INTEGRATING COMPUTERS INTO THE TEACHING OF CALCULUS: DIFFERENTIATING STUDENT NEEDS

JULIE RYAN
Institute of Education, University of Melbourne

The study undertaken here looked at difficulties associated with the first principles approach to the derivative of a function and concentrated in particular on the first five lessons in calculus as experienced by a typical group of nineteen year 10 students who were preparing to take calculus at year 11. A traditional teaching approach was contrasted with an alternative computer teaching approach and both approaches were analysed for success in terms of conceptual understanding, skill acquisition and student perceptions of whether the work was easy to understand. The traditional first principles approach was found to be too cognitively demanding for the students who demonstrated a 'rush to rule' for meaning. Students undergoing the computer treatment also demonstrated this rush to rule and therefore very gradual development is recommended for students in their first encounters with calculus.

CALCULUS AS A GATE-KEEPER IN MATHEMATICS

Calculus continues to be a significant component of senior secondary school mathematics and is still most often recommended as study for those students who wish to gain entry to a university mathematics course. The presentation of the early lessons of calculus appears to have changed little throughout this century but there is now a radical re-think with regard to how students should approach the ideas of calculus.

Since the intrinsic difficulties of calculus are usually part of the first encounter for the beginning student, it is no wonder that students (even those who have prided themselves on being on top of the subject) are somewhat bewildered by the intricacies of the underlying concepts:

> There is no part of mathematics for which the methods of approach and development are more important than the calculus, partly on account of the novelty of notation, but chiefly on account of intrinsic difficulties. These occur at the start, or more acutely at the start than at any later stage. (Mathematical Association, 1951, p.4)

Other mathematics educators too have exhorted teachers to take time with the early concepts (Austin, 1982; Orton, 1985). However, teachers and textbook writers have continued to introduce differential calculus to students with scant or token reference to underlying concepts followed by a relatively fast move to concentrating on facility with the processes (Ryan, 1990).

As some believe that calculus is "an obvious dividing line between those who claim to know a little about mathematics and those who do not" (Graham, Read, and King, 1973) and, as it is a significant or high status component of school mathematics, it can provide the final closed gate to the potential success and commitment of many secondary school students to mathematics. Because it "is a significant barrier on the road to a professional career" (Peterson, 1987), mathematics educators need to look closely at the intrinsic difficulties and the teaching methodology associated with beginning calculus. Allowing more time for gradual development may also have long term dividends for both student performance and satisfaction. Rather than a barrier for the majority of our students, calculus may prove to offer an exciting new direction in their mathematical growth - the stage where they are challenged to think in "new and different ways" (Orton, 1986).

To improve the presentation of calculus is a challenge, too, to mathematics educators - the potential of new technology offers the chance to radically re-design the introductory stages as well as expand the applications. We now have computer software to re-direct our approach so that, potentially, more students can gain greater understanding and confidence with it and not believe that this is where their mathematical education must end.

USING THE COMPUTER TO DEVELOP CONCEPTUAL UNDERSTANDING

Use of computer technology is certainly fashionable in mathematics education at the present time. For many educators the computer appears to offer a more interactive, exciting, real-world and individual learning format for students. In mathematics, the promise of greater access to the 'treasures' for more students who have historically rejected mathematics, through boredom or failure, is just too great to ignore. For the classroom teacher, with limited entry time to new technology, the challenge can sometimes only be taken up as an add-on to current practice; former teaching practice is not necessarily altered in the light of the new tool and students' preferred style is rarely matched with the new technology. Resistance by either student or teacher to new technology may not be a problem here - the type of interaction required by the 'best' software needs to be studied. I suspect that the whiz-bang technology can blur the underlying ideas just as much as the formal chalk-and-talk delivery if some respect for time for reflection and questioning is not built into the learning process.

New interactive computer packages. David Tall's *A Graphic Approach to the Calculus* was designed to "offer student and teacher alike the possibility to visualise and experiment with the fundamental concepts of calculus" (Tall, 1991, p.3). He believes that the traditional approach, where elementary ideas are presented first, actually sets up later cognitive difficulties for the student. He advocates a curriculum approach that allows the student to experience "the powerful ideas of the calculus from the beginning" (Tall, 1991, p.44).

His package, used in the study reported here, includes the MAGNIFY program (Figure 1) and the GRADIENT program (Figure 2). MAGNIFY allows students to investigate the *local straightness* of graphs of functions "thus the gradient of the graph can be given an immediate global impression, with the limiting processes implicitly contained in the magnification idea. No explicit discussion of limits is necessary" (Tall, 1991, p.48). The notion of the gradient of the tangent to a curve being a measure of the gradient of a curve may then be established in an intuitive sense initially, in preparation for the formal ideas of later analytical approaches.

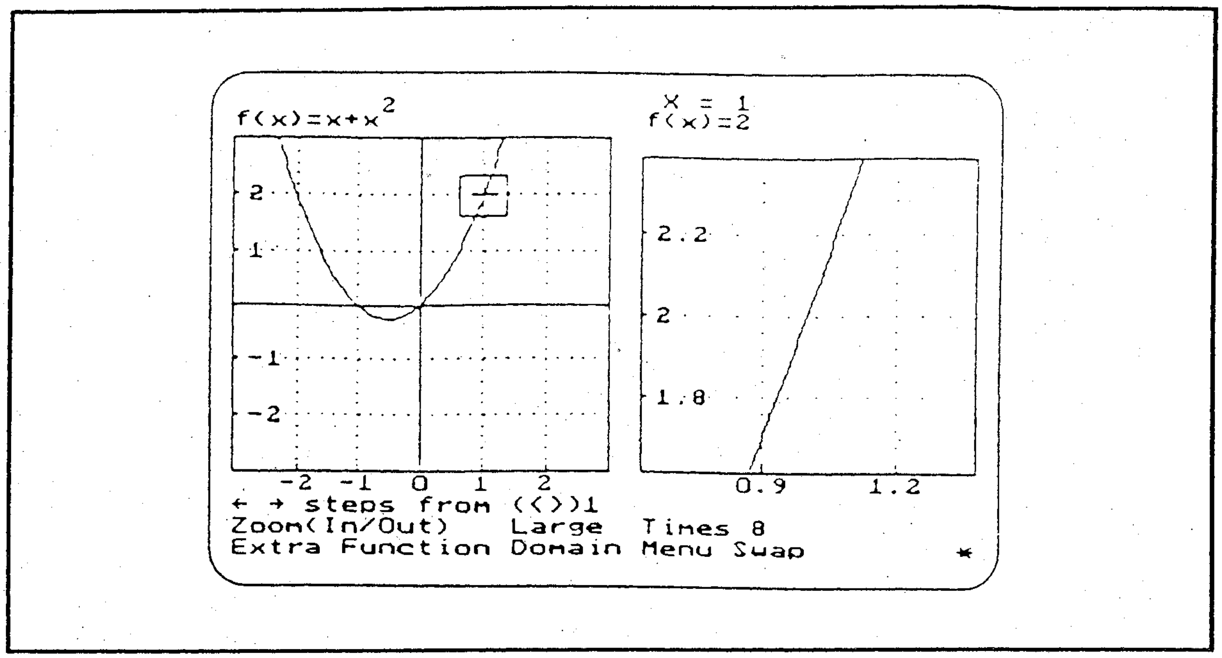

Figure 1: The Magnify program

The GRADIENT program (figure 2) point-plots the approximate *gradient function* and requires the student to name the derived function. GRADIENT also allows the student to investigate the limiting process at a point on the graph.

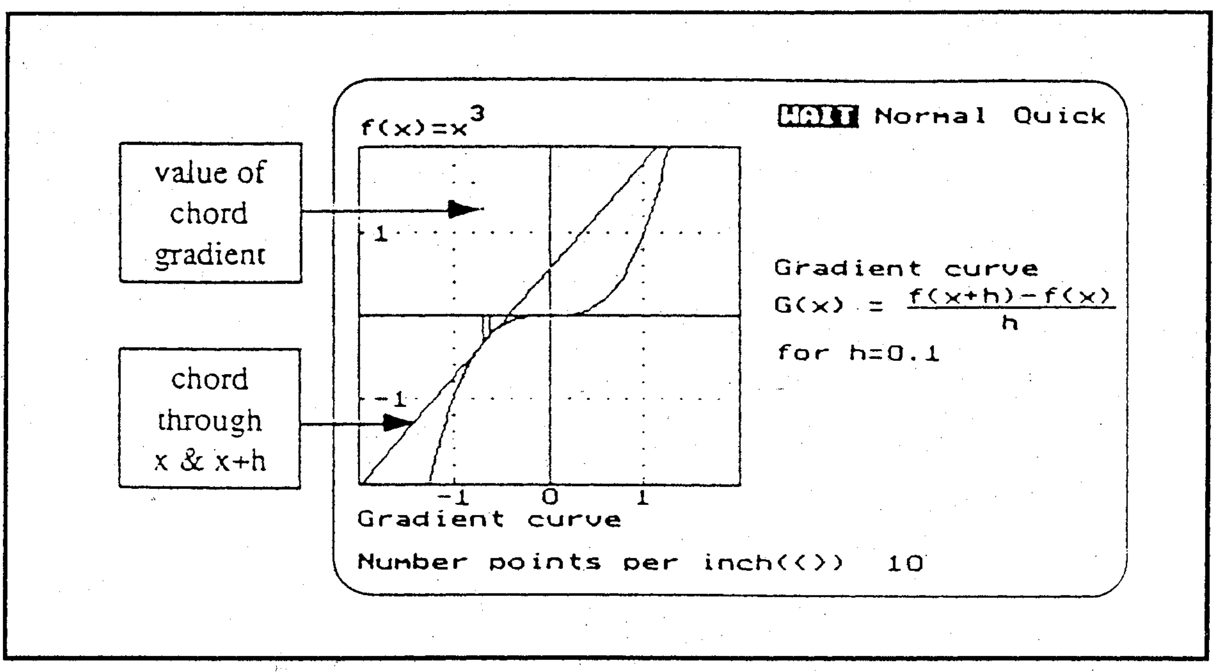

Figure 2: The Gradient program.

The exploratory nature of the activity is the key feature of Tall's *A Graphic Approach to the Calculus* programs. For students who have used the programs the underlying ideas of calculus are discovered quite intuitively and rules eventually developed *as the student is ready to organise them.*

THE STUDY

The study undertaken looked at difficulties associated with the first principles approach to the derivative of a function and concentrated in particular on the beginning lessons in calculus as experienced by a typical group of year 10 students who were preparing to take calculus at year 11. A traditional teaching approach was contrasted with an alternative computer teaching approach using Tall's package and both approaches were analysed for success in terms of conceptual understanding, skill acquisition and student perceptions of whether or not the work was easy to understand. As calculus has served as a critical filter for further study in mathematics, teaching methodology and student attitudes to the topic were also a focus of the study.

The study undertaken was in the form of a case study of the event "the first lessons in calculus" and used information gathered over a seven-day period encompassing a pre-test, a class lesson sequence of five lessons using two different teaching styles for contrast and a post-test. As the study aimed to focus in detail on students' early encounters with calculus, the first five class lessons were studied closely, both for what was learned and what the students believed they learned, under different learning and teaching styles. The two styles of teaching were contrasted for similarities and differences in student learning.

It is in the first few lessons that most teachers hope to establish a conceptual base for what comes later in differential calculus. It is already known that rule acquisition in calculus is the leading long-term outcome for most student learning, and that conceptual understanding is most often a casualty. Hence, the short-term concentration on the crucial first five lessons aimed to establish the source of early specific misconceptions and difficulties or the grounds for what appears to be 'concept avoidance' in students taking calculus for the first time.

The class: At the end of the academic year a class of twenty-five Year 10 students in a co-educational secondary school was divided into two groups, matched on the basis of their teacher's experience of their overall mathematical ability and test results throughout the year. Friendship groups or pairs were maintained to allow for mutual support in the studied groups, to maximise discussion and to minimise student anxiety. This class had never studied calculus, but all but one of the students hoped to go on to take the Year 11 mathematics subject "Change and Approximation" in the following year where calculus was a key component of the subject. They were seen to be a typical pre-calculus class with the normal mathematics preparation to that stage and were 'ripe' for the introduction of calculus.

SCOPE OF THE STUDY AND SOURCES OF DATA

The two groups formed were treated to a different style of class teaching over the five class lessons: traditional or non-traditional teaching using computer software.

The traditional group. The class teacher who had over twenty years' teaching experience undertook her teaching in what will be called *the traditional style.* This matched her teaching of introductory calculus in Year 11 up until this time. Her method can be characterised as a formal approach which was skills-based even though attempts were made to explain the underlying ideas to students. The teacher was at pains to resist newer techniques being presented to the second group even though she had recently become aware of them and hoped to vary her teaching in the future as a consequence. She was recognised in her school by students and colleagues as a leading and successful mathematics teacher. This presentation was therefore seen to be an example of 'best' or expert teaching in the traditional style.

The computer group. The second group was taught by the writer in the same class period in what will be called *the computer style* making use of Tall's *A Graphic Approach to the Calculus* for the IBM (Tall, 1991) in the school's computer laboratory. The method of teaching used here purported to establish a strong *intuitive* base for some of the basic concepts of differential calculus and incorporated visual learning and discussion leading to the students' establishing a rule for the derivative of a polynomial function. The method was underscored by an informal numerical approach and formal definitions were not used at any stage. The students were unfamiliar to the writer but appeared relaxed about taking part in the study - they had used the IBM computer to a limited degree in their Year 10 mathematics class. The methods used here had been trialled by the writer with groups of students at the tertiary-intake level and the writer was familiar with the materials used. This presentation was therefore seen to be an example of 'best' or expert teaching in the non-traditional style.

Students in the groups. Nineteen students attended seven class sessions for this study - nine students were in the traditional group and ten students were in the computer group.

Group	Students
Traditional (T)	9
Computer (C)	10

Table 1. Number of students in the study by treatment group.

Even though the class was the streamed 'top' group in the school, both groups formed had a similar number of upper (U), middle (M), and lower (L) ability students as classified by their teacher. The attempt to have similar groups from the same class subjected to the two different treatments allowed for reasonable similarity of background experience and range of student ability. Pre-test items verified these similarities.

Scope of the teaching. Both teachers initially came to agreement on the end-point of their lessons: the students were to be able to differentiate simple polynomial functions of the form $f(x) = ax^n$ by rule. Student understanding of the following concepts was to be established (differently) by both methodologies:

- *gradient of a curve at a point*
- *tangent to a curve at a point*
- *derivative function*
- *derivative at a point*

The traditional teaching would state and use the limit concept and limit notation explicitly and would present the formal definition of the first derivative of a function but the non-traditional computer teaching would use the limit concept intuitively or informally only and no definitions would be presented. Necessary notation (in particular, function notation) would be introduced formally to the traditional group but introduced only informally or incidentally through usage in the computer programs to the computer group.

Instrumentation. A pre-test questionnaire was administered to all nineteen subjects prior to the five class lessons. Records were kept by each student of any written class work in an exercise booklet provided to each student together with all distributed class material. At the end of each lesson the students were asked to fill out a short questionnaire which sought their reflections on the lesson. Both teachers' lesson notes were kept. A post-test was administered the day after the five class lessons.

Pre-test questionnaire (Q1). Items on this questionnaire had been trialled with groups of students at Years 10, 11 and the pre-tertiary level and included questions which sought to uncover students' notions of measures of curves. The results from this piloting showed some consistency between the three groups and with Tall's (1985) results which demonstrated students' difficulties with tangent to a curve. Another item from Hart (1981, p.130) was included to test the students' understanding of the gradient of a straight line - this item was also used to validate the matching of the two groups.

Teachers' notes. The notes of the teachers were kept to check what was taught in order to match student notes and to assist in the description of the methods used.

Student class notes. All notes taken by students and work undertaken in a written form were collected for verification of what was taught and for a search for possible sources of error either in the misreading of teacher notes or of consistent errors in set-work. Students were also encouraged to jot down in their booklets, during the lesson, notes on anything which was troubling them - this was to assist them in their reflections at the end of the lesson.

Reflections on the lesson. At the end of each lesson the students in both groups were given five minutes to fill in a short form asking them to describe what had happened in the lesson, what they found hard to understand, and what they found easy to understand. These items were used for comparison with what the teacher had meant to deliver and to trace student confidence and belief in their own progress.

Post-test questionnaire (Q2). The items on the questionnaire administered after the lesson sequence included questions:

- directly testing acquisition of the rule for the differentiation of $f(x) = ax^n$ with verbalisation used as a check of the rule; (skill)
- relating to ability to transfer the rule beyond their lesson experience to date; (transfer)
- seeking explanations for the meaning of the gradient of a curve; (concept)
- directly testing global understanding of derivative of a function; (concept)
- comparable to items on the pre-test to check for growth of understanding; (growth)
- seeking perceptions of the students relating to what was understood or not understood; (beliefs)

The information from the post-test was used to establish differences and similarities between both groups.

Limitations. A learning factor beyond the scope of this study relates to student pre-disposition to teaching or learning style. It was hoped that the group division would be random in this regard and match usual class distribution.

It may be that some students prefer a visual presentation rather than an analytical presentation and that this could be a key factor in effective individual student learning but, it was felt that as variety of teaching methods is currently promoted in whole-class teaching, and that it is not easy to access student-preferred learning styles or to match them to teaching style in normal practice in schools, it would be reasonable to ignore the factor in this study.

A recent study on visual versus analytical thinking favours "developing every topic with its analytical *as well as* with its visual aspects thus allowing each student to grasp the material in the way which is closer to his cognitive orientation" (Eisenberg and Dreyfus, 1986, p. 158) (italics added).

The possible novelty factor of the non-traditional teaching using computer software was disregarded as it was believed that the students were reasonably familiar and adept at using the particular computers and were in a familiar classroom setting.

FINDINGS AND DISCUSSION

Subtleties of the activities. There was evidence that the case study students in the computer group (C) were missing the subtleties of the activities. Use of different technology may require the teaching process as well as the learning process to be changed.

Teacher's aim (Computer group, lesson 2): to use the magnifying option of *A Graphic Approach to the Calculus* to demonstrate that for points on the graph of a polynomial function the curve approximates a straight line as we zoom in closer and closer to the point.	
S1(U,C):	On the computer we put in a function and we magnified a point so we could *see the point more easily.* (Italics added)
S3(U,C):	We used the computer to magnify places on the computer to *find a curved line.* (Italics added)

Student number one (S1), from the upper ability group (U) of the computer treatment (C), while wanting to "see the point", has not picked up that the neighboring 'straightness' of the curve around the point is the focus here. Student number three, S3, also from the upper ability group, wanting "to find a curved line", has not picked up the local straightness theme at all.

Other students in the group failed to mention the magnifying at all, whereas other students (from all ability groups, U,M,L) reported differently.

S5(U,C):	Magnify shows where the line is straight.
S8(M,C):	The magnifying was seeing which part of the graph was straight.
S10(L,C):	Magnifying, magnifies the line where it goes straight.

It may be that some of the upper ability students are operating on a more complex level and are actively seeking understanding rather than just picking up on key instructions.

Monitoring their own learning. There was also an indication that some students in the upper ability group (U) of students were able to identify when they were having a learning difficulty. At the same time, the middle and lower ability students (M, L) did not report any problem even though their descriptions of the lesson content did not indicate that they had taken in the lesson ideas fully.

Activity (Computer group, lesson 3):	determining gradient at (1,1) on $y = x^2$ using P(1,1) and successive Q(2,...), Q(1.9,...) ....Q(1.001,...).
Question:	What was hard to understand?
S2(U,C):	The tangent lines and what P and Q find.
S10(L,C):	Nothing, I found it easy.

(Traditional group, lesson 1):	limits of discontinuous functions.
Question:	What was hard to understand?
S3(U,T):	Undefined problems e.g. 1/0. How do we know when an equation is undefined?
S6(M,T):	Undefined problems e.g.1/0. I thought it would equal = 1. Limits were easy.
S9(L,T):	I found the limits easy.

Short-cuts: Meaning vs. formula. There was almost relief, as students recognised that they were struggling with ideas, when a rule or "short-cut" was provided.

Activity (Traditional lesson 3):	First principles derivation leading to the pattern for the derivative of a polynomial function.
S1(U,T):	I found the long way difficult to understand. The short cut (was easy).
S6(M,T):	Trying to remember the formula the long way (was hard). The quick way (was easy).

Most students found this lesson hard to understand. The complexity of "the formula" appears to have been a problem.

S2(U,T):	I didn't understand any of it really.
S3(U,T):	How to find the gradient of a curve itself (was hard to understand).
S1(U,T), S4(U,T), S5(U,T), S6(M,T), S9(L,T):	The formula m = f(x+h) - f(x)/ h (was hard).

This was the pivotal lesson in the sequence where the first principles derivation was meant to give meaning to the gradient of a curve at a point. Only S1 appears to have picked up some ideas here but she too reported that

S1(U,T):	*Getting used to* the formula ... (was difficult). (Italics added)

Use of rule. Skill acquisition in taking the derivative of a simple polynomial function was not markedly adversely affected by the computer treatment. While successful use of rule is eventually a priority it was surprising to see *how quickly* it had a tendency to take over as the meaning for what was being done.

Global understanding of derivative. The computer group only marginally outperformed the traditional group in recognising the nature of the gradient (positive, zero or negative) of a curve along its 'track'. This was surprising given that the major focus of the treatment was on f ' (x) being a *function.* The benefits might be longer-term, however.

SUMMARY AND RECOMMENDATIONS

Limited concept image for gradient. It was found that most students in the study had a limited concept image for gradient ("measure rise over run") and it is recommended that the development of the global ideas associated with the gradient of a straight line be a focus of learning *before* the idea of gradient of a curve is introduced in beginning calculus. Some students are still indicating fundamental problems with slope as a rate of change and are keying into the x-axis rather than the nature of the slope to state whether it is positive, zero or negative.

Tangent to a curve. It was found that the use of a tangent to a curve at a point to measure gradient of the curve was not a spontaneous intuition and it is recommended that more time be given to this notion in the first principles approach to differentiation. The traditional first principles approach was found to be too cognitively demanding for the students who demonstrated a 'rush to the rule' for meaning. Students undergoing the computer treatment also demonstrated this 'rush to rule'. It appears that it is very easy to foster instrumental understanding even with the use of dynamic and self-directed software where relational understanding is the teaching goal. The question is whether or not this suits the needs of some students; perhaps, *for some reason,* some students prefer to be able to *do something* early on rather than spend time on exploration. Decisions for teaching need to be well informed about student needs if new methodology is to be properly exploited.

In this study of student experiences in their first few days of calculus it appears that there are considerable learning difficulties in both modes of learning and that the introduction of "the rule" too soon may in fact be distracting the students from the underlying processes *even at this early stage.* Rather than the concepts being left behind, as facility with rule improves, perhaps the concepts are never taken on board at all. For many students the ability to use the rule appears to have offset their early conceptual struggles and the rule becoming the meaning (what you do is what it means) just takes over.

The computer group appears to have reached a reasonably similar position in terms of skill acquisition to the traditional group with less feeling of finding the work hard to understand - if anxiety can be reduced then future work in calculus may be more accessible.

REFERENCES

Austin, H.W. (1982). Calculus and its teaching: an accumulation effect. *International Journal of Mathematical Education in Science and Technology.* 13 (5), pp. 573-578.

Eisenberg, T. and Dreyfus, T. (1986). On visual versus analytical thinking in mathematics. In *Proceedings of the Tenth International Conference on the Psychology of Mathematics Education* (153-158). England: University of London Institute of Education.

Graham, A., Read, G.A., & King, V. (1973). Introducing differential calculus. *International Journal of Mathematical Education in Science and Technology.* 4, pp. 149-159.

Hart, K. (Ed.) (1981). *Children's understanding of mathematics 11-16.* London: John Murray.

Mathematical Association (1951). *The teaching of calculus in schools.* London: Bell & Sons.

Orton, A. (1985). When should we teach calculus? *Mathematics in School,* 14 (2), pp. 11 - 15.

Orton, A. (1986). Introducing calculus: an accumulation of teaching ideas? *International Journal of Mathematical Education in Science and Technology.* 17 (6), pp. 659-668.

Peterson, I. (1987). Calculus reform: Catching the wave? *Science News,* 132 (20), p. 317.

Ryan, J. (1990). *Students' early understandings in calculus: Does first principles come first?* Paper presented at the 13th Conference of the Mathematics Education Research Group of Australasia, University of Tasmania, Hobart.

Tall, D. (1985). Understanding the calculus. *Mathematics Teaching,* 110, pp. 49 - 53.

Tall, D. (1989). Concept images, generic organisers, computers, and curriculum change. *For the Learning of Mathematics,* 9 (3), pp. 37-42.

Tall, D. (1991). *A graphic approach to the calculus.* (Computer program for IBM compatibles). NY: Sunburst Communications Inc.