DOING AND CONSTRUING MATHEMATICS IN SCREEN-SPACE

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A preliminary analysis is offered of problems connected with the construal of images from and on screens, whether physical and dumb, like OHP screens, electronic and dumb like TV screens, physical and intelligent like software on computer screens, or intelligent like mental "screens". My interest is in relationships between doing and construing in the space accessed through mental and physical screens. Throwing images onto screens is a popular pastime, but there is a difference between "looking at" a screen and "looking through" a screen, with concomitant educational value.

INTRODUCTION

These notes are complementary to my presentation, which will take advantage of the opportunity to work on static and dynamic images, and to experience directly some of the aspects of images which I find vexing.

As we move from a world dominated by passively "dumb" screens (text on paper, posters on walls, images projected on screens, television), to a world dominated by actively "intelligent" screens (manipulable text, images, and animations, with the power to process these and to express one's thinking in mixed and multi-media), it might be worthwhile to try to learn from experience with presenting mathematics on mental and currently available physical screens, so that we can take maximal advantage of what mixed-media and computational tools for analysis can offer. In this paper I want to look at relationships between doing and construing in the space accessed through mental and physical screens.

A distinction of interest to me is that between

> students *engaging in* activity, summarised as *doing*

and

> student *learning from* activity, summarised as *construing*

Getting students active is relatively easy. Arranging that students learn from activity is quite another matter. They usually learn something, but what they learn is liable to be quite different from what the teacher intends. Distinctions between task-as-envisaged and task-as-set by the teacher, and between the task-as-perceived and the activity in which students actually engage, have been made by many authors, and I do not wish to rehearse them here. For me it all comes down to the fundamental questions:

> What are the students attending to? Where is student attention in the presence of "screened mathematics"? What is the role of their mental screen?

I am particularly interested in the role of awareness, that is, the extent to which students learn effectively both with, and without, having their attention drawn explicitly to what is "being learned", because I find that the distinctions between *outer* and *inner* activity, *explicit* and *implicit* learning, *overt* and *covert* curriculum, particularly vexing when it comes to still and animated screen images. Text is bad enough for permitting students to construe what they see in a variety of ways; images are even more open to ambiguity, to alternative stressing and ignoring (Gattegno, 1990). Yet at the same time, images can be remarkably effective.

Computers have been described as *conception* stretchers (Dennett, 1990), and one of the most significant ways in which conception stretching takes place is in the access to images and to image processing provided by computational devices. Dennett also suggested that:

> conceiving of something complex ... is a matter of learning your way around in a "space" you must construct in your mind.

How that mental space is related to the screen is of abiding interest to me, and of increasing interest in the educational community generally, as evidenced by the rapid increase in imagery-related papers at PME recently, and a sudden increase in the number of books about imagery in a variety of disciplines.

In attending to mental imagery, I find that I wish to depart from the semiotic orthodoxy which treats signifiers and signified as a dichotomous pair with an implied background referent. For me, symbols are not the mathematics, nor do they signify some Platonic or other mathematical referent; rather, they are an expression of seeing, of awareness, an integral part of thought experience. Diagrams are not the mathematics, nor do they signify some Platonic form; rather, they are an integral part of thought experience, an artificial snapshot of time-evolving sense-making. Words are not the mathematics, nor do they signify some "think" which is the mathematics; rather, they are an expression of a fleetingly one-dimensional stressing of certain features of thinking in process. Words and images which purport to signify, need not necessarily signify a signified, because words and images are at best fuzzy projections from a complexly experienced high-dimensional process space into at most one, sometimes two, and occasionally three physical dimensions (Mason, 1992). Even directing attention to signification may be a disservice, diverting attention away from possibly inexpressible but never the less crucial action which is part of thinking.

IMAGES ON SCREENS

I use the term "mental screen" to refer to internal experience of whatever form. The use of the term "screen" in the context of mental experience is *not* intended to summon an image of a homunculus watching an inner projection screen; indeed one of the features of the mental "screen" lies precisely in the all-encompassing, fully participatory experience that is so hard to describe metaphorically. Goodman (1990) distinguished "images of memory and imagination" from "optical and sensory images" though he emphasises that neither need be pictorial. But such a distinction seems to blur when it comes to mathematical images. Imagined and constructed generalised images sit side by side sensory images of mathematical apparatus and diagrams. Lacan (1985) links imagination not based on sensation with the symbolic:

> ... there is nothing in the nature of the wheel that will describe the pattern of marks that any one of its points makes on each turn. There is no cycloid in the imaginary. The cycloid is a discovery of the symbolic. (p. 208)

Goodman (1990) joins the ranks of the imagery-dubious when he observes that:

> Just as responsibilities are not things, but arise grammatically from speech and are more correctly given as having responsibility for such and such, so mental images arise grammatically from speech when what is intended is facility with describing, reasoning, identifying etc. (p. 363)

Although I am in general agreement with Goodman's overall perspective, my own experience of describing, reasoning, recognising and identifying is so strongly associated with rich inner experiences that I find the noun *imagery* sits easily and unproblematically as a summarising label for the whole.

Sometimes it is hard to distinguish between the mental experience and the physical screen; the latter acts as a window, and the mental screen becomes the world seen through that window. Animation is particularly engrossing: what is seen is not simply the screen image, but something more substantial, more general, more personal; a world beyond the window is contacted. This is what I mean by *looking through* as opposed to merely *looking at*. Just as a picture-frame is essential for focussing attention but often goes unnoticed, so in the merging of the physical and mental screens, the consequent framing focuses attention but often goes undetected.

The very pre-articulate, all-pervasive nature of mental imagery makes it critically important in human life generally and in education particularly. Arieti even coined the term *endocept* to refer to "that vague sense, that dim and faint form of knowing" (Neville, 1989) often associated with imagery. I am my self fully convinced of the importance of imagery in teaching and learning mathematics, and do not propose to put the case here. Rather, I want to look specifically at ways in which mental imagery is or can be invoked. Electronic screens permit us to throw a variety of images in front of students; students can become very involved in consequent *doing*, that is, in watching. They can also engage in discussing what was seen, and this may contribute to further construal. Students can also become entranced by video images, and even refer to them directly and indirectly in peer discussions. Television images, like pop music, form a background tapestry to their lives.

From a constructivist perspective, what are we to make of children assimilating words to pop songs and television advertisements without any apparent effort, and yet struggling to remember a few words in some foreign language, a few trigonometric relationships, or a mathematical diagram? Can we really get away with saying that some images are necessarily easily construed and recalled?

EVOKING MENTAL IMAGERY

Imagine a lemon, hanging on a tree. Watch it go from unripe green to ripe yellow, then imagine picking it.

What intentional acts were necessary, and what flowed without effort in that exercise? It is a common experience that a snatch of a tune can evoke much or all of the rest of the song, together with a variety of associations; we can hear in our heads but be unable to sing the tune out loud. So how *do* you go about accessing an image?

Describe the wall behind your bed at home.

What do you have to do to be able to carry out this exercise?

The point of these exercises is to remind you that you have frequent recourse to your imagistic powers, but despite that, you may have little direct awareness of how you invoke those powers. How then can you hope to evoke similar powers in students?

FROM STATIC TO DYNAMIC IMAGES

The encyclopaedists of the 18th century were taken with the newly released powers of printing combined with those of etching. The result of Diderot's compendium, and many others as well, was an exploration of the use of static diagrams referenced by the text and intended to augment rather than merely illustrate the text. At first all the diagrams had to be together in appendices. With technological improvements they could be included where desired in the text. Now we can initiate dynamic images from electronic text. But have we learned a great deal? Do we really appreciate the embedded bias of any collection of images, no matter how "accessible"?

Static Images

The power of effective imagery is unquestionable: the once common cliche

> a picture is worth a thousand words

is now fully integrated into our daily experience. We are close to being swamped by images in all media. We say:

> "If I just make this into a picture, a chart, a diagram, then they will see at once what I am trying to say."

I have a sense of this sentiment operating in almost every context: civil servants preparing reports for ministers; managers preparing business reports for senior managers, educational publishers preparing texts for schools, programmers developing educational software, and a host of others. But what is it that makes some images enduring and others transitory? There is a ubiquitous assumption that

> the right image thrown up on a physical screen will surely provide the students with an image that will inform their understanding.

This sentiment is no doubt derived from our assumptions about advertisements:

> the right image thrown up on a physical screen will surely provide the potential customer with an image that will inform their purchasing.

To this end, there has already been a significant increase in the use of cartoons for teacher mathematics and other subjects, and it has even reached the point that a presentation without a cartoon is considered unprofessional. We are also witnessing the reduction of complex ideas to a simple two-minute slot, a two-page spread in a text-book, or a series of brief work cards. These are products of the influence of television on society.

But there is more to pictures than casting them on a screen in front of people, and it certainly is not a matter of more is better. As one advertising executive is quoted (in Blonsky, 1985) as saying:

> Too often we hear - and ask - 'are we talking about the product, or are we doing imagery?' It's the wrong question. Everything is imagery ... We don't really remember facts, the figures, the classic copy-points of the strategy. What we do is get a net impression, a sense of what the brand is about. This is what we call imagery. (p. 507)

Despite the success of (some) advertisers, I remain convinced that to get much educational benefit, students need to be active in processing images; they need to *work on* the images, not just *look at* them. For example, they can be encouraged to reconstruct an *account of* what they have seen, and to construct coherence for themselves by *accounting for* it, in order to be able to reconstruct it again another time. The distinction is important, because accounting for something is so attractive that it gets mixed up with, and even displaces, what it is that is being accounted for. And if there is no agreement on the phenomenon, there is little sense in explanation. Advertisers who set their wares into a mini-plot seem to confirm this view.

The point about processing diagrams is that it is not just a matter of *looking at* them, but rather *looking through* them.

Imagine a general triangle. Now draw your image.

If you conclude that angles which differ by less than 10 degrees look much the same, there are in fact very few "general" triangles! It is not a matter of locating a single general triangle, but rather what you stress when you look at or imagine it. Drawing the image actually brings about different stress from just imagining. Bishop Berkeley is said by Dennett (1990) to have described conceiving of a general triangle as a matter of imagining some particular triangle and then noting parenthetically that the image had unintended features, and reminding oneself not to use the image in certain ways when employing it in the course of thinking. This is a crude form of looking through rather than at. Hilbert (Courant, 1981) referred to the use of the *generic example* in much the same way, and Mason and Pimm (1984) developed the notions of *seeing the general in the particular*, and *seeing the particular in the general* in the same vein. But as we know, getting students to see the general in the particular is not an easy matter.

Dynamic Images

Film in classrooms has never really taken off, largely because of the awkwardness of the medium. Video has helped somewhat, but is still used sparsely. Static images have developed significantly with a surfeit of illustrations, diagrams and colour trying to reach the attraction levels of magazines. Now commercial interests are strongly promoting multi-media hardware, with the implicit slogan:

> If it doesn't move it isn't as good as it could be.

or, in a modern version of the picture cliche,

a little animation is worth a thousand pictures.

We might well find ourselves assimilating this "value" without realising it, and thereby lose touch with ways of making significant use of both the static and the dynamic. Technological advances enculturate us to assimilating values with astonishing rapidity. For example, black and white film processing is now more expensive than colour (which has forced the former into the realm of do-it-yourself). Soon it may be more expensive to take stills than movies, and more expensive to send physical mail than electronic mail to anyone. Because it is possible, there is now a major industry in producing diagrams, animations, and simulations on video-disk, which allow viewers to "see" how a variety of physical and abstract things "work".

Is "to see" the same as "to understand"? The promoters appear to believe so. Is a well-formed animation sufficient to generate meaning and understanding? I take *meaning* to refer to a rich network of triggered connections with past experience, so that something seems *meaningful* when such a multiple triggering takes place. The connections are largely *intra*, that is, within the domain. Understanding refers to a rich similar network of accessible *inter* and *intra* connections, with emphasis on accessibility by the individual, and on both the *inter* and *intra* connections.

From a constructivist perspective, a text is certainly not sufficient in itself to generate meaning and understanding. But the success of advertising in particular, and television in general, makes this answer not quite so clear in regard to images.

I am sure that there is much more to be learned about using static images in teaching. It may even be that access to animations will teach us about the use of static pictures. But I want to caution against a wholesale movement into dynamic images without regard to lessons learned from the static.

Movement attracts attention by virtue of the genetic, adrenalin releasing, fight-flight response mechanisms which are triggered by our neural network. But movement can also be entrancing, attracting attention so completely that we are lost in the action. Children's response to television, and their concomitant demands for stimulation at school, provide but one manifestation. Miller (1990) notices that when a drive-in movie is viewed from long distance without access to the sound, attention is drawn to the rapidity of cuts and jumps from shot to shot, but these are not generally noticed when the sound is also available.

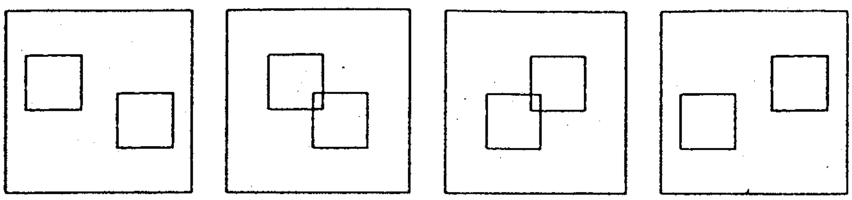
> When viewing a film from the correct distance, one becomes engrossed in the story and the continuity of meaning overwhelms the discontinuity of presentation ... what we lose when we view a film at a distance is the *meaning* of what is going on. (p. 190)

I have learned a great deal from working with static images such as posters, and also from working with animated film, with computer mice and tracker balls, and with dynamic mental imagery. I have worked at locating techniques to assist students to *work on* such visual images. I considered myself to be pro animation. Now I find myself questioning the push to a multitude of clever "visualisations", as computer scientists describe screen images, and the reason for my concern is that the engrossing feature of movement may act against the educational benefit.

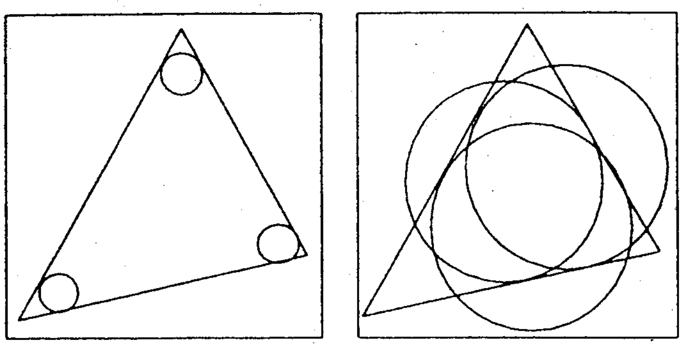
Static diagrams are notoriously unhelpful. Consider for example the difference between a picture, a diagram, and a sketch of something like of an engine. The picture is complete and total. All features receive the same stressing, so that the picture serves as a rest for the eyes as much as anything else. And pictures age so quickly. Diagrams also have a completeness which may render them inaccessible without considerable effort of reconstruction, whereas seeing a diagram built up spontaneously, or better, building it yourself, seems to be much more successful. Although an animation can provide the build up of a complex diagram, animation has its own entrancing features which contrast with participating in the building up of a diagram of one's own. It takes even more personal discipline to work at an animation than a static diagram, precisely because of the tendency to be swept up in the motion. It follows therefore that it is worthwhile developing working practices which focus attention on construal and reconstruction.

Fish and Scrivener (1990) draw attention to the importance of sketches, which are notable for their incompleteness. The evocative, incomplete sketch invites stressing and ignoring, active interpretation using both gestalt-evoked powers and conscious powers, in a way that complete diagrams and figures do not. Sketches employ the automatic gestalt-closure of perceptual completion. And there is a parallel with young children's preference for stories that are incomplete rather than complete.

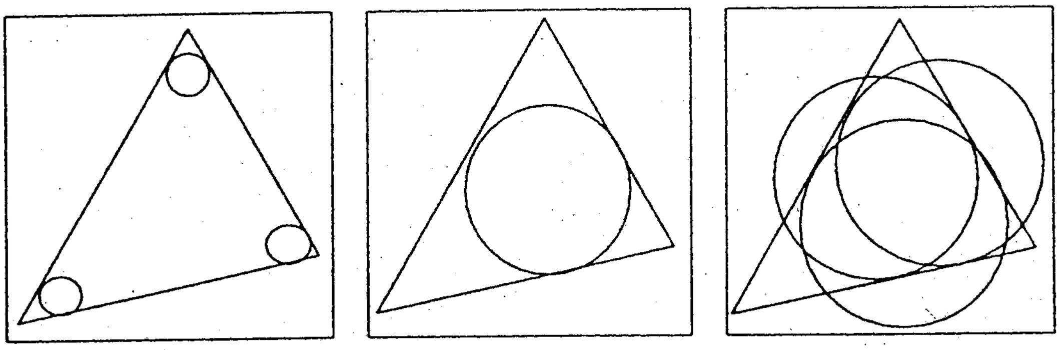
Part of the atomic quality of static diagrams is due to their being static, which signals something unchanging, undeveloping. Thinking of diagrams as snapshots of motion, of frames from a film sequence, calls upon inner powers to supply intermediate frames, and contributes to seeing through the diagram to a more fluid, general, abstract setting beyond. For example, the first sequence suggests movement in a variety of relativistic forms (does one square stay fixed, or does the frame?).



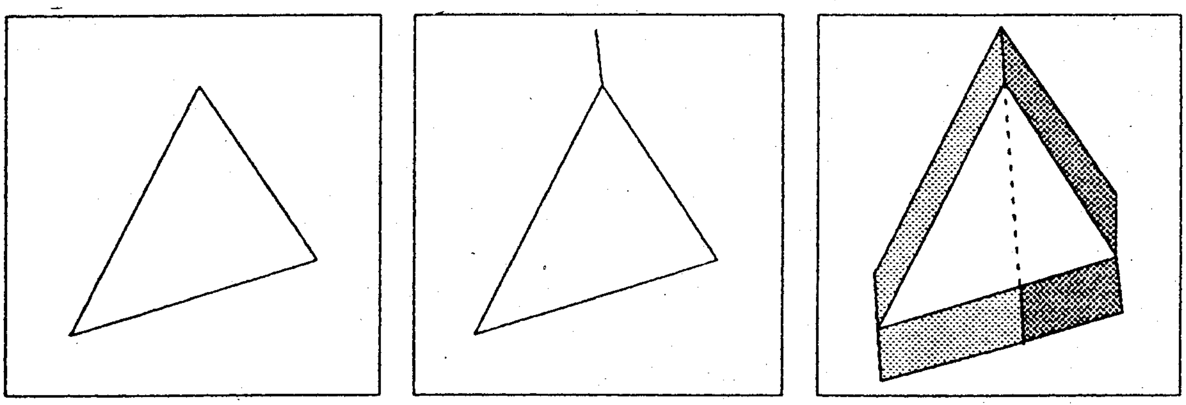
As a result of verbal descriptions by pupils of what they are seeing during their constructed movement, the next sequence of frames might lead to awareness of a particular intermediary position and thus to conjectures and explorations which would support meaning and understanding of a classical geometric result.



Interpolating an intermediate position focuses attention more directly on a known result, but depends for its support of understanding on trying to account for that frame rather than on "having constructed it"..



Choosing how much to depict, how much to describe, and how much to leave to the viewer is very much a personal preference, explored in some detail in Beeney et al (1982) and is the subject of a thesis in preparation by Hewitt (see for example Hewitt, 1991). The following sequence is based on a poster designed by Zavadowski (1991).



The underlying theorem is attributed to Pappus. Zavadowski uses the metonymic title Pappus' House or Pappus' Tent to provide pupils re-access to the theorem through the label. Quite different experience of the theorem (the seeing) arises from use of a poster, a diagram sequence, a teacher's verbal instructions to imagine, and use of a CABRI Geometre configuration to move around with a mouse.

The notion that animations lie behind static diagrams is actually quite an old idea. The following quotations are taken from Olive Whicher (1971).

> When movement arises between viewer and the object, there ensures a manifold possibility of alteration in the form seen. (p. 52)
>
> We must practise seeing the construction in *movement,* realising that the situation we happen to have is only one among infinitely many. (p. 52)
>
> ... The time is at hand when through active scientific thought mankind must find the way in clear consciousness to the worlds which lie beyond the sense world, from out of which the sense world is born. (p. 71)

One of the features of physical screens is that it is very hard to depict the abstract and the metaphorical. Television is first and foremost a particularising medium, focusing attention on detail because of the coarse grain resolution. As such, it acts against access to Whicher's valued "other worlds", confining attention to the sense world. It is easier to look at a TV screen than to look through it. High density television and computer screens may overcome this propensity, but it may also be the case that it will be some time before people treat electronic screens the way they treat photographs and paintings.

In my experience, the most important part of using an animation or poster for teaching is the processing of those images, the internalising and construing. It is a matter of learning to be *looking through* screens rather than *looking at* them which I think is crucial, not just to mathematics, but perhaps to education generally.

TYPES OF IMAGES

An image can be:

> as clear as your awareness of your front door viewed from outside as you approach it with bags of groceries in each hand: you know whether the handle is on the left or the right;
>
> as transient as a fleeting sense of recognition as someone walks past you;
>
> as vivid as a scene which often comes to mind, such as a particular beach, or a particular kitchen;
>
> as indefinite as your sense of yourself and where you are going in life: like a beacon on a hill that is often obscured by trees, providing direction for actions but remaining illusive and out of reach; to speak about it is to tumble it into substantiality like images of chairs and lemons and triangles and thereby turn it into something else;
>
> as detailed as a re constructible generic picture "of" an instance of a generality, from or through which the general theorem can be read;
>
> as multiple as a song which won't go away but which comes with an image of a favourite place, while you simultaneously consider what you will have to eat from a menu.

How we construct or obtain images is an important question. For example, in trying to invoke students' mental powers, it helps to have some awareness of how images might be structured. Images can be viewed as *eidetic* or *constructed*. An eidetic image comes fully fashioned from something sensed in the material world: the taste of a particular lemon, the smell of a particular rose, the colours of a particular sunset. A constructed image is built up from other images, which themselves may have been constructed or eidetic. Are all images constructed from "atomic" eidetic images? or can images be formed in some other way?

Images rapidly become abstracted. The word sunset may not trigger any particular sunset, but more a sense of sunset-ness. If you are told that you are going to the cinema, you have a sense of cinema-ness. The language of frames (Minsky, 1975) and scripts (Schank and Abelson, 1977) provide two ways of speaking about the image-based expectations which are triggered. If asked to describe "a cinema", you might work from a particularly vivid image of a specific cinema, or you might find yourself working from a generalised or abstracted image, perhaps composed of fragments of images of actual cinemas. As mathematics teachers, we would presumably like students to have access to similarly abstracted or generalised images rather than being confined to images of particular triangles or particular functions. In other words, we want our students to look through the particular to the general.

Are eidetic images at the heart of all imagery? Neonates sometimes display the behaviour of dreaming, yet it is hard to believe that all their dream-imagery is eidetic, based entirely on prior sense impressions; some artists display images which are hard to relate to prior eidetic images, even through long chains of constructions; mathematicians seem to make progress when they break out of established patterns and fashion new images, new ways of thinking, despite what Newton said about standing on the shoulders of giants.

Gavriel Salomon (1979) used the term *supplanted images* to describe images taken from pictures seen on screens. These are eidetic images from some visual medium, and distinct from those constructed by the person themself. For example, he praised a number of the computer animations in the Open University mathematics television programs because they seemed to provide students with dynamic images related to mathematical ideas, images which are very hard to construct for yourself from a static diagram.

But to construe computer animations it is necessary to recognise and be familiar with the elements of that animation; in other words, to have reconstructed them for oneself. Even if supplantation is possible there are still questions of how to assist students in gaining access to them when appropriate, and being able to use them, to "read" and develop them; to use them as access to the general or abstract ideas "beyond".

Perhaps all images are memory, perhaps even *just* memory. Even if images were *only* a form of memory, imagery would be of great importance in education, for it indicates a power of the mind which is so great that it cannot be modelled adequately by current one-dimensionally time-dependent machines. Using peripheral vision, you can be aware of the whole of a picture while attending to particular detail simultaneously. But you can also pick out features without consciously scanning. You can pick out someone you know in a crowd, and shown a pair of nearly identical pictures, it is not hard to detect differences, without doing a pixel-by-pixel scan of each one. To model this sort of data-access requires massively-parallel computing.

But are images just memory? Eric Love reports that:

> I found it hard to make some mathematical diagrams move in my mind. Then having constructed them in CABRI Geometre, I find I can now do much more with the diagram. And yet I am not simply remembering pictures from the screen, because the actual images are different. (Said when referring to a two-point perspective configuration).

Eric's account suggests to me that experience of dynamic images, particularly when linked to the musculature through your own movement of tracker-ball or mouse, or imaged, can strengthen what is available for processing on the mental screen. As important in education as images themselves, is the employment of those images, for thinking, and for expressing thinking to others. It may be that appropriate work with computer programs can evoke and strengthen processing powers in and on our mental screens.

MATHEMATICAL IMAGES

> Imagine a chiliagon (a 1000-sided regular polygon). Now imagine a 999-gon. What is the difference?

Dennett (1990, p 297) observes that for Descartes it was possible to *conceive* of a chiliagon and of a 999-gon, but that it was not possible to *imagine* them differently, and in this way he distinguished between *conception* and *imagination*. Dennett goes on to suggest that "to imagine" is an act that can be accomplished, can be embarked upon deliberately, whereas "to conceive of" is not. I find it rather difficult to sustain this distinction, for my imaginations of the chiliagon and the 999-gon have much more substance than just mental postcards. I have access to the number of sides as well, and hence to a whole range of calculations, even though I could not identify the difference between pictures of the two. Furthermore, I can work deliberately at juxtaposing the visual, aural, and other sense-based attributes which with my "knowledge of" and "knowledge about", make up the totality of my image. Tall and Vinner (1981) introduced the term *concept image* to refer to this totality of connections and associations, and the idea is developed in Tall (1991).

If a technical mathematical term, like *function*, or *circle*, or *fraction* is employed, then the very hearing of the term is likely to trigger metonymically a range of potential responses, including visual images, incantations that accompany techniques, forms of questions and approaches, awareness of connections and related ideas, associated feelings, access to past experience particularly of success or failure, etc. Many of these remain potential rather than activated, nascent rather than manifested. They will come to life, that is be fired (in the sense of a frame) or activated (in the sense of a script) only when further triggering takes place.

It follows that as teachers it is important that we gain access to our own imagery, our awarenesses (Gattegno, 1990), and then to locate ways of accessing those. One major access route is through images which are constructed or displayed. For example:

> Imagine a pair of circles. Introduce the common chord of their intersection. Now introduce a third circle intersecting the other two. Introduce the two further common chords. The three common chords intersect in a common point.

The final assertion signals a *seeing*, (the root meaning of theorem). It could have been left out, and participants invited to "say what they see", it could have been offered in other ways. The entire task could be presented in a variety of ways:

verbally, with or without attempts to evoke images;

through a static diagram;

through a few static diagrams with the invitation to see them as frames from a film;

through an animation;

through software such as CABRI geometre that enables the user to make changes, to explore the scope of generality by seeking what changes and what stays the same as some other things are changed.

Each of these media, these teacher-provided intermediaries between content and student, require experience and discipline if they are to be exploited successfully. Young children have to learn to read text, and to relate pictures on pages to things from the material world, for example, learning that the back of a depicted object is not depicted on the back of the page. People need to learn to read paintings in different styles in order to interpret them fully. In the same way, it may be necessary to learn how to read diagrams, frame sequences, and animations. Although children eventually work out that there are no little people inside a television set, do they work out how to make the most of the programs and the pictures that they see?

INNER AND OUTER MEANING

The power of imagery has been recognised for thousands of years. One source of its power lies in its often pre-articulate nature, directing and formatting attention without conscious awareness (*viz* Freud and Jung). Another is in the way it speaks to us metaphorically, and triggers new thoughts metonymically.

Tahta (1980, 1981) distinguished *inner* and *outer* meaning of tasks, and to these I add *meta* meaning. *Outer* meanings have to do with explicit content such as known mathematical results described in terms of a mathematical label which purports to summarise a mathematical story. To exploit the outer meaning of a task, it makes sense to engage students in story telling by reconstructing what they have seen by giving brief-but-vivid accounts of fragments from the task activity, and weaving that into a story which accounts for those fragments. Throughout this story weaving, negotiation with others plays a crucial role.

Inner meaning refers to global awareness such as multiplicity of definitions or of perspective, the linking together of previously disparate elements into one continuous family, the perception of an infinite class of elements as a single entity, the choosing of constraints and the effect of those constraints, the stressing of different points of view that yield an invariant result, the simultaneous holding of several points of view by means of one invariant. There are also personal meanings which are triggered metonymically, such as Kieren's encounter with a girl for whom 5/6 was a very important fraction because it reminded her of her family, her mother being absent. Many students find metonymic resonance with apparently entirely abstract geometry, such as being disturbed by a line moving about a plane, sometimes intersecting a circle, sometimes lying outside it, or by reflecting a figure about a line which is not a line of symmetry, or by encountering algebraic computations that get very messy before they again simplify.

Such poignant metonymy is part of a very mathematical encounter. It cannot be avoided; nor need it be dwelt in. Some people complain that mathematics denies such inner associative feelings; others that mathematics provides a refuge from such feelings. Neither of these extreme positions need be the subject of complaint. Acknowledgment without stressing provides a middle ground. Being sensitive to emotion triggered unexpectedly, and demonstrating an accepting but not indulging behaviour can do much for adolescents struggling to locate themselves in an emotionally charged world. Being on the look-out for metaphoric inner content of mathematical tasks can provide a distancing-effect, a shift of attention, for teachers who find themselves otherwise bored by the elementary mathematics they are charged with teaching.

Meta meaning refers to the opportunities provided by any mathematical exercise to observe one's own behaviour and propensities, and thereby perhaps to increase sensitivity so as to inform action in the future.

Tahta pointed out that in order to contact the inner meaning of a task, it is necessary to participate fully, to engage in the outer task as if it were the (complete) task, and the same is true in order to contact meta meaning. If part of you is kept separate, then you are not participating fully.

Being separate or whole is hard to talk about accurately. There are states of partial participation in which part of you is separate from the action in a negative and disruptive manner, perhaps through trying to see how to do the minimum possible, to "just do the task as set", even to try to find things wrong with the task as set. There are also states of full participation in which attention is split: part of you is separate but watching, in a productive and positive manner. The difference often lies in the presence, or absence, of judgement. When part of you is judging, criticising, justifying actions, then full participation is impossible, and contact with inner meaning of tasks is unlikely. When part of you is observing without judging, then such truly full participation is likely to lead to contact with inner meaning as well as a sense of the outer task.

For example, it is essential when doing calculations, whether arithmetic or algebraic, to have an inner monitor which every so often asks whether this calculation is going in a useful direction, and which is awake to slips made by the automated-technique-running self. Such a monitor develops over time, and is supported by reflective practices as described in Schoenfeld (1985), Mason, Burton and Stacey (1984), or indeed Schon (1983) and many other writers.

For years I have noticed that it is possible to generate a collective state in the room which is quite different from the usual bustle of a classroom, particularly when working directly with mental images. I have associated this state with participants trusting me, subordinating their own goals temporarily, and my use of gentle voice tones and an imperative speech pattern as illustrated in the example earlier. While reading Neville (1989) I recognised similarities with a variety of trance-like states, such as that sometimes generated when small children are listening to a story, or when people are ensconced in front of the television. It is as if they have been transported, as if they are not really present in the room but rather in a world entered through a fusion of physical and mental screen. Direct work with and on mental imagery has sometimes generated similar states.

In education, the word *trance* probably has a predominantly negative connotation, denoting lack of awareness rather than heightened awareness; yet the term *entranced* has positive aspects as well, to do with being taken out of oneself, to reaching unusual places or states. Neville connects such states with an increase in suggestibility, but goes to great lengths to argue that such suggestibility is not capable of making people operate against their own interests or desires. My own view is that if something has the potential for good, then it also has the potential for evil. Much more investigation of such states is required. But it may be that the changes in voice tones, from the hard forward thrusting tones of the controlling teacher, to the soft, entrancing and enticing tones of someone describing a mental image, increases the likelihood that participants can connect with inner and meta meanings as well as outer meanings in associated tasks.

Attention, almost exclusively in many published schemes of work, on outer tasks, is perfectly understandable in a climate in which business is of supreme value. But it is possible to account for the purposes of a task without focussing on pupils getting something very specific. One way is to use the language of opportunity: a task may provide opportunity for pupils to encounter one or more mathematical ideas, and one or more aspects of mathematical thinking. If teachers are aware of a range of possibilities that might arise from a task, they are more likely to be able to follow the thinking of individual pupils than if each task is seen as having but one specific aim. The ability to gain a more general view of the opportunities offered in a mathematical task is enhanced by deepening awarenesses of what mathematics is about, of what constitutes mathematical thinking, and how different aspects can emerge in the classroom through attending to doing and construing.

DOING AND CONSTRUING OF DOING AND CONSTRUING

In trying to exploit inner and outer tasks, there are many pitfalls, summed up for me in the words *doing* and *construing*, or in the slogan

> From doing to construing.

For example, a task could be set in game format so that participants encounter the use of strategy rather than gut reaction. However, since games can be played even moderately successfully without being aware of strategy, it is not enough simply to be *doing* the game. Even if someone draws attention to the possibility of a strategy, it may not be within your ken at the time, as you may be completely caught up in the doing. Yet once you become aware of the possibility of attending to strategy, and begin to try it, your whole view changes.

When a teacher presents something on a screen to students, students' construal powers are immediately activated. Sperber and Wilson (1986) came to the conclusion after much study that sense-making is invoked via the cultural assumption that when people act as if what they are saying or doing is relevant to you, then your construal is triggered. Choosing to show something on a screen certainly carries such an implication. Maturana and Varela (1988) suggest similarly that

> observing an interaction *as if* meaning is attributed, determines the course of the interaction.

If students see a task as meaningful to themselves then the course of their interaction will be quite different from the case in which they do not.

My original title was going to be

> (Doing and Construing)^2

because I wanted to look at the four notions which arise from the expansion

(Doing and Construing) (Doing and Construing)

= Doing Doing and Doing Construing and Construing Doing and Construing Construing

For example, the expression *doing doing,* by its repetition, summons up in me the full absorption of the student (or indeed the teacher) in the activity, in the outer task. In this fully absorbed state, the most we can say is that experience is being accumulated, which may form the basis for later reflection. Mistaking the overt, outer task as the content and aim of a lesson is to be caught up in *doing doing.*

At the other end of the spectrum, *construing construing* suggests potentially excessive emphasis on self awareness. If outer awareness is constantly on being aware of process, then the effectiveness of reflection and construal is likely to be lost. It is too direct. Inner meaning requires some attention on outer doing. However, spending time occasionally examining how it is that one goes about making sense of some mathematics, or of an article in mathematics education, is much more likely to be productive. It signals a shift of attention from making sense, to becoming aware of making sense, with a view to preparing to do better in the future.

Doing construing signals to me the incredible speed with which process can become content. Construing is going on all the time. Unfortunately, in mathematics classrooms, the nature of that construing is sometimes centred on self-evaluation in terms of success and failure rather than on mathematical story telling to account for surprising situations. Images are being construed, but if the construal amounts to "this is too hard for me", or something similar, then the effect is negative rather than positive. Sense making may result in students deciding that that topic, that subject, that teacher, even being in school, are not for them. It may be to dwell in metonymic resonances and so to miss the "teacher content" entirely. And none of these rule out success in performance, at least on local tests!

Construing doing signals to me the sort of state which I wish to promote: explicit attention to accounting for what I am doing as an integral part of learning; establishing a mathematically conjecturing atmosphere in which those who are uncertain take the initiative to express their thoughts while those who are certain listen carefully and reflect back what they hear; establishing the habit of sharing and negotiating such stories with peers and others, through first agreeing what it is that is to be accounted for.

IN CONCLUSION

I am convinced that our visually-dynamic dominated culture, far from making education an easier task, makes even more demands for teacher and researcher time to be devoted to making use of static and dynamic images, and although there hasn't been room here to mention it, to processing, analysing, and re-expressing those images with a variety of mental and electronic tools. Animated images in particular may decrease the time needed to encounter an idea, but there may be a corresponding increase in the time needed for construal. In other words, electronic screens may enhance learning, but may not make it less time consuming. The attraction, the entrancement of moving images is potentially a force for real development in education, but it is also potentially a force working against the type of inner and outer task construal which we associate with learning mathematics.

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