ACQUISITION OF CONCEPT OF VARIABLE IN A TRADITIONAL AND COMPUTER-INTENSIVE ALGEBRA CURRICULUM

MONIQUE A. M. BOERS
Swinburne University of Technology

INTRODUCTION

With the advent of computer technology and calculators that can perform symbolic algebra, graph functions and relations, and display tables of values the question arises: how should mathematics be taught in the presence of these technologies? In the United States one answer to this question was given by James Fey (1985) in the form of a curriculum, called the Computer-Intensive Algebra curriculum (here after referred to as CIA). In it the idea of algebra was radically changed by focussing on the concept of function and variable instead of equation and allowing an algebra system, DERIVE, to perform all symbol manipulations. For the curriculum a special program was developed that could graph functions, display tables of values, compare several tables of values, and perform certain simulations. The idea of the CIA was to teach students to use the available algebra tools as intelligently as possible and therefore to have more time available to study the concepts of algebra, variables and functions, and the modelling of problem situations.

The research reported here concerned itself with determining, whether the introduction of technologies in the algebra curriculum improved conceptual understanding of the concept of variable. To define what it means to understand variables after a first-year algebra course, algebraic activities and the role variables play in those activities were examined. This process resulted in a division of conceptual knowledge of variables into four aspects:

1. knowing the ways that symbols can be used to represent elements of a numerical domain;

2. interpreting variables in different contexts;

3. using variables in modelling problem situations and in translating one representation of a problem to another;

4. using variables in justification and proof.

These four aspects of understanding of variables were studied among students following a traditional algebra course (TA) and the CIA by interviewing six students from each course twice during the year and by giving a diagnostic exam at the end of the year to three CIA and two TA classes.

METHOD

Study Design

At the individual level six students from a TA (named T1 through T6) and six from a CIA class (named C1 through C6) were interviewed twice during the academic year: once half way and once near the end of the year. The interview students were selected from the average-and higher-ability students in each class. Ability was based on the January grade received in Algebra I during the year the research took place. Interviews were standardized, although routes of questioning could vary depending on the responses of the individual students. Interview questions were designed to measure the four aspects of conceptual knowledge of variables as described above. Interview I consisted of 8 questions and interview II of 6 questions respectively. Interviews were tape-recorded and lasted between 30 and 45 minutes. Examples of questions from interview I and II are given in Appendix I.

At the class level the pretest/posttest design was utilised. The pretest was the county wide Criterion Reference Test (CRT). This test measures competence in pre-algebra skills and was given in the first week of class to all ninth-grade algebra students in the county where the research took place. The post-test was given near the end of the school year to five classes: two TA classes having the same teacher (TA) and three CIA classes having three different teachers (CIA1, CIA2, and CIA3). It was specifically developed for this study. The post-test instead of being an achievement test, was diagnostic in character. It consisted of three parts, each having a different question format and probing one of the first three aspects of conceptual knowledge of variables. In this paper the results of the third part of the post-test will be given and discussed. The third part was mainly concerned with students' understanding of modelling and translating.

Post-test Analysis

Part III of the post-test consisted of eight free response questions. Each question was scored separately and partial credit was possible for some problems. Because different problems measured similar aspects of modelling and translating skills, the scores of some questions were combined into one score. This process resulted in four different scores and reduced the possibility that significant statistical differences would be found by chance. The first score, X1, reflected modelling skills. X2 was a score representing the ability to read and interpret tables of values and graphs. X3 gave insight into the students' ability to relate equations with their graphs, their understanding of the solution set of a system of two linear equations, and into the students' idea of the size of the solution set of a system of equations. The last score, X4, tested the students' ability to translate between a table of values and a graph (problem 17c, see Appendix II).

Because the three CIA classes were taught by different teachers and none of the teachers taught both curricula, possible effects of the teachers needed to be taken into account. Therefore a one-way analysis of variance was done with independent variable 'teacher' and dependent variables the four scores X1, X2, X3, X4, and the pre-test score.

RESULTS

The interviews and post-test revealed two different profiles of use and understanding of variables acquired by students in CIA and TA curricula. To reveal some of those differences the results for two of the four aspects of conceptual knowledge of variables will be discussed based on information gained from the interviews and post-test.

Interpreting Variables in Different Contexts

What is a variable? In both interviews the first question asked students "What is a variable?" from several angles by letting them talk about what algebra is, about the use of variables in algebra and in equations, the importance of variables, and how you would tell a friend what a variable is.

In their explanation of what a variable is all TA students mentioned the idea of an unknown number that you have to find. The main idea of variable for TA students was that of a static concept. What was most noticeable in their ideas of variables was that, for none of the TA students had the meaning of variables changed from the first to the second interview. The only change that had taken place was that T1 thought in the second interview that variables stood for large numbers, a view that T6 had expressed in the both interviews.

Another important idea of variable for TA students was that of a place holder. This was expressed by T3, T4, and T5. In answer to the question why variables are important they said: "if you did not have them in an equation or expression you might make a mistake in simplifying or solving." The idea was that "if you don't have them you'll make a mistake." In their opinion they seemed important in the process of solving.

In both interviews, the idea of variable expressed by CIA students was more varied than that of the TA students and made a distinct development from the first to the second interview. For CIA students variables could be input, output, varying quantities, generalisers, and missing numbers. Several CIA students gave practical examples of variables. By the time of the second interview, four of the six CIA students had a concept of variable resembling a generaliser, but at the same time understood variables could be missing numbers. Included in their concept of variable was the idea that two variables could be dependent on each other. The development of their idea of variable can be seen from the responses of one student C2. In the first interview this student mentioned two aspects of variables. First of all a variable was a substitution for a list of numbers, (a varying quantity) and secondly in an equation a variable was a missing number. The variable as a varying quantity she explained as follows:

> It's like a substitution of a number. Like if you have a list of numbers, like say they are from 1 to 10, and put the n and you can pick the number you want to use, and then you get 5n and you have a list of 1 to 10 and you want to pick 2, it would be 5 times 2. And if you didn't want to use all numbers you didn't have to. Or if you only want to use certain numbers at certain times you can do that.

In the second interview she (C2) responded to the question "Why variables are important" as follows:

> You can try all different numbers for variables, and if you just use numbers you have to do the problem all over again. You know with a variable all you have to do is write out that problem once and you can figure it out.

When this excerpt is compared with the response C2 gave in the first interview it shows that by the time of the second interview C2's idea of variable had sharpened. This change in idea of what a variable is from the first to the second interview could be detected in the responses of four of the six CIA students.

Variables in equations. From the interpretations that students gave for the equation $S = 6T$ (problem I6, Appendix I) several observations can be made. Firstly, a coefficient other than 1 in front of a variable seemed to confuse the students into using the variable as a label. If students interpreted variables as labels it was more often T than S.

Secondly, students had varying ideas about the dependency of S and T. Some thought that the two variables were dependent on one another and therefore saw S and T as varying quantities. Others saw only S as varying and thought that the number of teachers was fixed at 25, independent of the number of students. The last category of students thought of both S and T as labels. They were the students that reversed the interpretation of the equation. In the latter case S and T could be dependent on each other, but not always.

Thirdly, filling in numbers for the variables led four students to a correct interpretation of the equation. One student held on to his reversed interpretation after replacing the variables with numbers. For that student substituting a value for one of the variables changed the problem situation.

With respect to the difference in results between the two curricula the following remarks can be made. There were three CIA students who gave a correct interpretation of the equation, two of which needed numbers in order to do that. Only one TA student came to a correct interpretation after substituting numbers. Two CIA students saw S varying independently of T, which they thought to be fixed at 25. None of the TA students thought this way. One CIA student was unclear about his interpretation. Three TA students reversed the interpretation, none of the CIA students did that. One TA student could not give an interpretation because there was not enough information. And one TA student held two very different interpretations depending on whether you used numbers or not.

Variables in Modelling and Translating

Results from interviews. On questions similar to I3 (see Appendix I) where students were asked to model a problem situation, TA students seemed to remain thinking of variables as unknowns. In problem I3, when asked to represent the relationship between two varying quantities of the table of values in the form of an equation or rule, five of the six TA interview students wrote an equation for one instance of the table of values. For example, they wrote: $4x = 16$. Some commented that there was no need to write an equation, because they knew what the relationship was: "it was four times." When specifically asked to use two variables, one for the first and one for the second column, only one TA student succeeded in writing a correct equation. The other four gave a similar equation as before. TA students might not have thought of the first and second column of the table as standing for two varying quantities. Their choice and use of variable was consistent with the definition of variable in their curriculum. All CIA students were able to give the functional relationship for the table of values in problem I3.

Results from post-test. One-way analysis of variance on the pre-test, CRT, with the teachers as independent variable did not indicate any significant differences ($F(3, 97)=1.07$, $p<.37$) between the teachers, neither was there any differences between the X4 scores (ability to translate from table to graph, $F(3, 105)=.36$, $p<.78$). One-way analysis of variance on the other three scores with teachers as independent variable all reached significance (see Table 1 for the mean scores and standard deviations). For X1 the F-value was 2.62, $p<.055$, for X2 the F-value was 7.50, $p<.0001$, and for X3 the F-value was 6.94, $p<0.0003$. The degrees of freedom for all three analyses were 3 by 105. Because the latter three scores were significant, contrasts between the teachers were checked for their significance.

Table 1: Mean and standard deviation (in brackets) of CRT, X1, X2, X3, and X4 for the students of each teacher and the number of students (N) on which the mean and standard deviation are based.

SCORE	CIA1	CIA2	CIA3	TA	TOTAL
CRT	17.04 (5.0)	16.72 (5.4)	19.18 (4.9)	18.10 (4.5)	17.64 (4.9)
X1	1.89 (1.75)	2.69 (2.07)	2.81 (1.83)	1.70 (1.55)	2.18 (1.82)
X2	1.91 (.84)	2.32 (.53)	2.51 (.51)	1.66 (.84)	2.03 (.79)
X3	.13 (.26)	.35 (.34)	.58 (.67)	.14 (.29)	.26 (.42)
X4	.45 (.51)	.46 (.51)	.42 (.51)	.34 (.48)	.41 (.49)
N	29	26	19	35	109

Table 2: *T*-values and probabilities in brackets for contrasts in teachers on the scores X1, X2, and X3. For all *t*-values the degrees of freedom are 105. * indicates that the *t*-value is significant.

contrasts				***t*-values and probabilities**		
CIA1	CIA2	CIA3	TA	X1	X2	X3
-1	-1	-1	3	-2.08* (0.04)	-3.92* (0.000)	-2.62* (0.010)
0	-1	-1	2	2.61* (0.01)	4.60* (0.000)	3.66* (0.000)
-1	-1	2	0	1.11 (0.27)	2.08* (0.04)	3.32* (0.001)
-1	2	-1	0	0.78 (0.44)	0.61 (0.55)	-0.08 (0.93)
2	-1	-1	0	-2.03* (0.05)	-2.93* (0.004)	-3.61* (0.000)

In Table 2 the contrasts, their *t*-values and probabilities, based on the pooled variance estimate, are listed for each of the three scores X1, X2, and X3. Scheffé tests ($p<0.5$) indicated that the TA teacher differed significantly from the CIA2 and CIA3 teachers on the X2 score, and from the CIA3 teacher on the X3 score. CIA1 and CIA3 differed significantly from each other on X3, $p<0.5$. Table 2 further shows that the first two contrasts, were significant for all three scores. These two contrasts measured essentially the differences between the curricula. The last contrast, the difference between the teacher of

CIA1 and the other two CIA teachers, was also significant for all three teachers. Teacher of the CIA3 class taught the curriculum for the third time during the research, CIA1 and CIA2 teachers were teaching the curriculum for the first time. CIA1 teachers was observed to be much less organised than the other two CIA teachers. The TA teacher was considered to be a very good teacher in her school and county.

Conclusion. From the statistical analysis of the third part of the post-test one can draw the conclusion that on problems testing the ability to model problems situations (X1), the ability to read and interpret tables of values and graph (X2), and the ability to relate various aspects of equations with their graphs, CIA students significantly outperformed TA students. Some of the variance was due to teacher influence, but contrasting the curricula proved also significant.

CONCLUSION

The most important findings of this study are the gain four of the six CIA interview students made in their understanding of variables as generalisers and as varying quantities that are dependent on other quantities and the conclusion that CIA students were better able to model problem situations and read tables of values and graphs than TA students. It seems to indicate that technology can provide the opportunity for teachers to teach algebra with more emphasis on conceptual and less on procedural knowledge.

This research did not set out to prove that the TA curriculum was ineffective and it did not show that TA students had gross misunderstandings with respect to the concept of variable. Within the limits of their curriculum, TA students did reasonably well or at least not differently from what has been established in previous research (see, for example Küchemann, 1981). What has to be decided by secondary mathematics teachers and educators is: "What concept of variable do we want algebra students to learn?" the exclusive idea of variable as unknown, or a broader concept of variable including the idea of generaliser, missing number and dependency of varying quantities. The second question would be "What kind of skills do we want algebra students to learn?" the exclusive ability to solve equations or the ability to model problem situations, solve equations related to that model, and to be able to represent and interpret the model in other representations like tables of values and graphs. The CIA curriculum proves to be an option for the latter alternatives.

REFERENCES

Boers, M.A.M. (1990). *Understanding of variables and their uses acquired by students in a traditional and Computer-Intensive Algebra.* Unpublished doctoral dissertation, College Park, U.S.A: University of Maryland.

Fey, J., & Good, R. (1985). Rethinking the sequence and priorities of high school mathematics curricula. In C. Hirsch and M. Zweng (Eds.) *The secondary school mathematics curriculum.* Reston, VA: The National Council of Teachers of Mathematics.

Küchemann, D. (1981). Algebra. In K.M. Hart (Ed.) *Children's understanding of mathematics: 11-16.* London: John Murray Ltd.

Appendix I: Examples of Interview Problems

I3. Suppose a student did an experiment in science to determine the relationship between the number of tomato plants planted and the number of kilograms of tomatoes harvested. The student made the following table of results.

# of Plants	4	5	9	25	11
# of Kilograms of Tomatoes	16	20	36	100	?

(a) How many kilograms of tomatoes do you expect the student will harvest with 11 plants?

(b) What can you say about the relationship between the number of tomato plants and the number of kilograms of tomatoes?

(c) Can you express that relationship using variables?

(d) If the student harvested 24 kilograms of tomatoes, with how many plants did she start out?

I6. The relationship between the number of students and the number of teachers in a particular school is given as:

$$S = 25\,T$$

Where S stands for the number of students and T for the number of teachers. Explain what this relationship means to you?

Questions from interview II.

II1. This year you've taken algebra and you've worked in different ways with variables. Can you give some examples of how you have used variables in algebra this year?

If you had to tell a friend who had never taken algebra what a variable is, what would you tell him/her? How would you explain that?

What can you do in math now that you know about variables? Why do you think variables are important? Why do you think you learned about variables?

Have you any idea how you might use variables outside this class, for example in other school subjects, outside school? How variables might be useful in those situations?

Appendix II: One Question from the Third Part of the Posttest

17. A bicyclist is approaching a hill. When he is at the bottom of the hill, the time (T) is set to zero. As he is going up the hill the distance (D) between him and the bottom of the hill is measured every 10 seconds. Below in the table are the distances for the first ten measurements.

T in seconds	0	10	20	30	40	50	60	70	80	90	100
D in meters	0	100	190	280	360	430	480	520	550	575	590

(a) What is D for T = 50? ____________________

(b) During which 10 second period does the bicyclist cover the largest distance?

____

(c) Which graph below most accurately represents the relationship between time and distance for the
bicyclist? Check one alternative.