AN ANALYSIS OF CHILDREN'S UNDERSTANDING OF NUMERATION

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In this paper I report on a pilot study which has been carried out to investigate children's understanding of numeration. In particular I am interested in exploring how young children develop an understanding of the structure of ones, tens, hundreds, ... that is embodied in the mental, verbal and written notions of the number system that we use.

What is the relationship between a child's:

i) understanding and use of equivalent groups in counting;
ii) understanding of the process of regrouping in representations of number; and
iii) understanding of place value and the structure of the numeration system.

When and how does a child begin to understand the structure of the number system? How is structural flexibility developed so as to operate meaningfully with the number system (numeration sense)?

BACKGROUND

Increased conceptual understanding of numbers and facility with their manipulation which leads to flexibility in dealing with number, or in other words, the development of number sense has been advocated in recent reports and curriculum documents here and overseas (Cockcroft, 1986; NSW Department of Education, 1989; Australian Education Council, 1990; NCTM, 1989; National Research Council, 1989). At the same time these reports are suggesting a reduced emphasis on written algorithms for the four operations. This raises the question of how should we be planning school experiences which help children construct a system of numeration for themselves which involves 'number sense' and relates directly to their real life (and intuitive) experiences with number.

Constructivism has given us a philosophical rationale to support the more interactive, child-centered approach that many of us have been advocating through our professional development work. This theoretical position emphasises that children's mathematical understandings are constructed through their interactions with mathematical contexts and environments. Davis (1990) argues that constructivism has given us a 'formalized conceptualization' of the process of learning mathematics which could now be used to rethink the analysis of some earlier curriculum improvement projects.

Recent research (Bednarz and Janvier, 1988; Cobb and Wheatley, 1988; Denvir and Brown, 1986 a; b; Fuson, 1990; Kamii, 1985; 1986; 1989; Labinowicz, 1985; Ross, 1989) have contributed to our understanding of how children construct the numeration system for themselves so that it can become a tool for calculating. Fuson (1990) argues that present US textbook presentations of place value depend primarily on a skills analysis approach which contributes to the failure of children to build adequate multi-unit conceptual structures and so *multidigit addition and subtraction are learned as procedures carried out on columns of single digits, and meanings other than single-digit meanings are not constructed or are not accessed.*

(Fuson, 1990, p.274)

Ross (1989) suggests that teaching place value concepts separately as a prerequisite to algorithmic work, is ineffective if understanding is the goal. Although careful instruction with concrete material can facilitate the acquisition of the procedural knowledge to carry out computational algorithms, many children do not subsequently appear to develop relational understanding of the conventional algorithms for addition and subtraction of whole numbers. How do we get more children to demonstrate flexibility in partitioning numbers so that they might develop this relational understanding? Ross advocates a child-centered period of inventing procedures for solving two-digit number addition problems requiring renaming. This requires the development of a sense of number which includes the notion of the part-whole relationship and the idea that a number may have many names. Some experimental instructional studies (Kamii and Joseph, 1988; Cobb and Merkel, 1989) have shown that children, as young as first and second grade level can invent their own efficient algorithms.

Kamii (1985) argues that premature instruction in place value is detrimental to children making sense of numeration.

Given what we know about the developmental course of children's thinking, we ought to ask ourselves whether it would not be wiser to delay place value instruction until children have solidly constructed the number series (by repetition of the +1 operation) and can partition wholes in many different ways (part-whole relationships).

(Kamii, C., 1985, p63)

Young children first generate the numerals by recognising the pattern of using digits in their construction. They also learn through social transmission that the counting words for the teen numbers are not written as they sound. This does not involve an understanding of place value. Place value requires an understanding and integration of both the irregularly value-named system of number words and the positional base ten system for forming numerals. The base ten place value numeral system requires the mental construction of one group of ten out of ten ones and then the representation by a digit in the "tens place". This involves the construction of a 'second level' to the number system. This second level also involves the idea of multiplication as the groups of ten become the new units. Kamii (1985) explains that six and seven year old children are still in the process of constructing the number system with the relation "one more" and so it might be inferred that they are not ready to fully understand the symbolic notation of two-digit numbers. She suggests it is not possible for a child to construct the second level while the first level of the number system is still being built.

Hiebert and Wearne (1992) report on a numeration teaching programme which they developed for year 1 children. This 'conceptually based instruction' programme aims to help children construct connections between representations of number (physical, pictorial, verbal and symbolic) and use all of these representations with recording actions on number. Different forms of representation of quantity highlight different aspects of the structure of number (eg grouping with physical and pictorial models, place value with symbolic models). Understanding numeration involves *building connections between the key ideas of place value, such as quantifying sets of objects by grouping by 10, treating the groups as units... and using the structure of the written notation to capture the information about groupings.*

(Hiebert and Wearne, 1992, p.99)

Davis (1990) raises the issue of using other bases through his reference to the following problem which was given to children in the Madison Project.

Two small groups of children are asked to communicate messages back and forth, but they must pretend that nobody can count above three. One group of children - at the front of the room, say - is then given a pile of tongue depressors (let's say that you and I know that there are twenty-two tongue depressors in the pile). The children at the front must send messages to the other group of children (at the back of the room) so that the second group can assemble the same number of tongue depressors.

(Davis, 1990, p.97)

He says this is an example of focusing instruction on the basic task, and leaving it up to the children to invent a way to solve the problem. The construction of a recording system in this problem parallels the invention of place value numerals as an elegant solution to the problem of of how to record the very large numbers.

METHOD

The pilot study consisted of the intensive clinical interviewing of a cross-sectional sample of 40 children and took place in June, 1992. Four children were interviewed from each of the years Kindergarten to 4 in two country NSW schools. The children were selected by the class teacher on the basis of being representative of the spread of achievement levels in mathematics in the class (one chosen as a low achiever, two middle ranking achievers and one as a higher achiever). All interviews used the same tasks, asking the same initial follow-up questions. The follow-up questions were intended to elicit children's explanations about the strategies they used. Further questions depended on the responses. The thinking of children was probed until the strategy being used was clear or until it was obvious that no further explanation would be forthcoming. Children's explanations and visible strategies were recorded on an interview form. All interviews were audio taped and this used, when needed, to complete descriptions of responses. This cross-sectional analysis was carried out with children from 5 to 10 years of age. It must be remembered when carrying out any analysis that each child constructs whatever concepts he/she has on the basis of his/her experiences and so any model of a child's thinking must take these experiences into consideration.

INTERVIEW TASKS

There were four levels of tasks used in the pilot study. These levels reflect the increasingly complex nature of the number system as it is used to quantify larger and larger collections of objects and to record (orally and symbolically) the operations applied to the numbers. The levels of tasks are: counting; grouping; regrouping; and extended structure.

The interviews began with the counting tasks. The missing addend and removed item tasks (Steffe and Cobb, 1988; Wright, 1991) were used to determine whether the children could count the counting acts themselves, whether they were abstract counters. Other counting skills assessed were multiple counting and double counting.

As grouping is a operation which is central to the development of an understanding of numeration, tasks were used to determine the intuitive use children make of grouping in a number of situations. Partitioning problems were used to assess whether equivalent grouping is used in the sharing process, whether a collection is partitioned into equivalent groups by one-to-one or many-to-one correspondence. The problems involved situations with remainders (smaller numbers) and without remainders (larger numbers) in order to explore whether equivalence was established in these different situations. A counting task which assessed whether groups of ten objects were treated as units involved packs of lifesavers and individual lifesavers. The children established that there were ten lollies in each roll by counting the contents of transparently wrapped rolls and were then asked, in turn to quantify collections of fifteen and forty-three lifesavers consisting of standard arrangements of pregrouped rolls and individual lollies. Another counting task showed children an array (ten by six) of pictures of planes and asked how many planes there were altogether. An uncovering tens task (Cobb and Wheatley, 1988) was used to assess whether children coordinate counting by tens and ones. The relationship of grouping to recording with numerals was investigated with digit correspondence tasks (Ross, 1990). Another task investigated the children's use of the pattern of tens to locate numbers on the hundred square. A further task involved asking the children to close their eyes and to imagine the numbers from 1 to 100. This task was carried out earlier in the interview so that responses could not be influenced by experiences with other tasks. It was intended to assess the structure (or lack of structure) of the mental image that children have of these numbers.

The operation of regrouping was explored through several tasks. These tasks involved addition with pregrouped material, the possible use of the part-whole relationship, the regrouping of concrete representations of number to non-standard forms (more than nine individual objects in representation) and the renaming of numbers with equivalent symbolic names.

The extended structure of number was explored through tasks which extended the use of groups, to groups of groups, and the relationship to the system of ones, tens and hundreds. In a counting task the children were shown two bags, one roll and four individual lifesavers where all the wrappings were transparent. The roll of lifesavers contained 10 lollies and each bag contained 10 rolls. The children were asked to find how many lollies there were altogether. A similar task involved three bags, twelve rolls and five individual lifesavers (non-standard representation). Another task initially assessed whether children spontaneously used grouping in tens and hundreds to count large collections (pictorial presentation). This task further probed how the collection could be presented in order to facilitate the count and then assessed how the same collection, with the grouping of ten groupings of ten shown by circling, was interpreted (Bednarz and Janvier, 1988). In a further counting task, children were asked to find how many dots there were in an array of 10 000 (100 by 100), with spacing which separated blocks of one hundred and then of one thousand dots. Groups of five unifix cubes (towers) were used to explore children's generalization of the base ten system to a similar system based on a grouping number of five.

RESULTS

Missing addend task. Assess whether the counting acts themselves are counted - **abstract counting.**

Display 8 (5) shells.

How many shells are there here?

Place out 4 (3) shells which are screened from view.

There are 12 (8) shells altogether. How many are hidden?

Response Categories

Small numbers Large numbers

Year	1	2	3	4	5	6	7	8	Abstract counters
K	8								0
1	3	2	2	1	1	3		1	5
2	1				4	2		1	7
3					1	4	1	2	8
4					2			6	8

Response Descriptions:

1. Guess or no answer.
2. Counts-on using fingers (3).
3. Counts-on mentally (3).
4. Immediate knows fact (3).
5. Counts-on using fingers (4).
6. Counts-on mentally (4).
7. Counts backwards mentally (4).
8. Immediate knows fact (4).

Response examples:

Year 1

T. How many are hidden?

J. 4

T. How did you get that?

J. Because 8 plus 4 is 12.....would have been able to count on my fingers.

Year 1

H. Because 6 and 6 equals 12.....eight, nine, ten, eleven, twelve......so it is 4.

Year 2

R. Start from 9 and go 10, 11, 12.

Year 2

J. You have 8 there, and you break it up into two groups of four......another group of four makes twelve.

Year 3

S. If you have 8 out here, then you have to have 4 under there or it won't be 12.

Removed item task, assess whether the counting acts themselves are counted - **abstract counting.**
Assess use of ten.
Display a collection of ten counters.
How many shells are there here?
Hide 3 shells.
How many shells are under my hand?
Can you give me any other numbers that add to give 10?

Response categories
Removed item task Ten facts

Year	1	2	3	4	5	6
K	8					
1	3	3		2	2	
2	1	1	5	1	4	2
3			4	4	3	4
4			3	5	3	5

Response Descriptions:
1. Guess or no answer.
2. Counts-on using fingers (3).
3. Counts-on mentally (3).
4. Knows answer (3).
5. Calculates at least two further combinations which add to give ten.
6. Recalls at least two further combinations which add to give ten.

Partition
Grouping / concrete representation (composition of groups visible)
Discrete material
Directed - partition (no remainders)
Assess whether a collection can be partitioned into equivalent groups.
Is sharing 1:1 or many:1?
Give the same amount of lollies to each Lego person. We are to use all the lollies. (12 lollies shared between 3 people).

Response categories

Year	1	2	3	4
K	1	4	3	
1		4	4	
2		4	4	
3		5	2	1
4		2	5	1

Response Descriptions:
1. Did not share objects.
2. One to one correspondence.
3. Many to one correspondence, more than one deal carried out.
4. Many to one correspondence after answer calculated.

Partition
Grouping / concrete representation (composition of groups visible)
Discrete material
Directed - partition (larger number and remainders)
Assess whether a collection can be partitioned into equivalent groups.
Is sharing 1:1 or many:1?
I am going to give you some shells.
I want you to put these shells in the plates so that there is the same number of shells on each plate (26 shells with 6 plates).

Response categories

Year	1	2	3	4	5	6
K	2	3	2		1	
1	2	1	2		3	
2		1	3		4	
3			6		1	1
4			2		5	1

Response Descriptions:
1. Did not share objects.
2. One to one dealing, distributed all objects.
3. One to one dealing, acknowledged the remainder.
4. Many to one dealing, more than one deal carried out, distributed all objects.
5. Many to one dealing, more than one deal carried out, acknowledged the remainder.
6. Many to one dealing after answer calculated.

Multiple count - threes

Counts by threes up to:

Year	not able	6	9	12	15
K	8				
1	6	2			
2	2	2	1	2	1
3		4		1	3
4		1		2	5

Multiple count - fours

Counts by fours up to:

Year	not able	8	12	16	20	24
K	8					
1	5	2	1			
2	3		2	2		1
3	2	3			1	2
4		3	1			4

Multiple count - tens

Counts by tens up to:

Year	not able up to 90	100	100 +
K	8		
1	2	4	2
2	1	2	5
3		1	7
4			8

Double count with and without direct modelling

Assess use of double count to keep track of number of groups of 4.
eg 1 2 3 4/1, 5 6 7 8/2, 9 10 11 12/3
a) *See if you can think aloud as you do this next question.*
There are 12 children with 4 children sitting at each table.
How many tables are needed?
Give opportunity to draw picture or use material.
b) Provide 12 Lego people and 5 trucks made out of Lego blocks.
There are 12 Lego people and some trucks.
4 people go to work in each truck.
How many trucks are needed?

Response categories

Year	1	2	3	4	5
K	5	2	1		
1	1	6	1		
2		4	4		
3		2	3	1	2
4		3		1	4

Response Descriptions:
1. Not able to solve problem (meaningful model).
2. Solved problem with materials or by drawing pictures.
3. Solved problem mentally by building up groups of 4 until 12 (double count).
4. Solved problem mentally by taking away groups of 4.
5. Solved problem mentally by relating to known multiplication or division facts.

Compensation

Assess use of subitizing pattern when adding two single-digit numbers, pictorial representation.
Show a 6 and a 9 pattern board (twos pattern).
How many dots are here? Show the 6 board.
How many dots are here? Show the 9 board.
How many dots are there altogether?

Response categories

Year	1	2	3	4	5
K (4)	1	3			
1 (6)		5		1	
2	1	2	2	1	2
3 (7)		1		4	2
4		2	1		5

Response Descriptions:
1. Not able to find sum.
2. Count all by ones.
3. Count-on from 6 by ones.
4. Count-on from 9 by ones.
5. Subitize patterns in both numbers and add by partitioning and combining the numbers - compensation, bridging tens.

Response examples:
Year 3:
K. "Take one off there.... and put there.... and you still have 5.... 15."
Year 4:
B. "9 is near 10.... and another 6 is 15."
Year 4:
C. "6 and 6 plus 3 is 15."
Year 4:
D. "cause you take 1 from 6 and then that 1.... you add to the 9.... then you put the 5 on to give 15."

Counting pregrouped material
Grouping / opaque covering, one group of ten:
Directed
Assess level of counting, use of ten.
Show a roll of 10 lollies (transparent).
How many lollies are in this roll?
Show 4 **opaque** rolls, each containing 10 sweets.
How many lollies are in this roll?
How many lollies are in all these rolls?
1 roll and 5 loose sweets are displayed.
How many lollies?"

Response categories

Year	1	2	3	4	5
K	3	4	1		
1		2		5	1
2		1		3	4
3				2	6
4					8

Response Descriptions:
1. No response.
2. Counts in ones, guessing number in the package.
3. Counts in ones, correct number in the package.
4. Counts-on from ten by ones.
5. Counts ones (or subitizes) and adds ten.

Counting pregrouped material
Grouping / opaque covering, several groups of ten:
Directed
Assess level of counting, use of ten.
4 rolls and 3 loose sweets are displayed.
How many lollies are here altogether?

Response categories

Year	1	2	3	4	5	6	7
K	5		3				
1		2	2		1	3	
2		1	1			3	3
3						4	4
4						2	6

Response Descriptions:
1. No response.
2. Counts each package as one.
3. Counts in ones, guessing number in each package.
4. Counts in ones, correct number in each package.
5. Counts in tens and in ones without coordination.
6. Counts in tens and in ones with coordination.
7. Uses multiple of tens.

Regrouping
If you added 8 lollies to your collection there, how many lollies would you have altogether?
Assess whether child counts by ones and then trades or uses ten as a unit in the trade.

Response categories

Year	1	2	3	4	5
1	1	2			
2		4	1	2	
3		3	1	3	1
4		3		4	1

Response Descriptions:
1. Counts without coordination, incorrect result.
2. Counts-on from 43 by ones.
3. Counts-on from 48 by ones.
4. Adds the ones together and then regroups.
5. Adds ten and takes away 2, bridges tens.

Counting pregrouped material - pictorial.
Grouping / opaque covering, several groups of ten:
Pictorial, directed
Assess level of counting, use of ten.
Show a picture of 8 rolls and 6 separate lollies.
How many lollies in this roll?
How many lollies altogether?

Response categories

Year	1	2	3	4	5	6	7
K							
1 (6)		3		1		2	
2	1					7	
3 (7)						7	
4	1					7	

Response Descriptions:
1. Counts without coordination, incorrect result.
2. Counts tens as ones.
3. Counts ones as tens.
4. Counts all by ones, guessing number in each roll.
5. Counts all by ones, taking the number in each roll as ten.
6. Counts in tens and ones, with coordination.
7. Uses multiples of ten.

Counting objects in an array
Assess intuitive use of ten in counting when a ten by six array of pictures is presented.
Show an array of ten by six planes.
Can you tell me quickly how many planes are here?

Response categories

Year	1	2	3	4	5
1	2	1		1	
2	1	2		4	1
3		1	1	5	1
4			1	3	3

Response Descriptions:
1. Unable to count sucessfully.
2. Counts all by ones.
3. Attempts to count by sixes.
4. Counts by tens, repeated addition.
5. Uses multiplication.

Hundreds square
Assess how a number's location on the hundred square is found. What use is made of the pattern of tens.
Show a hundred square (0to 99).
Show me how you can get ten more than 36 quickly from the hundred square.
Can you show me ten less than 49?
Can you show me nine more than 67?

Response categories

Ten more than 36 — Ten less than 49

Year	1	2	3	4	5	6	7	8
1	3	1	1		3	1	1	
2	1		3	4	2		3	3
3	1		3	3	1		4	2
4	1		5	1	1		5	1

Response Descriptions:
1. Unable to locate number (addition).
2. Attempts to count-on by ones to get second number (addition).
3. Counts-on by ones to get second number as a result of addition.
4. Uses pattern of tens (addition).
5. Unable to locate number (subtraction).
6. Attempts to count-on by ones to get second number (subtraction).
7. Counts-back by ones to get second number as a result of subtraction.
8. Uses pattern of tens (subtraction).

Uncovering tens task - assess whether child can coordinate counting by tens and ones i.e. ten as an abstract composite unit.
Tens task - a board to which is affixed a sequence of Dienes longs and shorts is gradually uncovered and each time the cover is pulled back to show more material the child is asked: "how many are there now?"
10, 14, 34, 38, 41, 51, 53, 73

Response categories

Year	1	2	3	4	5	6	7
1	3	1	1		1	1	
2	2	1			1	2	2
3					2	4	2
4			1			3	4

Response Descriptions:
1. Attempts to count all by ones - counts tens by ones (sometimes unsucessfully).
2. Restarts each time to count-on from ten all by ones.
3. Miscounted on tens as ones.
4. Miscounted on ones as tens.
5. Start with ten and count-on by ones.
6. Does not count-on, restarts each time and collects units of the same rank together.
7. Counts-on by tens and ones as appropriate.

Addition involving regrouping
Regrouping - composition of groups hidden:
Addition/tens and missing addend/tens tasks. Assess use of ten in tasks that require regrouping - assess counting as: i) only by tens **or** by ones - **ten as a numerical composite** if even though it is known that a roll has ten lollies counting is by ones or counting tens as ones; ii)coordinating counting **tens as abstract singletons** and ones as abstract units, units of same rank are added with no notion of ten being composed of individual units; iii)constructing **tens as abstract composite units**, count by tens and ones starting in the middle of a decade; iv) constucting each number as so-many tens and so-

many ones and then adding (basic fact knowledge) units of the same rank, ie **abstracting collectable units**; v) **ten as an iterable unit**, a number is a numerical whole and units of ten and one at the same time.

3 rolls (opaque coverings) and 7 separate lollies are visible, the child is told 25 lollies are hidden beneath the cloth and asked to find how many lollies there are altogether.

Response categories

Year	1	2	3	4	5	6	7	8	9
1	7				1				
2	5	1			1				1
3	1	1		1	2	1		1	1
4	1	1				1	5		

Response Descriptions:

1. Unable to calculate total mentally.
2. Counts-on by ones (sometimes unsucessful) - ten as a numerical composite.
3. Counts-on by ones, counting tens as ones - ten as a numerical composite.
4. Counts-on by tens and then all by ones.
5. Counts-on by tens (or collects tens), adds 5 or 7, and then counts-on by ones.
6. Counts-on by tens and ones starting in the middle of a decade - ten as an abstract composite unit.
7. Adds units of the same rank.
8. Breaks up one number into parts and uses counting-on by ones and tens.
9. Breaks up numbers into parts in order to facilitate the addition process, use compensation or bridging tens - a number is a numerical whole and units of tens and ones at the same time.

Response examples:

Year 1:
J. "3 plus 2 is 5.... so 30 plus 20 is 50.... 57 with 7.... count-on 5 to give 62."
Year 2:
R. "Put 3 in to make ten.... then 50, 60.... then just put the two more to give 62."
Year 2:
J. "that's hard.... 30 plus 20 equals 50.... sixty something.... 57, 58, 59, 60, 61, 62 (using fingers)."
Year 3:
M. "25, 26, 27, 28, 29, 30,.... 40, 50, 60.... 61, 62."
Year 3:
K. "Adds all the ones together and there was 2 plus 10.... so 62."
Year 4:
R. "Added 30 and 20 to give 50.... 7 and 5 is 12.... you can't have 50 twelve so you get 60."

Structure - pictorial, discrete objects
Undirected
Determine pertinence of grouping and the significance of associated writing.
Responses: no need for grouping - either guess or count by ones; group to count quickly, recount the collection after grouping; use one order of groupings and see that writing is a code that is directly associated with these groupings; use a grouping of groupings.

Child presented with a picture of 143 marks randomly drawn.
Can you tell me quickly how many marks there are drawn there?
I am going to do the same thing later with a friend who will be here after you. Could you do something so that, when I show him the sheet, he will be able to tell me very quickly how many marks there are?
What did you do?
Now can you tell me quickly how many marks there are?
How do you know that?
Look at what the friend who came before you did (grouping of groupings is shown). What do you think of it?
Can we see quickly how many marks there are?
Suppose you have a younger brother / sister who you are going to help with his/her counting.
How would you explain the easy way to count those marks.

Response categories

	Spontaneous count			Suggested help				Pregrouped			
Year	1	2	3	4	5	6	7	8	9	10	11
1	5	3		3		1		3		1	
2	7	1		6		2		2	3	3	
3	3	5		4	1	3		1		6	1
4		8		3	3	2		1		4	3

Response Descriptions:
1. Does not do or guesses.
2. Attempts to count by ones.
3. Forms groups of ten to count.
4. No suggestions to help another person, labelling each mark or dividing in half.
5. Suggests grouping by a number other than ten.
6. Suggests grouping by tens.
7. Suggests grouping ten groups of tens.
8. Does not recognize and use tens when shown what a friend had done.
9. Recognizes but does not use successfully the groupings of tens.
10. Recognizes and uses the groupings of tens.
11. Recognizes and uses the groupings of tens and hundreds that were made by another person.

Assess use of groupings of tens, hundreds and thousands when quantifying large collections.
Show an array of 100x100 dots.
Can you tell me how many dots are here?

Response categories

Year	1	2	3	4	5
2	3	1	4		
3	3		4		1
4	2	1	2		3

Response Descriptions:
1. Unable to quantify.
2. Attempts to count by ones.
3. Determines the patterns of hundreds and unsucessfully attempts to count by hundreds.
4. Determines the patterns of hundreds and counts by hundreds to give 10 000.
5. Determines the patterns of hundreds, counts to give the thousand pattern and then counts to give 10 000.

Response examples:
Year 2:
J. Counted across the top (10) and down the side (10) of a square pattern. "100 in each square.... " Counted down the page.... "1, 2, 3, 4, 5, 6, 7, 8, 9, 10 hundred.... I know this is one thousand.... " counted across the bottom of page...."1, 2, 3, 4, 5, 6, 7, 8, 9, 10.... one million!"

Assess whether the mental picture of the numbers 1 to 100 is a single mental number line or a matrix of coordinated horizontal and vertical mental number lines.
Close your eyes. I want you to imagine the numbers from 1 to 100. Can you see a picture of these numbers?
Open your eyes.
Draw a picture of what you saw.

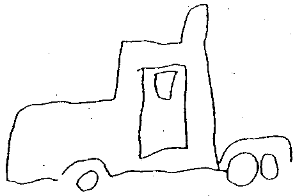
Response categories

Year	1	2	3	4	5	6	7	8
1 (6)	3	1	1		1			
2	1		2	1	1		3	
3 (7)	4	1		1			1	
4				3		1	2	2

Response Descriptions:
1. No mental picture given.
2. Pictures or marks, no pattern (100 objects, lots of squares).
3. Pictures or marks, array or line pattern (100 marks in a line, ten sticks)
4. Random selection of numerals, no pattern in display.
5. The numeral 100.
6. Numerals appearing one at a time in order to 100 (flashing).
7. Numerals in a long number line or a series of number lines for different multiple counting sequences.
8. Numerals in a 10 by 10 grid.

Year 1

Anthony



James

100

Lisa

100objects... all sorts of things... on the floor.

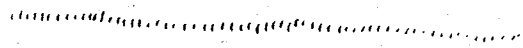
Year 2

Joshua

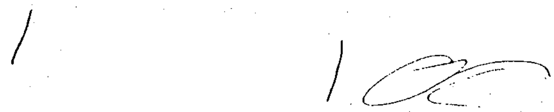
100 99 98 97 96 95

and on to 1.

Warren



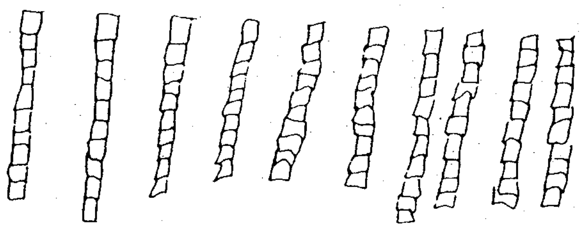
Gary



Hayley

1234

Mellissa



Oliver

10
20
30
40
50
60
70
80
90
100

Robert

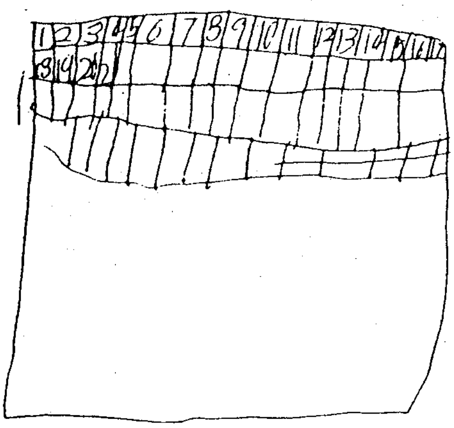


Figure 1: Examples of visualization:

Year 3
Michael

0 1 2 3 4 5 6 7 8 9 10

... and on to 100.

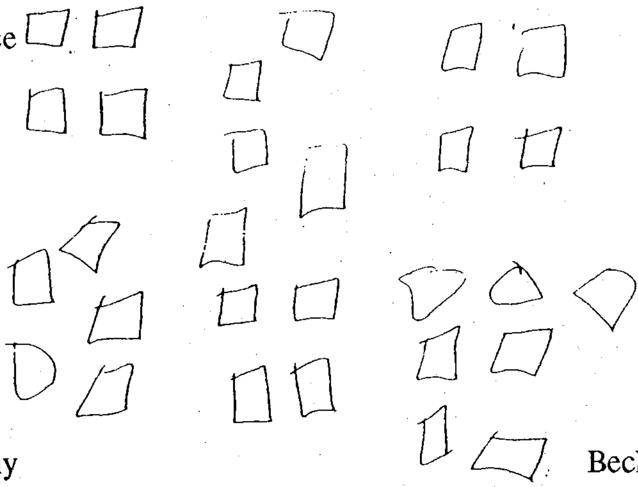
Amber

1 10 100 90

1 2 3 4 6 9 11 13 20

31 90 100

Candice



Year 4
Timothy

a

Becky

24 89

Cassie

1 5 10 6 20 30
9 2
8 4 16 18
3 7
11 12 19 17 33
22
94

Alex

1 2 3 4 5 6 7 8 9 10

10 20 30 40 50 60 70 80 90

5 10 15 20 25 30 35 40 45 50

55 60 65 70 75 80 85 90 95 100

Oliver

1 2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32

... and so on until 100.

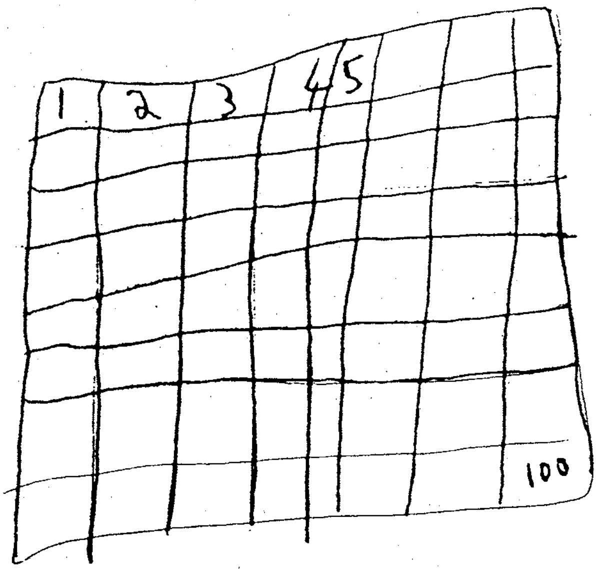
Rebecca

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

... up to 100.

Figure 1: Examples of visualization (continued)

Grant



David

Numbers flashing one at a time, in fives, up to 100.

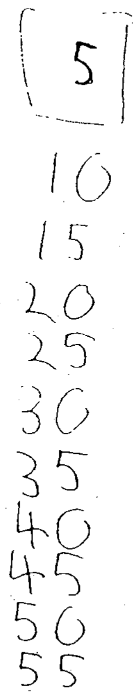


Figure 1: Examples of visualization (continued)

CONCLUSION

The missing addend and removed item tasks showed that all the Kindergarten and some Year 1 children were still in the process of constructing the number system with the relation 'one more'. These same children were equally likely to solve partition problems by one to one, or many to one, correspondence. Only two children could solve the quotition problem without direct modelling and they used a building-up mental strategy. They were more advanced with multiple counting using tens than with any other number and this was reflected by their developing skills for counting collections pregrouped as tens and ones. It should be noted that three of the Year 1 children successfully coordinated the counting in tens and ones, of a collection of 43 objects and two of them regrouped in their mental strategies when adding a single digit number to a two-digit number. At the same time these children do not use ten as an iterable unit, as shown by the uncovering tens task, do not subitize ten from visual patterns of two single-digit numbers and do not recognize the pattern of tens in the hundreds square.

Some of the children in Years 2 and 3 were shown to be using ten as an iterable unit while others used ten as abstract collectable units. This was reflected by the use of higher order strategies based on ten for solving addition with regrouping tasks and the use of the tens pattern in the hundreds square. It is interesting to note that none of the Year 4 children used the higher order strategies involving breaking numbers up into parts but rather depended upon adding units of the same rank. Is this because of instructional experiences? Most of the Year 2 and 3 children had poor multiple counting skills with numbers other than ten.

Overall there appears to be a lack of extended structure evident in a young child's understanding of number and a corresponding resistance to use the properties of ten in mental calculation. No children in our sample from years 1 to 4 spontaneously used grouping to find how many marks there were on the card (143 marks altogether) - they either said they did not know, guessed or counted by ones. Only two children recognized the grouping of ten groups of ten when shown 'what a friend had done to help him/her find quickly how many marks there were', even though the hundred was highlighted with a red circle.

The visualization of the numbers from 1 to 100 provided seven response categories. Of these responses only one was dynamic (flashing numbers), all others being static in nature. Approximately one third (8) of the grade 1 to 4 children did not visualize any picture, a third (10) visualized a picture with no structure and the remaining third (9) had structure of some kind in their picture. The results show that Year 2 children are beginning to 'see' structure in the number system. The grouping structure in the Year 2 responses is most clearly seen in the mental picture of ten ten-rods as illustrated by Melissa. The most highly developed visualizations of the structure are shown by the pictures illustrated by Grant and Oliver (hundred squares), Alex (patterns within the number sequence), and David (multiples of five as flashing numerals).

In this report only some of tasks that were used have been discussed. So far it appears that children as young as 5 year olds use equivalent groups when counting to solve partition and quotition tasks. Children in Kindergarten are building the first level of the number system using the 'one more' relation. Children in Years 2 and 3 are building the second level of the number system, important aspects being the ability to partition numbers in many different ways and the use of ten as an iterable unit. The construction of one group of ten out of ten ones, which then becomes a ten unit and then, the use of the two types of units (tens and ones) in regrouping is important for the development of number sense at this level. Many children in this study were still needing to develop this structural flexibility with the use of two-digit numbers. Understanding of the extended structure of the number system beyond two-digits was only displayed by one Year 3 and three Year 4 children.

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