IMAGES AND DEFINITIONS OF FUNCTIONS IN AUSTRALIAN SCHOOLS AND UNIVERSITIES

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The study of functions is central to modern mathematics, and occupies the major part of the time spent on this subject in the senior school. The concept itself is rich and diverse, able to be conceived of using a variety of images, singly and in combination. This study explores ways in which Australian secondary and tertiary mathematics students think about functions in terms of both images and definitions. It further investigates the extent to which these factors influence their ability to discriminate between functions and non-functions, and to solve problems involving functions and their properties. The results have implications for both the teaching of functions, and the use of mathematical computer software which facilitates the representation and manipulation of functions in a variety of forms.

A real-valued function F defined on a set D is a rule which assigns to each element x of D exactly one real number F(x).

(Walters and Wehrhahn (1987, p.20) in Barnes, 1988; 120)

A function is when you go out and have a good time, and relations - you visit them at Christmas.

(Student respondent to Function Questionnaire, 1991)

What is a mathematical function? How do students think about functions? If you were to ask your best and brightest students, what response would you get? This paper describes a recent study of 383 high ability secondary and tertiary Mathematics students from Queensland, New South Wales, South Australia and Western Australia and suggests that you might well be disappointed if you expected that most of them could distinguish functions from non-functions, or even say clearly what a function is. While we may not expect secondary students to be able to recite the formal definition above by heart, the requirements of tertiary mathematics may well be finding many of our students ill-prepared in this regard. At the same time, there is a surprising level of consistency across schools, sexes and even States in the ways in which students think about functions.

IMAGES AND DEFINITIONS OF FUNCTIONS

Teachers of mathematics are well aware that functions are the "bread and butter" of our discipline in the secondary school and beyond. Senior students are expected to be familiar with a range of common functions, including the linear, quadratic, trigonometric, exponential and logarithmic functions; to sketch, manipulate, differentiate and integrate them. The study of functions occupies the major part of the time spent on mathematics in the senior school.

While the "formal definition" (as stated above) may appear somewhat daunting and rigid, the ways in which functions are conceived of and used prove quite interesting. The concept of "function", not surprisingly, is one which is mathematically rich, capable of being thought of using a number of distinct "images". Numerous studies over the past decade, both in Australia and overseas, have investigated the ways in which different groups of people think about and use functions (see Dreyfuss and Eisenberg, 1982; Tall and Vinner, 1989; Vinner, 1983; Vinner and Dreyfuss, 1989). Barnes (1988; 121), in interviews with secondary and tertiary students in New South Wales, identified six frequently occurring images of functions:

- A graph or curve
- A relationship between two variables
- A "function machine" (input-output device)
- A set of ordered pairs or table of values
- An algebraic formula or equation
- The symbol f(x)

She also distinguished the image of function as a *"mapping between two sets"*, which was not widely used by students, but was seen as useful in thinking about the concept.

The SOLO taxonomy (Biggs and Collis, 1989) distinguishes levels of understanding in the acquisition of concepts which may be relevant here. Students who focus quickly upon a single property or characteristic of a concept are said to be *unistructural*; those who recognise several properties as relevant, but do not link these together may be thought of as *multistructural;* those who are able to see the relationships between the various properties or representations of a concept are said to be *relational.* Some may go beyond this level, forming new connections and seeing applications of the concept in new situations; such learners are said to be operating at an *extended abstract* level. These levels (together with an initial *preoperational* level) are considered to cycle through each of the developmental modes - sensori-motor, ikonic, concrete-symbolic, formal and post-formal operations. Learners who describe functions using multiple images would be considered to be operating at a higher cognitive level than those who use only a single image. More recent developments in SOLO theory (Biggs and Collis, 1991, Collis and Biggs, 1991) suggest that individuals who operate in a *multimodal* way (able to draw upon earlier modes of thinking) will be more effective as problem solvers and critical thinkers. In the present study, students able to draw upon *ikonic* images of function to supplement their more usual concrete-symbolic way of thinking may well prove more capable at analysing functions than those who tend to approach such situations unimodally.

In learning a concept such as function, students begin with an active process (substituting numerical values into expressions) and gradually come to view such expressions as objects in their own right. Such objects may then be manipulated and analysed for their own sake, eventually becoming the basis for new processes in the formation of new higher order concepts (as in the composition of functions) (see Sfard, 1991). This process is consistent with that described by Bruner (in Bruner and Anglin, 1973), who sees cognitive growth as moving through stages of representation which he described as *enactive*, *iconic* and *symbolic*.

The evolution of the function concept throughout secondary schooling follows this pattern. Although students begin with the enactive ideas of "function machine" or numerical substitution in junior secondary, they move quickly to the study of functions as objects, which appears to remain the focus for all future study. The valuable perception of function as process may well be lost for many students by the time they reach their senior years. Whether they are fixed at the analytic concrete-symbolic mode or the global ikonic mode, students unable to utilise both ways of thinking may be disadvantaged in thinking about and using functions effectively.

Recently, the use of computer technology has begun to impact upon senior mathematics classrooms. To a large extent this has been in the form of *graph plotting* software (such as *CAPgraph*, *Algebra Graf(x)* and hand-held plotters such as the *CASIO-7000G*). Such tools allow students to quickly and easily change the representation of functions from algebraic to graphical form, and their use might be expected to result in an increasing familiarity with the graphical form of functions. More recently still has seen the emergence of true *multiple representation software* (such as *ANUgraph* and *CC3 - the Calculus Calculator*) which allow both the representation and manipulation of functions in algebraic, graphical and tabular forms. The effects of such technology upon the ways in which our students visualise and use mathematical functions are likely to be significant, but remain, as yet, unknown.

THE SAMPLE

The sample for this study consisted of 383 high-level mathematics students - 160 first-year Mathematics students from universities in Queensland and New South Wales, and 223 Year 12 students attempting either the Mathematics I and II courses (the highest level possible in Queensland, South Australia and Western Australia) or the high-level Three Unit course in New South Wales. Three schools from New South Wales (N1-3), seven schools from Queensland (Q1-7), four from South Australia (S1-4) and one school from Western Australia (W1) made up the school sample.

THE QUESTIONNAIRE

The questionnaire consisted of three sections and was modelled upon studies conducted by Barnes (1988), Vinner and Dreyfuss (1989) and Ayers, Davis, Dubinsky and Lewin (1988). The first section required students to rank a list of six images in terms of their preferred way of thinking about functions. Students were also asked to describe (in their own words) what they thought a function was, and to decide if there was any difference between functions and relations in Mathematics.

The second section involved students in distinguishing functions from non-functions, when presented in both algebraic and graphical forms. Students were asked to decide if the given expressions and graphs showed y as a function of x, and to explain how they decided. The examples chosen highlight several of the difficulties described by Markovits, Eylon and Bruckheimer (1988) in their study of students' knowledge of functions. Problems associated with constant functions (for many students a contradiction in terms), discontinuities and "piecewise" functions have been included.

The third section of the questionnaire required students to attempt solutions to a number of problems involving functions. Two of the problems were chosen so as to present the ideas in a relatively unfamiliar context - one involving composition of non-numerical functions, the other involving difference equations; the other problem was a more "mathematical" question, involving composition of functions. These are described in more detail elsewhere (Arnold, in press).

THE RESULTS

Function Images and Definitions

The pattern of images which students chose as their preferred way of thinking about functions was surprisingly consistent across schools, states and universities. With the exception of the New South Wales schools, the most frequent function image described was that of an algebraic formula. The graphical image was the next most common choice (in the case of the New South Wales schools, this pattern was reversed). Less frequent were the images of "rule or relationship between two variables" and "function machine". Few of the respondents thought of functions primarily as "tables of values" or as mappings. These patterns of images were found to be consistent across the schools in each state group, and across the two universities - no statistically significant differences were found for these groups related to their image preferences.

When asked to describe in their own words what they understood by a function, the picture assumes quite a different countenance (see Figure 1). What are termed here "function descriptors" were largely made up of the images already described, but the pattern of distribution is quite different. In all groups except the South Australian schools (who preferred a graphical description), functions were described most commonly as some type of rule or relationship. This corresponds to the common usage of the term outside of its specific mathematical sense.

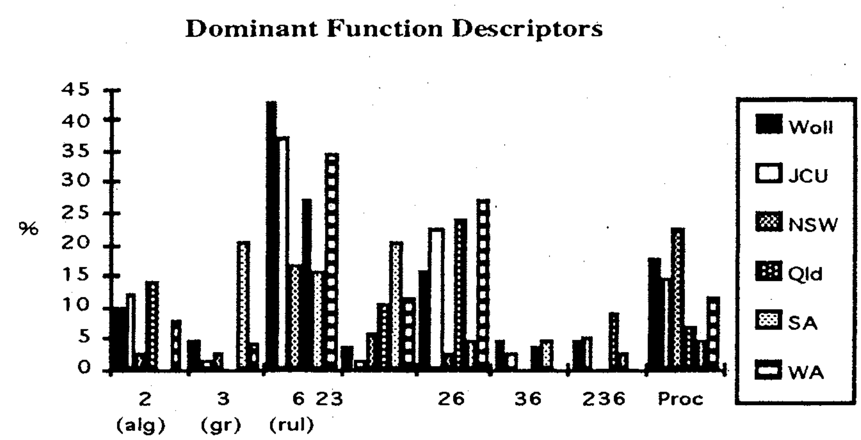

Figure 1: Dominant Function Descriptors.

Upon analysis of the entire sample's responses to the function definition question, ten different descriptors were found to account for over 95% of these answers. These are described in Table 1 with the code used to describe each in Figure 1. These descriptors, like the images above, proved surprisingly stable and consistent across the various groups which made up the sample (see Table 2). The two university groups exhibited identical orders for their most common descriptors, while across all groups the same five descriptors accounted for over 90% of responses.

Table 1: Most commonly occurring function descriptors (> 95%)

Function Descriptor	Code
1. Rule or relationship between variables	6
2. A rule which can be expressed algebraically	26
3. An algebraic formula	2
4. A formula which can be expressed as a graph	23
5. A process by which one number is changed to another	Proc (1)
6. A rule which can be expressed algebraically and graphically	236
7. A graph	3
8. A rule which can be graphed	36
9. A process which can be expressed using a formula	12
10. A rule which describes a process of change in variables.	16

How reliable might these descriptors be considered? No significant differences were found between the two universities, the seven Queensland schools or the four South Australian schools. The New South Wales schools were found to differ significantly [N = 36, df = 22, $c^2 = 41.25$, $p = .0077$], with one school (N2) exhibiting quite a different response pattern to the others, and a large number of non-attempts, particularly on the problems. This makes it difficult to comment further in this regard, although these schools were found to be consistent on sex, images and knowledge of the formal definition of function.

Table 2: Order of frequency of most common function descriptors (>90% of sample)

Woll.Uni	JCU	NSW school	Qld school	SA school	WA school
6	6	Process	6	3	6
26	26	6	26	23	26
2	2	12	2	6	23
236	236	23	23	Process	2
3	Process		236		
36			Process		

Further evidence for the stability of these function descriptors comes from comparison with their related images. Chi-squared analysis of contingency tables associating each of the three dominant images (algebraic formula, graph and rule) with function descriptors containing these images revealed significant differences. Those students whose function definition contained an algebraic element were significantly more likely to have chosen this as their preferred image [N

= 383, df = 1, χ^2 = 12.97, p = .0003]; those who used "graph" in their definition matched significantly with those who preferred a graphical image [N = 383, df = 1, χ^2 = 23.78, p = .0001], and similarly for the "rule or relationship" element [N = 383, df = 1, χ^2 = 7.95, p = .0048].

These results suggest that the images and definitions which are described here are stable and potentially useful factors in describing students thinking about functions in senior secondary and early tertiary studies.

One descriptor stands out as different to the others, in that it is not composed of one or more of the three dominant images. This is the definition of "function as process", which was found to be particularly strong in the New South Wales schools (excluding N2), the two university groups and, to a lesser extent, the South Australian schools. Although not as dominant a descriptor as some of the others (only 47 of the total sample of 383 described functions in this way), it proves interesting for a number of reasons. The idea of function as process was found to elicit significant differences when contrasted with sex (it was male-dominated [N = 383, df = 1, χ^2 = 6.63, p = .01]), with tertiary study (more than half of those who chose it came from the two university groups [N = 383, df = 1, χ^2 = 10.68, p = .002] and with knowledge of the function definition [N = 383, df = 1, χ^2 = 8.77, p = .0031]. It was also associated with significant success on one of the function problems, discussed below.

The final comment in this section concerns the occurrence of function definitions which utilise multiple images. Although the majority by far used single image definitions, double and triple image definitions were to be found in most groups, and more commonly in schools than universities. Once again, the relevance of multiple image definitions is discussed below in reference to the items involving function problems.

Knowledge of the Formal Definition of Function

While the images students use when thinking about functions may have been fairly stable across the various groups of this study, the same cannot be said about the knowledge of students regarding the nature of functions, and their abilities to distinguish functions from non-functions. For the purposes of this study, students were counted as "knowing" the function definition if they displayed knowledge of one or both of the following:

- The "algebraic" definition, that for each "x" there exists a unique "y" value;
- The "geometric" definition, often referred to as the "vertical line test", that a vertical line moved across the graph of a function should touch the curve at only one point for each point in the domain.

An early feature observed in processing the results from the Queensland schools was the lack of knowledge students displayed of either of these notions (Figure 2). Students from three of the seven schools showed no knowledge at all that a function in Mathematics was any more than "a relationship between things". When questioned regarding this, teachers generally saw no problem in this regard; while they used functions frequently, they did not see it as necessary for their students to be able to distinguish functions from non-functions.

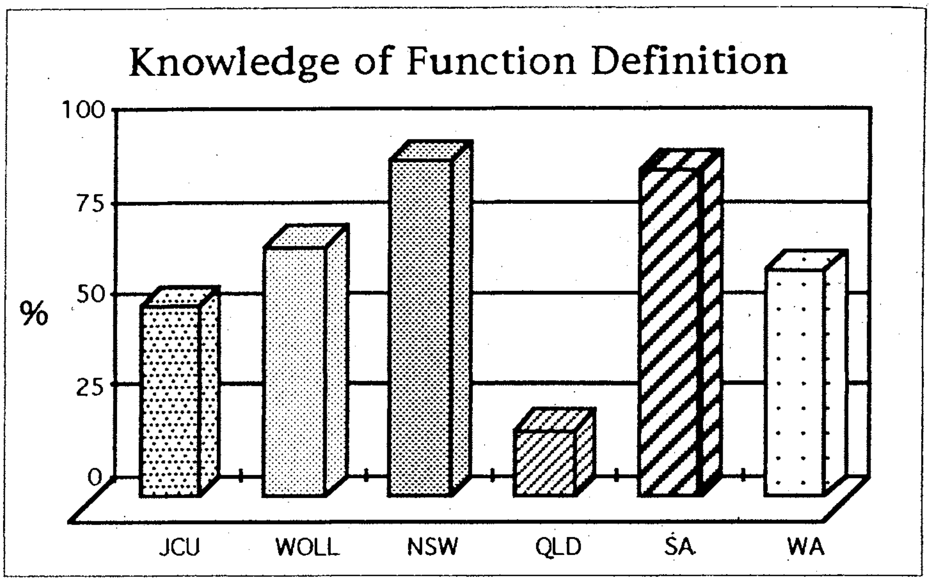

Figure 2: Percentage of State groups displaying knowledge of the Function Definition.

Of those who knew the formal definition, the majority overall used the algebraic form (46%), with 29% using the geometric form (the "vertical line test") and 25% displaying knowledge of both forms. This distribution, however, varied widely across the various groups. Males were found to be significantly more likely to use the algebraic form [$N = 208$, $df = 1$, $c^2 = 6.93$, $p = .0085$], while females were more likely to use the geometric form [$N = 208$, $df = 1$, $c^2 = 7.55$, $p = .006$]. The vertical line test was also found to be more prevalent among the school sample [$N = 208$, $df = 2$, $c^2 = 7.02$, $p = .0299$], and the algebraic form more commonly used by the university students [$N = 208$, $df = 2$, $c^2 = 6.58$, $p = .0373$].

Recognition of Functions and Non-Functions

It seems logical that, if students do not know the definition of a function, then they will have difficulty distinguishing examples from non-examples. This was found to be the case. The majority of students who did not know the definition decided on the basis of whether the example in question "looked familiar" (hence a circle was considered a function by many). Others looked for whether it "had an x and a y in it" (hence large numbers decided that $y = 4$ is not a function since there was no "x" term). Most often, the example was a function if it was perceived that, in some way, "y depended upon x".

In distinguishing those students who demonstrated knowledge of either the uniqueness property or the vertical line test from those who did not, there was found to be a clear advantage for the former in recognising functions and non-functions. Out of 8 questions, over 80% of those who knew the definition scored between 6 and 8, while the majority of those who did not know the definition scored between 3 and 5. All correctly identified the common examples given.

Statistically significant differences (p = .0001) were noted for most of the examples between those who knew the formal definition and those who did not, in both universities and schools. Knowledge of the definition proved no advantage in recognition of the familiar quadratic function, in both its algebraic and graphical forms. Nor did it advantage the university students in recognition of an unfamiliar function (in which y = 1 if x is rational, and y = 0 if x is irrational). It did, however, prove to be of some help to the school group.

Of the different forms of the function definition, the students who displayed knowledge of *both* algebraic and geometric forms were significantly more successful than others who knew the definition in only one form. Those using the geometric definition alone were found to be significantly more successful on most items than those using the algebraic form ($p < .01$).

Problems Involving Functions

The final section of the questionnaire involved students describing the solutions of problems involving function ideas in unfamiliar situations. Students describing multiple image definitions were significantly more successful than those using single images on all problems. Inspection of the contingency tables for each of the questions revealed that students using two and three images were more successful than those using only one. This finding is consistent with the SOLO Taxonomy analysis, which suggests that students using multiple images when thinking about a concept are operating at a higher cognitive level that those using only a single image. This argument is further supported by the finding that those students who displayed knowledge of the formal definition of function in both algebraic and geometric forms were significantly more successful on the item "RS" than those who used only a single form of the definition [$N = 208$, $df = 2$, $\chi^2 = 11.68$, $p = .0029$].

The perception of "function as process" was associated with significant success on the item involving interpretation of a difference equation. This was true for the entire sample [$N = 383$, $df = 2$, $\chi^2 = 7.04$, $p = .0296$] but especially so for the combined university sample [$N = 160$, $df = 2$, $\chi^2 = 10.82$, $p = .0045$].

IMPLICATIONS FOR TEACHING

This study was conceived as a pilot study for a more intensive investigation of these ideas. Nonetheless, certain factors appear to be sufficiently consistent to form the basis of recommendations for the current teaching of functions in the secondary school.

The Function Definition : Primarily important would appear to be the need to teach our students that not every relation is a function. Particularly in Queensland schools, inclusion of the teaching of at least the "vertical line test" and preferably both forms would appear to be a priority. The concept of function is central to modern mathematics, and it is unsatisfactory that our most capable students should be unaware of its nature, and frequently use the term inappropriately. Such students should be capable, not only of correctly distinguishing functions from non-functions, but of explaining the importance of the property of uniqueness in the mathematics they are studying.

Images of Functions : It is important, as teachers, to have some understanding of our students' thinking. From the results of this study, it is likely that most students will think of functions primarily as algebraic objects which can be graphed, and which describe a rule or relationship between things. While such an understanding is generally adequate for most purposes, it has a number of implications.

Students in this study seemed more likely to think of functions as *objects* rather than *processes.* There is a need to encourage an "active" perception, particularly when working with composition of functions and when solving problems which involve function ideas. The idea of *function as process* implies consideration of the domain and range and is probably closest to the image of "function machine" described above, in which students picture objects "going into" a function, and other objects "coming out".

Following from this is the implication that, in our teaching, we should attempt to provide students with as broad a base as possible when they are thinking about functions. Although the graphical image has some advantages over the algebraic, it is more useful still if students are encouraged to think about functions using multiple images wherever possible. The ability to mentally represent a given function variously as a graph, a formula, a table of values, a mapping or even as a "black box" must provide students with the means to view and attack given problem situations more intelligently and creatively than those who are fixed in a more narrow perception.

Functions and Computer Technology : A final implication follows on from the last. As the use of graph plotting software becomes more common in the teaching of senior mathematics, there is a real danger that it may cause a narrowing of the focus regarding function images, rather than the broadening recommended above. While the power of the computer will make it easy to switch from the algebraic form to the graphical, if not used carefully, two problems may arise.

First, students may come to depend more strongly upon a graphical representation when thinking about functions. This has some advantages, but may well prove to be a problem in tertiary studies of mathematics, when many functions are abstract and unable to be visualised. Students too dependent upon a visual image may well be disadvantaged in such cases. Secondly, the perception of "function as object" may be further reinforced, making students less likely to consider functions as active processes which act upon their domains and produce new objects.

The emergence of *multiple representation software* which allows the user to move between algebraic, graphical and tabular form may help to encourage this broadening process, as may the use of the spreadsheet as a tool for exploring algebraic relationships. Allowing the user to view the same mathematical function from various perspectives should increase understanding and improve the versatility of the thinking involved. Here, too, however, may be found the same dangers as described above. Teachers will need to assist their students to reflect upon the processes involved, rather than allow them to use the technology *automatically*, and without thought. The potential is great for computer tools such as these to improve the teaching and learning of mathematics in many ways, but there are dangers too which will need to be anticipated. Classroom teachers, through careful and reflective use of such tools, hold the answers to their effective use. Knowledge of the ways in which students think about and use functions seems likely to be inseparable from any consideration of the effects of such technology.

REFERENCES

Arnold, S.M. (in press). Images and definitions of functions in Australian schools and universities. *Australian Senior Mathematics Journal.*

Ayers, T., Davis, G., Dubinsky, E., & Lewin, P.(1988). Computer Experiences in Learning Composition of Functions. *Journal for Research in Mathematics Education,* 19 (3), pp. 246-259.

Barnes, M. (1988). Understanding the Function Concept: Some Results of Interviews with Secondary and Tertiary Students. *Journal of the Mathematics Education in Australasia,* May. pp. 24-33.

Biggs, J., & Collis, K.(1989). Towards a Model of School-Based Curriculum Development and Assessment Using the SOLO Taxonomy. *Australian Journal of Education,* 33 (2), pp. 151-163

Biggs, J.B., & Collis, K.F. (1991). Multimodal Learning and the Quality of Intelligent Behaviour. In H.H. Rowe, (Ed.) *Intelligence: reconceptualization and measurement* (57-76). Lawrence Erblaum Associates, Hillsdale, N.J. and ACER, Hawthorn , Vic.

Bruner, J., & Anglin, J.M. (1973). *Beyond the information given: Studies in the psychology of knowing* ... W.W. Norton. New York.

Collis, K.F., & Biggs, J.B. (1991). Developmental Determinants of Qualitative Aspects of School Learning. In G. Evans, (Ed.) *Learning and teaching cognitive skills* (185-207). Hawthorn, Victoria: ACER.

Dreyfuss, T., & Eisenberg, T. (1982). Intuitive functional concepts: A baseline study on intuitions. *Journal for Research in Mathematics Education,* 13 (5), pp. 360-380.

Markovits, Z., Eylon, B.S., & Bruckheimer, M. (1988). Difficulties Students have with the Function Concept. In A.F. Coxford and A.P. Shulte, (Eds.) *The ideas of algebra, K-12 (1988 Yearbook).* Reston, Va: NCTM.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics,* 22, pp. 1-36.

Tall, D. & Vinner, S. (1989). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 20, pp. 151-169.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology* 14(3) 293-305.

Vinner, S. & Dreyfuss, T. (1989). "Images and Definitions for the Concept of Function" *Journal for Research n Mathematics Education,* 20 (4), pp. 356-366.