Charting a learning progression for reasoning about angle situations

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As a multifaceted concept, the learning of angle concepts takes years to achieve and is beset with challenges. This paper explores how the processes of constructing and validating a learning progression in geometric reasoning can be used to generate targeted teaching advice to support the learning of angle concept. Data from 1090 Year 4 to Year 10 students' ability to reason about geometric properties and deduce angle magnitudes were analysed. Rasch analysis resulted in eight thinking zones being charted. Students' responses to the angle items within this larger data set were analysed with a focus on how reasoning about angles developed. The result is a five-stage framework for learning angle concepts.

Teaching that is informed by effective assessment data has a significant, proven effect on learning (Goss et al., 2015). Designing targeted teaching advice that can nurture mathematical reasoning has become even more vital in light of the 2018 Programme for International Student Assessment (PISA) results (Thomson et al., 2019). Australian students' mathematical problem solving ability is in a long-term decline, equivalent to the loss of more than a year's worth of schooling since 2003. Australian students are particularly weak in the content areas of geometry (Thomson et al., 2017), a discipline that is linked to measurement and spatial reasoning.

Understandings of measurement are embedded in all curriculum in the STEM (Science, Technology, Engineering and Mathematics) areas. Concepts such as length, volume and angle take years to learn and are beset with challenges. A case in point is the learning of angle measurement. The concept of angle can mean different things in different situations. When viewed as a static image, angle is defined as a geometric shape, a corner or two rays radiating from a point, then as a dynamic image, angle is a rotation and a measurement of turn. Research shows persistent student difficulties with angle concepts, including focusing on physical appearances such as the length of the arms or the radius of the arc marking the angle when comparing angles, inability to see angles from different perspectives and contexts, and errors in measuring the angle magnitudes using a protractor (Gibson et al., 2015; Mitchelmore & White, 2000). In the Australian Curriculum: Mathematics (Australian Curriculum Assessment and Reporting Authority [ACARA], n.d), the concept of angle is introduced under the sub-strands of geometric reasoning from Year 3 onwards. In Year 5, students are expected to use degrees and measure with a protractor and in Year 6, to find unknown angles. Year 7 refers to angle sums in triangles and quadrilaterals. The curriculum expectation is that the students will have the necessary understanding of angle and angle measurement to be able to reason about angle sizes by Year 6 and Year 7. It is presumed that teachers are able to make the necessary connections among and across content strands and teach for mathematical reasoning (Lowrie et al., 2012). International results obtained to date do not reflect such a reality.

With STEM becoming a key focus in education, research on learning progressions can help transform the teaching and learning of mathematical reasoning. In this paper, we survey and analyse Australian students' knowledge of and reasoning about angle measurement within a more comprehensive geometric learning progression.

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Theoretical Framework

Learning progressions are a set of empirically grounded and testable hypotheses about students' understanding of, and ability to use, specific discipline knowledge within a subject domain in increasingly sophisticated ways through appropriate instruction. They can relate to a specific instructional episode, develop a curriculum or in our case, charting mathematics learning that encompasses different but related aspects of mathematics. Our purpose is to equip teachers with the knowledge, confidence and disposition to go beyond narrow skill-based approaches to teach for understanding and mathematical reasoning.

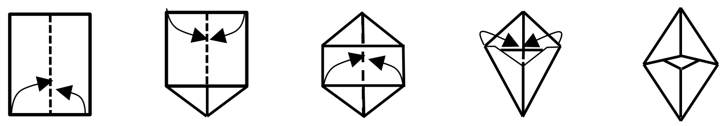
Reasoning is a cognitive process of developing lines of thinking or argument to either convince others or self of a particular claim, solve a problem or integrate a number of ideas into a more coherent whole (Brodie, 2010). Mathematical reasoning is about constructing mathematical conjectures, developing and evaluating mathematical arguments, and selecting and using various types of representations (National Council of Teachers of Mathematics [NCTM], 2000). Mathematical reasoning encompasses three core elements: (1) core knowledge needed to comprehend a situation, (2) processing skills needed to apply this knowledge, and (3) a capacity to communicate one's reasoning and solutions. Justifying and generalising are two key characteristics of mathematical reasoning (Brodie, 2010). To justify a position, individuals need to connect different mathematical ideas and arguments to support claims and conjectures. To generalise requires individuals to reconstruct core knowledge and skills when making sense of new situations. Both help improve reasoning skills, cement core knowledge and may lead to the development of new ideas.

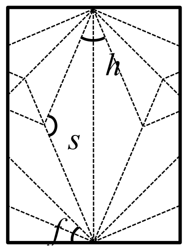
Engaging in mathematical reasoning is a social act, directed by a semiotic process (Bussi & Mariotti, 2008). Symbols (°, ∠), lines (∟, ⊥, ∡), shapes and objects serve as signs and artefacts for a particular purpose. An artefact (e.g., a folded piece of paper or written words) is a tool or an instrument that relates to a specific task to be used for a particular purpose. A sign is a product of a conjoint effort between it and the mind to communicate an intent, such as indication of a right angle. The use of signs and artefacts is never neutral but is intentional and highly subjective, linked to the learner's specific experience and requires the reorganisation of cognitive structures. From a cognitive perspective, how well a learner reasons mathematically is largely dependent on the degree of connectedness among multiple representations (artefacts), visualisation and mathematical discourse (Seah & Horne, 2019). Angle is multifaceted and can be represented in various ways. Visualisation of angle artefacts requires a dynamic neuronal interaction between perception and visual mental imagery. The viewers need to draw on past experiences and existing knowledge to make sense of the visualised artefacts. The context within which perception takes place plays a critical role in determining the type of imagery gaining attention. Individuals' beliefs about their own ability and how mathematics is practiced also play a critical role in this process. Context and beliefs are influenced by the mathematical narratives and routines learners experience. Words and terminologies produce certain visual images. For example, Gibson et al., (2015) found that whole-object word-learning bias led many pre-schoolers to judge angle size by the side length. This was also found with older children (Mitchelmore & White, 2000). During a mathematical discourse, communication can take a combination of linguistic, symbolic or diagrammatic forms. How they are being used reveal the users' thought processes and in turn shapes their thinking. Analysis of students' responses to angle measurement tasks will enable researchers to document and chart how students' reasoning about angle measurement progressed. This can then help design instructions that move students from where they are to the next level of their learning journey.

Method

Drawing on the work of Battista (2007), a draft geometric learning progression was developed that saw the development of geometric reasoning as moving through five levels of reasoning: visualising physical features, describing, analysing, and inferring geometric relationships, leading to engaging in formal deductive proof (Seah & Horne, 2019). The data presented here was taken from the Reframing Mathematical Future II study into the development of learning progression for mathematical reasoning. The participants were middle-years students from across Australia States and Territories. The first group – the trial data, was taken from two primary and four secondary schools across social strata and three States. They were asked to participate in trialling the assessment tasks to allow for a wider spread of data being collected. The trial school teachers administered the assessment tasks and returned the student work to the researchers. The results were marked by two markers and validated by a team of researchers to ascertain the usefulness of the scoring rubric and the accuracy of the data entry. The second group – the project data, came from 11 schools situated in lower socioeconomic regions with diverse populations across six States and Territories. The project school teachers marked the items and returned the raw score instead of individual forms to the researchers. They also received ongoing professional learning sessions and had access to a bank of teaching resources. There are two angle measurement tasks, Geometric Angles 1 and 2 (coded as GANG) reported here (Figure 1).

Geometric Angles 1
You will need the shape you made in class. The steps and diagrams below show how you made the shape.



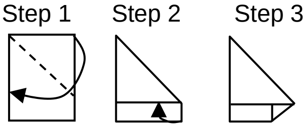


Step 1 Fold an A4 paper in half lengthwise to make a crease line in the middle of the page.
Step 2 Fold two corners to the middle at the bottom
Step 3 Fold two corners to middle at the top
Step 4 Fold the new corners on the sides at the bottom to the middle
Step 5 Do the same with the top

a [GANG1]
Phoebe made the same shape that you made using A4 paper. She said her shape is a rhombus.
Do you agree? Explain your reasoning.

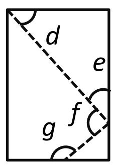
b [GANG2]
When Phoebe unfolds the paper, she found a number of crease lines. Find the marked angles on the crease line: Angle *f* = ____ Angle *h* = ____ Angle *s* =_____
Explain how you work out the angles.

Geometric Angles 2
A four-sided shape is folded from a sheet of A4 paper using the following instructions.



a [GANG3]
What is the name of this shape?
____
Explain your reasoning.

b [GANG4]
Unfold the paper and find the size of each marked angle.
Angle *d* = _____________ Angle *e* = _____________
Angle *f* = ______________ Angle *g* = _____________



Explain your reasoning.

Figure 1. Geometric angles task 1 and 2.

Note that both tasks were used in different forms rather than administered together. Both tasks begin with a question on geometric properties followed by deductions of angle magnitudes. In GANG1, the teacher was instructed to guide the students to first fold the shape and use it to answer the angle measurement question. In this way, the difficulty in following the origami instructions was avoided. As an artefact, the folded shape also served as a context and a tool to help students comprehend the diagram depicting the crease lines. In GANG3, students were shown the steps taken to fold a shape. No further information was given. Items GANG2 and GANG4 ask students to work out the magnitude of the angles formed by the crease lines. While the tasks GANG1 and GANG3 do not ask students specifically to use angle, angle properties are one component of shape classification. The focus in this paper is on reasoning about angle magnitude in GANG 2 and GANG 4.

Rasch partial credit model (Masters, 1982) using Winsteps 3.92.0 (Linacre, 2017) was used to analyse students' responses on the larger set of geometric reasoning tasks including these for the purpose of refining the marking rubrics and informing the drafting of an evidence based learning progression. Rasch analysis of the validity of the underlying construct through the idea of fit to the model produced eight thinking zones in geometric reasoning (Seah & Horne, 2019). To validate the zones, the research team interrogated student responses located at similar points on the scale to decide whether or not there were qualitative differences in the nature of adjacent responses with respect to the sophistication of reasoning involved and/or the extend of cognitive demand required (see Siemon & Callingham, 2019).

SCORE	**DESCRIPTION for GANG1**	**DESCRIPTION for GANG3**
0	No response or irrelevant response	
1	Disagree it is a rhombus based on appearance rather than properties	Diamond or other incorrect shape
2	Disagree it is a rhombus but claim it is a parallelogram with some properties	Quadrilateral because it has 4 sides OR because it looks like a kite
3	Agree it is rhombus but insufficient or incorrect properties to define it or claims it is a parallelogram and includes all properties	Kite OR unable to name, but gives side and/or angle properties of a kite
4	Agree it is rhombus. Explanation needs to include necessary and sufficient properties, that is, it has 4 equal sides, or it is a parallelogram with one of the following properties: • Adjacent sides equal • Diagonals bisect each other at right angles or diagonals bisect the angles • Two lines of symmetry	Kite because two pairs of adjacent equal sides are equal OR because at least a pair of opposite angles equal and at least one pair of adjacent sides the same length OR because it has a pair of opposite angles equal and a line of symmetry. May include other properties.
SCORE	**DESCRIPTION for GANG2**	**DESCRIPTION for GANG4**
0	No response or irrelevant response	
1	Incorrect angles	Incorrect with little/no reasoning, may include one correct angle
2	At least 2 angles correct but no reason given, or one angle correct with correct reasoning	At least two angles correct with an attempt at explaining reasoning
3	Two angles found correctly with sensible reasons or all angles correct with insufficient reasoning	Angles correct (d = e = 45°, f = 90° or right angle, g = 135°) but reasoning sparse and incomplete
4	All angles correct with clear reasons given relating to the folding and properties. F = 45°; h = 45°; s = 135° (e.g., Folding corner to centre creates half right angle; All angles around centre of side equal so any 2 make 45° or Four angles of quadrilateral add to 360°)	Angles correct. Reasoning includes justifies d as half of the right angle in corner or as angles in an isosceles triangle, and g on the basis that the four angles of the kite shape have to add to 360°

Figure 2. Geometric angles task scoring rubrics.

In the following, we focus on students' responses to the angle items to determine their usefulness and fit to the overall learning progression framework.

Findings

Based on 1041 students' responses from the larger study, the zones of geometric reasoning were established as precognition; recognition; emerging informal reasoning; informal and insufficient reasoning; emerging analytical reasoning; property based analytical reasoning; emerging deductive reasoning; and logical inference-based reasoning (Seah & Horne, 2019). Student responses were coded so that GANG3.1 meant a response at Level 1 on the rubric to the question GANG3. Table 1 shows how the responses to the GANG questions were spread across the zones (with Zone 8 being the highest level).

Table 1
Excerpt from the variable map for geometric reasoning (n=1041).

Zone 8			GANG3.4	GANG4.4
Zone 7	GANG1.4	GANG2.4		
Zone 6		GANG2.3 GANG2.2		GANG4.3
Zone 5				GANG4.2
Zone 4	GANG1.3 GANG1.2	GANG2.1	GANG3.3	
Zone 3	GANG1.1		GANG3.2	
Zone 2			GANG3.1	GANG4.1
Zone 1				

To validate these zones, the research team interrogated student responses located at similar points on the scale to decide whether or not there were qualitative differences in the nature of adjacent responses with respect to the sophistication of reasoning involved and/or the extent of cognitive demand required. For example, GANG1.2 (disagree it is a rhombus claiming it is a parallelogram) and GANG1.3 (agree that it is a rhombus with insufficient explanation about its properties) were located in zone 4, indicating similar level of thinking. Reasoning about a kite (GANG3.4 and GANG4.4) were located in the highest level (Zone 8), perhaps revealing students' lack of exposure to this concept. The angles on the rhombus were also easier to deduce than those on the kite.

Table 2
Breakdown of student responses on geometric properties (GANG1 and GANG3).

Score	Trial Data (*n=230)*				Project Data (*n=433)*			
GANG1	Yr 7 *n=83*	Yr 8 *n=90*	Yr 9 *n=31*	Yr 10 *n=26*	Yr 7 *n= 171*	Yr 8 *n= 204*	Yr 9 *n= 37*	Yr 10 *n= 21*
0	20.5	45.6	19.4	3.8	36.3	32.8	24.3	14.3
1	30.1	13.3	9.7	0	19.3	15.2	0	38.1
2	12.1	11.1	12.9	7.7	4.1	11.3	2.7	47.6
3	33.7	17.8	48.4	73.1	36.8	27.9	62.2	0
4	3.6	12.2	9.7	15.4	3.5	12.8	10.8	0
Score	Trial Data (*n=157)*				Project Data (*n=270)*			
GANG3	Yr 4 *n=31*	Yr 5 *n=59*	Yr 9 *n=30*	Yr 10 *n=37*	Yr 7 *n= 23*	Yr 8 *n= 113*	Yr 9 *n= 32*	Yr 10 *n= 102*
0	22.6	23.7	27.6	35.1	0	17.7	53.1	17.7
1	77.4	66.1	27.6	32.4	17.4	21.2	31.3	28.4
2	0	10.2	13.8	29.7	69.6	21.2	15.6	32.4
3	0	0	34.5	2.7	13	31.9	0	20.6
4	0	0	0	0	0	8	0	1

Because the Rasch model is probabilistic, an in-depth analysis of students' responses to the items were conducted. A total of 663 samples for Geometric Angles 1 task and 427 samples for Geometric Angles 2 were collected. Table 2 shows the breakdown of students' responses for reasoning about geometric properties. Both cohorts performed better in the rhombus item than the kite item. The majority of students found reasoning about geometric properties difficult and on average, around 25% of students did not respond. In GANG1, students tended to define a rhombus based on its orientation or what it looks like:

Year 7: I believe wasn't wide enough to become a rhombus and the shape is a diamond (score 1).

Year 10: It can be depending on how you look at it. It could be a diamond or rhombus (score 2).

Year 9: … when you hold the shape so that the pointed parts point from left to right you would see that it is in the shape of a rhombus (score 3).

Only a handful of students accurately defined a rhombus, as having '4 equal sides'; none included the square as part of the rhombus family. Further, when the term angle was used, it was to emphasize that a rhombus has no right angle, or incorrectly stating that the shape has 'four exactly the same sides with 4 acute angles'. In GANG3, 76.4% of students provided a 2D name to the folded shape, such as triangle (16%), irregular rectangle/square (31.2%), polygon (6.4%) and quadrilateral (8.9%). None of the trial school students were able to correctly state the properties of a kite.

Nevertheless, inability to reason about geometric properties did not appear to influence the deduction of angle magnitudes. Comparison of students' responses by year level shows that project schools' performance was slightly better and that the angles in the rhombus were easier to deduce than those in the kite (see Table 3). There was still a large number of no response or irrelevant responses received from the trial data (27.8% and 38.5% in GANG2 and GANG4 respectively).

Year 9: I worked this out by counting the crease of each angle (wrote 3, 3, 2 in GANG2).

Year 4: I measured each line and quartered it (wrote 2 cm, 3 cm, 4 cm, 5 cm in GANG4).

Other students (47.8% and 28.9% respectively) either wrote the name of the angles as acute or obtuse or were only able to give the magnitude of one angle.

Table 3
Breakdown of student responses on angle magnitudes (GANG2 and GANG4)

Score	Trial Data (*n=230*)				Project Data (*n=433)*			
GANG2	Yr 7	Yr 8	Yr 9	Yr 10	Yr 7	Yr 8	Yr 9	Yr 10
	n=83	*n=90*	*n=31*	*n=26*	*n= 171*	*n= 204*	*n= 37*	*n= 21*
0	14.5	44.4	29	11.5	47.4	39.7	35.1	9.2
1	67.5	35.6	35.5	42.3	33.9	27	35.1	9.5
2	6	7.8	22.6	7.7	7.6	9.3	5.4	14.3
3	3.6	10	9.7	23	6.4	10.3	10.8	4.8
4	8.4	2.2	3.2	15.4	4.7	13.7	13.5	61.9
Score	Trial Data (*n=157)*				Project Data (*n=270)*			
GANG4	Yr 4	Yr 5	Yr 9	Yr 10	Yr 7	Yr 8	Yr 9	Yr 10
	n=31	*n=59*	*n=30*	*n=37*	*n= 23*	*n= 113*	*n= 32*	*n= 102*
0	83.9	20.3	27.6	37.8	17.4	18.6	43.8	5
1	16.1	47.5	27.6	10.8	43.5	38.1	34.4	9.8
2	0	25.4	13.8	27	8.7	23	9.4	21.6
3	0	5.1	34.5	10.8	21.7	17.7	6.3	33.3
4	0	1.7	0	13.5	8.7	2.7	6.3	30.4

Many of the irrelevant responses for GANG4 were from the primary years 4 and 5, suggesting that these students may not have learned this concept. The trial data showed that some of the responses were far from the correct answers. Students either solved the problem based on physical appearance - 'it looks like a right angle… (90°, 43°, 180° for GANG2)', made obscure comments such as 'use a pencil (90°, 110°, 155°)', or wrote '60°, 70°, 90°, 140° the sum of all the angles = 360°' (GANG4). A further 29 trial school students admitted to using a protractor for GANG2. This may be due to the teacher's oversight or assumption that it was inaccessible by the students or because they have no strategies otherwise. Despite its availability, only two students were able to provide the correct answers.

A 45° angle was the easiest to deduce by using right angle as a benchmark. Using existing angle knowledge as benchmark did not always work however and the Year 9 and Year 10 students tended not to provide a reason for GANG4:

> Year 7: You work out the angles by knowing where 90° is and if the angle is smaller then you take a given between 0° and 90°. If the angle is bigger than 90° and smaller than 180° then you guess what the angle might be. I then checked with a protractor to see how far off I was (34°, 96.5°, 135°).
>
> Year 10: They all need to equal to 180 (wrote 20°, 30°, 130°)
>
> Year 9: *d* and *e* has the same size angle as you can see, *f* as everyone knows that it is 90° because it's a right angle and *g* is an obtuse, which is 180° (wrote 45°, 45°, 90°, 180°).

Discussion

Angle is the foundation for much of geometry and trigonometry and applicable in many daily activities, yet many students did not demonstrate understanding of the concept nor ability to reason about angle size. Looking within the overall geometric framework at the student responses to the question requiring reasoning about angle measurement in more detail gave an indication of the development of reasoning about angle.

Students operating in Zones 1 and 2 of the geometric learning progression usually did not show evidence of identifying the meaning of angle in any useful way. When they did use the term angle it was in reference to a right angle, often incorrectly. They did not use angle properties at all in identifying shapes. Students in Zone 3 were identifying right angles and in Zone 4 some of the students were referring to acute and obtuse angles though they were still not correctly giving many angle magnitudes with the exception of a right angle. By Zone 5 the students were attempting to reason about the angle magnitudes and were identifying the magnitudes of some of the angles correctly, usually in relation to a right angle. Diagrams, calculations and connecting language were beginning to appear. In Zone 6, the students were correctly identifying angle magnitudes, but their reasoning tended to relate just to the right angles and was incomplete. The few students who responded in Zones 7 and 8 were able to correctly identify the angles and explain their reasoning using a combination of diagrams and calculations integrated with words.

Relating this to the overall learning progression for geometric reasoning indicates that for reasoning about angle measurement there appeared to be five stages

1. Informal reasoning based on appearances (Zones 1-3): This encompasses the development of the concept from thinking of angles as lines or lengths through to identifying angles as corners and visually recognising 90° angles.
2. Informal and insufficient formal reasoning (Zone 4): Reasoning about magnitudes as being greater or less than a right angle and assigning magnitudes accordingly.
3. Emerging analytical reasoning (Zone 5): Deducing and arguing 45° angles in relation to a right angle, often with an accompanying diagram or calculation.

4. Relational-inferential property-based reasoning (Zone 6): Correctly identifying angles and giving reasoning for at least some of them usually with some attempt at using diagrams and connecting language, often with some calculation.
5. Emerging deductive and logical inference-based reasoning (Zone 7-8): Correct identification of angles reasoned with supporting diagrams, calculations and integrating connecting language.

The descriptions here are in the context of the questions that were asked. The final stage would be moving to full deductive reasoning and proof, but we have no evidence of this stage as the questions did not seek a response at this level. Nevertheless, the results show that many Australian students in our sample across all states are unable to do what the curriculum expects them to do. Learning progression research allows researchers to identify what learners can do, and what needs to be done to move their learning forward. The stages as described here contributed to the development of advice for teaching reasoning about angle measurement. Further research is needed to investigate this progression and expand it more fully to encompass the whole of angle measurement.

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