Why should we argue about the process if the outcome is the same? When communicational breaches remain unresolved

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This paper uses the commognitive framework to analyse how a group of four primary school students classify odd and even numbers. The findings show how students' reasoning is grounded in their personal uses of "odd" and "even". The students attend to different features of "oddness" and "evenness" and agree on *which* numbers are odd and even but disagree about *why*. The findings highlight the role that proving can play in signalling differences in reasoning within a group of students that may otherwise remain hidden. However, they also suggest students' awareness of the breach in communication may not be sufficient to engender a resolution, even when pedagogical moves toward this direction are made.

Mathematical proof is fundamental to the work of mathematicians, and many educators maintain that it should also be a fundamental part of school mathematics (CadwalladerOlsker, 2011). However, proving activity has been neglected in mathematics education (Stylianides, 2016), especially in the primary classroom. Accordingly, there have been recent calls recommending proving for all mathematical content areas and across the grades. For example, the PISA 2021 framework (OECD, 2018) highlights the centrality of mathematical reasoning and reforms in some countries' curriculum documents also now require proof and proving to be taught at all levels (e.g., Common Core State Standards Initiative [CCSSI], 2010; Department of Education [DfE], 2013; NCTM, 2000).

Although the fundamental purpose of a mathematical proof is to know whether a mathematical assertion or idea is true or false (CadwalladerOlsker, 2011), proving also has a more practical role in explaining and convincing others about our statements or theorems (Stylianides et al., 2017). It is through this practical role that proving has potential to support deep learning and sense-making. For instance, the NCTM's (2000) standards refer to proofs as offering "powerful ways of developing and expressing insights" through which "students should see and expect that mathematics makes sense." (p. 4).

What constitutes proof and proving at the primary level, however, is not entirely clear. Whilst it is unlikely that formal, deductive proofs expressed algebraically would be within reach of typical primary school students, Stylianides (2007) provided empirical accounts of how young students' informal arguments could be mapped onto corresponding formal proofs. These student arguments made use of manipulatives or diagrams to provide visual demonstrations of a generic example. Building from this research, Stylianides (2016) defined a proof as an argument, which is accepted by the classroom community and, uses and communicates reasoning in ways that are endorsable by the wider mathematical community but are also within reach of the classroom community. Nevertheless, even with a working definition of primary-level proofs, there is little research that explores *how* young students' arguments develop and become accepted within the classroom community. In this paper I utilise Sfard's (2008) *commognitive framework* to provide insights into how young students' arguments unfold as they *substantiate* (verify with evidence to prove why a reason is true) their classifications of numbers as even or odd.

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 251-258. Singapore: MERGA.

Theoretical Framework

Sfard (2008) defines mathematical discourse as a special form of communication, including self-communication (thinking), that is distinguishable via four interrelated characteristics: Its *keywords* (e.g., 'odd', 'even') and their use; its *visual mediators* (e.g., numerals, symbols, counters, pictures); its endorsed *narratives* (e.g., theorems, proofs, conjectures, definitions), and; its *routines* – discursive patterns, according to which mathematical tasks are being performed (e.g., the ways in which interlocutors substantiate oddness and evenness). Learning is seen as a lasting transformation in a learner's discourse, which is identifiable by changes in one or more of these four characteristics.

In terms of the keywords of interest to this study, 'odd' and 'even' are labels that function as nouns to denote discursive mathematical objects which may be *realized* in a multitude of ways; infinitely many numbers (e.g., 'odd' could be one, seventeen, one billion and one; 'even' could be two, forty-six, three million and eight) and each of these numbers could be realized as numerals (e.g., 1, 17; 2, 46), icons (e.g., an arrangements of dots) or symbolically as algebraic expressions (e.g., $2n+1$; $2n$). However, the illusory nature of mathematical objects (being products of our discourse as oppose to actual, tangible objects) entails that none of these *realizations* could be singled out as being 'the' object. During initial phases of learning, learners may have limited realizations of the signifiers 'even' and 'odd': Evidence of an expansion of realizations signals learning.

Another characteristic feature denoting the development of discourse is the level of *objectification*. Sfard (2008, p. 44) defines this as a process involving both *reification*– replacing talk about processes with talk about objects – and *alienation* – presenting phenomena impersonally, as if they were occurring independently from human participation. For example, when someone speaks of 'even' as "numbers that can be shared equally between two people", an activity (sharing) is indicated and the word 'even' acts as an adjective describing numbers. Whereas in the sentence "even plus even is even", 'even' has been objectified: The word is used as a noun that encapsulates all even numbers and realizations of even into one set, giving it separation from any activity and more permanence.

According to the commognitive framework, development occurs through the learner's exposure to, and participation in, the discourse he or she is supposed to individualise, and the support he or she receives from other participants. Encounters between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways, or perform the same mathematical tasks according to differing rules, have an indispensable role in this (Sfard, 2008, p. 162). Such encounters, termed *commognitive conflicts,* provide space for participants to consider new ways of talking, which is a prerequisite for experiencing a change in what they see. Sfard (2008, p. 258) maintains that resolving a commognitive conflict involves one of the interlocutors gradually accepting and adopting the incommensurable discourse and abandoning his or her own.

With regard to the group of students in focus, in this paper I ask, "What are the sources of commognitive conflict in the context of classifying odds and evens?" and "When and how can a commognitive conflict fail to give rise to a modification in students' discourse?"

Research Design

The present data is taken from a larger study aiming to investigate how students' arguments unfold and develop as they engage in proving activity. Year 4 students from two NZ schools were selected by their teachers to be withdrawn from their class to work in groups of four at a time with me (as a teacher-researcher) on three different tasks: (1) classifying numbers as odd or even; (2) proving conjectures about the sums of odds and evens; and (3) proving conjectures about the products of odds and evens. As the unit of investigation in this study was discourse, teachers selected students and groups according to whom they considered would be willing and able to engage in dialogue.

The data presented in this paper is taken from one group of four 8-year-old students as they participated in the first task. The students took turns to classify numbers presented on cards as odd or even and, as each card was classified, they were asked to substantiate their classifications and were encouraged to consider one another's questions and thinking. The cards displayed increased in complexity, from single digit numbers (shown as Numicon tiles or numerals) to six-digit numbers. Numicon tiles are visual representations of numbers 1-10 presented as dots within a frameless 2 x 5 rectangle (see Figure 1).

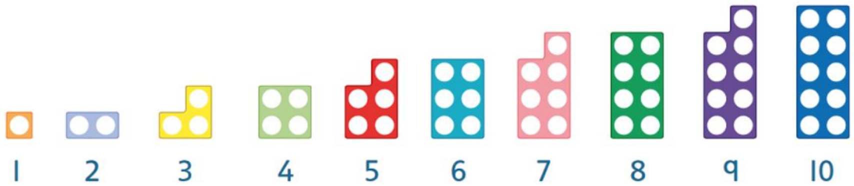

Figure 1. Numicon tiles.

The main aim of engaging students in this first task was to provide a baseline discourse for each student (i.e., what they already knew about 'odd' and 'even'), enabling me to track their learning (observable via the development of their discourse). The group sessions were audio and video-recorded, and their conversations were closely transcribed along with corresponding and relevant details about what the students did (e.g., gestures, facial expressions, actions, photos of their work). Here I conducted detailed discourse analysis utilising Sfard's (2008) commognitive framework to look for well-defined, repetitive patterns (routines) in students' discourses regarding their use of the words 'odd' and 'even' and their substantiating narratives about oddness and evenness. I also made use of *realization trees*. Whilst Sfard used "realization trees" (p. 165) to map personal realizations based on observations of the individual person implementing them, I constructed a *combined tree of realizations* for the group, mapping each interlocutor's observable realizations along specific branches, to help examine and clarify consistent and inconsistent uses of words within the group.

Findings and Discussion

Throughout the classification task all four students correctly classified the numbers and they agreed with the classifications made by their peers. Indeed, if the students had been asked to simply sort the cards into odd and even boxes with no justification, it would have been easy to assume that they held a common understanding about odd and even numbers. However, when I examined the routine ways the students substantiated their classifications, it became clear that their use of the words 'odd' and 'even' was different. Sfard (2008) states that interlocutors' word use (or what others might call 'word meaning') is important because "it is responsible for what the user is able to say about (and thus see) in the world." (p. 133).

Due to the scope of this paper, data that illustrate the students' word uses have been compressed in Table 1, rather than shown in their entirety. I include the turn number to provide the reader with an idea of turns elapsing and to enable me to refer to key turns within my analysis. A combined tree of realizations (Figure 2), constructed from the student group's discourse, is provided to visually illustrate the students' word uses.

Table 1
Student Substantiations for Classifying Numbers as Odd or Even

Card shown and teacher's question	*Jane, Zara and Robert's responses*	*Danny's response*
Why is 'six' even? 	[31] Jane: Because three plus three equals six and that's even.	[32] What the heck! That's not right! [34] That's so not right… three plus three… how does the three come here?
Why is 'nine' odd? 	[43] Zara: It's got a one there. (Pointing to the single one at the top of the Numicon piece.).	[45] So, even are always first, 'cos zero, two, four, six, eight, ten. And then the odd numbers are starting from one—one, three, five, seven, nine, eleven. It goes like that. So that nine is odd.
Why is 'four' even? 	[55] Robert: Because, two and two. [59] Robert: Mm… er, because two plus two. [73] Zara: Erm also if, if you had two people then you'd be able, they'd both get two each.	[58] But how did you get the two? [60] How did you get the two?! Where's the two? How did you get the two? [70] It's just the sequence. Same as the Fibonacci sequence and the other sequences. [79] I know something. So, it's like always like even-odd, even-odd, even-odd a number. [81] No odd can, odd can still be a fair share 'cos you can split it up into decimals. Like with seven you can make three point five.
Why is 'five' odd? 5	[90] Jane: Because a four and one. [92] Jane: Because the four is even, but five has like… [93] Zara: Instead of adding two on, you add on one and then it wouldn't be even. [94] Zara: So, two, two and one.	[91] But how did you get the four and one? [102] I know something. So, everything is like even-odd, even-odd, then even, then odd, so that's odd.
Why is 'twenty-two' even? 22	[119] Jane: Because eleven and eleven.	[120] All you need to know is like if it ends with a… if… This is like a simple way- if it ends with a zero, two, four, six or eight it is an even number… [122] And if it's one, three, five, seven, nine it's odd.

Jane, Zara, and Robert's substantiations of evenness and oddness

Table 1 shows that Jane substantiates evenness as referring to numbers formed by adding the same, or an even amount to itself (a double) to make a number [31, 119]. Her realization of the signifier 'even' is characterised by Branch 2a on the combined tree of realizations. Similarly, Robert substantiates the evenness of four by attending to "two plus two" [55, 59] and so his realization is also aligned with Branch 2a. Jane substantiates numbers as being 'odd' because they are an even number "and one" [90] (shown on Branch 2b). Zara has two substantiation routines for evenness: When two people have equal shares [73] (shown on Branch 2a) and, numbers that are formed by "adding two on" [93] (shown on Branch 2c). Her substantiations show she realises oddness as being unlike evenness because the structure deviates from 'adding two', to having "a one there" [43, 93, 94] (shown on Branch 2d). Whilst Zara's substantiations of oddness and evenness are not completely identical to Jane's and Robert's, the student group share a common branch (Branch 2) because their realizations of the signifiers 'even' and 'odd' attend to the symmetrical or asymmetrical *structure* of such numbers. I refer to their discourses about odd and even as being *structure-based*.

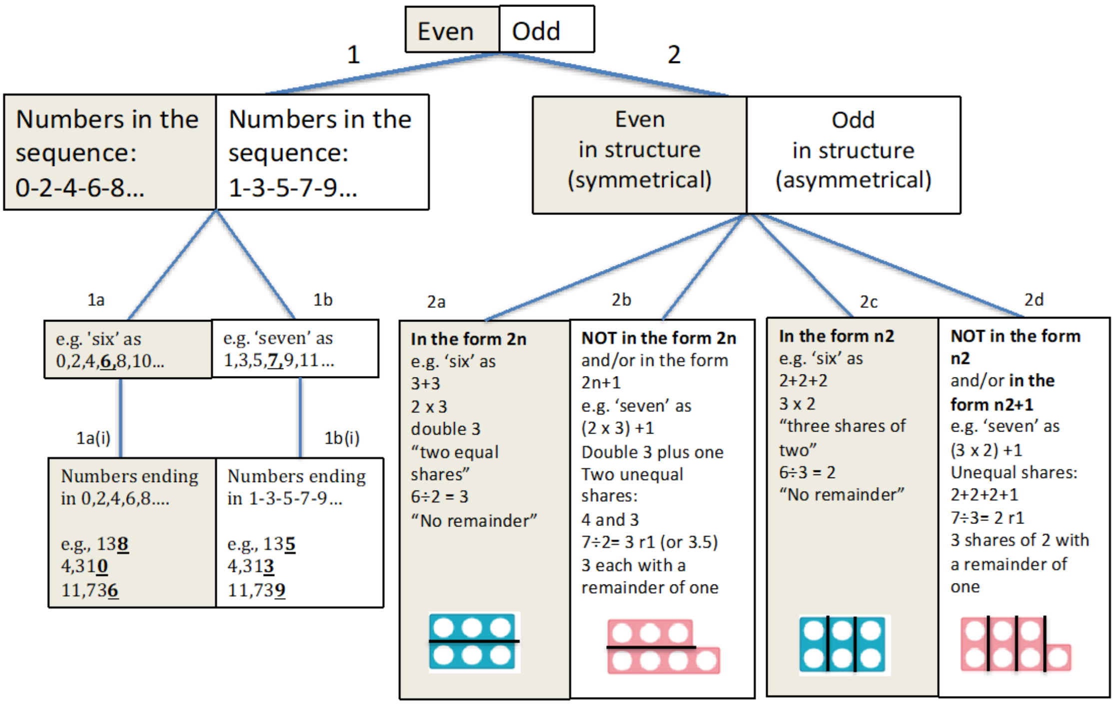

Figure 2. A combined tree of realizations showing the group's realizations of the signifiers 'odd' and 'even'.

Danny's substantiations of evenness and oddness

In contrast to the other students, Danny rejects (sometimes vehemently) structure-based substantiations of evenness and oddness [32, 34, 58, 60, 81, 91]. Even though the first three cards were presented as Numicon tiles, making the structure of these numbers visibly salient, his comments about Numicon 6 [32, 34] and Numicon 4 [58, 60] suggest he cannot even see the 'three plus three' in six nor see the 'double two' in four. However, Lavie and Sfard (2019) warn the researcher's tendency to look for things that children don't or can't do yet, means that they "remain oblivious to the possibility that the child's response to the [task situation]… may be about something else" (p. 423). Indeed, Danny's apparent bafflement and rejection of Jane's narrative is not necessarily because of an inability to see doubles but because, for him, his use of the word 'even' has nothing to do with a doubling criterion. Danny's routine uses of the words odd and even become apparent in his substantiation of nine as odd [45], and he repeats this substantiation for each subsequent number presented. For Danny, just as there is a Fibonacci sequence [69] and numbers within this set are '0,1,1,2,3,5,8…', the signifiers 'odd' and 'even' are sanctioned by sets of numbers in the sequence of ‘even-odd-even-odd-...’. His substantiations are shown by Branches 1a (for even) and 1b (for odd). When the task changes to include numbers with more than one digit, Danny elaborates on his substantiation of oddness and evenness, adding a ‘check the last digit’ [120; 122] procedure to his ‘check for place in with sequence’ procedure (shown on Branches 1a(i) and 1b(i)). Accordingly, I refer to Danny’s discourse about odd and even as being *sequence-based*.

Level of objectification

Having illustrated how Danny’s realizations of the signifiers odd and even are different to those of the other students, I now point to a further point of difference between their discourses in terms of the degree of objectification. Jane’s, Robert’s and Zara’s substantiating routines refer to numbers on the cards as specific concrete objects that serve as realizations of ‘odd’ and ‘even’, and these routines require an action. According to their routines, one is required to check if the specific numbers can; be *made* by a double [31, 55, 59, 119]; *make* two fair shares [73], or; be *grouped* in twos to prove evenness [93]. Oddness is proven where an even result is not possible or in instances where a remainder of one or unequal shares are *created* [43, 90, 93, 94]. These students also tend to use the words ‘odd’ and ‘even’ as adjectives; for example, when Jane substantiates the evenness of six, she describes the *even quantity* of “three plus three” [31]. In contrast, Danny’s discourse replaces talk about processes on concrete objects with talk about ‘even’ and ‘odd’ as *mathematical objects* existing in their own right, each as a condensed set of numbers reified from his known sequencing procedures. And when Danny uses the words ‘odd’ [81] and ‘even’ [45] they serve as nouns rather than adjectives; for example, “odd can still be a fair share”. In short, for Danny ‘odd’ and ‘even’ is the sequence itself, just like the Fibonacci sequence [70], whereas for the other students, these keywords appear as describing features derived from actions on specific numbers. These characteristics all provide evidence to suggest that Jane, Robert, and Zara are in the process of discovering generalizable features of odd and even, and show Danny’s sequence-based discourse on odd and even to be more objectified (and thus more entrenched) than the structure-based discourse of the other students.

The (unresolved) commognitive conflict

The exchanges in Table 1 present an example of what Sfard (2008) calls “commognitive conflict” (p. 161): The students have realized the signifiers ‘odd’ and ‘even’ in different ways and so are classifying numbers as odd or even according to different rules. The different branches of realizations (Figure 2) illustrate the differences in the students’ substantiations of evenness and oddness and thereby expose the source of the breach in communication: Jane, Zara, and Robert have a shared *structure-based* branch of realization for odd and even and so are able to communicate their process of classifying odds and evens effectively with one another whilst Danny’s *sequence-based* branch of realization for odd and even is disconnected from the others’, meaning he is unable to communicate his process effectively with the other three group members.

To support the students to resolve the commognitive conflict, the teacher-researcher makes several attempts to scaffold their participation in one-another’s discourses. An example of this can be seen in Table 2, where the teacher-researcher has assumed that Danny cannot see the double structure of even numbers and the asymmetry of odds, and she attempts to scaffold his participation in this structure-based discourse.

Table 2
Why is 'Five' Odd?

Line	Speaker	What was said	What action occurred
103.	Teacher:	And you were a bit confused weren't you. When you said, "Where's the four and the one come from?" Is that right? When she [Jane] said, "Five is four and one".	Directed to Danny.
104.	Danny:	I know, I know. I know why. Cos five is like split and there's a four and there's a one but that doesn't make any sense in high-school maths, 'cos it goes like...	Points to the Numicon 5 piece.
105.	Jane:	We're not doing high school maths.	
106.	Danny:	I know but, but it doesn't make any sense to like a really good, really good mathematicians 'cos it has no sense.	

This exchange is interesting as it shows signs of Danny attuning (albeit very slightly) to the discourse he initially rejected. And yet even when he eventually begins to endorse the other students' structure-based substantiations, he positions them being substandard to his own routine when he says theirs' "doesn't make sense" in "high school" (a more authoritative setting) and by "really good mathematicians" (people who have higher mathematical status) [104, 106]. In other words, he elevates his sequence-based substantiation routine as one that *does 'make sense'* and *is endorsed* by people and places of mathematical authority. By doing so, he maintains incommensurability between the two discourses. Hence, the teacher's pedagogical move to encourage Danny to make sense of the other students' did not result in him endorsing their substantiations.

For group members to resolve commognitive conflict, a "gradual mutual adjusting of their discursive ways" is required (Sfard, 2008, p. 145). However, during the entire classification task there was little evidence to suggest this occurring. The failure to resolve the conflict can be attributed to two factors. Firstly, the conflict was not about the outcome of classifying numbers as odd or even, it was about how the students substantiate oddness and evenness. For the students, deciding which numbers are odd and even was the goal of the task so they had no reason to resolve it because, on this, they agreed. Secondly, although the teacher encouraged the students to share their thinking and participate in each other's discourses, Jane, Robert, and Zara's structure-based routines were supporting them to make sense of and explain generalizable properties of even and odd, where Danny's routine way of substantiating oddness and evenness was too objectified for this purpose. And Danny rejected the other students' structure-based substantiation routines because his sequence-based routines were more entrenched and, not only did they work and produce the same outcome, they also were more efficient than the alternatives. From David's perspective the structure-based substantiations required a process (checking for symmetry in one way or another) and so were time-consuming and unnecessary when, with his substantiations, the last digit, simply and instantly, confirmed a number's membership in the set of even or the set of odd numbers. Accordingly, even when he eventually endorsed the structure-based substantiations, he maintained incommensurability between these and his own by positioning his routines as superior ones that worked in more authoritative contexts and with people who had more authority.

Conclusion and Implications

For effective interpersonal communication within the group to occur, group members need to build on one another's ideas using "the same means as those endorsed by his or her interlocutors" (Sfard, 2008, p. 173). This paper shows that whilst members of this group agreed on classifications of numbers as odd or even, they held different meanings of the keywords 'odd' and 'even' and accordingly substantiated oddness and evenness differently. Utilising the commognitive framework has helped highlight these distinctions in the students' reasoning that may otherwise be tacit. In terms of resolving the commognitive conflict, the students were unwilling to build on one another's ideas or reach a communicational agreement that rationalised these group decisions because they saw no incentive in doing so: They agreed on the classifications (which they interpreted as the goal of the task) and the alternative discourse did not serve them well with respect to this goal. Sfard (2007) notes that learners need good reason to change their routines, and I posit the classification task presented no such reason for any of the students to modify their substantiation routines.

The findings highlight the role proving activity can play in mathematics classrooms. In the absence of students' substantiations, a group consensus about an answer (in this case about which numbers are even and odd) may prematurely signal shared reasoning. Pressing students to publicly air their substantiations can bring differences in students' reasoning to the surface, which may otherwise be hidden. However, the findings also serve as a warning that common pedagogical moves to capitalise learning from mathematical disagreements by encouraging students to make sense of one-another's ideas may not necessarily result in students' adoption or even endorsement of them. Therefore, if a criterion for proof in the primary classroom is an argument accepted by the classroom community (Stylianides, 2016), the commognitive framework provides a useful lens to glean insights into barriers that a community may need to overcome in order to reach consensus.

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