CHILDREN'S SOLUTIONS TO PARTITION PROBLEMS

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Children's solutions to ten different multiplication and division word problem structures were analysed at four interview stages in a 2- year longitudinal study (Mulligan, 1992). The study followed 70 children from Year 2 into Year 3, from the time they had received no formal instruction in multiplication or division to the stage where they were being taught basic multiplication facts. A Teaching Experiment that encouraged children to represent a range of multiplication and division situations through language, modelling, drawing, symbolising and reflective writing, was conducted with 10 children in the later part of the Longitudinal study.

This paper reports the findings for the division Partition problems revealing that children very rarely used a sharing one-by-one (dealing) strategy at any stage in the Longitudinal Study or Teaching Experiment. Instead, a variety of counting and grouping strategies such as estimation and grouping, one-to-many correspondence and trial-and error grouping procedures was used. Knowledge of addition facts and skip (multiple) counting assisted children in forming equivalent groups.

INTRODUCTION

Over the past decade, researchers have analysed secondary students' solution processes to multiplication and division word problems based on differences in semantic structure, mathematical structure, size of quantities used, and student's intuitive models (Bell, Fischbein and Greer, 1984; Bell, Greer, Grimison and Mangan, 1989; Brown, 1981; De Corte, Verschaffel and Van Coillie, 1988; Fischbein et al., 1985; Nesher, 1988; Vergnaud, 1983, 1988). More recently, a growing number of studies on young children's solution strategies to multiplication and division problems has emerged (Anghileri, 1989; Boero, Ferrari and Ferrero, 1989; Brekke and Bell, 1992; Kouba, 1989; Mulligan, 1992; Olivier, Murray and Human, 1991; Steffe, 1988). These studies have provided complementary evidence that the semantic structure of the problem, an understanding of the problem context, and the development of counting, grouping and addition strategies may influence solution process.

Earlier studies by Gunderson (1953) on multiplication and division word problems, and by Zweng (1964) on division word problems, provided some evidence of Grade 2 children's responses to partition and quotition problem structures. Australian studies on division (Irons, 1977; Barr, 1980) and multiplication (English, 1982) word problems also focussed on differences in performance between problem structures but did not analyse the complexity of the solution strategies.

Davis and Pitkethly (1990) have investigated the cognitive aspects of pre-schooler's and Year 2 children's sharing and counting activities. The major features of the study suggest that dealing without counting is a widespread sharing strategy but that children needed to count after dealing to check for fair shares. Hunting and Davis(1991) conducted structured interviews with 75 pre-schoolers and sharing by dealing was demonstrated by 85% of the sample but children resorted to counting to check. However, Davis and Hunting (1990) found that with spontaneous sharing tasks in unstructured situations children did not use dealing at all. It seems that counting procedures were used instead of dealing.

The investigation of underlying intuitive models for division has shown that children employ three different models for partitive division: sharing by dealing, sharing by repeated taking away, and, sharing by repeated building-up (Kouba, 1989). It also appears that children may perceive partitive and quotitive division as more related than Fischbein et al.(1985) had suggested.

LONGITUDINAL STUDY

A purpose of this Longitudinal study was to analyse and classify the informal strategies children use in solving multiplication and division word problems (Appendix A). The results indicated that 75% of the children were able to solve the problems using a wide variety of strategies even though they had not received formal instruction in multiplication or division for most of the 2-year period. Performance level generally increased for each interview stage, but few differences were found between multiplication and division problems except for Cartesian and Factor problems. Most compelling was the evidence that the **semantic structure of the problem affected the choice of solution strategy,** more than it affected a change in performance. This was found to be consistent with the pattern of response identified in the Carpenter and Moser (1984) study on addition and subtraction word problems.

Solution strategies were classified for both multiplication and division problems at three levels:

(i) direct modelling with counting;
(ii) no direct modelling, with counting, additive or subtractive strategies; and
(iii) use of known or derived facts (addition, multiplication).

A wide range of counting strategies was classified as counting-all, skip counting and double counting. However the use of sharing one-by-one (dealing) was rarely found. Analysis of intuitive models revealed preference for a repeated addition model for multiplication, and a 'building-up' model for division rather than sharing or repeated subtraction.

PARTITION PROBLEMS

In order to determine whether children were influenced by different linguistic terms in the Partition problems, two different forms were used, with one problem type using the cue "**shared** equally": (a) There are 8 children and 2 tables in the classroom, how many children are seated at each table?; (b) 6 drinks were shared equally between 3 children. How many drinks did they have each? There were some differences found in performance between the two problem types with 75% of children gaining a correct solution for the (a) problem compared with 97% for the (b) problem at the final interview. When larger number combinations were used, again children performed much better on the (b) problem showing 83% correct compared with 55% for the (a) problem.

Table 1: Strategy Use for Partition (a)

Small and Large Number Problems : Interview Stages 1 to 4

STRATEGY TYPE	SMALL NO.				LARGE NO.			
	1	2	3	4	1	2	3	4
Direct Modelling								
Counting-all	6	6	5	2	4	10	6	8
Sharing			2			6	6	6
One-to-many	9				1			
Trial and error					13	13	6	17
Skip counting							3	3
Halving	3	1	2	2				3
No Direct Modelling								
Counting-all			2			1		
One-to-many	1	33	2					
Skip counting						1	2	3
Doubling	1		3					
Repeated subtraction	4	1	3	3				
Halving	34	6	26	38	3		3	6
Known Facts								
Addition	14	10	22	18				
Multiplication	3	3	6	5			2	6
Derived					1			2
Division			2	5				2
TOTAL % CORRECT	66	69	74	75	23	33	29	56

STRATEGY USE

A wide range of strategies was employed to solve the Partition problems and children generally relied on simple additive procedures or matching strategies. Children used one-to-many correspondence as a matching procedure for both Partition (a) and (b) problems. Table 1 compares the range of strategies used with halving shown as predominant at Stage 1, but one-to-many correspondence at Stage 2. Interview Stages 3 and 4 saw a return to the more efficient 'halving' process and the marked decrease in one-to-many correspondence. The use of trial and error grouping was identified for the large number combinations and was consistent with Kouba's (1989) findings for a similar partitive problem. On the other hand, one-by-one sharing was rarely found.

The Partition (b) problem (six drinks shared equally between 3 children. How many drinks each?) varied slightly in semantics from the Partition (a) problems, and a wide range of solution strategies was used. Table 2 shows one-to-many correspondence with and without modelling was preferred because children immediately represented 'six drinks and 3 children' as "two, two, two", "two for one, two for the next, two for the last." Because the number combination was easy to estimate the strategy of skip counting in 2's was also recognised. The large number combination (14 drinks shared between 7 children) was less easy to estimate, but doubling in 7's was used here. The sharing one-by-one was encouraged by the semantics in the problem "share" but the Quotition (b) problem , also using "shared" did not reveal any one-by-one sharing.

Table 2: Strategy Use for Partition (b)

Small and Large Number Problems : Interview Stages 1 to 4

STRATEGY TYPE	SMALL NO.				LARGE NO.			
	1	**2**	**3**	**4**	**1**	**2**	**3**	**4**
Direct Modelling								
Counting-all	1	1	2	3	4	4	8	6
Sharing	3	4		3	4	4	2	6
One-to-many	17	6	5	10	9	10	10	5
Trial and error		1				6		2
Skip counting	1	3	3		1	7	8	5
Repeated addition	1			2				
Doubling						1		
Halving								2
No Direct Modelling								
Counting-all						1		
Sharing		7	5	7	3	1	2	2
One-to-many	26	23	11	26	1	6	2	3
Skip counting	1	23	31	7	3	6	5	6
Repeated addition	4		3	7				
Doubling			3			3	5	
Repeated subtraction					1			
Halving		3	3	10		1	3	12
Known Facts								
Addition	3	3	9		4	7	14	17
Multiplication	3	3	6	15	1	3	6	15
Division			7					3
TOTAL % CORRECT	61	80	81	97	34	64	64	83

INTUITIVE MODELS

Following the analysis of solution strategies, underlying models of multiplication and division were investigated to complement the main findings of the study (Mulligan, 1991a). The underlying intuitive models for multiplication and division were found to be more complex and varied than previously found by Fischbein et al. (1985) with older pupils and were supported by evidence in the Kouba (1989) study on multiplication and division word problems with younger children.

Three underlying models were identified for division with small and large number combinations: sharing one-by-one, 'building-up' (additive), and 'building-down' (subtractive). Analysis of Partition, Quotition and Rate problems across the four interview stages showed a **widespread preference for the 'building-up' model.** This was based on an **underlying notion of addition or additive-type strategies.** The sharing one-by-one model was rarely observed.
Table 3 compares the percentage of correct responses for each intuitive model by problem structure. The three models were found across problem structures (except for sharing one-by-one), and across the four interview stages.

Table 3: Percentage of Correct Responses by Intuitive Model on Division Small No Problem: Interview Stages 1 to 4

PROBLEM STUCTURE		INTERVIEW STAGE 1				INTERVIEW STAGE 2				INTERVIEW STAGE 3				INTERVIEW STAGE 4			
		S	BD	BU	Tot	S	BD	BU	Tot	S	BD	BU	Tot	S	BD	BU	Tot
Part	(a)	0	27	39	66	0	32	37	69	2	31	41	74	0	51	24	75
	(b)	3	11	47	61	3	12	65	80	5	8	68	81	10	29	58	97
Rate	(a)	1	11	39	51	1	7	46	54	2	8	56	66	2	18	65	85
Quot	(a)	0	8	26	34	0	25	35	60	0	15	40	55	0	39	45	84
	(b)	1	9	37	47	0	23	41	64	0	16	53	69	0	26	67	93

S: **Sharing one by one**
BD: **Building down**
BU: **Building up**
Tot: **Total percentage of correct responses for each problem structure**

For the Partition (a) problem, 'building-down' was almost as prevalent as 'building-up' for Interview Stages 1 to 3. At Stage 4, though, the 'building-down' model was preferred; 51% compared with 24% ('building-up'), but this was an exception to the pattern found overall. The 'building-down' model found at Stage 4 could be explained by the use of the known division fact ($8 \div 2 = 4$). The reasonably high percentage of 'building-down' for Partition (a) in Interview Stages 1 to 3 could be attributed to this also. Similarly, the 'building-down' model was markedly increased at Stage 4 for the Partition (b) and both Quotition problems possibly for this reason. Comparison with the large number problems at Stage 4 showed that 'building-up' was preferred because children did not know the division fact ($28 \div 4 = 7$).

THE TEACHING EXPERIMENT

The Teaching Experiment focussed on representing a range of multiplication and division situations through language, modelling, drawing, symbolising and writing (Mulligan, 1991b). Children related their informal strategies to more formal symbolic representations by linking counting and additive recordings to multiplication and division. As well, children were able to find patterns and relationships between problems and devise their own problems showing understanding for the operation involved. Some direct teaching strategies were employed and these were influential in assisting children represent and solve the problems; relating skip counting to the problem situation, and using a hundred square to represent patterns. Further evidence of children's underlying intuitive models for multiplication and division in pictorial and symbolic form were consistent with the results of the Longitudinal study.

The small number Partition problem presented a simple combination (12 children and 6 tables. How many children at each table?), and did not include the term 'share' as it was not found to be influential in the Partition (b) problem in the Longitudinal Study. There was a wide variety of symbolic number sentences and pictorial diagrams represented and this was consistent with the wide range of strategies shown in the Longitudinal Study.

Samantha used the most abstract representation (Figure 1) where the division symbol was correctly used showing a quotient of 2 even though formal division had not been introduced in the classroom. However, another child used $12 - 2 = 6$ to represent the same picture.

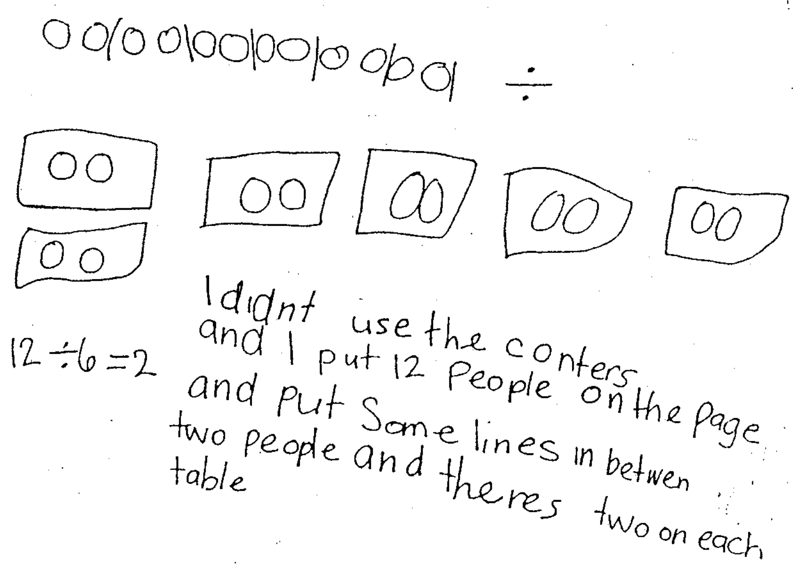

Figure 1

The multiplication fact $6 \times 2 = 12$ was used by four children meaning '6 tables with 2 at each', but the inverse, $2 \times 6 = 12$, meaning '2 children at each, times 6' was also verbalised and recorded. Moreover, repeated addition eg $2 + 2 + 2 + 2 + 2 + 2 = 12$ was found but no sharing one by one was used for the small or large number problem. The $3 \times 8 = 24$ model was most preferred for the large number problem, but in this case the division symbolism was used.

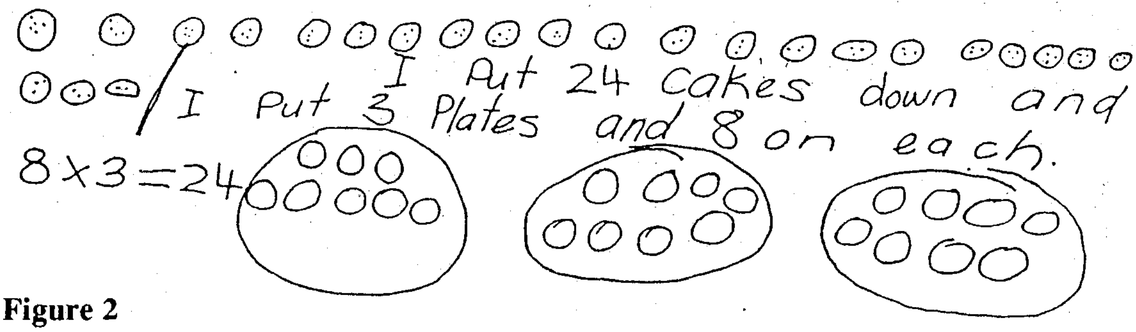

Figure 2

Estimation, one-to-many correspondence and trial and error grouping were shown with a variety of symbolic recordings: 3 x 8 = 24, 8 x 3 = 24, 24 -8 = 3, 24 -3 = 8. Three children avoided symbolising and simply described their picture. 'Building-up' and 'building-down' models were observed with 'building-down' being used where children drew the 24 cakes first. These representations were less sophisticated than the 'building-up' models. Figure 3 shows how Michelle counted-all first, before grouping.

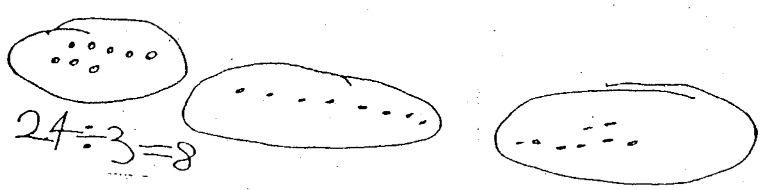

Figure 3

One explanation for the lack of a sharing one-by-one strategy may be attributed to the problem identifying the total dividend to be shared, whereby children did not need to count out the total to be shared. Their understanding of 'fair share' was demonstrated by the formation of equivalent groups and trial and error procedures to ensure that each group had the same amount. This constrasts with evidence from a number of studies with pre-school children showing that sharing by dealing was used widely (Davis and Hunting, 1990; Davis and Pilkethly; 1990; Hunting and Davis, 1991). However, some children who showed immature strategies based on counting-all and modelling also used sharing one-by-one, and by the final interview this strategy had changed to a more efficient one based on estimation of equivalent groups. It appears then, from the analysis of individual profiles that sharing one-by-one was used at early stages and that children progressed to more efficient division strategies with the development of skip (multiple)counting and addition skills.

The most compelling evidence in this analysis, that children used 'building-up' strategies based on addition and counting was not proposed by Fischbein et al. (1985) possibly because children in Grades 5 to 9 had been instructed in traditional models of sharing and repeated subtraction. The analysis of intuitive models has provided more insight into young children's developing notions of multiplication and division, and how these are inextricably linked to addition. Thus, these findings raise serious questions for teaching and learning methods that rely on the sharing and repeated subtraction models for division in the primary school. Furthermore, young children can be encouraged to develop efficient division strategies based on estimation, grouping and counting. The rigidity of traditional partitive and quotitive models needs to be redressed and the development of problem-solving strategies for multiplication and division contexts can be encouraged from the pre-school years.

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APPENDIX A

Multiplication and Division Word Problems

Multiplication	Division
i. Repeated Addition (a) There are 2 tables in the classroom and 4 children are seated at each table. How many children are there altogether? (b) Peter had 2 drinks at lunchtime every day for 3 days. How many drinks did he have altogether? (c) I have three 5c pieces. How much money do I have? **ii. Factor** John has 3 books and Sue has 4 times as many. How many books does Sue have? **iii. Rate** If you need 5c to buy one sticker, how much money do you need to buy 2 stickers? **iv. Cartesian Product** You can buy chicken chips or plain chips in small, medium or large packets. How many different choices (of packets) can you make? **v. Array** There are 4 lines of children with 3 children in each line. How many children are there altogether?	**i. Partition (Sharing)** (a) There are 8 children and 2 tables in the classroom. How many children are seated at each table? (b) 6 drinks were shared equally between 3 children. How many drinks did they have each? **ii. Factor** Simone has 9 books and this is 3 times as many as Lisa. How many books does Lisa have? **iii. Rate** Peter bought 4 lollies with 20c. If each lolly cost the same price, how much did one lolly cost? How much did 2 lollies cost? **iv. Quotition** (a) There are 16 children, and 2 children are seated at each table. How many tables are there? (b) 12 toys are shared equally between the children. If they each had 3 toys, how many children were there? **v. Sub-division** I have 3 apples to be shared evenly between 6 people. How much apple will each person get?