Design Principles for Raising Students' Awareness of Implicit Features of Ratio: Creating Opportunities to Make and Catch Mistakes

Sze Looi Chin
Nanyang Technological University
nie19.csl@e.ntu.edu.sg

Seeing a problem and immediately knowing how to solve it is typically desired in students. However, it can also lead to impulse thinking, where students are unconscious of critical features of concepts and consequently make negligent mistakes. This study investigated the design principles of a secondary mathematics teacher who designed instructional materials to raise her students' awareness of implicit features of ratio and proportionality. Analysis of her design and implementation revealed how she created opportunities for students to make mistakes in class and to "catch" them to address these implicit features. Implications of adopting design principles to manage impulse thinking are discussed, as well as an introduction of the notion of *catch tasks*.

Research on mathematics teachers' design work typically report on goals for developing students' conceptual understanding, procedural fluency, and metacognitive skills, amongst other things. To achieve these goals, design principles related to sequencing, choosing rich tasks, and forming connections are often used (Swan & Burkhardt, 2014). However, the goal of managing the pace of students' thinking (e.g., encouraging them to "slow down", to not act on impulse) is less reported, along with research on design principles that teachers use to achieve this goal. Although the ability to see a problem and immediately know how to solve it is desirable in students, acting on intuitions is not always conducive and can even be disadvantageous (Kahneman, 2012). This was the case for students in a class taught by the teacher in this study, Tanya (pseudonym). She noted that when her students are fixated on finding the solution and have prior knowledge of the content and procedures, they often act on impulse and are unaware of critical features of the task, consequently making negligent and costly mistakes that may impact future learning and assessment results. In the worst case, when *implicit features* that are tacit and rarely explicitly emphasised (e.g., proportionality is based on multiplicative reasoning, *not additive reasoning*) and errors (e.g., using addition to determine equivalent ratios) are not brought to students' attention and addressed, students can develop underlying misconceptions. Consequently, Tanya designed instructional materials with the goal of managing students' impulse thinking to raise their awareness of implicit features in the topic of ratio. The aim of this study is to examine the design principles she used in crafting her instructional materials to achieve this goal in the classroom.

Background

There is a growing consensus on the need for more research on mathematics teachers' design work (Kaur et al., 2022; Watson & Ohtani, 2015). While Brown (2009) conceived of teachers' design work as occurring predominantly during instruction, emergent research has documented the significant design that can occur before the lesson, which includes not only lesson planning but also the design of *instructional materials* (IM) (Kaur et al., 2022). These teacher-designed IMs generally help to guide the flow of the lesson, and as such, they typically consist of a multitude of tasks that are selected, modified, or created by the teacher for the purpose of achieving their intended curriculum (Remillard & Heck, 2014). Hence, to design these IMs requires teachers to make multiple deliberate design decisions related to gathering and sequencing tasks so that they will be effective for teaching.

Schoenfeld (2010) proposed that to make sense of how teachers make these kinds of decisions, researchers should investigate the interactions between teachers' resources, orientations, and goals, and noted that decision-making is ultimately goal-oriented. One goal that is seldom reported in

(2023). In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 163–170). Newcastle: MERGA.

mathematics education research is the goal of managing students' impulsive dispositions. This intuitive form of thinking, so-called *thinking fast* or *System 1* thinking (Kahneman, 2012), is automatic, well-rehearsed, and unconscious, but it becomes disadvantageous when individuals jump to conclusions without gathering accurate evidence and rational reasoning. In contrast, an analytic disposition, so-called *thinking slow* or *System 2* thinking, is deliberate, conscious, and logical; this is typically elicited when unexpected or unfamiliar situations arise that force individuals to reflect on and analyse the situation at hand. Lim and Wagler (2012) found that when students were familiar with the task situation and had some prior understanding, they tended to adopt an impulsive disposition, while not always producing correct solutions to mathematics problems. Despite its potential for supporting students' conceptual development, there is little research reported on what mathematics teachers do to engage and manage the pace of students' thinking through instructional design and in the classroom. If teachers do not consciously design opportunities for students to slow down to notice their errors, students are unlikely to notice important features of concepts and to realise their own mistakes (Watson & Ohtani, 2015).

Teachers' knowledge and their choice of examples and tasks is crucial for helping students to notice important features for solving problems (Ball et al., 2008; Goldenberg & Mason, 2008). While rich, collaborative, and higher-order tasks are usually touted for effective teaching (Swan & Burkhardt, 2014), typical problems like those commonly found in textbooks and examinations have also been shown to be promising for orchestrating productive classroom discussions. Choy and Dindyal (2021) demonstrated how this can be achieved when teachers perceive multiple affordances of a task and are therefore able to use seemingly typical tasks in rich and meaningful ways that extend beyond procedural solving. Furthermore, examples are an important resource for students' understanding of a concept, while non-examples serve a valuable role of demonstrating the boundaries of the concept (Mason & Watson, 2008), as "part of understanding a concept is knowing what it is *not* and when it does *not* apply", as stated by Lamon (2012, p. 5). Creating a so-called *cognitive conflict* can help students to identify and rectify their errors (de Bock et al., 2002). In particular, Warshauer (2014) suggested that providing opportunities for students to make mistakes and engage in struggle can be a productive way to strengthen their understanding of a concept. Thus, allowing students to see incorrect solutions and non-examples should also be a consideration for task selection.

It is evident that designing effective IMs can be complex, due to their central role in teaching and learning and the myriad of goals that are at times simultaneously desired, thus, teachers will necessarily employ several design principles (Kaur et al., 2022). By *design principles* (DP), I mean the guidelines for how teachers make decisions related to the selection, modification, creation, and sequencing of tasks that help to achieve their goals. Due to the recent emergence of research on teachers' design and scarcity of research on managing students' impulse thinking, there is little research that reports on teachers' DPs for managing impulse thinking and how they use these to support students' conceptual development. Hence, this study aimed to answer the following research question:

- What design principles does a secondary mathematics teacher use to design instructional materials to manage students' impulse thinking and raise their awareness of important features of concepts?

Methods

To answer the research question, the methodological approach chosen must be able to develop in-depth descriptions of the teachers' processes to explain how their design decisions help to raise students' awareness of critical features. This requires collecting data from multiple sources as the teacher designs, redesigns, and implements their IMs. Hence, a *qualitative case study approach* (Yin, 2014) was chosen to accomplish these research design requirements.

The data presented in this paper is a part of a larger study on secondary mathematics teachers' design of IMs. Four teachers engaged in professional learning discussions (PLD) on teaching the topic of ratio with an emphasis on proportionality. The teacher chosen for this case study, Tanya (pseudonym), is an *exemplifying case* (Bryman, 2016) of a teacher who designed IMs with the goal of slowing students down to raise their awareness of important features of ratio and proportionality. Throughout the 14 weeks of the study, Tanya repeatedly mentioned in design interviews and PLDs that her students often hurried through tasks in her lessons and consequently made mistakes without realising. Furthermore, as Tanya was also familiar with designing IMs over her 15 years of experience teaching mathematics across all secondary year levels in Singapore, her design would likely reveal a multitude of deliberate DPs that could explain how she intended to manage students' impulse thinking and reveal the features she deemed important for students to notice. The topic of her IMs was ratio and proportionality for 12- to 13-year-old students in their first year of secondary school.

The data collected in this study included recordings of Tanya's participation in four PLD sessions (40-90 min each), five versions of her IMs, four one-on-one semi-structured interviews (20-40 min) after each design draft, recordings of her implementation of the IMs across three lessons (60 min) and three post-lesson reflection interviews (15-20 min). Ongoing data analysis was conducted throughout the recording of her design process that traced her design decisions throughout the revisions (e.g., her modifications of a task) and sought increasingly detailed explanations for her decisions in interviews. This included questions about overarching goals and goals of specific tasks, expectations of how the tasks will unfold, and anticipation of students' responses. On the other hand, lesson observations and post-lesson reflection interviews focused on how the implementation aligned with her goals and expectations. The ongoing data analysis revealed her recurrent goal of managing students' impulse thinking.

To determine the DPs for achieving this goal, content analysis was initially conducted for each of the tasks in her IMs to determine the key mathematical ideas present in the task. The definition of ratio that Tanya presented in an annexe of her IMs to her students was modified from the textbook and stated: "Ratio is a way of *comparing* 2 or more quantities of the *same* kind that either have *no* units or are in the *same* unit. The ratio $a : b$, where a and b are positive numbers has *no units*. Note that a and b are *proportional*." After all recordings were manually transcribed, the transcripts were coded initially for when Tanya referred to her students' impulsive habits and her desires to slow them down, and the tasks within her IMs that would help her to address this. Subsequently, elaborations of how she wanted and anticipated her students would respond were coded for each task. This was used to identify the different features of ratio and proportionality that she deemed important and wanted students to notice, as well as features she expected they may not notice. These analysis steps combined were used to explain how her goals were manifested by her design decisions and the learning opportunities she wanted to afford her students through her design, hence, her design principles. The results of the analysis are presented in the following section.

Results

While there may be several DPs that collectively help to achieve Tanya's goal of raising students' awareness of important features in the topic of ratio and proportionality, two recurrent DPs emerged throughout the analysis of four sets of tasks in her IMs.

Design Principle 1 (DP1): Create Opportunities for Making Mistakes

In Example 1 (Figure 1), the aim was for students to determine equivalent ratios. As they had encountered the topic of ratios two years prior in primary school, the first two tasks acted as a review of how division and multiplication can be used to determine equivalence of ratios. However, in Question 3 the constant of proportionality to transform Ratio 1 into Ratio 2 using multiplication or division is not as immediately clear. Tanya had deliberately chosen the quantities of Ratio 2 to be +2 more than Ratio 1, to check if her students understood that transformations in ratio are based on *multiplicative reasoning*, *not additive reasoning* (Implicit Feature 1). As it turned out, during the lesson several students made the mistake of using additive reasoning to assert that the two ratios were equivalent. Although she was "stunned" (Reflection 1, #8) by her students' error, "I didn't expect that. I thought it was very clear cut that they would know that ratio is multiply or divide because it's not new to them" (Reflection 1, #13), it afforded her the chance to address this implicit feature during the lesson. She reminded students of the meaning of proportionality and asked them to write "Ratios cannot be added by the same number" in their IMs, an explicit statement of the implicit feature.

In a similar manner, Question 4 (Figure 1) was created by Tanya and introduced ratios with rational numbers, where the differing denominators would likely cause students to mistakenly multiply by different constants. Tanya's design allowed her to check if her students were aware of how to transform a ratio with differing denominators while maintaining the proportionality. While most of the students used the hint successfully, during the lesson there were still some students who made the vital mistake of assuming that multiplying each quantity in Ratio 1 by different constants was a valid way to transform the ratio. This consequently created the opportunity for Tanya to ask the class, "What's wrong here? … When you're simplifying ratio, can you multiply by different numbers? Cannot! Cannot by different numbers!" (Lesson 1, #91), drawing students' attention to the implicit rule that transformation of ratios requires multiplying or dividing by the *same constant, not different constants* (Implicit Feature 2).

In Example 3 (Figure 3), Questions 1 and 2 are typical problems that students would have encountered before in primary school and were not expected to pose any difficulty for students. Having led her students along for two tasks that affirm their use of multiplicative reasoning to determine an unknown value, the seemingly obvious solution to the subsequent question, Question 3, would be to assume the unknown value was the square root of the corresponding quantity, as squaring is also a multiplicative operation. Given that secondary students tend to have difficulty noticing the differences between linear and non-linear contexts (de Bock et al., 2002), Tanya's students were likely to make this mistake. Although most of them were quick to notice the critical difference between linear and quadratic operations, Question 3 would have caught students' attention and provided Tanya an opportunity to address that transformations in ratio are *linear, not quadratic* (Implicit Feature 3).

Likewise, Question 5 (Figure 3) was sequenced after a seemingly similar Question 4 and addressed another common mistake students make in ratio. In Question 4, the equivalent ratios were presented in fraction form and transforming the equation resulted in a solution that was finally presented as $y = 8$. However, in the subsequent Question 5, students' transformations of the ratios would lead them to the equation, $x = 6y$, which ultimately caused the class to stop and consider if the ratio $x: y$ should be $1: 6$ or $6: 1$. Almost the entire class (except two students) made the mistake of asserting the correct solution was $1: 6$ by taking the coefficients in the equation to be the corresponding quantities of the ratio. Following this lesson, Tanya reflected on this almost class-wide mistake, "I didn't expect that to happen … some of them catch it but there's still a group of them that still don't catch it" (Reflection 1, #39), meaning that some students eventually caught their own mistake by the end of the lesson. To explain their mistake, Tanya used the same feature students would have used in Question 4—a ratio $a: b$ in fraction form is represented as $\frac{a}{b}$, *not* $\frac{b}{a}$ (Implicit Feature 4). As they had confidently solved Question 4, if Tanya had not selected Question 5 from the textbook then it would have been likely that this significant misconception would not have otherwise been addressed.

Example 1: Without the use of calculator, determine if the following set of ratios are equivalent. Justify your conclusion using the table provided to help you.

#	Ratio 1	Ratio 2	Justification (Show your workings)
1	21 : 63	1:3	Yes / No 21 63
2	3 : 7	24 : 56	Yes / No 3 7
3	10 : 3	12 : 5	Yes / No 10 3
4	$2\frac{2}{5} : 1\frac{1}{4}$ Hint: Multiply by a common constant to convert both fractions into integers	9 : 5	Yes / No $2\frac{2}{5}$ $1\frac{1}{4}$

Figure 1. Example 1—determine equivalent ratios.

Example 2: Without the use of calculator, express each ratio in the simplest form.

#	Ratio	Justification (Show your workings)
1	144 : 132	144 132
2	$1\frac{1}{2} : 4\frac{1}{2}$	
3	$0.48 : 1\frac{1}{5}$ Hint: Convert $1\frac{1}{5}$ to a decimal then simplify from there	
4	$850g$ *is to* $3.4kg$	
5	1.4 : 7 : 6.3	

If two quantities that are **proportional,** they can be expressed in ratio from such as, **a : b = ka :** _____

Figure 2. Example 2—simplify ratios.

Across Examples 1 (Figure 1) and 3 (Figure 3), Tanya created several opportunities for students to make mistakes and to encounter cognitive conflicts, thereby raising their awareness of implicit features of ratio and proportionality that would not have otherwise been addressed. While it is more conventional to teach what a concept *is*, Tanya's design suggests that she believed "part of understanding a concept is knowing what it is *not* and when it does *not* apply" (Lamon, 2012, p. 5) and "[p]art of what it means to understand proportionality is to recognise valid and invalid transformations" (Lamon, 2012, p. 7). In these two sets of tasks, and in subsequent sets, being aware of the implicit features was critical for answering the questions correctly. If, and when, students fell into the trap of making a mistake, Tanya's design ensured that these mistakes could be used productively to bring implicit features to students' attention.

Design Principle 2 (DP2): Sequence Tasks to Lead To and Catch Mistakes

Beyond creating opportunities for mistakes to happen, analysis of Tanya's IMs also revealed how she strategically sequenced her tasks to *lead* and *catch* students in making mistakes. In Example 2 (Figure 2), Tanya included a variety of contexts for simplifying ratios, as well as the opportunity to make different types of mistakes. While the first three tasks were likely to be solvable by students, Questions 4 and 5 were intended to surprise students, causing them to slow down. Tanya wanted to remind her students that problems in ratio are not limited to numbers only, they can also contain units that need to be dealt with: "I want them to 'Eh?' Suddenly there's a need?' I wanted them to have a change of momentum" (Design 1, #31). Tanya later explained that her students "are interested in the answer … They'll probably [rush] with getting the right answer" (PLD 1, #191), hence the differing units would likely be overlooked, leading to students making the mistake of simplifying the ratio 850: 3.4 without recognising the need to convert the units. This was the only task in the worksheet with differing units, which suggests that it would be the sole opportunity in the lesson for students to make this mistake and for Tanya to remind students that ratio involves comparing two or more quantities of the same kind that have the *same units, not different units* (Implicit Feature 5).

In the final question in Example 2 (Figure 2), Tanya brought in a ratio with three quantities with decimals. This would indeed prompt students to reconsider how their existing procedures could be applied as they now had to maintain the proportionality between three quantities, two of which are decimals that required conversion. She noted that she wanted to use this task to "break the momentum" (Design 1, #34). If students were to be negligent of the decimals or unconscious of the need to maintain the proportionality between all quantities, Tanya's design of this task would remind students that transformation of ratios require multiplying or dividing by the same constant, *not different constants, even when there are more than two quantities* (Implicit Feature 6).

Example 3: Without the use of a calculator, answer the following questions

1. $3a : 7 = 18 : 21$, find a
2. $5 : 2b = 25 : 40$, find b
3. $3 : b = 3^2 : 5^2$, find b
4. $\frac{3y}{14} = \frac{12}{7}$, find y
5. $\frac{3x}{8} = \frac{9y}{4}$, find x : y

Figure 3. Example 3—find unknown value using ratio.

Example 4: Answer the following questions.

1. Find the simplest ratio of $76 to $84 to $20
2. If $x : y = 5 : 6$ and $y : z = 4 : 9$, find $x : y : z$
3. If $x : y = 3 : 4$ and $y : z = 5 : 8$, find $x : y : z$
4. If $p : q = \frac{2}{3} : 2$ and $p : r = \frac{1}{3} : \frac{1}{2}$, find $p : q : r$.

(*Convert the fractions to integer by multiply by a same constant*)

Figure 4. Example 4—ratios with three quantities.

In the last set of tasks (Figure 4), Tanya's way of leading students to make mistakes for the purpose of catching them is evident when examining the structure of the last three questions. In general, two ratios are presented as $x\colon y$ and $y\colon z$ to be used to determine $x\colon y\colon z$, with the ratio quantity y being common in both ratios, as can be seen in Questions 2 and 3. However, in the subsequent Question 4, Tanya deliberately made the common constant different, "I wanted to caught [sic] their moment, whether are they careful enough. Because when students do-do-do then they'll like 'ah!'. Because they'll just assume it's the same as the previous one, the sequence of the question. But I want to *catch* them and whether [they notice] this is another way of presenting the question" (Design 1, #77). Notably, because the sequence in Questions 2 and 3 are conventional, her design and sequencing of this question would likely lead and catch students making the mistake of focusing only on the numbers and not being conscious of the actual problem.

Design Principle 2 in Examples 2 and 4 can also be observed in Examples 1 and 3 and is necessarily two-fold, based on a deliberate sequencing of tasks. Firstly, Tanya created the opportunity for students to apply their existing procedures by beginning with tasks that could affirm their understanding of critical, yet explicit, features and solving procedures (e.g., using multiplication to determine unknown values), essentially allowing them to *think fast*. Once this procedure was anchored, subsequently within the middle or end of each set Tanya introduced tasks where these procedures may not be immediately applicable, where students may suddenly realise there was a need to *think slow*. Being aware of a specific critical, yet *implicit*, feature was necessary for solving these problems correctly. At these junctures, Tanya's sequencing was intended to cause students to either apply their previous understanding incorrectly, thereby making the mistake (e.g., squaring is multiplicative, so squaring and square-rooting are valid transformations) or pause to reflect on their understanding when confronted with cognitive conflicts ("Is there a difference between linear and quadratic?"). In both instances, DP2 will cause students to slow down, to "break the momentum" of their automatic and impulse thinking, and to increase her students' sensitivity to common mistakes and misconceptions they may have, while also demonstrating to them why they need to be more conscious of their thinking instead of fixating on solving the problem quickly.

Discussion

In this paper, I presented a case of a teacher who designed IMs with the goal of raising her students' awareness of important implicit features. As they were often impulsive in their solving and prone to neglecting certain implicit features of concepts that Tanya deemed critical, she employed two DPs that collectively help to slow students down to achieve her goal. The two design principles she employed were: (1) *to create opportunities for students to make mistakes*, and (2) *to sequence tasks to lead to and catch mistakes*. To elaborate on Tanya's two principles and to discuss their implications, I introduce the notion of *catch tasks*.

From the analysis of Tanya's design of IMs, it is evident that certain tasks have the power to cause students to make mistakes. I term these *catch tasks*, which describe tasks that deliberately aim to engender incorrect solutions with the purpose of *catching* students in making mistakes, thereby raising their awareness to implicit features of concepts that may otherwise not be addressed. While this may appear to be a counter-intuitive goal, creating a form of cognitive conflict by embedding implicit features into tasks so that students would answer questions incorrectly has its advantages. Like non-examples (Goldenberg & Mason, 2008), catch tasks emphasise the boundaries of a concept and highlight common errors. However, instead of demonstrating the incorrect reasoning, catch tasks *lead* students to making those errors in their *own working*. They embrace mistake-making and acknowledge it as a natural element of learning—a belief that may be understood by teachers but is seldom enacted and prioritised (Warshauer, 2014), at least not as prominently as Tanya has made it in her IMs.

As for selecting, modifying, or creating a catch task, as demonstrated by Tanya a catch task is most useful when it is grounded in an implicit feature of a concept, often a complement of a feature that is hidden and tacit, and sequenced strategically within a set. They need not be particularly complex or rich collaborative tasks like those suggested by Swan and Burkhardt (2014); indeed, Tanya's catch tasks are seemingly typical tasks, like those reported in Choy's and Dindyal's (2021) study. Tanya's use of these tasks demonstrate that typical tasks can certainly go beyond being used for procedural practice, they can also be used to raise students' awareness of implicit features and cause students to *think slow* to reflect on their understanding of a concept. The key to achieving this was Tanya's knowledge of the content and students (Ball et al., 2008) that allowed her to notice the affordances of these tasks as individual items within a set. Her sequencing of tasks allowed students to *think fast* (Kahneman, 2012), such that students would likely have an anchoring and familiar procedure, then the catch task would likely engender an impulsive (and likely incorrect) response. From the students' mistakes in the lesson, this study demonstrated the potential of catch tasks for raising students' awareness of mathematical structure (Watson & Ohtani, 2015), which was achieved by creating variation and using sequencing to cause mistakes, rather than demonstrating correct solutions.

Conclusion

Raising students' awareness of potentially implicit features of concepts is a valuable goal for teachers to have when designing lessons and IMs. In this paper, I presented a case of a teacher who achieved this by deliberately creating opportunities for students to make mistakes (DP1) and sequencing tasks to increase the likelihood for mistakes to happen (DP2). These two design principles productively engage students in a more conscious state of thinking but is contingent on the teacher's knowledge of the content and their students (KCS), and their ability to notice and harness the affordances of a task to be a catch task. While the notion of creating cognitive conflict has been historically recognised as a powerful way to help students to reflect on their understanding, the DPs underlying the notion of catch tasks afford opportunities for cognitive conflict to occur so that students who have the tendency to think fast may be forced to think slow to confront their errors. The deliberate and widespread use of catch tasks is not common amongst mathematics teachers in

Singapore, nor for teachers internationally. Despite appearing to be simply typical problems that may be commonly found in textbooks, catch tasks can be designed to develop conceptual ideas and to address students' errors. This presents a new possibility for teachers and educators when designing IMs. Hence, further research is needed to explore how catch tasks can be designed and implemented.

Acknowledgements

The data presented in this paper was collected as part of larger project funded by the Office of Educational Research, National Institute of Education, Nanyang Technological University, Singapore (Grant number: OER 31/19 BK). This work was also supported by the Nanyang Technological University Research Scholarship.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching. *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). Routledge.

Bryman, A. (2016). *Social research methods* (5th ed.). Oxford University Press.

Choy, B. H., & Dindyal, J. (2021). Productive teacher noticing and affordances of typical problems. *ZDM—Mathematics Education*, *53*, 195–213. https://doi.org/10.1007/s11858-020-01203-4

de Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics*, *50*, 311–334.

Goldenberg, P., & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics*, *69*(2), 183–194. https://doi.org/10.1007/s10649-008-9 143-3

Kahneman, D. (2012). *Thinking, fast and slow*. Farrar, Straus and Giroux.

Kaur, B., Leong, Y. H., & Attard, C. (2022). Teachers as designers of instructional tasks. *Mathematics Education Research Journal*, *34*(3), 483–489. https://doi.org/10.1007/s13394-022-00437-7

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (3rd ed.). Routledge Taylor & Francis Group.

Lim, K. H., & Wagler, A. (2012). Impulsive-analytic disposition in mathematical problem solving: A survey and a mathematics test. *Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia*, Singapore: MERGA.

Mason, J., & Watson, A. (2008). Mathematics as a constructive activity: Exploiting dimensions of possible variation. In M. P. Carlson, & C. Rasmussen (Eds.), *Making the connection: Research to practice in undergraduate mathematics education*. Mathematical Association of America.

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM—Mathematics Education*, *46*(5), 705–718. https://doi.org/10.1007/s11858-014-0600-4

Schoenfeld, A. H. (2010). *How we think: A theory of goal-orientated decision making and its educational applications*. Routledge.

Swan, M., & Burkhardt, H. (2014). Lesson design for formative assessment. *Educational Designer*, *2*(7), 1–41.

Warshauer, H. K. (2014). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education*, *18*(4), 375–400. https://doi.org/10.1007/s10857-014-9286-3

Watson, A., & Ohtani, M. (Eds.) (2015). *Task design in mathematics education: An ICMI study 22*. Springer. https://doi.org/10.1007/978-3-319-09629-2.

Yin, R. K. (2014). *Case study research: Design and methods* (5th ed.). SAGE Publications Inc.