mathematical problem that I hadn't thought of previously.	75	1	6	IT
23	It is not necessary for teachers to understand the source of students' errors; follow-up instruction will correct their difficulties.	75	3	4	IT
24	Listening carefully to the teacher explain a mathematics lesson is the most effective way to learn mathematics.	54	21	7	PIT
25	It is important to cover all the topics in the mathematics curriculum in the textbook sequence.	68	7	7	IT
26	If a students' explanation of a mathematical solution doesn't make sense to the teacher it is best to ignore it.	76	5	1	IT

Note: D = Strongly disagree or Disagree; U = Undecided; A = Strongly Agree or Agree; PS=Problem-solving; P=Platonist; I=Instrumental; L=learning; T=teaching

Beliefs about Teaching Mathematics

Ernest (1989) identified three different roles the teacher can take: an instructor (Instrumentalist), an explainer (Platonist), or a facilitator (Problem-solving). The instructor role aligns with traditional teaching methods (items 22, 23, 25, 26 in Table 2) which most academics disagreed with (at least 83%). A Platonist view of the teacher is an "explainer" helping students develop conceptual understanding and integrate knowledge (Ernest, 1989). "Listening to explanations as the most effective way to learn" (24) could be either an Instrumental or Platonist view: whilst 54 (66%) disagreed, 21 (26%) were undecided. A Platonist view requires teaching strategies that will support students constructing knowledge. Items 10-15 ask about these methods, though most could also be interpreted as Problem-solving. Academics were supportive of these, particularly "allowing students to struggle" (10) (78, or 95%, agreed) and "the importance of justifying statements" (13) (79, or 96%, agreed). With the other items, there were reasonable numbers who were undecided. For example, 13 (16%) were unsure of the value of a "nonthreatening environment" (15) and 22 (27%) were unsure of the "value of persistent questioning in learning" (12), while 54 (66%) had "developed an attitude of inquiry because of classroom experiences" (14). These responses indicate that, overall, respondents generally believed in a Problem-solving/Platonist view of teaching mathematics but there were aspects with which not all were comfortable, for example, the use of questioning, and developing an attitude of inquiry in the classroom.

Analysis and Discussion of Differences between Groups

There were differences in the responses related to participants' teaching responsibility and qualifications.

Differences Related to Teaching Responsibility

There were 60 (73%) respondents who only taught mathematics or statistics content courses (mathematicians), while 8 (10%) only taught pedagogy courses only (mathematics

educators) and 14 (17%) taught both discipline and pedagogy. Table 3 shows four beliefs with significant differences ($p<0.05$) in responses based on teaching responsibilities.

Those who taught both content and pedagogy had a larger mean agreement rating than mathematicians that investigating interesting problems in small groups was effective teaching (8), indicating their more Problem-solving view of learning. Mathematics educators had a lower mean agreement rating than mathematicians and those who teach both content and pedagogy for the importance of the "correct sequence" (18). Mathematics educators had a lower mean agreement rating than mathematicians about using an "expository style" (19), associated with an Instrumental view of learning. There was no significant difference between those who taught pedagogy only and those who taught content and pedagogy. Each group agreed that "mathematics is beautiful, creative, useful and a way of knowing and way of thinking," (9) but those who taught both mathematics content and pedagogy had a lower mean agreement rating than both the other groups.

These results indicate that those teaching mathematics pedagogy tend to have a more Problem-solving belief about learning whilst those teaching mathematics content have a stronger belief that some mathematical ideas need to be understood before other concepts can be learnt/understood, a more Platonist belief. Mathematicians tended to believe more in the value of expository teaching, an Instrumental view, than those who teach pedagogy.

Table 3
Differences in Beliefs Related to Teaching Responsibility

Abbreviated item and number	F (p-value)	Statistic	Teach content	Teach pedagogy	Teach content & pedagogy
Students need interesting problems to investigate in small groups. (8)	4.857	Mean	3.68^{a}	4.25^{ab}	4.43^{b}
	(0.010)	SE	0.122	0.250	0.173
Beautiful, creative, useful; a way of knowing & thinking. (9)	5.151	Mean	4.77^{b}	5.00^{b}	4.36^{a}
	(0.008)	SE	0.060	0.000	0.199
Content should be presented in the correct sequence. (18)	5.350	Mean	3.50^{b}	2.25^{a}	3.43^{b}
	(0.007)	SE	0.136	0.250	0.272
Mathematics should be presented in an expository style. (19)	5.030	Mean	3.37^{a}	2.38^{b}	2.79^{ab}
	(0.009)	SE	0.128	0.324	0.239

[a, b] Mean values within a row with unlike superscript letters are significantly different ($p<0.05$). For example, for item 9 the (Bonferroni-adjusted) t-test results show a small p comparing "teaching both content and pedagogy" with "teaching content", and with "teaching pedagogy", but not between "teaching content", and "teaching pedagogy."

Differences Related to Qualifications

As numbers in some individual qualification categories were small, the following groupings were formed: no PhD 15 (18%), PhD in education 12 (15%), PhD in mathematics 44 (54%) and PhD in mathematics and a GDE 11 (13%). The ANOVA indicated six beliefs that showed a statistically significant difference ($p<0.05$), see Table 4.

Mathematicians with a PhD in mathematics had a lower mean agreement rating than those with a GDE for "ignoring students' mathematical ideas can limit their learning" (2). Mathematicians had a lower mean agreement rating than each of the other groups that teachers should be "fascinated with how students think" (7), indicating less of a Problem-

solving view of learning. Mathematics educators with a PhD in education had a higher mean agreement rating than mathematicians and those with no PhD in "providing students with interesting problems to investigate in small groups" (8), indicating a stronger Problem-solving view of learning. Those with no PhD had a higher mean agreement rating for there being a "set amount of mathematical content to cover at each level" (17), a Platonist – Instrumental view. Mathematics educators with a PhD in education had a lower mean agreement rating than those with no PhD that "mathematics must be presented in the correct sequence" (18), less of a Platonist view. Mathematicians and those with no PhD had a stronger mean agreement rating than mathematics educators that "mathematics should be presented in an expository style" (19), the Instrumental view of teaching. This result suggests that gaining postgraduate education qualifications may lead people away from an Instrumental view of mathematics and traditional teaching strategies towards developing a more Problem-solving view of mathematics and mathematics teaching.

Table 4
Differences in Beliefs Related to Qualifications

Abbreviated item and number	F (p-value)	Stat-istic	PhD M, S	PhD Ed	PhD M & GDE	No PhD
Ignoring students' mathematical ideas can limit their learning. (2)	3.228	Mean	4.02^{a}	4.58^{ab}	4.82^{b}	4.20^{ab}
	(0.027)	SE	0.144	0.149	0.122	0.262
Teachers should be fascinated with students' thinking. (7)	8.868	Mean	3.68^{a}	4.50^{b}	4.55^{b}	4.73^{b}
	(0.000)	SE	0.152	0.151	0.157	0.118
Students with interesting problems to investigate in small groups. (8)	4.729	Mean	3.68^{b}	4.50^{a}	4.36^{ab}	3.53^{b}
	(0.004)	SE	0.121	0.195	0.203	0.322
An established amount of content to be covered at each level. (17)	3.090	Mean	3.41^{ab}	2.75^{a}	3.00^{ab}	3.87^{b}
	(0.032)	SE	0.157	0.329	0.270	0.256
Content should be presented in the correct sequence. (18)	3.451	Mean	3.57^{a}	2.50^{ab}	3.45^{ab}	3.40^{b}
	(0.020)	SE	0.154	0.314	0.312	0.254
Mathematics is best presented in an expository style. (19)	5.130	Mean	3.39^{b}	2.50^{a}	2.55^{ab}	3.53^{b}
	(0.003)	SE	0.135	0.359	0.247	0.236

[a, b] Mean values within a row with unlike superscript letters are significantly different ($p<0.05$). For example, for item 19 the (Bonferroni-adjusted) t-test results show a small p comparing "a PhD in mathematics or statistics" with "a PhD in education", but not between the others. M=mathematics, S=statistics, Ed=education.

Conclusions

Teachers' practices directly influence the beliefs about mathematics of the students in their classes (McLeod, 1992; Mosvold & Fauskanger, 2014). Hence it seems reasonable to assume that in ITE programs the practices of teacher educators – whether they are teaching content or teaching pedagogy – would influence the beliefs about mathematics teaching and learning held by pre-service teachers. It is important that pre-service teachers have experiences that will support the Problem-solving beliefs about mathematics in the *ACM*.

Generally the survey sent to mathematicians, statisticians and mathematics educators involved in ITE programs in Australia revealed those teaching future secondary mathematics teachers hold a Problem-solving view of mathematics as a discipline, but with some aspects

of Platonist views (Ernest, 1989). However, 10% agreed with statements of beliefs aligned with the Instrumental view of mathematics (Ernest, 1989), that is, that mathematics was "entirely distinct from other disciplines". Such beliefs are contrary to the aim of the *ACM*"recognis[ing] connections between the areas of mathematics and other disciplines" (ACARA, n.d.). Largely respondents believed in a Problem-solving /Platonist view of teaching mathematics, particularly "allowing students to struggle" (95% agreement) and "the importance of justifying statements" (96% agreement); but there were aspects with which not all were comfortable, for example, the use of questioning, and developing an attitude of inquiry in the classroom. These are teaching strategies commonly associated with problem solving (Van der Walle, Karp & Bay-Williams, 2016).

Some interesting differences were identified between respondents with/without educational qualifications, and with/without teaching responsibilities in pedagogy. While the analysis can say nothing about causality, it may be that postgraduate exposure to educational theories and perspectives supports the development of a Problem-solving view of mathematics learning, while specialisation in mathematics at the postgraduate level supports beliefs about correct content and sequencing. We are exploring these tentative claims in semi-structured interviews with a sample of survey respondents to better understand their beliefs about mathematics, and how mathematics is taught and learned. As a result we hope to support boundary dialogues (Goos & Bennison, 2018) between mathematicians and mathematics educators, aiming for greater coherence for the pre-service teachers who learn from mathematicians and mathematics educators.

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Engaging Pre-Service Non-Specialist Teachers in Teaching Mathematics Using Embodied Technology Tools

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We report on an initial analysis of survey data that was generated through a collaboration between the schools of Education and Information Technology in exploring pathways through which future teachers could envision mathematics as engaging and creative subject, while also enhancing their awareness of, and skills in, using digital technologies in teaching mathematics. We specifically share insights generated into students' attitudes towards, current understandings of, and expectations for uses of technology in teaching and learning mathematics in schools. We bring attention to a mismatch between pre-service teachers views of technology and those of technology educators.

The study reported in this paper was initiated under an umbrella of a large Australian multi-university project *Inspiring Mathematics and Science in Teacher Education (IMSITE).* The project aimed at enriching pre-service teacher education in science and mathematics by fostering genuine, lasting collaboration of relevant discipline and education scholars, and by institutionalising new ways of integrating the content and pedagogical expertise of education and discipline professionals. We analyse data collected as part of a collaboration between the schools of Education and Information Technology, in exploring pathways through which future teachers could envision mathematics as engaging and creative subject, while also enhancing their awareness of, and skills in, using digital technologies in teaching mathematics. We specifically share insights generated into students' attitudes towards and current understandings of and expectations for uses of technology in teaching and learning mathematics in schools.

Australian secondary pre-service teachers are becoming increasingly aware of a shortage of qualified mathematics teachers, with recent reports estimating that 21% of those who teach mathematics in middle years (teaching 12-15 year old students) in Australia teach the subject out-of-field (Weldon, 2016). This means that even for pre-service teachers whose specialisation area is not mathematics, it is rather likely that they will be, at some point in their teaching career, asked to teach middle years mathematics subjects. As a means of addressing this situation, universities are offering mathematics education courses for pre-service secondary teachers of other subject areas. At the University of Queensland, an elective course focuses on introducing the notion of teaching mathematics for conceptual understanding (Boaler, 2016; Carpenter & Lehrer, 1999), and engages pre-service teachers as learners in types of mathematical activities where they explore mathematical patterns and relationships and where memorising formulas and producing calculations is not positioned as central to mathematical activity.

Over the years, many pre-service teachers recognised the need to take the mathematics elective course and expressed the belief that this choice can positively impact their

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 527-534. Auckland: MERGA.

employability. In spite of such awareness, enrolling into a mathematics course—or imagining themselves as teachers of mathematics—is not unproblematic for many of these students. Anxiety towards mathematics broadly and mathematics teaching specifically remains a persistent issue (Haciomeroglu, 2014; Ho et al., 2000). For some of the pre-service teachers, their prior experiences from mathematics classrooms, beliefs about mathematics and how it should be taught, and identities as mathematics learners that they developed in the process of schooling (Boaler, 2002; Cobb, Gresalfi, & Hodge, 2009; Gresalfi & Cobb, 2006) present difficulties when attempting to productively re-engage with mathematics in teacher education courses. At the one hand, technologies are often conceptualised and trialled as a tool for engaging students' in curriculum (e.g., Norton, 2006). At the other hand, technology itself can be a source of additional anxiety for teachers and pre-service teachers (Duhaney, 2001; Wachira & Keengwe, 2011), and this in turn can instigate lack of interest and motivation to integrating technology when teaching mathematics. Bennison and Goos (2010) remind us of the importance of supporting and educating teachers in the use of technology, so that they experience opportunities to build up their confidence and develop positive beliefs about technology.

Background: Digital Technology in Mathematics Education

Even today, most of the research on technology for mathematics classrooms focuses on use of computers and calculators (Wachira & Keengwe, 2011). As a number of more hands-on alternatives become available, a number of distinctions about their uses in classrooms or by learners become relevant. We would like to focus in particular on the distinction between *passive*, *passive interactive*, and *interactive tangible technology*.

The category of passive technology includes those technologies where a user is viewed primarily as a consumer or recipient. Examples include online videos and PowerPoint slide shows, when intended uses do not extend beyond viewing the ready-made content. Passive interactive technology includes so called *amplifiers*, such as Excel spreadsheets and calculators, which allow users to perform the same actions that they performed with non-digital tools and technologies earlier, but provide significant improvements of the speed and organisation of these actions (Lee & Hollebrands, 2008). *Coding platforms* that allow students to create their own tools (e.g., to perform calculations and create graphs) are another example in this category. Finally, the term interactive tangible technology refers to those technologies that not only aim to develop students' computational thinking ability and deepen their learning of mathematics, but also allow for creation of physical representations of the mathematical concepts.

Within the latter category, visual programming tools, including robotics with Lego Mindstorm and Scratch programming language, have been documented to serve as adequate platforms for students' development of problem solving skills (Spector, Lockee, Smaldino, & Herring, 2013). These tools have generated unparalleled student interest, encouraged independent thinking, while at the same time increased the immediacy of relevant feedback. Examples of ways in which instructional activities with these tools facilitated learning include learners noticing a mistake or a faulty assumption quickly, and taking (often) independent steps to correct it. For instance, if the robot doesn't behave in the way the learner intended, inputs can be re-assessed, conjectures about their functioning readjusted, and new iteration or trial enacted.

Within such classroom activities, the focus is taken away from whether an incremental 'result' has been 'correct' and the premium is instead placed on figuring out, progressively, how to create the desired solution. Errors and mistakes have their legitimate place in the

students' activity as they become a means of increasing the insight into the problem situation and contributing to the resolution. While this set of values is highly compatible with mathematics teaching practices that aim for students' conceptual understanding, focus on correctness and speed remains ingrained as a focus in too many mathematics classrooms (cf. Boaler, 2016).

In the types of activities afforded by visual programing tools, students' construction of understandings from their lived experiences is almost palpable (Mikropoulos & Bellou, 2013). Robotics allow knowledge to be presented in a variety of different forms as an aid to substantive knowledge (Merrill, 2002). This creates possibilities for logical, objective truths and abstract problems to take on new meaning by becoming adaptive tangible experiences that provide tools for sense making (Núñez, Edwards, & Matos, 1999).

The Study Background

All 87 students undertaking a Graduate Diploma in education who enrolled in mathematics elective course during their one-year program were invited to participate in various aspects of data collection related to IMSITE study. Within the elective course, they were offered to participate in two optional, free, four-hour workshops that focused on (a) developing skills in technology for mathematics classroom use and (b) how technology can be used as an exciting pathway to mathematical learning. The workshops were conducted on campus on Saturdays and aimed at introducing new, practical, and engaging ways of exploring and using specific mathematical ideas in a classroom setting. The workshops were largely self-standing: The first one introduced Ev3 Mindstorm robots, while the second one focused on Scratch visual programing language and Makey-Makey technology.

We will limit our description to *Ev3 Mindstorms Robotics* (see Figure 1), which is a collaborative educational technology that uses simple visual blocks to program (Figure 1a), and technic Lego bricks with a variety of sensors that allow for construction of relatively sophisticated robots (Figure 1b) that execute programed code (Eguchi, 2010). It is currently used in many primary and secondary schools in Australia due to its modification flexibility, ability to interact with the world using sensors, and relatively easy-to-learn visual programing with blocks. It also allows teachers to later introduce more challenging programming through using a more traditional written code.

Figure 1. Ev3 Mindstorm Robotics (a) block code program, and (b) robot in action

Technologies that were the focus in the workshops are advantageous for their adaptability, as they can readily be reconfigured to expose a variety of mathematical

principles in a physical and dynamic way. Specifically, activities during the workshops included programing the robot to trace a square on the floor, estimate distances and angles, flexibly switch between metric units and units needed in programing the robot (e.g., number of rotations of wheels). Additional activities designed for classroom use included programing robots to move with uniform speed for differing numbers of seconds, as a means to explore graphs of linear functions.

After the course, students were expected to develop a lesson plan, and then implement that plan with support in a classroom during their pre-service placement in a state school.

Participants were recruited by an open invitation on an online student portal; emails and reminders in-class were given throughout the course. Although initially there was considerable interest from the 79 students in the course, in the end only five students could participate in the workshops with only two participating on both days. These numbers were much lower than anticipated, however we decided to still offer the program. Not only did we want to see how the pre-service students went about implementing robotics in their classrooms, but we decided that it would be imperative to understand what motivated the students to participate and what were the main barriers to participation to those who initially expressed interest in the activities.

At the end of the semester, after lectures had finished and in-service work was completed, a survey was sent out to the students. It was intended to generate insights into students' views of technology in the mathematics classroom and an understanding of the low participation numbers in the optional workshops. The survey had four main themes: attitudes towards mathematics, attitudes towards inclusion of technology in mathematics subject areas, technology and teaching in the future, and participation in workshops.

From a class of 79 students, eight completed and returned the survey. From the responses, we derived emerging themes (Harding & Whitehead, 2013) of attitudes towards technology and how these pre-service teachers currently perceive its role in mathematics, which we discuss in the remainder of this paper.

Results

Attitudes towards Mathematics

Our initial questions regarded students' enjoyment of mathematics. Our aim was to ascertain their level of anxiety towards mathematics, that might hinder further development in this area. All participants agreed or strongly agreed that they enjoy mathematics, and all but one participant felt confident in teaching mathematics in high school up to Year 9. We took this to indicate that the majority of participants who chose to fill out the survey, although their major was not in mathematics, believed they had reasonable mathematical background and were generally positive towards teaching mathematics in middle years classrooms.

Although respondents themselves were positively disposed to mathematics, they believed not many students would be so disposed. Respondents were asked to finish either or both statements "Overall, students dislike mathematics because…" or/and "Overall, students like mathematics because…" Interestingly, there wasn't a consensus between the respondents as to why students have a dislike of mathematics. Eight different types of factors were given as to students' unfavourable dispositions:

Relevance, mathematics is not grounded in subject areas that concern or interest students and "may seem irrelevant to what they think is important". If mathematics is not

grounded in what students perceive they will be doing in the future, then mathematics becomes an obsolete subject - something they have to get through.

Difficulty level, the unachievable measure which students believe they have to attain to succeed and do well in mathematics was discussed four times in the responses given. Respondents used phrases such as "they tell themselves they are too stupid to do it", "difficult to understand", and "they don't get the concepts".

Teaching styles, the way in which content and approach to problems is given by teachers isn't consistent, among each year level.

Teaching levels, teachers don't always explain concepts at the child's level, "students who do not 'get' maths from a young age are forever playing catch up", "from my prac experience, some students didn't listen to the teacher at all because their maths was not good in the beginning to learn new content" and "it's hard to find someone who explains things well i.e. at your level". The three responses indicate how teaching mathematics was perceived as difficult especially in classrooms with diverse students, where teachers had to ensure that everyone has access to mathematics required in classroom activities.

Maths assessment, respondents noted two different ways assessment may hinder students' enjoyment of mathematics. The first was success at exams, "they don't do well in maths exams so they start to dislike the maths." The second was related to the style of assessment "Maths is also frequently tested for procedural competency and not frequently placed in applied contexts that students are familiar with and as such produces high levels of anxiety about failing the subject"

Hindrance to independent thought, students may believe that mathematics does not allow for different views of the world "Because there is no room for individualism and interpretation" that other subjects allow. Students' belief that there is either a right or a wrong solution or way of doing something would shape their enjoyment of mathematics.

Required Practice, exercises are required to be performed outside the classroom, if you want to become better mathematician you need to practice.

Only for the gifted, "Viewed as the specialised or privileged knowledge of those with a 'maths gift or talent'"

The richness and complexity of reasons that the pre-service teachers could generate for why students might dislike mathematics indicates that, in their views, supporting students' mathematical learning would be a complex issue, often outside of teacher's control.

When commenting on "overall students like mathematics because…", not all respondents provided answers, and only 3 types of responses were given:

Successful, it is satisfying when one can solve a problem, obtain the solution, and move on to bigger challenges.

Interesting, teachers can make maths a "fun and interesting" subject to learn.

A right answer, a "perceived clear equity in results (eg. I got 20/25 right)"

The positive responses given to how students perceive mathematics were limited compared to the negative responses. The pre-service teachers did not suggest that mathematics in and of itself might elicit enjoyment. Instead they seem to believe that it is always up to the teacher to provide the fun during the lesson, and problems that would allow students to experience success. While this is a potentially useful perspective, the responsibility can be at times overwhelming for pre-service teachers.

Attitudes towards Inclusion of Technology in Mathematics Subject Areas

Pre-service teachers were asked if they planned to use technology in their classrooms and what type of technologies they would use. Figure 2 shows the distribution of the types of technologies discussed. Screen based, non-interactive technology and PowerPoint (also a non-interactive technology) were most frequently selected to be used. In contrast, data generating or collaborative work spaces, which can be seen as more tangible type of technologies, were at the low end of included technologies. Embodied tangible technologies such as Robotics and Makey-Makey were not discussed.

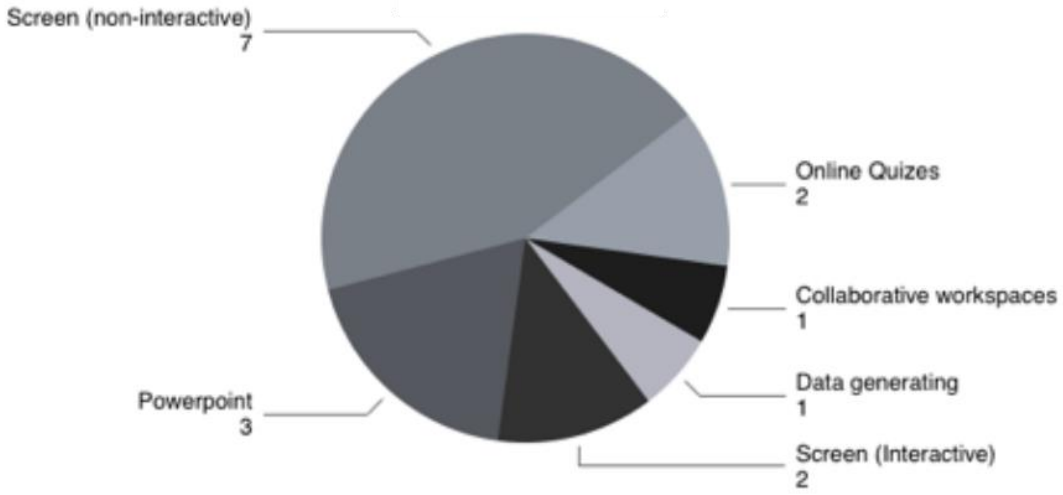

Figure 2. Technologies planned to be used in the classroom

Pre-service teachers were asked to evaluate their level of confidence and how interested they would be to learn more about technology in the classroom. Seven respondents stated that they are confident user of technology and seven said they would like to learn more about use of technology in the classroom.

When viewing these two results together, respondents viewing themselves as having a high level of confidence and utilising non-interactive screen-based technologies indicates that they view 'confidence in technology' in the sense of users rather than creators. This points to the need for design of activities that would support pre-service teachers in coming to view themselves as creators of technology. This would be a necessary step if we hope that pre-service teachers would create 'technology creation' goals for the learning of their future students.

Technology and Teaching in the Future

Respondents were asked to give their views of where they see the role of technology in the next 2-5 years, in relation to education and classroom teaching. The most responses were varied with most consensus that technology will be a larger part of day-to-day classroom. The majority of responses was made up of uses such as replacing paper. For example, one respondent saw technology to become "integral but not a dominant component - it does not cater for all learning styles". In their responses, pre-service teachers indicated that they view technology as a means for passive use, not as something they can proactively control, shape, and integrate as an interactive tool.

Limitations

When viewing the outcomes of the data, it is important to note that the participants in the survey are not necessarily indicative of the population of pre-service non-specialist teachers studying mathematics in Australia. However, the insights generated are indicative of some currently existing views. Further explorations would be essential to elaborate the range and prevalence of different perspectives that could inform design of effective teacher education interventions and programs.

Discussion

One of the reasons why we find this to be an especially interesting case, is because the students' responses portray how the pre-service teachers conceptualised technology primarily, and at times exclusively, as a tool for passive content delivery. This is in a stark contrast to the valuation of technology that is widespread amongst technology educators, where students are to be supported in developing computational thinking skills and building their own tools. If the views of these two communities continue to misalign, the potential for technologies to enhance core curriculum areas will remain under realised. Although classrooms are full of technologies, these technologies are being used to either marginally enhance content delivery (e.g., by viewing videos over internet streaming) or as amplifiers to allow students to perform task quicker and easier than they would be able to do with a pen and paper. While these kinds of interactions with technology might build users confidence, and the level of comfort they perceive around the use of technology, they rarely significantly alter the range and depth of mathematical (and other disciplinary) ideas that are accessible to students in the classroom.

Interestingly, through optional course additions, we have not been able to equip the pre-service teachers with understanding that interactive technology can help students create their own tools and help them understand abstract mathematical principles. It is also interesting to note that research previously discussed revealed in multitude of ways how different technologies help in mathematical understanding by making it fun, interesting, adaptive tangible experiences. Finding ways to support pre-service teachers in both engaging in similar experiences and in designing such experiences for their students is of utmost importance.

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Noticing Mathematical Pattern and Structure Embodied in Young Children's Play

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This position paper proposes that a relationship between young children's embodied mathematical concepts and their awareness of mathematical pattern and structure (AMPS) (Mulligan & Mitchelmore, 2009) develops through play. Theoretical perspectives on the development of schematic patterns, the embodiment of mathematical understandings, and the development of AMPS are outlined. We propose AMPS may underlie children's embodied actions in play. Thus, the practice of professionals' 'noticing' is central to supporting children's development of mathematical concepts. Implications for further research, including the development of an observational framework to notice AMPS through play, are discussed.

Young children may reveal their mathematical ideas naturally through play, providing a valuable context for early childhood educators to respond authentically to children's mathematical curiosities. However, educators' awareness of the importance of interpreting mathematical possibilities, and their perceived lack of confidence and corresponding mathematical content knowledge can inhibit their response to these playful encounters (Cohrssen, 2015; Lee, 2107). Early Childhood Australia (ECA) recognises the difficulty some educators experience in responding to mathematical concepts young children engage with, acknowledging that many feel "less comfortable having conversations with children that enable [them] to assess their mathematical thinking during play" (Cohrssen, 2015). However, intentional observational practice is advocated in the Early Years Learning Framework [EYLF] (Department for Education, Employment and Workplace Relations [DEEWR], 2009). Thus, there is a need to support educators to notice the complexity of mathematical concepts children are naturally exploring, and to actively engage in questioning and dialogue to elicit children's reasoning to inform future directions for learning (Cohrssen, 2015).

Children's natural ability to notice, develop and utilise mathematical ideas to make meaning of experiences has been well documented through the practice of observing their play (Marcus, Perry, Dockett & MacDonald, 2016). This research found that young children "noticed … explored … [and] talked about the mathematics they encountered … highlighting the importance of conversations with children" (Marcus et al. 2016, pp. 441; 445). Play has been widely recognised as an enabling context whereby the breadth and depth of children's true competencies can be observed (Rinaldi 2013; Van Hoorn, Nourot, Scales & Alward, 2015). Through play, young children's mathematical understandings can be interpreted as an embodied representation of their thinking (Thom, 2017). Research into embodied cognition recognises the role of physical movement and playful engagement with learning environments as critical factors in the formation of preverbal mathematical concepts (Kim,

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Roth & Thom, 2010; Nunez, Edwards, Matos, 1999). Bautista, Roth and Thom (2012) recognise that movement alone reveals thinking, proposing that there is a relationship between movement and "the emergence of abstract mathematical knowledge" (p. 363). The authors contend that abstraction of mathematical understanding is the result of engaging physically with "rhythmic patterns [that] emerge in corporeal-kinetic dimensions" (Bautista et al, 2012, p. 368), concluding that these cognitive structures are a consequence of engaging physically with the world. For example, as children move around boundaries of play spaces, they experience this as an iteration of steps, and larger spaces are experienced as requiring more paces. Movement patterns such as this naturally emerge from young children's play as they repeat and iterate known actions, 'schemes' that reveal "organised patterns of behaviour" (Piaget, 1952, cited in Ginsburg & Opper, 1969, pp. 20-21). Athey (2007) identified eight common dynamic behavioural patterns, termed 'action schemas'. These observable actions have been likened to the development of "conceptual clusters … [cognitive systems] that illustrate aspects of spatial thought" (Hayes, 1979, cited in Athey, 2007, p. 2). Therefore, the ways that children interpret and engage with spatial features of play spaces can be analysed through observing their pattern of actions (Athey, 2007).

Pattern and structure, are widely recognised as foundational in the development of mathematical understandings in the early years (Mulligan & Mitchelmore, 2009; Papic, Mulligan & Mitchelmore, 2011). Specifically, an awareness of mathematical pattern and structure (AMPS) (Mulligan et al., 2009) has been found to be critical in the development of pre-algebraic reasoning, supporting the abstraction and generalisation of mathematical concepts (Kieran, Pang, Schifter & Ng, 2016; Mulligan & Mitchelmore, 2013; Papic et al., 2011). For example, young children reveal an awareness of mathematical pattern and structure when they notice similarities between objects; when they can recognise what is the 'same' across a variety of experiences; or predict what may come next in a sequence of events, reasoning and predicting about change. Mathematical thinking emerges from all aspects of life (Ernest, 1991, 1994), therefore an awareness of pattern and structure could be considered to underlie life experiences (McCluskey, Mulligan & Mitchelmore, 2013). Thus, an implication for early childhood education is that the development of mathematical pattern and structure could be observable in the children's embodied actions.

The intention of this position paper is to provide a rationale for exploring the relationship between the embodiment of mathematical understandings and the development of children's awareness of mathematical pattern and structure. Theoretical perspectives into the development of schematic patterns, the embodiment of mathematical understandings, and children's awareness of mathematical pattern and structure are presented to propose that an awareness of mathematical pattern and structure may underlie children's play. The development of educators' pedagogical content knowledge and the notion of active noticing will also be discussed. Two research questions are raised; does AMPS underlie young children's embodied actions in play? and, what effective elements of practice could support educators' active noticing of children's AMPS through play?

Background

Viewing the Child as Mathematically Competent

Early Childhood Australia's (ECA) *Strategic Plan 2014-2017* identifies the need to realise each child's innate capacity to grow and learn, advocating for children in all Australian contexts to be viewed as already capable and competent (ECA, 2014). This has implications for practice whereby educators' focus becomes geared towards observing what

children can do rather than noticing what they can't do (ECA, 2012). A strength-based approach advocates for early childhood practitioners' to notice and base future learning directions upon children's current capabilities (Connor, 2011; Department of Education and Early Child Development [DEECD], 2012; ECA, 2012). Thus, assessment practices that capture, document and reflect children's existing 'thinking in action' is an integral aspect of ongoing planning for learning in early childhood (Cohrssen, 2015).

This view of the child as mathematically capable is iterated in the *Early Years Learning Framework [EYLF*], (Department of Employment, Education and Workplace Relations [DEEWR], 2009); this national document guides the practice of early childhood educators, who work with children from birth to 5 years of age across Australia. The EYLF acknowledges that "all children demonstrate their learning in different ways …[therefore] … approaches to assessment are … relevant and responsive to the physical and intellectual capabilities of each child", and these observations are connected to learning outcomes (DEEWR, 2009, p.17). However, the broadly stated outcomes in the EYLF provide minimal reference to the specific type and depth of mathematical thinking, processes and concepts children engage with. For example, outcomes 4.2 and 4.3, draw attention to children "communicating mathematical ideas and concepts; using patterns, mathematical language and symbols; contributing constructively to mathematical discussions and arguments; making connections; solving problems; applying generalisations; trying out strategies; and transferring knowledge" (DEEWR, 2009, pp. 35-36). Thus, specific information, regarding how mathematical ideas and concepts transform as children develop greater conceptual awareness, is not evident or elaborated upon in the EYLF. Similarly, identifying the possible depth and range of young children's mathematical thinking is not well supported through the descriptions and analysis of learning stories in the accompanying resource, *Educators: Belonging Being & Becoming* (DEEWR, 2010).

Educators' Noticing of Mathematical Features of Children's Play

The Department of Education and Child Development (DECD) in South Australia has produced *Implementation guidelines for indicators of preschool numeracy and literacy* to support early childhood educators identifying mathematical features of children's interactions (DECD, 2015). This document refers to a *Numeracy Chart* that is multi-layered, whereby mathematical processes (behavioural) and four broad indicators (conceptual) articulate how a "child sees, interacts with and explores their world" [mathematically], the indicators are "interconnected and observable," relating broadly to young children's developing mathematical senses (DECD, 2015, p. 9). Key elements and examples of practice in the document support professional learning and further discourse around identifying children's mathematical thinking consequently strengthening educators' pedagogical content knowledge. However, the children's thinking in the documented examples is not captured over time, presented as isolated exemplars. Thus, the development of mathematical understanding, and pathways that delineate changes in awareness of specific mathematical concepts, is not revealed.

In their *Evidence Pape*r on *Assessment for Learning and Development* in early childhood contexts, Flottman, Stewart and Tayler (2011) state that "non-judgemental assessments [that are] evidence based … dynamic and ongoing …systematic and rigorous" (p. 5), are essential elements in effective assessment for young children's learning through play. Ongoing, formative assessment strategies underlie the intentional, reflective practice affirmed in the EYLF (DEEWR, 2009; DEEWR, 2010). However, there is a need to support educators to notice the complexity of mathematical concepts children are naturally exploring, and to

actively engage in questioning to elicit children's reasoning. "Early numeracy skills predict [later] achievement in mathematics … [and] … greater growth in numeracy skills [is] related to greater maths-specific talk amongst teachers" (Reid, 2016, p. 7). Therefore, the development of systematic, evidence-based practices and professional learning resources are needed to support early childhood educators' noticing and responding to the depth of young children's mathematical thinking (Lee, 2017).

Theoretical Perspectives

This paper considers three theoretical perspectives-theories concerning schematic patterns, embodied cognition, and the awareness of mathematical pattern and structure [AMPS] (Mulligan et al., 2009), to explain the development of children's mathematical understandings revealed through their embodied actions in play. In presenting these perspectives we propose that an inherent relationship exists between them, which then leads tentatively to the formation of a multi-dimensional theoretical framework. Approaches to supporting professional learning to develop educators' pedagogical content knowledge and active noticing of children's mathematical understandings will also be discussed.

Action Schemas, Embodied Cognition, and Early Childhood Mathematics Education

Piaget asserts that young children display "order and coherence" through the development of patterned schemes, which "refer to the basic structure underlying the child's overt actions … [there is] structure of behaviour; that is an abstraction of the features common to a variety of acts which differ in detail … [observable as a] regularity of behaviour" (Piaget, 1952, cited in Ginsburg et al., 1969, pp 20-21). Chris Athey (2007) built upon Piaget's theories regarding common patterns of behaviour, categorising these into eight observable action schemas that relate to how children engage spatially and naturally within their environment. These being; "dynamic vertical; dynamic back and forth; circular rotation; going over, under or on top; round a boundary; enveloping or containing; going through a boundary; and [externalised] thought" (Athey, 2007, p. 3). The schemas progress through four distinct but interrelated stages, "sensorimotor behaviour, symbolic representation, functional development, to thought", interestingly the final stage, the expression of thought, which is also one of the action schemas, emerges from a coordination of the other action patterns (Athey, 2007, p. 2). However, the structural elements underlying the development of the action schemas at the sensorimotor level requires deeper understanding to interpret 'order and coherence' underlying these behavioural patterns and to connect this with the perspective of embodied cognition.

Thought is not always expressed verbally; non-verbal actions are readily observable through children's play and can reveal children's patterns of thinking mathematically (Athey, 2007; Ginsburg et al., 1969; Kim et al., 2010; Piaget, 1926, 1952). Children's expression of mathematical concepts, initially formed as a bodily sense of knowing, have been studied through the field of embodied cognition (Kim, et al., 2010; Meltzoff, 1999; Merleau-Ponty, 2002). Underlying the theory behind embodiment is the premise that bodily intelligence emerges through the child's movement and sensory engagement with environments as "movement is the mother tongue," the first outward expression children engage with, from which all other cognitions are derived (Smith & Gasser, 2005, p. 29). Bautista, Roth and Thom (2012) "propose that kinetic movement constitutes thinking itself," referring to dynamic action as "thinking in movement" (p. 380), whereby

mathematical concepts are "in the flesh" (p. 364) and thus experienced through the movement of the body. Observable features of children's dynamic movement include actions such as "rhythmic patterns … beat gestures … body position and object orientation" (Bautista, et al., 2012, p. 368). Therefore, children's movement in action, iterated over time would reveal patterns of commonalities, emerging across experiences; and these similarities could reveal an underlying coordinated structure (Athey 2007; Ginsburg et al., 1969; Piaget, 1952).

Children's Development of an Awareness of Mathematical Pattern and Structure (AMPS)

Having a deep, fluent understanding of how mathematical concepts develop requires educators to have an awareness of the structure of concepts observed (Mason, Stephens & Watson, 2009). Structural understanding involves reasoning about the relationships between patterns to recognise and engage with similarities between and across concepts (Wood, 2002). Patterns have predictable elements that repeat (Mulligan et al., 2009). Thus, reasoning about the relationships between the repeating elements, and across different types of patterns, leads to generalisations about structural features underlying all mathematical concepts (Mason et al., 2009; Mulligan et al., 2009). Mulligan, Mitchelmore and Stephanou (2015) identified these underlying mathematical structures as: sequences; shape and alignment; equal spacing; structured counting; and partitioning. An interview-based Pattern and Structure Assessment [PASA] (Mulligan et al., 2015) can be implemented to measure individual children's levels of AMPS and underlying structural development. Longitudinal research with children aged 5 to 7 years has found that attention to these structural features of mathematical concepts supports the growth in children's level of awareness of mathematical pattern and structure [AMPS] (Mulligan et al., 2015). However, formal methods of assessing children's understandings can remove children from contexts and experiences that are familiar and meaningful to them (Cohrssen, 2015; Macmillan, 2009; Papic, 2015; Van Hoorn et al., 2015) and thus may not indicate the depth of children's existing mathematical capabilities revealed through familiar contexts such as play.

Professional Learning Perspectives: Developing Early Childhood Educators' Practice of Noticing Mathematical Features of Children's Play

There is an expressed need to support educators' awareness of young children's embodiment of mathematical thinking (Thom, 2017). In a recent reconceptualization of early childhood educators' mathematical pedagogical content knowledge (PCK), three essential underlying constructs were identified: '*noticing'* everyday mathematical opportunities; '*interpreting'* the mathematical content inherent in these situations; and '*enhancing'* this to deepen the children's mathematical thinking (Lee, 2017, pp. 232-233). These three interrelated aspects of PCK are developed through educators' "knowledge of children's development [of] mathematical concepts as well as their own knowledge of strategies … [and] are effective only when teachers have a depth of personal understanding about mathematical content … to mathematize children's informal experiences" (Lee, 2017, p. 241). In Lee's study (2017) it was found that the educators' ability to notice geometric/spatial elements of the children's play was substantially less than noticing aspects of number/measurement, and that "ability to notice [did] not necessarily translate into effective execution of interpretation" (p. 240). Strengthening educators' mathematical knowledge was

identified as key in supporting the development of all three interrelated aspects of PCK, that is: noticing, interpreting and enhancing young children's mathematical thinking (Lee, 2017).

Summary and Recommendations

This paper presents a theoretical proposition that an awareness of mathematical pattern and structure [AMPS] (Mulligan et al., 2009) could underlie the development of children's embodied mathematical cognition. Thus, identifying children's use of, or attention to, pattern and structure evidenced from observing their dynamic movement through play could provide a more integrated view of their developing mathematical understandings. As this relationship is yet to be fully described, the development of a multi-dimensional observational framework is planned. The aim is to focus educators' attention to noticing children's pattern of actions (Athey, 2007). This will be overlayed with the five mathematical structures indicating levels of AMPS (Mulligan et al., 2015). This could reveal insight into how to examine the emergence of AMPS (Mulligan et al., 2009) underlying embodied actions in children's play.

Educators' practice of interpreting children's developing mathematical understanding through noticing patterns of dynamic movement expressed in their play has important implications for early childhood mathematics education (Lee, 2017). Interpreting embodied expressions is reliant upon educators' own awareness of recognising young children's mathematical thinking as concepts in action and knowing how to respond to differing levels of children's awareness. This will enable educators to connect with and develop these non-verbal understandings (Cohrssen, 2015; Lee, 2017; Marcus et al., 2016; Thom, 2017). Therefore, further research into the design of professional learning that supports educators' PCK may explore how pattern and structure could underlie children's embodied action, revealing a more coherent picture of young children's mathematical development noticed through play.

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Experimenting with Reform-Orientated Approaches: Difficulties and Advantages Experienced by Primary Mathematics Teachers

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There have been many attempts to reform mathematics teaching in Australia to encourage teachers to use more cognitively demanding tasks that focus on problem solving and reasoning. However, there is limited specific advice for teachers on how best to do this. This paper reports on one set of survey items that examines 52 teachers' responses to experimenting with a reform-orientated approach and challenging tasks through the EPMC project. Findings indicate that the EPMC approach was different to most teachers practice, and despite the difficulties experienced by both students and teachers, both responded positively and reported the approach effectively supported student learning and in developing growth mindsets.

With growing evidence supporting the notion that students learn best when they are presented with academically challenging tasks that focus on problem solving and reasoning (NCTM, 2014; Kilpatrick, Swafford, & Findell, 2001) many countries around the world have made curricular changes to give problem solving a more central role. However, despite reforms in mathematics education and resources that encourage teachers to utilise more cognitively demanding problem-solving tasks, it appears that many teachers are reluctant to use such tasks and experience difficulties incorporating problem solving into classroom practice (Sullivan, Clarke, Clarke, & O'Shea, 2010). In exploring the reluctance of teachers to pose more cognitively demanding tasks, literature suggest it may arise from fear of student reactions to being challenged and a lack of time to plan such lessons (Sullivan et al., 2014). Stacey (2016) described the difficulty experienced by teachers as not only mathematical, but as pedagogical and personal; as teachers strive to meet a range of student learning needs, take appropriate risks and invite more student autonomy. Furthermore, Stein, Grover and Henningsen (1996) found that when teachers incorporate cognitively demanding tasks, they are often transformed into less demanding tasks during instruction. Consequently, it appears that teachers may need more support and opportunities to incorporate challenging problem solving tasks in their classroom.

The data reported below were collected as part of the Encouraging Persistence Maintaining Challenging (EPMC) project (Sullivan, Borcek, Walker, & Rennie, 2016). This project was a teacher professional learning (TPL) initiative based on the notion that one way to encourage innovation is to offer teachers specific suggestions of learning sequences involving engaging and challenging problem solving experiences that prompt the experimentation of alternate approaches.

The EPMC Project as a form of Reform-Orientated Approach

It is acknowledged that tasks play a vital role in student learning (Anthony & Walshaw, 2009). However, as Stein and Lane (1996) found, engaging students in high levels of cognitive thinking and reasoning is dependent on the how problems are set up and implemented in the mathematics classroom. Furthermore, Marshall and Horton (2011) concluded after examining the order of instruction of over 100 lessons, that students thought more deeply about the content when given the opportunity to explore the concepts prior to any explanation (whether this be by the teacher or students). The implication is that teachers should provide students with opportunities to develop ideas for themselves to maximise learning opportunities. Despite consistent advice to incorporate more challenging problem solving tasks that teach through

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problem solving, there is limited specific advice for teachers on *how* best to do this (Sullivan et. al., 2016).

One example of a TPL initiative that attempted to support learning through problem solving is the EPMC project. The project encourages teachers to consider an alternative approach premised on the assumption that students learn mathematics best when they engage in building connections between mathematical ideas for themselves, prior to any instruction, by working on unfamiliar challenging tasks. Essentially the EPMC approach facilitates students in moving from initial confusion, to eventual clarity and understanding. As a result, the EPMC approach is quite different to traditional approaches that utilise more explicit approaches to instruction, such as 'teacher telling'. Supported by empirical, conceptual and theoretical developments in mathematics education over the past three decades, the following key aspects of the EPMC approach (often described as a reformist approach) were summarised by Sullivan et. al. (2016) to involve:

- Approaches that teach *through* problem solving that use problem solving as a context for learning new concepts and developing the four proficiencies (understanding, problem solving, reasoning and fluency). Problem tasks are presented with minimal instruction from the teacher (and no explicit instruction) where students learn through the problem-solving experience, as well as from listening to others justify and explain their solutions and strategies. (see Schroeder & Lester, 1989)
- Adaptations of the task to differentiate the learning experience, including enabling and extending prompts that support students' thinking and to access the main learning task. (see Sullivan, Mousley, & Jorgensen, 2009)
- Approaches to reviewing student work including anticipating students' mathematical responses, monitoring student responses, purposefully selecting students' responses to display and have them explain their solutions and thinking, purposefully sequencing those responses so that the reporting is cumulative and connecting the student responses. (see Stein, Engle, Smith, & Hughes, 2008)
- Providing opportunities to consolidate the learning by posing further tasks that are appropriately varied. (see Dooley, 2012; Kullberg, Runesson, & Mårtensson, 2013)
- Tasks that are open-ended, have multiple entry points, allow varied solution strategies, require the students to justify and explain their solutions, require complex and non-algorithmic thinking, involve some level of anxiety for the students due to the unpredictable nature of the solution process required, and demand considerable cognitive effort. (see Stein, Grover, & Henningsen, 1996)
- Developing classroom cultures that fosters growth mindsets, where effort is valued and recognised to lead to success, mistakes and confusion are promoted as part of learning, challenge is welcomed, and persistence encouraged. (see Dweck, 2000)

The research reported in this paper is one aspect of my research exploring the factors that influence teachers to utilise reform-orientated approaches and challenging tasks. Teachers knowledge, beliefs, opportunities and constraints (including professional learning opportunities) are examined to explore teachers decisions on tasks and approaches that influence student thinking and learning. The theoretical model by Carpenter and Fennema (1991) provides a framework for this study emphasising that tasks and instructional approaches do not directly influence student learning, but instead influence student thinking and behaviour, which in turn, influences student learning.

In acknowledgement that practising teachers often find it challenging to implement reform-orientated approaches that encourage teaching through problem solving (Sullivan et al., 2009), the present study explored the following research question through the EPMC project: *When experimenting with a reform-orientated approach as part of a project, what are the difficulties*

and advantages experienced by primary mathematics teachers that can influence their decisions to incorporate reform-orientated approaches?

Data Collection and Analysis

The data reported below were collected as part of the EPMC project that adopted a design research approach which "attempts to support arguments constructed around the results of active innovation and intervention in classrooms" (Kelly, 2003, p.3). The intervention involved the suggestions of challenging tasks with a specific approach to teaching, and the innovation was the notion of activating cognition through embracing confusion. The project was iterative for two reasons: (1) up to 14 suggestions were implemented sequentially by teachers and (2) this approach was repeated to involve different mathematical content and demographic of teachers. The current paper focuses on one iteration of the project relating to geometric reasoning tasks, with an emphasis on the learning of angles.

As part of the TPL initiative, there were two professional learning days (at the start and end of the iteration) where teachers completed an online survey using Qualtrics (2015) at the beginning of each day. Whilst survey items for this study were incorporated into both days, only data from the second professional learning (PL) day are presented in this paper, after the implementation of the EPMC tasks and approach. Open response items were incorporated into the EPMC survey and designed to prompt self-reports of instructional practices and experiences implementing the EPMC approach. Participants involved 54 Australian teachers of Years 4, 5, and 6 classes (students aged 9-12 years old) from both public and catholic schools in rural and metropolitan areas. Teachers were from a range of socio economic backgrounds and years' experience teaching. Survey responses were analysed using inductive methods and Braun and Clarke's (2016) six phases of thematic analysis. Through Excel, data were inspected, coded, themes identified and reviewed for later refinement. Themes for survey item (a) were compared to the EPMC approach, reinspected and refined.

Results

The set of survey items presented to teachers on the second face to face PL day were:

> The set of angles suggestions asked you to:
>
> - Present students with tasks prior to any explicit instruction on the underlying concepts or without explaining how students should approach the tasks;
> - Differentiate the experience for students who need it;
> - Review students' strategies drawing on suggestions from various students;
> - Pose further similar tasks to consolidate the students' learning.
>
> a) In what ways is this approach similar to what you usually do?
>
> b) Assuming that you use the suggestions in this way, what difficulties did you experience?
>
> c) Assuming that you use the suggestions in this way, what do you see as the advantages of this approach?

Survey results are presented in three section according to survey item (a), (b) and (c). Note that due to teachers often describing more than one theme, the number of themes exceed the number of teachers in each section.

Reported Problem-Solving Practice

The survey item initially present instructional practices consistent with the EPMC approach and item (a) asked teachers to compare and identify similarities to their own approach, prior to

their exposure to the EPMC professional learning. The most frequent practices reported by the 54 teachers were differentiating the learning experience with 31 teachers describing *differentiation* (e.g., grouping, offering different tasks, enabling and extending prompts), and reviewing students' solutions and strategies with 28 teachers describing *reviewing student work* (either throughout the lesson or more commonly at the end of the lesson).

There were 17 teachers who stated their approach was *completely different* and described explicit approaches. Some representative responses were:

> I would usually provide some form of explicit teaching of skills/concepts before sending them off to do challenging tasks. Having completed this work over the past little while with Peter, I have a much greater appreciation for the way of presenting tasks without explicit instruction on the underlying concepts allows students a better opportunity to construct their own understandings allowing for a far richer learning outcome.

> I find I usually do more explicit teaching with teacher modelling… I found this approach to be really uncomfortable for me to stand back and just watch and not help those who were struggling. But I got used to it and I saw they weren't really struggling, they were problem solving and thinking! I don't give enough think time. I was amazed to see how well my students did using this approach.

Interestingly, some teachers reported using explicit approaches prior to the EPMC project, however, appear surprised when describing the EPMC approach to be effective without explicit approaches. There were an additional nine teachers who described *explicit approaches* with differentiation and/or reviewing.

There were 12 teachers who described *open-ended problem solving tasks* and 11 teachers described using *consolidating tasks*. Interestingly, of the 11 teachers who described using consolidating tasks, most of these teachers were familiar with the project and approach, having previous involvement in the project or other PL experiences. One teacher explained: *we have been implementing Challenging Tasks for two years now and this style of lesson is becoming our regular practice for all topics.* Ten teachers who described *no explicit instruction or modelling* also stated previous involvement in the EPMC project or related PL.

Overall, there were nine teachers who described their practice to be the same as listed in the survey item and who were also familiar with the EPMC project. One teacher commented *It is the same approach to many of my lessons but not all. This is mainly due to the Masters I have been involved in with Peter and so therefore my approach has changed to include more of these types of lessons.* It appears that the EPMC project and other professional learning opportunities have had an impact on some teachers' practice in that they describe using this approach in their classrooms. In contrast, some teachers new to the project described quite a different approach and reported a change in their practice, knowledge and dispositions.

Difficulties Experienced when Implementing a Reform-Orientated Approach

Consistent themes surfaced when analysing question (b) exploring the difficulties of a reform-orientated approach, and findings are presented in Table 1. The most common difficulty reported by 17 teachers was refraining from telling the students how to solve the task, including deconstructing the problem and 'teacher telling' of the procedures and concepts. Some responses include:

> I did find it difficult to not really get in there and support some students. But when I didn't, I found they often figured things out on their own. Or, after 5 minutes' think time, when they shared ideas, the kids just learned from each other

> The complete lack of explanation at first, many students want or are used to being told what to do at first. Students didn't like that feeling of not knowing what to do at first. But they got used to it relatively quickly

Despite teachers reporting it difficult to refrain from using explicit approaches, teachers reported that students responded productively to the EPMC approach.

Table 1
Themes from Survey Item (b) Exploring the Difficulties of the EPMC Approach

Themes	Total
Not telling	17
Fixed mindsets	15
Tasks	11
Differentiation	9
Time	8
None	5
Reviewing	4

The next prevalent difficulty reported by 15 teachers was related to fixed mindsets and trying to establish a classroom culture that promotes a growth mindset. Representative responses include:

> Tackling the fixed mindset of some of my students. They have a very fixed idea about their abilities. They have become so used to maths sessions being structured a certain and very traditional way that to be asked to solve problems without some time dedicated to explicit teaching was highly unusual to them.

> Some of the difficulties experienced included getting the students to adjust and have a go at these tasks without the fear of making mistakes.

These responses connect to the lack of teacher telling and students' response to feeling confused, challenged and having to find solutions for themselves. It is reasonable to assume that students' fixed mindsets were connected, in part, to teacher telling.

There were 11 teachers who reported difficulties related to tasks, with some teachers reporting them as too hard for their Year 3 students (which can be expected due to half the tasks being targeted for Years 5 and 6), and other teachers reporting *ambiguous wording of some of the tasks*, or *the tasks were too easy for my high achieving students*.

Differentiation was another theme described by nine teachers. Some teachers were not sure how long to wait before giving the enabling prompts and others stated they experienced: *difficulty in extending kids that needed a really deep extension, due to my own weakness in maths*. Time was also a prominent theme. Eight teachers reported that the end of year was too busy with little time to complete all the tasks due to reporting and assessment. Teachers also stated that the lessons required more time than they were accustomed to.

There were five teachers who reported they experienced no difficulties and four teachers who described difficulty with choosing when to review students' work, and how to draw out the learning as some students *struggled to explain* their thinking, solutions and strategies.

Advantages Experienced when Implementing a Reform-Orientated Approach

Table 2 presents the findings to survey item (c) exploring the advantages experienced by the teachers when implementing a reform-orientated approach. When asked to describe the advantages experienced when using the EPMC approach, majority of teachers (33) described students engaged and effectively learning. Representative responses include:

> Great for students in helping make connections between different concepts

> Critical thinking. Students are building the knowledge for themselves

> Allowing students to authentically construct their own deep understandings of the concepts

One of the key aspects of the EPMC approach is providing opportunities for students to connect ideas for themselves and constructing knowledge, and it appears most teachers saw the value in this approach and experienced students learning this way.

Table 2
Themes from Survey Item (c) Exploring the Advantages of the EPMC approach

Themes	Total
Effective learning and engagement	33
Growth mindsets	32
Reviewing students' work and students learning from others	22
Differentiation	19
Challenging problem-solving tasks	19
Student led learning and independent thinking	13

The next prominent theme was related to growth mindsets. Some responses include:

> Students did become more persistent with each lesson and not giving up after the first few minutes.

> Once the students adapted to the process and understood this method they were very enthusiastic about the approach and became confident sharing their responses whether they were right or wrong.

> I felt a shift in some students in their mentality towards challenge and confusion. Many students realised their potential as problem solvers through this approach.

Many teachers reported difficulty supporting students who had a fixed mindset. Despite this challenge, it appears most teachers experienced success with shifting students' fixed mindset to a growth mindset after some time. It appears that teachers perceive the EPMC approach to be effective in supporting their students to build persistence, welcome challenge, see mistakes as part of learning, and build confidence in the process.

Reviewing students' work and students learning from one another was also a common theme among the teachers, with 22 teachers describing this theme. One teacher reported it was *Amazing to see when reviewing students' strategies how many students went back and understood the task.* Another teacher commented that *students listened to the suggestions of others and this helped them to deepen their own understandings.* It appears these teachers experienced the process of reviewing students' work a valuable aspect of student learning, with students learning through each other and the problem-solving experience, rather than explicitly being taught the concepts and procedures.

Differentiation, including the use of enabling and extending prompt that differentiated the learning experience, was described by 19 teachers. Some representative responses include:

> Enabling and extending prompts allowed me to cater for all students and they all experienced success.

> I liked how all students began the task and then could either be supported or challenged. This built confidence in the students as they were happier to work through the challenges.

> The high-end students didn't have an end point, they were consistently extended through the prompts and it challenged the students that always "get it" to explain their thinking deeper.

Despite concerns related to teachers experiencing difficulty meeting a range of student needs (Stacey, 2016), it appears that these teachers perceived the EPMC tasks and approach to be effective in differentiating the learning experience. Teachers often described the use of enabling and extending prompts to facilitate differentiation whilst also highlighting the advantages of all students working on the same task and beginning at the same level. Overall, teachers described the approach to differentiation to be effective at catering for all students, promoting successful learning and encouraging positive mindsets (including confidence and enjoyment) among their students.

There were a further 19 teachers who described advantages related to the use of open-ended challenging problem solving tasks. Some responses include:

> The tasks were clear and easy to teach, despite the initial feelings of confusion students were all able to contribute and learn no matter their level, the tasks were non-repetitive and engaging.

> It enabled students to be challenged and find a variety of ways to solve the problem.

Not only did teachers report ease of implementation when using challenging and open-ended problem solving tasks, but they saw advantages in tasks that prompted a variety of solutions and strategies to explore the underlying concepts. Teachers also reported that students *enjoyed the challenge* and *embraced the challenges and wanted to find the answer*. Interestingly, when using cognitively demanding tasks, teachers described positive responses from the students in that they were not fearing challenge, but rather welcoming it. These findings conflict with the concerns related to teachers fearing student negative reactions to challenge and that students prefer to be told what to do (Sullivan et al., 2014).

Connected to the above themes was the final theme *Student led learning and Independent thinking*. 13 teachers used words that described students' thinking *for themselves* and building their *own understanding*, describing students having ownership over their learning.

Discussion and Conclusion

Extending from previous research that explored the cognitive demand and features of problem solving tasks in primary mathematics classrooms (McCormick, 2016), this paper sought to gain an insight into teachers' experiences implementing a reform-orientated approach. Overall, many teachers described a different approach to teaching mathematics with some similar themes (but commonly a different approach) including differentiation and reviewing students' work. There were some teachers who described a similar approach to teaching who had previous involvement in the project and related professional development opportunities. When experimenting with a reform-orientated approach, many teachers reported difficulty refraining from telling the students what to do or teaching the concepts, and encouraging students who had a fixed mindset. When reflecting on the advantages of the EPMC approach most teachers reported the approach to be engaging and effectively support students develop deep understanding of the concepts and procedures. Furthermore, the teachers reported that the EPMC approach helped to develop growth mindsets with their students, including students welcoming challenge and confusion, encouraging persistence and seeing mistakes as part of learning. Despite concerns related to the reluctance of teachers to incorporate challenging problem solving tasks and constraints experienced (Sullivan et al., 2010), it appears that the EPMC approach supported teachers to experiment with alternate approaches, and both students and teachers responded positively to the EPMC approach. Some teachers reported that their *students also commented they prefer to do Maths this way.*

Interestingly, the EPMC project appears to have had an impact on some teachers practice in that they described incorporating reform-orientated approaches in their mathematics classrooms after involvement in the project. Previous EPMC survey items explored this phenomenon, and unpublished findings suggest that teachers reported the documentation (that detailed the tasks and approach) was well organised, thorough, high quality, well linked to the curriculum, and easy to implement and incorporate into planning. Some teachers also stated that they found modelling and trialling the tasks during the first day to be effective in preparing them for trying the new approach and tasks. These finding could be of interest to schools and systems in designing professional development programs for significant and sustained improvement in student learning. For a more accurate portrayal of teacher practice this research will augment survey data with case study data, to further explore teachers' beliefs, knowledge, opportunities and constraints related to problem-solving practice and factors that influence effective practice, including professional development opportunities.

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Principals' Perceptions and Expectations of Primary Teachers with a Specialisation in Mathematics

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This study explored the perspectives of primary principals, as they consider the prospect of employing new teachers with a 'specialisation' in mathematics. Structured interviews were conducted with six NSW principals across school sectors. Analysis of the data revealed the nature of 'specialist' roles in a school depended heavily on current funding arrangements and the levels of existing expertise. The traits that principals wanted new specialists to have formed three strong themes: knowledge for teaching mathematics, personal attributes, and relationships with others – with leadership qualities anticipated across all three. We raise questions about the preparation of graduates to meet the expectations of schools.

Traditionally, the primary school teacher in Australia is considered a generalist. While this remains the case, we have entered a new phase in education history where new graduates are expected to also qualify with a 'specialisation'. The trigger for this change has been declining performance of Australian school students in international comparative tests of mathematics and science, and a perceived need to lift the competency of primary school teachers in these subjects. In 2014, the Teacher Education Ministerial Advisory Group (TEMAG) recommended to the Australian government that all primary teachers graduate with at least one subject specialisation, giving priority to mathematics, science and languages (TEMAG, 2014). Subsequently, the Australian Institute of Teaching and School Leadership (AITSL) mandated that by 2019, primary teacher specialisations be delivered by every initial teacher education (ITE) provider (AITSL, 2015a).

AITSL provided minimal guidance for the interpretation and implementation of the new Program Standard 4.4 by for the state regulatory bodies - the key statement being that graduates should "… demonstrate expert content knowledge and pedagogical content knowledge and highly effective classroom teaching in their area of specialisation" (AITSL, 2015b, p.14). The expectation primary teachers graduating with mathematics specialisation will improve student numeracy is made clear.

The success of this policy-driven initiative is likely to depend on several key factors, one of these being the ways in which a school supports and utilises new teachers with a specialisation in mathematics. This study took place before these new teachers entered the teacher workforce. It investigates the perceptions and expectations of school principals who will, over the next few years, encounter newly graduated generalist teachers with a specialisation in mathematics.

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Literature Review

Prospective primary teachers in Australia are known to hold reservations concerning their teaching of mathematics (Lomas, Grootenboer, & Attard, 2012; Maasepp & Bobis, 2014). Some would be happy if they were not required to teach it (Williams, 2009), but Pezaro (2017) argues that specialists are not the answer to teachers' lack of confidence in a subject area. She advocates that primary teachers remain as generalists because generalists have more time with their students and are better able to integrate content across subjects. However, she sees the value of having teachers able to coach their less confident colleagues. If a teacher does not fully understand a concept, they are not comfortable in teaching it and can generate student misconceptions (Betts & Frost, 2000).

Recognition of a secondary teacher as a subject specialist is based on their formal tertiary qualifications. The specialist label however, is problematic when used in relation to primary teachers. In some countries, specialist primary teachers are trained and employed like secondary teachers, only having responsibility for teaching their area (or areas) of specialisation. In Singapore for example, primary teachers graduate with a combination of two subject specialisations and these are the only subjects they teach (Khamid, 2016).

In Australia, most primary teachers are employed as generalists. A survey of 401 principals of NSW primary schools however, found that 73% had used subject specialists (Ardzejewska, McMaugh, & Coutts, 2010), subject specialists being defined as auxiliary teachers employed to teach in only one curriculum subject area. Of these subject specialists, about 40% taught Science and Technology and 30% taught Creative Arts. Just 4% were English specialists and no principal said they used a mathematics specialist. This was because principals viewed mathematics and English as the core teaching areas of generalist classroom teachers, mathematics being essential for numeracy and English for literacy. Principals' main considerations leading to their use of a subject specialist were found to be the perceived lack of expertise in the subject at their school, teachers' willingness to teach it, and their desire for the school to improve in that area.

In recent years, the term 'primary mathematics specialist' has been equated with mathematics leadership (Driscoll, 2017). A mathematics/numeracy leader is someone who has a role in improving mathematics teaching at their school (Jorgensen, 2016). These teachers may have obtained the role through receiving in-service training to improve the mathematics content and pedagogical knowledge of teacher colleagues (Driscoll, 2017; Jorgensen, 2016). Balancing classroom teaching with subject responsibility can be difficult (Driscoll, 2017), particularly when they are early career teachers (Jorgensen, 2016).

There is debate as to whether specialist subject expertise should be developed within ITE programs or should only be developed after a primary teacher has had generalist classroom teaching experience. McMaster & Cavanagh (2016) posit that pre-service teachers can benefit from a specialist professional experience placement in mathematics even prior to a generalist placement within their ITE program. In their policy framework NESA (2016) suggests the provision of specialisation be supported by targeted professional experience with "mentoring by supportive accomplished teachers in the subject areas" (p. 2).

It is widely recognised that effective mathematics teaching in schools requires more than just the professional development of individual teachers. Teachers share improved practices in communities. The leadership of the school principal is vitally important to the development and resourcing of these communities, thereby ensuring on-going improvement in mathematics outcomes for students (Gaffney, 2012). There is anecdotal evidence that school principals perceive a need for graduate teachers with additional expertise in teaching mathematics, but we are not aware currently of any research into principals' views.

The Study

The 2017 study was a preliminary investigation of primary principals' views about employing teachers with a specialisation in mathematics. In particular, it sought insight into their expectations of these new graduates and the roles they might fulfil in schools.

Context and Participants

At the time of the interviews, no teachers had graduated with a specialisation in mathematics under the new policy, so the principals had no experience of working with teachers possessing this qualification. It was apparent that the principals had little or no knowledge of the requirements placed on ITE providers by the NSW Education Standards Authority regarding the preparation required by ITE programs for the mathematics specialisation (NESA, 2016). The principals volunteered to participate because of their interest in developing and maintaining a strong mathematics leadership team at their school, dedicated to improving the mathematics outcomes of their students.

The schools were deliberately selected to provide variety in sector, student population, location, proportion of language background (LBOTE) and socio-economic levels (ICSEA), using 2016 data found in the MySchool website (See Table 1).

Table 1
Demographic Data of the Principals' Schools from https://www.myschool.edu.au

Principal (pseudonym)	School Sector	Student Population	Location	LBOTE	ICSEA
Andrew	Government	medium size, co-educational K-6	Metropolitan	78%	957
Bethany	Government	small-medium size, co-educational K-6	Regional	5%	1000
Cynthia	Catholic system	large size, co-educational K-6	Metropolitan	74%	1029
David	Government	small size, co-educational K-6	Outer-metropolitan	9%	1036
Enid	Government	medium size, co-educational K-6	Metropolitan	26%	1161
Felicity	Independent	medium size, girls only, Junior school (K-6) within a K-12 school	Metropolitan	19%	1193

The Interviews

A written set of ten interview questions was given to the principals for their consideration prior to their decision to participate in the study. On agreeing to participate, Enid chose to provide written answers to these questions. The other five principals agreed to be interviewed individually by the first author, at a time and place of their choosing. The interviews typically lasted for approximately 20 minutes. They were audio-recorded and transcribed. Most of interview questions concerned mathematics leadership roles, organisational matters and relevant funding arrangements. This paper focuses only on the last two interview questions that were about teachers with a specialisation:

Q.9. If you had the opportunity to employ a teacher who has a specialisation in mathematics, would you do this? Why?

Q.10. If you were seeking to employ a mathematics specialist who is also a classroom teacher, what would you list as the essential attributes? what would you list as desirable attributes?

Analysis

A first reading of the six interview transcripts for Q9 revealed that the responses were quite specific to the school context. Therefore, the analysis approach was simply to summarise key points and look for similarities and difference across the schools. However, the responses to question 10, seeking the desired traits of newly graduated classroom teachers with a specialisation, were more complex and detailed. Therefore, inductive analysis was applied, involving multiple readings, coding of phrases and sentences, and clustering of codes into categories (Braun & Clarke, 2006).

Results

The findings from analysis have been organised into two sections, determined by the two interview questions. During the interviews, principals used the terms 'mathematics' and 'numeracy' interchangeably, so we have not made any distinction between them.

The Need for Employing a Specialist Mathematics Teacher

All the principals in this study had prioritised mathematics in their schools' current strategic direction. However, their perceived need to employ a teacher with a specialisation in mathematics depended on the particular circumstances in their school, with the main determinants being; a) the number of existing staff with additional training in mathematics content and pedagogy, and, b) access to funding for staff training from external sources. The level of available support funding was related to the ICSEA value for the school. Schools with high support needs may have funding allocated for an *additional* staff member to fill a specialist support role. The principals of the larger schools (Andrew and Cynthia) were managing funding from short-term numeracy programs to enable on-going professional development of staff. This funding is not generally available to schools like Enid's that have a high ICSEA value.

In the absence of a funded numeracy program, principals had classroom teachers who supported their colleagues in mathematics teaching. Bethany, working at a regional school, felt "very blessed" that she currently had two teachers who had received professional development as trainers in previous numeracy programs, acknowledging that other schools in the region were not as fortunate. At his small school, David said how "very lucky" he was to have a new early career teacher who was enthusiastic about mathematics and shared her mathematics expertise with others. He supported her self-identified professional development outside school hours.

When possible, schools without funded numeracy programs made use of external consultants. Felicity (independent school, high ICSEA) arranges training for her staff through a numeracy consultant from the Association of Independent Schools who works with teachers in their classrooms. David (small school) gave his staff a one-off professional development day with a private numeracy consultant, which was made affordable by sharing the session with staff from nearby schools in an informal community of schools.

When asked specifically about employing a general classroom teacher with a specialisation in mathematics, David, Beth and Felicity expressed enthusiasm. However, the following conversation between the interviewer, Cynthia (large Catholic school), and the diocese 'numeracy educator' (Cathy) who happened to also be present, revealed a preference for 'in the job training'.

Cynthia: Well, I'd rather them be a specialist in mathematics than say creative arts. Let's be realistic here. If you've got a really strong background in a curriculum area that's always a great advantage…Well obviously, unless I'm advertising for a creative arts teacher.

Interviewer: But if it was a general teaching position, a classroom teacher?

Cathy: It's hard isn't it?

Cynthia: I don't know. I don't know that I would - there's so many things that go into having a good CV, anything across the board to get an interview.

Cathy: It is true we like to grow them, don't we?

Cynthia: The reality is…

Cathy: Grow them in the context.

Andrew, who also develops mathematics leaders from within his staff, mentioned looking for new teachers who are open to being mentored by the mathematics leaders at his school.

Attributes of Teachers with a Specialisation in Mathematics

Table 2
Grouping of Attributes the Six Principals Considered Essential or Desirable

Category	**Principals who mentioned this attribute**
Knowledge for teaching mathematics:	
Curriculum	Andrew, Bethany, Cynthia, David, Enid, Felicity
How children learn	Andrew, Cynthia
Teaching approaches	Andrew, Cynthia, Felicity
Topic connections	Andrew
Mathematics education language	Cynthia
Use of mathematical representations	Cynthia
Current research	Bethany
Mathematics resources	David, Enid
Personal attribute:	
Passionate	David, Felicity
Helpful	Bethany, David
Sharing	Enid
Personable, respectful	Cynthia
Articulates concepts	Cynthia, David
Approachable, dedicated, flexible	David
Open to learning	Andrew
Builds relationships with:	
Colleagues	Bethany, Enid, Felicity
Children	Andrew, Bethany, Cynthia
Parents	Andrew, Bethany
Community	Bethany, David

As expected, the principals mentioned attributes that were not specific to the teaching of mathematics. Cynthia, Enid and Felicity mentioned before anything else, that the teachers must have good general classroom teaching skills; "How proficient they are as teachers

themselves first and foremost" (Cynthia). Attributes other than good general classroom teaching skills, fell into three clusters as shown in Table 2: knowledge for teaching mathematics, personal attributes, and the ability to build relationships with others.

a) Knowledge for teaching mathematics

This category includes traits such as knowledge of the mathematics curriculum, how children learn, teaching approaches (a problem-solving approach was specifically mentioned by two principals), current research and good teaching resources. For example, "The curriculum knowledge number one. They need to have a very high level of understanding…" (David); and, "Having a really firm understanding of what the research says around best practice…" (Bethany); and, "…a really clear understanding of how children learn and are able to articulate it" (Cynthia).

Embedded within several comments about Knowledge, was the implication of leadership: "…talk at staff meetings about things like resources" (David); and, "Sometimes, taking the lead and saying let's try it this way" (Bethany); and, "…assume responsibility for the curriculum" (Felicity).

b) Personality

All the principals believed that the graduate's personality would be of importance. They specified traits such as being helpful, approachable, passionate and flexible. For example: "… someone who is passionate about it…be willing to assist…" (David); and, "…happy to roll up their sleeves, be in there as an additional person to support…" (Bethany). Some of these traits related to an ability to mentor others: "I have worked with people over the years who have a wonderful knowledge themselves but were not able to bring people along at the level they were at" (Cynthia). Andrew mentioned the importance of young teachers being open to learning from more experienced mathematics leaders.

c) Relationships

The principals spoke of relationships with children, parents, teacher colleagues, and the community as being critical for having a lasting influence. Mentoring and leadership expectations were framed in productive relationships. For example: "…they are people who have really strong capacity to build relationships very quickly with children" (Bethany); and, "…directing things in certain ways that create a long-term effect change for children" (Bethany); and, "…the links between the classroom and the lounge rooms of those kids involved is most important" (Andrew); and, "ability to work with a team to develop mathematics teaching in the school" (Enid).

Discussion

The decision as to whether to employ a graduate teacher with a specialisation in mathematics is strongly influenced by the school's current circumstances - particularly the funding they have for additional staff, and the number of 'good' maths teachers already at the school. This is because formal numeracy leadership positions in Australian primary schools are only possible through funding that is surplus to the usual funding models (Jorgenson, 2016). An important point is that, even when schools had funds to employ an additional teacher as a 'mathematics specialist', the role of this person was to provide professional development and support for other teachers, *not* teaching the mathematics for them. This is consistent with the 2010 survey of Australian principals by Ardzejewska, McMaugh, & Coutts (2010), and literature on the nature of mathematics leadership (Driscoll, 2017; Jorgensen, 2016). It supports the notion that in Australia, English and Mathematics are

considered the core responsibility of each primary classroom teacher and highlights the need to support the teachers who struggle to teach mathematics effectively (Lomas, Grootenboer, & Attard, 2012; Maasepp & Bobis, 2014). When funding was not available, principals still saw the need to have one or more teachers with particular strength in mathematics who could address the professional learning needs of other teachers.

However, when the conversation with principals moved away from existing arrangements in their schools to the future prospects of employing a new general-primary graduate with a specialisation in mathematics, the traits they emphasised where much less predictable from previous research. Given the widely-established concerns about the depth of mathematics content knowledge of primary teachers, we were surprised that only one principal mentioned it. Perhaps it was assumed that all such graduates would have high-level competence in mathematics. Instead, the principals spoke of *knowledge for teaching mathematics*, such as deep knowledge of the curriculum, how content progresses, and how children learn. Several principals extended this to being able to articulate their knowledge clearly, so they could share it with other teachers.

Through their emphasis on personal qualities and skill with forming productive relationships, the principals made it very clear that they expected 'new specialists' to extend their influence outside their own classrooms, to work with other teachers, and reach into the school's community. Expectations for mentoring and leadership permeated all three categories of traits: sharing of knowledge for teaching; enthusiastic and approachable people; and, forming productive relationship to effect change. Interestingly, AITSL's paper on graduate outcomes for primary specialisations (AITSL, 2017) specifies *content knowledge*, *pedagogical content knowledge*, and *highly effective classroom practice* as requirements, but makes no mention of leadership qualities. A "… capacity to share knowledge with other teachers" is listed as one possible additional feature (AITSL, 2017, p 1). The more detailed NSW policy framework goes a step further by suggesting the ITE providers might consider "… focussing on both academic and personal attributes including enthusiasm for the learning area" (NESA, 2016, p.2).

Conclusion

It should be remembered, that although the principals in our study came from a variety of school contexts, the views of only six principals cannot be considered as representative of the perceptions of principals across the state of NSW and may give little indication of the situation in different parts of Australia. Yet the findings add to the scarce literature on this topic, by raising some interesting issues and questions.

There appears to be a mismatch between the AITSL policy guidelines for primary specialisation, and the needs and expectations of schools. The policy focusses on academic traits and practice inside the classroom. The principals emphasise personal traits and relationships outside the classroom. ITE providers, of course, attend to the academic preparation of their graduates, along with the practical preparation provided through professional experience placements. How well do ITE providers attend to the personal and inter-personal qualities of their students? Is it their responsibility to do so? Given the strong expectations of schools for 'specialist' teachers in mathematics to provide support, mentoring and leadership for other teachers, should graduates be explicitly prepared for such roles, or should we be trying to change the needs and expectations of the schools?

On the basis of this study we advocate the urgent need for extensive research into the multiple perspectives of policy-makers, ITE providers, schools and the graduates with a mathematics specialisation. The mathematics education community has an unprecedented

opportunity for sweeping reform in primary mathematics, operating through the imminent 'flood' of specialist graduates. However, we may be about to 'get it horribly wrong'.

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Defining the Characteristics of Critical Mathematical Thinking

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In this paper, we report on the interim findings of a study that seeks to identify the characteristics of children's Critical Mathematical Thinking CMT. Characteristics of CMT were initially generated from a synthesis of relevant research literature and then validated using a case study methodology via trials in early childhood classrooms. This paper provides a framework for CMT distilled from the literature and an illustrative case study of one student to provide tentative evidence that young children's use of CMT capabilities can be identified. The long term aim of this line of research is to explore the potential to promote CMT capabilities in a targeted manner.

It has been well established, that children begin to use mathematical thinking skills from a young age (e.g., Bobis, Clarke, Clarke, Thomas, Young-Loveridge, & Gould, 2005; Doig & Ompok, 2010). Evidence for this claim is primarily found in studies that have looked at the way young children learn mathematics (Sarama & Clements, 2009; Clarke, Clarke & Roche, 2011). Consistent with this perspective, current advice about the development of mathematical thinking capabilities in early learning contexts is that instruction should adopt an investigative approach to promote deep understanding and connections between mathematical ideas (Clements, 2001; Sarama, Lange, Clements, & Wolfe, 2012). The need to adopt investigative approaches, as a means of promoting mathematical thinking capabilities, is further supported by curriculum documents and educational policy (ACARA, 2016; Australian Government Department of Education Employment and Workplace, 2009). As a result, educators of young learners have worked to create mathematical learning experiences that focus on open-ended approaches to support creativity, imagination and reflexivity in addition to conceptual development. While early childhood educators have been provided with direction on the content and processes to be taught, as well appropriate pedagogical approaches, there has been limited advice from research literature about how to make judgments about levels of development for students' mathematical thinking capabilities. As such, limited means is available to assist early childhood teachers in identifying and describing students' mathematical thinking capabilities. It is essential that tools for assessing the capabilities be developed, in order to provide feedback to students and teachers about a student's progress – informing teachers' decisions about appropriate approaches to instruction.

The purpose of this paper is to outline a framework within which the characteristics of children's critical mathematical thinking are outlined and described. Consistent with this purpose, we will address the following research question.

What are the observable characteristics of young children's critical mathematical thinking?

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In attending to this question, we will (1) provide a synthesis of current literature related to children's critical mathematical thinking; (2) extend the synthesis to define *critical* mathematical thinking; (3) present a case study as an illustrative example of categories within a framework for critical mathematical thinking; and (4) discuss the potential for further research.

Mathematical Thinking

Advancing children's mathematical thinking has been a focus of an expanding body of research in recent years (e.g., Carpenter, Franke, Johnson, Turrou, & Wager 2017; Breen & O'Shea, 2010; Fraivillig, Murphy, & Fuson, 1999). While perspectives in the filed are wide ranging, conceptions of mathematical thinking, appear to coalesce around a number of central principles: children require mathematical knowledge (Burton 1984); a basic understanding of mathematical concepts (Burton, 1984); and opportunities to engage in mathematical learning in different ways, all within a learning environment that fosters mathematical development (Ginsburg, Cannon, Eisenband, & Pappas, 2006). M*athematical thinking* refers more to the "doing" of mathematics rather than the memorising of formulas or the application of procedures (Stein, Grover, & Henningsen, 1996) and so involves problem solving, reasoning and critical thinking. Key characteristics that demonstrate mathematical thinking have been synthesized from relevant research literature as, connecting procedures, tacking complex problems in novel ways, reasoning and sense-making (Table 1, # 1-4).

Research conducted by Cengiz, Kline, & Grant, (2011) investigated the types of tasks used to extend thinking with children in grades 1 – 4. Strategies observed by these researchers found that teachers invited children to provide an evaluation of their learning that would allow for reflection and sharing of ideas or strategies (Table 1, #5). Strategies to encourage reasoning were also researched and found that teacher probing questions such as "What makes you say that? How do you know? Why do you suppose that?" (Cengiz, Kline, & Grant, 2011) elicited children's thinking. Thus, how tasks are designed is critical for uncovering children's mathematical thinking - requiring children to reason and think mathematically (Stein Grover & Henningsen,1996). These include open-ended tasks that have multiple answers, many modes of representation and in particular the opportunity for children to explain and justify their thinking.

How tasks are implemented, however, is a critical aspect of eliciting children's mathematical thinking. A study by Fraivillig, Murphy and Fuson (1999) of first grade children and teachers looked at teacher practices and found teachers were using eliciting, supporting and extending strategies with the children to facilitate their mathematical thinking. The study found that with teachers promoting children's thinking, the following mathematical thinking capabilities emerged from the learners: describing solutions (#7); elaborating on an idea (#7); clarifying own solutions (#7); generalizing across concepts (#1); noting relationships (#1); and considerations of alternate solutions (#6). Similarities between the research examined thus far, has found that in order for children to think mathematically, teachers have a significant role as a guide, for the thinking to emerge. The characteristics of mathematical thinking is summarised in Table 1.

Table 1
Characteristics that Demonstrate Mathematical Thinking

1	Connecting procedures/ noting relationships/ generalizing across concepts	(Fraivillig, Murphey, & Fuson, 1999; Stein, Grover, & Henningsen, 1996)

2	Tackling complex problems in novel ways	(Stein, Grover, & Henningsen, 1996)
3	Reasoning	(Cengiz, Kline, & Grant, 2011; Jacobs, Lamb, & Philipp, 2010; Stein, Grover, & Henningsen, 1996)
4	Sense-making	(Jacobs, Lamb, & Philipp, 2010; Stein, Grover, & Henningsen, 1996)
5	Evaluating	(Cengiz, Kline, & Grant, 2011)
6	Considering other methods/strategies/ alternate solutions	(Cengiz, Kline, & Grant, , 2011; Fraivillig, Murphey, & Fuson, 1999)
7	Describing solutions/ Clarification of solutions/Elaborating on ideas	(Fraivillig, Murphey, & Fuson, 1999)

Critical Mathematical Thinking

Stenberg (1986) identified the construct of critical thinking as a lens to gain more in-depth insight into children's mathematical thinking. According to Sternberg, critical thinking includes "mental processes, strategies, and representations people use to solve problems, make decisions, and learn new concepts" (p.3). Additionally, critical thinking includes building knowledge, comparing and identifying differences, supporting ideas with reasons and examples and considering alternative solutions (Florea & Hurjui, 2015). The importance of critical thinking can be located in many educational documents and international assessments such as New Media Consortium (NMC)/ Consortium for School Networking (CoSN) Horizon Report: 2016 K - 12 Edition (Johnson, Adams Becker, Cummins, Estrada, Freeman, & Hall, 2016) and the Programme for International Students Assessment (PISA) (Organisation for Economic Co-operation and Development, 2018).

While mathematical thinking and critical thinking have considerable common ground, they differ at the level of detail. An alignment between mathematical thinking and critical thinking is presented in Table 2. An additional column, titled *Critical Mathematical Thinking Capabilities* provides detail of additional observable features of this alignment.

Table 2
Aligning Critical and Mathematical Thinking

Mathematical Thinking characteristics from Table 1	Capabilities of Critical Thinking		Critical Mathematical Thinking Capabilities
Connecting procedures/ noting relationships	Generate and evaluate knowledge; Apply new ideas to specific contexts	➔	Uses mathematical and other understandings to generate, evaluate, connect and create new ideas
Tackling complex problems in novel ways	Seek possibilities; Consider alternatives; Imagination; Innovation; Test	➔	Identifies and performs many ways to solve mathematical problems
Reasoning	Reason; Logic	➔	Provides reasons or judgments

Sense-making	Clarify concepts and ideas; Interpret; Sequence; Generalise	➔	Uses mathematical strategies to prove the answer is possible
Evaluating	Use evidence to support an argument; Draw reasoned conclusions; Evaluate	➔	Self-evaluates, using mathematical evidence and reasoning
Considering other methods/strategies/ alternate solutions	Solve problems; Resourcefulness; Analyse; Compare	➔	Identifies and performs many ways to solve mathematical problems
Describing solutions/ Clarification of solutions/Elaborating on ideas	Thinking skills and strategies; Explain; Question; Infer; Hypothesise; Appraise	➔	Builds on ideas through explanation, questioning, inferencing, hypothesising and appraising

The alignment between mathematical thinking and capabilities of critical thinking is displayed in Table 2 and results in an overall definition of *Critical Mathematical Thinking* (CMT). The following points provide the summary of CMT:

- Using the knowledge of mathematics and mathematical processes to:
- Use mathematical and other understandings to generate, evaluate, connect and create new ideas;
- Identify and performs many ways to solve mathematical problems;
- Provide reasons or judgments;
- Use mathematical strategies to prove the answer is possible;
- Self-evaluate, using mathematical evidence and reasoning; and
- Build on ideas through explanation, questioning, inferencing, hypothesising and appraising.

Methodological Approach

To address the research question about characteristics of young children's critical mathematical thinking, a case study methodology was used to gain insights into one child's sophisticated mathematical strategies when engaging with high-level open-ended tasks. Case study was selected as data was drawn from a bounded system (Stake, 1995); one early childhood classroom.

Participant Selection

The data for this study includes four major sources: (1) classroom observation field notes; (2) semi-structured focus group interviews; and (3) interviews of Kindergarten (1st year of formal schooling in NSW) classroom teachers; and (4) interviews of Kindergarten children. The timing of this research was during the beginning of a school year.

Four mathematics lessons, led by the teacher, were observed. These lessons included the entire class and were based on activities related to whole number patterns and algebraic thinking. The CMT capabilities listed in Table 2 was used by the researcher as a lens to identify children that presented CMT capabilities. Each child in the class was questioned by the research about their learning during the lesson by using probing or prompting questions (Rigelman, 2007) based on the CMT capabilities. Summaries of their responses were video recorded. Field notes included the observations made by the researcher during the class

lesson informed by the CMT framework. Based on observation notes and in class questioning, 38 children that showed potential CMT took part in 3 separate focus groups. The focus groups were based on the mathematics learning the children were participating in during their classroom teacher led lessons. The mathematics included: patterns and algebra, addition and subtraction and two-dimensional shapes. The researcher posed questions in relation to the teacher designed mathematical task, in order to ascertain students levels of mathematical reasoning. Examples of such questions included: How did you work that out?; What would happen if…?; Is there another way to do this?.

One child, Jordan, was selected for the illustrative case study, reported here, on the basis of the high level of interest he displayed in investigative tasks during the observed lesson and the insightfulness demonstrated during follow-up focus group interviews. Jordan's interview was based on a semi-structured interview protocol that included 8 open-ended questions aimed at prompting responses indicative of critical mathematical thinking. Each question was designed to allow for specific CMT characteristics, as identified in Table 2, to emerge.

The researcher video recorded and transcribed the interview with Jordan. Each response was mapped against the CMT capabilities to determine the scope of Jordan's development in this area.

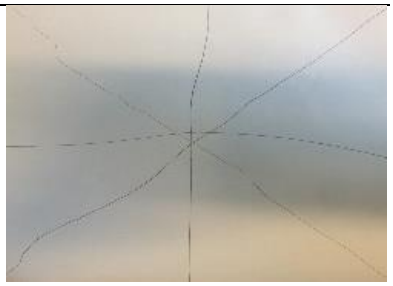
Results - Jordan's Critical Mathematical Thinking

In this section, an illustrative example is presented based on three out of eight questions from Jordan's interview. These three questions were selected for discussion because they provided clearly observable characteristics of critical mathematical thinking. Jordan's responses are mapped against the Key CMT characteristics (Table 2).

Question 1

Question 1 required Jordan to find the middle of wall to hang a picture frame. The manipulatives given to Jordan included an A3 sheet of paper and a small laminated picture frame. No additional resources were provided.

Table 3
Jordan's Responses to the Question 1 of the Semi-Structured One-On-One Assessment Instrument

CMT Question Instructions	Jordan's response	Jordan's work Sample for Question 1
This is a framed photograph of my son Joey. (Hold up real framed photograph.) I have a blank wall at home and I would like to hang this photograph in the middle of that wall. Let's imagine this A3 piece of paper (hold up A3 paper) is the blank	"Are you going to give me a ruler? You can't fold a wall so you can't fold this paper. I will draw a line (diagonally) here and another line (diagonally) here and just to prove it to you I will draw another line this way (horizontally) and another line this way (vertically), that is the middle (pointed to where the lines intersect)".	

wall and here is a smaller picture frame (hold up small picture frame).
How can you find the exact place to hang up Joey's photograph?
Key CMT characteristics:
Uses mathematical and other understandings to generate, evaluate, connect and create new ideas
Identifies and performs many ways to solve mathematical problems
Provides reasons or judgments
Uses mathematical strategies to prove the answer is possible
Self-evaluates, using mathematical evidence and reasoning
Identifies and performs many ways to solve mathematical problems

After some probing from the researcher, including directions to use the A3 sheet of paper to draw or write on, Jordan was able to produce a mathematical strategy for question 1 by drawing 4 lines that intersect in the centre of the page. The alignment of Jordan's response with the CMT capabilities are as follows:

- Estimating: Determining where the midpoint of the paper was to begin to draw a line. Using informal measurement to draw cross sections
- Grasping principles/noting relationships: Using known understanding and skills of drawing lines, half, in a new context of 'centre'
- Offering opinions: Questioning if a wall can be folded or if a rule was to be provided
- Reasoning: Proving that where all lines intersected was the centre

Question 2

This task required Jordan to consider the associative property of addition. The example provided included one-digit numbers. The number sentences, 3 + 3 and 4 + 2, were typed on a card for Jordan to view.

Table 4
Jordan's Responses to the Question 2 of the Semi-Structured One-On-One Assessment Instrument

CMT Question Instructions	Jordan's response
Why is 3 + 3 the same as 4 + 2. Change the numbers to 2 digit numbers if appropriate.	"If you put 4 + 2, then you take one away and you put it with a 3, it is even 3+ 3=6"
Can you tell me why 3 + 3 is the same as 4 + 2? Why/why not? Provide two reasons why they are equal.	"They both equal 6"
Can you tell me another way you can work this out?	

The response provided by Jordan for question 2 involved the compensation strategy – subtracting from one digit and adding to another digit. Jordan displayed CMT capabilities in line with those expected of a Year 3 or 4 child, according to the Australian Curriculum: mathematics content descriptors (ACARA, 2016). Alignment of CMT capabilities displays Jordan's understandings by:

- Classifying: Arranging numbers to allow for compensation strategy discussion
- Grasping principles/noting relationships: Understandings the value of place value and applying it to a new situation
- Offering opinions with reason: Articulating that if you take one digit away and place it elsewhere it still has value

Question 3

Jordan was asked to consider a way to identify the number of tiles required to cover a floor surface. The researcher provided a photograph of a cubby house for Jordan to view and tiles that measured 10 cm x 10 cm.

Table 5
Jordan's Responses to the Question 3 of the Semi-Structured One-On-One Assessment Instrument

CMT Question Instructions	Jordan's response
I have just finished building a cubby house for my children at home (show picture of the cubby house). I would like to put these tiles down on the floor of the cubby house (show rectangular tile). How can I work out how many tiles I need?	"Well you know how big your tile is so you can see how many fit across and then you can buy one row at a time to save money".

Jordan displayed the capacity to make reference to the tile being used as a repetitive unit of measurement, to determine the number of required tiles. His reference to money provides an insight into Jordan's understanding of finances and quantity. There is also an initial reference to the concept of area with his discussion on rows by saying, "how many fit across". The CMT capabilities displayed in Jordan's responses included:

- Assuming: Considering that many will fit in a row and that there will be many rows and using the tile as an informal unit of measure
- Grasping Principles: Showing understanding of area and ways to find out the area in a real-life context
- Offering opinions with reasons: The inclusion of money provides a real-life situation

Conclusion

This paper has addressed the research question: What are some characteristics of young children's critical mathematical thinking? In the case presented here, Jordan's responses demonstrated evidence of CMT characteristics such as reasoning, noting relationships and classifying. These observed characteristics of CMT are in agreement with previous research findings that young children are capable of CMT (Bobis et al., 2005; Doig & Ompok, 2010). This study takes a small step to extending previous knowledge in the area of children's critical mathematical thinking by providing a framework that shows the potential to act as the basis for developing tools for determining students' CMT capability.

This research is an initial step in advancing understanding of young children's higher order thinking in mathematics. Thus, the study, has implications for teachers and policy makers who may review the way in which mathematical learning and assessment is designed for young children. While this study provides evidence of how a child's responses can display critical mathematical thinking and a range of mathematical strategies, no attempt is made to generalise as the purpose of the paper is to seek to understand. These initial findings, however, will inform the larger study from which the data used here was sourced and provide further scope for investigating how CMT can be assessed in young children.

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It's More Than the Videos: Examining the Factors That Impact Upon Students' Uptake of the Flipped Classroom Approach in a Senior Secondary Mathematics Classroom

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Senior secondary mathematics in Australia is generally characterised by a challenging prescribed curriculum, textbook usage, high homework expectations and externally imposed assessment tasks. An increasing number of senior secondary mathematics teachers are incorporating a flipped classroom approach into their teaching as a means of addressing this challenging teaching space. This paper looks at a case study undertaken with a Grade 12 class where the teacher utilised a flipped classroom approach to teach Mathematics 2. The results showed that it was the holistic approach and commitment and dedication of the teacher that primarily influenced students' uptake. The study has implications for other teachers who may be considering implementing a flipped classroom approach, particularly in terms of the commitment required.

With its focus on high stakes assessment and mandated syllabus, teaching senior secondary mathematics is a challenging task. In order to meet the demands of a crowded syllabus, some senior secondary mathematics teachers have adopted a flipped classroom approach in their mathematics classrooms. This approach, which has been credited to Bergman and Sams (2012), typically involves the recording and narration of video tutorials, which replace 'traditional' homework practices and frees up class time for more focused teaching, rather than direct instruction. Advocates of the approach report that it allows for differentiated teaching for a range of student abilities, increased student motivation and autonomy and increased student-teacher interaction (e.g., Abeysekera & Dawson, 2015; Bergman & Sams, 2012; Muir, 2016). Despite maintaining that "it's not about the videos" (Bergman & Sams, 2012, p. 95), but rather the increased class time the videos can facilitate, much of the reported research in the area has focused on the affordances of the approach (e.g., Straw, Quinlan, Harland, & Walker, 2015) and often in the context of tertiary settings. Some authors have argued that it's more about good teaching practice that incorporates constructivist principles (e.g., Strayer, 2012), leading one to question as to whether or not it's just 'good teaching'. The authors' previous work has examined the affordances of the approach in terms of creating conditions that motivate students to engage with the approach. In common with Abeysekera and Dawson's (2015) motivation factors, Muir (2016) found that a sense of relatedness with the teacher was a strong motivating factor for students to engage with the approach. This paper specifically examines the role of the teacher in implementing a flipped classroom and the impact this has on motivating students to engage with the approach. Classroom observations, teacher and student interviews were conducted in order to answer the following research questions: How is the flipped classroom experienced in a senior secondary mathematics class? What impact does the teacher have on students' uptake of the approach?

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Review of the Literature

The terms flipped classroom and flipped learning are not interchangeable, and according to the Flipped Learning Network (FLN, 2014), flipped learning only truly occurs when 'four pillars' are applied in practice. These four pillars are flexible environments, learning culture, intentional content and professional educator. Of particular relevance to this paper is the role of the professional educator, where the instructor is described as an active observer who offers timely and relevant feedback and assessment, connectedness, reflection and revision, and who intentionally designs content to promote critical and higher-order thinking (FLN, 2014).

In a recent synthesis of research into mathematics flipped classrooms, Lo, Hew, and Chen (2017) examined classroom studies in which pre-class instructional videos were provided prior to face-to-face meetings. They examined the types of out-of-class and in-class instructional activities used, the effect of flipped learning on student achievement, the participant perceptions of flipped classroom benefits and the main challenges of flipped classroom implementations. Along with highlighting the limited research undertaken in secondary school settings and Australia, their synthesis showed that the top three most frequently reported benefits of flipped learning were instructor feedback, peer-assisted learning and more in-class time to apply concepts during activities (Lo et al., 2017). On-demand accessibility of video lectures and preparing students for class were also reported as positive benefits, along with the use of differentiated instructional activities. The two major challenges reported with implementing the approach were the students' unfamiliarity with flipped learning and the instructors' significant start-up effort. Other studies have reported similar findings (e.g., Muir & Geiger, 2015; Straw, et al., 2015), with student reports indicating that in contrast with traditional practices experienced in the past, the flipped classroom approach provided them with an increased level of satisfaction with the relevancy of materials provided and greater engagement with, and autonomy over their learning (Muir, 2016). Affordances such as self-paced learning (Goodwin & Miller, 2013; Muir, 2016), improved student-teacher interaction (Goodwin & Miller, 2013) and accessibility (Muir, 2016; Straw, et al., 2015) were also reported as positively influencing students' motivation to engage with the approach.

Theoretical Framework

Engagement, Motivation and Individual Needs

Engagement is a multi-faceted concept that is typically described as including behavioural, emotional and cognitive aspects (Fredericks, Blumenfeld, & Paris, 2004). Behavioural engagement concerns involvement in learning and academic tasks, while cognitive engagement involves behaviours such as being strategic or self-regulating, and use of learning strategies such as rehearsal, summarising, and elaboration to remember, organise, and understand the material (Fredericks, et al., 2004; Corno & Mandinach, 1983). Emotional engagement refers to students' affective responses and includes feelings and attitudes such as interest and anxiety (Fredericks, et al. 2004). Reeve (2013) identified a fourth dimension of engagement: agentic engagement, which involves students' self-learning with the teacher providing instructional support. Teacher support has been shown to influence behavioural, emotional and cognitive engagement (Fredericks, et al., 2004), with teacher involvement being positively associated with engagement, and that in turn, higher student engagement can result in great teacher involvement (Skinner & Belmont, 1993). Student engagement has

also been shown to increase when teachers support autonomy and cater for students' needs for competence and relatedness, which is more likely in classrooms where teachers and peers create a supportive environment (Fredericks, et al., 2004). The three basic needs of competence, autonomy and relatedness form the basis of self-determination theory (Ryan & Deci, 2000), and have been shown to be catered for in the context of a flipped classroom approach (Abeysekera & Dawson, 2015).

Methodology

The study used a mixed-methods approach (Creswell, 2003) to investigate the enactment of a flipped classroom in a Grade 12 senior secondary class, taught by Mr Simmons (pseudonyms used for school, teacher and students throughout). Data were gathered from a student online survey, classroom observations and teacher and student focus group interviews. The online survey consisted of 42 Likert scale items that required students to indicate levels of agreement and 10 open-ended questions. Likert scale items included general statements about accessing online resources (e.g., 'I have used online tutorials/videos not prepared by my teacher to help me with my mathematics this year'), pragmatic items about usage (e.g., 'I usually watch all the tutorials/videos from beginning to end'), and motivational aspects (e.g., 'I would not watch the tutorials/videos if my teacher had not prepared them'). Twenty six of the 27 students in the class completed the online survey approximately two weeks prior to the classroom observations to help inform the focus group interviews. A total of 20 students participated in eight focus group interviews which were semi-structured in nature, of 20-30 minutes duration, audio-taped and fully transcribed. An audio-recorded semi-structured interview was also conducted with Mr Simmons following the classroom observations. Quantitative data from the Likert scale items were analysed using descriptive statistics to report on percentage level of agreement with the statements. Open-ended survey responses and interview transcripts were analysed using reflexive iteration (Srivastava, 2009), whereby each sentence was initially open-coded and common themes identified. These themes included, but were not restricted to, affordances of the approach as identified in the literature, and evidence of meeting students' needs of autonomy, relatedness and competence.

Context

The case study school 'Barton Anglican School' is a co-educational independent K-12 Anglican School with a student population of 1600. Mr Simmons had taught at the school for approximately 10 years, and this was his second year of both flipping his mathematics class and teaching this cohort of students. He was currently teaching Year 12 mathematics 2 unit, which was a Higher School Certificate (HSC) subject and included topics such as plane geometry, integration, differential calculus and probability (NSW Board of Studies, 2014). The lessons observed were from Topic 9, 'Applications of Series' and involved calculating compound interests and superannuation tables.

For each topic Mr Simmons would prepare a 'roadmap' (see Figure 1) which included notes for each week, readings, videos and individual student directions and was accessed through the school's intranet system. Students could work individually through the roadmap for each topic. Typically, each week would require students to watch one or more videos, complete exercises from the textbook and complete quizzes or practice exam questions. The video tutorials would be watched before class, with class time spent individually working through exercises. During class lessons Mr Simmons would individually work with students,

occasionally instruct whole class on specific problems or applications and often tutor small groups who were at a similar level or having common difficulties.

Date	Work covered	Homework	Homework problems
1 May	Devices and headphones in class tomorrow HY practice paper solutions distributed. Mark them, marks to tr. Venn diagrams pp 9-10	Cambridge (12) Ex 8D Q 1,4,7,10,13 Mark 2010 HY paper Video on population estimation (capture-recapture) p. 11 (5 mins)	Quentin Q 12 Callum Q 12, Q6C Liam Q 12 Q 12 Travis Q 12
3 May	p. 17 example link Parabolas follow up: Jim, Gavin	Notes p18 Q2, 4-12 Notes p19 Q1-4, 8, 9. Videos pp20-22 Multi-stage events and product rule. Page 22 is heavy going. Approach with a fresh mind and you may need to watch it more than once. (total 34 minutes)	Michael p 19 Q 4 Travis p 19 Q4

Figure 1. 'Roadmap'.

Results and Discussion

Experiences of the Flipped Classroom

In order to understand how the flipped classroom was implemented and experienced by the participants in this study, data relevant to the enactment of the approach in practice was extracted from observations, survey and interviews. In the lessons observed, students came to class having watched the allocated video tutorials, usually the night before, and worked through the lesson following a plan projected on the whiteboard. Leo summed up a typical lesson as follows:

> … what we normally do is he has an overhead where he shows us a couple of pages of the topic and he'll kind of work through a couple of examples with us and then do a proof, talk through the formulas and how it all works so we get a greater understanding of it and then after that we go through and then work on some questions on our own to see if we can do it by ourselves and then just ask him questions as he kind of goes around the classroom and just making sure everyone's down pat with [everything].

Classroom observations showed that it was evident that students had come to class prepared, having watched the video prior and were able to start work immediately. There was a classroom culture of expectation that students would watch the video, with class work dependent upon this:

> … a lot of it is very dependent on watching the videos, because if 25 of us watch the video and there's like three kids who don't he's not going to spend a whole other 25 minutes explaining it to those kids … when he's already produced [the video] [Quentin]

Mr Simmons explained that in contrast to the past, more class time was devoted to helping individuals and that the videos also allowed for greater focus and less repetition:

> I'm spending a lot less time delivering content in class and I'd like to think with that time I've gained I'm able to spend that in hopefully a quality way in helping students. Like this morning for example the help was mainly individual but there was a little bit of group stuff, particularly the group at the front who were all stuck on the same thing and I thought it was efficient to do it all together.

> [students are] able to go back and relearn content, with it not always being up to me. So if a student says I've forgotten how to do quadratic equations by substitution I can say "there's a video for that, go back and watch it."

Although it was not possible to observe students' interaction with the videos at home, the interviews provided an insight into their use. All students interviewed indicated that they set aside time to watch the videos, away from distractions, and that it was not a passive exercise. The following quote from David was typical of the responses received:

> I do it at my desk in my room, with the worksheet in front of me, copying it down and you've usually got to rewind it a couple of times and think of what is assumed knowledge and revise a little bit on that … and if I had difficulty understanding it I'd write that down and either email Mr Simmons or ask him the next day or I'd try to do questions and then go back to the video to see if I did it right or not.

In the survey, 93% of students agreed or strongly agreed that they usually watched the videos from beginning to end, with 92% of them agreeing that they were about the right length, despite some being up to 35 minutes long.

Motivations for Engaging with the Approach

Data from the survey and interviews indicated that students were able to identify a number of affordances with the approach which influenced their motivation to watch the videos, come to class prepared and participate in class. While students certainly believed that the approach assisted them academically and cognitively, it seemed they were particularly motivated to watch the videos and engage in class because of their teacher. Over two years, Mr Simmons had developed a strong relationship with this cohort of students, making a *sense of relatedness* particularly influential on their motivation to engage. The survey results showed 100% agreement with the statement, 'I relate well to my teacher', 96% of students agreed or strongly agreed that they were 'motivated to watch the videos because my teacher prepared them' and 89% agreed that Mr Simmons 'enjoys making the videos'. Interview data showed that there were nine direct references to Mr Simmons' dedication, including statements such as "I don't know a teacher who is as dedicated to their class or each of their classes like Mr Simmons" [Grant] and "I'm sure the other guys here can say the same, but I don't think I've known a teacher who puts in as much work with his students than Mr Simmons has" [Chris]. The students genuinely expressed an appreciation for the work involved with the approach, with at least 10 students emphasising this, as illustrated by the following interview responses:

> It would be unfair not to show respect for him when I know that [he] is putting in that work because he wants me to do well … I want to put that work back in [Leo]

> He's always one of those teachers that you want to well for just because he puts so much time and effort into you and he wants you to succeed [Michael]

Mr Simmons was quite modest about his role and emphasised that it was "… not about me. It's about the content and trying to explain the content as best I can". He did not include an image of himself in his video tutorials or attempt to make them entertaining or humorous. He was, however, meticulous about the quality of the videos in terms of being relevant,

accurate and helpful. For example, he cited a recent example which required him to re-record a video because he was not happy with it:

> Sometimes I realise I've made mistakes .... I did one recently where I recorded a whole video one night at school and it was about [how] to use geometric series to simplify a recurring decimal. So if you've got 0.232323 how you'd write that as a sum which you can then use series techniques to write as a fraction. I did the whole thing - it was a 15 minute video - got home and I thought I've done it wrong. That was not the best way to explain it. So I came back in the next day and redid it.

In the survey and interviews, students were specifically asked about their use of other online videos and tutorials and whether or not they thought it was important that the class teacher prepared the videos. In response to the statement 'I would not watch the tutorials/videos if my teacher had not prepared them', only 40% of students agreed and 33% were undecided. Open-ended survey responses indicated that several students had accessed online videos through Khan Academy with the general consensus being that while some maybe useful, the material prepared by their teacher was more specific and relevant to their learning, and they had the opportunity to follow up in class the following day. Further probing in the interviews revealed that while students were open to other teachers preparing the videos, "they'd need to be of the same standard as him … I'm not sure if there is someone like that …" [Grant] and "… it kind of builds a relationship between teacher and student being able to have the video there, but also still being in class with him. I guess that's important because if it was someone else making the video, there'd kind of be a disconnect" [Mitch].

Students also reported that Mr Simmons' flipped classroom approach helped developed their *sense of autonomy*. The autonomy to go at one's own pace in terms of "not having to wait for others" [Michael] and "allowing you to take your time with stuff that you may need to focus more on rather than having to get through it at the same pace as everyone else" [Grant] was referred to frequently in the interviews. The pause and rewind affordance of the videos also promoted autonomy in that:

> I can pause at points if I don't get something. I can rewind – I can do basically whatever I want with the video in terms of manipulating the order of instructions … skipping bits, going back … that you can't really do in a classroom if the teacher's teaching you … I think it gives you a lot more independence, a lot more freedom to actually learn at your own pace and get the stuff that you may feel less confident with [Grant]

Grant's comment also shows the link between the *sense of autonomy* and *sense of competence*. As previously mentioned, both the teacher and the students felt that it was important that the watching of the videos were not passive, but involved interaction in terms of note-taking, working through examples and recording questions for later follow up. Directions were often given which recommended that students pause the tutorials at certain points to work through particular examples:

> There's heaps of occasions in the videos where Mr Simmons will say pause here if you want to work this out yourself and you can do things yourself based on how you might think – it gives you the tools to work things out yourself … you definitely have to interact and work with it [Michael]

Together with recognising the time and effort Mr Simmons put into the videos, students also recognised that the content of the videos was designed to help them learn and increase their competence. Ben, for example, stated that "There's never been one where I've gone in and watched a video and gone wow, that was disappointing or I don't understand – I come out of every single video saying OK, I know what I need to do tomorrow in class".

Mr Simmons not only prepared content that was relevant and specific to their course, he also carefully planned and structured the order in which content was presented. Quinn, for

example, stated that "he plans it all out so that you watch the video at the right point in the course, and he makes them when he thinks we're going to need them"; another strategy which catered for students' *sense of competence*.

The flipped classroom approach has been criticised for its seemingly strong emphasis on procedures, however the students in this class were confident that understanding and application was also emphasised. Grant, for example, stated that:

> It's a bit procedural, but you do end up getting a good idea of how the actual theory works because at the start he explains it to you … he'll usually spend time explaining the theory behind using xy, ab, … in the actual equation. Then after he's explained all that and how to do in that way, then we'll move onto a couple of different examples to show how you can apply it to different problems.

The survey results also indicated students thought understanding was important. There was 100% agreement with the statements: 'The tutorials/videos helped me to understand a concept' and 'My teacher wants us to understand the work, not just memorise it'.

Overall the teacher and students were advocates of the approach, particularly in comparison with more traditional approaches experienced in the past. In the survey, 93% of students strongly agreed or agreed that they 'Prefer to learn mathematics using this approach', with 100% agreement that they would recommend the approach to others. Students cited the video tutorials as being both complementary but also preferable to the text book in that "it makes it easier to understand when the content is difficult, as it is being explained by someone who understands the content already" and "the videos created by my teacher sometimes include better methods of solving problems that aren't included in the textbook" (open-ended survey responses). While they did not see the videos as replacing their teacher, they cited advantages such as having a resource readily available and revisiting concepts, as affordances conducive to their learning.

Conclusions

The results showed that the enactment of a flipped classroom approach in this study consisted of a carefully designed program where students were provided with clear directions, relevant resources, and dedicated teacher support. While acknowledging that the students in the study elected the subject and wanted to achieve good results, the approach adopted by Mr Simmons also facilitated their motivation and engagement. This required a considerable time and dedication. The students consistently reported that they had developed a strong *sense of relatedness* with Mr Simmons, both in terms of their relationship with him and his ability to provide them with content and resources that were relevant and accessible. Consistent with previous findings (e.g., Muir, 2016), many of these students reported that they were motivated to watch the videos because of all the time and effort put in by their teacher, thus feeling a sense of obligation to 'reward him for his efforts'. References made by students to relevance and understanding also provided evidence that they were cognitively engaged (Fredericks, et al., 2004) with the materials, demonstrated behaviours such as being strategic or self-regulating, and used learning strategies such as rehearsal, summarising, and elaboration to remember, organise, and understand the material (Fredericks, et al., 2004; Corno & Mandinach, 1983). The autonomous nature of the approach also facilitated cognitive engagement and agentic engagement (Reeve, 2013) as students were involved in self-learning, with teacher instructional support.

Senior secondary mathematics is a challenging domain where teachers are required to teach complex mathematical topics, follow a set curriculum and prepare students for externally imposed high stakes assessment tasks. The flipped classroom approach enacted in this study has demonstrated how a teacher can adapt traditional classroom instruction to

maximise opportunities for students to be in control of their learning. This has practical implications for other teachers in similar contexts who may consider adopting such an approach, with the understanding that it can require a considerable commitment of time on behalf of the teacher, yet it seems the benefits are worth the investment. The paper adds to the limited research on flipped mathematics classrooms in secondary settings through its focus on the role of the teacher, rather than the pragmatics of video production, and through listening to the students' voices in identifying the factors which motivated them to engage with the approach.

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Determinants of Success in Learning Mathematics: A Study of Post-Secondary Students in New Zealand

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In this paper, I report on doctoral research in which I investigated the relationships between student approaches to learning, conceptions of mathematics, mathematical self-efficacy, personal factors and examination results. Using seventy-three post-secondary mathematics students, some pertinent findings were: self-belief in selection processes predicted high examination results; deep approaches and cohesive conceptions correlated positively with examination results; participants with low prior mathematics performed better than individuals with advanced secondary qualifications. These findings could pose practical teaching implications in mathematics education.

Introduction

My study examined the relationships between students' mathematical self-efficacy (MSE), student approaches to learning (SAL), student conceptions of mathematics (SCM), personal factors (age, prior mathematics) and mathematics examination results. This study is important as it serves to highlight personal, affective as well as cognitive determinants that influence mathematical performances of post-secondary mathematics students. It also serves as a conduit for mathematics education researchers and post-secondary mathematics educators to conceptualise student learning and enhance teaching programmes. The key constructs of this study were prior mathematics, age, student approaches to learning, conceptions of mathematics, mathematical self-efficacy, and mathematical performances.

Personal Factors: Prior Mathematics and Age

Prior knowledge is defined as the highest mathematics qualification gained at school. A New Zealand report by Engler (2010) argued that gaining mastery of the skills taught in secondary mathematics could improve advancement in tertiary education. In order to achieve higher levels of university performance, students should achieve a level of understanding that leads to proficiency in the use of those skills and knowledge. As expected, attaining the highest mathematics secondary education (Year 13) is advantageous for future success in tertiary education (Henderson & Broadbridge, 2009).

In particular, young people (15-24 years old) were targeted by the New Zealand Tertiary Education Commission (2013) as a priority group for increasing Science, Technology, Engineering and Mathematics (STEM)-related qualifications. As such, raising mathematics performances of young people could contribute to STEM-related careers in NZ. An empirical study of tertiary students by Carmichael and Taylor (2005) found that while there was no significant difference in mathematical performances of traditional students (18-25 years old), non-traditional students (over 25 years old) could perform better than the younger counterparts due to greater self-efficacy levels. Furthermore, a study by Miller-Reilly (2006) showed that academic support helped non-traditional learners to develop greater confidence in learning mathematics and improved their grades. Non-traditional students tended to have

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a better academic preparation in foundation studies (Liston & O'Donoghue, 2010), develop a sense of confidence and enjoyment in learning (Carmichael & Taylor, 2005; Miller-Reilly, 2006) and accept challenges in learning (Forgasz & Leder, 2000). Therefore, non-traditional students could perform better in their studies since they were more mature and committed to learning.

Cognitive Factor: Student Approaches to Learning

SAL, originally coined by Marton and Säljö (1976), refer to co-existence of intention and process of learning. A deep approach involves the motive of intrinsic interest and strategy to maximise meaning, whereas a surface approach to learning is driven by one's fear of failure and a process of rote-learning. An achieving approach or organised effort which overlaps with a deep approach, is driven by one's need for obtaining good grades and how one makes use of space and time to achieve a task. These learning approaches affect the quality of learning outcomes.

Affective Factor: Student Conceptions of Mathematics

Student conception of mathematics, as a form of belief within the affective system, can be described as stable traits and mathematical world views (DeBellis & Goldin, 2006). An international study of 1182 mathematics students by Wood et al. (2012) reported that students at SCM level 1 perceived mathematics to be about numbers and components (53%); at SCM level 2, mathematics is considered to be about modelling and abstraction (34%); and at SCM level 3, mathematics is perceived to be relevant to life (6%). The SCM level 1 as a study of numbers, components or techniques that could be used to solve problems overlaps with 'fragmented conception' of Maths as a set of numbers, rules and formulae which could be applied to solve problems (Crawford, Gordon, Nicholas, & Prosser, 1994). In contrast, cohesive conception, whereby mathematics is a complex logical system which could be used to solve complex problems and provides insights used for understanding the world, was identical in meaning to how mathematical modelling is used to solve real life problems (SCM level 2) and mathematics is applicable in people's lives (SCM level 3). These high level cohesive conceptions tended to be formed by mathematicians, who were familiar with mathematical applications. Moreover, Crawford et al. (1994) reported that fragmented conception was related to a surface approach and unsuccessful outcomes whereas cohesive conception of mathematics corresponded with a deep approach and positive outcomes.

Affective Factor: Mathematical Self-Efficacy

Another construct of this investigation is mathematical self-efficacy. According to Bandura (1997), mathematical self-efficacy is a personal judgement of one's ability to do mathematics. Self-efficacy produces learning outcomes through major processes known as cognitive, motivational and selection processes. Firstly, cognitive processes are described as thinking processes which involves the acquisition, organization and use of information. As a function of self-appraisal of capabilities, goal setting resides in forethought which translates into purposive actions. People with high self-efficacy mediate through cognitive processes by visualising success, which in turn provides cognitive support and guides for attainment. The stronger the self-efficacy, the higher the goals individuals set themselves to attain performances. Secondly, self-efficacy plays a key role in self-regulating motivation. Motivational processes include causal attributions, outcome expectancy, and cognized goals, corresponding with the attribution theory, expectancy-value theory and goal theory. In causal

attribution, people with high self-efficacy will attribute poor outcomes to lack of effort whereas those with low self-efficacy attribute failure to low ability. Further, expectancy theory states that people expect their behaviour and actions to bring about valued outcomes. People with high self-efficacy are more likely to persevere and attain successful outcomes. Also, goal setting is governed by the cognitive processes of motivation. Those with strong self-efficacy will endeavour to reach their goals through effort and persistence. Thirdly, driven by selection processes, people are partly the product of their environment because they choose the social and physical environment and types of activities that they judge themselves to be capable of handling. In theory, these metacognitive processes determine self-efficacy and indirectly affects the outcomes of learning.

In metacognitive terms pertaining to self-efficacy, self-belief for self-regulated learning promotes both skill mastery and learning strategies. According to Bandura (1997), self-regulation entails skills and strategies for planning and organizing instructional activities, utilising resources, adjusting one's own motivation and using metacognitive skills to evaluate the adequacy of one's strategies and knowledge. Students who have strong belief in using self-regulation strategies tend to have better mastery of mathematics skills and performances because they develop learning strategies such as, orienting oneself before an assignment, collecting relevant resources, integrating ideas and monitoring progress in learning. As such, these strategies would enable individuals to steer their learning processes, to self-regulate their motivation for learning and amount of effort.

Many researchers have shown that self-efficacy predicts success in mathematics performance (Pajares & Kranzler, 1995; Pajares & Miller, 1994; Skaalvik & Skaavik, 2011).A study of middle and high school mathematics students has found that self-efficacy was a better predictor of mathematics achievement than prior achievement (Skaalvik & Skaavik, 2011). This result was also evident for higher education students of calculus in a study by Hall and Ponton (2005) who found that university calculus students who reported high self-efficacy gained better results than other remedial students who also had low prior experience and/or achievement. Pajares and Kranzler (1995) concluded that students had high self-efficacy because they exhibited more effort and perseverance in challenging problem-solving situations. Although these findings were reported in different educational settings, these studies serve to conceptualise the role of self-efficacy in learning mathematics.

Mathematical Performances

Mathematical performances are measured by mathematics examination scores. The mathematics examination is an appropriate product of learning given that summative assessments fulfil a broad range of learning, ranging from mathematical calculations and comprehension to applications of knowledge in the course learning outcomes. If a student attains 50% marks and above in an examination, it indicates success in the course.

To date, there has been insufficient research in post-secondary education, which relates the constructs of SCM, MSE and SAL to mathematics performances. In order to advance research in student learning within the fields of psychology of learning mathematics and affect, my research questions are as follows:

Q1. What is the relationship between mathematics self-efficacies in five areas (problem-solving, cognitive, motivational, selection processes, and self-regulated learning), deep approaches to learning/organised effort/surface approaches to learning, as well as conceptions levels 1,2 3 in relations to examination results?

Q2. Which factor(s) is/are the most salient predictor(s) of mathematics performances?

Q3.To what extent do age differences, course type and highest level of secondary mathematics determine success in learning mathematics?

Method

Seventy-three (37% of cohort) mathematics students in a New Zealand tertiary institution participated in this study. The sample consisted of males (80%, N=58) and females (20%, N=15). Their ages were 18-25 years old (73%, N=53) and over 25 years old (27%, N=20). Given that some data were missing, at secondary levels, the majority had achieved National Certificate of Educational Achievement (NCEA) Level 3 (30%, N=22) or an overseas qualifications (29%, N=21). Some had completed NCEA Mathematics Level 1 (8%, N=6), NCEA Mathematics Level 2 (15%, N=11) and Mathematics at Cambridge and International Baccalaureate (IB) levels (7%, N=5). The engineering participants were enrolled in Pre-Degree Engineering Mathematics (N=5), Engineering Mathematics 1 at diploma level (N=47) and Engineering Mathematics 2 at degree level (N=6). The business participants were enrolled in Programming Precepts (N=7) and Business Statistical Analysis (N=8).

Using five-point Likert style questionnaires (Likert, 1931), the Refined Self-efficacy Scale (Marat, 2005), Conceptions of Mathematics Form (Wood, Petocz, & Reid, 2012) and the Shortened Experiences of Teaching and Learning Questionnaire (Hounsell et al., 2005) were distributed in March-May. The mathematics examination results were recorded in July. After data collection, the IBM SPSS 22 statistical software was used to carry out correlational studies, multivariate regression and general linear model analyses.

Findings

Q1. Relationships Between Sub-Constructs of MSE, SAL, SCM and Results

Using the categorisations of the strength of correlations (i.e., strong correlations range from R =.7 to .9, moderate to be .4 to .6, weak as ranging from .1 to .3 (Dancey & Reidy, 2004), moderate correlations were found between examination results and self-efficacy in problem-solving and self-efficacy in using motivational, cognitive, selection strategies (see Table 1); deep approaches and organised effort. Weak and positive correlations (two-tail significance) were reported between results and self-belief for self-regulated learning; results and deep approaches; results and Level 2 SCM (see Table 2). The highest mean scores were ‘Self-belief in using motivation strategies’ (3.66), ‘Deep Approaches to Learning’ (3.96) and ‘Level 1 SCM’ (3.98) and ‘Level 2 SCM’ (3.96).

Table 1
Mean, Standard Deviation and Correlations of MSE and Examination Results (N=73)

	1	2	3	4	5	6
1.Examination results	1					
2.Self-efficacy in solving mathematical problems	.43**	1				
3. Self-belief in using motivation processes	.41**	.65**	1			

4. Self-belief in using cognitive processes	.48**	.77**	.80**	1		
5. Self-belief in using selection processes	.52**	.62**	.73**	.83**	1	
6.Self-belief for self-regulated learning	.39**	.71**	.72**	.86**	.87**	1
Mean	51.08	3.38	3.66	3.38	3.46	3.48
Standard deviation	23.24	0.59	0.69	0.65	0.77	0.67

Table 2
Mean, Standard Deviation and Correlations of SCM, SAL and Examination Results (N=73)

	1	7	8	9	10	11	12
1.Examination results	1						
7. SCM Level 1	0.08	1					
8. SCM Level 2	.23*	.69**	1				
9. SCM Level 3	0.14	.33**	.48**	1			
10.A deep approach	.27*	0.04	0.21	0.23	1		
11.A surface approach	-0.11	-0.16	-.27*	-0.03	-0.12	1	
12. An organised effort	0.16	-0.01	0.02	0.08	.63**	0.14	1
Mean	51.08	3.98	3.96	3.44	3.96	3.16	3.77
Standard deviation	23.24	0.87	0.80	0.87	0.64	0.91	0.91

Notes. **p< 0.01. * p<0.05.

Q2. Factors Predicting Performance

Considering all the predictors (See Table 3), the most significant predictor was self-belief in selection processes given that the regression assumptions were not violated (Hair, Black, Babin, Anderson, & Tatham, 2006). The F ratio of the model mean square to error mean square was 4.702 (df=7, Sign=0.000).The model (Beta=0.589, t=2.413, p=0.000) accounts for 34.7% (R square) of the variation of results.

Table 3
ANOVA[a]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	13345.505	7	1906.501	4.702	.000[b]
	Residual	25141.434	62	405.507		
	Total	38486.939	69			

a. Dependent Variable: Mathematics examination results. b. Predictors: (Constant), Self-belief for self-regulated learning, SCM Level 2, A deep approach, Self-efficacy in solving mathematical problems, Self-belief in using motivation strategies, Self-belief in using selection processes, Self-belief in using cognitive processes.

Q.3 Personal Factors Predicting Performance

53% of the participants (N=73) passed the mathematics examination. A higher percentage (81%; N=24) of traditional students passed the examination compared to non-traditional students (75%; N=15). The average examination scores of those who attained NCEA levels 1, 2 and 3 were 63 marks, 44 marks and 51 marks respectively. Firstly, age differences (18-25 years old and over 25 years old) were not significant factors of examination results (F=2.632, p=.111). Secondly, the univariate variance of analyses showed significant effects (Sign < 0.05) of current mathematics course and mathematics background (Mardia, 1980). The univariate general linear model 2-way ANOVA table showed the F value (3.452) and low significance value (0.014). The estimated marginal means and significant (F=4.002, p=0.007) and pairwise comparisons revealed that participants who were studying Engineering Mathematics 2 (84 marks) and pre-degree Engineering Mathematics (74 marks), had completed mathematics at NCEA level 1 (65 marks), Cambridge and IB (65 marks) and overseas students (68 marks), were more likely to score higher examination marks than those with NCEA Level 2 (47 marks) and Level 3 (50 marks).

Discussion

The correlational analysis showed that examination results were positively associated with high scores in a cohesive conception of mathematics, a deep approach to learning and an organised effort. Firstly, inconsistent with previous literature (Crawford et. al, 1994; Wood et. al, 2012), the participants had high scores in a deep approach, an organised effort and SCM level 2. As mathematics was taught by engineering and business lecturers, their teaching outcomes entailed teaching mathematical procedures as well applications in engineering and business situations. In these courses, the participants were expected to adopt a mathematical belief that mathematics is about modelling mathematical concepts, a cohesive conception which would underpin a deep approach to learning. If the participants were familiar with surface learning, they would carry out procedural calculations and rely on memorisation of facts, which is driven by a fragmented belief about mathematics. Secondly, in line with previous research findings (Crawford et. al, 1994), the sample data revealed positive correlations between high scores in a deep approach and examination results; between high scores in SCM level 2 (or cohesive conceptions) and examinations results. However, the low pass rate in examination suggested that examination results did not reflect the participants' perceived importance of a deep approach and a cohesive conception.

Besides a deep approach to learning, the five domains of self-efficacy correlated positively with strong mathematical performances. Previous literature has also revealed the performance-enhancing role of self-efficacy (Hall & Ponton, 2005; Pajares & Kranzler, 1995; Pajares & Miller, 1994; Skaalvik & Skaavik, 2011). However, my regression data revealed that considering five self-efficacy domains, SAL and SCM sub-constructs, student beliefs in using selection processes were the best predictor of examination results. According to Bandura (1997), individuals develop self-efficacy through the optimal use of resources to accomplish certain tasks. By having strong beliefs about using selection strategies (e.g., time management, effort), individuals can adapt to the teaching and learning environment and are equipped with the necessary means for task completion by developing positive study strategies (e.g. time management, note taking; critical thinking). By gaining more control of one's learning, students develop effective use of self-regulation strategies (e.g. study

independently, concentration), which may enable them to make more effort and stimulate their intellectual curiosity in learning tasks.

Another potential determinant of success is prior mathematics background. The univariate data was inconsistent with other studies (Engler, 2010; Henderson & Broadbridge, 2009) as the participants with low mathematics background (NCEA Level 1 Mathematics) scored better than participants with NCEA levels 2 and 3 mathematics qualifications (equivalent to Years 12 and 13). This finding suggested that despite higher secondary qualifications, some participants were less prepared for post-secondary mathematics. Prior to studying a tertiary mathematics course, some participants had completed a refresher mathematics course in order to master basic mathematics skills. Therefore, this finding indicated that participants with the lowest level of secondary mathematics qualifications were likely to succeed in mathematics if they were given early interventions.

Contrary to my expectation, age differences were not a determinant of mathematical success. This result did not match past literature (Forgasz & Leder, 2000, Carmichael & Taylor, 2005; Miller-Reilly, 2006; Liston & O'Donoghue, 2010). This was because the non-traditional participants in the sample were under-represented. While some participants were determined to study mathematics in order to meet their career goals, other participants were easily susceptible to dropping out of the course, which, in turn, could lead to a dramatic reduction of course completion rate. Hence, given this inconsistency, further investigation was warranted to improve generalisability of future research.

Limitations

I found that self-reports of student learning using questionnaires could pose certain limitations even though it was advantageous to gather data efficiently in large lectures. Firstly, the scales could limit the participants to respond according to the fixed categories of key constructs, as established by the researcher. Secondly, reports about student beliefs and learning approaches could be limited by one's interpretation of the scale and rely on the individual to recall their experience in learning mathematics. Thirdly, at times, self-reports could be self-promoting as participants were keen to over-estimate their judgements of their own capabilities and of deep learning. In order to improve research trustworthiness, these limitations should be taken into account in future quantitative studies.

Conclusion

My research contributes to the field of mathematics education by advancing our understanding of some cognitive, affective, and personal factors that influence success in post-secondary mathematics education. In this study, some key determinants of success were self-beliefs in using selection processes, a deep approach to learning and a cohesive student conception of mathematics whereas age differences and prior mathematics were not. These findings pose further questions about the ways in which mathematics practitioners promote mathematical self-efficacy as well as deep learning strategies for enhancing mathematical achievement of post-secondary students.

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School location and socio-economic status and patterns of participation and achievement in Year 12 enabling mathematics

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There is national and international pressure for schools to increase student engagement and skills in more challenging mathematics, in particular for disadvantaged students. This study repurposes school level data to examine patterns of participation and achievement in advanced year 12 mathematics. It confirms that school socio-economic status (SES) is strongly tied to participation and achievement in these subjects, and that non-metropolitan schools tend to perform more poorly than metropolitan schools in these areas. However, it suggests that a non-metropolitan location mitigates against the apparent impact of SES, pointing the way for potentially fruitful lines of future inquiry.

In an increasingly technocentric world, it is argued that deep understanding of mathematics is a critical right for all, and there is a need to equip all citizens with increasingly sophisticated mathematical skills (Center for Curriculum Redesign, 2013). The rise of the Science, Technology, Education and Mathematics (STEM) movement has seen Mathematics positioned as the essential foundation underpinning all aspects of STEM and contributing to the betterment of society (Office of the Chief Scientist, 2013). There is a call to grow a workforce with strong STEM skills, particularly in mathematics and related fields such as data analysis, coding and engineering (Australian Industry Group [AIG], 2015; Office of the Chief Scientist, 2014). A necessary part of achieving this growth is addressing the under-representation of students from disadvantaged or rural backgrounds pursuing STEM career pathways. In response, Australia has set national goals to increase participation and achievement in the more challenging STEM subjects, including mathematics, with a particular focus on addressing equity issues (Education Council, 2015).

Unfortunately, as the global and national emphasis on mathematics has grown, achievement and engagement in secondary school mathematics education in Australia has waned. Since 2003, the Programme for International Student Assessment (PISA) reveals a downward trend in the mathematical literacy of Australian 15 year olds, both relative to other nations and in absolute terms (Thomson, De Bortoli, & Underwood, 2017). Moreover, McPhan, Morony, Pegg, Cooksey and Lynch (2008) point to a worrying trend of senior secondary students turning away from higher-level mathematics subjects, preferring instead to study the more basic mathematics courses.

At the same time, there is significant disparity in the mathematical performance of Australian students from families of different SES, and different geographic locations. PISA testing suggests that students from the highest SES quartile are on average three years ahead of students from the lowest SES quartile (Thomson, De Bortoli, & Underwood, 2017) and students from metropolitan areas outperform students from non-metropolitan areas by a year or more. Similar patterns are revealed through the National Assessment Programme's numeracy testing. (Australian Curriculum Assessment and Reporting Authority [ACARA], 2017).

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The present study explored patterns of inequity in the participation and achievement of Year 12 students in enabling mathematics in government secondary schools in Victoria, Australia. Enabling mathematics subjects are defined here as those that are explicitly identified as prerequisites from further study in tertiary mathematics or mathematics related fields. This study was part of a wider research programme repurposing data routinely collected from all government secondary schools offering the Victorian Certificate of Education (VCE) to measure the success of schools in various aspects of STEM education. This paper focuses on Year 12 enabling mathematics education and addresses three research questions:

1. What is the relationship between school socio-economic status (the status of families sending children to the school) and student participation and achievement?
2. What is the relationship between the location of a school (metropolitan or non-metropolitan) and student participation and achievement? and
3. Is there an interaction effect of socio-economic status combined with location on student participation and achievement?

Method

. This paper presents analyses of patterns of participation and achievement in Year 12 enabling mathematics subjects offered within the VCE. As part of their VCE, students are required to complete a Year 12 English subject, and at least three other Year 12 subjects (Victorian Curriculum and Assessment Authority [VCAA], 2017). Typically, students complete five or six year 12 subjects. Studying mathematics is not compulsory to earn a VCE. (VCAA, 2016). While many university courses recommend that students complete a mathematics subject at Year 12 level, only Mathematical Methods and Specialist Mathematics are listed explicitly by any Victorian tertiary institution as a pre-requisite for entry into any of their courses (Victorian Tertiary Admissions Centre [VTAC], 2016) so these were categorised as enabling mathematics subjects for this study. Mathematical Methods includes the study of calculus, probability and statistics. Specialist Mathematics is designed to be taken in conjunction with Mathematical Methods, extending its content to look at topics such as complex numbers, vectors, and statistical inference

Study Data

Location and demographic information, enrolment numbers and median study scores were obtained from the Victorian Department of Education and Training (DET), for every Victorian government secondary school offering a VCE program during 2014, 2015 and 2016 (N=286). Sampling across these three recent years mitigates against cohort effects while also producing contemporary baseline findings on which to base future comparisons.

Outcome Variables

Schools from different locations and serving communities of different socio-economic status were compared using three outcome variables: Subjects Provided, Enrolment Proportion, and Achievement Level.

Subjects Provided. The Subjects Provided variable tracks which of the Year 12 enabling mathematics subjects schools had students studying across the three years.

Enrolment Proportions. Enrolment Proportions for each enabling mathematics subject were calculated for each school providing that subject by dividing the number of enrolments in a particular mathematics subject by the total number of Year 12 enrolments and then averaging this result across the three years.

Achievement Levels. Achievement Levels were calculated for each subject in each school running that subject in all three years by averaging the median school Year 12 study scores from each of the three years. These study scores are standardised by the VCAA by ranking student performance in each subject and then allocating normalised student study scores according to rank, with a maximum of 50, a set mean of 30 and standard deviation of 7 (VCAA, 2017). Given this, it is legitimate to compare study scores from school to school and year to year.

Explanatory Variables

Two explanatory variables are considered in this study: Student Family Occupation and Education Index (SFOE) and School Location.

SFOE. SFOE is the DET measure of SES (DET, 2016). SFOE is calculated for each school by DET using both parental education levels and occupation categories as recorded in school enrolment details. The higher the SFOE, the lower the SES, and the greater the disadvantage of families at the school. In some analyses the SFOE is analysed in quartiles, with the first SFOE quartile including the higher SES schools and the fourth SFOE quartile the lower SES schools.

School Location. Schools were categorised as either metropolitan (N=164), if located in a local government area (LGA) within the Greater Melbourne area, or non-metropolitan (N=122), if located in a LGA in any other region in Victoria (Victorian Government, 2017). Consequently, schools classified as non-metropolitan include schools in regional cities as well as rural and remote locations.

Analysis

As this study used data from the entire population of interest, sampling error was not a risk and therefore calculations of statistical significance were not required. The focus was on the practical significance of the statistics only (Cohen, Manion, & Morrison, 2011).

Descriptive statistics were used to summarise patterns of participation and achievement in the enabling mathematics subjects across location and SES categories. The proportions of schools providing the two enabling mathematics subjects, and the means and ranges of enrolment proportions and achievement levels were compared by school location and SFOE quartile. The relationships between both enrolment proportions, achievement level, and SFOE were further investigated using Spearman's rho correlation coefficients. Coefficients were calculated for all schools, for metropolitan schools and for non-metropolitan schools respectively to examine differences in these relationships based on location.

Results

Year 12 Enabling Mathematics Subjects Provided

Tables 1 and 2 show that almost all schools, independent of location or SES, delivered Mathematical Methods during the three years. However, there is a significant difference in the proportion of schools providing Specialist Mathematics. Metropolitan schools and high

SES schools delivered Specialist Mathematics at a rate of 86% (141 out of 164 schools) and 94% (68 out of 72 schools) respectively. For non-metropolitan and low SES schools this rate dropped to 70% (86 out of 122 schools) and 66% (47 out of 71 schools) respectively.

Enrolment Proportions

Table 1 shows that the enrolment proportion for each of the VCE mathematics subjects varies with location. It shows that the enrolment proportion in Mathematical Methods is 0.051 in metropolitan schools compared to 0.037 in non-metropolitan schools. There is also a large difference in enrolment proportions in Specialist Mathematics, being 0.017 in metropolitan schools and 0.010 in non-metropolitan schools. Table 2 shows that the highest SES schools also have greater enrolment proportions compared to the lowest SES schools in Mathematical Methods (0.064 compared to 0.039 respectively) and Specialist Mathematics (0.022 compared to 0.014 respectively).

Table 1
Enrolments in VCE Mathematics Subjects as a Proportion of all VCE Subject Enrolments by School Location.

	All Schools		Metropolitan Schools (N=164)		Non-Metropolitan Schools (N=122)	
	N	Mean (Range)	N	Mean (Range)	N	Mean (Range)
Mathematical Methods	280	0.045 (0.004 – 0.155)	162	0.051 (0.004 – 0.155)	118	0.037 (0.004-0.103)
Specialist Mathematics	227	0.014 (0.001 – 0.075)	141	0.017 (0.001 – 0.075)	86	0.010 (0.001-0.028)

Table 2
Enrolments in VCE Mathematics Subjects as a Proportion of all VCE Subject Enrolments by SFOE Quartile.

	1st Quartile SFOE Schools (Highest SES) (N=72)		2nd Quartile SFOE Schools (N=71)		3rd Quartile SFOE Schools (N=72)		4th Quartile SFOE Schools (Lowest SES) (N=71)	
	N	Mean (Range)	N	Mean (Range)	N	Mean (Range)	N	Mean (Range)
Mathematical Methods	72	0.064 (0.012-0.155)	70	0.040 (0.004-0.098)	70	0.037 (0.004-0.103)	68	0.039 (0.006-0.094)
Specialist Mathematics	68	0.022 (0.001-0.075)	55	0.011 (0.002-0.030)	57	0.009 (0.001-0.024)	47	0.014 (0.002-0.055)

As can be seen in Table 3, there is a weak negative correlation between enrolment proportions and SFOE in both the enabling mathematics. However, when calculating coefficients using data from only metropolitan schools, the strength of these correlations increases. Conversely, in non-metropolitan schools, these correlations become negligible.

Table 3
Spearman's rho correlation coefficients for SFOE and Year 12 Mathematics Subject Enrolment Proportions by all Schools, Metropolitan Schools and Non-Metropolitan Schools

	All Schools	Metropolitan Schools	Non-Metropolitan Schools
	rho (N)	rho (N)	rho (N)
Mathematical Methods	-0.34 (284)	-0.5 (162)	-0.02 (122)
Specialist Mathematics	-0.29 (227)	-0.33 (141)	-0.08 (86)

Achievement Level

Table 4 shows that metropolitan schools outperform non-metropolitan schools in both Mathematical Methods and Specialist Mathematics, however the average difference was only 1.67 and 1.82 study score points respectively. In contrast, Table 5 shows the difference in achievement levels between highest and lowest SES schools was starker. On average, the highest SES schools outperformed the lowest SES schools by 4.01 and 3.54 points in Mathematical Methods and Specialist Mathematics respectively.

Table 4
Comparison of schools' achievement levels in VCE Year 12 mathematics subjects, where results were available for 2014, 2015 and 2016, by location

	All Schools		Metropolitan Schools		Non-Metropolitan Schools	
	N	Mean (Range)	N	Mean (Range)	N	Mean (Range)
Mathematical Methods	221	26.57 (16.96-37.32)	139	27.19 (18.44-37.32)	82	25.52 (16.96-33.09)
Specialist Mathematics	222	26.66 (17.00-46.00)	139	27.34 (17.08-46.00)	83	25.52 (17.00-37.00)

Table 5
Comparison of Schools' Achievement Levels in VCE Year 12 Mathematics Subjects, where Results were available for 2014, 2015 and 2016, by SFOE Quartile

	1st Quartile SFOE Schools (Highest SES) (N=72)		2nd Quartile SFOE Schools (N=71)		3rd Quartile SFOE Schools (N=72)		4th Quartile SFOE Schools (Lowest SES) (N=71)	
	N	Mean (Range)	N	Mean (Range)	N	Mean (Range)	N	Mean (Range)
Mathematical Methods	68	28.75 (18.80-37.32)	54	26.36 (21.21-33.09)	52	25.59 (16.96-32.34)	47	24.74 (18.44-31.00)

Specialist Mathematics	68	28.32 (17.00-36.31)	54	26.14 (17.71-33.00)	53	26.73 (18.00-46.00)	47	24.78 (17.08-35.00)

Table 6 shows there is a negative correlation between SFOE and achievement across all schools in both subjects. However, the strength of these correlations increases when considering only metropolitan schools, with a strong negative and moderate negative correlation in Mathematical Methods and Specialist Mathematics respectively. Conversely, in non-metropolitan schools there is only a weak negative correlation in Specialist Mathematics, and a negligible correlation in Mathematical Methods.

Table 6
Spearman's rho correlation coefficients for SFOE and VCE Year 12 Mathematics Subject Achievement Levels for all Schools, Metropolitan Schools and Non-Metropolitan Schools

	All Schools	Metropolitan Schools	Non-Metropolitan Schools
	rho (N)	rho (N)	rho (N)
Mathematical Methods	-.482 (280)	-.613 (162)	-.193 (118)
Specialist Mathematics	-.390 (226)	-.453 (141)	-.242 (85)

Discussion and Conclusion

This study explored the impact of school SES and school location on the provision of, enrolment in, and achievement in, Year 12 enabling mathematics subjects. It did this using school-level data from all Victorian government secondary schools offering either subject across the three years of 2014, 2015 and 2016.

On average, students from lower SES schools were less likely to have access to enabling mathematics subjects, were less likely to enrol in these subjects, and achieved less well in these subjects, compared with students attending higher SES schools. Similar patterns were observed in the participation and achievement levels of non-metropolitan schools versus metropolitan schools, with metropolitan schools generally attracting more students to, and on average achieving better results in, enabling mathematics subjects than schools outside the greater Melbourne metropolitan area. These results mirror the findings of previous studies of the impact of SES and geographic location on the mathematical literacy of secondary school students (ACARA, 2017; McConney & Perry, 2010; Thomson, De Bortoli & Underwood, 2017).

More revelatory is that this study suggests that a non-metropolitan location can mitigate the apparent influence of school SES. In metropolitan schools, as school SES decreased, participation and achievement in enabling mathematics tended to decrease. However, in the non-metropolitan schools, SES appeared to have little to no impact on the enabling mathematics subjects delivered, nor the proportions of students enrolling in these subjects, nor the average achievement levels in enabling mathematics. Importantly, some non-metropolitan schools dramatically outperformed other schools. While non-metropolitan schools on average underperformed relative to the metropolitan schools, their performance was more varied and SES did not appear to explain that variability.

So, what are the factors, independent of SES, influencing non-metropolitan school performance in the enabling mathematics? Past research hints at possible explanations. Many researchers, including Marginson (2013), highlight the difficulties of recruiting

qualified mathematics teachers to rural and remote areas. Without quality mathematics teachers, schools may not be able to offer the more advanced mathematics courses, let alone attract students to enrol in them or adequately prepare students to perform well. McPhan (2008) identified student reticence to participate in composite and distance classes in rural schools as a reason why students do not take up advanced senior mathematics classes, and participating in such class formats may go some way in explaining the lower mathematics achievement levels of students in some non-metropolitan schools. Other authors have suggested that rural students (and their parents) have lower expectations of continuing on to tertiary study (Centre for Education Statistics and Evaluation, 2013), so they may be less motivated to participate and achieve in enabling mathematics subjects. However, while this research may help explain why country schools tend to perform less well in senior mathematics than their city cousins, it does not explain why some non-metropolitan schools perform unexpectedly well.

Existing research suggests few possible explanations for this aberrant excellence in mathematics. Some research suggests that strong family-school connections and supportive relationships with school communities can positively affect the educational outcomes of rural students (Barley & Beesley, 2007; Semke & Sheridan, 2012). Possibly the high-performing non-metropolitan schools identified in the current study have been able to exploit their location and perhaps smaller size to better foster such relationships. Related to this may be that some non-metropolitan schools are better able to make use of rich local community resources afforded non-metropolitan schools, such as agriculture, industry and the natural environment, to provide relevant contexts for mathematics learning, thus improving student engagement and achievement.

Whatever the explanation, these findings have concerning practical implications for students attending our low SES and non-metropolitan schools. Low participation and achievement in enabling mathematics subjects mean that many students from these schools are automatically ruled out of access to some tertiary courses in engineering, computer science and biomedical science (VTAC, 2016), all of which lead to careers with growing demand for workers (AIG, 2015).

This study re-purposed school level data from the Victorian DET to uncover broad patterns of participation and achievement in the enabling mathematics subjects and to set a baseline for future research. As such, it does not reveal anything of the role student characteristics, such as gender, indigeneity or ethnicity, may have in moderating the relationships observed in this study, yet these variables are likely to inter-relate with the variables discussed in this paper (Thomson, De Bortoli & Underwood, 2017). Finally, while this study reveals relationships between school SES and location and enabling mathematics participation and achievement, the data analysed in this study do not explain why these relationships exist.

Further research is needed to seek an explanation for these relationships. In particular, developing an understanding as to why some non-metropolitan schools perform much better than expected in mathematics education promises to not only provide a model for improving enabling mathematics education in other non-metropolitan schools, but it could also identify ways in which metropolitan schools might minimise the influence of disadvantage. Case studies should be made of high performing non-metropolitan schools at all SES levels, with particular focus on staffing, resourcing, community connection, and student and parent expectations. This research could help identify positive school leadership and mathematics education practices for other schools to consider.

Acknowledgements

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Examining a teacher's use of multiple representations in the teaching of percentages: A commognitive perspective

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Learning Mathematics in Primary Schools is often mediated through the use of multiple representations. However, teachers may not pay enough attention to the way they use these representations. Given that the translations among representations may not always be smooth, it may be insightful to examine how teachers mediate learning through the use of multiple representations. In this paper, I will share key ideas in commognition before I present a case study of how Hannah, a teacher, mediate learning of percentages in her class. I will also introduce the idea of a 'Mediation Flowchart' and demonstrate how it can be used to describe and analyse a teacher's use of multiple representations.

Representation is one of the five process standards stated in the principles and standard of school mathematics (National Council of Teachers of Mathematics (NCTM), 2000). Representation is both a process and a product (NCTM, 2000). As a product, representation refer to external form of representation (Goldin, 1998) such as symbols, graphs and diagrams. As a process, it is seen as the internal thinking in the teachers and pupils' mind when working with representations. Representation can then be viewed as a useful means for communicating mathematical ideas. More specifically, pupils demonstrate their ability to connect mathematical ideas when they are able to translate among different representations of the concepts fluently, resulting in deeper and meaningful mathematical understanding (NCTM, 2000). Mathematics communication and connection are important mathematical processes under Singapore's Mathematics framework (Ministry of Education (MOE), 2012). Hence, a study on the use of mathematical representations would also improve mathematics communication and connection.

Although the use of multiple representations is an integral part of mathematics teaching and learning and teachers are also encouraged to integrate a variety of multiple representations into their teaching (Goldin 1998; NCTM 2000), several studies have raised issues on the use of multiple representations in the teaching and learning of mathematics. One issue is that teachers often use representations in isolation (Dreher & Kuntze, 2015; Goldin & Shteingold, 2001; NCTM, 2000). When representations are not connected fluently, mathematics communication will be affected and the lack of representational fluency may hinder deep and meaningful mathematics learning (Goldin 1998; NCTM 2000). In addition, the translation between representations is also often challanging (Pape & Tchoshanov, 2001), especially in topics such as fractions and percentages.

Percentages is an essential topic in the Singapore primary school mathematics syllabus (MOE, 2012). However, many pupils do not have a good understanding of percentages (Zambo, 2008). Moreover, teachers may not have a clear understanding of this topic. For instance, in a study done by Koay (1998), she found that many pre-service teachers in Singapore did not have a good understanding of the percentages topic. Her findings, and others like hers, suggest that the teaching and learning of percentages should be more closely examined. However, there are only a few studies (e.g., See Parker & Leinhardt, 1995) which

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 591-598. Auckland: MERGA.

focus on the teaching and learning of percentages as most studies focus on fractions and decimals instead. More importantly, there are not many studies which explore the use and interplay of representations in the teaching and learning of percentages, given that the use of representations may play a critical role in teaching the topic. In addition, there is also no recent study on the use of representations in the teaching and learning of percentages. Therefore, this study aims to shed some light on the use and interplay of representations and contribute towards a better understanding of how multiple representations can be used in the teaching of percentages.

A Commognitive Perspective of Learning and Teaching

This paper positions the interaction between the teachers and the pupils when using multiple representations within a participationist view of learning. A participationist perspective of learning reflects a shift from the acquisitionist perspective. According to Sfard (2001), an acquisitionist perspective describes learning as a mental action such as learning new concepts and forming new schemas. Some researchers challenged the acquisitionist perspective which did not consider the social cultural context which learning takes place (Sfard, 2015). On the other hand, participationist perspective views "learning is first and foremost about the development of ways in which an individual participates in well-established communal activities" (Sfard, 2001, p. 10). In other words, participationist perspective focuses on the interaction between the learner and the rest of his community. Participationism is able to complement the acquisitionist perspective in analysing pupils' learning (Sfard, 2001). In addition, an important aspect of examining the use of multiple representations is to investigate how they are used within a social context (Pape & Tchoshanov, 2001). Hence, examining the use of multiple representations using the participationist perspective will provide new insight to current research on the use of multiple representations.

Sfard (2008) introduces the commognitive perspective to analyse mathematical communication and thinking. Commognition, which is formed using the words 'communicating' and 'cognition', stems from a participationist perspective that views thinking as a form of communication. In this section, I will first introduce the key terms from the commognitive framework used in the study, as summarised in Figure 1.

According to Sfard (2008), mathematical discourses are categorised using four characteristics: keywords, visual mediation, narratives and routines. Keywords are important in mathematical discourses because they help to convey meaning to the participants. Visual mediators are visible objects used in the communication such as symbols or iconic representations. Next, narratives involve a set of spoken and written utterances which describes mathematical objects and the relationships among them. The narratives are subject to endorsement, or rejection based on their substantiation procedure. Endorsed narratives, for example, theorems and proofs, are labelled as true. Endorsed narratives are created when there are elaborated realizing procedures between the signifiers and their realisations. Signifiers are words, symbols or other form of representations used in utterances by the participants and its realisations are objects that are operated upon their signifiers to produce narratives. Realisation can be visual or vocal (Spoken Words). Visual realisations may be represented using symbols, concrete objects, icons, gestures or written words. The last characteristic is the use of routines. Routines are sets of metarules that describe repetitive discursive action.

Routines can be further categorized into explorations, deeds and rituals (Sfard, 2008). The goal of the use of explorations is the production of endorsed narratives. Exploration can

also be divided into three different types: construction, substantiation and recall (Sfard, 2008). Construction of narratives will result in the construction of new endorsable narratives. Substantiation are actions which determine whether the narratives should be endorsed, and recalling act is the process of recalling previously endorsed narratives. Deeds are defined as a set of rules that produce or change the physical object involved in discourse. Ritual is a routine which primary goal is to create and sustain relationship with others.

For example, in the teaching of addition of unlike fractions, 2/3 + 1/4, the class may be involved in the use of different types of routines. The pupils may need to recall previously endorsed narratives such as definition of like and unlike fractions (Recalling). Instead of only stating the algorithm to be performed, teachers may be involved in substantiation of narratives such as explaining the importance of converting unlike fraction to like fraction (Substantiation). Eventually, the use of recalling act and substantiation will lead to the creation of new endorsed narratives 2/3 + 1/4 = 11/12 (Construction). The use of deed may include the conversion of fractions to their equivalent forms (Deed). Inevitably, ritual such as the use of teachers' questioning will be used during the interaction between the teacher and the pupils (Ritual).

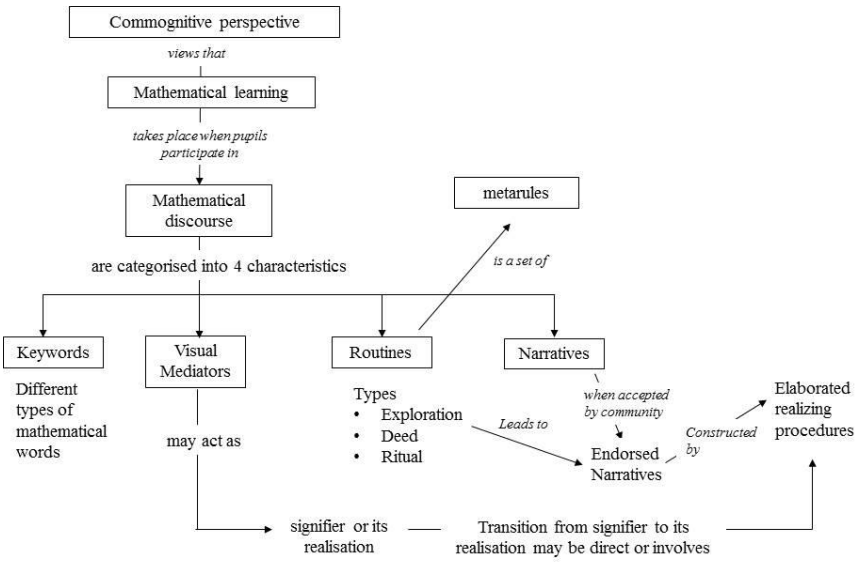

Figure 1. Summary of commognitive terms used in the study.
Adapted with permission from Choy (2015, p. 28)

Method

This study explores how the commognition framework can be used to analyse the transitions among multiple representations in the teaching of Mathematics. The participants of the study included an experienced teacher, Mrs Hannah (pseudonym) and her class of seven pupils at Primary 5 level from a Singapore public primary school. At the time of this study, Mrs Hannah had 12 years of teaching experience in primary school. She received teacher training at the Institute of Education (Singapore) and graduated with a Postgraduate Diploma in Education. During her teaching years, she had taught different profiles of pupils. The seven students in this study were identified based on their results and their behavioural needs at the end of their Primary Four academic year. These pupils have outlier scores (lowest) across all subjects and were grouped to form a small class so that they would be able to receive more attention and assistance from the teacher.

This study consists of four main phases: Pre-data collection, data collection, data condensation and data analysis. During pre-data collection, the necessary ethics clearance were made. Next, for data collection, six consecutive lessons on Mrs Hannah's teaching of percentages were recorded. The average duration of each lesson is 40 minutes. Due to the huge amount of data collected, I went through a process of data condensation. I watched the six videos and identified the relevant teaching moments which may be relevant to the study. I wrote brief comments about the teaching moments (Example: Mrs Hannah connect 1% to 1 building block to 1 base ten cube.) More examples of brief comments can be found in Appendix E in Chia (2017). I categorised the brief comments into five categories. The five categories are connecting different representations, focus on percentage symbols and the use of base 100, choice of example, using pupils' common mistake and using pupils' existing knowledge of decimals and fractions. The recordings of the selected teaching moments for the first two categories were transcribed. I analysed the transcripts from a commognitive perspective and selected an episode from Mrs Hannah's fourth lesson which reflects a rich use of representations to illustrate how Mrs Hannah's use of multiple representation can be analysed with the use of a mediation flowchart. The mediation flowchart is my extension of a figure displaying the different types of signifiers' realisation in mathematical discourse (Sfard, 2008, p. 155). In the next section, I describe a pedagogically significant moment, which happened in the fourth lesson, and illustrate how the mediation flowchart can highlight the interplay between the different representations used by Mrs Hannah.

Results and Discussion

The episode described in this paper is selected from Mrs Hannah's fourth lesson. Prior to the fourth lesson, Mrs Hannah had introduced pupils to associate percentages with 100 squares. Pupils had experience learning using unit blocks and 10 × 10 square grids. She taught pupils the procedure for converting percentages to fractions and vice versa by converting the denominator to 100. For example, 25% = 25/100 =1/4. In the episode, Mrs Hannah began the discourse by revising the conversion from fraction to percentage by changing the denominator of the fraction to 100.

1.	Mrs Hannah:	Question 1, you have 1/25. Remember, let's recall what we have learnt about percentage. What do you know about percentage? Percentage is how many squares?
2.	Josh:	Hundred square
3.	Mrs Hannah:	Thank you. Josh. We learnt that percentage is equal to 100 squares. In your mind, you should picture these 100 squares. Out of 100, how many squares must you colour? So that is percentage. So 1 out of 25, can I make it into 100?
4.	Josh:	Yes, times 4
5.	Mrs Hannah:	Woah, Josh is so fast. Very good. Do you just multiply by 4 this way? [*Mrs Hannah wrote 1/25 × 4.*]
6.	Kate:	No
7.	Mrs Hannah:	What should I do? Thank you Kate. She says you must multiply the factor 4 to both the numerator and denominator. So that's one method going about doing it. You get 4/100. [*Mrs Hannah completes the working 1/25 = 4/100 as she talks.*] Ruth do you think you can help us along to change this to percentage, or perhaps you change it to decimal first? This is something which we do last week. If you can remember. Or anyone? Ruth looks so nervous. Is there anyone else who can help her?
8.	Mitch:	4% [*Mrs Hannah completes the working 1/25 = 4/100 = 4% as she talks.*]

9.	Mrs Hannah:	How do you get 4%? In your mind, how do you read this?
10.	Mitch:	4 out of 100
11.	Mrs Hannah:	[*Mrs Hannah circled 4/100 and extended an arrow out of the circle and wrote 4 out of 100.*] That's right. You must be able to read this as 4 out of 100. 4 squares out of 100 squares. 4 squares out of 100 squares will be 4 percent. Remember what I say about percentage. Percentage is about 100 squares. So it is 4 out of 100 which is 4%. Very good, Mitch.

As can be seen from the transcript, Mrs Hannah first elicited responses from the pupils that percentage is associated with 100 squares. Next, Mrs Hannah explained that 100 squares can be represented by the denominator 100, which the pupils had learnt previously. She highlighted to the pupils that they should convert the denominator to 100 when converting fractions to percentages. As seen from the above example of 1/25, the class first converted 1/25 to 4/100. Next, by replacing the denominator '/100' with the '%' sign, the class converted 4/100 to 4%. I will now provide a fine-grained analysis of how Mrs Hannah mediated the use of different representations through the lens of commognition—keywords, visual realisations, endorsed narratives, and routines.

Mediation using keywords. Keywords used in this segment can be categorised into three different categories: 'mathematical terms (percentage)', 'everyday words' and 'other mathematical terms' (Shuard & Rothery, 1984). Examples of 'mathematical terms (percentage)' are 'percentage' and 'out of 25'. 'Everyday words' are example such as 'equal to' and 'other mathematical terms' refers to words like 'numerator' and 'denominator'. In this segment, keywords, in both spoken and written forms are used to mediate between symbolic representations and algebraic representations. In turn 3, the use of mathematical terms such as percentage, 100 squares and out of 100, are used to mediate between symbolic representations, '1/25' and its iconic representation which is 10 × 10 square grids. In the case of 1/25, pupils may not be able to visualise the fraction as 100 squares directly. Using everyday words, pupils would realise that they need to 'make it' into 100 squares. After converting 1/25 to 4/100, similarly, the use of spoken and written forms of '4 out of 100' would be used to mediate between the two symbolic representations of 1/25, 4/100 and 4%. Figure 2. on the next page provides a visual flow chart of the direction of mediation in the episode.

Visual realisations of 1/25. The flowchart shows the different realisations of 1/25 in the form of concrete objects, iconic representations, spoken and written words and algebraic symbols. Base 10 blocks and 10 × 10 square grids were used in the previous lessons. Hence, pupils may make reference to these representations to make meaningful connections to the new representations used in this lesson. In this segment, the use of the phrase '____ out of ____' is frequently used by Mrs Hannah both in written and spoken form. Algebraic symbols includes 4/100, an equivalent fraction of 1/25, and 4%. The sequence of the appearance of the different realisations is presented from top to bottom with the full arrows showing the direction of mediation. These arrows also connect the signifier-realisation pairs through the mediation process which was mainly through the use of written and spoken words. The numbers and directions of the arrows reflect that the realising procedure is a non-straightforward, complicated one. There are mainly four signifier-realisation pairs as summarised in the Figure 3 below.

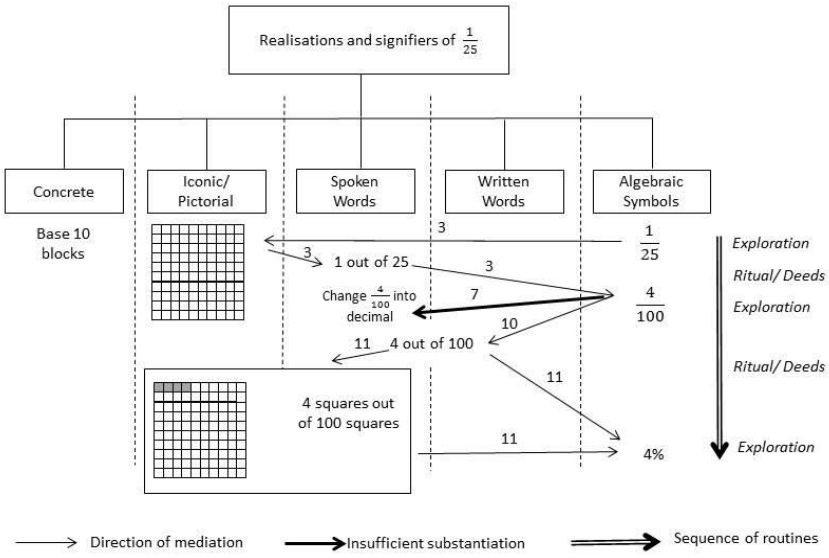

Figure 2. Mediation of algebraic-symbolic and iconic representations using written and spoken words in the episode.

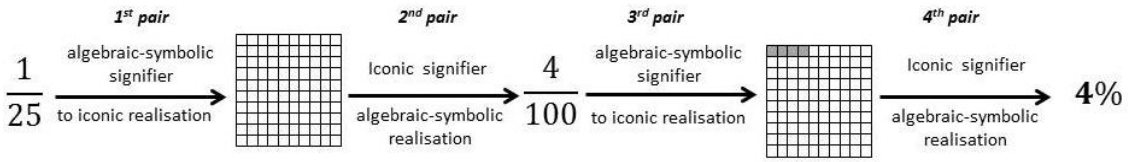

Figure 3. Signifier-realisation pairs in this episode.

Endorsed narratives. The realising procedures which translate the signifiers to their realisations lead to the creation of endorsed narratives. The first two signifier-realisation pairs lead to the endorsed narratives, 1/25 = 4/100. Using the third signifier-realisation pair, the class was able to conclude that 4/100 can be expressed using the iconic representation of 100 squares with only four squares being shaded. Lastly, in the fourth signifier-realisation pairs, we can equate 4 squares out of 100 squares to 4%. Combining the endorsed narratives found in this segment, we can express the realisation as 1/25 = 4/100 = 4%. Mathematical communication is fluent when there is coherence between the use of the keywords and narratives by the participants. In this segment, the interplay of keywords and narrative suggests that the realisation of the signifiers in the four signifiers-realisation pairs are examples of fluent communication which lead to the production of endorsed narratives. However, in turn 7, teacher asked the pupils to convert 4/100 into decimal. There is an absent of realisation of 4/100 in its decimal form in the lesson as reflected in Figure 2. This is a non-example of fluent mathematical communication.

Types of routines. The endorsed narratives, 1/25 = 4/100 = 4% is the product of the interchange among the act of different routines – explorations, deeds and rituals. Although explorations are the only types of routines which lead to endorsed narrative, the use of deeds and rituals are also important in developing act of explorations (Sfard, 2008). In this segment, Mrs Hannah had used different types of routines to improve the fluency when

connecting the different signifiers and realisations of 1/25 (Goldin & Shteingold, 2001; NCTM 2000). Table 1 provides a summary of the sequence and explanation of the type of routines used by Mrs Hannah in Segment 1.

Table 1
Summary of Types of Routines Used in Segment 1

Turn	Types of routines	Task
1-3	Exploring through Recalling	Pupils recalled that percentage is associated with 100 squares.
3	Deed	Act of colouring squares in 10 × 10 square grids to represent 1/25.
3-4	Ritual	Pupils responded to Mrs Hannah by explaining how they converted 1/25 to 4/100.
7	Deed	Mrs Hannah explained the conversion of 1/25 to 4/100.
7	Exploration through construction	Mrs Hannah explained the endorsed narratives 1/25 =4/100.
9	Ritual	Mrs Hannah provided scaffolding by asking them to read 4% as 4 out of 100.
11	Exploration through substantiation	Mrs Hannah explained the endorsed narratives 4/100 = 4%.

Mrs Hannah began the lesson using endorsed narratives from the previous lessons that associate percentages with 100 squares. The pupils were involved in exploration through recalling that percentages is associated with 100 squares. After that, the pupils carried out the deed of picturing the number of coloured squares to represent 1/25. Through the use of ritual, Mrs Hannah also prompted the pupils to convert 1/25 into denominator 100 and carried out the deed of multiplying both the numerator and denominator by 4 to convert 1/25 into denominator 100. The use of deeds and rituals had led to the extension of the previous endorsed narratives that percentage is associated with 100 squares and led to exploration through construction that 1/25= 4/100. Using the new endorsed narrative, 1/25=4/100, Mrs Hannah continued to teach her pupils to convert 1/25 into percentages. As Mitch had answered 4% in turn 8, Mrs Hannah substantiated Mitch's constructed narratives through the use of ritual. She questioned Mitch how he had read 4/100. After Mitch replied '4 out of 100', she substantiated his constructed narratives by explaining that 4 out of 100 is the same as 4 squares out of 100 squares which is 4%. From Segment 1, Mrs Hannah had used rituals and deeds to lead to exploration which produces endorsed narratives. The various modes of routines used is also an evident of Mrs Hannah's numeracy fluency (Sfard, 2008; Thomas, 2008).

There are several key findings from the analysis of Mrs Hannah's discourse in this episode. First, key words can be used to mediate between different representations. In turn 3, Mrs Hannah used mathematical terms and everyday words to connect different representations of 1/25. When the realising procedure that translates a signifier to its realisation is elaborated, the translation between the representations will be fluent. This can be seen in the creation of the four signifier-realisation pairs in the episode (See Figure 2). In turn 7, the absence of an elaborated realising procedure between 4/100 and its decimal form

provides an example of a non-fluent transition between representations. As demonstrated in this episode, the use of different types of routines in a mathematical discourse helps to improve the fluency in the translation of different representations. Lastly, the use of a mediation flowchart serves as a tool to make the representations visible for analysis to take place. In particular, the use of arrows in the flowchart helps to identify and connect signifier-realisation pairs. Any missing or incomplete realising procedures are represented using bolded arrows. Two more episodes of analysis can be found in Chia (2017).

Concluding Remarks

Notwithstanding the limitations of a single case study, this study has demonstrated how classroom discourse can be analysed from a commognitive perspective. The use of a commognitive perspective increases teachers' awareness when using multiple representations. The coherence among the different characteristics of mathematics discourse affects the fluent use of representations. With the aim of improving teaching and learning, both researchers and teachers can better reflect on their use of representations and language during teaching by making their use of multiple representations more visible using the mediation flowchart. Through visual representations of teachers' thinking, teachers can identify gaps in their use of multiple representations and suggest alternative teaching strategies. Although it remains to be seen whether such commognitive analysis can lead to teaching and learning, this study has shed some important insights into the complexity of mathematical communication through multiple representation.

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Middle School Pre-Service Teachers' Mathematics Content Knowledge and Mathematical Pedagogy Content Knowledge: Assessing and Relating

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This paper examines the mathematics content knowledge of graduate entry middle school mathematics pre-service teachers at the beginning (n=105; 80%) and end of a mathematics curriculum course. It was found that mathematics content knowledge at the commencement of the course was not strong. An intervention was designed to take account of content while preparing teachers to teach the material with specific pedagogical models. On a test of similar difficulty level, the marks approximately doubled, but in areas of upper secondary mathematics, significant deficits remained. Content knowledge at the end of the course was highly predictive of measures of mathematics pedagogical knowledge including how to diagnose student errors and plan learning support. The finding have implications for teacher preparation at the study institution and potentially more broadly.

It is typical in Australia to have two pathways to mathematics teaching: an undergraduate pathway and a graduate pathway. The first pathway is via an undergraduate degree in which prospective teachers are expected to complete six university mathematics courses. The second pathway is via a graduate entry and this is the pathway that is the subject of this study. While there is some variation across the nation, in the study state the criteria for entry to middle years mathematics teacher education programs are based on the successful completion of at least four university-based subjects rich in mathematics. A number of authors (e.g., Burghes & Geach, 2011; Tatto et al, 2008) have cautioned against the use of proxy measures to evaluate the content knowledge levels of pre-service teachers. With this context in mind the paper examines middle school pre-service teachers' mathematical content knowledge (MCK) at the start and end of a mathematics curriculum intervention and relates this to a measure of pedagogical content knowledge (PCK) at the end of the intervention.

Literature review

It has been convincingly argued that high school mathematics teachers with a strong knowledge of the mathematics they are teaching are more likely to be effective in developing this knowledge in their classrooms (Australian Academy of Science, 2015; Cai, Mok, Reddy, & Stacey, 2016; Krainer, Hsieh, Peck, & Tatto, 2015). The Teacher Education and Development Study in Mathematics (TEDS-M) (Tatto et al., 2008, p. 19) summed up this argument: "Knowledge of content to be taught is a crucial factor in influencing the quality of teaching." The effects of depth of content knowledge and its relationship to effective teaching have been well researched, particularly since Shulman (1987) defined mathematics content knowledge and pedagogical content knowledge. Many scholars have subsequently refined understandings of the relationship between content knowledge and effective teaching of mathematics (e.g., Beswick & Goos, 2012; Chapman, 2015). So intertwined are these key factors that Beswick and Goos noted the "interconnectedness of MCK and PCK and the difficulty in distinguishing between them" (p. 72). In this study, the term mathematics pedagogical content knowledge (MPCK), which includes mathematics curriculum knowledge, knowledge of planning for mathematics teaching, and enacting mathematics for teaching, as defined by Tatto et al. (2008, p. 39), is used.

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Australian Teaching Standards

Given the research affirming the importance of teachers' content knowledge, it is not surprising that standards for graduating mathematics teachers stipulate that prospective teachers know their content (e.g., Australian Institute for Teaching and School Leadership [AITSL], 2014). The Australian curriculum (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2012) outlines the content for Australian school children. The number and algebra strand of Year 7 includes fraction operations, ratio, and solving simple linear equations. By Year 10, more advanced students are expected to be fluent with surds, exponential and logarithmic expressions, and non-linear algebra including working with parabolas, hyperbolas, circles, and exponential functions (ACARA, 2012). Clearly, there is a great deal of mathematics between these fraction operations and logarithmic expressions; yet, a number of authors have expressed concerns about the content levels of Australian high school teachers, and therefore their ability to teach to these standards (e.g., Beswick, Callingham, & Watson, 2012; Hine, 2015).

Mathematics Curriculum Course Assessment Protocols

Henderson and Rodrigues (2008), Hine (2015), and Kotzee (2012) claim that it has become a tradition in teacher education courses at Western universities to focus on big-picture curriculum issues. Of pre-service teachers in the UK, Burghes and Geach (2011) noted a lack of relevance between theoretical studies undertaken (such as views on theories of learning assessed via essays) and school-based work. The focus on more general curriculum knowledge can be justified if it is assumed that pre-service teachers enter middle school mathematics curriculum units with a reasonable depth of mathematical knowledge. A review of assessment protocols reported by universities indicated that in Australia, almost without exception, middle years' mathematics curriculum courses are assessed via essays and the production of mathematics teaching resources. Producing teaching resources can be directly linked to teaching the subject of the resource, but essays and reflections tend to be more abstractly focused and related to generic considerations rather than the detailed pedagogy associated with specific mathematics concepts. One Australian anomaly to this general pattern of essays, report writing, and resource construction is the study institution, which had a 60%, 3-hour closed-book examination and a 40% case study research assignment involving testing, planning an intervention, and implementing the intervention.

Research Questions

With the background above in mind the research questions for this study are as follows:

1) What was the starting content knowledge (MCK) and proficiency in basic middle school computation as measured by a written test (Author, 2017, scale).

2) What relationship exists between MCK measured at the commencement and end of the course and key components of mathematical pedagogical content knowledge (MPCK) measured at the end of the course?

Method

Overview of Methodology

The method is correlational in so much as the associations between MCK and measures of MPCK are examined. The software package SPSS was used to calculate descriptive statistics and calculate correlations that enable the reader to assess the relationships between the tests that form the basis of the data presented in this paper. The participants were most of a cohort

(n=105/131 [80%] pre-test and n=128/131 post-test) of an intake of graduate diploma in mathematics education at a reputable Australian university.

The Intervention

Three Australian professional standards for teachers (AITSL, 2014) were the target of the intervention. The first was "Know content and how to teach it" (p. 3). This was interpreted that upon graduation most of the pre-service teachers would have a reasonable grasp of a significant spread of middle years' number and algebra content. Further, they would be able to articulate this knowledge as well as a range of specific pedagogies to provide learning support, in this way demonstrating aspects of the third standard, "Plan for and implement effective teaching and learning." (AITSL, 2014, p. 3). Finally, in the assignment (40%) and for aspects of the final written test (60%), pre-service teachers were required to assess student learning and provide feedback, thus demonstrating some knowledge of the fifth standard, "Assess, provide feedback and report on student learning" (AITSL, 2014, p. 3). The course was run over 7 weeks with 28 contact hours. Learning support in the form of a 500-page text (Norton, 2014a) was supplied (given in pdf format) for the teaching of concepts from counting to simultaneous equations. A second text (2014b) covering the specific pedagogy for teaching quadratic equations was also supplied. These texts were auxiliary to lecture-captured lectures (seven) and workshops (seven) and were supported by optional video production, of which multiple copies were placed in the libraries (Norton, 2014a, b). The intent was to deepen schematic knowledge of middle school mathematics while learning specific pedagogy including error analysis.

Testing Instruments

The data reported come from two written tests where calculators were not permitted and 1 hour was allowed for completion. The intake (pre-test) test was comprised of 31 questions based on the number and algebra content the pre-service teachers were preparing to teach. Item descriptions in the results section illustrate the content validity of this assessment in that each question can readily be mapped to the current middle school mathematics curriculum. The content ranged from whole-number operations, which is primary mathematics, to quadratic equation conventions, which is Year 10 advanced content. With few exceptions, this test is simply a test of fluency with middle school content. In this regard the test items are similar to those used by Tatto et al. (2008) to assess the knowing and applying cognitive domains of mathematics. Burghes (2007) used similar items. The test was conducted during the first tutorial of the course. The Cronbach's alpha score of the pre-test MCK was 0.903 indicating very high internal consistency. The test described above is unlike the Literacy and Numeracy Test for Initial Teacher Education Students (ACER, 2017) which for most questions allows calculators and has content demands at about Year 7 level and lower.

The post-test represented the final assessment of the course; it was conducted in the 8th week of the course and had a duration of 3 hours, since MPCK was tested in addition to MCK. This test had two sections. Part A had a virtual replication of the pre-test, except that for each question a scenario was presented and the pre-service teacher was asked to identify student error and then provide a correct solution. For example, the pre-test asked for a large subtraction in context, and the replication of this was simply minor modifications in context and numbers used. The mathematics underpinning each was essentially identical.

Question 20 of the post-test assessed pre-service teachers' diagnostic capability with regard to linear algebra conventions. The mathematics with respect to context and procedures required was virtually the same as Question 21 of the pre-test. The responses below contain errors that the pre-service teachers had to describe, provide the correct solutions for, and in some instances suggest and justify grades and provide learning support. Figure 1 illustrates one

such example of student error.

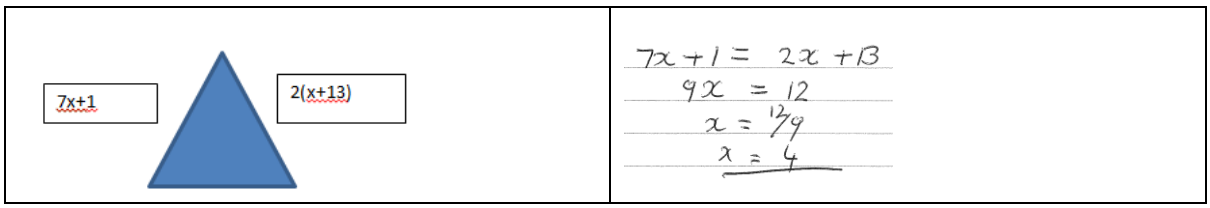

Figure 1. Example of algebra convention post-test assessment of MPCK (describe student thinking) and MCK (present the correct solution).

For all questions in the post-test, pre-service teachers were asked to describe children's errors and provide the correct solution. Analysing or evaluating students' solutions and diagnosing students' responses falls into the MPCK defined by Tatto et al. (2008). The Cronbach's Alpha statistic for the post-test MCK 27 items was 0.881 indicating strong reliability. The scale measuring diagnostic capability was based on the same 27 items on the post-test and had a Cronbach's Alpha of 0.839. The method of testing enabled both the content (MCK) and an ability to diagnose and describe children's errors, an aspect of MPCK, to be documented over a range of middle years' concepts.

In the second section of the post-test there were additional dimensions including assessing pre-service teachers' planning to provide learning support and justification of grading of children's written mathematical work. There were four questions of this nature related to teaching whole-number division, fraction subtraction, quadratic problem solving, and quadratic conventions. There was one generic MPCK question asking pre-service teachers to describe factors that challenge children's learning of mathematics and how the resultant incidence of misconceptions can be reduced through specific learning support. By way of example with respect to assessing specific MPCK, Question 3, related to assessing and providing learning support in the topic of quadratics, illustrates the question format. "*The base of a right angle triangle is 4 cm longer than its height, and the hypotenuse is 4 cm longer than the base. Find the height*." The Year 10 student provided the response shown below in Figure 2:

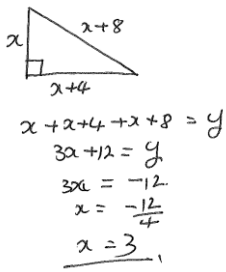

Figure 2. Year 10 student sample solution related to working with quadratics requiring pre-service teachers to diagnose children's misconceptions, grade the response, and provide learning support.

For the question illustrated in Figure 3, the pre-service teachers were asked to grade the work (out of 5) and provide a justification for the grade including identifying any procedural or conceptual errors in the solution to justify the grade. The second part of this question required the pre-service teacher to provide a step-by-step solution using "completing the square" method. Questions such as this enable scoring on three aspect dimensions: provision of the correct solution (an aspect of MCK), diagnostic capability (an aspect of MPCK), and provision of learning support (an aspect of MPCK). This last aspect of the test aligns with "Enacting Mathematics for Teaching and Learning," in particular, "Explaining or representing mathematical concepts or procedures" (Tatto et al., 2008, p. 39). It also demonstrates aspects

of the AITSL (2014) standard "know content and how to teach" (p. 3). The scale designed to assess pre-service teachers' capacity to provide specific learning support had a Cronbach's Alpha score of 0.608.

Analysis

SPSS was used to calculate descriptive data (means, minimum, maximum, standard deviations) and correlational data sufficient to answer the research questions.

Results and Analysis

Descriptive summary data on the tests assist in answering the research questions. Detailed success rates on particular items add meaning to these statistics, presented in Table 1.

Table 1.

Descriptive Statistics Related to the Test Instruments

Test	Minimum score	Maximum score	Mean score	Standard deviation
Pre-test content /31 (n=105)	1	29	10.34 (33%)	6.641
Post-test content /53 (n=128)	9	53	35.62 (67%)	10.82
Post-test diagnosis/27 (n=128)	9	27	21.61 (80%)	4.28
Post-test learning support (n=128) /25	7	25	15.15 (60%)	5.70

The pre-service test data presented in Table 1 indicate that while there is a considerable spread and some students have attained nearly perfect scores on the pre-test, the means are not flattering. Table 2 documents MCK in middle school content areas assessed. The measured MCK doubled subsequent to the course intervention.

Table 2.

Summaries of Average Success on Different Content Domains for the Pre-test (n=105)

Content domain/total score	Minimum score	Maximum score	Mean score	Standard deviation
Whole number computation /4	1	4	2.22 (55%)	1.194
Fraction computation /6	0	6	2.99(59%)	1.975
Index and logarithm conventions /9	0	9	2.29(25%)	2.072
Linear algebra conventions/6	0	6	1.82(30%)	1.692
Quadratic conventions /6	0	6	0.99(16.5%)	1.417

Data in Table 2 indicate that abstractness resulted in greater challenges. Specific results give the reader added insight with respect to the nature of the pre-test. Whole-number computation included subtracting 2,147 from 50,000 (85% success rate; Year 6 Standard; ACARA, 2012) and dividing 18,354 by 23 (40% success rate; Year 6 standard). The best done fraction problem necessitated the addition of mixed numbers ($3\frac{5}{7} + 2\frac{4}{6}$.: 58% success rate; Year 7) and the most difficult computation was to find the area of a silicon chip "*0.2mm wide and 0.3mm long*" (39% success rate-Year 7 standard). Question 19 involved problem-solving by repeated division by two; "*Say I had 64 biscuits and ate half of the number/amount each time. How many/much biscuits would I have on the 8th feed?*" had a success rate of 66% (Year 9 standard). The most difficult index/log question was "*Solve for x in:* $4^x = 8$" (success rate 7%; Year 10A standard). The most successfully done linear algebra question was to solve for "n" in $5n+2=9n-26$ (44% success rate; Year 8 standard) and the most difficult was related to simultaneous equations: "*There are 10 more men than women at a party. If one more woman joined the party, there would be twice as many men as women. How many men and how many women at the party?"* (success rate 19%; Year 10 standard). The most successfully completed quadratic question was "*factorise* x^2*-x-12"* (28% success rate; Year 10 standard) and the most challenging question was to state the equation of a graphed quadratic where the roots, turning point, and y intercept were clearly identified (6% success rate; Year 10A standard).

Table 3 indicates the correlations between the four scales; this helps to answer the second research question.

Table 3.

Correlation Coefficients between Tests of Content Knowledge, Diagnostic Capability, and Descriptions of Provisions of Learning Support

Test	Pre-test content	Post-test content	Post-test diagnosis	Post-test learning support
Pre-test content	1	.791	.709	.640
Post-test content	.791	1	.877	.694
Post-test diagnosis	.709	.877	1	.646
Post-test learning support	.640	.694	.646	1

Each correlation was significant at 0.01 level (2 –tailed).

The correlation coefficients between pre- and post-test MCK was reasonably high at .791. This means that about 63% in variance of post-test content is predicted by pre-test content scores. Unsurprisingly, post-intervention MCK was more strongly correlated to diagnostic capability than demonstrated content at the commencement of the course, having a correlation coefficient of .877. The correlation between post-test MCK and capacity to describe learning support was relatively strong at .694; this means about 48% of the variance in provision of learning support was explained by the post-test content score. The correlation statistics associated with learning support can be explained by the lower reliability Cronbach's Alpha score for that scale (.608). The limited number of items used to test learning support provision may have been a factor in these weaker relationships.

Discussion

The data in Table 1 on pre-test scores in MCK and the detail of pre-test scores documented in Table 2 effectively answer the first research question. The level of MCK of the enrolling pre-service teachers was varied, but overall very poor. In this regard the data support earlier questioning of the wisdom of using proxy measures such as numbers of courses completed to assume reasonable content levels prior to enrolment (Burghes & Geach, 2011; Tatto et al., 2008). The relatively high correlation between pre- and post-test MCK supports the notion that selecting candidates with strong mathematics into courses is a worthy endeavour; however, as noted above, it is best to avoid the use of proxy methods to assess MCK. Assuming that MCK is central to the enactment of effective teaching, as suggested by a range of authors (e.g., Cai et al., 2016; Krainer et al., 2015; Tatto et al., 2008) the teacher-preparation processes might consider providing learning support and focusing on ensuring pre-service teachers graduate with reasonable levels of MCK for middle school teaching. The content associated with simultaneous equations, index notation, and quadratics was in urgent need of support in the study institution.

The correlations presented in Table 3 suggest that MCK, especially MCK at the end of a content and specific mathematics pedagogy course, is highly predictive of capacity to diagnose student thinking, an important aspect of MPCK (Tatto et al., 2008). In this study MCK was moderately correlated with the measure of provision of learning support, another aspect of MPCK. This reduced predictability is likely explained by the reduced Cronbach's Alpha reliability of the post-test learning support test. The practicalities of testing that constrained the testing of MPCK were manifested in the relatively small number of items in this aspect of the scale. The test was already 3 hours in duration, and one purpose of the test was to assess a relatively broad range of middle years' mathematics with respect to MCK and diagnostic capability. Testing the provision of learning support is a time-consuming process and this was a major factor necessitating a relatively small number of items in this section of the final test. These correlational results of the study add empirical evidence informing the model-building accounting for teacher knowledge.

Conclusions

The findings have implications for the study institution in that a review of the structure of the middle years' mathematics curriculum course, indeed the entire middle years' teacher preparation program, warrants consideration. The first interpretation of the finishing MCK mean of 67% is that it is potentially encouraging. Closer examination reveals that it was easier to teach pre-service teachers whole-number and fraction computation than algebra and logarithms, suggesting these topic areas warrant special attention. What has not been clearly articulated earlier in the paper is that the focus of the intervention was relatively narrow, since the focus of the curriculum course and its subsequent assessment was on number and algebra. This focus was justified by the author because it was considered this aspect of middle school mathematics was critical. Not to address gaps in content in these concept areas could create pressure for the newly graduated teachers to teach themselves the content, as well as how to teach it, during school employment. In addition, at least half the cohort was enrolling in senior mathematics curriculum subjects. For them, middle school content consolidation would seem a worthy pre-course preparation for the teaching of senior calculus and statistics.

The relationship between knowing mathematics and knowing how to teach it is complicated. The manner in which MCK and MPCK was measured and the correlational relationship between the measures adds to the empirical data that informs these models. It is the responsibility of readers in other institutions to consider if the data from this institution have any relevance to their circumstances.

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Mathematics Anxiety: Year 7 and 8 Student Perceptions

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Studies, such as Programme for International Student Assessment 2012, indicate that there are gender based differences in measures of mathematics anxiety, self-concept and self-efficacy among students. In this study we explore self-efficacy, self-concept and mathematics anxiety in a sample of Year 7 and 8 South Australian students to examine if these differences still exist. The findings indicate that high levels of mathematics anxiety is present among Year 7 and 8 students and that gender based differences are also evident in both self-efficacy and anxiety.

Outputs from the office of the chief scientist have, over the past five years, repeatedly referenced the need for greater student participation in STEM (Science, Technology, Engineering & Mathematics) subjects. However, studies such as Mack and Wilson (2015) continue to highlight how the numbers of students opting for STEM subjects continues to decline. They identified that since 2001 there has been no substantial growth in science participation despite numerous initiatives to address this. They also note that mathematics participation continues to decline and where there is participation in mathematics, students tend to opt for elementary mathematics rather than intermediate or advanced mathematics. There are many reasons cited for such changes including expressed dissatisfaction with mathematics (Hine, 2017), pressure to choose subjects most likely to yield higher Australian Tertiary Admission Rank (ATAR) scores (Mathematical Association of New South Wales, 2014), students' self-efficacy in regard in their mathematics performance and the presence/absence of mathematics anxiety (Sax, Kanny, Riggers-Piehl, Whang, and Paulson, 2015). The aim of this paper is to explore year 7 and 8 student dispositions towards studying mathematics, by examining students mathematics; self-efficacy, self-concept and anxiety. In addition, gender differences that were identified in the 2012 Programme for International Student Assessment (PISA) survey will be examined to see if they are still evident in the project schools.

Literature

According to the 2012 PISA (Organisation for Economic Co-operation and Development (OECD), 2013) survey almost one third of fifteen year olds reported some level of mathematics anxiety. Also, variation in performance in mathematics is explained by mathematics anxiety in 14% of OECD countries. This suggests that while it is important to analyse and understand

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subject choices at senior secondary level, the issues in regard to avoiding mathematics begin much earlier. The transition from primary to secondary school is well documented as a time of upheaval and distress for many students (Hanewald, 2013; Maguire & Yu, 2015). It is a time when anxiety can manifest itself, impacting on general engagement and hence academic performance (Howard & Johnson, 2004; Hanewald, 2013). This transition can lead to the development of negative perceptions and attitudes towards mathematics and towards school in general (Attard, 2012)

Mathematics anxiety, described as a fear or state of discomfort when faced with mathematics tasks/problems (Hembree, 1990; Hoffman, 2010), is widely accepted as an issue in mathematics education which can hinder the true ability of students. Hoffman (2010) discusses the negative correlation between mathematics anxiety and achievement and also how this anxiety can be triggered by factors such as low self-efficacy and previous lack of success. The concept of self-efficacy, which stems from Bandura's social learning theory (Bandura (1977), affects choices of both activity and behaviour which impacts on how much effort and persistence one applies (Brown & O'Keeffe, 2016). Someone with high self-efficacy is deemed to be more likely to show greater interest in and commitment to working with problems and greater effort and perseverance as they have a "heightened sense of optimism that they can ultimately succeed" (Pajares, 1996, p.326). This is important given that non-cognitive characteristics such as effort and perseverance have been known to predict student success in education in general. For example, Pajares and Miller (1994) found that self-efficacy beliefs about problem solving are a strong predictive indicator of performance. They found that self-efficacy is a greater predictor than factors such as gender or mathematics background or variables such as mathematics anxiety, self-concept, or perceived usefulness of mathematics. PISA 2012 (OECD, 2013) and more recently Bettinger, Ludvigsen, Rege, Solli and Yeager (2018) reiterate that a student's self-efficacy is a predictor of their perseverance and hence on their overall performance in mathematics.

The 2012 PISA survey looked specifically at mathematics self-efficacy and mathematics anxiety, (along with mathematics self-concept among students and student engagement). The data indicated that "almost "30% of students reported that they feel helpless when doing mathematics problems" (OECD, 2013, p.80). Of this 30% it was clear that girls and socio-economically disadvantaged students are more likely to have lower self-efficacy levels. This PISA survey found that girls were less confident at calculation tasks, such as how much cheaper a TV would be after a 30% discount, than they were at abstract/classroom tasks such as solving a linear or a quadratic equation (OECD, 2013, p.83). The findings also indicate that girls were less likely to be confident (75% were confident or very confident) than boys (84%) with such calculation tasks (as evident in the responses to specific survey items). This gender gap was even more evident with tasks that the OCED describe as being associated with 'stereotypical gender roles'.

PISA 2012 also highlighted that 43% of students believed they were not good at mathematics, despite 59% reporting that they get good grades. The data also suggests gender differences for self-concept and self-efficacy, with more boys believing they are good at mathematics than girls. Similar outcomes were found for mathematics anxiety. Students also reported feeling anxious about mathematics class (59%), homework (35%) mathematical problems (31% reported feeling nervous, 30% helpless) and about getting poor grades (61%) (OECD, 2013, p.90). Gender was also a factor (in 56 of the 65 OECD countries, including Australia) with mathematics anxiety, with girls recording higher levels of mathematics anxiety than their male counterparts. The findings also suggest that students in 2012 were more likely to be anxious about mathematics than those in the 2003 survey, with 13 countries, including

Australia, exhibiting statistically significance increase in the mathematics anxiety recorded by their students.

Methodology

A student survey was distributed online to all students (approximately 1780 students) involved in the study. A total of 1,240 Year 7 and 8 students completed the survey; 618 Year 7 students (approximately 880 students sent survey) and 622 Year 8 students (approximately 900 students sent survey). The survey was designed and distributed by the Department for Education and Child Development (DECD), South Australia and the sections relevant to this paper, mathematics self-efficacy, self-concept and anxiety were based on the PISA 2012 survey. Students were asked to respond on a Likert scale of 1 (Strongly Disagree) to 5 (Strongly Agree) and a total of 13 questions were included, as presented in Table 1 below:

Table 1
Survey items

	Secondary school student survey statements	Primary school student survey statements
Mathematics Self-Efficacy	I feel good about myself when I do mathematics.	I am good at mathematics.
	I would consider a career in mathematics.	When I leave school I will need mathematics for my future work
	I can usually give good answers to test questions on mathematics topics.	I learn mathematics quickly.
	Knowing mathematics will help me make good decisions in the future.	Knowing mathematics will help me make good decisions in the future.
Mathematics Self-Concept	I can understand most subjects well, but mathematics is difficult for me.	I can understand most subjects well, but mathematics is difficult for me.
	I am sure I could do advanced work in mathematics.	I am sure I can solve challenging problems in mathematics.
	When I am being taught mathematics, I can understand the concepts very well.	I have always believed that mathematics is one of my best subjects.
	I get good marks in mathematics.	I get good marks in mathematics.
Mathematics Anxiety	I often worry that it will be difficult for me in Mathematics classes.	I often worry that it will be difficult for me in Mathematics classes.
	I don't enjoy trying to solve mathematics problems.	I like to solve mathematics problems.
	I get nervous doing Mathematics problems.	I get nervous doing Mathematics problems.
	I feel helpless when doing a Mathematics problem.	I feel helpless when doing a Mathematics problem.
	I worry that I will get poor marks in Mathematics.	I worry that I will get poor marks in Mathematics.

The questions in Table 1 were part of larger initial survey given to all students involved in a STEM project. The project is, in part, examining; "What impacts on students' understandings and dispositions around STEM are evident from their involvement in the Years 7 and 8 STEM Collaborative Inquiry Project?" The questions were completed by students in Years 7 and 8, across 36 schools in South Australia (5 High Schools, 1 R-12 School, 2 Community Schools and 28 Primary Schools). This paper examines the student responses to the questions in relation

to gender, some reference to the relevant socio-economic status (SES) data is also discussed. The results will be compared to the PISA 2012 results to identify if patterns are similar in the current sample of schools. Statistical comparisons between data sets will not be made given that the survey was undertaken with Year 7 and 8 students while PISA is completed by 15 year olds (Year 9). While not part of this paper a post survey will then examine if being involved in a STEM collaborative inquiry project has any impact on these and is able to narrow any evident gaps.

Findings

Mathematics Self-Efficacy

Table 2 presents the summary data for all Year 7 students (primary school) and Year 8 (secondary school) for their responses to the four self-efficacy statements. The maximum score for each statement was 5 (Strongly Agree) and the minimum 1 (Strongly Disagree). Both male and female primary students report positive self-efficacy scores across the four questions [3.28, 4.05], however the female primary students are less likely to think they are good at mathematics and to believe that they learn mathematics quickly. While similar scores were found with the Year 8 cohort, ranging from 2.64 to 3.90, statistically significant differences between female and male responses were not found in the same questions (see Table 2).

Table 2
Year 7 & 8 Student Self-Efficacy Values by Gender

	Year	**Gender**	**n**	**$\bar{x}$***	***SD****	***p-value***
I am good at mathematics.	**7**	**F**	325	3.28	1.165	
		M	293	3.72	1.127	0.00
I feel good about myself when I do mathematics	**8**	**F**	343	3.33	1.216	
		M	287	3.45	1.258	.251
When I leave school I will need mathematics for my future work	**7**	**F**	324	3.98	1.066	
		M	296	4.14	.977	.062
I would consider a career in mathematics	**8**	**F**	347	2.64	1.107	
		M	291	2.92	1.284	.003
I learn mathematics quickly.	**7**	**F**	325	3.37	1.149	
		M	295	3.69	1.117	0.00
I can usually give good answers to test questions on mathematics topics	**8**	**F**	347	3.22	1.164	
		M	292	3.46	1.158	.013
Knowing mathematics will help me make good decisions in the future.	**7**	**F**	328	3.88	.953	
		M	296	4.05	.990	0.30
Knowing mathematics will help me make good decisions in the future	**8**	**F**	348	3.83	1.023	
		M	292	3.90	1.057	.373

* $\bar{x}$ = mean **SD= Standard Deviation

The data suggests that Year 8 male students are less inclined to believe they are good at Mathematics than Year 7 students, with little difference between the female students. All Year 8 students, both male and female, perceive mathematics to be less likely for their future career than their Year 7 counterparts. This question resulted in the greatest difference between cohorts,

with male scores reducing by over one unit (-1.22) and female scores by an even greater margin (-1.34). Examining this data by SES indicates only one instance of statistically significance differences, with low SES students being less likely to believe they are good at mathematics in both Year 7 (p=0.003) and in Year 8 (p=0.035) than the high SES students. The lowest scoring statement across SES was gain the need for mathematics in future work/careers.

The results are similar to those from the PISA 2012 survey. For example, for the statement "*I learn mathematics quickly*", 52% of the Primary students responded as Agree or Strongly Agree, the same as the average for OECD countries for the same statement and similar to that for Australian average of 54%, (OECD, 2013). The positive response to the statement "*I am good at mathematics*" was also similar, with 53% of students indicating Agree or Strongly Agree, 10% less than the Australian PISA 2012 equivalent and just 4% lower than the OECD average (57%). However, 19% of students indicated that they Disagree or Strongly Disagree that they are good at mathematics, much lower than the 43% OECD average for the PISA 2012 equivalent.

Mathematics Self-Concept

Table 3 presents the summary data for the four self-concept questions for all students. Female Year 7 and Year 8 students are more likely to find mathematics difficult than the male students, with the gap between genders closing by Year 8. Male students are also more likely to believe in their own ability to do challenging/advanced mathematics tasks and in their ability to do learn mathematics, with a statistically significance difference between genders at both Year 7 and in Year 8 (see Table 3). While all students report positive scores for the statement about getting good grades [3.31, 3.70], the gap between genders reduces by Year 8, from 0.31 in Year 7 (with a *p-value* is 0.000) to 0.14 in Year 8 (*p-value* is 0.132).

Table 3
Year 7 & 8 Student Self-Concept Values by Gender

	Year	**Gender**	**n**	$\bar{x}$	***SD***	***p-value***
I can understand most subjects well, but mathematics is difficult for me.	**7**	**F**	329	2.68	1.209	
		M	295	2.47	1.236	.028
I can understand most subjects well, but mathematics is difficult for me.	**8**	**F**	347	2.67	1.398	
		M	294	2.61	1.245	.514
I am sure I can solve challenging problems in mathematics.	**7**	**F**	324	3.44	1.076	
		M	295	3.74	1.016	.001
I am sure I could do advanced work in mathematics.	**8**	**F**	346	2.91	1.240	
		M	292	3.22	1.227	.002
I get good marks in mathematics.	**7**	**F**	326	3.39	1.094	
		M	293	3.70	1.050	.000
I get good marks in mathematics.	**8**	**F**	347	3.31	1.209	
		M	292	3.45	1.125	.132
I have always believed that mathematics is one of my best subjects.	**7**	**F**	329	2.90	1.282	
		M	295	3.49	1.293	.000
When I am being taught mathematics, I can understand the concepts very well.	**8**	**F**	346	3.25	1.153	
		M	292	3.48	1.171	.014

Further analysis of this data indicates that there are also socio-economic differences between students. Year 7 students from low SES schools are less likely to believe they can solve challenging problems (*p-value* is 0.018) or to receive good grades for mathematics (*p-value* is 0.003). The difference between cohorts is reduced by Year 8, with students from low SES schools responding more positively to the statement about good grades in Year 8 ($\bar{x}$ =3.43) than in Year 7 ($\bar{x}$ =3.19). The biggest change is also evident in responses to this statement, with little difference between Year 8 low ($\bar{x}$ =3.43) and high ($\bar{x}$ =3.42) SES. However, Year 8 students from low SES students are less likely to believe they understand what they are being taught in mathematics (*p-value* = 0.025).

Again, comparing like statements to the PISA 2012 counterparts presents similar patterns. For example, for the statement "*I have always believed that mathematics is one of my best subjects*", 40% of Primary school students Agreed or Strongly Agreed which is similar to both the OECD (38%) and Australian (40%). However, the response to the statement "*I get good marks in mathematics*", present lower percentage scores. 53% of Primary students and 50% of Secondary students Agreed or Strongly Agreed with this statement. While this is similar to the OECD average of 59% (OECD, 2013), it is more than 10% lower than the Australian average for PISA 2012 which is 64.5%.

Mathematics Anxiety

The mathematics anxiety data represents the greatest difference by gender in this dataset, with female students consistently exhibiting higher levels of anxiety than their male counterparts. Each of the statements related to mathematics anxiety reflect a statistically significant difference between Year 7 males and females (see Table 4). Year 8 students exhibit similar responses, with all statements except for *I feel helpless when doing a Mathematics problem* present statistically significance differences between gender.

Table 4
Year 7 & 8 Mathematics Anxiety Values: Gender

	Year	**Gender**	**n**	$\bar{x}$	***SD****	***p-value***
I often worry that it will be difficult for me in Mathematics classes.	**7**	**F**	331	3.09	1.202	
		M	297	2.71	1.278	.000
	8	**F**	347	2.83	1.277	
		M	293	2.64	1.243	.057
I like to solve mathematics problems.	**7**	**F**	332	3.25	1.162	
		M	296	3.58	1.135	.000
	8	**F**	347	3.11	1.244	
		M	292	3.28	1.232	.071
I get nervous doing Mathematics problems.	**7**	**F**	330	2.98	1.229	
		M	295	2.57	1.273	.000
	8	**F**	346	2.77	1.306	
		M	286	2.45	1.266	.002
I feel helpless when doing a Mathematics problem.	**7**	**F**	331	2.55	1.149	
		M	296	2.31	1.238	.013
	8	**F**	348	2.53	1.291	
		M	290	2.39	1.210	.175

I worry that I will get poor marks in Mathematics.	7	**F**	329	3.14	1.383	
		M	295	2.65	1.336	.000
	8	**F**	347	3.11	1.368	
		M	293	2.75	1.306	.001

The overall combined mean anxiety scores were also calculated; the maximum 'anxiety score' a student could receive was 25 (five questions answered on a Likert scale 1-5, with statement 2 reversed) which would equate to being extremely anxious and the minimum score is 5 (not very anxious). For female Year 7 students the overall combined mean anxiety score is 14.46 (n=325, SD = 4.9), whereas the combined mean score for male Year 7 students is 12.66 (n=293, SD 4.94). This indicates a statistically significant difference (p= 0.00) between genders, which female Year 7 students likely to be more anxious than their male counterparts. Similarly the combined mean anxiety score for female Year 8 students is 14.1079 (n=343, SD = 4.7), whereas the combined mean score for male Year 8 students is 12.84 (n=279, SD 4.39). This also represents a statistically significant difference (p= 0.01) between genders.

In Table 5 below the equivalent questions from the DECD and the PISA 2012 surveys show a similar pattern, with the South Australian students showing less concern with the difficulty of mathematics and getting good grades but very similar responses in regard to nervousness and helplessness.

Table 5
Percentage of Students with Agree or Strongly Agree Responses to Mathematics Anxiety Statements

	Survey Year 7	**Survey Year 8**	**Australian average**	**OECD average**
I often worry that it will be difficult for me in Mathematics classes.	33	38	59.7	59
I get nervous doing Mathematics problems.	31	27	28.9	31
I feel helpless when doing a Mathematics problem.	18	21	24.6	30
I worry that I will get poor marks in Mathematics.	35	37	61.8	61

Summary and Conclusion

Between the 2003 and 2012 PISA the Australian male student data for self-efficacy and self-concept remained high and unchanged, while that for mathematics anxiety remained unchanged but lower than the OECD average. While, for female Australian students self-concept, which declined from 2003 to 2012, and self-efficacy were both lower than the OECD average and anxiety remained high in 2003 and grew again in 2012.

The South Australian data presented in this paper suggests that male students remain more confident than their female counterparts and while the students present with varying levels of mathematics anxiety, the data suggests this is a deeper issue with female students. While some studies may suggest the gender gap is declining, the data presented here suggests this is still an issue in the South Australian context, particularly in Year 7. The data also indicates that the gender differences are reduced in Year 8. However, this appears this has more to do with males losing confidence than the girls 'catching-up' suggesting more still needs to be done to address issues such as self-efficacy, self-concept and anxiety. This is not surprising as it is in line with Attard's (2012) work on transition from primary to secondary school, however it is significant

to the South Australian context, particularly as SA moves towards integrating Year 7 into secondary school settings (in line with other Australian states).

In summary, it would seem that mathematics anxiety is still an issue for about one third of students in South Australian schools and that gender based differences still persist in mathematics anxiety as well as self-efficacy and self-concept. These differences follow a similar pattern to those identified in the PISA 2012 survey. While this is only a sample of schools the results would indicate that further work is needed to overcome these challenges.

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Investigating Mathematics Students' Motivational Beliefs and Perceptions: An Exploratory Study

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The purpose of this study was to explore the factorial structure of motivation and perception items from a student survey utilised as part of the Reframing Mathematical Futures II (RMFII) Project. Data was collected in 2017 from 442 students in Years 7 to 10 from various different States across Australia. An exploratory factor analysis identified four factors which were consistent with the studies the items were adapted from: Intrinsic and Cognitive Value of Mathematics, Instrumental Value of Mathematics, Mathematics Effort, and Social Impact of School Mathematics. An analysis of variance (ANOVA) also revealed that there were statistically significant differences between Year Level and State for some of these factors.

For many years, researchers in the field of mathematics education have acknowledged the significant role that affective factors play in the teaching and learning of mathematics (Goldin 2002). Although defined in many different ways, the affective research area within this field focuses on "the interplay between cognitive and emotional aspects in mathematics education" (Di Martino & Zan, 2010, p. 1). Based on McLeod's (1992) work, the affective domain is also seen as composed of three major constructs - *beliefs, attitudes,* and *emotions* – with each representing "increased levels of affective involvement, decreased levels of cognitive involvement, increasing levels of intensity of response, and decreasing levels of response stability" (p. 579).

While there are many interpretations for each construct in the literature, they are often considered difficult to define, particularly due to their overlapping nature (Di Martino & Zan, 2010). For example, early definitions of attitudes by Neale (1969) and Hart (1989) embedded beliefs about mathematics as a key element of this construct along with its usefulness to the learner. Hart (1989) also considered "one's emotional reaction to mathematics" (p. 39) in his definition of attitudes. However, in alignment with McLeod's (1992) initial interpretation, Goldin (2002) conceptualised each construct as follows:

> (1) emotions (rapidly changing states of feeling, mild to very intense, that are usually local or embedded in context), (2) attitudes (moderately stable predispositions toward ways of feeling in classes of situations, involving a balance of affect and cognition), (3) beliefs (internal representations to which the holder attributes truth, validity, or applicability, usually stable and highly cognitive, may be highly structured). (p. 61)

While the three constructs of beliefs, attitudes, and emotions have been widely studied within the mathematics domain, they do not cover the entire field of affective research (Zan, Brown, Evans, & Hannula, 2006). *Motivation* is another construct which has had significant implications for student achievement in mathematics although it has not been a prominent field of study within this context (Hannula, 2006; Middleton & Spanias, 1999). As with its other affective counterparts, motivation has also been defined in various different ways in the literature. While Middleton and Spanias (1999) proposed that "motivations are reasons individuals have for behaving in a given manner in a given situation" (p. 66), different theoretical perspectives have varying interpretations for how these reasons may arise. For

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example, the behaviourist perspective sees motivation as resulting from external incentives, such as for rewards or to avoid punishment, whereas the social cognitivist view sees motivation as resulting from a sense of self and self-efficacy (Churchill et al., 2013). Each perspective has been referred to in the literature as extrinsic and intrinsic motivation respectively with the latter seen as valuable in promoting pedagogically desirable behaviours in mathematics such as persistence and risk taking (Middleton & Spanias, 1999).

Regardless of the challenges faced in defining the aforementioned affective variables, there are a number of instruments designed to measure constructs such as attitude and motivation in mathematics. However, as with their definitions, some constructs are measured as part of others. For example, Tapia and Marsh (2004) developed an instrument to explore the construct of attitudes called the Attitudes Towards Mathematics Inventory. They conceptualised attitudes as having four underlying dimensions, one of which was motivation. Additionally, the Fennema-Sherman Mathematics Attitudes Scales, one of the most popular instruments in mathematics education, also views motivation as a sub-set of attitudes, with 12 items on this construct forming one of the nine scales (Fennema-Sherman, 1976). As can be seen from the aforementioned instruments, development of a scale measuring an affective construct is no easy task. Although referring to attitudes, Taylor (1992) makes an important point in that the formation of a construct "is a complex process involving the interaction of many factors. It cannot be explained simply or completely" (p. 12).

With this in mind, the research presented in this paper will examine the motivations and perceptions items from a student survey which was utilised as part of a larger project, and will outline key findings with respect to the variables explored.

Aims

The aims of the study were to investigate:

The factorial structure of the motivations and perceptions items

The existence of statistically significant differences between the derived factors and the independent variables Year Level and State.

Methods

Data Source and Sample

An online survey was undertaken as part of the Reframing Mathematical Futures (RMFII) Project, which aims to find ways to improve the teaching and learning of mathematics for students in Year 7 to 10. The purpose of the survey was to examine students' views regarding their learning experiences in mathematics. The participants came from Australian State and Catholic schools involved in the RMFII project across various Australian States. A total of 442 Year 7 to 10 students from eleven schools across Victoria, New South Wales, Queensland, Northern Territory, South Australia, and Tasmania responded to the survey.

Instrument

The survey consisted of 95 items and was designed by adapting items from instruments developed in prior studies (Dweck, Chiu, & Hong, 1995; Frenzel, Goetz, Pekrun, & Watt, 2010; PISA, 2006; Watt 2004; 2010; Wyn, Turnbull, & Grimshaw, 2014; You, Ritchey, Furlong, Shochet, & Boman, 2011). The survey examined the following constructs: *Mathematics Learning Climate, Friends Perceptions of Mathematics, Perceptions of NAPLAN, Homework, Mathematics Motivations and Perceptions, Gender Perceptions of Mathematics, Personal*

Goals in Mathematics, Mindset, Perceptions of School, Perceptions of Mathematics Teaching, and Mathematics Career.

For the purposes of this paper only the 2017 Mathematics Motivations and Perceptions item responses will be examined. There are a total of 21 items adapted from Watt (2004; 2010) and PISA (2006) examining factors that influence students' perceptions of mathematics and their beliefs about themselves as mathematics learners.

Data Collection

A link to the online survey was provided to participating students by their teachers from February 2017 and it was completed either in the students' own time at home or during class time. The survey was anonymous and students and their respective parents were made aware of the purpose of the survey.

Results

An initial data screening was carried out to test for univariate normality, multivariate outliers (Mahalanobis' distance criterion), homogeneity of variance-covariance matrices (using Box's M tests), and multicollinearity and singularity (tested in the ANOVA analysis). Descriptive statistics normality tests (normal probability plot, detrended normal plot, Kolmogorov-Smirnov statistic with a Lilliefors significance level, Shapiro-Wilks statistic, skewness and kurtosis) showed that assumptions of univariate normality were not violated. Mahalanobis' distance was calculated and a new variable was added to the data file. There were fewer than twenty outlying cases, which is acceptable in a sample of 442 students. These outliers were therefore retained in the data set. Box's M Test of homogeneity of the variance-covariance matrices was not significant at the 0.001 alpha level and we therefore concluded that we have homogeneity of variance. The questionnaire items were subjected to an Exploratory Factor Analysis (EFA) by using SPSSwin. Reliability tests were also conducted. An Analysis of Variance (ANOVA) statistical test was used to investigate statistically significant differences by Year Level and by State.

Exploratory Factor Analysis (EFA)

Given the exploratory nature of the study and that the structure could vary, three factor analyses – one for each of the possible combinations between the three Year Levels (7, 8, and 9) categories (Year 10 was not used because of the relatively small number of students in that category) with sufficient student numbers - were performed in order to investigate possible differences between Year Levels. Since no differences were observed in the three initial analyses, a final factor analysis using data from 438 complete students' responses to the 21 items forming the questionnaire, indicates that the data satisfy the underlying assumptions of the factor analysis and that together four factors (each with eigenvalues greater than 1) explain 72.4% of the variance, with 44.5% attributed to the first factor – *Intrinsic and Cognitive Value of Mathematics* (see Table 1).

Further, according to Coakes and Steed (1999), if the Kaiser–Meyer–Olkin (KMO) measure of sampling adequacy is greater than 0.6 and the Bartlett's test of sphericity (BTS) is significant then factorability of the correlation matrix is assumed. A matrix that is factorable should include several sizable correlations. For this reason (Tabachnick & Fidell, 1996) it is helpful to examine matrices for partial correlations where pairwise correlations are adjusted for effects of all other variables. The Kaiser–Meyer–Olkin (KMO) measure of sampling adequacy in this study is

greater is 0.92 and the Bartlett's test of sphericity (BTS) is significant at 0.001 level, so factorability of the correlation matrix has been assumed.

Reliability analysis yield satisfactory Cronbach's alpha values for each factor: Factor 1, 0.93; Factor 2, 0.90; Factor 3, 0.85 and Factor, 0.80. This indicates a strong degree of internal consistency in each factor.

Table 1
Rotated Factor Matrix (Varimax Rotation)

	Factor			
	1	2	3	4
Factor 1: Intrinsic and Cognitive Value of Mathematics				
Q46 I find mathematics enjoyable	.866			
Q45 I find mathematics interesting	.856			
Q48 I would like to find out more about some of the things we deal with in our mathematics class	.820			
Q51 Being good at mathematics is an important part of who I am	.807			
Q44 I like mathematics more than other subjects	.806			
Q49 I want to know all about mathematics	.803			
Q47 After a mathematics class, I look forward to what we are going to do in the next lesson	.784			
Q52 It is important for me to be someone who is good at solving mathematics problems	.738			
Q50 Being someone who is good at mathematics is important to me	.709			
Factor 2: Instrumental Value of Mathematics				
Q33 What I learn in mathematics is important for me because I need this for what I want to study later on		.854		
Q35 Studying mathematics is worthwhile for me because what I learn will improve my career prospects		.849		
Q32 Making an effort in mathematics is worth it because this will help me with what I want to do		.835		
Q34 I study mathematics because I know it is useful for me		.795		
Q36 I will learn many things in mathematics that will help me get a job		.787		
Factor 3: Mathematics Effort				
Q40 It worries me that mathematics courses are harder than other courses			.856	
Q41 I am concerned that I won't be able to handle the stress that goes along with studying mathematics			.827	
Q39 Achieving in mathematics sounds like it really requires more effort than I'm willing to put in			.735	

Q37 When I think about the hard work needed to get through in mathematics, I am not sure that it is going to be worth it in the end	.696	
Q38 Considering what I want to do with my life, studying mathematics is just not worth the effort	.535	
Factor 4: Social Impact of School Mathematics		
Q42 I'm concerned that working hard in mathematics classes might mean I lose some of my close friends		.903
Q43 I worry about losing some valuable friendships if I'm studying mathematics and my friends are not		.899

The naming of the *four factors* was guided by the relevant literature and the nature of the questionnaire items associated with each factor. This resulted in the following four factors (F1-F4) described below:

F1: Intrinsic and Cognitive Value of Mathematics. The first component consists of nine items, which examine the intrinsic and cognitive value of mathematics. Three of these items were adapted from Watt (2004) and examine students' intrinsic value of mathematics (i.e., how likeable or enjoyable students find the subject). Three items were adapted from Watt's (2010) STEP study which examine the attainment value of mathematics (i.e., how important it is to do well in mathematics). The final three items were adapted from Frenzel, Goetz, Pekrun, and Watt (2010) and examine students' interest in mathematics. All nine items explore mathematical value in terms of personal enjoyment, importance, or interest hence the construct has been labelled to encompass these factors (intrinsic and cognitive value).

F2: Instrumental Value of Mathematics. The second component consists of five items that examine the instrumental value of mathematics (i.e., that the learning of mathematics is valuable for students' futures). The five items have all been adapted from the PISA (2006) questionnaire and specifically examined students' instrumental motivation to learn science subject(s) – the term science subject(s) was replaced with mathematics. These items were the only items taken from the PISA (2006) questionnaire and have loaded to develop a construct consistent with the original study.

F3: Mathematics Effort. The third component consists of five items and examines students' perceptions of the effort required in mathematics. The items were adapted from Watt's (2010) STEP study and examine the "costs" associated with mathematics. Three items examine the Effort Costs and two items examine the Psychological Costs associated with mathematics. As the latter two items can be related to the greater effort expended in mathematics (harder and more stressful) the five items have been grouped together under the overall construct of Mathematics Effort.

F4: Social Impact of School Mathematics. The fourth component consists of two items examining the social impact of school mathematics. These items were adapted from the instrument used in Watt's (2010) STEP study, which specifically examined the Social Cost perceived by students as a result of studying or working hard in mathematics. Consistent with this study, the two items have loaded to form the construct labelled here.

Analysis of Variance (ANOVA) by Year Level

The existence of statistically significant differences on each of the four derived factors by Year Level was investigated by conducting an Analysis of Variance ANOVA statistical test.

The dependent variables (DVs) were the four factors derived from the EFA and the independent variables (IVs) were Year Level (Levels 7-9) and State. Year 10 students' responses have not been used in this analysis because of the relatively small number of students. Effect sizes were calculated using eta squared (η2). In our interpretation of effect sizes we have been guided by Cohen, Manion and Morrison's (2018) proposal that 0.1 represents a small effect size, 0.3 represents a medium effect size, and 0.5 represents a large effect size. We have significant univariate main effects for the following variables:

Factor 3: Mathematics Effort [$F(2, 392) = 6.85$, $p < 0.001$, $\eta 2 = 0.1$]. Effect sizes were calculated using eta squared (η2). The effect size was 0.1 (small effect). A Games-Howell post hoc multiple comparisons test was performed. The purpose of the post hoc tests is to determine which Year Levels are statistical significant different from each other. The Games-Howell test has been used because the Year Level sizes differ. It was found that Year 8 and Year 9 students' scores had significantly different mean values ($p < 0.001$) for Factor 3: Mathematics Effort. It was also found that Grade 7 and Grade 9 students' scores were statistically significantly different ($p < 0.001$) for Factor 3. The mean scores indicate that Year 9 students had a higher mean than Year 7 and Year 8 students. Also, Year 8 students had a marginally higher mean than Year 7 students.

ANOVA by State

The existence of statistically significant differences on each of the four derived factors by State was investigated by conducting an Analysis of Variance (ANOVA) statistical test. The dependent variables (DVs) were the four factors derived from the EFA and the independent variable (IVs) *State.* We have significant univariate main effects for the following variables:

Factor 1: Intrinsic and Cognitive Value of Mathematics [$F(5, 392) = 4.99$, $p < 0.001$, $\eta 2 = .06$]. Effect sizes were calculated using eta squared (η^2). The effect size was .06 (small effect). A Games-Howell post hoc multiple comparisons test was performed in order to explore the differences for each factor. It was found that the New South Wales and the Queensland students' scores had significantly different mean values ($p < 0.01$) for Factor 1. Also, Queensland students had a higher mean than New South Wales students.

Factor 2: Instrumental Value of Mathematics [$F(5, 392) = 4.69$, $p < 0.001$, $\eta 2 = .02$]. Effect sizes were calculated using eta squared (η^2). The effect size was .02 (small effect). A Games-Howell post hoc multiple comparisons test was performed in order to explore the differences for each factor. It was found that the New South Wales and the Queensland students' scores had significantly different mean values ($p < 0.01$) for Factor 2. Also, Queensland students had a higher mean than NSW students.

Factor 3: Mathematics Effort [$F(5, 392) = 2.38$, $p < 0.05$, $\eta 2 = .01$]. Effect sizes were calculated using eta squared (η^2). The effect size was .01 (small effect). A Games-Howell post hoc multiple comparisons test was performed in order to explore the differences for each factor. It was found that the New South Wales and the Queensland students' scores had significantly different mean values ($p < 0.01$) for Factor 3. Also, New South Wales students had a higher mean than Queensland students.

Discussion and Conclusion

Examination of the survey items using an exploratory factor analysis identified four factors, each with eigenvalues > 1 that together explained 74.6% of the variance. The 21 items analysed within this paper were also found to load on factors consistent with those of the studies they

were sourced from (Frenzel et al., 2010; PISA, 2006; Watt 2004; 2010). Each factor explored a different aspect of students' motivations and perceptions regarding mathematics and their beliefs about themselves as mathematics learners. The factors were labelled *Intrinsic and Cognitive Value of Mathematics, Instrumental Value of Mathematics, Mathematics Effort,* and *Social Impact of School Mathematics.*

In addition to exploring the factorial structure of the survey, this study also aimed to examine if there were any statistically significant differences between Year Levels and States on the identified factors. Using an Analysis of Variance (ANOVA), the results revealed that there were statistically significant differences for the factor *Mathematics Effort* between Year Levels, and for the factors, *Intrinsic and Cognitive Value of Mathematics, Instrumental Value of Mathematics,* and *Mathematics Effort* between States.

Further examination using post hoc tests for the Year Level variable showed that Year 9 students had significantly higher mean scores for *Mathematics Effort* when compared to Year 8 students and Year 7 students. These findings are not surprising considering that mathematics becomes more complex as students move into higher year levels and students' may perceive that studying mathematics requires more effort as a result. Comparing the Australian Curriculum Year 9 mathematics content descriptors with those of Year 8 and Year 7, there are many new concepts learned at this higher year level that have not been previously introduced in the prior years (e.g., trigonometry, Pythagoras theorem, non-linear relations), whereas Year 8 students build upon and explore similar concepts to students in Year 7 (ACARA, 2010 to present). The results are also consistent with an Australian study conducted by Watt (2004) who found that, from the end of Grade 7 through to Grade 10, students perceived mathematics as requiring slightly more effort.

Post hoc tests for the State variable showed that the significant differences for *Intrinsic and Cognitive Value of Mathematics, Instrumental Value of Mathematics,* and *Mathematics Effort* were between students from Queensland and students from New South Wales. Students from Queensland scored significantly higher mean values for the first two factors compared to their New South Wales counterparts, but scored significantly lower mean values for *Mathematics Effort*. Thus, Queensland students see mathematics as more interesting and enjoyable, useful for their future careers, and requiring less effort than New South Wales students. Although Yates (2011) commented that different Australian States have approached the curriculum differently depending on what is valued, it is difficult to explain the results between Queensland and New South Wales based on this alone. There may be many other contextual factors than can play a role in developing students' perceptions and beliefs regarding mathematics. For example, one key finding from a review by Middleton and Spanias (1999) highlighted that "motivations towards mathematics are developed early . . . and are influenced greatly by teacher actions and attitudes" (p. 80). Fredricks and Eccles (2002) also suggested that decreases in mathematics task values over time in their study could be explained by the increased competitiveness and evaluation methods used in classrooms as students progress into higher year levels.

In summary, the results from the study have confirmed that the survey items continue to be valid and reliable in the mathematics context as the factors developed were consistent with the studies they were adapted from. The findings also highlight the need for further investigations to examine how students' motivations and perceptions of mathematics develop and differ across the different States in Australia. Having a more representative sample of students from each state across a variety of different Year Levels could provide greater insights into how students' perceptions of mathematics change in different contexts over time.

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Characteristics of Spatial Visualisation: Perspectives from Area of Composite Shapes

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This paper attempted to make explicit some of the underlying characteristics of spatial visualisation using the concept of area of composite shapes. By engaging students with metric-free tasks, we identify the type of perceptual and visual/spatial manoeuvres that they deploy in such situations. Interview data collected from three students in Grade 7, 8, and 9 are used to exemplify three key constituents of spatial visualisation: figure-ground perception, global and local perception, and gesturing. An observable discontinuity was discovered in coordinating different pieces of spatial information after disembedding the parts that constitute the whole. This paper concludes with pedagogical implications.

Imagine the task of finding the area of the region bounded by a square and an inscribed circle or the area of the region bounded by a circle containing a square. As you think about this experimental scenario, consider some of the things that may have come to your mind's eye as you visualized the actions without physically undertaking the task. Such situations may also arise in determining the nets of a cube or in finding the number of lines of symmetry of a figure. The specific spatial ability that is thought to underlie such processing is referred to as *spatial visualisation*. In the mathematics curriculum, a range of such spatial manoeuvres may be encountered especially as part of the geometry and measurement strand.

Spatial visualisation is an umbrella term that includes a range of visual/spatial manoeuvres (Carroll, 1993; Clements & Battista, 1992; Lowrie, Logan, & Ramful, 2017; McGee, 1979; Yakimanskaya, 1991) unlike mental rotation and spatial orientation, which have well-defined conceptual boundaries. Battista (2007) refers to spatial visualisation as "the ability to 'see', inspect, and reflect on spatial objects, images, relationships and transformations" (p. 843). Joining parts of a shape to construct its configuration and folding 2D nets to form 3D objects may constitutively involve different mental operations and both seem to fit the above definition of spatial visualisation. Similarly, imagining cross sections of given objects and anticipating the result of cutting a section of an object may also fit into this category. It appears that the etymology of the term 'spatial visualisation' as involving a visualisation and a spatial component tend to lead to an elusive interpretation of spatial visualisation as spatial reasoning itself. The important and open question then is: what is and what is not spatial visualisation? What may possibly provide partial answers to this question is the unpacking of the ways in which we operate with images in various tasks. Such a research endeavour may potentially elucidate the different kinds of manipulations that currently fall under the label of spatial visualisation.

This study is part of a spatial reasoning research programme that is attempting to unpack the ways in which spatial reasoning plays out in the Mathematics curriculum, more

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specifically in the Geometry and Measurement strand. This paper focuses on spatial visualisation as it occurs in finding the area of composite shapes. Much of the research on composing and decomposing shapes appear to emanate from early childhood and primary education (Clements, 2004; Sinclair & Bruce, 2015). This is not surprising given that such young children are at a stage where they are building the foundational concepts for geometric and spatial thinking. As students move into secondary school, understanding shape composition and decomposition becomes an important aspect of measurement of area. Students are required to have a sound understanding of the geometric aspect of shapes before they can apply the measurement concepts of area (Huang & Witz, 2013). In an attempt to understand how students generate, retain, and process visual/spatial images, and identify the characteristic of spatial visualisation, students engaged with tasks involving the segmentation of areas of composite shapes. Explicitly, the objective of this study is to characterise the type of perceptual and spatial manoeuvres that students deploy in finding the area of composite shapes, with the aim of tracking down the spatial manoeuvres that form part of spatial visualisation.

Analytical Framework

Visual Perception

Visual perception is associated with "the ability to see and interpret" (Hoffer, 1977, cited in, Gal & Linchevski, 2010) and has a strong conceptual link with spatial visualisation. Gal and Linchevski (2010) assert that "organization of perceptual data, recognition, and representation of objects in mind—all are a *sine qua non* base of visualisation" (p. 167). In this study, the composite area tasks required the participants to extract objects from the visual scene or to recognise shapes or objects (as entities). After the perceptual recognition and visual pattern recognition stages, the problem solver may represent this knowledge in terms of verbal/pictorial form or hierarchically in terms of mental images. Gal and Linchevski assert that: "[t]he mental picture that exists in the observer's mind consists of mental cuttings of the original physical (drawn) configuration in question. These mental cuttings separate the decomposed configuration into sub-units" (p. 177). Thus, the representation that we hold of a spatial task may consist of a complex configuration of mental images, parsed by our individual interpretation. It is also important to make the distinction between spatial and visual mental images as basic units of spatial visualisation. Spatial mental images contain information about the location, size, and orientation of entities while visual mental images represent shape information including other physical attributes such as colour and depth (Kosslyn, 1994, cited in, Sima, Schultheis, & Barkowsky, 2013).

Figure-Ground Perception

Visual perception includes, among others, figure-ground perception, perception of spatial relationships, and visual discrimination (Del Grande, 1990; Kovacs & Julesz, 1993). Figure-ground perception is emphasised given its salience in the current study. Figure-ground perception refers to the visual act of prioritising attention on a specific component/shape in a given configuration. Thus, a particular component/shape is foregrounded while others are left in the background. Another mental operation that may be involved in such a visual act is disembedding (Kovacs & Julesz, 1993). This mental operation allows an individual part from a partitioned whole to be lifted from its referent

whole while keeping in mind its relation to the whole. Thus, both the part and the whole can be discerned as separate entities.

Global and Local Perception

Another concept that was relevant in this study is global and local perception (Enns & Kingstone, 1995). Global perception refers to perceiving the overall structure of the scene or image being processed, identifying the spatial relationship among elements and linking them together. Local perception focuses on boundaries of images, contrasts, and individual elements (Nayar, Franchak, Adolph, & Kiorpes, 2015). Nayar et al. (2015) suggested that adults often perceive images in a global form and will identify holistic shapes based on imagined edges "rather than a collection of…local elements" (p. 39). Nayar et al. found that children as young as 10 years of age can move between local and global perception and processing.

Gestures

Gestures give evidence of the spatial representations that individuals hold while solving tasks. Hostetter and Alibali (2008) assert that co-gestures are commonly used by students in working with spatial tasks. Logan, Lowrie, and Diezmann (2014) found that primary school-aged children utilised gesture as a support mechanism for their cognitive processing when engaging with spatial tasks. This tangible form of image making acted as an insight into their spatial thinking and strategy use, providing observable details of some cognitive and conceptual aspects of their learning (Alibali, 2005).

The Context of the Study

This study is situated within a Government Partnerships for Development (GPFD) project funded by the Department of Foreign Affairs and Trade (DFAT) involving University of Canberra mathematics educators working with Indonesian mathematics teachers in a disadvantaged community. Specifically, this study was part of a teacher professional development programme, where the Indonesian mathematics teachers were engaged in classroom action research related to spatial reasoning.

Participants

The three students who are the subject of this paper were high-ability students in their classroom context and were from Grade 7 (S1-female), 8 (S2-female), and 9 (S3-male). Given the preliminary nature of this study, the focus was on high-ability students to have a proxy of the accessibility of the chosen spatial tasks. Similarly, one student from each of the three grade levels was chosen for exploratory purposes. The three students were given a pre-test prior to the interviews to gauge their prerequisite knowledge of area. All of them were proficient in measuring the area of basic shapes such as rectangle, square, circle, and various triangles.

Task Design

The composite area tasks were designed to engage students in coordinating spatial information as they occur in the interpretation of the area of 2D shapes. The intent was to create a context where students would manipulate visual/spatial images which are regarded as the basic units of spatial visualisation. The tasks were metric-free (see Figure 1) and

therefore required students to describe the procedure rather than work with measurements such as the area formula. The tasks were organised into three levels: finding area by segmenting polygonal shapes (Level 1 - e.g., Tasks 1a & 1b); finding area by working with the difference among polygonal shapes (Level 2 - e.g., Tasks 2a & 2b); and finding area by working with differences involving circles (Level 3 - e.g., Tasks 3a-d). Figure 1 exemplifies sample tasks from each of the three levels.

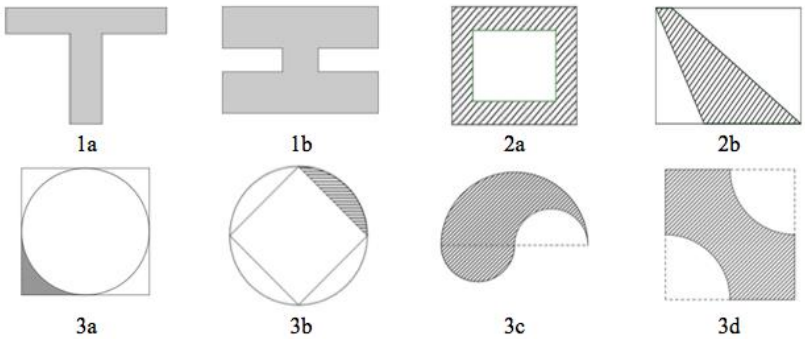

Figure 1. Sample tasks at Levels 1, 2, and 3.

Data Collection and Analysis

The interviews took place at a school in a remote area in West Nusa Tenggara, Indonesia. The interviews were conducted after school hours, and they were anonymously video recorded (faces were not captured). Interview guidelines were developed by the researchers in collaboration with the two teachers working with the participants. After the teachers conducted the interviews, the video recordings were uploaded onto a password-protected shared Google drive. Retrospective analysis of video data focused on the identification of the spatial manoeuvres that the students deployed along with the gestures they made. The figural processing of the tasks was scrutinized, following the written inscriptions that participants made on the paper provided. The interviewers' strategy of occasionally asking students to explain twice how they solved the tasks was beneficial.

Results and Discussion

Three distinct manoeuvres underlying spatial visualisation were identified: (i) figure-ground perception; (ii) global and local perception, and (iii) gestures. The data is presented along the three themes to display their incidence in the students' attempts to find the area of the composite shapes.

Figure-Ground Perception

Figure-ground perception as a visual act was clearly apparent through the explanations and gestures that the students made with their fingers as they foregrounded particular shapes and put others in the background. Figures 2(a) and 2(b) illustrate how students focused on particular shapes in their attempts to find the shaded area in Task 2b. The students used their fingers to show the shapes they considered (which are indicated by the bold outline in Figure 2).

S3: Method 1 is to find the area of the trapezium [trace the outline 2(a)], then subtract with the area of this triangle [trace the outline 2(b)]. The second method, find the area of this trapezium [trace the outline 2(c)], then subtract the area of this triangle [trace the outline 2(d)].

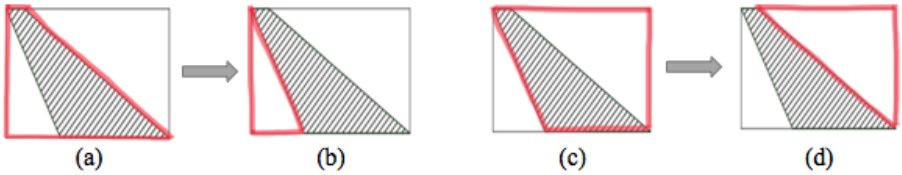

Figure 2. Foregrounding and backgrounding particular shapes by S3 (Methods 1 and 2).

Manipulation of spatial/visual images. An example is provided to illustrate how student S1 handled spatial/visual images. In her first approach to Task 2b, S1 coordinated the three segments as follows (see Figure 3):

S1: First, we find the area of the rectangle [trace the outline 3(a)], then find the area of triangle A [trace the outline 3(b)], then triangle B [trace the outline 3(c)]. Add the area of triangle A and the area of triangle B. So, the area of this (shaded part) is the area of the rectangle subtract the area of A and B.

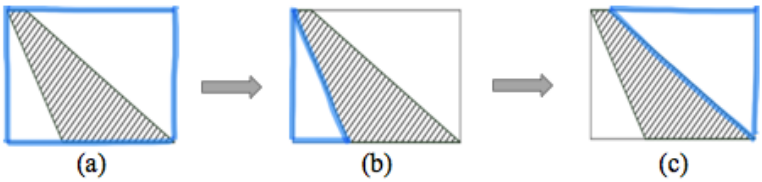

Figure 3. Segmentation of space by S1 in Task 2b using triangles.

When the teacher asked S1 if she had another method, she used two trapeziums within the picture to provide a highly spatial explanation (see Figure 4):

S1: Find the area of trapezium A [trace the outline 4(a)], then find the area of trapezium B [trace the outline 4(b)], and find the area of the rectangle [trace the outline 4(c)]. To find the area of the shaded region is the area of trapezium A, add the area of trapezium B. The result of this addition subtracts with the area of the rectangle.

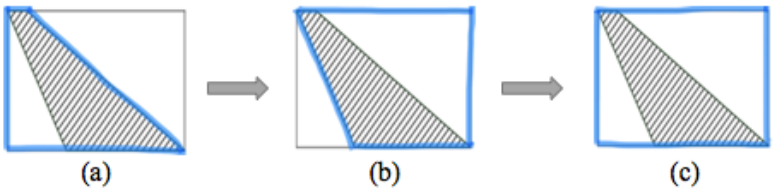

Figure 4. Segmentation of space by S1 in Task 2b using trapeziums.

In this example, the student may have visually superimposed Figures 4(a) and 4(b) and realised that she filled the whole space in the rectangle but counted the shaded region twice. Thus, she subtracted the area of the rectangle to find the area of the shaded region.

Global and Local Perception

Two major patterns were identified from the students' strategy of finding the area of the shaded region of composites shapes, namely, global and local approaches. The local

approach focused on visualising and decomposing smaller parts of the task and particular segments of the shapes. For instance, in Figure 5(b), S1 first split the circle into four parts and focused on one quarter of the circle and a smaller square to explain: “the area of the shaded region can be found from the area of a small square subtract the area of a quarter of the circle.” Similarly, in Figure 5(d), she attempted to split the shaded region internally, drawing a line in the semicircle region to show a smaller circle.

The global approach was identified when students interpreted the task holistically and visualised or added lines or shapes to enclose the graphic. For example, in Figure 5(a), S3 drew lines outside the T shape in Task 1a to construct a bigger rectangle. In Figure 5(c), he drew a curved line to show an enclosing circle and then drew a vertical line to split the circle into four parts. In summary, it appears that S1 tended to approach the tasks locally while S3 tended to use both local and global strategies.

Figure 5. Sample global and local approaches to find area.

(a) Global approach-S3 (b) Local approach-S1 (c) Global approach-S3 (d) Local approach-S1

Gestures

Two types of gestures were evident from the video records: (i) rotation of the composite shapes, and (ii) movement of fingers on the boundary of the shapes (the students would occasionally use their pen to trace such movement). The given shape was rotated to position it in such a way that it allowed them to identify and disembed known shapes or to perform horizontal or vertical segmentation. The rotating action was also apparent when the students were attempting to identify shapes within a shape as illustrated in Figure 6.

(a) (b) (c) (d)

Figure 6. Illustration of rotating the shapes to identify shapes within shapes.

Student S2 first rotated the shape to position it horizontally to highlight: “a right triangle” (Figure 6(a)). Then she rotated the shape three more times in Figures 6(b)-(d) and mentioned: “there are three triangles and one rectangle”.

The gestures indicate that the students were gathering perceptual information from the drawn objects by their sensory system (Gal & Linchevski, 2010). The trace of the fingers (with or without a pen) made explicit the focus of their attention. On many occasions, students S1 and S3 were observed stopping momentarily to focus on the composite shape while rotating it in different directions, particularly when prompted to find a second method to obtain the area of the shaded region. Among the three students, S1 tended to use more gestures.

Disembedding and Assembling a Composite Shape as a Part-Part-Whole Structure

All three students could readily disembed the different shapes from the given composite. However, the apparent discontinuity that one of them (S2) experienced was in assembling the spatial information together as a part-part-whole structure after disembedding the parts. Two examples are provided (from Tasks 2b and 3d) to show this distinct aspect of spatial visualisation in coordinating different pieces of spatial information. In Task 2b, although S2 produced three distinct spatial/visual images, she could not relate them in a part-part-whole structure.

> S2: We find the area of this triangle [trace the outline 7(a)], then find the area of this triangle [trace the outline 7(b)]. Then we find the area of the rectangle [trace the outline 7(c)]. After that, we add those three shapes. Then divided by two because this (pointing to the shaded area) is about a half of the rectangle.

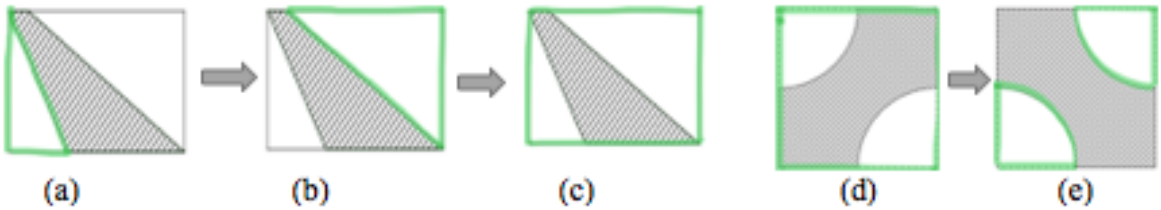

Figure 7. Approach used by S2.

Similarly, in Task 3d (see Figure 1), she could not coordinate the different parts as reflected in the following script:

> T: What shapes can you see?
>
> S2: A square [point 7(d)] and quarters of circle [point 7(e)].
>
> T: If I ask you to find the area of the shaded region, how would you find it?
>
> S2: First, find the area of both the quarter circle [point 7(e)]. Then we find the area of the square, this outside part [point 7(d)], then add them.

Conclusion and Implications

This study provides illustrative examples of the characteristic spatial manoeuvres that underlie spatial visualisation in working with areas of composite shapes. The measurement aspect of area was purposively put in the background to be able to identify the ways in which students manipulate spatial/visual images. The paper attempted to capture at a fine-grained level of detail the perceptual processes that may be in operation in holding spatial/visual images. In particular, Gal and Linchevski's (2010) articulation of perception was found to be pertinent in focussing on students' actions in spatial tasks. Additionally, this study prompted the authors to bring in the concept of local and global perception from Enns and Kingstone (1995). Different students see different things in a particular spatial task. While some operate at the local level, others tend to be more global in approach.

Although the processing of spatial information is different from that of numerical information, there are some parallel strategies that may exist between the two. This study showed instances of the part-part-whole structure as students handled different pieces of spatial/visual information. The part-part-whole structure is well established in the domain of numbers (Baroody, 1999). Although limited to three students, the findings of this study

provide much motivation to continue to unpack the salience of spatial reasoning in its different forms in the mathematics curriculum.

Reflecting on the ways in which the students interacted with the metric-free tasks, the following suggestion is provided that can potentially enhance students' spatial visualisation experiences. Students should be given non-metric experiences to interpret area as the amount of covering in composite shapes and articulate part-part-whole relationships. Textbooks can be helpful in promoting non-metric area tasks. Premature introduction to the quantitative approach to area may lead to an overreliance on numbers. Spatial experiences with area in the form of global and local interpretation may not be naturally occurring and hence may need instructional prompts.

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Diagram Effective or Diagram Dependent?

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This paper will focus on two students who depended on diagrammatic representations in both a Fraction Screening Test and in a subsequent Structured Interview. One student attempted to use diagrams, with limited success, to identify the correct relationships, and consequently struggled to generalise her strategies as she responded to the questions presented in the interview. The other student used diagrams more effectively, and was able to move from a reliance on diagrams to using a partially multiplicative solution strategy. When supported by strong number knowledge, diagrams are an effective means for helping to solve reverse fraction tasks but may hinder students attempts to generalise their thinking.

The links between fractional knowledge and readiness for algebra have been highlighted by many researchers such as Wu (2001); Jacobs, Franke, Carpenter, Levi, and Battey, (2007); and Empson, Levi, and Carpenter, (2011). Siegler et al. (2012) used longitudinal data from both the United States and United Kingdom to show that competence with fractions and division in fifth or sixth grade is a uniquely accurate predictor of students' attainment in algebra and overall mathematics performance five or six years later. This paper focuses on the final stage of an Australian research study that investigated the links between fractional competence and algebraic thinking. For our research, emergent algebraic thinking is defined in terms of students' capacity to identify an equivalence relationship between a given collection of objects and the fraction this collection represents of an unknown whole, and then to operate multiplicatively on both to find the whole. We also anticipated that some students would be able to generalise their solutions, providing even more convincing evidence of algebraic thinking.

Researchers have advocated the use of diagrams as a problem-solving strategy for students solving unfamiliar problems. Diezman and English, (2001) stated that a diagram is a visual representation that presents information in a spatial layout. The appropriateness of a diagram for the solution of a problem depends on how well it represents that problem's structure. Booth and Thomas (2000) suggested that while diagrams are useful for some students, other students may not see the structure of the problem in diagrams or may be unfamiliar with the use of diagrams in the problem-solving process.

This paper will focus on two students, each of whom initially appeared to depend on diagrammatic representations when solving the Structured Interview tasks. We address the research question: *What aspects of the use of diagrams helps or hinders the development of emergent algebraic thinking?*

The Study

In this research middle years' students completed two paper and pencil tests: the Fraction Screening Test and an Algebraic Thinking Questionnaire (Pearn & Stephens, 2015). Later a Structured Interview was used with 45 students from two schools, 19 Year 5 and 6 (10-12 years

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old) students and 26 Year 8 (14 years old) students. One of the students reported in this paper was in Year 5, the other in Year 8. Responses across the Structured Interview tasks revealed that while some students struggled to move on from the additive strategies they used in paper and pencil tests, others used more robust generalisations (Pearn & Stephens, 2017).

The three reverse fraction tasks from the Fraction Screening Test (Pearn & Stephens, 2015; 2017) provided an initial lens into the different types of students' strategies. These are called *reverse fraction tasks* as students need to find the number of objects representing the whole when given the number of objects representing a given fractional part.

Reverse Fraction Task 1	Reverse Fraction Task 2	Reverse Fraction Task 3
This collection of 10 counters is $^{2}/_{3}$ of the number of counters I started with.  How many counters did I start with? Explain how you decided your answer is correct.	Susie's CD collection is $^{4}/_{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? _____ Show all your working.	This collection of 14 counters is $^{7}/_{6}$ of the number of counters I started with.  How many counters did I start with? Explain how you decided that your answer is correct.

Figure 1. The three reverse fraction tasks from the Fraction Screening Test.

Students were chosen to be interviewed only if they had successfully solved at least two of the three reverse fraction tasks. The Structured Interview was designed to investigate whether students who had relied on the use of diagrams or a mix of additive and multiplicative strategies, could because of carefully chosen questions, adopt more consistent multiplicative and generalisable strategies, that are precursors to algebra.

The Structured Interview, included reverse fraction tasks similar to those in the Fraction Screening Test but with progressive levels of abstraction, starting from particular instances and becoming progressively more generalised. The first three questions of the Structured Interview are shown in Figure 2, using the same three fractions as before, without diagrams and with different quantities representing each fraction.

1. Imagine that I gave you 12 counters which is 2/3 of the number of counters I started with. How many counters did I start with? Explain your thinking.	2. Susie has 8 CDs. Her CD collection is 4/7 of her friend Kay's. How many CDs does Kay have? _____ Explain your thinking.	3. Imagine that I gave you 21 counters which is 7/6 of the number of counters I started with How many counters did I start with? Explain your thinking.

Figure 2. Questions 1- 3, Structured Interview

In a second set of three questions (4, 5, and 6), the first part used a new quantity with the same fraction; and the second part started with: "If I gave you *any* number of counters which is also a (given fraction) of the number I started with, what would you need to do to find the number of counters I started with?" Question 4, in Figure 3, is such a question.

4a. If I gave you 18 counters, which is 2/3 of the number of counters I started with, how would you find the number of counters I started with?	4b. If I gave you *any* number of counters, which is also 2/3 of the number I started with, what would you need to do to find the number of counters I started with?

Figure 3. Questions 4a and 4b, Structured Interview.

Students who satisfactorily completed the first six questions of the Structured Interview were asked Question 7 (Figure 4), which required them to use a generalisable method.

What if I gave you *any* number of counters, and they represented *any* fraction of the number of counters I started with, how would you work out the number of counters I started with? Can you tell me what you would do? Please write your explanation in your own words.

Figure 4. Question 7, Structured Interview.

In the Structured Interview, we noted whether students who had relied on additive or subtractive methods, with or without a diagram, used multiplicative methods once the diagrams were no longer provided. We were interested to see whether these deliberately graduated interview questions prompted students to adopt more generalisable methods

Administration of the Interview

The Structured Interview was conducted at each school with four experienced interviewers. At the start of the interview students were shown a copy of their responses to the three paper and pencil reverse fraction tasks. This was then left on the table for students to refer to, if required. The record of interview consisted of interviewers' notes and a three-page document which included the questions and space for students to record their answers and explain their thinking. Students were encouraged to think about, and articulate, their response before writing anything on paper. Students unable to answer Questions 4b, 5b, or 6b were not given Question 7. Each interview took approximately 15 minutes. Students were free to correct their written responses or to exit the interview at any point.

The students' solution strategies for each Structured Interview question were classified using five categories established using the process of the thematic analysis approach suggested by Braun & Clarke (2006). *Diagram dependent* strategies include the use of explicit partitioning of diagrams before using additive or subtractive strategies. *Additive/Subtractive* strategies include those where the student has used addition or subtraction without explicit partitioning of a diagram. Students find the number of objects needed to represent the unit fraction and then use counting or repeated addition to find the number of objects needed to represent the whole. Students using a *partially multiplicative* strategy use both multiplicative and additive methods, by calculating the missing fractional part and then adding it onto the original quantity. Students using *fully multiplicative* strategies find the quantity represented by the unit fraction using division and then multiply that quantity of the unit fraction to find the whole. Students using *advanced multiplicative* methods use appropriate algebraic notation to find the whole, or a one-step method to find the whole by, for example, dividing the given quantity by the known fraction.

Results: Two Case Studies of Gloria and Violet

Reverse Fraction Tasks

In her written response to Reverse Fraction Task 1, Gloria (Figure 5, left), a Year 5 student, used an *additive strategy* to mentally add on another five circles to correctly determine the whole collection was 15. For the same task Violet, Year 8, used a *fully multiplicative* method as shown in the right-hand side of Figure 5. She circled five of the dots given in the diagram and wrote the symbol for one-half above the circled dots. While her written explanation was

brief it demonstrated that she knew that one-third was represented by five dots and she multiplied five by three to find three-thirds or one-whole.

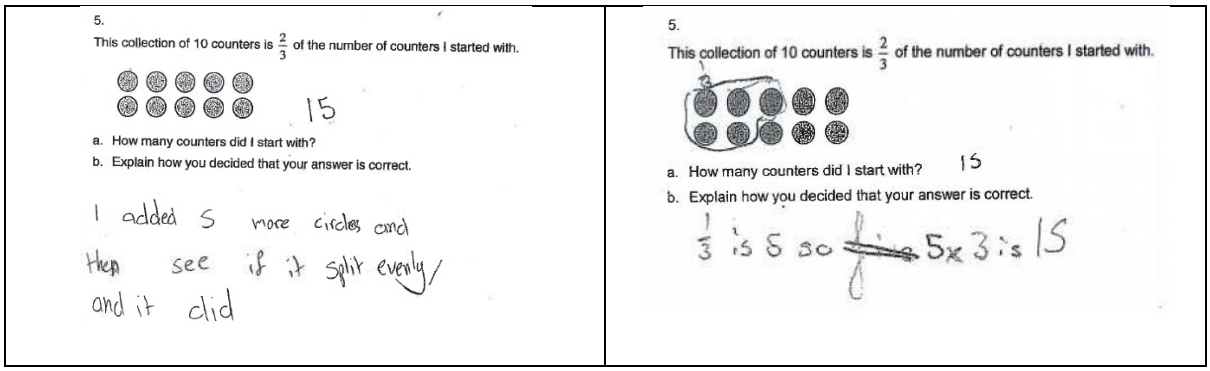

Figure 5. Gloria's and Violet's responses to Reverse Fraction Task 1.

In Reverse Fraction Task 2 both students used a *partially multiplicative* solution strategy. Gloria (left) wrote her solution in words as shown in Figure 6. Violet (right) drew her own diagrams to solve this task. She initially drew four groups of three circles to represent four-sevenths then drew another three groups of three circles to represent the extra three-sevenths needed to represent seven-sevenths or the whole.

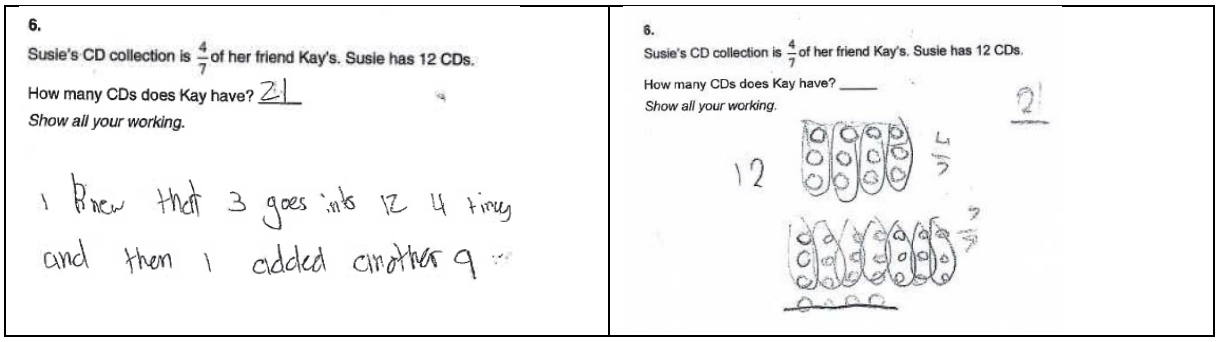

Figure 6. Gloria's solution to Reverse Fraction Task 2.

For Reverse Fraction Task 3 Gloria (left) successfully used a *partially multiplicative* strategy and calculated that one-seventh was represented by two counters which she subtracted from the 14 to get 12 as shown in Figure 7. Violet (right) had several attempts at circling sets of dots in her attempt to solve Reverse Fraction Task 3. Her written explanation appears to indicate she is using a *multiplicative strategy* but there is an element of uncertainty in her response as she has written: "started with 12?"

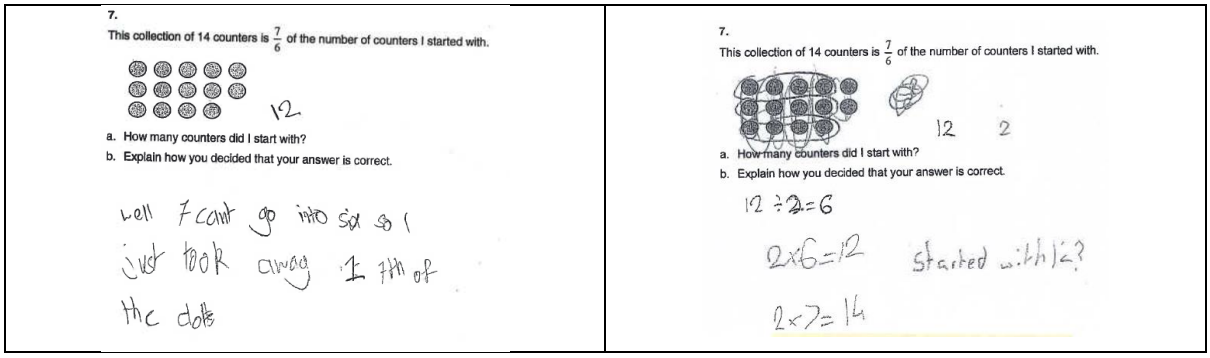

Figure 7. Gloria and Violet's responses to Reverse Fraction Task 3.

Gloria consistently used a *partially multiplicative* strategy for each of the three reverse fraction tasks. Violet used a variety of strategies but needed diagrams for all three tasks. The Structured Interview provided additional opportunities to explore the robustness and limitations of the methods used by these two students in the Fraction Screening Test.

Structured Interview

Gloria successfully solved Question 1 (Figure 2) and gave a *partially multiplicative* response as she halved the number representing two-thirds to find the number of counters representing one-third and finally added both amounts together to get three-thirds or the whole. For the same question Violet used a *diagrammatic* approach. She drew three rows of six circles then drew around two of those rows to indicate two-thirds. She correctly wrote that the initial number was 18 counters.

In Question 2 Gloria initially drew eight circles (left-hand side of Figure 8) and initially tried to place the eight circles into seven equal groups. She then reread the question and drew the eight circles in four groups of two and circled each pair (left-hand side of Figure 8) before adding three more pairs of circles. While she did not write that the total was 14 she stated verbally that the answer was 14 CDs.

Violet's initial attempt at using a diagram in Question 2 is incorrect (shown on the right-hand side of Figure 8). She initially drew an array of five rows of four circles then crossed out one circle from each row. She then added two more rows of three circles to make seven rows of three circles. Violet then drew an additional array of seven rows of three circles before attempting to draw around groups of seven circles. She drew around three groups of seven circles, four groups of eight circles and one group of five circles. These attempts are evidence Violet's struggle with multiplication facts. At this stage, she was encouraged by the interviewer to re-read the question. Violet then correctly drew a row of eight circles to represent the four-sevenths, and divided these eight circles into four equal groups, writing the fraction four-sevenths beside the drawing. She then added a further four groups of two circles underneath the first diagram, crossed out one group of two circles, leaving three groups with two circles to represent three-sevenths and correctly stated that the total is 14 CDs. This *partially multiplicative* method is consistent with her solution to Reverse Fraction 2 shown in Figure 6.

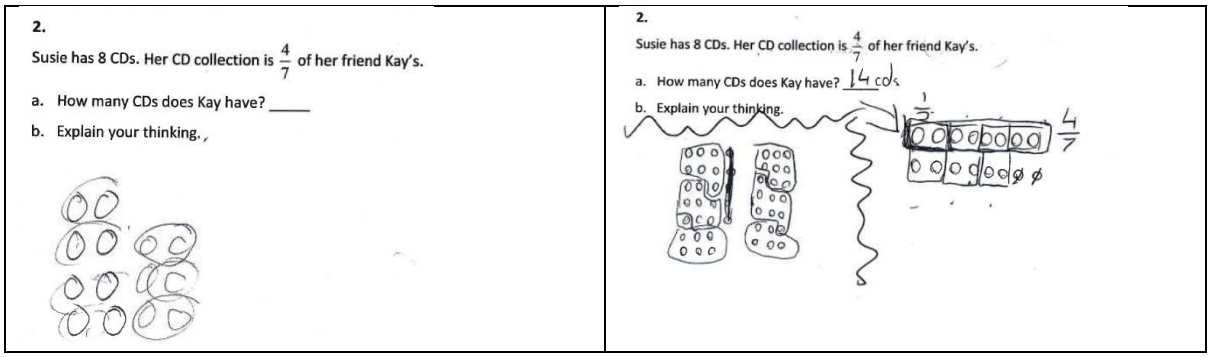

Figure 8. Gloria and Violet's responses to Question 2 of the Structured Interview.

While Gloria was unable to give a correct response for Question 3 Violet correctly partitioned the 21 dots into seven groups of three, recognised that three dots represented one-sixth and correctly stated that there were 18 counters in the whole group. Gloria confidently

answered Question 4 of the Structured Interview using a *partially multiplicative* solution strategy and said: “Half of 18 is 9 so if I add the nine to 18 I get 27 counters. When asked what she would do if she was given ‘any number of counters which was $^2/_3$ of the number’, Gloria confidently responded: “You would halve the number and then add it to that number”.

As shown in Figure 9, Violet constructed four diagrams but the first, second and fourth are incorrect. In the first diagram, she draws three rows of six circles and draws around each row stating that each row represents one-third. In the second diagram she again draws three rows of six circles and attempts to divide these in two parts but unfortunately, she ends up with one group of seven dots and one of 11 dots. The fourth diagram shows three rows of seven dots, which she divided into three groups of seven, then wrote the answer as 14, which is two of the groups of seven or two-thirds of the 21 dots she drew. The third diagram shows three groups of nine, but only after many corrections have been made. She then correctly decided that three-thirds was 27. After finally succeeding with Question 4a using the third diagram in Figure 9, Violet correctly uses an additive solution for Question 4b which asked about ‘*any* number of counters’ representing two-thirds saying: “You would halve the number and then add the result to the number you started with”.

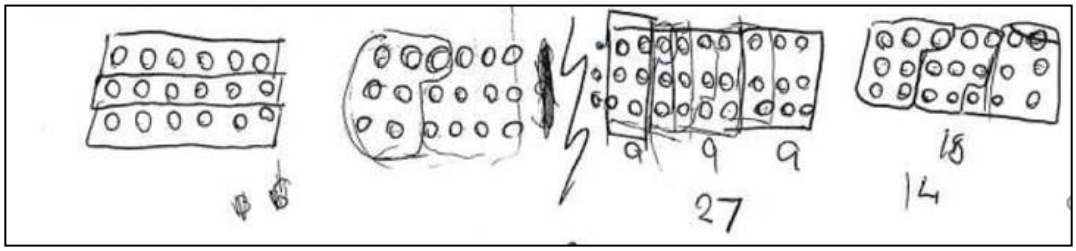

Figure 9. Violet’s responses to Question 4 of the Structured Interview.

Gloria used a *partially multiplicative* strategy for Question 5 (see Figure 11). She drew five rows of four circles, then drew vertical lines highlighting the columns of five to show one-seventh of the whole group. She then verbally added on three more groups of five (15) to the original 20 to get 35 CDs. She explained and then wrote, that to calculate the number of CDs needed in the general case of any number of CDs: “whatever number that you have you have to put it into 4 groups and then add another 3 of the groups”.

In Question 5, Violet (right-hand side of Figure 10) correctly finds one quarter of 20 by halving, and halving again, but represents this as three equivalent expressions. She started to draw a diagram which she then scribbled out before using a *partially multiplicative* method to correctly determine three-sevenths as 15 (5 × 3) and then calculate the whole by adding three-sevenths to the original four-sevenths, which she wrote as 15 + 20 = 35.

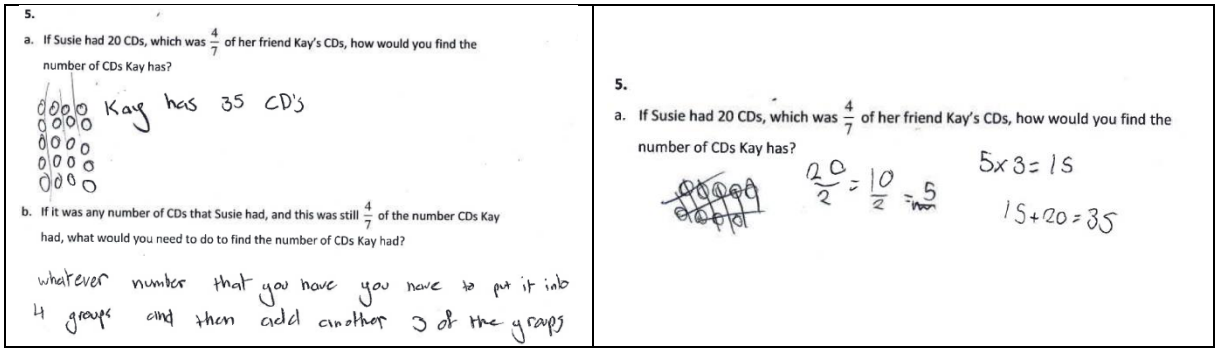

Figure 10. Gloria and Violet’s responses to Question 5, Structured Interview.

After completing Question 5, Violet was unable to complete any further questions and the interview was discontinued. Gloria continued with Questions 6 and 7, successfully responding to Question 6a by drawing seven circles to represent the 70 counters and stated that there would be seven groups with 10 counters in each group. To find the number of counters in the whole group she said that she would need to remove one group of ten to get the answer 60. For the general case of 'any number of counters' representing the fraction seven-sixths in Question 6b she stated: "Put it into 7 groups. However, many in that group take it away from the original number". While this demonstrates her use of the *partially multiplicative* strategy as she calculates one-sixth and subtracts that number of counters away from seven-sixths to find six-sixths. Her successful *subtractive strategy* still appears to rely on a diagram to assist in using this method.

Gloria's response for Question 7 involving '*any* fraction' with '*any* number of counters representing that fraction' showed that she used the same *partially multiplicative* strategy she had used for the previous tasks when she stated: "Whatever the numerator is put it into however many groups you (need) then either add or subtract that number". While this strategy may work for fractions like two-thirds and seven-sixths, with other fractions it is unclear how many times that number might need to be added or subtracted.

Discussion

Gloria's use of diagrammatic strategies draws attention to a clearly established pattern of representing a whole as a composite of its fractional parts. The underlying conception is that of part-part-whole. Diagrams, often with circling, are used to identify, usually successfully, the component relationships; recognising that it is necessary to deduce the value of the unit fraction, in order to scale up (or down) the number of fractional parts to make a whole. Apart from the first multiplicative step to create a unit fraction, all other operations are performed additively. When presented with a known fraction representing 'any number' Gloria explained how the separate parts or components can be combined to make a whole. But in Question 7 when presented with '*any* fraction', Gloria's clearly understood part-part-whole strategies cannot be effectively generalised in the way that a fully multiplicative strategy can be generalised: "Whatever the numerator is, put it into however many groups. You then either add or subtract that number". However, Gloria's confident use of part-part-whole strategies gives her a clear advantage over Violet who needed diagrams to aid her attempted calculations when solving the interview questions.

Violet has difficulty in creating an appropriate diagram to represent the number relationships as required by the Structured Interview tasks, often requiring several attempts. She provided a partially multiplicative solution for the partly generalised task where '*any* number' of counters' represented two-thirds but could not offer a solution for the generalised version of '*any* number of counters' for either four-seventh or seven-sixths. As the numbers changed and became bigger for the two-thirds questions Violet's diagrams became more complex requiring several attempts to partition the numbers.

Conclusion and Implications

As the fractions become less familiar, and the numbers larger, students like Violet who rely on diagrams to partition the numbers, encounter greater difficulty. Proficient multiplicative facts and thinking are needed to partition the numbers to find the appropriate unit fraction and from there to scale up to the whole. When dependence on diagrams is not supported by strong

number knowledge, success becomes limited. Diagrams can be effective when students know the number of objects and the fraction they are representing provided these representations reflect proficient multiplicative thinking. Diagrammatic representations may also help students when thinking about how a given fraction may be scaled up or down to give a whole. However, students who depend on diagrams to scale up or down appear to have difficulty in moving away from part-part-whole *additive or subtractive* strategies. Their diagram dependence seems to prevent them from recognising an underlying multiplicative structure from which a truly generalised solution can be constructed. In this respect, the reliance on diagrams when linked with additive or subtractive strategies may hinder emergent algebraic thinking in the form of a truly generalised solution.

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The Effects of Mathematics Anxiety on Primary Students

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Mathematics anxiety is a known problem in mathematics education. This paper reports on a study looking at mathematics anxiety in a primary school classroom (year 6). Students were given a mathematics test and a following anxiety questionnaire to assess their levels of anxiety, to try to better understand issues that caused anxiety and to understand ways that teachers might help reduce anxiety. Results indicated that assessing anxiety itself could be pedagogically valuable. Results also offered some suggestions for reducing anxiety.

Mathematics anxiety is a significant issue in schools, universities and the work force. Beyond difficulty with poor mathematical ability (Beilock et al., 2010), is a genuine feeling of discomfort, even a phobia (Krinzinger, Kaufmann, & Willmes, 2009).

One of the largest factors in the success of mathematics education is how students feel. When students are relaxed and comfortable, success appears to come naturally, but when students feel stressed, rushed or anxious, the results are very different. Although Maloney and Beilock (2012) stated that stress can boost performance as a physiological response, personal experiences of teaching upper primary students suggest that too much stress reduces performances.

Using the literature, this research paper will define mathematics anxiety, explore whether there is any difference between the genders and if so what that might mean. It will look at the role of the teacher with respect to mathematic anxiety. The review will discuss the available testing of mathematics anxiety, and will try and link the information to primary age students wherever possible.

Literature Review

According to Lyons and Beilock (2012), mathematics anxiety is characterised by feelings of tension, apprehension, and fear about performing math. Wilson (2013) used the work of Dreger and Atkin (1957) and Richardson and Suinn (1972) to define mathematical anxiety as an emotional reaction and feelings of tension and anxiety when doing arithmetic. Among the early researchers of mathematics anxiety, Dreger and Atkin (1957, p. 344), identified "emotional reactions to arithmetic and mathematics". Richardson and Suinn, (1972, p. 551) elaborated "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (Wilson, 2013, p.667).

Mathematics anxiety can present itself in a variety of ways. It can present as a dislike of mathematics or as a worry or pure fear (Wigfield & Meece, 1988; Hart, 1989) due to external pressures placed on a person, such as in a testing situation (Ma, 1999; Krinzinger et al., 2009; Sheffield & Hunt, 2006; Wilson, 2013). Whyte and Anthony (2012) explain mathematics anxiety as the actual situational stress experienced that is specific to personally stressful or fearful circumstances. Research also notes that maths anxiety can affect individuals in varying ways, inducing cognitive, affective, or physical reactions.

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It appears that the teacher and inappropriate teaching practices have a significant impact on the level of mathematics anxiety in students (Hasbee, Sam, Nur, & Tan, 2009; Uusimaki & Nason, 2004; Vinson, 2001). Research found a positive correlation between the level of mathematics anxiety in the teacher and the impact that this played on the students (Beilock et al., 2010). When the students had a teacher, who was anxious in his/her own mathematic ability, there was a downward trend in performance.

If the teacher makes an impact in the classroom, it is the responsibility of the universities and teaching institutions to produce good teachers. It must be incumbent on the institutions to address this issue. One possibility was explored by the University of Birmingham, where they realised there was an issue in their technology program and the student's capabilities in mathematics. Per Metje, Frank and Croft (2007) decided to ask two questions about their students:

> 1) What is the appropriate starting level for this group of students?
>
> 2) How can the students' fear of mathematics be alleviated? (p.80).

Those questions surely need to become the starting point for teacher education in mathematics. Teachers in a classroom could also implement a questionnaire at the beginning of each year, particularly at a primary level to gauge mathematics anxiety. There appears to be no research into assessing how primary students feel at the beginning of the school year, thus the teachers could make appropriate teaching changes that could positively impact students' learning.

Research by Schulz (2005) suggested that the school and teacher play a large part in providing support for the assumption that:

> self-concept and anxiety are relative measures strongly influenced by the school context of the individual. Self-judgements of 'being good at maths' or emotional distress related to this subject depend on self-comparisons with peers and the demands within schools and study programmes (Schulz, 2005, p. 22).

If the school has such a large part in the formation of the student and the levels of mathematics anxiety, how is it measured meaningfully?

There appears to be no research into assessing how primary students feel at the beginning of the school year, but it seems likely that teachers could use such knowledge to make appropriate teaching changes that could positively impact students' learning.

There are two studies regarding mathematics anxiety in primary students that we would like to mention: Krinzinger et al. (2009) examined primary students' anxiety, and Henderson (2012) looked at teacher training at university level in student primary teachers' development of mathematics subject knowledge and the effects of cognition and affect intertwined. Krinzinger and co-authors (2009) explored many ways of assessing mathematics anxiety of a primary student aged 7 to 9 years old. They suggested physiological responses, such as heart rate could be a useful indication (Krinzinger et al., 2009), but that self-report could also be useful.

Research Design

This study employed a similar process to that Beilock and Willingham (2014), who asked questions (modified to the age group) about individual mathematics anxiety (self-efficacy) and then had participants complete a standardised test to see if there was correlation between reported anxiety and test scores. For those students who display mathematic anxiety, a

questionnaire about where and how they think they developed mathematics anxiety might shed some light on their experience and provide guidance for teachers.

The purpose of this paper is to describe how Dave (a pseudonym), an in-service teacher who has 9 Years of teaching experience explored the levels, (if any) of mathematics anxiety whin his classroom. Dave taught Year 6 students at a Catholic primary school in Melbourne. He hoped to gain a better understanding of the anxiety levels of his students, and from that information he would hope to modify his teaching methods to reduce anxiety levels (if they exist).

Therefore, his specific research question for this case study is: "To what level, if any, does mathematics anxiety exist within Dave's classroom?" Given the complex nature of the phenomenon of maths anxiety, and the aim of the study to access the level of mathematics anxiety within Dave's classroom, a design research methodology (Cobb et al. 2003, p. 11) was appropriate to investigate the causes of this anxiety. Design based research embraces the complexity of classroom settings and takes into consideration the complexity of maths anxiety. In this study, we examine the level of mathematics anxiety by: a) iteratively studying the students' interactions with mathematical knowledge; b) establish cycles of iteration, ongoing reflection and feedback to refine learning environment and diminishing students' mathematical anxiety.

Methods

The research study participants were a cohort of Year 6 students of a Catholic Primary School in Melbourne. A total of 26 students (16 Males & 10 Females) ranging between the ages of 11-13 participated in the research study. The data was collected in the participants' setting.

Ethics approval of informed consent procedures was received from the university's ethics committee. The year 6 students conducted an online mathematics test, which consisted of 30 questions covering a wide range of mathematical areas. It included four questions about statistics and probability, nine questions about measurement and geometry, and seventeen questions about number and algebra. Twenty-eight questions were multiple choice (with four options) and two questions were to be answered directly. After the students completed the test, they were asked to use Google Forms to answer to a questionnaire on their feelings about mathematics. The results were collated into an Excel File.

The students were familiar with this form of testing. They used an iPad or a computer to complete the test and the questionnaire. They had access to paper for working outs, but these were not collected.

The questionnaire had 10 questions. Two (#1, #3) were five-point Likert type questions asking the students how they felt during the test and when their teachers says to get out maths books. One question (#2) asked them to predict their result for the test, another (#4) asked them to choose from a list of adjectives describing their feelings about maths. One question asked whether they felt confident doing maths (five-point Likert-type from "need help" to "capable on my own"). Questions #6 and #7 asked students to identify the areas where they struggled or felt confident. Question #8 asked students how the teacher could best help during a mathematics lesson. The final questions asked for age and name (which were not used during the report). Students were also asked which were the most difficult questions. The data were analysed using a mixed methodology approach.

Dave hoped that if there were significant levels of mathematics anxiety, then further research could help explain the causes--whether parental impact, teacher impact, a

significant traumatic event attached to mathematics or some other factor. And if there were reasons, could a whole-school approach help improve overall performance?

Results

The median score on the test was 17, with the most frequent score being 16. The students were asked to predict how they scored. The values for the answers were in groups of 5, with the difference calculated but the top or bottom score in that range. On average, the students in the class predicted that they would achieve a better result by 1.27 points. Only two students made predictions that were significantly different from their scores. Overall this demonstrates that the students have a fair feel on how they thought that they would achieve in the test.

When the students were asked to indicate which question they thought was difficult (Figure 1), from the 30 questions, some students had multiple difficult questions, while one student said that they were all equally difficult. Of the 30 responses, 15 or 50% of the questions were answered correctly by all students. No student got 100% on the test.

Only 1 student reported feeling "completely confident" when doing the test on the questionnaire. He got a score of 28, the top score, and he was one of 2 students who also reported feeling "completely confident" on question #3 ('how do you feel when the teacher says “get out your maths book”"). At the other end of the anxiety spectrum, there were 4 students who scored 2, (close to feeling really anxious/upset) when doing this test. Three of the students got 16 out of 30, while the other student achieved 12 out of 30. All four of the students used the words stressed or overwhelmed to describe how they felt during the test, but only 1 student explained further, stating “It is stressful in test situations”. That student went on to explain that one way a teacher could help would be to not indicate that it was a test. The word test itself caused panic for this student. These 4 students were pretty accurate in predicting how they performed. Two were right within their range, while the other 2 students were only one question out. This shows that some students who don’t feel confident in maths have a pretty good feel of where they are mathematically. This creates a very good building block for teaching and their own learning as their expectations are not too high or too low, hence becoming unrealistic. Working with one of the students for the past year, he has become a lot more accepting of his mathematical ability, knowing that this is an area he needs to continually focus on. At the beginning of the year, almost every time that a mathematics lesson occurred, he would present with high stress and regularly state “I can’t do this” or similar words. He needed a lot of reassurance and positive reinforcement. This is verified with his statement of “by going through it thoroughly” when commenting on how a teacher can help him. Dave had not seen him get stressed out in a mathematics lesson in a while, but judging by his response, testing was stressful. He also responded with a 3 (midpoint between anxious and confident) when the teacher said get out your maths books, but also selected the word 'overwhelmed.'

One of the four students intrigued Dave when looking at his results. The student got 16 out of 30, about where Dave expected. In class, he displayed no reluctance to do maths. He would have a go at answering any question asked of him, and in general his results were in the middle range, or just below middle. He is happy with his successes and willingly took on advice to improve. Therefore, it was a surprise to see him score a 2 on how he felt doing this test, and score a 1 (really anxious/upset) on the "take out your maths book" question. He also used the word stressed to describe how he felt and was relieved when it was all over. When Dave spoke with his co-teacher, she was surprised at the results too, as that student outwardly did not obviously display anxiety.

If we take that anyone who scored a 2 or below on the questions about how they felt with the test or how they feel when the teacher asks to take out your maths books as displaying some form of mathematics anxiety, then 7 out of the 26 or 27% of students presented as such. Their results suggest that their anxiety does have an impact on their test results. The 7 students averaged 13.9/30 and they were all in the bottom half of the class results. Each one of them used either of the words 'stressed' or 'overwhelmed' when describing how they felt during a maths test. It is interesting to note that only 2 of the students chose 2 on whether or not they need help. The rest of the students chose 3. Dave's impression of the 2 students is that they are unlikely to come and ask for help in mathematics lessons. They will greatly appreciate the help when you get to them, but they are not very proactive in asking for help. Again, if the survey was conducted at the beginning of the year, then this could increase awareness.

According to the students, the two most difficult question on the test were:

Qu14 - 'What is 36.15 ÷ 5 equivalent to?'

- 3615 ÷ 5000
- 3615 ÷ 500
- 3615 ÷50
- 361.5 ÷ 500

Qu18 – 'Two items cost $65.75 and $42.83. How much would you pay if there is a discount of 30% on both items?'

- $32.58
- $16.05
- $76.01
- $44.37

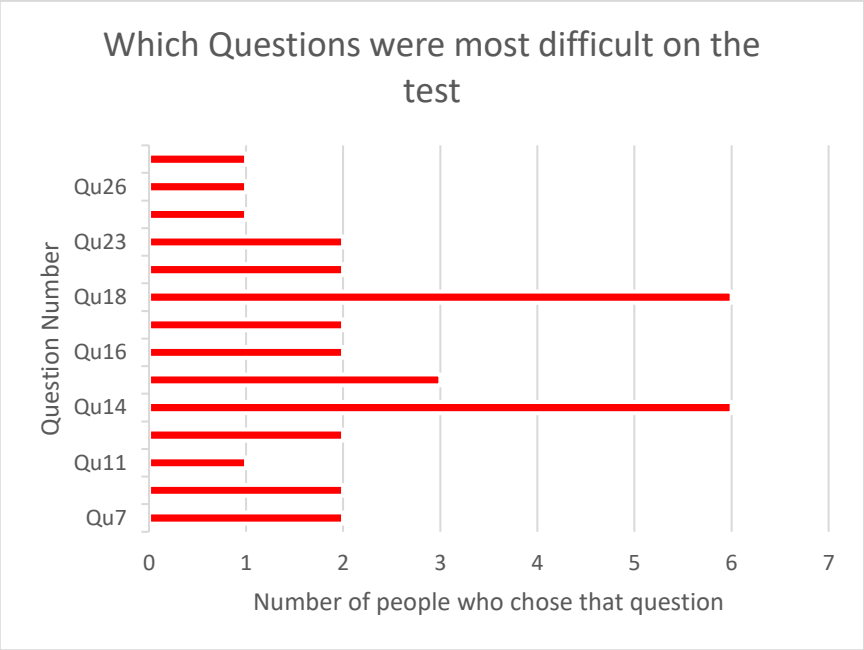

Figure 1. The most difficult questions of the test.

42% of the class got Question 14 correct, while only 31% of the class got Question 18 correct. Of the 7 students identifying themselves with some form of anxiety, no one got question 14 correct and only 1 got question 18 correct.

Both questions had been covered in class throughout the year. The first question is place value and the second is percentage and estimation. With support, Dave assumed most of his students would be able to answer these questions correctly as the issue is more comprehension rather than mathematical skill. For question 14, the students saw it as a division problem. For the class' least favourite aspects of mathematics, division was mentioned 8 out of 26 responses. This is good to know as a teacher, and that it requires work as a class. If the students were feeling less anxious, would they realise that it is more a place value question? This answer could only come from a one on one discussion with each child.

For question 18, if the students used estimation, then finding the answer is quite easy in a multiple-choice situation, especially when there is no response close to the answer of $76.01. It has been discussed throughout the year on numerous occasions to estimate if not sure. $60+$40 = $100. Take away 30% of $100 would leave $70. The only answer close is $76.01. Do the anxious students look at the maths and go it is too hard? Do they freeze up. In regards to testing, are looking at how to work it out, mathematics comprehension or real-life problem solving? Do we as teachers need to place a bigger emphasis on the real-life aspect of mathematics, especially for our more anxious students?

All of the 7 students who reported feeling anxiety were in the bottom half of the results in the test scores, but they were not the lowest scores. It is interesting to note that the 7 students identified as anxious during mathematics tests or lessons struggled a lot more in the number and algebra areas of mathematics. Admittedly there were more questions asked compared to the other areas, perhaps offering greater differentiation in the questioning, but this does present as a clear area of weakness. A reason behind this difference could be in the type of questions presented. For Statistics and Probability, there were 2 graphs shown, which they had to answer questions from. For Measurement and Geometry, there were 2 questions

relating to timetables and 2 questions with a diagram. None of the Number and Algebra questions had graphs or other images. The difference between questions that had some kind of picture or not was striking in this sample for students who identify as suffering some form of mathematical anxiety. For the non-anxious students, there was a smaller difference in performance with questions that were picture based or not.

Finally, the results suggest that poor performance in the test is not a direct indication of mathematics anxiety. The student, whose performance was the least successful, did not demonstrate anxiety. He was not confident at all, but when asked how he felt when the teachers ask to pull out your maths book, he replied with "not fussed." He scored his feelings as 3/5 for both the test and the maths lessons. His expectations of his results were extremely unrealistic, missing the target by 11 points. He only got 5/30 on the test, but thought that he was in the 16-20 bracket. This is with him even considering similar performances on similar tests held previously[6]. Additionally, three students who performed better than average on the test, with one of them scoring the top results, used the words stress and overwhelmed when doing maths. Only one of those students didn't use a positive word to describe how he feels during a maths test. Does this mean that some form of stress can help performance?

Conclusion

One indication from the results was that the use of more pictures while teaching might help those who are little more anxious. With respect to the result that questions with graphs or pictures caused less anxiety, does this mean that anxious students are more comfortable with the visual question in mathematics that with questions that are number based? Does dyscalculia, which "means to count badly and is used to describe people who have difficulties with numbers" Cohen and Walsh, (2007, p. 946) play a part in the anxiety levels? As Soares and Patel explain (2015):

> Children with dyscalculia tend to be less accurate in single-digit subtraction and multiplication than controls and also significantly slower on addition, subtraction, and multiplication. They may also depend more on "immature strategies," such as counting on their fingers to solve problems. The majority of dyscalculic children have problems with both knowledge of facts and knowledge of arithmetical procedures. Difficulty with basic arithmetic is a common characteristic, but dyscalculics appear to perform poorly on tasks requiring an understanding of basic numerical concepts, especially the concept of numerocity. This affects even very simple tasks such as counting or comparing numerical magnitudes (p. 20).

Although this study had a low sample size from a small timeframe, it is clear that there is some anxiety in the classroom when it comes to mathematics.

There might be some value in administrating this questionnaire at the beginning of the year to students in the higher levels of Primary School. This could give an indication of how they are feeling regarding mathematics and could point towards problem areas, providing direction for teaching in mathematics. It could be used as a screening test for the students, giving the teachers some indication on how students feel in mathematics. It would allow the teacher to be more perceptive to how each student internalises their feelings. If we had known how this student felt, would that have enabled us to move him from being an average student to being above average?

[6] Of course, this is based on self-report, and this is the kind of situation where self-report is problematic--is this boy telling the truth about his feelings, or is he projecting a confidence he doesn't actually feel?

As a school, this study might help the teachers to be more aware or mathematics anxiety and it will hopefully lead to an open discussion so it can address the needs of the students.

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The experiences of homeschooling parents when teaching mathematics

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This paper compares and contrasts the experiences of two parents who have chosen to homeschool their children in mathematics despite having difficulties with mathematics at school themselves. It describes the strategies these two parents used to overcome their lack of content knowledge and/or mathematics anxiety and are illustrative examples of how these deficits can be overcome with sufficient motivation.

This paper compares and contrasts the experiences of two parents who have chosen to homeschool their children in mathematics despite having difficulties with mathematics at school themselves.

Homeschooling is a form of education where the parent makes all the decisions about planning, implementation and assessment of their children's education (Department of Education and Training, n.d.). It is different from other distance education programs where others set the curriculum and assessment. In 2016 it was estimated that approximately 17,000 children were registered to be homeschooled (Home School Legal Defense Association [HSLDA], 2017). Although all states and territories in Australia require homeschooling parents to register their children, according to Green (2012) there are many parents who choose not to do this. The number of children who are actually homeschooled in Australia, therefore, could be as high as 50,000 (Green, 2012). Whatever the true numbers are, the number of registered homeschooled children appears to be increasing (Drabsch, 2013).

Parents choose to home school their children for a variety of reasons. These include religious dictates, a desire to protect the child from influences of which they disapprove, a strong belief in the supremacy of the parents' role in a child's education, objections to the socialisation process that occurs in schools, a desire to protect their child from bullying, to provide an individual curriculum for a child with special needs, distance, and travel (Drabsch, 2013).

Parental influences on their children's education

Bandura (1977) describes self-efficacy as "the conviction that one can successfully execute the behaviour required to produce the [desired] outcomes" (p. 193). Mathematics self-efficacy is the judgement of one's capabilities to do mathematics successfully. For students of mathematics, it has been demonstrated that mathematics self-efficacy and achievement are not only strongly correlated, but that self-efficacy is predictive of mathematics achievement (Ayotala & Adedeji, 2009; Liu & Koirala, 2009).

It has been shown that positive parental attitudes to mathematics positively affect the attitudes and beliefs of their children (Soni & Kumari, 2015) and that parental involvement in their children's education has positive effects on their children's outcomes (Cai, 2005; Desforges & Abouchaar, 2003). Unfortunately, however, it is also known that, in general, parents of traditionally schooled children play a more important role in their children's

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language education than in their mathematics education (Cannon & Ginsburg, 2008), possibly due to their own perceived lack of efficacy (Marshall & Swan, 2010). Parents may also hold negative beliefs about mathematics believing that mathematics is difficult, boring, and/or for males (Lim, 2002). Parents may also have had unpleasant experiences in mathematics during their own schooling resulting in low self-efficacy. Parental attitudes and beliefs about mathematics may be passed onto their children (Burnett & Wichman, 1997).

This paper reports on the strategies and the motivations of two homeschooling parents who for varying reasons did not have good experiences of mathematics at school.

Methodology

These cases are drawn from a wider exploratory study of parents' self-efficacy in the mathematics education of their homeschooled children. To obtain potential participants, homeschooling associations in all territories and states in Australia that had an internet presence were contacted and asked to send out an information email to their membership with an anonymous survey link to Qualtrics (https://www.qualtrics.com/au). Once participants had filled out the survey, they were then asked to go to another link and add their contact details if they wished to be interviewed. This process retained anonymity of the participants. As a result, 80 completed surveys were received and 45 volunteered for interview. The participants were selected at random and interviewed until saturation was achieved (eight interviews). Owing to the recruitment process, it is not known how many information letters were sent out to potential participants and from which locations the participants originated. The interviews were transcribed and were coded and analysed using thematic analysis (Braun & Clarke, 2006; Charmaz, 2006).

This paper is based on two interviews. These interviews were selected because, unlike the other interview participants, these participants did not have good experiences of mathematics at school. These interviews illustrate how parents may respond to the challenge of homeschooling their children in mathematics, and how adult mathematics learners in general may respond when sufficiently motivated.

Results

Olivia (not her real name)

At the time of the interview, Olivia homeschooled one child aged 11, assisted with the learning of another primary school child, and had an older child who is now at high school but had been previously homeschooled. Olivia has bad memories of mathematics at school.

> I don't think I learnt anything. All I remember is sitting in maths class at primary school totally filled with dread. Because I didn't know any answers and I didn't know how to work them out and I was too shy to ask the teacher for help…I was terrified I was going to be asked a question in front of the whole class because I knew I wouldn't know the answer. And then in high school I stopped going altogether to maths classes.

Olivia's response to her own experience was to gain a determination that she would not pass her lack of self-efficacy on to her own children.

> I grew up spending most of my early adult life thinking I couldn't do maths and that I just didn't have that ability. I was a bit dumb at maths. And like so many people I just thought it was boring and difficult and I didn't want my own children to have any of these misconceptions. And I didn't want them to struggle with it so I just worked really hard at finding a way to understand it myself and then finding a way to love it and then help the kids and for them it was just easy. My [older child] went on

> to high school and …she's gone on to advanced level maths and she gets all A's, she's really thriving with maths so that gave me a lot of confidence…[that] my approach worked.

Olivia was also keenly aware of the effects of parental attitudes to mathematics on their children.

> I do often think that a lot of people don't understand the need for maths or that it's really not that necessarily difficult or tedious. I come across people a lot who aren't really into it and of course that rubs off on the kids. We know lots of kids that don't do that much maths and I think in general it would be good if people had a better attitude towards it.

She was also keen that her children should want to learn.

> We started off with natural learning and we like the children to be child-led but there's many essentials we just have to learn in life so I just make sure their life is set up and they're exposed to things and conversations that make them realise they do actually want to learn that.

Olivia used manipulatives including "fingers and objects" when she introduced her children to early mathematical concepts. She also liked to approach new topics from the perspective of "maths history". She also stated that the family practices their multiplication tables in the car and that she constantly monitored the children's progress. As the children became older Olivia used conventional text books and second-hand mathematics books that she had found. She did not take recommendations from other people in this selection, but instead went to a "curriculum supply place" and searched for a series of mathematics books that "looked aesthetically pleasing" and were "easy to read, pretty simple to understand to someone without a maths background and the lessons were self-explanatory." By these means she felt that she had found a program that matched the Australian Curriculum and did not have to find a program on her own. Whereas she did sometimes work with the children they generally worked independently. In addition, a neighbour who was "passionate about maths" occasionally assisted with the children's learning. She also stated that her son used the Khan Academy extensively because "He enjoys it".

Emma (not her real name)

At the time of interview, Emma homeschooled three children ranging in age from nine to sixteen years of age. Owing to unspecified circumstances, Emma had spent much of her youth living in places other than her home and so had missed "a lot of school". She did, however, study externally and as a result, had had to teach herself mathematical topics such as logarithms. Despite this history, she felt that she was "pretty good with numbers" and she had "gotten into jobs and things based on numerical testings and things." When it came to teaching mathematics to her children, however, she knew that there were many gaps in her knowledge. Her solution was to do extensive work in building up her own content knowledge.

> Last year I spent the year working through the Khan Academy. They have maths missions on it – I spent the year working on that from kindergarten to advanced Year 12. I got up every morning at 4 in the morning and spent two hours doing maths which is kind of fun…I got up to like integral calculus …and I just finished that in February so I'm heaps more confident now. [When my child asks for help] I'm like 'Oh, I know that!' Because I just didn't have the knowledge before, like I had confidence but I had no knowledge because I missed too much at school.

When her children were younger she used workbooks with them but also supplemented their mathematics learning through the use of play including board games, manipulatives such as counters, and including the children in practical activities such as measurement and budgeting with their own money. She was aware of the affective factors that may influence

her children's learning: "You have to be confident with maths and the kid's kind of have to be prepared to do a bit of maths because it takes a while; it's not fast".

Because of her own successful experiences with teaching herself mathematics it was her aim that her younger child, like her older children, should be able to "teach himself maths from his book." She also used the example of her oldest child.

> And you know like my Year 12 [who] had friends who have done exactly the same maths program at school and…one of them in particular using the same text book actually bombed out and said it was all the teacher and I said ironically [my child] had the same book and doesn't have a teacher and has managed to master the curriculum – just using those books.

She then went on to relate that same child had had good NAPLAN results in the past and was currently studying ATAR accredited mathematics subjects.

Despite her stated aim that the children should become independent, Emma did not have a completely hands-off attitude to her children's mathematical learning. She set a timetable for them each year although her oldest child was working towards doing this for herself. She also looked at the work her children would be doing before they completed it themselves. She marked her children's mathematical work and required them to redo questions where they had made errors so that they "don't make silly mistakes anymore." She encouraged her youngest child to talk through his work aloud which she has found helpful for him, "which would be harder in a classroom because you can't talk." She also chose her children's text books and encouraged them to go online for extra help when required. When choosing their books, she looked for books that had a high emphasis on problem solving and books that came with a solutions manual.

Summary and Discussion

The parents described in this paper both had negative experiences at school but for very different reasons. Whereas Emma had had a disrupted school life but showed high self-efficacy in mathematics, Olivia had a severe dislike of mathematics leading to avoidance of mathematics classes. In this Olivia is illustrative of the research that shows the importance of mathematical experiences in primary school; unpleasant experiences at this time often lead to a life-long dislike and fear of mathematics (Relich, 1996).

Both parents expected their children to work independently, however, they also invested considerable time into their children's mathematical learning. In this they reflect the widespread view that mathematics is important and useful (Coben et al., 2003). They both spent time working with their children and they both reviewed their children's work. Emma encouraged her children in the practical side of mathematics (measurement and money) and Olivia, aware of her own deficits, encouraged the children to "want" to learn mathematics and spent time looking for resources that she herself could understand.

Emma was remarkable for the time she took to address her lack of content knowledge. Olivia was notable for her level of self-awareness and for her knowledge of the importance of affective factors in mathematics and this was reflected in her determination to give her children positive experiences in mathematics. Her opinion that adults can regard mathematics as difficult and tedious is supported by the research literature (Lim, 2002). Emma also addressed the idea that success in mathematics can take time to achieve.

Both participants reported that their children were performing well at mathematics. As evidence Emma referred to her older child's current work to achieve ATAR accredited mathematics subjects and previous performance in NAPLAN tests. Olivia talked about her daughter's success regarding her grades in a traditional high school, and her son's voluntary time spent on a mathematics website.

There is much debate in the literature on the comparative success of homeschooled children in mathematics (e.g., Kunzman & Gaither, 2013). There is some literature to suggest that homeschooled children do well in academic achievement but this is confounded by the Socio-Economic Status (SES) of the participants. Much of the literature relies on studies that are volunteer based, and it appears in at least some of the studies that many of the participants belonged to families where, in general, the parents are well educated and well off financially and had a stay at home mother who took responsibility for the children's education (Kunzman & Gaither, 2013). In this study the SES characteristics of the participants are not known and this is a limitation of the study.

What has been determined, however, is that parental participation is a major factor in their children's academic achievement (Desforges & Abouchaar, 2003). Emma and Olivia both expected their children to work independently, but both of them also worked with their children to different degrees, either learning the work with their children, assessing their work, finding the resources or assisting when problems occurred.

The question arises as to why these parents were prepared to undertake such effort to either upgrade their lack of content knowledge or to avoid the transference of mathematics anxiety to their children. It is not unusual for adults to have negative views of mathematics and this can lead to the reluctance of adults to study further in this area (Klinger, 2011). This reluctance to learn mathematics, however, can be overcome with sufficient motivation; that is, people may become purposeful in the pursuit of their desired goals (Wlodkowski & Ginsberg, 2017). In his description of adult learners Knowles (1970) stated that as people mature, their readiness to learn becomes "oriented increasingly to the developmental tasks of [their] social roles" (p. 55). Hence adults might be ready to learn mathematics in their roles as employees in the workplace (Coben et al., 2003) or may return to formal mathematics education in order to fulfil their current employment needs or to gain a qualification so that they can fulfil their desires for later employment (Ali, 2013). In this context, Emma and Olivia's roles as homeschooling parents desire to do the best for their children, as evidenced by their own mathematical development and their children's mathematical achievements, gave them the motivation to do what they felt necessary to enable them to fulfil these roles.

This paper examined the experiences of two parents who homeschool their children in mathematics but it has wider implications for other adults (including pre-service teachers) who, for various reasons, study mathematics, even if they have had poor experiences at school. What motivates these adults, and to what extent can this motivation overcome previous mathematics anxiety and lack of content? More research in this area would be worthwhile.

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Examining Nondominant Student and Teacher Agency in a U.S. High School Mathematics Classroom

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Shifting classrooms towards places where students' ideas are at the centre of the discussions has been challenging for teachers to put into practice (Hiebert & Wearne, 2003). This study explores the attempts I made as a researcher-teacher to promote equitable student-driven whole class discussions in a racially and socioeconomically diverse classroom. Using ethnographic methods, I examined the research question: *What are the successes and challenges a teacher encounters when attempting to promote student agency in an Integrated Mathematics 1 classroom?* I used one transcript from the end of the year to illustrate examples of student and teacher participation and areas for improvement.

Existing mathematics classroom practices currently perpetuate global social inequities (Apple, 1995) by privileging ideas and dominant ways of being from Eurocentric groups of people (Lipka, 1994). This limits the number of students who are successful under traditional instructional models (Tate, 1997). Mathematics classrooms tend to prioritize rote practice of procedural skills over critical thinking or conceptual understanding (Hiebert & Wearne, 2003). In the U.S., students in low-SES communities or in urban areas receive less exposure to problem-based lessons or whole class discussions than their more affluent counterparts (Lubienski, 2000). Yet many STEM education practitioners and researchers have argued that reform-based practices, such as student exploration of procedural and conceptual ideas, can strengthen learners' mathematical understandings (e.g. Moschkovich, 1999; Stein, Grover, & Henningsen, 1996). The recent Common Core Standards in the U.S. state that all learners should speak about mathematics in ways that students believe are meaningful and rigorous (Yonezawa, 2015).

Asking students to explain what they know can strengthen mathematical conceptions for all participants in the classroom (Kazemi & Stipek, 2001). This constant press to justify one's thinking is most effective when it becomes part of the normative expectation of the classroom practice (Yackel & Cobb, 1996). Sociocultural practices, such as whole class discussions and small group collaboration, require a slow release of scaffolds as students learn how to take responsibility for their own learning processes (Hufferd-Ackles, Fuson, & Sherin, 2004). This is especially important for students who have been marginalized from traditional mathematics who may lack the confidence and skills required for verbal participation (Ball, 1993; Planas, & Gregorió, 2004). I use the term marginalised synonymously with nondominant or underserved to describe any person outside of the dominant norm regarding race, class, gender, language, ability, sexuality, educational background, and other underserved groups.

In this paper I support student-centred discussions of ideas, finding that instructional practices that empower students through responsive instruction is possible within an integrated high school mathematics classroom. Drawing on a teacher action research project that I implemented as a researcher-teacher (Ball, 1993) in two Integrated Mathematics 1 classes, I describe how the classroom community can make student agency a tool for

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learning, while also attending to the challenges that prevent some teachers from shifting mathematics education towards student-centred spaces where students can critically engage with mathematical ideas.

Relevant Literature

Sociocultural Learning Theory

In recent years there have been a growing number of research studies where scholars have touched upon the ways sociocultural learning practices can support mathematical understanding. I build off Vygotsky's (1978) sociocultural learning theory, claiming that one constructs individual ideas after interacting with others, to examine the ways whole class discussions support learners' development. Mathematics educators have pushed for a balance between procedural and conceptual fluency by urging practitioners to press students to discover multiple solution strategies (Stein et al., 1996) and explain their thinking regarding those methods (Kazemi & Stipek, 2001). Discussion over students' ideas occurs most organically during open-ended problems where there are multiple entry points into the problem (Boaler, 1998). An interactive environment requires the teacher to become a facilitator of the discussion rather than a transmitter of information (Truxaw & DeFranco, 2008).

I define equitable whole class discussions as places where all students are free to express mathematical ideas, not just those who have traditionally been successful in mathematics. This study draws on the work of Cohen and colleagues (1999) in relation to the idea of eliminating status so that all learners feel comfortable sharing their mathematical ideas out loud, regardless of traditional positions of power. Additionally, the work of Hufferd-Ackles and colleagues' (2004) is drawn upon, with the idea of a math-talk learning community which stresses the importance of utilising students' mathematical ideas to guide discussions. Whole class discussions are effective when all learners have opportunities to share their mathematical ideas, not just those who are fluent in the language of instruction or those who score high on achievement tests (Ball, 1993; Planas, & Gregorió, 2004).

Agency

Classrooms that promote student interactions allow more opportunities for students to have agency over their own learning processes. I use Turner's definition of mathematical agency: students who construct rigorous mathematical understandings and who participate in mathematics in personally and socially meaningful ways (Turner, 2003). Holland and colleagues (2008) refer to the different ways people take up roles in various situations as the agency they have in each of their Figured Worlds. They explain that from one context to the next, people choose to act in certain ways depending on how they identify with those situations. Thus, the behaviours students choose to engage in fluctuates depending on how personally and socially meaningful the structures (tasks, norms, activities) are in the classroom. Dialectically, the structures also change depending on how participants choose to interact (Varelas, Tucker-Raymond, & Richards, 2015).

Because of this fluidity, I acknowledge Solórzano and Solórzano's (1995) contribution, that human agency is the confidence and skills to act on one's own behalf. Therefore, learners who lack confidence in their mathematical abilities may be less inclined to contribute to whole class discussions in mathematically meaningful ways. This is especially true for groups of people who have traditionally been marginalized from mathematics.

Hence, I rely on Yackel and Cobb's (1996) distinction between social and mathematical norms to separate when participants contributed to the social expectations of the classroom and when learners expressed rigorous understandings of the mathematical material.

Methods

This study explored the ways students from a diverse range of ethnic and socioeconomic backgrounds contributed to the whole class discussions by focusing on the agency employed by the students and me, the researcher-teacher. I simultaneously implemented instruction and collected artefacts in a U.S. classroom to better understand what was required to implement reform-based practices. This paper addresses findings regarding the research question, what are the successes and challenges a teacher encounters when attempting to promote student agency in an Integrated Mathematics 1 classroom? In exploring the comments from one of the transcribed audio recordings, I aim to show how these utterances 1) related to the ways students and I chose to employ agency in one of the two Integrated 1 classrooms and 2) influenced how I interacted with and facilitated lessons for my students.

Action Research

Lampert (1985), Ball (1993), and Chazan (2000) were three prominent action-researchers who focused on problem-solving discussions during an era of mathematics education reform. Each of the researchers facilitated classroom discussions around cognitively demanding problems for the purpose of deepening students' conceptual understandings. Gutstein (2003), Frankenstein (1990), and Brantlinger (2013) taught mathematics using social and political issues. Their goal was to provide a space for students to discuss and reflect on social inequalities within our global society. Lessons that develop students' critical consciousness use mathematics to examine social injustices, such as comparing proportions of liquor stores to movie theatres in different neighbourhoods.

My contribution to the field combines the two goals of deepening students' mathematical understandings and teaching for social justice. I extend Lampert, Ball and Chazan's work of deepening students' mathematical understandings by shifting my lens from the mathematical richness of the problem being discussed to examining the ways students offered up their own conjectures to a problem, as stated in the Common Core standards (Common Core, 2010). Rather than focusing on topics of discussion designed to raise students' critical consciousness, like Gutstein, Brantlinger, and Frankenstein, I expanded upon their pedagogical beliefs by choosing instructional methods that gave all students opportunities to use their voice. Overall, I believed that creating an environment where all students felt comfortable making mistakes, asking questions, and critiquing each other's reasoning (Lampert, 1990) could strengthen their mathematical understanding and simultaneously empower those who have historically been left out of traditional mathematics curriculum (Gutierréz, 2002).

Context

I was the researcher-teacher for two Integrated Mathematics 1 classes at a racially and economically diverse public high school in northern California in the United States. This paper will focus on the findings from one of these two classes. The racial demographics in the Integrated Math 1 class were: 52% Black, 29% Latino, 16% Asian/Pacific (Samoan, Filipino and Chinese), and 3% white. Seventy-three percent of students were on free-and-reduced lunch. There was a mixture of 9th -12th grade students with a majority of ninth

graders. The number of students in one period of Integrated Math 1 fluctuated around thirty-five. The upperclassmen were students who had not yet passed algebra and were required to retake the class leading to a wide distribution of skills and knowledge among students in the class.

The range of skills, ages, and racial and class backgrounds contributed to the ways students chose to engage with the social expectations of the classroom. Fifty percent of the class wrote on their mathographies that they have never felt successful in mathematics. Some students expressed that they did not feel fluent in mathematics by saying comments such as, "I need to go back to third grade. I don't know how to multiply. That's why it doesn't make sense when you're talking about writing equations," or "I just don't apply myself in this class." These perceptions of mathematical and academic status created tensions that challenged my goal to shift the classroom to an interactive learning space. The range of skills, ages, and racial and class backgrounds varied.

The district was in its second year of implementing Integrated Mathematics classes using the Carnegie Learning curriculum (Bartle, 2012). My lesson-planning process focused on preparing students for the department chapter test. I closely followed the district's pacing guide using the suggested text while supplementing some activities with tasks from the Integrated Mathematics Project (IMP) curriculum (Fendel, Resek, Alper, & Fraser, 2003), Discovering Algebra (Kamischke & Murdock, 2007) texts, or Rethinking Mathematics (Gutstein, 2006). I used problems that allowed opportunities for students to discuss their mathematical understandings in small group and whole class interactions.

Data Collection and Analysis

I collected daily audio recordings, daily lesson plans, and I wrote daily field notes for the purpose of documenting the ways students talked about mathematics. Field notes were first used to categorise themes that emerged from the data (Strauss & Corbin, 1990) and the patterns from these were used determine ten days of audio recordings to transcribe. Strategic sampling (Merriam, 2008) was used to identify three whole class discussions from the beginning, middle and end of each semester. I transcribed four more discussions from second semester where I noticed students contributing who did not usually participate. I coded these ten transcriptions to find examples of student agency – moments where students took initiative to contribute to the discussion without being required by the teacher. I use one example from the last quarter of the school year to illustrate some of the sociomathematical norms that were evident in our classroom.

Results

In this example, Martin (pseudonyms were used for all participants) stood up front and wrote his conjecture for the transformation used in *Figure 1* below. He wrote, 'the shape rotated 180 degrees.' Jasmine stood up front and recorded the names of people who participated. Tiana read the prompt from her seat. I started by standing in the back of the room:

> Tiana: Describe in detail the steps you took to find a, b, c, d, e in figure A.
>
> T: It's prime. Say prime.
>
> Tiana: Oh, a prime, b prime, c prime, d prime, and e prime.
>
> T: Thank you Tiana. Martin, what type of transformation do you think happened from this shape to this shape? What do you think happened?

Martin: I'm going to stick with what I said first. One hundred and eighty degree rotation, that's what happened.

T [to class]: Do you agree or disagree, 180-degree rotation?

[Audience members raised their hands]

T: Yeah. Martin, why do you think that?

Martin: It's because if you rotate it, if you have like, something to rotate, it would go exactly the opposite of this, but facing this way, I mean, not exactly, but facing this way on the other side of the graph.

T: Okay, one of our tools is transparency paper. Can you show us a 180-degree rotation? (Audio, 4/18/16).

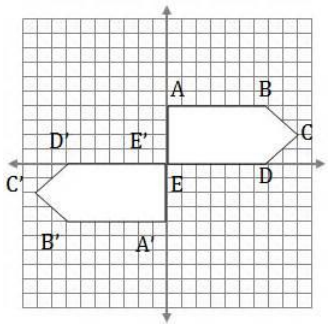

Figure 1. Picture of task during 'Transformations' discussion.

In the above transcription, Martin started the discussion by standing up in front, pointing to the graph and his written answer. After Martin explained why he believed the transformation was a 180-degree rotation, Jerry and KC joined in by sharing different conjectures with the class.

Jerry: You can reflect over the x-axis and the y-axis.

[Martin wrote, 'reflection' on the white board]

T: Reflect over the… reflect over the x-axis and the y-axis. So reflect and reflect.

[T walked up front to label the graph with arrows to visually depict what Jerry said]

Jasmine: And it's a rotation?

T: You could say either one, reflection or rotation.

Martin: So … to say it exactly, wouldn't [it] be over the y-axis?

T [used patty paper on the white board to explore Martin's understanding]: If you reflect it over the y, it would still be...

Martin: Oh! Then reflect it to the x (audio, 4/18/16).

Jasmine, Martin and I stood up front paraphrasing Jerry's idea using words, pictures and patty paper. By revoicing Jerry's input, I hoped that his ideas were valued and that the audience internalized the variety of solutions.

Last, KC offered a third description from the audience, "I was going to say it can also be a reflection over y then x" (audio, 4/18/16). I was happy to see multiple students willingly share their own ideas. KC, Jerry, and Martin took the initiative to offer their thoughts, without waiting to be prompted by the teacher. Additionally, Jasmine's facilitation of her peers' ideas was also a way for

her to play a central role in the discussion. She chose to stand up in front of the class with a marker providing a visual for the audience to pay attention to while three students spoke. Jasmine did not exhibit the confidence or skills to share her own mathematical ideas.

Discussion

Some of my goals to create an environment where students had opportunities to employ agency over their own learning processes came into fruition. As the teacher, I chose to stand on the side to prompt students to take the lead in the discussion. I positioned the audience as authorities over the knowledge by asking them to evaluate the correctness of Martin's conjecture, "do you agree or disagree, 180-degree rotation?" I also encouraged Martin to explain why he thought that transformation made sense. I purposefully chose to create an environment prioritising student sense making with the hope that students would be agents of their own learning processes. I used moves such as physical positioning and prompting to cultivate an organic discussion of mathematical ideas, similar to how mathematicians interact when determining correctness of proofs (Lampert, 1990).

Students employed agency when they volunteered their ideas without any prompting. Martin started the discussion by stating the type of transformation he thought was displayed. Jerry and KC offered two alternative transformations. Although the multiple transformations offered lacked rigor, students' contributions in the whole class discussion contained some mathematical power because the students drove the conversation (Hufferd-Ackles et al., 2004). Jasmine played an important role in the discussion by standing up in front of the class as a scribe. Although her participation did not offer mathematical insight, she chose to engage in the social expectations (Yackel & Cobb, 1996) of a student-centred classroom.

There are challenges involved when interacting in a spontaneous dialogue. After reflecting on what occurred, I realized there was more I could have done to facilitate an effective discussion of ideas. For instance, I took authority over the knowledge when I responded to Jasmine's question about multiple answers when I said, "you could say either one, reflection or rotation." In hindsight, I could have asked the audience to respond to her question. This would have created an opportunity for more students to share their thoughts in addition to the five who spoke out loud. Second, KC's statement, "I was going to say it can also be a reflection over y then x," would be more precise if KC used the terms, "x-axis" and "y-axis." Lastly, the task itself was not very cognitively demanding. This interaction would be strengthened if there was more student authority over the knowledge (Gutstein, 2003; Gutierrez, 2002), academic language (Common Core, 2010), increased cognitive demand (Stein et al., 1996) or cultural relevance (Ladson-Billings, 1995) of the task.

Conclusion

Shift Towards Student-Centred Discussions

The classroom excerpt shared above illustrates ways that I attempted to shift the Integrated Math 1 classroom from a traditional teacher-centred space to a community where students had authority over the knowledge being discussed. Some of the norms that the students and I co-created to promote discussion were evident in the transcript, such as students up front using their ideas to guide the discussion. It was an expectation that more than one idea was discussed. As the teacher, my role was to press students to justify their ideas using complete sentences, tools (patty paper) and diagrams. My decisions were based on supporting students in purposeful ways.

Challenges

The above-mentioned challenges of implementing effective whole class discussions are real. The spontaneity of the discussion and the low cognitive demand of the task only partially matched my overall goal to encourage students to make sense of the math being discussed. The transcript is not an exemplar of an ideal situation. Rather, it exists as a demonstration of real classroom norms that were established with the goal of encouraging students' ideas to be discussed. This example can be strengthened and extended in other classrooms that share the same goal of striving to cultivate student agency.

Limitations

The evidence captured on the audio recording or written into my daily field notes limits my assessment of student agency. Paying attention only to spoken ideas does not capture the entirety of student thinking. One must remain conscious of which students feel confident and skilled enough to share their ideas out loud. My decisions and the classroom structures further marginalized some students, which limited the agency they were able to employ. With this limitation in mind, I continue to search for ways to support teachers and students to co-create classroom norms that all learners negotiate as they find productive ways to interact in their math classroom environments.

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Measuring Mental Computational Fluency With Addition: A Novel Approach

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Measuring computational fluency, an aspect of procedural fluency, is complex. Many attempts to measure this construct have emphasised accuracy and efficiency at the expense of flexibility and appropriate strategy choice. Efforts to account for these latter constructs through assessing children's computational reasoning using structured interviews (e.g., MAI), are necessarily time-intensive. In this paper, we introduce a novel measure of Mental Computational Fluency with Addition (MCF-A) that attempts to incorporate these aspects by requiring children to reason from the perspective of another child. We describe results of a pilot study using the MCF-A with 169 Year 3 and 4 students.

Computational fluency has been described as receiving more attention and attracting more controversy than perhaps any other topic within mathematics education (Boerst & Schielack, 2003). Whether computational fluency should be defined broadly, and incorporate conceptual understanding, or narrowly, and focus exclusively on quick and accurate recall of basic facts and procedures, has been a particular point of contention amongst educational researchers, with the latter definition frequently prevailing (Clarke, Nelson & Shanley, 2016). At least in part in response to this debate, the National Council of Teachers of Mathematics (NCTM) has consistently sought to explain and clarify what is meant by the term. In its Principles and Standards for School Mathematics released nearly two decades ago, NCTM describes computational fluency as a "connection between conceptual understanding and computational proficiency" (NCTM, 2000, p. 35). The Standards go on to emphasise the interplay between concepts and procedures for achieving computational fluency: "computational methods that are over-practiced without understanding are often forgotten or remembered incorrectly... on the other hand, understanding without fluency can inhibit the problem-solving process" (p. 35). Despite substantial empirical support for its stance (e.g., the mutual interdependency of conceptual and procedural knowledge; Rittle-Johnson, 2017), the NCTM felt the need to further clarify what was meant by 'fluency' in a 2014 position paper, in response to continued inconsistency in the terms interpretation and operationalisation. It introduces the term "procedural fluency" to emphasise that fluency is relevant to all areas of mathematics, not just mental computation. Procedural fluency is defined as "the ability to apply procedures accurately, efficiently, and flexibly… and to recognize when one strategy or procedure is more appropriate to apply than another" (NCTM, 2014, p. 1). This definition mirrors that of the National Research Council (2001), that emphasises the importance of students using procedures "flexibility, accurately, efficiently and appropriately" (p. 116).

This paper briefly evaluates attempts within educational research to measure fluency against these four aspects highlighted by the NCTM (i.e., accuracy, efficiency, flexibility, and strategy appropriateness), before putting forward a novel measure of computational fluency that we assert embodies all of these components. Data from a pilot study trialling this new measure, which we term Mental Computational Fluency with Addition (MCF-A),

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is then presented and discussed. The paper concludes by discussing the practical implications of the measures potential use in classrooms, as well as highlighting possible directions for future research.

Attempts to Measure Computational Fluency

As noted earlier, computational fluency has often been defined narrowly, leading to corresponding measures to focus exclusively on speed and accuracy of number fact recall.

Calhoon, Emerson, Flores and Houchins (2007) refer to the NCTM Standards when defining computational fluency, however rather than describe the link between conceptual understanding and computational proficiency, they focus on the statement that describes computational fluency as "having efficient and accurate methods for computing" (NCTM, 2000, p. 152). They subsequently interpret this to mean that "to be computationally fluent, students must correctly answer mathematics problems at an identified level of difficulty within a given time period" (Calhoon et al., 2007, p. 292). Consistent with this narrower understanding of fluency, in their study of 224 high school students with mathematics disabilities, Calhoon et al. employ the Mathematics Operations Test–Revised (MOT-R) as their fluency measure. This test includes 50 items covering the four operations in whole numbers and decimals/ fractions, covering curriculum expectations from Year 2 to Year 6. The items are organised according to topic and difficulty, and students are given 10 minutes to complete as many items are possible. In their study, Calhoon et al. score the MOT-R such that performance is measured by the number of questions correctly answered in the time allocated.

Foster (2017) actually referred to the NCTM position paper when describing computational fluency as the ability to perform a procedure "accurately, efficiently and flexibly" (NCTM, 2014, p. 1). However, in his two studies of 377 high school students which focussed on the number and algebra domain, fluency was again measured narrowly by assessing performance on four routine tasks, to be completed within an approximate time limit of 10 minutes. Students were given one point for a correct answer, and their overall performance was marked out of four.

Codding, Burns and Lukito (2011) undertook a meta-analysis examining the impact of interventions on basic fact fluency. Their analysis included 17 single case-study designs covering 55 participants, all of whom were elementary school students identified as "struggling with computation fluency" (p. 38). Although the authors do not attempt to define explicitly the term computational fluency in their study, part of their inclusion criteria for their meta-analysis was that studies include the metric of digits correct per minute in their results, and this became the sole outcome variable the authors' evaluated. Consequently, it would appear that Codding et al. (2011), and perhaps the studies that comprised their meta-analysis, have again emphasised the accuracy and efficiency aspects of fluency at the expense of flexibility and strategy choice.

Beyond neglecting flexibility and appropriate strategy choice, there are additional issues with the above measures of computational fluency worthy of comment. In particular, whether the notion of efficiency can be simply equated with speed of recall, or performance on a timed-assessment, is problematic. For instance, Star (2005) makes the point that what is meant by the term the "most efficient strategy" is quite nuanced and context dependent. He asks: "Is the most efficient strategy the one that is the quickest or easiest to do, the one with the fewest steps, the one that avoids the use of fractions, or the one that the solver likes the best?" (p. 409). Star goes on to propose that there are "subtle interactions among the

problem's characteristics, one's knowledge of procedures, and one's problem-solving goals" that lead an individual to implement a particular procedure in a particular context (p. 409).

In more general terms, whether time-limited tests have actually been devised to capture 'efficiency' and serve as a proxy measure for the fluency construct, or whether they have simply been loosely interpreted as measuring fluency post hoc, is also contentious. Clarke et al. (2016) argue that an assessment that is timed is frequently interpreted as a fluency measure, when it is actually intended as a broader measure of mathematical performance. The authors highlight that often time-limits are imposed on performance assessments simply as an artefact of ensuring the measure possesses particular design characteristics to enable it "to be administered frequently and repeatedly over time" (p. 79). Arguably, this remains problematic because measures which rely on timed tests may also provoke maths anxiety, meaning that we are underestimating the true ability of maths anxious individuals when we rely on these measures (Ashcraft & Moore, 2009). We suggest that this provides a powerful reason to move away from using time-based assessments if it can be demonstrated that there are other reliable means of probing the phenomenon of interest.

It should be acknowledged that we ourselves have recently completed a study that claimed to measure fluency by focussing on the completion of routine tasks within a given time limit, with one point given for a correct answer, and overall performance marked out of 25 (Russo & Hopkins, 2018). Consequently, our above critique should certainly not be interpreted as an attempt to undermine the careful work undertaken by our colleagues; rather, we acknowledge from our own firsthand experience that measuring computational fluency in a manner that comprehensively captures the construct as defined by the NCTM (2014) position paper is a complex and difficult business.

Narrow interpretations and measurements of computational fluency, such as those described above, including our own work, are perhaps what have led to concerns than an emphasis on fluency may constitute a "threat to reform approaches to the learning of mathematics" (Foster, 2017, p. 122). However, there have been attempts to assess computational fluency in a manner that values flexibility and strategy choice, in addition to accuracy and efficiency. For example, in the early 2000s, a team of Australian researchers developed the Early Numeracy Interview, which comprehensively assessed the mathematical knowledge expected of students up to around Year 4 (according to most curriculum standards) against pre-identified growth points (Clarke et al., 2002). More recently, the interview was updated and renamed the Mathematics Assessment Interview (MAI; Gervasoni & Perry, 2015). Four of the nine sections of the interview relate to number (Counting, Place Value, Addition and Subtraction and Multiplication), and the interview's emphasis on mathematical reasoning, appropriate strategy use, and mental calculation in the context of solving arithmetic problems means that these number strands can be perhaps broadly equated with assessing computational fluency.

Although the MAI and its predecessor are valuable tools for classroom teachers to assess students' knowledge and skills, and focus teaching accordingly, there are at least three concerns with co-opting the tool as a measure of computational fluency for research purposes. Firstly, the interview is undertaken individually with children, and takes between 30 and 45 minutes depending on a child's particular responses (Gervasoni & Perry, 2015). It is therefore time and resource intensive to administer for research purposes. Secondly, these interviews were developed first and foremost as a tool for assessing students in the early years (Gervasoni & Sullivan, 2007). Consequently, there are comparatively limited items covering multi-digit reasoning. Finally, the purpose of the interview as primarily a

formative assessment tool means that many students only complete a small subset of the items, limiting its utility as a measure in a research context.

It should be noted that the large amount of not missing at random data inherent in the MAI data collection process does not necessarily render it unsuitable as a measure of computational fluency, with sophisticated multiple imputation techniques available to researchers to correct for this issue (Rose & Fraser, 2008). However, the a priori requirement that such statistically complex adjustments be made certainly impacts the likelihood that it will have general appeal as a measure beyond studies which have already collected large amounts of MAI data. Specifically, a measure relying on large amounts of imputation would lack transparency, require considerable statistical expertise to calculate, be complex to interpret, and may face reliability issues in the face of researchers making idiosyncratic adjustments to their own individual data sets (e.g., choosing unique sets of imputation variables).

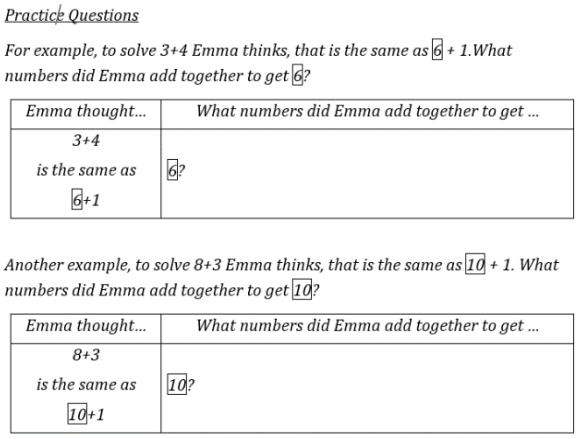
Consequently, there remains a need to develop a reliable, valid and cost-effective measure of computational fluency which captures all aspects of the definition of fluency put forward by the NCTM, particularly the neglected flexibility and strategy choice components. In the following section, we introduce our MCF-A measure and describe how it meets the aforementioned need.

Introducing a Measure of Mental Computational Fluency with Addition (MCF-A)

The MCF-A is a 25-item measure of computational fluency in the context of solving addition problems, developed primarily for students in Year 3 and above. At the beginning of the MCF-A, children are given the following information:

> You will have about 30 minutes to have a go at the 25 questions on this assessment. As always, we just want you to try your best. Before beginning the assessment, we will do two practice questions together as a whole class.
>
> Emma is good at adding numbers. She uses clever strategies to make adding easier. Your job is to try and think like Emma did. Explain what Emma did to get the number in the box.

Children are then presented with the two practice questions, which are completed as a whole class (see Figure 1).



Practice Questions

For example, to solve 3+4 Emma thinks, that is the same as $\boxed{6}$ + 1.What numbers did Emma add together to get $\boxed{6}$?

Emma thought…	*What numbers did Emma add together to get …*
3+4 is the same as $\boxed{6}$+1	$\boxed{6}$?

Another example, to solve 8+3 Emma thinks, that is the same as $\boxed{10}$ + 1. What numbers did Emma add together to get $\boxed{10}$?

Emma thought…	*What numbers did Emma add together to get …*
8+3 is the same as $\boxed{10}$+1	$\boxed{10}$?

Figure 1. Practice questions for the MCF-A.

Children have to first determine the strategy used by Emma, and represent their thinking by recording how Emma obtained the number in the box. Consider the first practice question. A child might realise that, when confronted with the problem 3 + 4, there are several potential solution strategies. One might directly retrieve the solution ("7") count-on from the larger number ("5, 6 ,7"), count-on from the first number ("4, 5, 6, 7"), use the near doubles strategy subtracting from the known doubles fact ("4 + 4 – 1"), or use the near doubles strategy adding to the known doubles fact ("3 + 3 + 1"). However, it is clear that because Emma has realised "3 + 4 is the same as 6 + 1", Emma has used the near doubles add strategy. The child would record "3 + 3" in the space provided, indicating that this is how Emma obtained the intermediate solution 6.

Now consider the second practice question. When confronted with the problem 8 + 3, a child may again realise there are a range of possible solution strategies. However, the fact that the child knows that Emma has noted that "8 + 3 is the same as 10 + 1" means that he or she can be confident that Emma has bridged through 10. Unlike in the first practice question, when only one intermediate step was consistent with the information presented to students (i.e., 3 + 3), there are in fact two possible means by which Emma could have bridged through 10: 8 + 2 or 7 + 3. This question, therefore, has two acceptable answers.

Across the 25 items included in the MCF-A, a range of mental computation strategies are covered to capture the thinking of Emma, including: doubles, near doubles, number bonds equalling 10 (and multiples of 10), bridging through 10 (and multiples of 10), compensation (overshoot), compensation (change both numbers), jump strategy and split strategy. Moreover, the items contain a mix of 1-digit, 2-digit and 3-digit addends, and are presented in anticipated order of difficulty (least to most difficult). In terms of calculating a total score of computational fluency, children are awarded one point for a correct answer, with the measure being scored out of 25.

We contend that the MCF-A effectively operationalises all four aspects of the computational fluency definitions put forward by the NCTM (2001, 2014) and National Research Council (2001). Firstly, it requires *flexibility*, as children need to generate a range of potential solution strategies when considering the original addition problem. Secondly, it emphasises *strategy appropriateness*, as children select one strategy from a range of alternatives that appropriately mimics Emma's thinking, and leads them to the same intermediate step as Emma. Thirdly, it demands knowledge of *efficient* strategies, because the assessment has been designed such that the intermediate step used by Emma represents an efficient means of solving the addition problem. Finally, the child has to *accurately* recall or calculate the appropriate number fact corresponding to the intermediate step arrived at by Emma. For example, a child who incorrectly recalls that 3 + 3 = 5 will not be able to identify the strategy used by Emma in solving the first practice problem.

Method

One hundred and sixty-nine children (boys = 84; girls = 85) in Years 3 and 4 from three Melbourne public Primary schools completed the MCF-A as part of a larger study into addition strategies. The schools covered a broad range of demographics, with one school community relatively advantaged, one relatively disadvantaged, and one school similar to the national average. The MCF-A was administered to participants in Terms 2 and Terms 3 in large groups (5 to 20 students) by the first author. Although students were told they would be given "about 30 minutes" to complete the assessment, this was intended and enacted as a 'soft' time limit, with students being given additional time if they required it. From anecdotal observations made by the first author, any students requesting more time appeared to make

very little progress with the assessment beyond what they had achieved in the first 30 minutes, suggesting that imposing a 'hard' time limit would not have yielded different results.

Results

Table 1
Descriptive Statistics for MCF-A

Group	N	Mean	SD	Median	Range	Participation Rate
Low SES school	31	11.13	6.53	11	0-23	38%
Medium SES school	60	10.40	6.56	10	0-22	71%
High SES school	78	13.00	6.37	15	0-22	53%
Year 3 students	79	11.46	6.50	11	0-23	45%
Year 4 students	90	11.98	6.60	13	0-23	66%
Overall	*169*	*11.73*	*6.54*	*12*	*0-23*	*54%*

Table 1 displays the mean scores, median scores, range and standard deviations for the MCF-A measure, delineated by different student groups. The overall mean score was 11.7, and the median 12, indicating that the middle performing student was correct on about half the items. Although the differences between mean scores across year levels might be lower than expected, and differences in mean scores across schools at times counter-intuitive (i.e., the low SES school outperforming the medium SES school), there was evidence that this was a consequence of differing participation rates amongst the groups of children (higher achieving students were more likely to consent to participate in the study).

Table 2
Individual Items on the MCF-A

No.	Item	Correct response(s)	% Correct	No.	Item	Correct response(s)	% Correct
1[a]	4+5=8+1	4+4	87.0	14[a b]	48+45=90+3	45+45; 48+42	39.6
2[b]	4+7=10+1	4+6; 3+7	74.0	15[a]	36+37=72+1	36+36	26.0
3[e]	9+3=13–1	10+3	17.8	16[e]	44+49=94–1	50+44	19.5
4[a]	5+8+5=10+8	5+5	73.4	17[c]	76+21=96+1	76+20	43.8
5[a]	6+3+6=12+3	6+6	75.1	18[d]	56+33=80+9	50+30	45.6
6[b]	8+3+2=10+3	8+2	71.0	19[e]	22+49=21+50	22-1	29.0
7[e]	8+19=28–1	8+20	21.9	20[c]	23+44=64+3	20+44	35.5
8[b]	4+28=30+2	2+28; 26+4	62.7	21[b]	67+45+43=110+45	67+43	30.2
9[d]	10+11=20+1	10+10	78.1	22[d]	45+67+82=180+14	40+60+80	24.9

10[c]	28+13=38+3	28+10	56.2	23[a,b]	235+238=470+3	235+235; 232+238	28.4
11[d]	21+26=40+7	20+20	59.2	24[d,b]	456+356=800+12	450+350; 456+344; 444+356	24.9
12[b]	11+3+9=20+3	11+9	65.1	25[b]	955+445=1000+400	955+45	22.6
13[b,d]	6+18+34=40+18	6+34; 30+10	62.1				

Note. a = doubles or near doubles strategies; b = number bonds equalling (multiples of) 10 or bridging strategies; c = jump strategy; d = split strategy; e = compensation strategies.

Table 2 presents the full list of items from the MCF-A, alongside the corresponding strategy, solutions and percentage of students obtaining a correct score on the item. Recall that the items were presented in anticipated order of difficulty, such that we would expect Item 1 to have the highest percentage of correct scores, and Item 25 the lowest percentage. This was generally the case, with the exception of the compensation items, which were more difficult than anticipated. For example, consider Item 3, which we expected to be the third easiest item, and for approximately three-quarters of students to respond with a correct answer. In fact, less than one in five students were able to correctly respond to the item: "Emma thought 9+3 is the same as 13-1; how did she get the 13?" by identifying that Emma had added 10 to the 3 to make 13, before 'paying back' the 1 to get her final answer (that is, she applied the compensation – overshoot strategy). This suggests that the compensation items (Items 3, 7, 16 and 19) should have been included later in the assessment.

Finally, the Cronbach Alpha for the MCF-A measure was excellent (α= 0.92). Moreover, Cronbach Alpha could not have been improved through removing any of the 25 items.

Discussion and Future Directions

As highlighted in our introduction, previous efforts to measure computational fluency for research purposes have either defined the construct narrowly (e.g., Calhoon et al., 2007), or been time-intensive to administer (e.g., Gervasoni & Sullivan, 2007). We have attempted to demonstrate that it is possible to create a reliable, time-efficient measure of computational fluency with addition (MCF-A) that operationalises all four aspects of the definition put forward by the NCTM (2001, 2014); that is, flexibility, strategy appropriateness, efficiency and accuracy. There are, however, a number of future research directions that warrant consideration.

First, although we have presented evidence that the MCF-A is a valid and reliable measure, the next step would be to expose it to a more rigorous evaluation of its properties as a measure through Rasch analysis. This will allow the MCF-A to be further refined and improved. Second, there is a need to apply these same design principles to develop a measure of computational fluency within other number domains, such as multiplication. Third, the correlation between scores on the MCF-A and other less comprehensive measures of computational fluency (e.g., recall of addition facts) should be examined. We are intending to explore these three ideas in the near future.

Finally, in terms of its practical implications, although the MCF-A measure described in this paper has predominantly been developed with a research purpose in mind, it may be of value to classroom teachers as a formative assessment tool. As noted by Russell (2000) teachers "cannot simply accept any student method" but rather they "need to analyse what a student's procedure reveals about their underlying understanding so that (they) can plan the next steps in instruction" (p. 157). The MCF-A meets this need through offering insight into which addition strategies students are able to execute with a high degree of proficiency, and which strategies require further exploration and consolidation. For instance, within our sample, results indicate that many students need more exposure and practice with compensation strategies in particular. Thus, although certainly not intended to replace tools designed for more comprehensive formative assessment (e.g., MAI), the MCF-A is arguably of more value to teachers than other standardised measures used as proxies for fluency (e.g., MOT-R), as the latter tend to solely emphasise outcomes (i.e., correct or incorrect) over process (e.g., strategy choice).

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Sense-Making in Mathematics: Towards a Dialogical Framing

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This paper presents a new theoretical viewpoint blended from the perspectives that mathematical meaning is extracted (from objects falling under a particular concept) and that mathematical meaning is given (to objects that an individual interacts with). It is elaborated that neither uni-directional framing (whether involving extracting meaning or giving meaning) provides a comprehensive account of the complex emergence of evolving forms of meaning. It is argued for a framing that construes sense-making in mathematics as dialogical: where what meaning one extracts is a function of what meaning is given to, and vice versa.

Sense-making in mathematics has been a critical theme in research on mathematics knowing, learning, and teaching. Schoenfeld (1992), for instance, discussed mathematics as an act of sense-making and underlined sense-making activities as vital for students coming to understand and use mathematics in meaningful ways. Von Glasersfeld (1995), on the other hand, regarded students as active sense-makers in mathematical concept formation, that is, students actively seek comprehensibility of a mathematical concept. Though consideration of sense-making in mathematics has a long tradition in, and is undoubtedly an essential topic of, mathematics education, the notion of sense-making is somewhat ambiguous, often framed in opposing perspectives. Two of those perspectives are the substance of this paper that are grounded in a division of two strands of mathematical concept formation (i.e., abstraction-from-actions approaches and abstraction-from-objects approaches). Recently, Scheiner (2016) moved the discussion from simple comparison towards a synergy of theoretical frameworks that acknowledges the complementarity of the two strands of mathematical concept formation. Specifically, Scheiner (2016) blended theoretical frameworks on two fundamental kinds of abstraction (reflective abstraction and structural abstraction) and their respective forms of sense-making (extracting meaning and giving meaning). This blending argues strongly against dismissing abstraction from objects as irrelevant for mathematical concept formation, and instead aims to overcome misleading dichotomies of abstraction from actions and abstraction from objects, as Piaget (1977/2001) put forth.

This paper contributes to the current conversation of the relation between extracting meaning and giving meaning. The paper makes a case for a dialogical framing of these two forms of sense-making that has the potential to account for the complex dynamics involved in mathematical concept formation, dynamics which cannot be accounted for considering extracting meaning and giving meaning separately. In doing so, some theoretical assertions are outlined that orient the general discussion of concept formation and sense-making. Afterwards, explicit and implicit assumptions underlying the respective forms of sense-making are examined. Then, the dialogical framing of extracting meaning and giving meaning is delineated, revealing the complex dynamics involved in mathematical concept formation.

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Theoretical Orientations

The theoretical foundation for coordinating the two strands of mathematical concept formation, as presented in Scheiner (2016), relies on and projects several theoretical insights revealed by Frege (1892a, 1892b) that have informed a variety of theoretical perspectives on mathematical knowing, thinking, and learning (see Arzarello, Bazzini, & Chiappini, 2001; Duval, 2006; Radford, 2002). In particular, the theoretical foundation in Scheiner (2016) shares Frege's (1892a) assertion that a mathematical concept is not directly accessible through the concept itself but only through objects that act as proxies for it. However, mathematical objects (unlike objects of natural sciences) cannot be apprehended by human senses (we cannot, for instance, 'see' the object), but only via some 'mode of presentation' (Frege, 1892b) – that is, objects need to be expressed by using signs or other semiotic means such as a gestures, pictures, or linguistic expression (Radford, 2002).

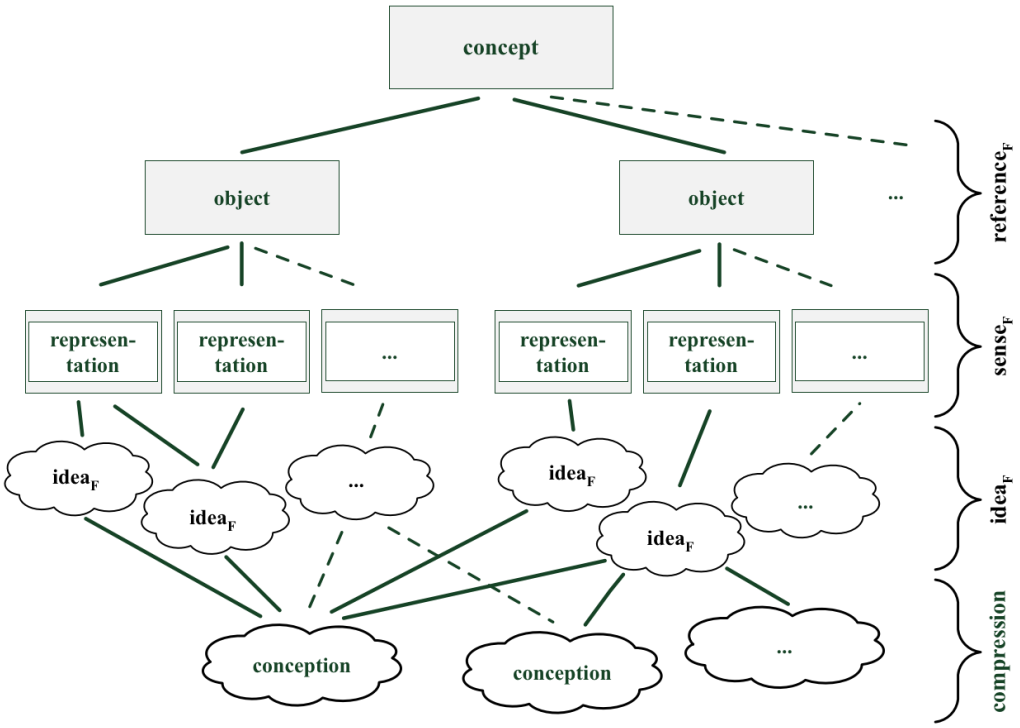

Figure 5. On reference$_F$, sense$_F$, and idea$_F$ (reproduced from Scheiner, 2016, p. 179).

The 'mode of presentation' (or 'way of presentation') of an object is to be distinguished from the object that is represented, as individuals often confuse a sense$_F$ ('Sinn') of an expression (or representation) with the reference$_F$ ('Bedeutung') of an expression (or representation) (the subscript F indicates that these terms refer to Frege, 1892b). The reference$_F$ of an expression is the object it refers to, whereas the sense$_F$ is the way in which the object is given to the mind, or in other words, it is the thought ('Gedanke') expressed by the expression (or representation) (Frege, 1892b). The expression 'a = b', for instance, is informative, in contrast to the expression 'a = a', as the sense$_F$ of 'a' differs from the sense$_F$ of 'b'. The upshot of this is, senses$_F$ capture the epistemological and cognitive significance of expressions. This implies one of Frege's decisive assertions, that an object can only be apprehended via a sense$_F$ of an expression (or representation): the sense$_F$ orients how a

person thinks of the object being referred to. Thus, it seems reasonable to understand Frege's (1892b) notion of an idea_F ('Vorstellung') as the manner in which a person makes sense_F of the world. Ideas_F can interact with each other and form more compressed knowledge structures, called conceptions. A general outline of this view is provided in Figure 1.

There are several ways that individuals can make sense of a mathematical concept; the focus here is on Pinto's (1998) distinction between extracting meaning and giving meaning with respect to sense-making of a formal concept definition: "Extracting meaning involves working within the content, routinizing it, using it, and building its meaning as a formal construct. Giving meaning means taking one's personal concept imagery as a starting point to build new knowledge." (Pinto, 1998, pp. 298-299)

Gray, Pinto, Pitta, and Tall (1999) stated that in giving meaning a person attempts to build from their own perspective, trying to give meaning to mathematics from current cognitive structures. Tall (2013) elucidated that these two approaches are related to a 'natural approach', that builds on the concept image, and a 'formal approach', that builds formal theorems based on the formal definition. Scheiner (2016) linked extracting meaning to the manipulation of objects and reflection on the variations in modes of presentation when objects are manipulated. These cognitive processes are often associated with Piaget's (1977/2001) reflective abstraction, that is, abstraction through coordination of actions on mental objects (e.g., Dubinsky, 1991). Giving meaning, on the other hand, was related to attaching an idea_F to a mode of presentation. That is, an individual gives meaning to the objects one interacts with from the perspective an individual has taken.

On Extracting Meaning

A common assumption is that the meaning of a mathematical concept is an inherent quality of objects that fall under a particular concept, and that this quality is to be extracted. This extraction of meaning is realised through the manipulation of objects and reflection of variations of senses_F when objects are manipulated. These cognitive processes are often associated with reflective abstraction, that is, reflecting on the coordination of actions on mental objects (see Piaget, 1977/2001). Similarly, Duval (2006) argued that via systematic variation of one representation of an object and reflecting on resulting variations in another representation of the same object, an individual can recognise what is mathematically relevant and separate the sense_F of a representation from the reference_F of a representation. Such a view asserts that individuals internalise extracted mathematical structures and relations associated with their actions and reflections of their actions on objects. It gives the impression that individuals construct mental models (ideas_F or conceptions) that correspond to an ideal realm (objects or concepts), though it might be read as taking a 'trivial constructivist' position (von Glasersfeld, 1989): the view that a necessary condition of knowledge is that individuals construct, constitute, make, or produce their own understanding (see Ernest, 2010). More importantly, such a view seems to suggest a 'conception-to-concept direction of fit' (see Scheiner, 2017) that is, mathematical concept formation is regarded as individuals constructing conceptions that best reflect a (seemingly given) mathematical concept (see Figure 2).

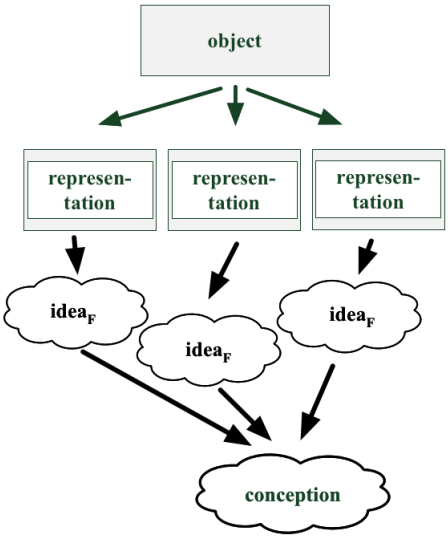

Figure 2. From object to idea$_F$ to conception.

On Giving Meaning

In the attempt to coordinate abstraction-from-actions and abstraction-from-objects approaches, a new understanding of abstraction emerged: abstraction is not so much the extraction of a previously unnoticed meaning of a concept (or the recognition of structure common to various objects), but rather a process of giving meaning to the objects an individual interacts with from the perspective an individual has taken. Abstraction, as such, is more focused on "the richness of the particular [that] is embodied not in the concept as such but rather in the objects that falling under the concept [...]. This view gives primacy to meaningful, richly contextualised forms of (mathematical) structure over formal (mathematical) structures" (Scheiner, 2016, p. 175). This is to say, individuals give meaning to the objects they interact with by attaching ideas$_F$ to objects or, more precisely, by attaching ideas$_F$ to the senses$_F$ expressed by the representations in which an object actualises. Recent research investigating the contextuality, complementarity, and complexity of this sense-making strategy (see Scheiner & Pinto, 2018) asserted that in contrast to Frege (1892b), who construed sense$_F$ in a disembodied fashion as a way an object is given to an individual, an individual assigns sense$_F$ to object. However, what sense$_F$ is assigned to an object is a function of what idea$_F$ is activated in the immediate context. In this view, ideas$_F$ direct forming the modes of presentation under which an individual refers to an object. As such, it is a person's complex system of ideas$_F$ that directs forming a sense$_F$, rather than merely the object a representation refers to. This research also indicated that individuals might even give meaning to objects that are yet to become. This means that although an object does not have a being prior to the individual's attempts to know it, an individual might create a new idea$_F$ that directs their thinking to potential objects, or more precisely: an individual might create an idea$_F$ that allows assigning a new sense$_F$ to objects that are yet to become. That is, individuals might ascribe meaning beyond what is apparent. It is proposed that the creation of such ideas$_F$ is of the nature of what Koestler (1964) described as 'bisociation', and Fauconnier and Turner (2002) elaborated as 'conceptual blending'; that is, to construct a partial match between frames from established conceptual domains, in order to project selectively from those domains into a novel hybrid frame, comprised of unique (emergent) structure of its own (see Figure 3).

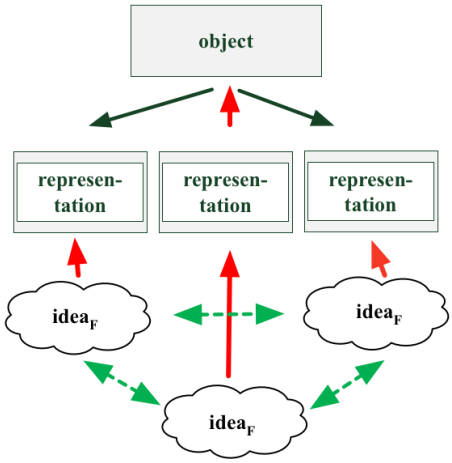

Figure 3. Transforming ideas$_F$ to give (new) meaning to an object.

The key insight here is that unrelated ideas$_F$ can be transformed into new ideas$_F$ that allow 'setting the mind' not only to actual instances, but also to potential instances that might become 'reality' in the future. In such cases, conceptual development is not merely meant to reflect an actual concept, but rather to create a concept: a view that suggests a 'concept-to-conception direction of fit' (see Scheiner, 2017) that is, mathematical concept formation is regarded as individuals creating a concept that best fits their conceptions. Similarly, Lakoff and Johnson (1980), drew attention to the power of (new) metaphors to create a (new) reality rather than simply to provide a way of conceptualising a pre-existing reality: "changes in our conceptual system do change what is real for us and affect how we perceive the world and act upon those perceptions" (pp. 145-146.). It is reasonable to assume that students transform ideas$_F$ to express a yet-to-be-realised state of a concept. This accentuates Tall's (2013) assertion that the "whole development of mathematical thinking is presented as a combination of compression and blending of knowledge structures to produce crystalline concepts that can lead to imaginative new ways of thinking mathematically in new contexts" (p. 28).

Towards a Dialogical Framing

Each of the previous two sections articulated a particular form of sense-making: extracting meaning from objects (via manipulating objects and reflecting on the variations) and giving meaning to objects (via attaching existing and new ideas$_F$ to objects). These two forms of sense-making seem to differ in their directions of fit: extracting meaning involves individuals' attempts to construct conceptions that aim to fit a concept (conception-to-concept direction of fit), whereas giving meaning involves individuals' attempts to create a concept that aims to fit their conceptions (concept-to-conception direction of fit) (for a detailed discussion, see Scheiner, 2017).

Instead of construing extracting meaning and giving meaning as independent processes that point in two opposing directions, it is argued here for a bi-directional theoretical framing of mathematical concept formation. Specifically, it is argued for a dialogical framing of extracting meaning and giving meaning, asserting that extracting meaning and giving meaning are interdependent (rather than independent): what meaning one extracts is very much a function of what meaning is given to, and vice versa (see Figure 4). This dialogical framing can better account for the complex emergence of evolving forms of meaning:

meaning not only emerges (from Latin emergere, ‘to become visible’) via reflection on manipulations of objects, but also evolves (from Latin evolvere, ‘to become more complex’) via transforming previously constructed ideas_F (see Scheiner, 2017).

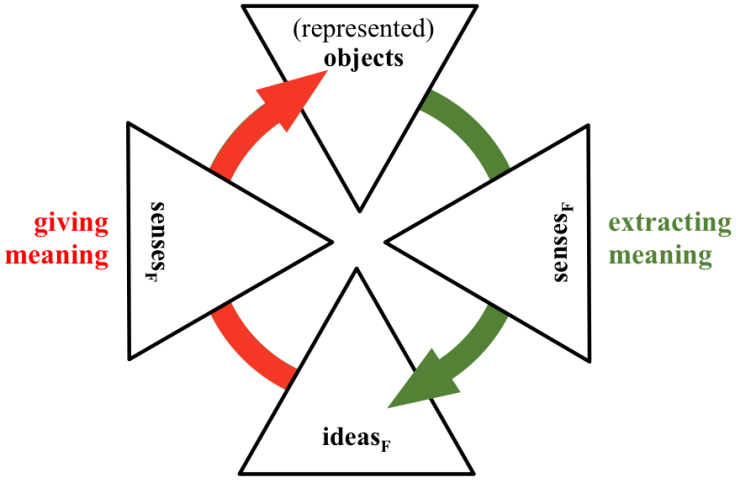

Figure 4. On the dialogue of extracting meaning and ascribing meaning.

The dialogical framing of extracting meaning and giving meaning acknowledges the complex emergence of evolving forms of meaning that cannot be accounted for by viewing extracting meaning or giving meaning as separate. Extracting meaning and giving meaning, though they have value in their own right, are restricted, and restricting, in their accounts of mathematical concept formation. This is due to the ‘hidden determinisms' inherent in the two approaches: extracting meaning assumes that what dictates meaning is the concept itself; while giving meaning advocates an individual's conceptions as the determinants of all meaning. The dialogical framing, in contrast, is not deterministic but bi-directional: mathematical concept formation involves processes that direct from conception to concept as much as it involves processes that direct from concept to conception. As such, the dialogical framing is more than a matter of recasting the concept-conception divide: it underlines that concept and conception are not static and apart but fluid and co-specifying.

Figure 5 is an alternative to the reductionist view taken in respective approaches of extracting meaning (see Figure 2) and giving meaning (see Figure 3), both being rather uni-directional and deterministic in orientation. The dialogical framing provides new interpretative possibilities regarding the complex dynamics in mathematical concept formation, allowing for a move beyond simplistic assertions about linearity and determinism (that were transposed from analytical science and analytical philosophy onto discussions of mathematical concept formation). Figure 5 attends to the complexity in mathematical concept formation and speaks to the nonlinear, emergent characters of evolving forms of mathematical meaning (see Pirie & Kieren, 1994; Schoenfeld, Smith, & Arcavi, 1993).

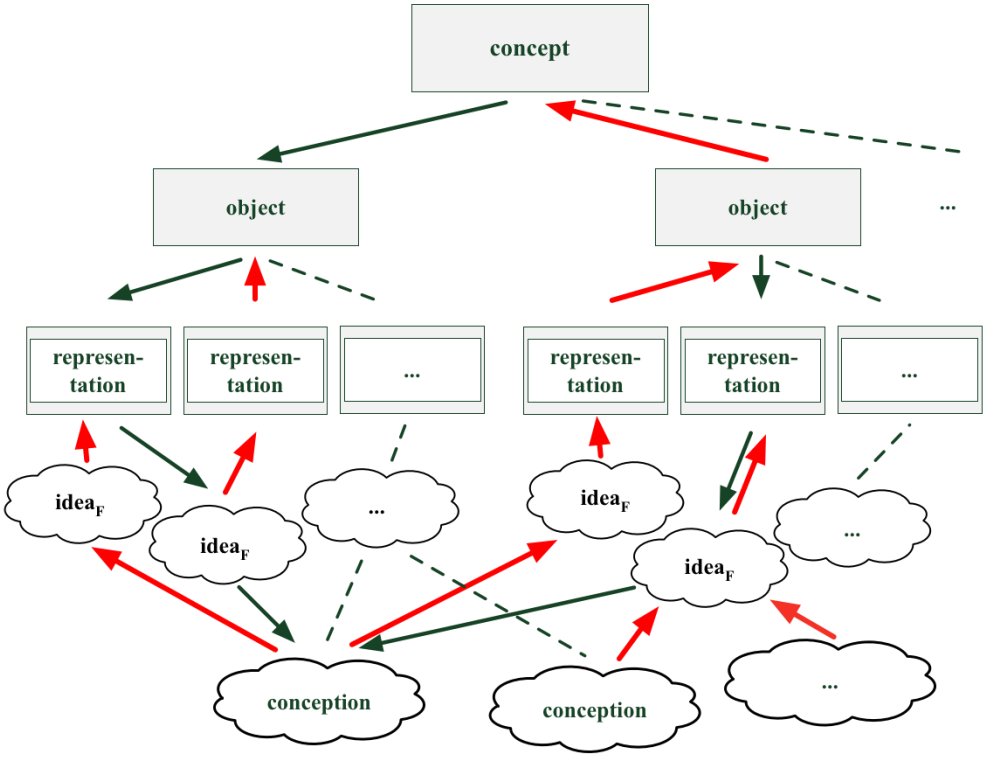

Figure 5. A complexivist frame: on the complex interaction between concept and conception.

Conclusion

This paper presents a new theoretical perspective blended from the existing perspectives that mathematical meaning is extracted (from objects falling under a particular concept) and that mathematical meaning is given (to objects that an individual interacts with by that individual). This blending seeks to frame mathematical concept formation as bi-directional (where what meaning one extracts is a function of what meaning is given to, and vice versa) and to recast the concept-conception divide (by viewing concept and conception as fluid and co-specifying instead of static and apart). In doing so, the dialogical framing presents a view of mathematical concept formation that is complex, dynamic, non-linear, and possessed of emergent characteristics.

This theoretical contribution makes the case that neither uni-directional framing of mathematical concept formation (whether involving extracting meaning or giving meaning) provides a comprehensive account of the complex emergence of evolving forms of meaning. It is argued for an alternative framing that acknowledges mathematical concept formation as both directed from concept to conception and from conception to concept. Mathematical concept formation, then, is construed as an ongoing, intertwined process of extracting meaning and giving meaning, in which conceptions shape, and are shaped by, the concepts with which an individual interacts.

This dialogical framing brings a greater insight: that any attempt to frame cognition in terms of mind over matter or matter over mind is misleading, as cognition is bi-directional: from the outside in (mind- to-world direction of fit) and from the inside out (world-to-mind direction of fit). That is, mind and world are engaged in a co-creative interaction: mind is shaped by the world and mind shapes the world. Such a world is subjectively articulated, in that its objectivity is relative to how it has been shaped by the knower (see Reason, 1998).

Such a dialogical framing is not so much a unification of any monism (that sees, for instance, mind as situated within its world), nor of any dualism (that sees mind apart from the world), but rather is an acknowledgment that mind is an integral part of the world, and as such both mind and world are in a constant state of flux, changing in the ever-unfolding process of extracting meaning and giving meaning.

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Perceiving and Reasoning about Geometric Objects in the Middle Years

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Although spatial thinking and reasoning is recognised as a key component for promoting STEM discipline, very little research has been done on its promotion in middle years in Australia. How do students perceive three dimensional geometrical objects? Are they able to recognise the objects from different perspectives and explain their reasoning for their drawing of the object? In surveying and analysing 709 students' responses, this study found that classroom experiences can enhance students' spatial and reasoning skills.

There is a common assumption that students have distinctive learning styles, that some students are verbal learners and other visual learners. This ill-informed belief led many to see spatial ability as fixed, with clear gender differences that are hardwired biologically (Newcombe & Stieff, 2011). Evidence shows that males and females use different strategies when visualising objects (Kozhevnikov, Kosslyn, & Shephard, 2005). Gender differences could also be caused by socio-cultural factor (Yilmaz, 2009) and seems to only appear in adults (Li, 2014). Spatial ability is certainly a trainable skill. Living in a three-dimensional (3D) world, the capacity to reason spatially is crucial for human existence. Spatial reasoning plays a critical role for developing Science, Technology, Engineering and Mathematics related disciplines (Wai, Lubinski, & Benbow, 2009). Early spatial skill training such as visualising and analysing shapes with different orientations has been found to improve subsequent arithmetic competence and predict success in engaging in mathematics reasoning task in the middle years (Casey, Lombardi, Pollock, Fineman, & Pezaris, 2017; Verdine, Golinkoff, Hirsh-Pasek, & Newcombe, 2017). Despite its importance, the construct of spatial reasoning is not as well understood. There is a lack of agreement on the process and steps of its development (Yilmaz, 2009). How students learn to cultivate such ability is unclear. Although the Australian Curriculum: Mathematics has integrated spatial reasoning as part of the construct of numeracy, questions remain on how to nurture such an ability.

The *Trends in International Mathematics and Science Study* (TIMSS) reported that Australian students are particularly weak in the content areas of geometry, a discipline that builds on spatial reasoning and visualisation ability (Thomson, Wernert, O'Grady, & Rodrigues, 2017). In particular, very little research is done on how students work with 3D objects. This negatively impact students' understanding of measurement concepts as much of students' difficulties with volume and surface area measurement are due to an inability to visualise and reason 3D objects (Lieberman, 2009). There is clearly a need to investigate the relationships between visualisation, language and representation in the construction of 3D knowledge.

Literature Review

Spatial reasoning is the ability to make sense of spatial relationships between shapes and objects. This thinking encompasses an understanding of the feature, size, orientation, location, direction or trajectory of geometric shapes and objects and being able to make

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spatial transformations. Spatial reasoning is not a monolithic construct and there is a lack of consensus about its components due to inconsistent naming of the factors (Yilmaz, 2009). Initial psychometric studies viewed spatial abilities as consisting of two main factors: *Spatial visualisation* - the ability to imagine the rotating, folding, or any changes made to the position of objects and *spatial orientation* – one's ability to imagine or 'view' an object from different perspectives (McGee, 1979). Additional factors such as speed, flexibility and spatial relations were added subsequently by other researchers (see Yilmaz, 2009 for a review on this).

From a mathematics education research perspective, the lack of standardisation of how terms should be defined makes researching and teaching spatial ability problematic (Ramful, Lowrie, & Logan, 2017). For Ramful and his colleagues, spatial ability is best viewed in terms of mental rotation, spatial orientation and spatial visualisation to capture much of the middle school mathematics curricula requirements. Mental rotation refers to one's ability to imagine how 2D and 3D objects would appear after they are turned around. Spatial orientation is the ability to imagine how an object looks from a different vantage point. Spatial visualisation, for them, refers to any spatial tasks that do not involve mental rotation or orientation. While the definitions provided useful boundaries for designing multiple choice test items. There may be other skills at play that the test did not address.

Indeed, the concept of spatial visualisation may be the most difficult to define because of its lack of specificity. For Phillips, Norris and Macnab (2010), visualisation is a cognitive process in which objects are interpreted within the person's existing network of beliefs, experiences, and understanding. It takes place when an image is viewed and interpreted for the purpose of understanding something other than the object itself. For example, looking at a net consisting of four equilateral triangles and identifying it as a tetrahedron. To visualise the object requires individuals to introspect possible images similar to a visualisation object and interpret it within the person's existing network of beliefs, experiences and understanding. Since our visual sensory input is constantly bombarded with different imageries, our visual cognition makes a distinction between spatial images (relating to information about the location, size, and orientation of an image) and visual images (such as shapes, colour and depth) (Sima, Schultheis, & Barkowsky, 2013). These two distinct types of visualising style reflect different ways of the brain generate mental images and process visual-spatial information. Those who focus on visual images, the object visualisers, tend to encode images globally as a single perceptual unit based on actual appearances (Kozhevnikov et al., 2005). They generate detailed pictorial images of objects and process the information holistically. They are faster and more accurate when performing recognition and memory tasks. When asked to interpret and reconstruct 3D objects using 2D format, object visualisers often reproduce images that resembles the actual object. Conversely, those who focus heavily on spatial images, the spatial visualisers, tend to encode and process images analytically, using spatial relations to generate schematic and abstract images from what they see. These individuals are better able to interpret and analyse abstract representations.

Language, also plays an important role in influencing our visual spatial perception. How one sees an object is influenced by ones' definitions of that object. For example, if a student defines a hexagon as a shape with many corners, or 'roundish', s/he is likely to call an octagon a hexagon. Similarly, if one's sole experience with 3D objects is prism, s/he is likely to call a triangular pyramid a triangular prism. The way a student perceives and talks about geometric visual representations reveals their thought processes and shapes their thinking (Sfard, 2008). Changing how students visualise 3D object necessitates changing their

discourse. As stated earlier, investigation of spatial ability tends to rely on multiple choice items to test specific spatial factors defined by the researchers. This makes the identification of other possible spatial abilities difficult. How students apply their spatial skills use in problem situations is unclear. Accordingly, in this study, we present students with a picture showing a dog facing three geometric objects. The students were asked first to name the objects, then draw what the dog sees and explain their reasoning. We ask: Can students comprehend the concept of left, and right? And can they name the 3D objects in the picture? But more importantly, what spatial skills did they employ to produce an image of what the dog sees? And what was their rationale?

The descriptive data from the question will show students' use of *keywords* to name the object and its components and the *narratives*, written utterances, students made to justify their reasoning. The diagrams allow researchers to determine what information best captured individual students' attention when visualising the objects, whether they focus on the spatial or visual images. By comparing their drawing with the reasoning, we seek to identify other possible spatial abilities beyond mental rotation and mental orientation.

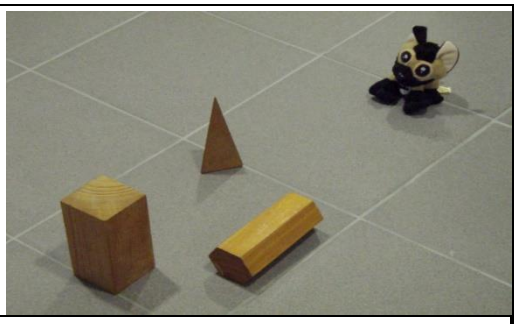
Method

[GPERS1] The dog in the picture is looking at three wooden shapes.

a. [GPERS1] Name the wooden shape
   - On your left _____
   - On your right _____
   - In the middle at the top _____

b. [GPERS2] Draw what the dog sees

c. [GPERS3] Explain how you decided what to draw and where to draw it.



Score	Description (GPERS1)
0	No response or irrelevant response
1	Names at least two objects in terms of their faces (e.g., square, rectangle, hexagon or triangle), may name one object correctly (e.g., cuboid or rectangular prism) or names faces or objects from dog's perspectives
2	Names at least two objects correctly relative to the students' perspective
3	All objects named correctly relative to the student's perspective (i.e., cuboid or rectangular prism, hexagonal prism and triangular pyramid)
Score	**Description (GPERS2)**
0	No response or irrelevant response
1	Draws shapes (2D) or objects (3D) from the student's perspective
2	Draws at least one correct shape from the dog's perspective (see below)
3	Draws three correct shapes (see below) but may not be correctly positioned or oriented from the dog's perspective.
4	Draws three correct shapes in the correct position from the dog's perspective (i.e., a hexagon on the left, a rectangle in the middle and a triangle on the right)
Score	**Description (GPERS3)**
0	No response or irrelevant response
1	Explanation relates to what the student sees
2	Explanation relates to correct shapes or correct position but not both
3	Reasonable explanation provided for naming all shapes and their position

Figure 1. The dog's perspective task (GPERS) and marking rubric.

This study is part of a larger study, Reframing Mathematical Futures II (FMFII), where we have been developing a learning assessment framework to assist teachers to teach reasoning in geometric measurement. It is based on the premise that an evidence-based validated set of an assessment tools and learning tasks can be used to nurture students mathematical reasoning ability (Siemon et al., 2017). Figure 1 shows the item GPERS with its marking rubric, designed to assess students' ability to visualise geometric objects from different perspective. The item is part of the assessment forms designed to assess students' geometric and spatial reasoning. The data collected contributed to the identification of eight distinct thinking zones through Rasch analysis (see Siemon et al., 2017 for more details).

The participants were middle-years students from across Australia States and Territories. Two groups of cohorts were involved. The first set of data – the trial data, was taken from 436 Year 4 - 10 students from three primary and seven high schools across social strata and States to allow for a wider spread of data being collected. The teachers were asked to administer the assessment tasks and return the student work. The trial results were marked by two markers and validated by a team of researchers to ascertain the usefulness of the scoring rubric and the accuracy of the data entry. The second set of data – the project data, was taken from 273 Year 8-10 students from six high schools situated in lower socioeconomic regions with diverse populations. The project school teachers were asked to mark and return the raw score instead of individual forms to the researchers. The project school teachers received two 3 days face-to-face professional learning sessions on spatial and geometric reasoning prior to the implementation of the assessment tasks. They also had access to a bank of teaching resources and four on-site visits to support their teaching effort.

Findings

Table 1 show the overall percentage breakdown of student responses for GPERS and Table 2 show the breakdown according to each year level. No Year 7 data had been obtained from the project schools at the time when this data was analysed. The project schools clearly outperform trial school students in a number of areas. This is very encouraging as it shows that the professional learning the teachers in project schools received may have contributed to better awareness and attention given to the teaching of geometry in school. While gender difference was not the aim of our investigation, we nonetheless found no significant differences at the $p < 0.05$ level.

Table 1
Overall results expressed as percentages for the Perspective Task GPERS.

Score	Trial Schools (n=436)			Project schools (n=273)		
	GPERS1	GPERS2	GPERS3	GPERS1	GPERS2	GPERS3
0	12.6	15.4	37.6	9.9	1.1	35.9
1	42.9	18.6	46.3	28.9	14.3	30.4
2	17.7	17.7	14.2	15	7.7	16.9
3	26.8	30.5	1.8	46.2	12.1	16.9
4		17.9			64.8	

Overall, students' knowledge of 3D objects was poor as 55.5% trial schools and 38.8% project school students were unable to correctly name the 3D objects in the photo (GPERS1). This difficulty was due to a lack of experience rather than based just on year level (e.g., see Table 2 Year 10's result between trial schools and project school).

Table 2
Percentage breakdown for GPERS1, GPERS2, and GPERS3 according to year level.

Score	Trial Schools						Project Schools		
	Yr 4 *n* =31	Yr 5 *n* = 59	Yr 7 *n* = 111	Yr 8 *n* = 74	Yr 9 *n* = 79	Yr 10 *n* = 82	Yr 8 *n* = 87	Yr 9 *n* = 93	Yr 10 *n* = 93
	GPERS1								
0	0	8.5	7.2	12.2	15.2	25.6	4.6	11.8	12.9
1	58.1	61	43.2	25.7	342	47.6	23	44.1	19.4
2	19.4	17	22.5	16.2	15.2	14.6	26.4	6.5	12.9
3	22.6	13.6	27	46	35.4	12.2	46	37.6	54.8
	GPERS2								
0	12.9	6.8	10.8	13.5	12.7	32.9	2.3	0	1.1
1	29	33.9	23.4	10.8	2.5	19.5	24.1	18.3	1.1
2	32.3	22	18	10.8	20.3	12.2	5.8	15.1	2.2
3	25.8	28.8	28.8	31.1	40.5	25.6	18.4	12.9	5.4
4	0	8.5	18.9	33.8	24.1	9.8	49.4	56.8	90.3
	GPERS3								
0	38.7	25.4	29.7	40.5	34.2	57.3	43.7	36.6	28
1	61.3	62.7	47.8	39.2	48.1	31.7	33.3	25.8	32.3
2	0	11.9	18	18.9	15.2	11	9.2	8.6	32.3
3	0	0	4.5	1.4	2.6	0	13.8	29	7.5

Except for Year 10 in the project schools, more than 50% of the students in both cohort were unable to name the three objects correctly (GPERS1). Students who score 1 may have named the objects based on the dog's perspective. Analysis of the trial school data showed that 46.6% of the students name the squared/rectangular based prism correctly; a further 12% wrote rectangle. 45% named hexagonal prism correctly and 8.7% wrote hexagon. The triangular based pyramid presented the most challenge with 24% named it correctly, a further 14% used the term pyramid and 2% wrote square based pyramid. For some, this may have been a difficulty in their knowledge of left and right, however, only 3% gave a complete reversal, mixing the right and the left.

Analysis of *keywords* used in the trial schools showed that 13% of the cohort did not respond to the question. Around 47% named the rectangular prism, 45% named the hexagonal prism, and 24% name the triangular based pyramid correctly. Misspelling words such as 'prisim' 'prizem', 'prymand', 'pryrimid', 'peyment', were not counted as errors. Some students used 2D shape names for the 3D objects (17% rectangle/square; 15% hexagon and triangle) and 10% named triangular pyramid as triangular prism. Others named the objects by joining known terms, such as 'hexagonal cilender', 'rectangular hexagon', 'rectangular square', 'pentagon cilender', and 'rectangular cylinder'.

When drawing the objects from the dog's perspective (GPERS2), the project schools showed a clear improvement through Year 8, 9 and 10 with 90% of the Year 10 students able to successfully complete the task. The greatest error was that the students drew the objects from their perspective but claimed that that was what the dog saw or reversed the order saying the dog was seeing the objects from the other side. This account for between 25% and 40% of the trial school cohort and 5 - 18% of the project data.

Students have varying degrees of experience with drawing 3D objects. Many trial schools' students tried to incorporate all the components in their drawing. Eight students (2

Yr 4, 4 Yr 5 and 1 each from Yr 8 and 10) drew a bird eye view of the scene including the dog, similar to the drawing on the left in Figure 2. The drawing on the right shows that the student may have seen drawings of a square or rectangular prism before but little experience with the other objects. Even with the prism, this student has included all faces although the dog would be unable to see them all. The hexagonal prism shows both ends for the same reason. With the triangular pyramid he knows that there are three triangular faces on the side but was unable to depict it in 3D.

Figure 2. Students' representation of objects showing all components.

When comparing the drawings by year levels, Year 4 students tended to produce a wider range of drawings, from 2D shapes or 3D objects drawn on one plane, that show depth, to drawing an octagon as a hexagon and mixing the positions of the objects. From Year 5 onwards, depth and dimensionality became important features. While the rubric did not specify 'depth' as a criterion (placing the rectangular prism to the rear), many trial schools students demonstrated this in their drawing. Three students (1 Yr 4 and 2 Yr 7) included the tile lines although only the Year 7 students provided an explanation that they were used to either get the proportions or position correct (see row 3 in Figure 2).

Indeed, analysing trial schools' students' drawing and their explanations led to six strategies: mental rotation, physical rotation of page, mental reflection, perspective drawing, position and depth, and 2D perspective of 3D object. Figure 3 shows examples of student drawing and their explanations on how they drew what the dog saw. Some samples used more than one strategy.

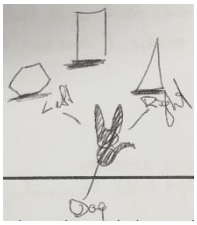
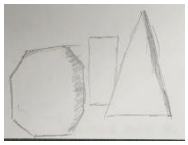
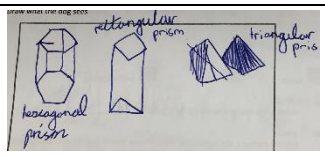
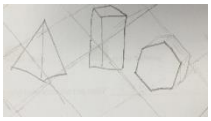
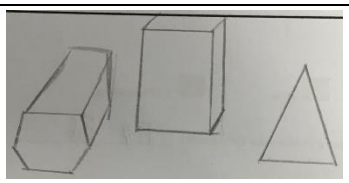
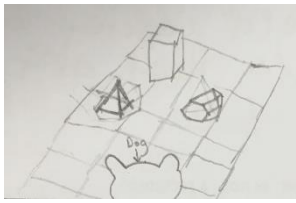
Student's drawing	**Strategy used and student's justification**
i.  ii. 	<u>Mental rotation (with position and depth)</u> i. I did it as if the dog was me. I put myself in the dog position and figured it out as if the shapes were right in front of me. ii. I put myself in the dogs positon and he can only see straight so they look 2D (*sic*) (<u>with 2D perspective of 3D object</u>).
	<u>Physically rotate page</u> I turned the paper around and figured what shapes it would look like from the dogs perspective (*sic*).
	<u>Mental reflection</u> I just flipped it around to make a reverse picture. (incorrectly reflected)
	<u>Perspective drawing (mental or physical rotation)</u> I decided to draw some of the shapes 3D because the dog would be able to see the top of the rectangular prism a little bit and the dog would be able to see the top and right side of the hexagonal prism, but the triangle he would only be able to see the face.

Figure 3. Students' representation of the dog's perspective.

There is a difference in the sophistication of the drawing and explanations as students attempt to indicate depth and dimensionality, as seen in row 1 and 4 of Figure 3. The student's explanation of the perspective drawing (row 4) is correct as the dog would be able to see the top and right side of the hexagonal prism, and at least two sides of the rectangular prism but only one face of the pyramid. Equally, sophistication of drawing may not necessarily reflect the thinking that showed in the explanation (Figure 4).



<u>Correct justification, incorrect drawing</u>

If the hexagonal prisim is on my right, the dog is facing the other way so it is on its left, this means that the pyramid that is on my left would be on its right and tha rectangular prisim stays in the same position (*sic*).

Figure 4. A correct justification with incorrect drawing of the dog's perspective.

Students do not seem to be used to justifying their action (GPERS3) even when they were able to successfully complete the drawing. Many did not response but for those who did, the explanation tended to be superficial such as 'I imagined I was the dog', 'I decided to draw it like that because'. Less than 5% of the trial school cohort and between 7 - 29% of project school students were able to give an adequate explanation.

Discussion

In this study, we presented the data collected on a task design to determine students' ability to reason about a situation involving 3D objects. Given the scarcity of research conducted in this area among Australian students, the data should provide valuable information to shape instructional design and future research direction.

With regards to whether the students can comprehend the concept of left and right, it appears that this was not an issue for most students, although 3% of the students reversed the objects on the left and right demonstrating some confusion. They were spread across year levels. Students' attempts at naming the objects revealed a lack of knowledge of the correct geometric terms as less than half of the cohort were able to correctly name at least one object. Spelling was also an issue together with a confusion between prisms, pyramids, cylinders and cones and between hexagon, pentagon and octagon. Many students showed a willingness to create new terms using a mixture of words they knew (rectangular square).

The spatial skills students used to assist them in drawing what the dog saw included: mental rotation, physical rotation of page, mental reflection, perspective drawing, recognition of position and depth, and a 2D perspective of 3D object. These showed in both the student drawing and their explanations to justify their actions. While the literature refers to mental rotation (see Yilmaz, 2009 for a review on this), it is of interest here that many students in their explanation actually use other strategies including physical rotation to answer a mental rotation question. Research in this area has often used multiple choice type tasks to assess spatial skills such as mental rotation. Yet it is only in situations where students are asked to explain their actions that one can fully appreciate their use of spatial skills. However, we acknowledge that verbal explanation does not always match action as we found with some students in this study. Combining both the drawing and the explanation allows students' spatial skills to be better understood.

The data show that students in the project schools were far more successful in completing the drawing from the dog's perspective (over 50% compared to under 35%). When asked to explain their reasoning, a large number of students did not respond (37% overall) or gave

very superficial reasoning (46% in trial schools and 30% in project schools). Only 2% (trial schools) and 17% (project schools) of students gave complete reasoning. This suggest that classroom experiences in handling 3D objects and reasoning about them can assist in contributing to students' development of spatial skills. Similarly, a classroom culture where explanation and reasoning are required contributed to students developing reasoning abilities. This is not a sociocultural artefact since the project schools were from low socioeconomic backgrounds whereas the trial schools were from a diverse range of economic areas.

There is much still to be learned about visualisation and its impact on spatial reasoning. Clearly, more research is needed in this area of spatial skills and reasoning with specific focus on the impact of classroom experiences on learning, teacher knowledge, and the contribution made by the classroom discourse and the culture of the classroom and how these contribute to the development of visualisation and spatial reasoning.

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Contribution to the Development of Self-Regulated Learning Through Merging Music and Mathematics

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This paper reports on a study into pre-school children's self-regulated learning. Analysis is provided through findings taken from a larger study, the core aim of which was to contribute to developing and strengthening self-regulated learning competencies through merging Traditional African music and mathematics. An Action-Research-embedded-in-Design-Research approach was employed to design and implement this new mode of learning. Enactivism, the basic assumptions of which are shared understanding and joint action through engagement, provided the theoretical framework. Findings suggested that the intervention contributed to an increased ability to attend to tasks through watching, listening and self-initiation of the relevant actions

This paper reports on an intervention programme designed to help develop young mathematics learners' self-regulated learning skills. The aim of the programme was to assist students to become more active agents of their own learning. Rather than simply being told what to do, students value "opportunities to be thinking, creative agents" (Boaler and Greeno (2000), (p. 11). However, children who experience problems manipulating information in their working memory, or in being able to apply cognitive strategies flexibly, or who struggle to inhibit inappropriate learning behaviours, are likely to experience conceptual difficulties with various aspects of mathematics (Clark, Pritchard & Woodward (2010)).

In this paper I focus on aspects of my Master's research, for which I designed and developed a programme which merged music and mathematics. I am a professional musician and teacher of young children. I have more recently been involved in an Early Number Fun [ENF] programme initiated by the South African Numeracy Chair Project [SANCP (2016)]. The ENF programme was designed for local Grade R teachers and their learners. In South Africa, Grade R is the reception year, that is, the year before children begin formal schooling. It caters for five to six-year old's. In the course of my participation with these teachers and learners, I noticed that many of the children appeared to struggle with their self-regulated learning [SRL] competencies. In particular the children seemed to find it difficult to exercise inhibitory control, an important aspect of SRL, a difficulty which can negatively impact learning.

Given that SRL is seen to be an important element of 'doing mathematics' (after Stein and Smith, 1998), I respond in this paper to the following research question: What might a programme built around the use of African music principles contribute to children's ongoing development of SRL?

There are two important points to note. Firstly, music and mathematics share some common features: numbers and counting, patterning, sequencing and memorising being just some of these. Secondly, music, particularly for young children, is generally embraced as a 'happy' and 'fun' experience. Mathematics, by contrast, is often perceived as a difficult and daunting subject. According to the South African Curriculum and Assessment Policy Statement (CAPS) however, "learning Mathematics should be based on the principles of integration and play-based learning" (DBE, 2012, p. 14).

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I argue that, as a ‘door of entry,’ music provides a good foundation upon which to base a play-based intervention for young mathematics learners. Play-based learning appeals to any young child and literature provides evidence that music is found to help develop SRL, a requirement also in learning mathematics.

For my intervention programme, I chose Traditional *African* music over Western music. The programme took place in South African state schools where most of the children attending Grade R came from homes where an African language was spoken (IsiXhosa, in the Eastern Cape), albeit that, on entering the school, these children were required to switch to English as the medium of instruction. Apart from relating to an important aspect of the children’s cultural roots, a particular value of African music for my study stems from its strong emphasis on rhythms, and on repetition as a basis on which to build improvisation, plus the fact that it is generally performed in groups. These three elements were built into the intervention programme.

Further, one of the ways in which African music is captured is through very simple block notation. This form of notation is visually accessible to pre-literate children. Figure 1 below provides an example of an African block notated rhythm and repeated pattern.

X		XX	X	X		XX	X

Figure 1. Block notation (where ‘X’ denotes the sounds).

Literature Review

I chose enactivism for my theoretical framework. Enactivism embraces “an enactive approach to cognition,” with “learning equivalent to action” (Brown & Coles, 2012, p. 217). It represents a combination of a constructivist approach to learning and the view that cognition and environment are linked, thereby implying that learning is best achieved through interactive groups where shared actions contribute to a sense of belonging. Not only did enactivism fit my programme design intentions but it aligned also with my goal of encouraging learner interaction and reflection throughout the programme. Further, and of importance, enactivism supports the “development of the *whole* child by encouraging teachers and learners to reflect deeply on their practice to understand the purpose of all actions" (Hamilton, 2006, p. 6). This aspect corresponds well with South Africa’s CAPS which emphasizes that mathematical learning in Grade R “should promote the *holistic* development of the child” (DBE, 2012, p. 14). Implied here is the need to be aware of students’ social, emotional and cultural backgrounds.

SRL is a “cornerstone of early childhood development” (Gillespie & Seibel, 2006, p.34). It involves taking control, being able to switch skills, and adjust to change, exercising inhibitory control (in other words resisting impulsiveness). Bodrova and Leong (2008) describe it, as “a deep internal mechanism that enables children … to engage in mindful, intentional, and thoughtful behaviors” (p. 1). Such behaviours are requirements for problem solving, for planning, and for behavioural regulation, all essential elements for learning mathematics. “Lack of inhibition and poor working memory, resulting in problems with switching and evaluation of new strategies for dealing with a particular task” which impedes children’s mathematical proficiencies (Bull & Scerif, 2001, p.273).

I developed a programme to try to start addressing such problems. The programme drew on the beats and rhythms of African music through using the children’s body movements

(clapping, tapping, stamping). These I believed would help embody and thus facilitate their understanding of such mathematical concepts as patterning, and sequencing (such as is shown in Figure 1). Much has been written on links between music and mathematics. So, for example, Hallam (2015) noted that although the evidence for the impact of musical activity on mathematics performance is mixed, there is *positive* evidence from intervention studies with children, particularly where musical concepts are used to support the understanding of fractions. The Arts Education Partnership in Washington DC (AEP, 2011) too has reported on an extensive body of research aimed at identifying high quality, evidence-based studies documenting young people's learning outcomes associated with education in and through music. In merging music and mathematics, it was established that the development of executive functioning, which strongly overlaps with SRL, is promoted. The report argued that results showed conclusively that music education enhances working memory, promotes better thinking skills, and strengthens perseverance. It went so far as to argue that music even advances mathematics achievement. Zuk, Benjamin, Kenyon and Gaab (2014) similarly argued that musical training may "promote the development and maintenance of executive function, which could mediate the reported links between musical training and heightened academic achievement" (p. 7). In line with this premise, Australian mathematics education academics, Still and Bobis (2017) too stated that "music and mathematics are theoretically connected in areas such as harmony, with evidence of this dating from the time of Pythagoras", further indicating that "mathematical qualities are also inherent in other aspects of music, such as rhythm, tempo and melody" (2017, p. 712).

It should be noted that the above were all based on Western music. I was not able to locate any studies on SRL development using African music, a further motivation to me in designing and trialling my own *African-based* intervention.

Methodology

The research arose through my SANCP involvement with Grade R teachers. Two of the teachers participating in SANCP's ENF programme expressed interest in my idea of running a pilot project in their Grade R mathematics classrooms in which I would use African music as a means of helping develop children's SRL skills. In this sense the two teachers and their learners constituted an 'opportunity sample'.

I designed 16 interactive sessions of approximately 30 minutes each over a period of seven weeks for each school. My research approach blended Design Research and Action Research. In trialling the intervention programme with the first class of Grade R learners, Design Research, with its iterative process for initial planning and adaptations, pre-dominated. With my ongoing interaction and activities within the classes, and the participatory nature of the programme, my approach then took on a more Action Research orientation. Thus, the resultant overall structure of my intervention was a combination of Design Research and Action Research. This led me to coin the phrase: *Action-Research-embedded-in-Design-Research*. Figure 2 illustrates the criteria for this embedment.

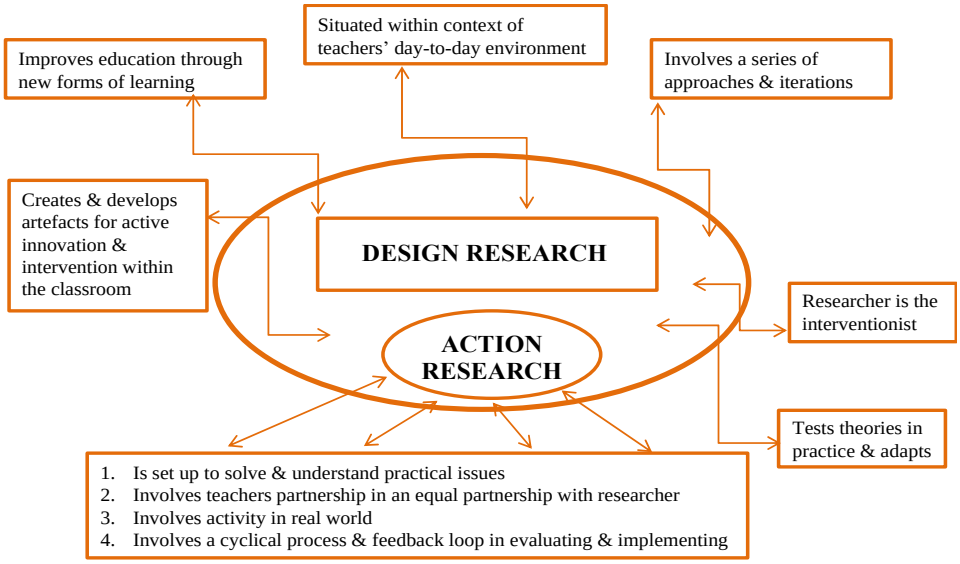

Figure 2. Action-Research-embedded-in-Design-Research characteristics.

I began by first obtaining ethical clearance to engage with two of the participating teachers from two different state schools, one urban school and one in a nearby township. (The word 'township' in South Africa refers to urban residential areas for lower income communities). The intervention was based on the principal of informed consent together with the assurance to participants of their right to withdraw at any time. I also gained parental approval to make video recordings of learners in each class.

The two Grade R classes from which I gathered data were mixed-gender, (28 and 29 children respectively). I made video recordings of the children's physical actions and re-actions throughout the intervention period. These, I later transcribed into text with supporting video photographs. Based on my reading of the relevant literature on SRL I identified the following indicators which I used to analyse the video data:

Thinking, and reasoning about one's self-control;
Taking control by planning, monitoring and evaluating one's own learning behaviour;
Exercising inhibitory control (i.e. resisting impulsiveness);
Being able to switch skills and adjust to change.

At the conclusion of the programmes in each school, I conducted 'mini' informal group interviews with the children where I gave them the opportunity to respond to questions about the sessions such as: "Did you enjoy the programme"? and "What did you learn"? These responses also contributed to my data.

Development of the Intervention

To gain an initial sense of the possible changes in the children's levels of SRL, I started the programme with an introductory session involving three activities. I got them to play games of 'snap' (using playing cards) to assess working memory; I got them to identify shapes, (triangles, squares, circles); and I got them to copy block notation patterns. As the sessions forming the intervention programme got underway the emphasis on music merging with mathematics took shape. While the aim was to work with African notation, there had to be some recognisable lead-in likely to appeal to young children. The drum, as a musical

instrument was what I used, and then later a shaker (or maraca). In keeping with the ENF Grade R teacher workshops I purposely modelled, where possible, the programme around known mathematical principles. Thus, was born a drum inside a square; a shaker inside a circle, and so on (Figure 3).

Figure 3. Drum inside a square: Shaker inside a circle.

The children and I used these instruments to produce a class ensemble. We used large yogurt containers, which the children and I called 'Yogi' drums, and hand cream containers with a few seeds inside, which we used as shakers. Once the children were familiar with the drum and shaker, I then formally introduced them to block notation. This they embraced with the same enthusiasm as the drums and shakers, even so far as to anticipate, for example, 'the X inside a *square'*. Importantly, the children were able to write an X with ease, whether on a chalkboard or on paper. Figure 4 illustrates four blocks with the letter X denoting the beats. The idea was for the children to copy the rhythms and, initially to clap and count one count to *each* block, and to notice that where there is no X, there is no clap, (so denoting a silent count or beat).

X	X		X

Figure 4. Block notation for 4 counts (or beats).

The children worked and played in groups of two to six according to the need of the activity and the class layout. Different groups were given different rhythms, either as a written example to read or through a dictated clapping or drumming rhythm. Figure 5 illustrates block notation for two different rhythms. These include, what we labelled: 'running' and 'walking' sounds, (where running sounds are twice as quick as walking sounds). Developing and exercising SRL, required a high concentration level from the children to hold on to the different patterns.

Group 1	X			X
Group 2	X	X	X	X

Figure 5. Block notation for two group rhythms.

The introduction of block notation opened up many opportunities for creating class ensemble rhythmic playing, and for leadership development, by, for example, conducting

the class. It also led to the children writing and composing their own rhythms for performance opportunities, something they embraced with great enthusiasm.

Findings and Discussion

The four programme sessions I focus on for this paper, I regard as particular 'Aha' moments and evidence of the emergence of SRL skills. These are Session 12 from the urban school; Sessions 1, 9 and 11 from the township school.

Urban School 'Aha' Moment.

Session 12 provided a clear example of SRL from Ben (pseudonym) who, when I called for a volunteer to lead the drums in the class ensemble, very quickly put up his hand to take on the task. Prior to this moment, whenever individuals were encouraged to take on a leadership role (for example conducting the class, leading the clappers, the shakers or the counters), the individuals concerned would take up their place standing beside me. Standing on the floor, due to their height, meant they were not always clearly visible. On this occasion, Ben, of his own volition, picked up his drum, tucked it under his arm, picked up his chair, and carried it over and set it down beside me. He then climbed up onto his chair and turned beaming at his peers, in readiness to lead them and the other drummers. Not only was this an excellent example of what SRL looked like, it also showed the planning that must have taken place within Ben in anticipation that he might be chosen to lead. He would have needed to think the whole process through, from start to finish. Photo 1 below shows Ben beaming down at his peers.

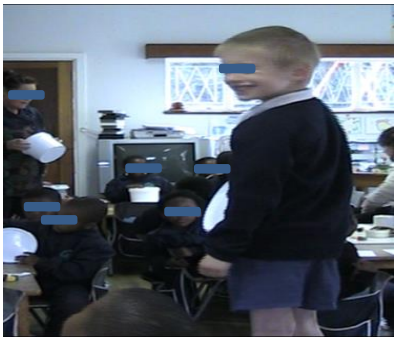

Photo 1. Ben the chair leader.

Township School 'Aha' Moments

A discussion on numbers and counting set the wheels in motion for the first session of the programme in the township school, I asked the children to show me how they could count, (aloud), and clap a certain number of counts. After splitting the class into different groups, to clap and count a different set of numbers, I asked the children if they could suggest a different body movement to that of clapping. Some suggested tapping; others indicated clicking with their fingers. One little boy, Joe (pseudonym), came up with a combination of clap, (the conventional way), followed by turning the backs of the hands to clap the back of each hand together. This was then taken up by the rest of the class. Thus, right at the start of the programme, there was this indication of SRL.

Session Nine in the same class, involved the introduction of groups of five children around an A3 chalkboard, learning how to compose different four block rhythms. Each group

was provided with a piece of chalk. Once the concept had been explained, each group was asked to come up with their own rhythm and write it on their boards. My initial intention was for the children to work out two different rhythms. The outcome exceeded my expectations as noted in Photos 2 and 3 where one child holding her group's board shows three rhythms. Photo 3 shows two boards with different sized block notated rhythms. The end of the session witnessed the children eagerly demonstrating their masterpieces by clapping and counting each composition individually, which was subsequently taken up by the whole class. The outcome of the session evidenced the children's ability to discuss and plan the compositions of different rhythms and to then transcribe these into block notation.

Photos 2 and 3. Block notation rhythm 1 and rhythm 2.

By Session Eleven, Joe, who throughout had shown commitment to the sessions with his attentiveness and readiness to respond to questions and to interact, showed some frustration with his peers 'lack of attention' to what I had to say. He jumped up onto his chair and raised one arm up high, hand outstretched, and demanded, very loudly: "All eyes on me." His action was followed by a stunned silence. I translated this as an 'Aha' moment, for, as Photo 4 illustrates, the follow-on effect was that other children within the class took up the same peer instruction to 'stop the talking and to listen'. This powerful means of children demanding focusing, thinking, planning and strategizing from their peers represents, I believe, evidence of a self-regulated development.

Photo 4. All eyes on me.

Concluding Remarks

The examples in the preceding section illustrate just a few instances of emerging SRL competency. An indication of the children's increased confidence was particularly noticeable, when at the end of the programme, in each of the two classes, on being asked what they had learnt, many children jostled to share their experiences e.g. "I learnt about drums"; "I learnt shakers"; "I learnt about Yogi drums"; "I learnt an X and about drums in a square"; I learnt about "a rectangle and a circle"; and "about running"; "and walking"; and about "block notation"; "reading and writing on the board"; "playing drums and shakers";

"I wrote numbers"; "I counted to 16 and to 30"; "I made my own rhythms". These enthusiastic comments showed a marked development over the kinds of monosyllabic responses they had offered at the start of the intervention.

Referring back to my research question: 'What might a programme built around the use of African music principles contribute to children's ongoing development of SRL?' I feel confident that my research provides evidence that the merge of music with mathematics intervention afforded a platform for young children to 'play' and in so doing develop aspects of the kinds of SRL skills required in the mathematics classroom.

Acknowledgements

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Preservice Teacher Mathematics Education: Online vs. Blended vs. Face to Face! Is this the whole story?

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Negative experiences of university mathematics education are often laid at the feet of online or blended learning. However, data collected as part of a five-year project at two universities suggests that there is much more to consider in determining the quality of preservice teacher mathematics education courses. This paper outlines a methodology that investigates the experiences of pre-service education teachers (PSTs) in relation to their journey of learning how to teach primary mathematics delivered via a variety of modes. Results indicated that the mode of delivery is not the critical factor as course design, teacher knowledge, and building rapport seem to be more influential in student success.

In recent years mathematics education has been in the spotlight in Australia with the most recent TIMMS (2016) and PISA (2016) results for Australian students indicating a continuing decline in mathematical performance (when compared internationally). Poor scores on these tests are contributing to the current zeitgeist that university mathematics education courses in Australia are failing future school students. Our roles for the past five years has been as mathematics educators – at two different universities (Author 2), teaching mathematics education to 1st and 2nd year pre-service teachers (PSTs), in a range of modes - fully online (Author 2), blended (face-to-face and online components) and solely face-to-face (F2F). The success or otherwise of university courses are often attributed to the mode of delivery with proponents of the various forms of delivery (online, blended or F2F) citing success in courses as primarily a consequence of delivery mode. In particular, negative PST experiences of university mathematics courses are often laid at the feet of online, and to a lesser extent, blended learning (Larkin, 2017). However, our experiences suggest that there is much more to consider in the puzzle of successful university mathematics education. This paper may assist other preservice mathematics educators when planning learning experiences for their students.

What the literature tells us

Concerns about mathematics education are not new; however, in some sense, a perfect storm impacts on the PSTs in this research who are: 1st or 2nd year students; studying university courses offered largely in online or blended mode; often anxious about teaching mathematics; and, often returning to education after completing secondary schooling 10-20 years earlier. Data suggest that many PSTs fail to enjoy or recognise the personal relevance of mathematics. Chubb (2014) writes of the broader disenfranchisement within mathematics education contributing to the decline in the number of students studying mathematics in Senior Secondary School or at university. These findings point to the need to ensure that PST's graduate with high levels of mathematical content knowledge (MCK), mathematical pedagogical knowledge (MPK), and with positive attitudes towards mathematics. The development of positive attitudes towards mathematics is identified as a core requirement of mathematics education courses given the persistence of negative attitudes held by PSTs that

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 693-700. Auckland: MERGA.

prove highly resistant to change (Grootenboer, 2008) and which often exhibit themselves in the form of anxiety. Even more problematic is that, whilst mathematics anxiety exists in almost all educational contexts, it appears to be much more prevalent in primary, pre-service, mathematics education students (Peker, 2009).

A further dimension to consider is the increasing use of online components as part of a contemporary university experience. This, in part, is due to economic imperatives as it is often more financially viable to offer online courses to large cohorts and also an acknowledgement of the changed landscape for PSTs who are likely to be juggling demands imposed by work, family, and study. From a university's perspective, the increased use of online components is rationalised as an appropriate response to the perceived needs of university students for "anywhere, anytime" learning. Although research findings are mixed, they generally indicate that the use of online lectures contributes to both cognitive and affective positive outcomes for students. However, some evidence suggests areas of challenge including infrastructure issues; technical quality of online components, and time management. In addition, the use of online lectures and/or online tutorials can heighten anxiety for students who are not "tech savvy". A final consideration in online PST mathematics education is the potential "loss of relational contact" (Kim, 2011, p. 763) with negative impacts on PST attitudes towards mathematics. Given the considerations raised in the literature regarding PST mathematics education, we sought to answer the following question:

What is the impact of mode of delivery (online, blended, or F2F), and course teaching personnel, on undergraduate PSTs experience of Primary mathematics education courses?

Method

As reflective educators, we use a design-experiment approach to continually improve the mathematics education courses we teach. In brief, a design-based experiment is concerned with the study of learning in specific contexts and then extending knowledge by generating models of successful innovation. The design-based experiment cycle of data collection and reflection is an authentic research approach as teaching academics are best placed to identify changes that need to be made to improve learning and teaching (Cohen, Manion & Morrison, 2002).

Although this research is not conducted as an experiment, it is worthwhile to note that there is a great deal of homogeneity between the four cohorts discussed in this paper and therefore some measure of control of some of the educational variables is possible. For example: both universities offered Bachelor of Primary Education Degrees registered by the same accrediting authority; the content of the courses remained largely consistent; each course was supported by an online Learning Management System (course profile, course readings, lecture notes, tutorial notes, additional resources); the PSTs were 1st or 2nd year students and comprised a mixture of immediate school leavers and mature age students; and each cohort received three hours of "contact" with the teaching team each week. Provided below is a brief description of each of the four cohorts in the study.

Cohort A (2011): This cohort had no physical F2F access as all course elements were delivered online. These students received a weekly one-hour recorded lecture (audio with accompanying PowerPoint slides), and a two-hour Wimba tutorial (Wimba is a proprietary software and provides a virtual classroom with chat, voice, interactive white board, and breakout rooms for small group activities).

Cohort B (2015): This cohort had a mixture of a one-hour online lecture (with embedded video of the lecturer), a one-hour F2F workshop, and a one-hour F2F tutorial. This delivery

mode is referred to in this paper as the (1+1+1) model. Each one-hour interactive workshop was conducted with the entire cohort. The tutorials consisted of a maximum of 30 students in a classroom environment (group work at tables).

Cohort C (2016): This cohort received all contact hours in F2F mode. The three hours were comprised of a one-hour lecture, one-hour workshop and one-hour tutorial. It was thus similar to the 2015 model with the difference being the online lecture was replaced with a F2F lecture that was also recorded for later access by the students.

Cohort D (2017): This cohort is identical to Cohort B in terms of delivery mode, with the difference being a new lecturer (Author 1) delivering the F2F components. As in 2015, these students received the (1+1+1) delivery mode.

Thus, the significant variable for three of the four cohorts was the mode of delivery of the 'contact' hours and the variable for the fourth cohort was a change in personnel delivering the face-to-face components. The data set for this study consists of end-of-course Student Evaluation of Teaching (SET) quantitative scores and qualitative feedback; online, in-course surveys (ICS) during two of the courses (2015; 2016); and, our reflections on the course delivery over the period under investigation. Each end of semester survey sought to gather information from students regarding their experience of the course. Although there have been concerns expressed in the literature regarding the reliability and validity of SET evaluations (Larkin, 2017; Rowan, 2013), Likert scale SET scores are used here as blunt indicators of course quality (Table 1).

Table 1
*End of Course Feedback (2011-2017) and *In Course Surveys (2015 / 2016)*

Cohort	Number of Students	Number of Responses	Response Rate
Fully Online (2011)	223	124	56%
Blended (2015)	136	53	39%
Blended (2015)*	136	45*	33%
Face to Face (2016)	130	31	24%
Face to Face (2016)*	130	66*	51%
Blended – New Lecturer (2017)	153	49	32%

Although the criteria differ slightly between the two universities (See Tables 2 and 3), each end of semester survey evaluated similar aspects of the courses i.e. learning expectations, course structure, individual treatment, assessment feedback etc. In addition to the end of semester feedback, qualitative data regarding student experience of the 2015 and 2016 cohorts were collected via anonymous, in-course surveys (ICS) conducted mid-way through each course. These surveys included a number of open-ended questions regarding the mode of lecture delivery.

In order to make sense of the data collected in the SET and ICS, Thematic Analysis was used. According to Braun and Clark (2006) Thematic Analysis is a "method for identifying, analysing and reporting patterns (themes) within data. It minimally organizes and describes your data set in (rich) detail" (p.79). Broadly speaking, Thematic Analysis involves a range of processes (i.e. familiarisation, generating codes, and then searching for, reviewing, defining and communicating themes). Although the overall process appears prescriptive, it is important to acknowledge that the various processes are guidelines to be applied flexibly to each research context (e.g. in this research, the familiarization process did not involve transcription and we were already very familiar with the data as it is a common component

of our twice-yearly evaluation practices). A second observation regarding Thematic Analysis is that it is a recursive rather than linear process and thus, there is movement back and forth between the phases (e.g. we commenced with the 2011 data which was then revisited after we had processed the 2015 and 2016 data).

Impact of the Different Delivery Modes

Findings from each of the four discrete cohorts are presented below. As each course offering had a particular targeted modification (i.e. mode of delivery or team personnel), the response of PSTs to these modifications are analysed first.

Cohort A – 2011 (Author 2 Only - Fully Online: University One)

As can be seen from the data in Table 2, when compared with other courses, and other faculties, the PSTs evaluated the course as being very successful.

Table 2:
Student Evaluation of Teaching (Cohort A-Fully Online: University One – Identifying data omitted).

Comparative Means Questions with a scale of 5, and from the SEC survey	No. Ans	Resp. Rate	Mean							Std Dev	% Pstve
			Class	Course	Dept	Disc	Faculty	Campus			
SEC01: The learning outcomes and the expected standards of this course were clear to me	124/223	56%	4.48	4.48	4.01	3.99	3.99	3.93	3.93	0.78	89.5
SEC02: Course content was presented in ways which greatly assisted my learning.	124/223	56%	4.47	4.47	3.92	3.87	3.87	3.75	3.76	0.91	86.3
SEC03: Where relevant, teaching in this course helped me to learn effectively.	124/223	56%	4.48	4.48	3.86	3.84	3.84	3.70	3.71	0.86	87.9
SEC04: I was able to obtain individual help in this course when I needed it.	123/223	55%	4.58	4.58	4.00	3.98	3.98	3.79	3.79	0.74	88.6
SEC05: I have learned and understood the subject materials in this course.	123/223	55%	4.41	4.41	4.12	4.11	4.11	3.99	3.99	0.87	88.6
SEC06: The learning resources (library, study guides, handouts, text, web resources, etc) were adequate for my study in this course.	123/223	55%	4.41	4.41	3.98	3.96	3.96	3.86	3.86	0.91	86.2
SEC07: I have learned to make connections between this subject and others.	123/223	55%	4.32	4.32	4.10	4.08	4.08	3.96	3.97	0.85	84.6
SEC08: Overall I have learned a lot in this course.	122/223	55%	4.57	4.57	4.10	4.08	4.08	3.99	4.00	0.81	91.8
SEC09: The workload in this course was too high.	123/223	55%	3.02	3.02	2.86	2.87	2.87	2.98	2.97	1.07	30.1
SEC10: The assessment tasks were appropriate to the aims of this course.	124/223	56%	4.27	4.27	4.02	4.01	4.01	3.94	3.94	1.03	83.9
SEC11: The criteria used to assess student work were clear.	123/223	55%	4.37	4.37	3.81	3.81	3.81	3.77	3.78	0.96	87.8
SEC12: Feedback from assignments was timely.	123/223	55%	4.07	4.07	3.76	3.74	3.74	3.69	3.69	0.93	74.8
SEC13: My understanding of the subject has improved as a result of feedback from assignments.	123/223	55%	4.07	4.07	3.67	3.66	3.66	3.58	3.59	1.02	69.1
SEC14: Overall I was satisfied with the quality of this course.	122/223	55%	4.47	4.47	3.89	3.87	3.87	3.74	3.75	0.86	86.9

The major innovation in the course was the provision of a weekly Wimba (online classroom) tutorial with much of the feedback (37 comments) discussing various positive aspects of this innovation. Sample comments included *"Wimba tutorials- opportunity to feel part of a community of learners"* {SET2011} and *"Wimba classes were great – live opportunity to see resources and to ask questions and gain from other class members"* {SET2011}. This innovation enabled the provision of a pedagogical space for me to connect synchronously with the PSTs online, where effective mathematics pedagogy could be demonstrated. I was therefore able to replicate many of the affordances of a F2F teaching environment in this virtual space.

Cohort B – 2015 (Author 1 and 2 - Blended 1-1-1: University Two)

Cohort B comprised PSTs at University Two who received very similar content to the earlier cohort. The major innovation in this course was the blended delivery (online lectures and F2F workshops and tutorials). A further modification was the inclusion of video of the lecturer in the online component (as opposed to just audio and PowerPoint slides in previous offerings of this course in 2013 and 2014). Once again, the course was evaluated favourably

(See Table 3) via end of semester student evaluations. Feedback from the ICS also indicated a positive experience suggesting that the use of video was very important for their overall success. Sample ICS responses report both: gains in attention - *"I have noticed that I pay more attention to the video lectures as opposed to my zoning out on the non-video lectures. It's not you, it's me. I just learn better visually"*{ICS2015}; and also a heightened sense of connection with me as the lecturer - *"I prefer the video as it feels like I am at a real lecture and it feels more personal. Without the video I feel like that, as a student, I don't really mean anything"*{ICS2015}.

Table 3
Student Evaluation of Teaching (Cohorts B, C & D: University Two).

Descriptor This staff member…	Mean (2015 n = 53) Blended*	Mean (2016 n = 31) Face to face*	Mean (2017 n=49) Blended* (New Lecturer)
Presented material in an organised way	4.8	4.8	4.8
Presented material in an interesting way	4.7	4.9	4.9
Treated me with respect	4.8	4.9	5.0
Showed good subject matter knowledge	4.9	4.9	5.0
Overall I am satisfied with the teaching	4.8	4.9	4.9

*Results are mean scores from across three campuses rounded to the nearest tenth. n = number of responses

Cohort C – 2016 (Author 1 and 2 - Fully Face-to-Face: University Two)

Cohort C comprised students at University Two and the major innovation in this course was the delivery of all lectures, workshops and tutorials in F2F mode. As was the case in each of the two previous course offerings, end of semester student feedback (Table 3) indicated a positive course experience. In addition, data from the 66 In-course survey respondents generally indicating a preference for the full F2F course experience (54 positive, 6 neutral and 6 indicating a preference for the blended mode). Sample ICS responses included "*The F2F lectures are more interactive and the questions people ask are often interesting*"{ICS2016}; and *"I prefer F2F as it gives you more of a chance to engage with the lecturer/content and to ask questions and to have them answered"* {ICS2016}. One of the neutral PST responses noted that *"Both online and F2F have benefits but I prefer the F2F model as I can stay motivated and keep coming to the lectures instead of missing the lecture videos in Mathematics One"*{ICS2016}.

Cohort D – 2017 (Blended 1+1+1 with New Lecturer: University Two)

Cohort D comprised PSTs at University Two who received an identical mode of delivery to that outlined earlier for Cohort B i.e. one-hour online lecture, one-hour F2F workshop and one-hour tutorial. The only difference for this course offering was that all F2F teaching, including weekly workshops and tutorials, was delivered by a new Lecturer (Author 1). The first author had previously worked on the course as a tutor and was experienced in the course assessment and practices in earlier course offerings. The previous lecturer (Author 2) remained involved in the course as course convenor. Author 2 had taken a step back from course teaching due to commitments to a large national research project. Despite the change in the teaching team, the course was still evaluated very favourably (See Table 3) via end of

semester student evaluations and maintained the high scores obtained in previous course offerings.

Feedback from the ECS indicated that the structure and specific combination of teaching approaches were beneficial to student learning outcomes. With specific reference to the online lecturers, sample responses included – *"The online lectures were structured brilliantly and the workshops/tutorials are useful in complementing this information"*{SET2017}. Furthermore, students felt the combination of face to face and online components was balanced and offered enjoyment in learning the mathematical content that some students find difficult to comprehend and appreciate – *"The one hour lecture followed by a one hour workshop and concluding with an one hour tutorial provides a balanced learning experience where we get plenty of hands-on learning"*{SET2017}; *"I enjoyed how there was a variety of ways the content was taught- online, workshops and tutorials etc."*{SET2017}. An important point to note about this cohort is the emphasis in the end of semester feedback on the structure of the course and not on the personnel involved in delivering the course.

Although each of four cohorts received a very different mode of course delivery (online, blended or F2F) or team personnel, the combined data indicates that the PSTs in each respective cohort judged each course as very successful. Thus, regardless of the mode, the student satisfaction scores remained well above average across the four cohorts and in the very high 4.7-5.0 range for Cohort B, C, & D. Clearly then, this quantitative data indicates that neither the mode of delivery nor the team personnel are the sole factor in the success of the course. Interestingly, there was little mention in the end of semester feedback by either Cohort B PSTs of the use of online lectures, or by Cohort C PSTs of the delivery of F2F lectures. This lack of end of course commentary, regarding mode of delivery, is a critical point and provides further qualitative evidence that the mode of delivery is only one of the determining factors in the success or otherwise of the courses. The question therefore remains - If not mode of delivery or team personnel, what factors contributed to the success of the course over a five-year period?

Factors Impacting on Student Success

The major themes that emerged from the data collected from the four cohorts were: a) course structure and dialogue; b) MCK and PCK of lecturers; and c) rapport with the lecturer and with the discipline of mathematics.

Structure and Dialogue

One possible explanation for the success of the three courses was careful attention to structure and dialogue, as informed by Transactional Distance Theory (TDT). Moore and Kearsley (1993) suggests that TDT accounts for the psychological and communications space that occurs between learners, which is shaped by the learning environment and by the patterns of activity of individuals within the environment. TDT is influenced by two core factors: the structure of the program and the dialogue that exists between the teacher and the learner. These both impact on the level of autonomy required by each individual PST to successfully complete the course. Structure and dialogue can be manipulated to cater for, in this case, PSTs studying mathematics education via various modes of course delivery. Structure refers to the extent to which an educational program, or course within a program, can be responsive to the learning needs of individual PSTs. Dialogue refers to the interplay of words and actions between teacher and learner, and learner and learner, when one gives

instruction and the other responds. Much of the SET feedback across the four cohorts indicated that the courses were successful because of their structure (e.g. the course management system [Blackboard in all cases] and the availability of digital mathematics resources; and secondly on how dialogue was managed within the courses (e.g. F2F interactions [either physical or virtual or both], email correspondence, or course announcements. Of course, the mode of lecture delivery is one component of the overall course structure; however, as noted earlier, mode of delivery was barely mentioned in the end of semester feedback. Instead, the majority of feedback focused on the well-organised structure of the course as well as the following two factors.

Lecturer Content and Pedagogy Knowledge

There is a clear expectation from professional mathematics bodies that, upon graduation, PSTs are knowledgeable about best practice in mathematics education, including knowledge of students, knowledge of mathematics, and knowledge of students' learning of mathematics (Frid, Goos, & Sparrow, 2008/2009). Given these expectations, a key teaching goal is ensuring that the PSTs are competent and confident mathematics teachers. It is thus pleasing to see reflected in the SET scores of 4.9 or 5.0 (Table 3), an acknowledgement of our MCK and MPK. PSTs typically commented positively on the interconnected nature of our mathematics knowledge *"Very knowledgeable about all content and teaches in a very interesting way to cater for all learners"*{SET2015}; *"Current and up to date understanding of mathematics education – great for confidence in the classroom"*{SET2015}; and *"Teacher displayed high level of confidence in her content knowledge and was able to explain mathematical concepts, theories and ideas at both very basic and complex levels to ensure student understanding"*{SET2017}.

Knowing PSTs as Individuals

Although our focus in these courses is, by design, mathematics education; our teaching of mathematics exists within the broader framework of training PSTs to be primary school teachers, not solely specialist primary school mathematics teachers. The third piece of the "success puzzle" therefore transcends just mathematics education and instead reflects the fact that our teaching philosophy, regardless of mode of delivery, is highly relational i.e. it recognises that learning can only occur when a positive learning relationship has been established between the learners and the lecturers. Therefore, we always prioritise the building of rapport and PST engagement. PST feedback suggest that these endeavours were successful - *"Very interesting, clear, approachable and builds a rapport with students, engaging, knowledgeable, understanding and helpful"*{SET2015}; *"XXX positive attitude towards mathematics and passion for learning mathematics radiates and encourages students to learn"*{SET2016}; and *"XXX is a highly approachable, enthusiastic tutor, who genuinely wants the best for every student that she teachers"*{SET2017}.

Conclusion

Based on our five-year experience of delivery of mathematics education across two universities, we are confident that the mode of delivery is not the critical factor in the overall success of the courses. This is an important contribution given that the views of many mathematics educators often focus on the mode of delivery as the determining factor in the success or otherwise of the mathematics education courses they teach. Based on the data in this project, this is clearly not the case. A second observation is that success is often seen by

educators as a consequence of what might be termed "the great teacher" construct. This has been the experience of Author 2 who, in presenting previously on the issue of mode of delivery, has been challenged by colleagues that the success of the course is due to the teacher and not due to the structure of the course guided by TDT principles. Whilst we acknowledge that a specific teacher can play an important role in the effectiveness of courses; as this course has been successful, with different lecturers, we are confident that the teacher is not the sole determining factor in course success. Rather than relying on the mode of delivery or the specific personnel to explain success, the data suggests that it is more important for academics, when planning and delivering PST mathematics education courses to: focus on course design in terms of how the course is structured and the mode of delivery; ensure high personal levels of MCK and PCK and ensure students are engaged in the mathematical content (especially considering that many undergraduates commence mathematics education courses with a deficit view); and finally to commit to establishing and maintaining student rapport both with the teaching team and also with the discipline of mathematics to provide maximum opportunities for student learning. It is perhaps convenient to explain teaching success as dependent on external factors (mode of delivery, time allocations, etc.) or internal factors (i.e. teacher charisma) that are both difficult to change; however, we suggest that considerations of course structure, dialogue, teacher knowledge and building student relationships are much more important, and critically, are within the power of all academics to encompass in their mathematics education courses.

Acknowledgements

A book chapter (Larkin, 2017) is based on the Cohort B data. We refer to some of those findings in this paper.

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Aligning Online Mathematical Problem Solving with the Australian Curriculum

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The current Australian Curriculum mandates that technology be utilised to support students to "investigate, create and communicate mathematical ideas and concepts" (ACARA, 2018). However, research reports suggest that use of digital technologies in Australian primary mathematics often focusses on the lower-order drilling of algorithms and basic facts. In this study, an online environment provided the medium for Year 5 students to engage in collaborative mathematical problem solving. The students' online dialogue, along with uploaded diagrams and spreadsheets, provide evidence of their development of the Australian Curriculum: Mathematics' proficiencies of Problem Solving and Reasoning.

Introduction

Since the proliferation of the Personal Computer, much has been made of the potential for technology to transform teaching and learning within primary and secondary education. The Australian Curriculum mandates that technology be utilised to support students to "investigate, create and communicate mathematical ideas and concepts" (ACARA, 2018). For this technology to assist learning, teachers need to think carefully about when and how it is used. Niess (2005), for example, argued that:

> for technology to become an integral component or tool for learning, science and mathematics preservice teachers must also develop an overarching conception of their subject matter with respect to technology and what it means to teach with technology—a technology PCK (TPCK) Pedagogical Content Knowledge, Technological Pedagogical Content Knowledge)] (p. 510).

Currently, it is rare to observe digital technologies-based mathematics teaching occurring in the symbiotic manner to which Niess (2005) refers. Instead, as discussed below, we see a focus on drill and practice-based activities. This paper reports an approach that may offer opportunities for teachers to make more effective use of technology integration within their primary mathematics teaching. The research question addressed is:

How does student engagement with online mathematical problem solving align with the Australian Curriculum?

A brief review provides background related to technology use in mathematics education. This is followed by details of the study, data analysis, then results, discussion and implications.

Digital Technology use in Mathematics Education

Zbiek, Heid, Blume, and Dick (2007) draw attention to research driven, historical, alternative approaches to technology integration in the mathematics classroom. The most common approach, they note, develops technical (or skill and procedure focused) proficiencies while the other promotes conceptual development (finding patterns, conjecturing, generalizing, connecting representations, predicting). However, they suggest that while technology may 'free' students from laborious computation, in many cases, students must have conceptual understanding of the mathematics to successfully operate the technology to execute the required operation. Much of the work of Zbeik et al. (2007) refers to the use of sophisticated computer algebra systems and hence allows us only limited understanding within the primary

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school mathematics setting. As Day (2013), Kuiper and de Pater-Sneep (2014), Turvey (2006) have reported, a large proportion of time spent on technology integration within the primary mathematics classroom is often assigned to drill-and-practice mathematics software that is conveniently and freely available on the Internet. Day's (2013) study of 118 primary schools in Western Australia shows that teachers, school administrators and pre-service teachers all believe that ICT integration has the potential to lead to conceptual knowledge development. However, the evidence presented suggests that the resources most commonly used do not target this goal. She found that 80% to 90% of ICT integration for mathematics referred to 'the Internet' and typically involved students playing games that encourage practice of routine procedures. Day's (2013) findings support the claims of Herrington and Kervin (2007) who stated that, technology was often employed for all the wrong reasons, for example: pressure from school administrators and the belief that students need to be entertained. While Sinclair and Yerushalmy (2016) note reports of primary teachers sharing mathematical thinking and pedagogy online but this did not extend to primary students. Changes to mathematical discourse, whether linguistic or non-linguistic, may correspond to changes in students' mathematical thinking and access to mathematical software potentially enriches the possibilities for mixed discourse.

The conclusions of this earlier research suggest that achieving the Australian Curriculum goals will entail pedagogy promoting discourse and deeper conceptual understanding of mathematics. This paper informs such an approach to the use of ICT in primary mathematics.

Background and Method

Context

The study was conducted over a nine-week period in a Melbourne state primary school with an Index of Community Socio-Educational Advantage closely matching the state average. Participants were Year 5 students: 26 boys and 28 girls (10 to 12 years old). They were allocated to 10 mixed ability groups of 3 to 6 students within an online space. Groups were created based on prior judgments, by their teacher, classifying students as below, at or above level in mathematics. The context of the online work was Edmodo, a freely available online computer supported collaborative learning environment. Throughout the intervention students were supported by the first author with a weekly classroom session that typically took the format of a short review of online interaction from the previous week, followed, in weeks one to seven, by discussion of a new problem to be solved, this included technical information that might help students use available software (Edmodo, MS Excel, MS Word etc). Early on, time was spent outlining appropriate online behaviours and promoting discussion that goes beyond superficial chat. The last two weeks of the intervention were different. No discussion preceded online mathematical problem solving; students immediately started their work in the online space. Work in weeks 8 and 9 was therefore less influenced by the researcher (facilitator).

Coding and Analysis

The mathematical proficiency strand of Problem Solving as prescribed by the Australian Curriculum: Mathematics (ACARA, 2018) is summarised in Figure 1 as progressive steps towards problem solutions. Figure 2 summarises the description of the Reasoning proficiency of the Australian Curriculum: Mathematics (2018). This also sets out sequential strategies although each could be evident with different levels of sophistication. These formed frameworks used for the analysis of students' online work. The purpose of using the Australian Curriculum descriptors as a source for coding and analysis of the data was to explicitly look for evidence of if and how student engagement with online mathematical problem solving

aligned with the Australian Curriculum. This research considers all data shared via Edmodo: both the text of students' online discussion and artefacts (graphs, diagrams, tables, pictures)

Analysis of data was supported using qualitative analysis software, NVIVO (International, 2015). Data consisted of individual students' online work. Classroom discussion was not recorded. The themes from Figures 2 and 3 were designated as nodes. Each file (discussion or added artefact) was separately multi-coded for problem solving abilities, and then again for the analysis of reasoning. This indicated whether students were likely to engage in skill or conceptual development during online discussions or while constructing supporting artefacts. Coding was undertaken by two researchers independently of one another. Over 85% inter-rater reliability was achieved, and the remainder was agreed upon through discussion.

Mathematics	Problem Solving: Students Developing ability to:	Make Choices
		Interpret (understand the problem)
		Formulate (Use Procedure)
		Model and Investigate Problem Situations

Figure 1. Australian Curriculum Framework for Analysis of Problem Solving.

Mathematics	Reasoning: Students Developing ability to engage in:	Analysing
		Evaluating
		Explaining
		Generalising (use of explicit equation formula)
		Inferring (placing understanding in new context)
		Justifying
		Proving

Figure 2. Australian Curriculum Framework for Analysis of Reasoning.

Extracts of two examples of set problems, with layout compressed, are shown in Figure 3.

Week 3: Wallpaper patterns are an example of symmetry in our everyday lives.
Record your groups' understanding of the different types of symmetry in this space …
You can find a very good explanation of these in the folder attached to week 3.
Now your group will design a wallpaper 'block'. You will need to decide how to incorporate symmetry into your block. Your group will create (and upload to this space) one sheet of wallpaper using Microsoft Word. …Most importantly… Your group will write a summary of which types of symmetry you have used and how you have incorporated these into your wallpaper designs. These will be posted within this forum.

Week 4: What is the biggest breed of dog?
Research a variety of dogs using your netbook. Decide what 'biggest' means. Provide a definition. Your group will have to decide whether they think 'biggest' means heaviest, tallest, longest etc How do breeders measure this? Create a graph in Excel representing the data you have found.
Horizontal axis (x axis) should be breed of dog and vertical axis (y axis) should be height/ weight/ length etc.
Upload the graph that you have made to this message board.
Which dog, according to your definition, is the 'biggest'? Discuss any other facts that you can 'read' from the graph that your group has created?
Think about another measurement you can use to define 'biggest' e.g. If you defined 'biggest' as height of the dog last time, you might like to use weight this time. Create a new graph.

Figure 3. Extract of two examples of problems set for students (Symons, 2017)

Results and Discussion

Problem Solving

Coding and analysis of students' online discussion data and uploaded artefacts shows that individual students made use of the online environment in subtly different ways. Table 1 shows that when students used Excel, the files that they created and uploaded almost all involved formulation (the use of procedures), interpretation (evidence that the students either fully or partly understood the problem), making choices and modelling and/ or investigation of some aspect of the problem. Of the 97 Excel files 95 displayed evidence of the development of these themes.

Table 1
Frequency of examples of Problem Solving Categories evident in uploaded artefacts

Category	Excel Artefacts	Paint Artefacts	Word Artefacts
Formulate (Use Procedure)	96	3	59
Interpret (Understand the Problem)	95	3	59
Make Choices	96	3	59
Model and Investigate Problem Situations	96	3	60

Of the 65 Word Files uploaded, 60 showed evidence of at least one of the mathematical problem-solving categories. In the five files where these themes were not detected students were providing general (non-mathematical) background information about the problem. The three Paint artefacts showed evidence of all four problem solving categories.

Table 2 provides an indication of differences between student use of the various software for different mathematics. While across the nine weeks the overall focus was on problem solving, as indicated, each week the problem drew on one Australian Curriculum mathematics content area: Measurement and Geometry (M&G), Statistics and Probability (S&P) or Number and Algebra (N&A). Students were encouraged to use any software, depending on which they believed would best support their thinking and communication of ideas. Uploading supporting artefacts was introduced to students in week 2, hence. the absence of artefacts in the first week.

Table 2
Comparison of Student use of Software in Uploaded Artefacts Across Problems

	Excel Artefact	Paint Artefact	Word Artefact
Week 1 - Toilet Roll (M&G)	0	0	0
Week 2 - 10 Hour Day (N A)	0	0	1
Week 3 – Symmetry (M&G)	0	0	30
Week 4 - Biggest Dog (S&P)	26	0	8
Week 5 - Animal Ages (N&A)	30	0	5
Week 6 – Shapes (M&G)	0	3	15
Week 7 - Pet Names (S&P)	10	0	0

Week 8 - Mr M's iPhone (N&A)	27	0	0
Week 9 – Geese (N&A)	3	0	6

It is apparent that when students developed and communicated their thinking in a problem where M&G was a focus, while we might have expected them to use Paint, students preferred to use Microsoft Word. This was the package with which they were most familiar. In week 3, students were asked to create a panel of wallpaper, representing their understanding of symmetry. Of the 30 artefacts uploaded all utilized Microsoft Word for this problem. When students were asked to investigate four sided shapes in week 6, again most chose to use Microsoft Word. In this problem, while the auto-shapes function within Word was heavily employed students also organized their thinking by using tables in Word

Table 3 (below), represents how students engaged in the four categories of MPS in online discussion compared to their level of engagement with these categories when constructing artefacts. It indicates that students more commonly showed evidence of engaging with these important categories when representing their mathematical thinking through creating a representation within their uploaded artefacts. It is interesting that in the 'Interpret' category the distribution is more evenly shared between artefacts and online discussion. This indicates that students unpacked and discussed their ideas with each other in order to ensure they fully understood the problem.

Table 3
Comparison of Development of Problem Solving Concepts in Artefacts Vs Online Discussion

Category	Examples within all Uploaded Artefacts	Examples within Online Discussion	% of Developing Concepts in Uploaded Artefacts	% of Developing Concepts in Online Discussion (Excluding Artefacts)
Formulate	158	54	75	25
Interpret	157	109	59	41
Make Choices	158	103	61	39
Model and Investigate Problem	159	67	70	30

Reasoning

Analysis of data suggests that the types of reasoning engaged in over the period of the intervention changed from week to week. It appears that students were able to engage in analysis, evaluation and explanation throughout, however it is not until week four that consistent evidence of students engaging in generalizing, inferring and proving occurs. The evidence here suggests that students' development of reasoning skills progressed over the course of the intervention. A comparison of week 2, when students devised timetables for 10-hour days with 100 minutes per hour and week 8, when students investigated an iPhone that progressively halved its battery life, suggests improvement in reasoning skills. By week 8 students were giving evidence of their thinking not just results. In weeks eight and nine students did not have the prompt of classroom discussion prior to their online work. Therefore, the fact that students showed evidence of all areas of reasoning in these weeks was encouraging.

Table 5 (below) shows the degree to which students who were assessed (by their teacher) as below level, at level and above level, in mathematics, engaged in reasoning throughout the

intervention. Across the three groups there was a fairly consistent strong level of analysis, evaluation and explaining occurring. This is interesting because it might be assumed that the students described as *below level* (by their teachers) would show less ability to engage in all areas of reasoning. Additionally, the lower ability students' engagement with the remaining skills (except 'generalizing') was of a similar level to that of their peers. All students demonstrated fewer instances of generalizing, inferring and proving. This is not unexpected given that these aspects of reasoning are considered to involve more sophisticated processes.

Table 4

Reasoning used Across Nine Weeks Evidenced by Online Discussion and Artefacts

	Analysing	Evaluating	Explaining	Generalising	Inferring	Justifying	Proving
Week 1	1	3	22	0	0	13	0
Week 2	3	4	6	0	0	1	0
Week 3	1	4	35	0	0	2	0
Week 4	32	43	64	0	2	6	0
Week 5	23	26	37	3	7	8	6
Week 6	12	15	24	0	1	1	0
Week 7	12	20	30	1	0	3	1
Week 8	35	38	50	3	5	17	5
Week 9	11	18	21	5	2	11	3

Table 5

Reasoning Categories and Teacher Allocated 'mathematical Ability'

	Below Level	At Level	Above Level
Analysing	36	51	43
Evaluating	48	72	51
Explaining	80	116	93
Generalising	0	4	8
Inferring	3	7	7
Justifying	18	26	18
Proving	4	3	8

Table 6 (below) splits the "ability levels" into girls and boys. There is evidence that the ability of boys to reason increased according to the ability group they had been assigned. For example, the *Below Level Boys* exhibited 8 instances of *Analysing*, the *At Level Boys* exhibited 13 instances of *Analysing* and the *Above Level Boys* exhibited 32 examples of *Analysing*. It is worth noting here that students had been evenly distributed across the three ability classifications. Thus, the tendency of this pattern to be replicated across the various reasoning categories is important. The reasoning of girls did not follow the pattern of increasing according to teacher assigned ability group. *At Level* and *Below level* girls showed evidence of a greater volume and variety of approaches to reasoning. This may indicate that the procedural tests conducted for the purpose of allocating ability groups may not provide teachers with adequate information about their students' ability to engage in mathematical reasoning. This is an issue for further research.

Table 6
Comparison of reasoning between Genders

	Below Level Boys	Below level Girls	At Level Boys	At Level Girls	Above level boys	Above Level Girls
Analysing	8	28	13	38	32	11
Evaluating	9	39	20	52	39	12
Explaining	16	64	29	87	67	26
Generalising	0	0	1	3	7	1
Inferring	0	3	4	3	3	4
Justifying	3	15	11	15	13	5
Proving	1	3	2	1	7	1

Conclusion and Implications

Across the world, curricula are now requiring teachers to teach and assess problem solving and reasoning. It is no longer enough to require students to 'do mathematics' rather, there is an increasing expectation that students should be able to demonstrate an ability think, behave and communicate mathematically (Boaler, 2008). In the USA, The Common Core Standards for mathematics, released in 2010 (NGA Center, 2010) for the first time included *Standards for Mathematical Practice*. These detail the level to which students should be able make sense of, persevere with, reason, argue and critique, model and choose the appropriate tools and strategies when engaging in mathematical activity. In Australia, our national curriculum placed a new emphasis on problem solving, reasoning and communicating mathematics.

Traditional modes of assessment that privilege summative above formative approaches and focus on a student's ability to perform procedures, for example, through the use of regular mathematical content driven pre- and post-tests, provide limited information about the student's ability to engage mathematically. These traditional methods of assessment may provide the teacher with a skewed view of the range of abilities and levels of understandings of their students. Students who may correctly execute basic computation and procedures, sometimes beyond the level expected of them, may struggle to apply these skills to problem-based contexts. Conversely, some students who may not perform as well on the narrowly targeted tests may demonstrate ability to engage in the mathematical communication, investigation and reasoning associated with collaborative problem solving.

The online approach taken to problem solving in this study can circumvent some of these issues. The online platform provided a detailed record of interactions and supported the production of meaning making artefacts so necessary for teachers to make judgments about students' problem solving and reasoning.

The results of this study have shown that when engaged in MPS in the online environment students are more likely to use software that they are familiar with, even when other software might be more suited to a task. We have shown that when making mathematical meaning online student thinking associated with MPS is evident in the artefacts that they create, while their interpretation both of their own work and each other's work is seen in their online discussion. When considering student mathematical reasoning we have found that only after a number of weeks engaged in the scaffolded online MPS process were students starting to consistently display higher-order reasoning skills.

At best, a teacher within a classroom will have the opportunity to observe each group of students for a few moments within a session. From the online environment, a teacher can

review all interaction and discussion. In a traditional classroom where discussion is encouraged, it can be very difficult for the teacher to know that the discussion ace in each small group is productive and related to the mathematical problem being investigated. The approach offered in this study allows teachers easy access to data related to the amount of time each group remained 'on task'.

Whilst there have been intentions for the integration of digital technologies within mathematics instruction to allow for the communication, representation and investigation of mathematical ideas and concepts over many curricula, for many years, as Day (2013) has reported this is rarely achieved in Australian primary mathematics classrooms.

The data from this study supports the value of online mathematical problem solving with upper primary students as a strategy for achieving the goals of the Australian Curriculum Mathematics (ACARA, 2014) in the proficiencies of problem solving and reasoning.

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Participatory Task Science: The reSolve story

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In this paper we introduce the term "task science" to encompass the range of activities involved in designing tasks for school mathematics. We argue that task science is enriched by the participation of teachers, but more particularly that participating in task science is a powerful form of professional learning. We describe the role and design of task science in the reSolve: Maths by Inquiry project, and give examples of how teachers' involvement in each phase of the process was both critical in developing the resources and promoted rich professional learning.

Tasks have long been recognised as central to mathematics teaching (Anthony & Walshaw, 2009; Jones & Pepin, 2016). Just as it is through experiments that students obtain a sense of what it is to do science, it is through tasks that students get their sense of what it is to do mathematics (Henningsen & Stein, 1997). Hence, the tasks in which students engage need to provide opportunities for students to encounter new and challenging ideas, and inquire into and solve meaningful problems.

Well-designed tasks have the potential to fulfil a number of important functions in school mathematics education. These include:

- Shaping students' mathematical understanding (Stein, Grover, & Henningsen, 1996);
- Providing students with opportunities to learn (Goos, 2014);
- Promoting cognitive activation (Russo & Hopkins, 2017);
- Developing students' mathematical reasoning and giving opportunities for students to communicate their reasoning (Choppin, 2011; Choy, 2016);
- Encouraging generalisation (Papadopoulos & Iatridou, 2010);
- Promoting positive dispositions towards mathematics (Attard, 2013);
- Enabling teachers to better understand and act on students' thinking (Didis, Erbas, Cetinkaya, Cakiroglu, & Alacaci, 2016); and
- Prompting students to make inferences and connect ideas (Fielding-Wells, O'Brien, & Makar, 2017)

Using good tasks is therefore critically important if teachers are to help students realise the range of goals for mathematics education described in, for example, the Australian Curriculum: Mathematics (Australian Curriculum Assessment and Reporting Authority, 2018). We suggest that there are two inherent dangers in separating the roles of teaching and task design: the first is that the task designers may lose sight of the perspective of teachers; the second is that teachers may fail to see the intent of the designer. We therefore describe a process in which teachers are involved at all stages of task design and discuss the implications for both task quality and teacher learning.

In the first section we describe the reSolve: Maths by Inquiry project, and particularly the resource development process that we undertook with teachers. We describe this as *task science*

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rather than *task design* in order to emphasise the full range of activities involved in prioritising, conceptualising, selecting, designing, and implementing tasks in the school mathematics classroom. In the second section we focus on the impact that engaging in task science has on the professional learning of teachers. Using data obtained from participant reflections, we describe how the process of engaging in task science impacted on the personal domain, the domain of practice, and the domain of consequence (Clarke and Hollingsworth, 2002). We conclude by suggesting that participatory task science opens up a range of opportunities for improving professional learning at scale and could become a productive field of research in its own right.

Task science in the reSolve: Maths by Inquiry project

The reSolve: Maths by Inquiry project.

reSolve: Maths by Inquiry is an Australian Government funded project designed to promote a spirit of inquiry in students from Foundation to Year 10. The project is managed by the Australian Academy of Science in collaboration with the Australian Association of Mathematics Teachers. It has two specific but overlapping aims: the first is the development of a coherent suite of resources promoting mathematical inquiry; the second is the engagement of the profession. The suite of resources includes professional learning resources focused on important elements of inquiry, highlighted by exemplary classroom resources addressing key components of the Australian Curriculum: Mathematics (ACARA, 2018). Engagement of the profession occurs through a cadre of almost 300 Champions across all states and territories of Australia, each of whom undertakes a 12-month professional learning and networking program.

The project's philosophy is built around what we term the reSolve: Maths by Inquiry Protocol (Thornton, 2017). The Protocol articulates those elements of the mathematics, tasks and learning environment that we believe will promote a spirit of inquiry. The three key elements in the Protocol are:

- reSolve mathematics is purposeful
- reSolve tasks are challenging yet accessible
- reSolve classrooms promote a knowledge-building culture

By *mathematics that is purposeful* we wish to challenge perceptions that mathematics is merely a body of disconnected facts or procedures described in a curriculum document. We highlight connections between mathematical ideas and between mathematics and the real world by focusing on important mathematical ideas that give students power in their lives. We seek to acknowledge mathematics as a creative and imaginative endeavour, continually changing and developing in a technological society.

By *tasks that are challenging yet accessible* we wish to challenge perceptions that mathematics is for the few, and assert that it ought to be both challenging and accessible for all. reSolve tasks seek to activate existing knowledge and to develop new knowledge through the exploration of relationships between key ideas. The tasks are designed to engage students in sustained inquiry, problem solving, decision making and communication with the goal of optimising their mathematical development. They use evidence of students' progress to inform feedback and subsequent teaching action, and provide prompts and activities that meet a range of student capabilities.

By *knowledge-building culture* we wish to challenge a view that mathematics is best learnt through demonstration, reproduction and repetition. We seek to promote environments that sustain higher order thinking through the active role of both teachers and students and that build success through collaborative inquiry, action and reflection. We seek to challenge

existing student ideas or misconceptions and use mistakes as opportunities for learning. We seek to build positive dispositions such as productive struggle and the confidence to take risks.

The reSolve resource development process.

The resource development process employed in the reSolve project follows a five-stage design thinking process (Figure 1), which we refer to as *task science*. The process of task science is actualised on two levels. The first is the level of the reSolve Design Team (RDT) comprising writers employed to oversee the process and to prepare resources to publication standard. The second level is the Collaborative Design Team (CDT), which includes up to 15 invited teachers and one or two invited academics. At least one CDT has been established in each Australian state and territory.

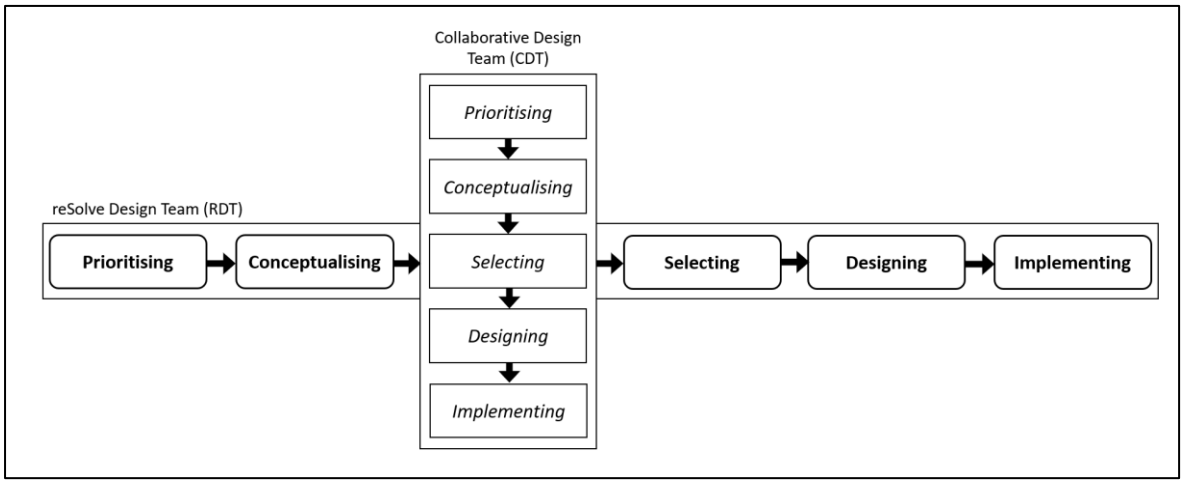

Figure 1. The reSolve: Maths by Inquiry task science process.

The first stage of the process, which occurs at the design level, is *prioritising* a particular mathematical focus for the tasks. The RDT identifies areas of the Australian Curriculum: Mathematics that may be underserved in currently available resources or for which we see a strong need for elaboration. For example, we identified multiplicative thinking as a topic that was well researched (e.g. Siemon, Breed, Dole, Izard, & Virgona, 2006), but for which good resources were not widely available. From here, we *conceptualise* a learning progression for the topic which aligns with the curriculum and draws from relevant research. The learning progression attempts to articulate the key developmental growth points for students, which then provides a focus for the collaborative design process to follow.

Each member of the CDT is then sent a paper outlining the thinking of the RDT, along with the proposed learning progression and accompanying research articles. The RDT, teachers and invited academics then meet for a two-day workshop where the process of design commences on the second level with the CDT.

On the first day, the teachers use the learning progressions to *prioritise* a focus for resources for the given topic. They are specifically asked to consider three questions:

- What do students experience in this topic?
- What is missing?
- What is there too much of?

For multiplicative thinking, the CDT felt that there was a large selection of resources developing the idea of the array, but that the idea of *for each* (Cartesian product) was not well resourced. The CDT is then asked to *conceptualise* by brainstorming ideas that might bring these priorities into sharp focus. The group then *selects* a number of these that show particular promise in terms of their capacity to address the needs identified in the prioritisation exercise.

The second day is focused on the process of developing ideas and *designing* tasks that build on these ideas. Prototype tasks are developed that attempt to capture the reSolve spirit of inquiry and that enact the three central elements of the Protocol. Teachers then take these tasks back to their own classrooms to *implement* them as a proof of concept. For multiplicative thinking, a sequence of resources developing the 'for each' idea was conceptualised for Year 2 and then Year 4. In year 2 students play a version of "Go fish" in which they find robots that have unique combinations of heads, bodies and legs, leading to an investigation of how many are possible. In year 4 the ideas are extended to a more formalised notion of Cartesian product in which students are challenged to design their own avatar and represent Cartesian product as a tree diagram.

At this stage the tasks are then passed back to the RDT who make decisions on which of the prototype resources should be selected for further development and refinement. They are then carefully documented and designed, paying particular attention to the choice of contexts, choice of examples, and optimal sequencing to best enact the student learning progression. They are also put into a form that is consistent with other reSolve resources, and made available for implementation through widespread trialling and focused feedback designed to capture teachers' views on how the resources enhance students' engagement and understanding. The design and implementation are repeated as often as necessary to take account of feedback from the field.

While the primary goal of the process described above is to enhance the quality of resources, informal comments from participating teachers highlighted the value of the process in their own professional learning. Accordingly, we designed a small-scale qualitative study to examine the impact of the process on teacher learning.

The impact on teacher learning

Theoretical background

We used the Clarke-Hollingsworth interconnected model of professional growth (Clarke & Hollingsworth, 2002) as a framework for structuring our thinking and methodology. The model identifies four domains that encompass the professional world of teachers: the personal domain (teacher knowledge, beliefs and attitudes), the domain of practice (professional experimentation), the domain of consequence (salient outcomes) and the external domain (sources of information, stimulus or support) (see Figure 2). These four domains of the model are *change* domains, that is, any professional change observed can be located within one or more of these domains. For example, a teacher who tries a new pedagogical strategy is engaging in a form of professional experimentation and so this change is located within the domain of practice. Changed expectations of students is a change in beliefs and attitudes and will therefore be situated in the personal domain.

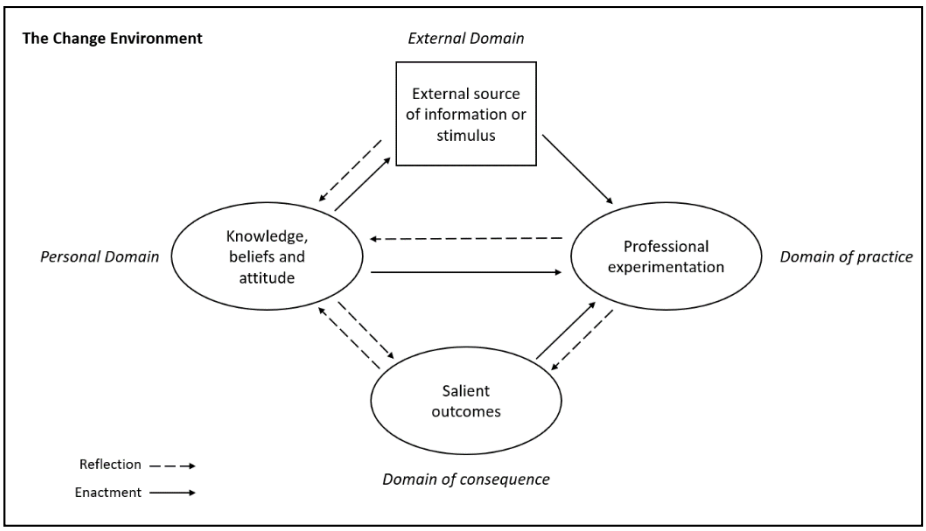

Figure 2. The Clarke-Hollingsworth interconnected model of professional growth

For the purposes of this study we interpreted these change domains as:

External Domain: The reSolve Protocol, and teachers' collaboration with other members of the CDT, (Wilkie & Clarke, 2015) informed by the documentation prepared by the reSolve team and invited academics.

Personal Domain: The reported impact on teachers' knowledge, attitudes and understanding towards inquiry.

Domain of Practice: Teachers' reporting of their experimenting with ideas and developing tasks in collaboration with others in the workshop; their feedback on how they modified tasks based on trialling in classrooms.

Domain of Consequence: The value of the process in developing reSolve resources; the impact of the process on teachers' self-efficacy and observations of students' engagement and understanding.

Methodology

We designed a survey around the change domains identified above. Teachers were asked to comment on how they were professionally challenged and affirmed within each domain, and were then asked whether they had noted changes in the domains or if they anticipated future changes to surface. Teachers were also asked to comment on the value of their participation in the workshop as professional learning and if they would look for further opportunities to participate in similar experiences in the future. The survey was sent to 84 teachers and 29 responses were received. These responses were collated and summaries were sent to the original 84 teachers for confirmation. The teachers were asked to add further comments if they believed the summaries missed information that they believed to be important.

Results

Each of the three authors independently read the teachers' responses and suggested how they might be categorised. Further discussion led to the development of five locations in which changes might be observed: personal, teaching, students, community and resources. These were combined with the domains identified by Clarke and Hollingsworth to construct a two-dimensional model through which we summarised the key observations from the teachers' responses to the survey. The model is shown in Table 1, along with indicative quotes to illustrate the changes in the domains.

Table 1
Categorisation of Teacher Responses with Indicative Quotes

	Personal Domain	**Domain of Practice**	**Domain of Consequence**
Personal	"I though maths was black and white as there was one answer for each question. It opened my mind to open-ended questions and the endless amount of thinking a question can generate."	"…successful mathematics learning requires a pedagogical shift from transmissive to challenging teaching where students are active (not passive) in their learning."	"It made me think deeply about the different aspects of teaching multiplications and ways to develop rich tasks."
Teaching	"It encouraged me to deepen my content-knowledge and pedagogical content knowledge in early years foundational maths skills."	"The importance of developing a culture of learning where ideas are challenged (not people)."	"…inspired! The ideas shared at the workshop were so great and creative and fun. That's what I want to see for my students."
Students	"The expectation of the students and how I have to challenge the more capable ones."	"Constantly pushing students outside of their comfort zones in mathematics."	"The students engage in mathematics to a greater extent. They are confident to take risks and enjoy the activities."
Community	"Continuing to hold myself and colleagues to a high standard."	"I have created a sequence of lessons myself for my teaching and colleagues."	"We changed the way we planned and taught maths in Prep."
Resources	"Quality maths lessons/investigations take a while to develop."	"I now will incorporate all the reSolve resources for my students and I will share them with colleagues."	"We have continued our work of designing rich mathematical tasks for students ensuring they have opportunity to explore, engage and experiment with mathematical concepts."

Discussion

Table 1 shows that teachers' involvement in the process of task science did not just achieve one goal, but many. In the personal domain, it impacted on: their sense of efficacy as a teacher; their mathematical and pedagogical content knowledge; the expectations they had of their students and colleagues, and; their understanding of the complexity of task science. In the domain of practice, it impacted on: their approach to teaching; the culture of the classroom, including challenging students to take risks, and; their commitment to work with colleagues to design tasks and to select and use rich tasks in their own teaching. Finally, in the domain of consequence, it: provoked deep thinking about mathematics and teaching; inspired creativity; engaged students cognitively and affectively; challenged teachers in their school context, and; stimulated ongoing task design among colleagues.

Three things stood out in the intensity and frequency of the teachers' responses.

1. The power of the external domain

While the value of high quality professional learning opportunities, often conducted by externally sourced experts, is well documented in the literature, the importance of professional collaboration and collegiality in the external domain (Wilkie & Clarke, 2015) is less frequently

described. Almost every participant in this study commented on the value of this collegiality, and described how the process of participatory task science positioned them as designers and developers of tasks, rather than as merely consumers and implementers.

2. The extent to which participatory task science affirmed and challenged already expert teachers

Although the teachers invited to participate in the workshops were already highly regarded and knowledgeable, many commented quite animatedly about the extent to which they were affirmed in their existing practice. Their self-efficacy was enhanced as they saw their ideas valued and shared in a national project. They expressed a deep commitment and desire to be involved in professional learning and to share the experience with their colleagues.

3. Participants' knowledge of the big picture

Participants frequently referred to their increased knowledge of how students develop understanding of particular aspects of mathematics. They commented that they saw more clearly how mathematical ideas were connected across grade levels as well as across content strands. The process of prioritising and conceptualising drew into sharp focus the stages of the learning progression, highlighting the gaps and excesses in existing resources.

Conclusions

We have described the process of participatory task science as enacted in the reSolve: Maths by Inquiry project. We suggest that the active involvement of teachers at all stages of task design is essential if we are to realise the full range of potential outcomes of well-designed tasks. This helps avoid the parallel dangers that task designers may lose sight of the perspective of teachers, and that teachers may fail to see the intent of the designer.

Equally significantly the act of participating in the workshops had a profound impact on participants' knowledge, practice and self-efficacy. Participants' reflections on their involvement in the process highlighted their enhanced sense of worth as a teacher, their increased knowledge of student learning progressions and their increased commitment to work with colleagues in a similar process. Indeed, a common response at the end of each two-day workshop was that this was the best professional learning they had ever done.

We have termed the process described in the paper *task science*. By using this term, we have tried to capture both the rigour and richness of the process. We suggest that this is much more than task design, a term which fails to capture the subtleties arising when tasks are enacted by expert teachers in the classroom. We suggest that this conception of participatory task science opens up a range of possibilities for further research into its impact on teacher learning, particularly into how it might be implemented at scale for sustained improvements in mathematics education.

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The Role of the Story in Enabling Meaningful Mathematical Engagement in the Classroom

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We draw on our work with designed instructional sequence on Fractions as Measures across three international contexts and explore the functions of the design features related to the stories embedded within the sequence. We discuss functions that relate to how stories support students' meaningful engagement in classroom mathematical activities, focusing, in particular, on students from under-resourced environments. While doing so, we highlight the mediating role of teachers who adapt designed resources to the specific needs of their classrooms. We clarify how instructional sequences can provide guidance for teachers' adaptations by making design rationales for mathematics and functions of the story explicit.

Educators and education researchers across subject areas recognise the learning potential of stories and storytelling experiences (Phillips, 2000). They found children to be drawn to narrative and demonstrated that storytelling, among other benefits, enhances children's imagination as well as their cognitive skills, contributes significantly to all aspects of language development, supports and extends their social lives (Britsch, 1992; Cooper, Collins, & Saxby, 1992; Raines & Isbell, 1994). Furthermore, stories can be used to explore complex notions such as social justice and active citizenship even with very young children (Phillips, 2012a, 2012b). We link to this literature and demonstrate how stories and storytelling can productively support classroom *mathematical* learning experiences, and help to guide children's reinvention of specific mathematical ideas.

Drawing on Benjamin (1955/1999), Nussbaum (1997), and Greene's (1995) work, Phillips (2012b) highlights how storytelling enables listeners to "connect with the characters and accompany the teller on the journey of experience" (p. 142). She posits that as a result, two things become possible. First, children can be led to *reach the understandings of humanity* "via the cultivation of sympathetic imagination that storytelling fosters" (ibid). Second, by having the capacity to "captivate people to see and feel the perspective of another" (ibid), stories *motivate relations, possibilities, and actions*. Part of our intent in this paper is to illustrate how these very issues matter, and can be leveraged, in teaching and learning mathematics in classrooms, and in designing *mathematics teachers' resources* (Visnovska & Cortina, 2018).

We pursue this intent by presenting what, at the moment, is a web of ideas and theoretical connections closely related to the various functions that stories served across our experiences of using a designed instructional sequence on Fractions as Measures (Cortina, Visnovska, & Zuniga, 2014) in different classroom and professional development design experiments in

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Mexico, Australia, and South Africa. We focus on the students, in particular from under-resourced environments, and document functions related to how stories support students' meaningful engagement in classroom mathematical activities. In doing so, we are mindful of the mediating role of teachers and consider supports that the sequence provides for their work of making reasoned adaptations, specifically in the space of stories. We now open the space of stories in our design work to closer scrutiny.

Stories and the Theory of Realistic Mathematics Education

The perspectives from research on storytelling deeply resonate with authors' experiences from mathematics classrooms and our own conceptualisations of how students' learning can be supported. Partly, this resonance stems from the shared grounding in humanistic philosophies, from foregrounding the students' active role in learning, and the expectation that effective pedagogies are inevitably those that are responsive to students' actual needs.

Our work as instructional designers is guided by the theory of Realistic Mathematics Education (RME; Gravemeijer, 1994), and rooted in Freudenthal's (1973) interpretation of mathematics as a human activity. In Freudenthal's view, mathematics is highly relevant to many human endeavours, and students should be given the opportunity to *reinvent* mathematics by organising or mathematising either real world situations or mathematical relationships and processes. In developing this position, Freudenthal emphasised that students should encounter the material they are to mathematise as being *experientially real* for them.

Problem situations and tools that are experientially real are those with which students can immediately engage in personally meaningful mathematical activity (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). In psychological terms, these would be problems and representations with which students can readily become imagistically involved (Thompson, 1996). Consequently, for a problem situation or a tool to be regarded as experientially real, it does not necessarily need to come from or be relevant to all students' everyday experiences. However, it has to be possible for all students to construe the problem and representations as personally meaningful and mathematically engaging, *with teacher guidance and support*. We would like to point out that we find it is unreasonable to expect students to construe in this way the problems that, in their view, can only be considered meaningful in school.

This is where we see the tenets of RME and the research on benefits of stories to children's learning overlap. Stories are spaces in which children can come to relate to problems that can be addressed by mathematical means. When children engage with the virtual world of stories (Gee, 2007), identify and connect with the characters, and come to see the world from their perspective, the storyteller can motivate the classroom to the actions of mathematising and problem solving. In this way, the story becomes a means of presenting the listeners with the purpose and need for mathematical actions.

Purposes and Functions of the Story

We start our conceptualisation of the purposes and functions of story and storytelling in mathematics teaching and learning by discussing the initial design work on the instructional sequence on Fractions as Measures. We then address how attending explicitly to the purposes and functions of the story within the designed sequence allowed for successful adaptations of the sequence to new settings, and new instructional purposes.

The Initial Story

An important aspect of the instructional sequence on Fractions as Measures involves having students use an unconventional unit to measure lengths, namely an unmarked measuring *stick*, which later comes to play the role of a reference unit. The stick is used to engage the students in reasoning about the function of such a measurement unit (its utilitarian purpose) and the relative size of subunits of measure, which are each the size of a unit fraction, and physically separate from the stick (e.g., $1/5 > 1/6$). It is also used to help students gauge, and reason about, the size of a measure as being shorter than, as long as, or longer than the length of the stick (e.g., $5/6 < 1$; $4/4 = 1$; $6/5 > 1$).

The design decision of basing the instructional sequence in measurement activities with informal units of measure was based on conceptual and phenomenological analyses of fractions that we reported in detail elsewhere (Cortina, Visnovska, & Zuniga, 2015). Pursuit of mathematical learning goals necessitated that we, the designers, (a) consider how teachers could make it *reasonable* for the students to use the stick as a measurement tool, and (b) support teachers in this endeavour (Cobb, Zhao, & Visnovska, 2008). Without a good story, students could rightly wonder why they should engage with such activities, given that outside of the classroom, people use a ruler or a measuring tape to accomplish what the teacher would be asking them to do.

When first trialling the sequence in Mexico, we thus aimed to engage students in an *interactive legend* about how people measured before the metric system was invented. We capitalised on the rich historical heritage of the region, in which the school was situated. The main characters of the story we created were the Acahay, a group of wise elders, women and men, who lived in a legendary Mayan city that has been long lost: Napiniaca. Our aim was that the students would experience the instructional sequence as an inquiry journey into the challenges that the ancient Acahay faced, as they struggled to come up with better ways of accounting for the lengths of things (what tools to use, how to name and symbolise them).

This example illustrates how RME design heuristics oriented us to develop meaningful narratives, within the broader storyline, so that each instructional activity could then be introduced as a genuine problem that story characters faced. Students' identification with characters in the story was, in turn, intended to motivate and warrant students' effort and persistence while advising the characters on the resolution of challenging problems.

We contrast the outlined approach with that of using a story as a 'hook' to encourage students' 'buy in' to subsequent mathematics activities. Here students are enticed to engage with fun or enjoyable, familiar contexts, which then abruptly transition to mathematics activities with only a vague connection to the 'hook' and where the initial 'hook' is no longer of consequence. From the students' perspective, the mathematics they are asked to do, while connected to some aspect of the world beyond school, is essentially detached from real life considerations. Doing school mathematics and solving a problem that requires mathematisation in the real world (outside of the classroom) are then essentially two different practices for the children.

Story and Mathematics

While the broader storyline can be used to situate day-to-day classroom mathematical activities and provide connections across them, story and storytelling can also play an important part in supporting meaning-making and organising activities when teachers pursue specific mathematical goals and connections within a lesson (cf. Sleep, 2012). We return to RME and the guidance the theory provides for designing viable instructional starting points

- or how students can be productively introduced to new problem situations. In addition to the requirement we mentioned earlier, that problem situations would become *experientially real* to students during their introductory classroom discussions, an additional characteristic is key to our present purpose. The problem situations need to, when introduced by the teacher in her classroom, *trigger students' informal ways of reasoning* that can become a basis for developing increasingly sophisticated mathematical ways of knowing in a particular domain (Cobb et al., 1997). In other words, the problems and inscriptions need to be a means of achieving learning goals in the lessons.

We illustrate how, in the Fractions as Measures sequence, the story becomes a key means for the teacher to elicit informal ways of reasoning *before* more formal mathematical ideas and innovations that present the short-term learning goals would be introduced. In a number of initial activities in this sequence, the focus is on supporting students' realisation of how the tools (or inscriptions) that were previously legitimately used for measuring (or symbolising) are, in new situations, no longer suitable. Such realisation is intended to provide students with the *need* for innovations, which can then be either devised by the students or introduced by the teacher.

Within the instructional sequence, guiding the reinvention of fractions is expected to start with students recognising a need for a standardised unit of measure. Students are first asked to measure objects in the classroom using parts of their bodies (e.g., their hands) so that they come to experience and recognise that measuring in this way can be problematic. To support this recognition, a story is told in which one Acahay daughter took a measure for a clay pot, ordered by a villager, with her hand, which led to her mother making the pot (using her own hand to measure) the wrong size. This is presented as a puzzle for students to figure out why the pot was not the right size. Once such recognition is accomplished, the standard unit of measurement (the stick) is introduced as a resource that allows consistent measurement of the lengths of things, by different people and at different times.

As a next step, the students are supported to become mindful of the limitations of solely using the stick for 'accurate' measurements. This time, a story can be used in which the villagers (students) measured their height with the stick and were all claimed to be the same height – five sticks and a bit tall. Students typically vehemently disagree with such a conclusion and recognise the limitation of their measurement tool. At this point, they learn that Acahay elders solved this problem by introducing smaller length measures, *smalls*, which are created by using the stick in the pattern: *small of two* (*small of three*, etc.) is a rod of such a length that, when used to measure the stick, it measures exactly two (three, etc.) rods (Figure 1). Each small thus represents a unit fraction of the length of the stick.

Figure 1. Small of three rod with such a length that three iterations of the rod cover the same length as the *stick* (reference unit).

We would like to highlight that the situations in the story are intended to support students to reason intuitively, based on the experiences generated through in-class activities, about why measurement tools used by Acahay were insufficient for certain purposes. While intuitive, this reasoning is inherently mathematical and helps to orient students' attention to aspects of the situation on which further mathematics will be built. In addition, any innovation - and fractions in particular - are then introduced as a solution to a problem which

all students have already recognised. Importantly, we learned from the teachers with whom we worked in Mexico and Australia that once they saw some of their students produce the initial intuitive arguments (e.g., why measuring with hands can be problematic), they found they wanted to support all their students to reason in these ways. Seeing how a story functioned in their classroom, they keenly developed similar stories to provide their students with additional opportunities for mathematical reasoning, thus supporting them in accomplishing specific learning goals (Visnovska & Cortina, 2017, 2018).

It is important to clarify that it was not our intention to design a story that would 'work' irrespective of local aspects of the classroom context. Creating characters with which students can identify and purposes for which they would be keen to engage in problem solving would indeed need to be informed by local knowledge. However, the Fractions as Measures sequence provides a strong mathematically driven frame within which to develop locally adapted stories to support student learning of the intended mathematics. As a consequence, only some of this work can be done by designers, much is left for teachers. Indeed, the notion that underlines our design work is that of implementation as a conjecture-driven adaptation. We thus aim to design resources that make local adaptations possible. This necessitates provision of guidance for resource users in terms of the sequence rationale, so that in reasoned adaptations, the story can still be used to drive both the initial and the sustained mathematical engagement. We now discuss the kinds of adaptations that were made when the sequence was used in South Africa, the purposes that necessitated these adaptations, and how the sequence facilitated this work.

Sequence Adaptations

The South African Numeracy Chair Project is mandated to research innovative, sustainable and practical solutions to the challenges of numeracy education in South Africa, and particularly in low socioeconomic status schools in the Eastern Cape. As part of this mandate we (the latter two authors) trialled the Fractions as Measures sequence in three Grade 3 classes in a local school as a possible approach to help students to better comprehend the multiple meanings of fractions. In the research reported here, we sought to find out specifically whether the sequence promotes an understanding of the relative sizes of unit fractions.

In South Africa, language proficiency is acknowledged to be a key contributing factor to students' continued poor performance in mathematics (Graven & Venkat, 2017). The issue of the language of instruction affects access to mathematical knowledge for the majority of students. Policy advocates for mother tongue instruction, particularly in the early years, and allows schools to select which of the 11 official languages to use as the language of learning and teaching (LoLT, Department of Basic Education, 2010). English, however, remains overwhelmingly the preferred LoLT, with Afrikaans a distant second, while the remaining 9 indigenous languages together account for the LoLT of less than 10% of students, despite being the native languages of 82.8% of all students (DBE, 2010; Robertson & Graven, 2015). These young students are thus faced with the "dual burden … [of] mastering their LoLT while at the same time gaining epistemological access to mathematics through the LoLT" (Robertson & Graven, 2015, p. 286).

This was the reality of the students participating in this research in the Eastern Cape. The vast majority of the 105 Grade 3 students who participated in the implementation of the Fractions as Measure instructional sequence across five lessons were isiXhosa-speakers, who were learning in either English (two classes of 35 students) or Afrikaans (one class of 35). We were mindful of this 'dual burden' when adapting the Fractions as Measures

sequence to this context. Vygotsky (1978) speaks of the transition learners must make from spontaneous everyday language to scientific language and the importance of the role of a mediator in moving from spontaneous to scientific concepts. The story, the activity sequence, and the teacher serve as mediators in supporting this transition. In our experience we saw the support of a story driving mathematical activities, as especially important in our second language learning context because it supported classroom talk and the use of exploratory language. This was important for enabling students to develop meaningful understanding of scientific concepts that students were learning about in a second language. This paved the way for better access to the meaning of the dense and structurally complex language of mathematics (Hammill, 2010). The Fractions as Measures instructional sequence was well suited to providing resources for addressing these complexities meaningfully in the classroom. The story through which mathematics was to be introduced allowed for adaptations that leveraged and increased student talk, while maintaining the coherent storyline of students' mathematical learning. This supported later development of student fluency in formulating explanations of generalised mathematical relationships by working through the imagery in the story, and using the increasingly abstract ways to symbolise fraction quantities (concluding with conventional $5/6 < 1 < 6/5$).

The students were all familiar with the well-established social practice of engaging with stories in the context of literacy and language teaching. They all demonstrated an appropriate everyday command of the LoLT, but, concurring with national data, teachers indicated that many did not read at a grade-appropriate level. In these classrooms, it was therefore important to work on mathematical ideas without reliance on students reading a text or working from written worksheets. Storytelling in the context of the Fractions as Measures sequence provided a suitable means of conveying and negotiating the mathematical questions and ideas that we wanted students to consider.

While discussing stories is a common literacy teaching practice in South Africa, using stories as instructional starting points for teaching specific mathematical ideas in a classroom setting is certainly not part of typical mathematics teaching (see Hoadley, 2007 for discussion of primary mathematics pedagogy in South Africa). Bringing these two together provided a space where students could engage in a familiar story discussion space with everyday language around complex mathematical ideas including the relative size of unit fractions and the concept of fraction as measure.

To illustrate the reasoning this discussion space made available, we provide examples of students reasoning about the relative sizes of the 'smalls' (unit fractions of the stick) that were typical during whole class discussions in late classroom sessions. Students used colourful straws and cut these to create smalls up to small of ten, in a process of trial and error. When asked to compare lengths of the smalls they had created, they were able to respond with explanations. Two examples of student explanations are given below:

Student 1:	Small of nine is smaller [than small of three] because it [nine] is a bigger number and it [small of nine] must fit nine times [that is, more times onto the stick].
Student 2:	A small of two is bigger [than small of five] because a small of two fits in [the stick] two times and a small of five fits in [the stick] five times.

Students had many opportunities to informally construct such explanations. They were consistently asked to predict the size of the next small they were to create (i.e., *Will small of 3 be longer or shorter than small of 2 that you just made?*). They were also asked to collaborate on formulating and writing down explanations for such comparisons in small groups. As illustrated above, students did not base their reasoning on comparing the lengths of physical rods. Instead, they called on the imagery of iterating specific smalls along the

stick when measuring it (e.g., the image that "small of two fits in two times"). In this way, they came to think about unit fractions as quantities that can be compared and ordered meaningfully. Importantly, they linked these discussions to the kinds of thinking with which the story characters engaged in addressing genuine problems they faced.

To support the students' access to these mathematical ideas, we needed to construct the story so that the language would not present a barrier to students' understanding and active participation. We first adapted the context of the story so that it happened in an African village. We replaced all names and native words in the original story with more familiar sounding words and isiXhosa names to support students' ease of comprehension. Such modifications were possible, and indeed expected by initial designers. They maintained coherence with the designed sequence in that (a) the time and place in which the story happened provided a rationale for use of informal units of measure, (b) familiar names were used to support students' identification with the story characters, and (c) adaptations were made to avoid break-downs in the story that use of new or incomprehensible names, words, or situations could generate.

During storytelling episodes, we intentionally used voice, gesture, and visual aids to support students in making sense of spoken language. The story was told using props, miming the actions of the main characters, and including a student to act as the child character in the story. We highlight the dialogic nature of this practice and contrast it with 'story-reading'. While not a focus of this paper, students showed impressive gains in their understanding of the inverse order relation of unit fractions in a post assessment.

Our experiences of the Fraction as Measure sequence, combined with the student improvements on post assessments led us to conclude that student meaning making showed strong coherence with: everyday experiences of measuring and the relative size of objects, the story and its problem, and mathematical coherence in understanding the relative size of unit fractions. Through the story, the teacher supported students in engaging meaningfully with the mathematical ideas that were the aim of the sequence. This resulted in increased overall classroom communication, richer responses from more students, and deeper engagement with the mathematical ideas.

Summary and Conclusions

We opened the conversation about functions of stories within instructional sequences that are aimed at guiding students' reinvention of key mathematical ideas. We documented how several of these functions relate to RME heuristics for setting viable instructional starting points. We stress that instructional resources should be designed as *teachers' resources* (Visnovska & Cortina, 2018), and thus allow for and support reasoned adaptations by teachers and others who use them. This paper illustrates how stories can be designed in this way and accompanied by the rationale for specific design decisions.

Sleep (2012) refers to the coherence and connectedness in how mathematics is being taught, or sequenced in curricular documents, as a mathematical storyline. We fully agree that such coherence in design is essential if our goal is for the students to experience the mathematics they learn as coherent. We would also like to extend this argument and point out that when a coherent mathematical storyline is combined with a coherent take on experiences that generate the need for intended mathematics, the classroom-based reinvention of mathematics can become a mesmerising storytelling journey.

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Teacher Learning in the Use of Technology for Teaching Mathematics in an Online Learning Community: A Sociocultural Perspective

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This study investigates the potential offered by online learning communities (OLC) for teacher professional learning in use of technology for mathematics teaching practices. The paper analyses social learning interactions as teachers engage in OLC. Drawing on two case studies of teachers in Indonesia, this study has captured aspects of social learning interactions in the OLC which affected their teaching practices with technology. The findings show that teacher participation in the OLC caused changes to their instructional practices with technology. The teachers showed how to use technology either as a partner or an extension of self to extend student's mental thinking and cognitive capacities.

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2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 725. Auckland: MERGA.

Exploring the Application of the Pythagorean Theorem in Junior Secondary Mathematics

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In recent years, research exploring student understanding and use of the Pythagorean theorem, has been extended beyond the mechanical approach of using the well-known formula, to include visual/geometrical applications. The use of technology and media-rich platforms has also been utilised to emphasise the visual aspects of the Theorem. At our school students were afforded the opportunity to visit the topic, both in terms of geometry and integration with area formula. The aim was to use assignments, which are notorious for less time restraints, to increase familiarity with the theorem and move toward viewing its many guises. It will be shown that the two assignments elicited opportunities for meaningful conversation about the applications of the theorem in problem solving and provided comprehensive assessment of student knowledge of the Pythagorean theorem. It is also important to note the Pythagorean Theorem was added to the year eight schedule despite not appearing in ACARA curriculum content descriptors until year nine.

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2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 726. Auckland: MERGA.

Improving Access to Mathematical Concepts: Lessons from Korean Practice

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Strategies that support the growth of deep mathematical understanding among our students continues to be an important theme among researchers and teachers of mathematics. The present study is premised on the assumption that representation of concepts provide a powerful theoretical lens for analyzing teaching that aims to extend the depth of students' mathematical understandings. Drawing on a representational perspective, we analysed key episodes of teaching from two Korean mathematics lessons (Years 2 and 8). Four significant findings emerge from a preliminary data analyses. Firstly, both teachers provided clear instructions to access students' prior knowledge. Secondly, teachers use effective open-ended tasks to elicit student engagement. Thirdly, participating teachers sustained student engagement with prior mathematics concepts by challenging them constantly. Finally, there is indirect evidence that teachers accessed high levels of content and pedagogical content knowledge in order to drive the representation-based learning activities. We suggest that, at least in the case of our two teachers, Korean mathematics practices place high premium on prior knowledge access, exploration and extension, and flexible utilization of that knowledge in the problem space.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 727. Auckland: MERGA.

The Mathematical Content in the Planning of an Experienced Teacher

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Planning is an important instructional practice in the teaching mathematics. In this paper, we report the factors that an experienced teacher of mathematics considers in the planning of his lessons in mathematics at the secondary level. While the documented lesson plan shows very few details, this experienced teacher demonstrates a very rich *lesson image* of how his lessons will play out and which factors he should consider in planning the content he has to teach.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 728. Auckland: MERGA.

The Transition from Primary to Secondary School Mathematics in Australia: Can Spatial Reasoning Help?

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The negative impact of the transition from primary to secondary school in Australia on students' mathematics performance is well documented. Meanwhile spatial ability has been shown to correlate with, and predict later mathematics performance in children. We sought to determine if mathematics and spatial performance in grade six would be associated with mathematics performance in grade eight. Students (N = 10) completed spatial reasoning and mathematics measures in grades five, six, and eight. Spatial reasoning and mathematics in grade six predicted mathematics performance in grade eight. Results suggest that spatial training may provide a scaffold for primary school students preparing for the altogether different demands of the secondary school mathematics classroom in Australia.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 729. Auckland: MERGA.

Addressing Barriers to Communication in a Mathematical Inquiry Community

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This paper explores teaching practices that support students experiencing auditory and oratory communication barriers to access more equitable discourse within a mathematical community of inquiry. It draws on two branches of research into effective teacher practices; the branch of discourse-intensive approaches, and the research base of Deaf Studies focusing on the status of sign language within mathematics education. The findings of the case study reported here suggest that promoting the status of visual communication can provide more equitable access to communication and participation within a mathematical inquiry community.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 730. Auckland: MERGA.

The Challenge of Determining the Meaning of Approximate Number System in Early Mathematic Education

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Many studies claimed that approximate number system (ANS) has improve the individual achievement in mathematics. However, most of these researchers failed to reach agreement in defining about ANS. Therefore, a brief explanation related to the meaning of ANS will be discussed. The discussion are based on the definitions used by researchers in their research. The articles in this research were selected randomly as long as it links to ANS definition. We conclude that there is a difficulty in determining the meaning of the ANS due to the overwhelming variety of deficiencies given. Surprisingly, the challenge is getting harder when we try to determine the meaning of ANS to be appropriate in early childhood mathematics education. The complexities and problems occur because of each researcher defines ANS in their study depending on the age of the participant involved.

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2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 731. Auckland: MERGA.

Examining Statistical Literacy Skills Within High-Stakes Assessment

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This study investigates statistical items from high stakes tests (i.e., PISA, TIMSS, and Indonesian National Exams (UN[7])) based on five skills of statistical literacy. The five skills comprised of *understanding*, *interpreting*, *evaluating*, *communicating*, and *decision-making*. The findings of this study revealed that most of the statistical items in TIMSS and UN assess *understanding*, while in PISA assess all skills except *decision-making*. Furthermore, TIMSS and PISA item features are more diverse than UN's since it only provides multiple choice. This study provides evidence that the three existing high-stakes assessments are not adequate to understand the Indonesian students' statistical literacy.

[7] The UN included two school levels, namely assessment at the end of Junior High School (SMP-students aged 13-15) and at the end of Senior High School (SMA-students aged 16-18)

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 732. Auckland: MERGA.

How do Primary Students Improve their Reasoning? The Influence of Argumentative Activities

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This study aims at developing an argumentative activity to help students improve their reasoning at primary level. There were 113 grade 5 students in Taiwan to participate in this study and all students were divided into two groups. Fifty-six students were in the experimental group and experienced the argumentative activity, but fifty-seven students were in the control group without any relevant activities. The results showed that students in the experimental group could be improved in the geometric argumentative test and the qualitative data also showed that students' levels of reasoning have been improved. Finally, this study provides some suggestions for the future studies.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 733. Auckland: MERGA.

What is Mathematics Education for Babies and Toddlers?

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This paper introduces a newly-funded Australian Research Council *Discovery Early Career Researcher Award* (DECRA) project titled, 'What is mathematics education for babies and toddlers?' The project aims to investigate mathematics education for children aged under three years by examining the beliefs and practices of the educators who work with these children. This study will generate new knowledge about when and how mathematical experiences are provided for children in Under 3s settings. This new knowledge will contribute to the current agenda of enhancing Science, Technology, Engineering, and Mathematics (STEM) participation and outcomes by elucidating the early mathematics education base upon which STEM education can build. Findings will inform the development of professional learning materials for educators in Under 3s settings, and will enhance pedagogical approaches to support high-quality mathematics education for very young children.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 734. Auckland: MERGA.

Genericism: Do we Want our Maths Curriculum Courses to Follow this Model and do we have a Choice?

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This paper reports on the responses of over 100 middle years' pre-service teachers to a mathematics curriculum course. The course attempted to develop content knowledge and specific pedagogy in a truncated semester. Most pre-service teachers valued the opportunity to revise and deepen their knowledge of mathematics and to be exposed to detailed specific pedagogy. Courses of this form are not necessary for program accreditation and are potentially at odds with the wider goals of the institution to offer flexible, cost-effective programs. It is possible that in the absence of academic governance changes, such courses may well disappear from the Australian teacher education landscape.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 735. Auckland: MERGA.

Where Should I Begin? Self-regulation Strategies for Learning Mathematics

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Students in schools today are recognised as active participants in their own learning. As active participants students are expected to take responsibility for the cognitive and metacognitive aspects of their learning including understanding what learning is expected as well as knowing how, when, where and why they learn best. To this end students must become self-regulatory (Zimmerman & Labuhn, 2011). The first phase of a two-phase study included identifying what student recognised as important strategies to support their mathematical learning. 167 Year 7/8 students (11 – 12 year olds) participated in the first phase. Each student completed a written survey. They rated strategies identified in research as supporting successful learning in mathematics. These strategies included; having time to think about a problem before starting to solve it, discussing what the problem is asking and how you might answer it, and watching/listening to the teacher demonstrate how to solve the problem. All strategies could be considered examples of the Forethought phase – the first phase of self-regulation. As well as stating why the identified strategy was important students gave each strategy a rating of one (high level of importance) to five (low level of importance). These ratings, when averaged showed students recognised thinking about a problem before starting to solve it as the most important and discussing what the problem is asking and how you might answer it as the least important.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 736. Auckland: MERGA.

Exploring Mathematics Using Computational Thinking: The ScratchMaths Pilot Project

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The introduction of the Australian Curriculum: Digital Technologies provides an opportunity for teachers to integrate computational thinking with the teaching of mathematics. In this paper, we report on a pilot program with 15 primary school teachers who participated in a professional development program to learn to integrate these two curricular areas. The program, based on ScratchMaths resources developed in the UK: (a) was successful in introducing novice teachers to basic coding ideas; (b) resulted in highly engaged students as teachers trialled the resources in their classrooms; and (c) highlighted the need for teachers to make mathematics explicit as students engage in the activities.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 737. Auckland: MERGA.

Walking the Line Between Order and Chaos: A Teacher-Researcher's Reflection on Teaching with Challenging Tasks in Primary Classrooms

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To support a developing dialogue around the experience of teaching with challenging tasks, I document my experience of teaching 84 lessons involving challenging tasks to three grades of year 1 and 2 students as part of a research project. Adopting a 'practitioner inquiry' lens, I analyse my reflective journal to reveal four themes: classroom management, maintaining and managing cognitive demand, time management, and tensions between discussion objectives. Implications for teacher professional-learning are briefly discussed.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 738. Auckland: MERGA.

Networked Learning Community: Building Teachers' Capacity to Implement Mathematical Tasks

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How mathematical tasks are implemented in the classrooms affect the way students engage with the concepts and mathematical processes. For teachers to leverage the affordances of the tasks, they need opportunities to dialogue about the content and instructional practices with other teachers. Networked learning community (NLC) facilitates teachers to learn from and with one another to deepen their understanding of using tasks in their lessons. Data from personal narratives by teachers, field notes recorded during NLC meetings and student artefacts reveal that teachers in the NLC were more confident to implement and extend the tasks to engage students in mathematical communication and reasoning.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 739. Auckland: MERGA.

Can Zone Theory Open Space to Observe Student Agency in Learning?

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Based on previous findings, I suggest that student agency may play a significant part in how learning unfolds. Valsiner argued that children are active participants in their own development and can change their environment to achieve their goals. More specifically, I suggest that students are instrumental in choosing how they respond and take advantage of mediation. Using Zone Theory as an analysis lens I use the constructs of the ZFM/ZPA complex and canalisation to explore how the two zones interact to canalise a child's actions along a certain learning pathway and perhaps speak a little to the agency that arises as a result.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 740. Auckland: MERGA.

Beyond the Edutainment: The Possibility of Implementing Discovery Based Digital Games to Develop Mathematical Understanding

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In contrast with many research that focused on digital edutainment games in improving students' mathematical procedural skills, this paper evaluated the potential of discovery-based digital games to support students in building mathematical understanding. By analysing the students' interaction in two discovery-based games, this study found the discovery-based games did have affordance to promote students' development of mathematical understanding. Furthermore, this study also highlighted this affordance should not be taken for granted since students might only employ 'trial and error' strategy in games. Therefore, the researchers of this study urged that teachers' pedagogical efforts in a digital game-based learning activity are essential.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 741. Auckland: MERGA.

Pre-Service Teachers' Experiences with ACER's Numeracy Test for Initial Teacher Education Students

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In this paper, we discuss pre-service teachers' experiences with the Australian Council for Educational Research's Literacy and Numeracy Test for Initial Teacher Education Students (LANTITE). We report on findings from a questionnaire completed by 120 pre-service teachers at a prestigious Australian university about their experiences preparing for and completing the numeracy portion of the LANTITE. Specifically, we address the ways that the participants prepared for the test, their experiences with the test, and their impressions of the test and their results. Our findings provide insight about students' experiences and thus provide useful information to universities who offer teacher preparation programs.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 742. Auckland: MERGA.

Supporting Inquiry-Based Teaching in Qatari Mathematics Classrooms

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Inquiry-based approaches to teaching are seen to increase the engagement of students (e.g. Artigue, Dillon, Harlen, & Lena, 2012). As part of an international research project that aimed to introduce inquiry-based learning in mathematics and science with students from grades 4 to 9, we worked with a team of Qatari professional development specialists in adapting two pedagogical tools, Exploratory Talk and WebQuests. A key aim of the research project was to evaluate any impact these tools might have on transforming the pedagogy. Issues raised included meeting the diverse needs of students and moving towards a more student-centred approach.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 743. Auckland: MERGA.

A Test to Test Test-Readiness - Improving Teacher Education Students' Numeracy Skills Development

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In order to meet accreditation requirements, teacher education students in Australia must pass the Literacy and Numeracy Test for Initial Teacher Education (LANTITE). However, many students attend the test underprepared. We previously reported on the development of an online system at the University of Notre Dame Australia (UNDA) to better equip students for the LANTITE. In this short communication, we will report on a collaboration between UNDA and Western Sydney University, which extends the project by developing a new online diagnostic test, with enhanced opportunities for learning analytics evaluation.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 744. Auckland: MERGA.

Using Mobile Technology Applications (Apps) when Teaching and Learning Geometry in Junior Secondary School Mathematics Education in Sri Lanka

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The purpose of this study is to explore the factors influencing the use of mobile Apps by pre-service mathematics teachers with a focus on the pedagogy approaches when teaching geometry to year 10 students in Sri Lanka. In the last decade, there has been a rapid development of mobile Apps and their adoption for mathematics education has been widely discussed in the research literature (Calder & Larkin, 2016; Carr, 2012) and the interpretive paradigm is selected as the philosophical and methodological underpinning of the study. Study has followed mixed method approach with two phases and suggested data collection methods are case studies, survey, per-post geometry test, interviews and documentary analysis. The purposive sample of 60 mathematics pre-service teachers, two mathematic lecturers from two pre-service teacher education institutes in Sri Lanka will be participated in the study. The data analysis will involve constant comparative methods, descriptive statistic and ANOVA.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 745. Auckland: MERGA.

Analysing Instruction as a Coordination of Dimensions of Mathematical Progression: The Case of Blair

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Within a design research project, a distinctive approach to data analysis has been developed, which tracks the progression of instruction over sequences of tasks as a coordination of adjustments on a few key dimensions of mathematical progression. The paper presents an example analysis, drawn from a teaching experiment involving intervention with a low-attaining primary student over 20 weeks. For instruction developing multiplicative strategies, five key dimensions are identified: range, orientation, setting, notation, and attention to structuring and strategies. The analysis is recommended for illuminating responsive instruction, and for informing the design of a learning trajectory.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 746. Auckland: MERGA.

Students' use of Mathematical Evidence in Guided Mathematical Inquiry

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Extended, guided mathematical inquiries require students to address questions with potential for multiple interpretations of a question and question context. Students typically adopt diverse approaches to solving the inquiry question, resulting in multiple possible solutions. Students need to be able to explain and justify their solutions and their chosen pathway to solution, by providing and drawing on mathematical evidence. Making decisions about what evidence is needed and how to collect it falls on the students and this becomes a challenging component of the inquiry that, if not supported, can result in shallow mathematical coverage of a topic. In a study currently in progress, a bank of video-taped inquiry lessons from the authors' previous research is being analysed with involvement from the teachers conducting the lessons. This data, as well as input from a focus group of expert inquiry teachers, has been drawn upon to present a developing framework of the ways in which students engage with mathematical evidence through varied stages of an inquiry. The framework is illustrated with examples from a Year 3 inquiry.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 747. Auckland: MERGA.

Teaching Measurements in Australian Primary Schools

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The use of novel educational approaches has been suggested as a way to counter the declining mathematics achievement in Australian children (ACARA, 2009; Attard, 2013; Roseno et al., 2015; Sullivan, 2011). To develop an educational resource within the geometric shapes and measures domain, we created an online survey to evaluate mathematics resources currently used to teach volume and capacity, its integration with other learning areas, and the use of technology and games in Australian primary schools. Approximately 100 Year 3 and/or 4 teachers will be recruited through social media and Qualtrics. Preliminary analyses will be presented during the conference.

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Attard, C. (2013). "If I had to pick any subject, it wouldn't be maths": Foundations for engagement with mathematics during the middle years. *Mathematics Education Research Journal*, *25*(4), 569-587. doi:10.1007/s13394-013-0081-8

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Sullivan, P. (2011). *Teaching Mathematics: Using research-informed strategies*. Retrieved from https://research.acer.edu.au/cgi/viewcontent.cgi?article=1022&context=aer

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 748. Auckland: MERGA.

The Australian General Public's Views of Gender and Mathematics: A Comparison of Findings from Binary and Non-Binary Studies

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In this short communication, we discuss preliminary findings from a study in which Australian and Canadian members of the general public were queried about their views of gender and mathematics. This study is a replication of prior research by Helen Forgasz and Gilah Leder, but with a significant change to the instrument: All of the questions were re-worded to be non-binary (e.g. "For which gender…") instead of binary (e.g. "girls or boys"). We compare the findings from the Australian datasets in each study and consider how the changes to the instrument may have influenced the findings.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 749. Auckland: MERGA.

Making Maths a HIIT at School: A Whole School Approach

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Declining levels of physical fitness in children are linked to an increased risk of developing poor physical and mental health. Similarly, the declining levels of engagement, interest and achievement in mathematics in young people is concerning. Making Maths a HIIT at School is an eight-week physically active mathematics intervention. A two-arm controlled trial was used to test the feasibility of the program for use in the whole school (across K-6). Analysis using linear mixed models revealed significant intervention effects for student engagement. Teacher feedback was very supportive and highlighted the potential of integrating high intensity activity in mathematics.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 750. Auckland: MERGA.

A Preliminary Illustration of Mathematical Inquiry Norms in a Primary Classroom

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Norms are cognitive and social structures that are negotiated explicitly and implicitly as expectations of appropriate behaviour in a mathematics classroom. Yackel and Cobb's (1996) classic work identified key social and socio-mathematical norms in a primary classroom. However, the problems used in this and subsequent research on norms in mathematics classrooms have primarily focused on well-defined problems with a single, correct answer and/or procedural efficiency. Problems addressed in mathematical inquiry often involve complex, ambiguous tasks with a method that is ill-defined and a solution that is assessed on the quality of evidence that students use to convince their audience. Mathematical inquiry norms are unique in that they often conflict with expectations developed implicitly in traditional classroom mathematics lessons. A study is underway that aims to identify norms used by primary teachers experienced with mathematical inquiry. An initial tentative set of diverse and complex norms has been developed through drawing on a database archive of classroom videos of mathematical inquiry lessons from the authors' previous research. In this presentation, we provide a short excerpt from a primary classroom video to illustrate a subset of identified norms that engage with social, dispositional, mathematical and inquiry-based expectations in a classroom as students address a mathematical inquiry question.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458–477.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 751. Auckland: MERGA.

Mathematics Education: What Students Want to Learn and What Lecturers Think Students Need to Learn?

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This research study explores how students and lecturers view and evaluate mathematical content and pedagogical content knowledge within current initial teacher education programmes. Essentially, we examine the dichotomy between what students want to learn about, what lecturers think students need to learn about, and how these two points of view intersect with 21st Century learning practices. Data from the student evaluation of paper reports from undergraduate and graduate mathematics education papers within the initial teacher education programmes are used in this study. Lecturer voice was based on reflective discourse from two lecturers within the programme's mathematics education papers. This presentation will highlight student and lecturer voice, and evaluate these within a context as outlined by the OECD's framework for 21st Century learning skills and competencies.

2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). Making waves, opening spaces *(Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia)* pp. 752. Auckland: MERGA.

Development of a Theoretical Framework: Designing Online Challenging Mathematics Tasks for Pre-Service Teacher Education

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The purpose of this project is to strengthen Australian and German pre-service teachers' mathematical knowledge, particularly their higher order thinking capabilities such as mathematical problem solving – a significant issue in Australian and German education (e.g., Finkel, 2018; Kunter et al., 2011). The project brings together different theoretical aspects of mathematics education that have not been connected in previous research: the notion of challenge (Sullivan, 2011); the perspective of task design in digital environments (Geiger, 2017); frames for technology rich pedagogy (Pierce & Stacey, 2010); and, quality of online learning environments (Collet, Bruder & Ströbele, 2008). In this presentation, we will outline our preliminary work in developing a framework for designing challenging online mathematical tasks. This will include, illustrative examples of tasks designed in alignment with the framework for both Australian and German preservice teacher education.

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Designing a Pedagogical Program to Support Spatial Reasoning in the Primary School

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This presentation provides an overview of the development and implementation of a Spatial Reasoning Mathematics Program (SRMP) in Grades 3 to 4, drawn from a larger study of spatial reasoning and mathematics learning*. Classroom teachers are integral to the development and implementation of the SRMP, supported by the project team and professional learning. Key components of the SRMP connect students' experiences of 2-dimensional and 3-dimensional patterns and structures with spatial reasoning tasks. These tasks focus on collinearity, transformation, perspective taking and mapping. Students are assessed on spatial reasoning, general mathematics ability and pattern and structure, as well as tracking of individual profiles of learning. Preliminary findings of the project will be reported.

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Connecting Research with Outreach

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The schools outreach program of the Australian Mathematical Sciences Institute (AMSI) has been providing professional development and support to schools for many years. AMSI's current outreach program, CHOOSEMATHS, includes research components exploring student attitudes to mathematics and the needs of teachers when attempting to improve these attitudes and student performance. The results of this research, along with that of others in the field, is then fed back into the outreach elements of the program. At the halfway mark of the CHOOSEMATHS project this session outlines what has been learnt so far.

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The 'Art' of Ratio: the Potential of a Hands-On Ratio Task

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The importance of ratio and proportional reasoning is regularly emphasised in the literature. However, the teaching and learning of these concepts continues to be considered an area of mathematics that presents considerable challenge. Hands-on tasks are well documented as having potential to impact positively on student conceptual development of abstract concepts such as ratio. This communication presents a pilot study which examined a hands-on task designed to encourage student visualisation of ratio. Teacher perceptions of this task were positive, indicating the potential of this task to impact on students' engagement and understanding of ratio.

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A Teacher's Challenge in Developing Mathematics Talk for Sense-Making in and Through a Second Language

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In this presentation we use a socio-linguistic lens to illuminate the challenge a Grade 4 teacher confronts when teaching fractions in English to students with limited English language proficiency. We analyse transcript data of the classroom talk in one lesson, together with interview data, to highlight the struggle both the teacher and learners have when they are unable to access their major source of linguistic capital - their native language. The question we address is: How might socio-linguistics shed light on a mathematics teacher's challenge of facilitating talk for mathematical sense-making in and through a second language?

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Provision of Private Tutoring While School Teachers Provide Enough Support in Mathematics Education

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Private tutoring seems to be a business, which exists in developing and developed countries and is conducted to supplement the mainstream school education system. However, some parents provide tutors while children have enough support from school teachers. The current study aimed to identify whether private tutoring positively or negatively affect school children and their mathematics education. The results show that private tutoring has mixed effects depending on how a child feels about and makes use of such support. Hence, the outcome is not always positive. Also, higher achievers are not necessarily the students who have tutors.

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Exploring the Potential of Structured Sequences of Challenging Mathematics Experiences

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It is widely accepted that "student learning is greatest in classrooms where the tasks consistently encourage higher-level student thinking and reasoning" (National Council of Teachers of Mathematics, 2014, p.17). Unfortunately, many teachers are reportedly hesitant to integrate such tasks into their classrooms (Cheeseman, Clarke, Roche & Wilson, 2013), raising concerns about some teachers' capacities to activate higher-level thinking in their students. The need to encourage all teachers to implement such experiences prompted Sullivan, Borcek, Walker and Rennie (2016) to explore an approach that initiated learning through challenging tasks. They found that student learning is facilitated when a particular lesson structure is enacted. This structure involves initiating learning through an appropriate challenging task, differentiating that challenge, and "consolidating the learning through task variations" (p.159). Further explorations by Russo and Hopkins (in press) revealed that variations to this lesson structure could also have positive learning outcomes for students when challenging tasks are utilised.

While higher-level student thinking can be activated within thoughtfully constructed lessons such as those proposed by Sullivan et al. (2016) and Russo and Hopkins (in press), the potential of implementing structured *sequences* of challenging mathematics experiences remains underexplored. The round table will begin by outlining the rationale and theoretical underpinnings of a research project that aims to explore the impact of sequences of connected, cumulative, and challenging tasks on student learning and teacher knowledge of mathematics and pedagogy. Data from the first stage of this exploratory study will provide the stimulus for discussion amongst roundtable participants.

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Mentoring to Develop Mathematical Inquiry Communities

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The aim of this Round Table is to invite discussion across the research community about the challenges encountered by teacher educators as they work with teachers to develop ambitious pedagogy embedded in culturally responsive and sustaining equitable practices in our school communities. We will initiate the discussion by outlining a professional learning and development project-Developing Mathematical Inquiry Communities (DMIC). This project promotes connected, rich mathematical thinking and reasoning, inquiry learning, high expectations and inclusion. We will discuss the importance of challenging the status quo in which persistent underachievement and inequitable outcomes are maintained in our mathematics classrooms through the structural inequities and institutionalised schooling practices and the resulting deficit thinking on the part of teachers and students themselves. The discussion will then be opened for all to share their experiences with working with teachers in professional learning and development mathematics initiatives which promote equitable practices, and inclusion. Beginning questions to consider are listed below:

What changes in beliefs and values enable teachers to engage with changes in their pedagogy?

How can this be achieved through mentoring, or coaching in PD and in classrooms?

How can teachers be convinced that ambitious culturally responsive teaching is worthwhile?

What challenges do mentors or coaches face convincing teachers that ambitious and culturally responsive teaching is worthwhile? What actions can be taken?

What research areas have you worked in linked to ambitious and culturally responsive teaching and mentoring and coaching? What advice would you give us?

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Connections in the Teaching of Mathematics

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The National Council of Teachers of Mathematics (NCTM) put forward connections as one of the five process standards in the Principles and Standards for School Mathematics (NCTM, 2000). Indeed, it was stated that, "When students connect mathematical ideas, their understanding is deeper" (p. 64). How do we help students to experience connections? Amongst others, Coxford (1995) enumerated the following: linking conceptual and procedural knowledge, using mathematics in other curriculum areas, and using mathematics in daily life activities. This can only be achieved if concomitantly teachers demonstrate a deep understanding of mathematics are able to make connections themselves. Askew, Brown, Rhodes, Wiliam, and Johnson (1997) identified connectionist orientated teachers in the context of the Effective Teachers of Numeracy project in the U.K. However, the idea of teachers with connectionist orientations need further exploration. How do we help teachers to become better at making connections? Accordingly, how can teachers enhance their students' experiences to make connections? In this Round Table, we propose to discuss this idea of connections in the teaching and learning of mathematics. We will use some examples from our own project as a basis for discussion to illustrate how teachers make connections. We welcome participants with interest in this area from all levels of teaching of mathematics.

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Research on Gender and Mathematics Education: Where to From Here?

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Gender issues in mathematics learning outcomes have been on the research agenda since the early 1970s. Knowledge in the field has expanded, yet participation rates in the most challenging mathematics subjects and in related careers persist, as do attitudinal differences leading to future success, and the gender stereotyping of mathematics as a male domain (Forgasz, Leder, & Tan, 2014). Based on the dichotomy of "male" and "female", the terminology used in early research was "sex differences". To highlight that psycho-social factors were implicated, the terminology later changed to "gender" differences. More recently, researchers have been challenged by a range of sex and gender self-identifiers. Since 2017, for example, AERA uses the following question for membership demographics:

Which best describes your gender identity?

Female/Woman Male/Man

Transgender Female/Transgender Woman Transgender Male/Transgender Man

Another gender identity (please specify): _____

Australian statistical data gathered on sex or on gender will use: "male", "female", and "X" (The Australian Bureau of Statistics, 2016). At this round table, we will discuss the implications of the current gender identification challenge in the field of gender and mathematics learning and more broadly.

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A Lecturer's Thoughts on how Students Learn Mathematical Modelling

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In the 1970s a move to formalise mathematical modelling in education began (Kaiser, Lederich & Rau 2010). There is a growing body of evidence for its implementation at secondary school level (Blum 2011, Blum 2015, Maaß 2005, Stillman et al 2010) though little is still known about its effective implementation at tertiary level. In general research focused on lecturer's actions and goals and their relation to student learning at university level is lacking (Jokawariski 2016). Using the learning theories Social Constructivism, Behaviorism and Participatory Orientation the presenter will discuss and present preliminary findings addressing the question "How does a lecturer think students learn mathematical modelling?" A case study was carried out involving a lecturer of a first year engineering mathematical modelling course. Data was collected through video recordings and field notes of lecturer interviews and observations. It was analysed for specific learning goals, instructional activities and learning processes mentioned and used by lecturer. Preliminary results show the lecturer identified observation, active participation, discussion, reflection, purposeful thinking and use of visual aids as behaviours indicative that students are involved in a process of learning for mathematical modelling. The lecturer held a strong belief that role modelling was an important learning tool supporting behavorism as a learning theory. The lecturer also valued social constructivism and has a participatory orientation regarding active participation as a valuable learning experience. An implication of the study is the awareness that learning behaviors don't happen in isolation and are normally most effective when combined with another behavior.

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