The reification of the array: The case of multi-digit multiplication

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The array is a powerful tool that builds students' understanding in multiplication. Students' interactions with the array changes through the course of an instructional sequence, which can be viewed as a process of *reification*. In this paper, I report the findings of a research study conducted with 45 Year 5 students in Sydney. The study explored students' changing use of the array through the course of an instructional sequence on multi-digit multiplication. Design Research methods were used to track students' use of different forms of the array and the functions that these forms served. Three key stages were identified in the process of reifying the array in multi-digit multiplication.

Mathematical representations play an important role in the development of understanding (Hiebert & Carpenter, 1992; Pirie & Kieren, 1994). Representations make visible that which is abstract, thus making more abstract concepts accessible to students (Gravemeijer, 2004). Students' abilities to work with mathematical representations flexibly and their capacities to interpret and connect representations are key to the process of building mathematical understanding (Goldin & Shteingold, 2001; Gravemeijer, 1999). As students interact with formal, external representations they can more easily observe connections and relationships between mathematical concepts. Those observed connections and relationships form students' own internal representations of concepts (Goldin & Shteingold, 2001).

The array has been recognised as a powerful representation that allows access to the important theoretical constructs of multiplication (Barmby et al., 2009; Battista et al., 1998). This two-dimensional representation of multiplication highlights equal groupings and shows how the composite units build on each other to produce a whole (Steffe, 1994). Curriculum documentation presents the array in various forms as an important tool in the teaching of single- and multi-digit multiplication (ACARA, 2017).

There has been substantial research on the array with single-digit multiplication but there exists limited research on its usage in multi-digit multiplication (Barmby et al., 2009; Young-Loveridge & Mills, 2009). Despite this limitation, studies have shown that the array affords students access to important multi-digit multiplication understandings, including the distributive property (Barmby et al., 2009, Izsak, 2004; Young-Loveridge & Mills, 2009) and the associativity property (Ding et al., 2013). What is less evident in the literature is how students' interaction with the array in multi-digit multiplication evolves over the course of an instructional sequence as their understanding of the multiplicative structure develops.

In this study, I examined the power of the array as a *representation of* a contextual situation through to a *representation for* mathematical reasoning as enacted by the students through their mathematical activity over the course of an instructional sequence. Gravemeijer (1999) described this changing use of representations as a process of *reification*, where mathematical activity takes on object-like character as a result of student activity. According to Gravemeijer (1999), there are two stages to the process of reification. First, students' activity is bound in the context of the problem, a stage Gravemeijer refers to as the *referential level*. The second stage is the *general level*, where students' interpretations and solutions operate separately to the contextual imagery.

To address the research gap, and to inform curriculum design and teaching practice, the following question focused the research: *How does students' use of the array develop from*

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a representation of a contextual situation to a representation that is used for more generalised mathematical reasoning in multi-digit multiplication?

Theoretical Framework

The theory of Realistic Mathematics Education (RME) was used to guide the design of this research. RME is founded on the belief that mathematics is not a closed body of knowledge to be transmitted. Rather, it is an exercise in which learners are active participants (Van den Heuvel-Panhuizen, 2003), whereby one 'reinvents' conventional mathematics for themselves (Gravemeijer, 2004). In the context of the classroom, students engage in tasks that require them to develop their own tools and strategies as they solve experientially real problems. This is the process of mathematisation. Students form and organise new knowledge and develop their own mathematical insights (Van den Heuvel-Panhuizen & Drijvers, 2014). The aim of RME is to support students' progressive mathematisation, or level-raising (Gravemeijer et al., 2003). To achieve this, learning experiences are based on three important design heuristics: experientially real contexts, guided reinvention (a process where students reinvent conventional mathematics through active teacher guidance), and emergent modelling. Most relevant to this paper is the heuristic of emergent modelling, which describes how students' interactions with models develop and change through the course of an instructional sequence.

In RME, a *model* is a broad term that encompasses varied representations of mathematical concepts and structures (Van den Heuvel-Panhuizen, 2003). Models are not designed as ready-made representations trying to make mathematical concepts concrete. Rather, models are developed out of contexts (Gravemeijer, 2004) and support students in the process of progressive mathematisation. The model, as Van den Heuvel-Panhuizen (2003) explains, serves as a bridge. On one side of the bridge are the informal understandings bound within the context of the problem, and on the other side are the formalised mathematical concepts. It is students' interactions with the model that allow them to cross this bridge.

The nature of the model changes through students' activity. It moves from a *model of* a situation to a *model for* mathematical reasoning (Gravemeijer, 2004; Van den Heuvel-Panhuizen, 2003) with different forms of a model serving different functions (Saxe, 2002, 2004). Initially the model is closely connected to the context of the problem: it is a model of a particular situation and students use it to make sense of the problem at hand. As students work with the model over multiple experiences, they build an appreciation for the mathematical concept or structure that the model embodies. Their understanding of the model becomes more generalised, and it becomes a model for mathematical reasoning. The model is *reified*. As Gravemeijer (2004) explains,

> the model first comes to the fore as a model of the students' situated informal strategies. Then, over time, the model gradually takes on a life of its own. The model becomes an entity in its own right and starts to serve as a model for more formal, yet personally meaningful, mathematical reasoning (p. 117).

Methods

The methodological approach for this research needed to allow the researcher to observe first-hand students' reasoning and interaction with the array. To meet this aim, Design Research methods were employed (as described by Cobb & Gravemeijer, 2008). Three research phases were enacted: preparatory thought experiments, teaching experiments and a retrospective analysis.

The preparatory phase formed the foundation of the project. A detailed analysis of relevant literature was the basis for anticipatory thought experiments (Gravemeijer, 2004). This phase clarified the learning goals, documented the starting points for instruction and then, from this, delineated a predicted learning pathway.

The teaching experiment phase of the project was conducted in two different Year 5 classes. Both classes were from non-government schools in Sydney; 23 students in the first class and 22 in the second class, comprised a sample size of 45 students. The researcher adopted the role of the teacher in each teaching experiment. This approach allowed the learning environment and teaching across both experiments to be controlled and enabled the researcher to experience first-hand the events of the classroom, thus enriching the ongoing cycles of data analysis and experimentation. The students' regular teacher was also present in the classroom and helped facilitate student activity. The same instructional sequence was taught in both classes. The sequence was implemented over a two-week period and comprised of four teaching episodes. Each teaching episode spanned two or three one-hour lessons and was characterised by a focus on a distinct mathematical concept, presented through the context of a problem. Each teaching episode is described later in the results section of this paper.

The retrospective analysis situated the classroom learning process into the "broader theoretical context as a paradigmatic case of a more encompassing phenomenon" (Cobb & Gravemeijer, 2008, p. 83). It was in this phase that a grounded theory (Glaser & Strauss, 1968) on the reification process of the array was formed.

Data collection and analysis

The data collected needed to elicit evidence of students' reasoning with the array, shifts in their reasoning, and how these shifts were supported and organised. Based on this, three key forms of data were collected: student work samples from the teaching episodes, transcribed video recordings of classroom activity, and field notes compiled by the researcher and class teacher during classroom lessons.

The analysis of data occurred over two phases of the research. The Constant Comparative method (Glaser & Strauss, 1968), adapted to the needs of Design Research as illustrated by Cobb and Whitenack (1996), was used during the teaching experiments. Students' use of the array was tracked across the teaching episodes and descriptions of students' usage were grouped in two ways: according to the individual students and then according to the solution method used. This enabled observation of whether the model held power for individual students, which would be evident through the moving from a *model of* the contextualised situation to a *model for* more generalised mathematical reasoning.

The second round of data analysis was conducted as part of the retrospective analysis, which mapped the process of the array moving from a *model of* a contextualised situation to a *model for* more generalised mathematical reasoning. Saxe's (2004) form-function framework was used to explore the different forms of the array used by students and what function each form of the array served. The framework helped explain how students' use of the array shifted over the course of the instructional sequence, to serve differing functions. The *form* of the array was defined as specific visual features, and its *function* was defined as the way the students chose to interact with the array in their work. Three forms of the array were observed across the instructional sequence: arrays with all individual parts visible, a pre-partitioned array, and an open array. The dataset was grouped according to the three forms of the array so that commonalities could be identified, and so that shifts in students' form-function use over the course of the instructional sequence could be noted. The dataset was then re-grouped, this time based on the array's function. Grouping in this way served to confirm the commonalities that were identified, students' evolving use of the array, and to highlight any anomalies. The final step in the analysis was to explore the form and function of the array based on students' diverse conceptions and strategies. To do this, data were grouped based on the form and function of the array that the students chose to use as they developed solutions to the problems.

Results

The results section describes the visual form of the array used and the student-chosen function that each array served. The students' use of context is also recorded as the process of reification is mapped. Examples of students' work is used to illustrate each teaching episode. The work of these students was typical of what was observed across both classes.

Teaching Episode 1 – Zoe and Lucille

The first teaching episode introduced the students to the context of a bakery that sold cupcakes. Students were presented with the following narrative: A baker makes and sells eight different flavours of cupcakes. The cakes are baked in a tray that has four rows with six cakes in each row. He bakes one tray of each flavour. How many cupcakes does he bake each day?

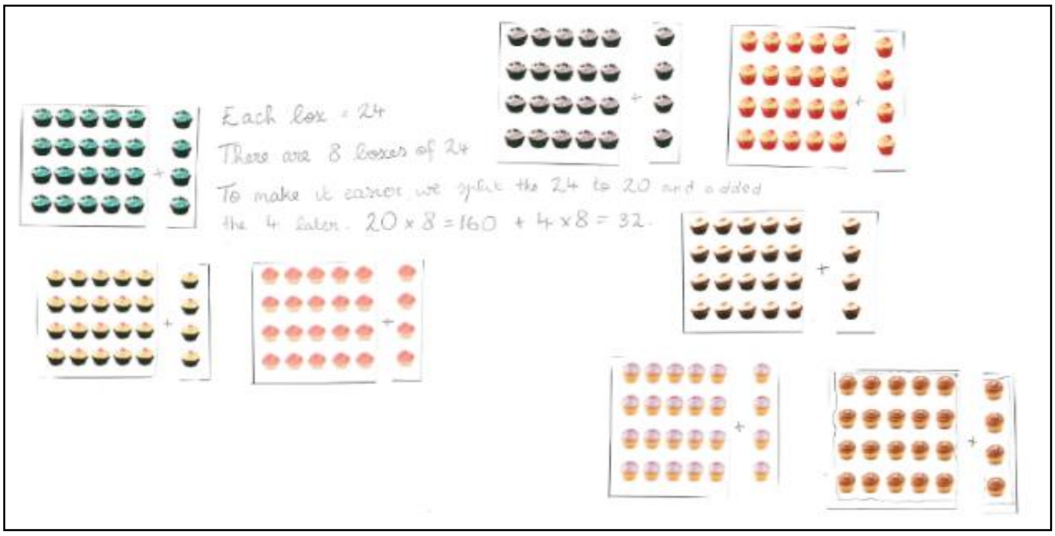

Figure 1. Zoe and Lucille's work sample from Teaching Episode 1

Zoe and Lucille's solution and justification were bound within the context of the problem. They represented their strategy using arrays which were presented as actual cakes. The function of the array in this form was to support calculation. Each tray was considered individually and was partitioned into groups of 20 and 4 (Figure 1). Lucille's verbal explanation of their strategy highlighted that the context was relevant to their thinking as they solved the problem: *But it is not like you are really cutting a row off, like, you can't. They are in a tray, so, yeah, you can't actually do it. But it is just how we worked it out.*

Teaching Episode 2 – Ryan and Dylan

The class was shown a picture of 16 filled cupcake boxes sitting on a bench in a 4 × 4 array, and students were told that each box held 12 cakes. The array was somewhat abstracted as the individual cakes were not visible. However, a further diagram shown to students revealed that the cakes in each box were configured in three rows of 4.

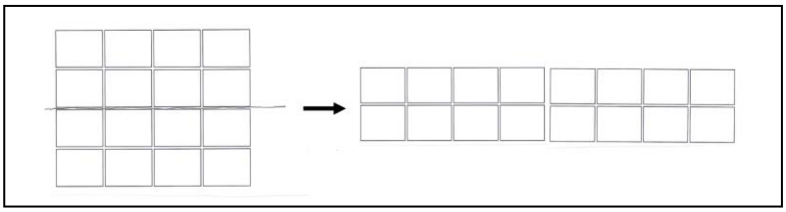

Figure 2. Ryan and Dylan's work sample in Teaching Episode 2

Ryan and Dylan partitioned the array based on place value and noted that the result for this collection of cakes, 16 × 12, was the same as the total number of cakes in the first teaching episode, an array of 24 × 8. This led into investigating a second mathematical goal—why did 16 × 12 = 24 × 8? Recognising that 16 could be halved to make 8 and that 12 could be doubled to give 24, the pair divided the array in half and rearranged it to transform 16 × 12 into 24 × 8 (Figure 2). Through their mathematical activity, they established a new function for the array: the array could be manipulated. The array moved from a static tool to a dynamic one. Through their working, the boys reasoning remained connected to the context, as illustrated in the following comment from Ryan: *You could join two of the boxes together to make 24 then… wait, that's 8 groups…yeah…that's 8 groups because 8 twos are 16.*

Teaching Episode 3 - Amelie

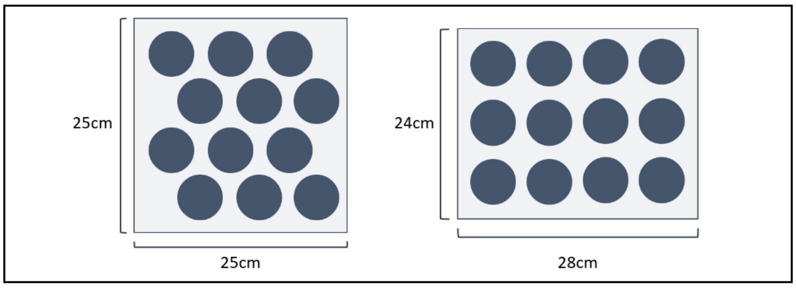

Figure 3. Comparing the area of two trays in Teaching Episode 3

In the third teaching episode, the students were shown the trays inside different cupcake boxes, and they discussed why one array was skewed and the other was not (see Figure 3). Students hypothesised that the skewed array was smaller in area and therefore would be cheaper to make. This hypothesis was the focus of the teaching episode.

Amelie, a student from Class 2, reasoned that 28 × 24 would be bigger, arguing that 2 could be taken from 28 and added to 24 resulting in the "equivalent" equation 26 × 26 (which is larger than 25 × 25). While 28 × 24 was indeed bigger, Amelie came to see that her reasoning was incorrect, and a new mathematical goal emerged: why was 28 × 24 not equivalent to 26 × 26? To achieve this goal, Amelie worked independently from the context of the problem. She regressed from an open array to a more familiar form of the array, a grid array with all parts visible. The function of the array in this form was a sense making tool for the multiplicative structure. Amelie created a 28 × 24 array from grid paper, then cut off two columns from the 28 and taped them to the bottom of the array (Figure 4). She noticed that a 2 × 2 corner was missing, which left her puzzled. To understand what was happening, Amelie explored some other calculations, including 12 × 8. She recognised that, when attempting to form a square, a square corner with the dimensions of the number removed would be missing. This process helped Amelie realise that additive compensation could not be used in a multiplicative context.

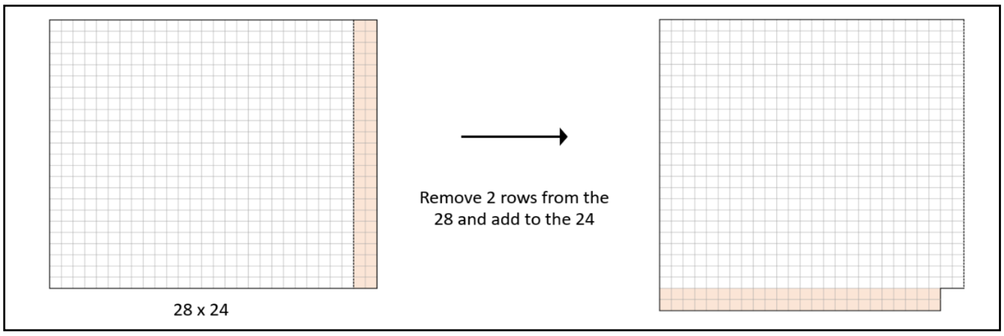

Figure 4. Amelie's strategy for comparing 28 × 4 and 26 × 26

Teaching Episode 4 – Hannah and Ava

The final teaching episode continued the narrative of the bakery and presented students with a multi-step problem: the total cost of 24 trays of cakes packed into boxes of 12 and sold at $28 per box. This context could not be easily represented as an array, as the problem presented a rate-based context. The intention was to see if students' strategies were limited by the context, or if they moved beyond the context to use the array as a calculation tool.

The majority of students from both classes used the array, partitioning it into place value parts to form simpler calculations. This is illustrated by Hannah and Ava's work. The girls reasoned that partitioning into place value parts created calculations that were easy to perform. Hannah and Ava were working abstractly with the array and made no reference to the context of the problem in their recording or justifications.

Abstract thinking, disconnected from the context, was evident in most students' work. While in earlier teaching episodes students referred to calculating with 'boxes', in this episode a shift was made to calculating with numbers: *We timesed [sic] 64 by 20 which is really just like doing 64 times 2 and then adding a zero. And then we just timesed [sic] 64 by 2, and then doubled again to get 64 times 4.* Abstract thinking was realised through students' mental calculations, as illustrated by one student's solution to 32 × 25: *25 is a friendly number because you just multiply it by 4 to get 100, so you divide the 32 by 4 to get 8, so it is just the same as 8 × 100.*

Discussion

The process of the array's reification can be understood by examining the forms of the array that students selected to use and the function that each form served. Students chose to use different forms of the array within a single problem and oscillated between multiple forms across the instructional sequence. At the start of the instructional sequence and when a problem was first posed, students used a form of the array that was closely connected to the context of the problem. In the same way, their interactions with the array were contextually bound. The function of the array in this form was to support calculations. This is indicative of the *referential* level in the process of reification (Gravemeijer, 2004). By the end of the sequence, students' reasoning with the array was more abstract and generalised and removed from the context of the problem; they had progressed to the *general* level in the reification process (Gravemeijer, 2004).

An interim level was observed in this process which I have termed *structuring*. In the process of making sense of the multiplicative structure, students worked independently of the context with a previously understood form of the array. They were no longer working at the referential level, nor were they generalising. The array had not yet become a tool for more formal mathematical reasoning as students were not engaged in reflection, explanation and justification. Student activity was focused on sense-making through an exploration of the multiplicative structure, removed from the context of the problem.

Central to this process of structuring was the flexibility for students to move between different forms of the array. On several occasions when using a form of the array connected to the context of the problem, students were faced with their own insufficient or incomplete internal representations (Goldin & Shteingold, 2001). In these instances, students would 'fold back' (Pirie & Kieren, 1994) to the simpler form of the external representation: the array with all parts visible, as illustrated by Amelie's working. Students would use this form of the array to explore the multiplicative structure and to make sense of what was happening mathematically. The evidence suggested that students were creating new connections and strengthening existing connections between their internal representations and, in so doing, building a deeper, or 'thicker', mathematical understanding (Pirie & Kieren, 1994). This process of thickening understanding was removed from the context of the problem.

At this structuring level, students also needed to work independently of the context of the problem in order to make sense of the mathematical properties of the array. As powerful as a context can be in enabling students' access to mathematical ideas, it can also be a hindrance. Students' strategies can be bound within the context of a problem (Ambrose et al., 2003) and the array may not be recognised as a multiplicative representation. This does not mean that contextual situations should not be used to introduce mathematical content. However, students must have the opportunity to work independently of the context in order to connect the array representation with the mathematical concept being explored. It is the array representation, not the context, that highlights important theoretical properties of multiplication.

Conclusion

The process of reification of the array in multi-digit multiplication highlights how students progress from using the representation as a model of a particular situation to using the array for more generalised mathematical reasoning. Mapping this process contributes to the growing body of knowledge on how the array supports the development of understanding and provides guidance to curriculum designers and practitioners. Students need the opportunity to use the array to explore the multiplicative structure. In their explorations, students should not be restricted to one form of the array. Rather, flexibility is needed. Students should be afforded the opportunity to select and use different forms of the array, recognising that different forms will serve different functions.

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