# STUDENTS' MANIPULATION OF ALGEBRA SYMBOLS AND THEIR AWARENESS OF THE CORRESPONDING CONCEPTUAL RELATIONSHIPS


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## INTRODUCTION

> ... substantial evidence exists to indicate that the learning of algebra is addressed by many students as a problem of learning to manipulate symbols in accordance with certain transformation rules (i.e. syntactically) without reference to the meaning of the expressions or transformations (i.e. the semantics). This is of course not surprising, since most algebra syllabuses in the past have paid considerable attention to the syntactic aspects of algebra, precisely because of the central role that symbolic representation plays in algebra work, because power over such representation is crucial to successful performance in algebra, and because the symbolism is both new to the students and an obvious feature of this area of study. (Booth, 1989, p. 58)

Large scale studies in the early 1980s by the CSMS (Concepts in Secondary Maths and Science) and SESM (Strategies and Errors in Secondary Mathematics) project teams in the UK were carried out on thirteen to fifteen year olds in order to ascertain student understanding of the basic concepts and procedures of elementary algebra (Hart, 1981; Booth, 1982). The results confirmed the existence of many difficulties experienced by students in this age range. As a result of these studies and others conducted in the Algebra Learning Project at the University of Georgia (Wagner et al., 1984), teachers now have access to a variety of findings on the diagnosis of learners' errors in elementary algebra. Despite this accessibility, however, many teachers remain as unaware of the difficulties faced by their students as they did almost twenty years ago:

> Too often, when teachers find errors in a child's work, they mark the example wrong, assume that the child did not master the basic facts, and prescribe further drill. Careful analysis of errors through observation and interviews with individual children is essential. (Pincus, 1975, p. 581)

A survey of the teaching strategies employed by over 800 teachers of elementary algebra revealed teaching practices almost totally geared towards rote learning of symbol manipulation (Oliver, 1984). At least two models have been designed in an effort to improve on this situation by avoiding errors pupils make and misconceptions they develop before or during the traditional teaching of elementary algebra (Booth, 1984, Oliver, 1984) and both models have been relatively successful in the short term.

Research has also shown that when commencing algebra instruction, students often possess well established beliefs, some of which are invariably incorrect (Perso, 1992; Claxton, 1987; Osborne, 1984). These conceptual errors are more often than not responsible for students' mistakes (Balacheff, 1984; Bell, 1984) and it is clear that teachers need to be aware of these errors if they are to be effective in teaching. Onslow's view was that:

Unless teachers are alert to children's errors and misconceptions, situations for overcoming them are easily overlooked. (1986, p. 218)

While Driver and Oldham's constructivist model of teaching incorporates a stage devoted to determining students' errors (1986, p.113).

The purpose of this ongoing study was to examine just a small element of the complex area involving students' errors and misunderstandings in elementary algebra, namely the relationship between students' thinking, based on their visual perception of the physical movement of algebraic symbols and their awareness of the conceptual link thus denoted.

Other matters of interest to the researchers included: students' approaches to reading and transforming relationships expressed by formulae; students' ability to see and express generalisations; the behaviour of capable students when faced with problems beyond their manipulations; the performance of students on a number of paradigmatic problems to identify points of breakdown, and the capability of average students to acquire and retain effective strategies for checking their work on algebra problems.

## METHOD

One hundred and thirty-seven pupils of five classes drawn from years 8(1), 9(2) and 10(2) in an Australian senior high school took a one-hour written test (in two parts, A and B). Interviews were conducted with two samples of 10 pupils, one beforehand to guide the test design, and one afterwards, to clarify the methods being used.

The main target tasks may be illustrated by the following item:

Which of the following formulae do you think is most likely to be correct? Underline your answer. Give reasons:

Pull of the earth on a satellite at height h.

$P = kh \qquad P = kh^2 \qquad P = \frac{k}{h} \qquad P = \frac{k}{h^2}$

This involves *reading* the formulae and *recognising* the functional relationships expressed, (a mathematical task), so that it is then possible to consider which of the relations is most likely to fit the physical quantities (this involves some awareness of physical principles).

A different type of task was to read the formula $V = \pi r^2 h$ and to extract from it the various functional relationships - such as the relation between V and r, if h is fixed.

The target *transformation* task was to transform a formula such as $V = \pi r^2 h$ or $S = \pi r^2 h(r-h)$ to give h. Such manipulations are typically performed by the (implicit) application of the transformations of $A + B = C$ (to $A = C - B$ or $B = C - A$), or of $PQ = R$ (to $P = R/Q$ or $Q = R/P$); where the components may be numbers, single letters, or 'chunks' such as $\pi r^2$ or $(r - h)$. Other algebraic laws such as the distributive, associative and commutative laws may be required; and much may be subsumed under the principle of 'doing the same to both sides'. Particular known hazards relate to false commuting or reversing of subtraction or division, and the reversal of order exposed by the students and professors problem, in which six students to every professor is symbolised as 6S = P. This last has been shown to arise in some cases from over-literal translation of the verbal statement, and in others from a perception of S being the big number, by association with the 6, with a degenerate reading of the = sign as indicating a correspondence rather than an equality of numbers (Clement, 1982).

The test questions include some designed to expose these known errors. Others are focused on the basic multiplicative transformations of PQ = R, looking both for methods used and for the possible effect of contexts. We shall discuss first the questions demanding reading and writing of algebra, then those concerning transformations.

## RESULTS AND DISCUSSION

### (a) *Reading and writing algebra*

Questions covered (1) the translation of a simple verbally described situation involving adding and subtracting, into an algebraic formula; (2) testing for possible false commuting of subtraction, for the 'equals is makes' error and for the 'students and professors' reversal in additive cases; (3) forming an expression for perimeter containing both letters and numbers from an annotated diagram; (4) reading functional relationships from $V = \pi r^2 h$; and (5) choosing plausible formulae for physical situations, as illustrated above.

The first formula construction question was:

1a Clare has earned D dollars. She spent C dollars for clothes and F dollars for food, and has L dollars left. Complete this expression for L, using C, D and F:

L = ____________

(Tick one box)

Are you sure ☐ ☐ ☐

No Fairly Yes

1b Also write a formula to give F in terms of C, D and L.

F = ____________

(Tick one box)

Are you sure ☐ ☐ ☐

No Fairly Yes

(Pupils' degrees of sureness were elicited as shown; these will be analysed later.)

Success rates for the five classes (Classes 8, 9L, 9T, 10L, 10U) were 30, 57, 50, 69 and 70% respectively – showing an increase over the three years. A further 10% of responses were correct except for the omission of brackets, and a steady 5 - 8% each year reversed subtraction, for example writing L = (C + F) - D instead of L = D - (C + F). In classes 8 and 9L, 6 - 7% gave numerical responses, assuming some actual possible amounts of dollars. The 20% or so each year of unclassified responses included D - C, F or D - C and F, D ÷ C, F, and there were omissions declining from about 20% in Class 8 to zero in class 10U.

The second part of the question required a more difficult transformation of the data. In the first part, the formula could be constructed in the order of events; we start with dollars earned, subtract what was spent and finish with the amount left over. The second part requires extracting what was spent on one of the two items, and counting what was left over in the same way as the other spending item. In the year 8 class, the success rate is still 30%, but the omission rate doubles, to 40%; in the other classes, success falls sharply, to 43, 23, 43, 40% respectively.

Question B2 asks for the perimeters of two annotated figures (a) a triangle with sides marked x cm, y cm, 8cm, and (b) a parallelogram with sides x cm, 5 cm, x cm, 5 cm. The error of juxtaposition (xy8) was seen in a few cases (less than 10%). More noticeable was the large minority (20-25%) in certain classes (8, 9L, 10L) who gave numerical answers, by assuming values for the lengths x and y. These pupils appeared not to accept the possibility of an expression as an answer in this case. These were somewhat larger percentages than those failing to give a formula in the previously discussed question – it appears that the mixed expression x + y + 8 is less acceptable than the wholly literal one D - C- F.

(b) *Commutativity problems*

A question asking for the value of x in 8 - 5 = x - 8 showed the false commutative response (5) in 10-15% of cases, with no decrease with increasing age. This misconception clearly does not get treatment adequately in the curriculum. In a question asking for an expression for the time taken by a bus travelling 28 km at 65 km per hour, the correct answer 28 ÷ 65, was given by substantially fewer pupils than gave 65 ÷ 28; in interviews, where pupils were directed to estimate the expected size of the answer, correction often took place, though a few pupils continued to assert the equivalence of these expressions.

(c) *The students-and-professors reversal in additive problems*

Two questions had the potential to expose this error. The first was:

'I have m dollars and you have k dollars. I have $6 more than you.'

Which equation must be true? (Underline one of the answers below.)

$6k = m$ $6m = k$ $k + 6 = m$ $m = 6 = k$ $6 - m = k$

Correct responses averaged about 50%, and the reversal error about 25%. The second was:

We are told that *a* and *b* are numbers and $a = 28 + b$; which of the following must be true? (Underline one of the answers below.)

a *a* is larger than *b*
b *b* is larger than *a*
c You can't tell which number is larger
d *a* is equal to 28

Success rates here were about 60% (*a* is larger than *b*) with very few reversals, but there were some 15% each of '$a = 28$' (equals is makes) and of 'you cannot tell'.

*(d)* *Reading a Complex Formula*

Extracting the functional relations from $V = \pi r^2 h$ produced very low levels of success, except for the first question (r constant, what happens to V when h is doubled); the correct response here (doubled) was given by 50-60% – but it could be argued that this is the obvious response. There were hardly any correct responses to the remaining questions (h fixed, r doubled, V?; V fixed, r doubled, h?; transform the formula to give h.) One of the obstacles here is the kind of acceptance of lack of closure required to envisage a doubling of a general unknown value, and its effect.

*(e)* *Reading and Judging Functional Relations*

These were questions asking which was the most likely correct function for each of four physical situations.

They were:

The pull of the earth of a satellite at height h

$kh$ $\quad$ $kh^2$ $\quad$ $\frac{k}{h}$ $\quad$ $\frac{k}{h^2}$

Force needed on pedals to ride bike at speed v

$kv$ $\quad$ $kv^2$ $\quad$ $\frac{k}{v}$ $\quad$ $\frac{k}{v^2}$

Number of marbles of diameter d in a kilogram

$dk$ $\quad$ $\frac{k}{d}$ $\quad$ $\frac{k}{d^2}$ $\quad$ $\frac{k}{d^3}$

Stopping distance of a car with speed v mph

$kv + \frac{k^1}{v^2}$ $\quad$ $kv + k^1 v^2$

These questions were found very hard; almost no acceptable reasons were given, and there was a majority of omissions. Ignoring the subtleties of linear or square, thus considering only the direction of the relationships, the satellite questions collected, in all, 43 (incorrect) choices of kh or $kh^2$, and 20 of $\frac{k}{h}$ or $\frac{k}{h^2}$. The bike question showed 45 choices of the correct direction of relationship (kv, $kv^2$) and 32 incorrect ones. In the absence of valid reasons it is not possible to distinguish errors in the algebraic reading from incorrect choices of relationships. These questions must be redesigned.

*(f)* *Transformations*

Attention was focused mainly on the very commonly occurring three-term multiplicative formulae. The Resistance, Current and Voltage relations, Speed, Distance and Time, and the similar relation between three numbers A, B and C were used. The last appeared twice, once using division signs, (as the contextual questions), and once using the fraction bar to denote division. One example is given.

The current, voltage and resistance in an electric circuit are connected by the formula: $R = V \div C$

Complete the formula for finding the voltage, given the current and the resistance.

V = -------------------

Which of these are correct formulae? Put √ or x.

a. $R = C \div V$ ____ b. $V = R \div C$ ____ c. $C = V \div R$ ____

The age and class trend in the results was of a modest steady increase through classes 8, 9L, 9T, 10L, and a drop in class 10U to a level close to that of class 8. Combining the four question items and the five classes gives overall percentage success rates as follows: for D S T 51%; C V R 51%; A B C with ÷ 49%; A B C with fractional bar, 45%. (The drop in the last figure comes mainly from classes 9T, 10L.) It was hypothesised that the DST context would be more familiar and hence more supportive, than CVR; indeed, it was expected that the latter result would be depressed compared with the numerical case, ABC. However, the results refute these hypotheses, showing that the fraction bar notation is *less* easy to handle than the division sign, and that both contexts are equally supportive.

*(g)* *Mode of Thinking*

The interviews suggested that four modes of thinking were used to effect these transformations; and in the written test pupils were asked about these.

Look back at the questions on C, V, R (no.2). Tick the one or two of the following statements which is closest to the way you were thinking.

| | | |
|---|---|---|
| a | I thought which would be the biggest number, so the others would be divided into it | ☐ |
| b | I tried some actual numbers in my head | ☐ |
| c | I just remembered the formulae | ☐ |
| d | I thought the one on top on the right hand side would go underneath on the other side | ☐ |

The last mode was shown in interviews only in the use of cross multiplication (with insertion of 1 where necessary, e.g., to make $A = \frac{B}{C}$ into $\frac{A}{1} = \frac{B}{C}$. However, it was conjectured that this visualisation of physical movement of the symbols was the dominant mode for experts, and we were particularly interested in its appearance in our sample.

The relative frequency of responses of the four modes (with no great differences between the classes) was 27%, 41%, 21%, 11%. Interviews also confirmed the tendency to consider sizes and actual numbers; and this result relates to the relatively less successful transformation of the fraction-bar version of the ABC question discussed above.

It is also important to consider whether there is any relation between the mode of thinking and success rates. Tentative partial figures, (subject to further analysis) are, for modes ABCD on the CVR question, 2.2, 2.0, 1.8, 1.6 (mean items correct out of 3); and for the ABC fraction bar question 0.9, 1.3, 1.6, 1.7 (these are pupils who said they used this mode for the CVR question; they were not asked about the ABC question). This matter is worthy of fuller investigation.

We may compare these responses with those of a sample of ten 'experts' – teachers of science and mathematics enrolled in an in-service Masters' degree course. These responded to the ABC fraction bar question, (all correctly, except for one error by one person), and to the 'mode of thinking' question. Of the ten, seven reported using mode D, two mode C and one 'none of these'. To check how far this difference in response was likely to be due to the difference in presentation of the two questions, a further small sample of nine novice mathematics teachers was given both the CVR and ABC fraction bar questions, and after doing each, was asked the 'mode of thinking' question. In all cases, they reported the same mode of thinking for the two questions; four mode D, four C and one both C and D. Some of these subjects also reported imagining the CVR relation written with a fraction bar (so that they could use mode D).

Questions which arise from this study are the following. How far is the eventual adoption of vision symbol-moving methods associated with a loss of the ability to recover the awareness of the underlying mathematical operations? Conversely, should the move towards visual methods be encouraged in school, or guided in such a way as to preserve the links?

We hope to explore these in a more extensive project.

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