ABSTRACT THINKING IN RATES OF CHANGE AND DERIVATIVE

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Responses to problems involving rates of change were compared across four data collections throughout an introductory calculus course given to a group of first year university students, all of whom had studied calculus at school. The course focused on derivative as instantaneous rate of change, and employed a method based on examining graphs of physical situations. The number of students who could symbolise rates in non-complex situations increased dramatically, but no improvement was seen in complex items or in items which required algebraic modelling. The results point to the critical role of a developed concept of a variable in learning calculus, and are interpreted by showing the inadequacies of abstract-apart concepts as opposed to abstract-general ones.

Advances in technology, qualifications of teachers and mathematical competence of students have brought under fire the traditional place of calculus courses. There seems to be some concern about the large numbers of students taking calculus and the rote, manipulative learning that takes place (Barnes, 1988; Grimison, 1988, Steen 1988; White 1990). The value of skill based calculus courses has come under fire even more with the development of computers and calculators which perform most (if not all) of the manipulative procedures taught in such courses (Steen, 1988).

Replacing the traditional introduction to calculus via limits with more informal approaches has been widely endorsed. One such approach suggests that initially concepts should be introduced intuitively so that the introduction to differentiation is based largely on numerical and graphical explorations assisted by an electronic calculator or computer (Barnes, 1988, 1992; Orton, 1983; Tall, 1986; Wilkins, 1987).

Hiebert and Lefevre (1986) say conceptual knowledge is characterised by relationships between pieces of knowledge and, thus, often plays a part in the choice of procedure for a given mathematical problem. The degree of abstractness of a relationship can vary in that abstractness increases as knowledge becomes freed from specific contexts. A useful distinction arose from comparing their conception of "abstractness" with the everyday meaning of the word and Skemp's (1971) description of the process of abstracting. The Webster Dictionary presents the following definition for "abstract":

Abstract (ver): **To consider apart from particular instances; to form a general notion of.**

The two key words (general and apart) lead to the identification of two ways of looking at how abstract mathematical objects are used and related. Using mathematical symbols may be an abstract operation if the symbols have no concrete reference: they are "apart". The only context for the symbols is the symbols themselves. In fact, in some cases no other meaning for the symbols may exist. An example of "abstract-apart" is knowing how to manipulate algebraic symbols without having any sense of what the letters stand for. On the other hand, "abstract-general" indicates that the mathematical objects involved are seen as generalisations of a variety of situations and so can be used appropriately in different looking situations. For example, an abstract-general concept of addition would be something like "addition is the result of combining sets of like objects" and so embraces fractions, decimals, algebraic symbols, complex numbers, vectors... Such a concept could also see "addition" in cues other than add or plus. On the other hand, an extreme example of an abstract-apart concept would be only being able to see addition being required when a plus sign (or a limited number of equivalent key words) is clearly evident.

Relationships between pieces of knowledge may be a key factor for conceptual knowledge, but does not always indicate abstract-general thinking. In the abstract-apart case, relationships are formed between mathematical objects which have no conceptual base. Any relationship between the objects is "superficial" because it can only be formed on the basis of what the mathematical objects look like. In the abstract-general situation, "meaningful" relationships can be formed because the mathematical objects have meaning beyond the symbols themselves. A key factor to indicate the difference between superficial and meaningful relationships is the cues that prompt the appropriate links between pieces of information. Abstract-apart needs definite, visible cues because abstract-apart cannot generalise to different looking situations. Abstract-general, on the other hand, can link all sorts of situations regardless of how dissimilar looking they are.

Our definition of "understanding" a concept is having an abstract-general notion of it. The addition example suggests that abstract-general and abstract-apart are not a dichotomy because there are many possible levels of generality between the two extremes described. Hence, there is a continuum between the two. Tall and Vinner's (1981) concept image and concept definition can be seen as an example of the continuum. Other applications of the continuum can be found in Mitchelmore (1992).

The purpose of the present study was to investigate the effects of a course of study based on interpreting rates of change using graphs on some first year tertiary students' understanding in rates of change and derivative. Experience had suggested that many first year tertiary students would have received a skill oriented course in high school and be proficient in routine procedures, but lack many of the underlying concepts. In particular, a preliminary study (White, 1989) suggested that translating a rate of change to a derivative is a crucial link in solving rates of change items because abstract-general notions are involved. The research was not a teaching experiment; the teaching was merely providing an appropriate environment for observing any changes in student thinking.

METHOD

Sample

The sample was forty first year full time students enrolled in Mathematics I at a smaller New South Wales university. A prerequisite for the course was a satisfactory result in the final high school exam for a mathematics course which contains a large component of calculus. It should be noted that none of the students had finished in the top 10% in that final exam; the students were in most cases "average".

Instrument

After trialling a number of items with students and experts in the field of calculus, four test items were chosen. Items 1 and 2 required the direct translation of a rate of change into a derivative, with Item 1 being judged by experts as the more difficult. Items 3 and 4 involved symbolising derivatives to maximise/minimise a situation, with Item 3 being judged the more difficult.

Each of the four items were structured in four versions (a, b, c, d) so that the expected correct response for each version was basically the same. The difference between the versions was that each required successively less symbolisation. Hence, (a) required symbolising all rates to an appropriate derivative, whereas (d) had all information presented in symbolic form. As mentioned earlier, being able to symbolise rates of change was suggested as a crucial factor for understanding derivative. The four versions allowed for the symbolising aspects of responses to the items to be isolated. For clarification, the four parts for Item 1 and the part (a)s for the other three items are:

Item 1

(a) In the special theory of relativity, the mass of a particle moving at velocity v is given by:

$$m = \frac{m0}{\sqrt{1 - \frac{v2}{c2}}}$$ where c = speed of light and m0 is the rest mass.

At what rate is the mass changing when the velocity is $\frac{1}{2}c$ and the acceleration is 0.01c per sec?

(b) In the special theory of relativity, the mass of a particle moving at velocity v is given by:

$$m = \frac{m0}{\sqrt{1 - \frac{v2}{c2}}}$$ where c = speed of light and m0 is the rest mass.

Find $\frac{dm}{dt}$ when $v = \frac{1}{2}c$ and the acceleration is 0.01c per sec.

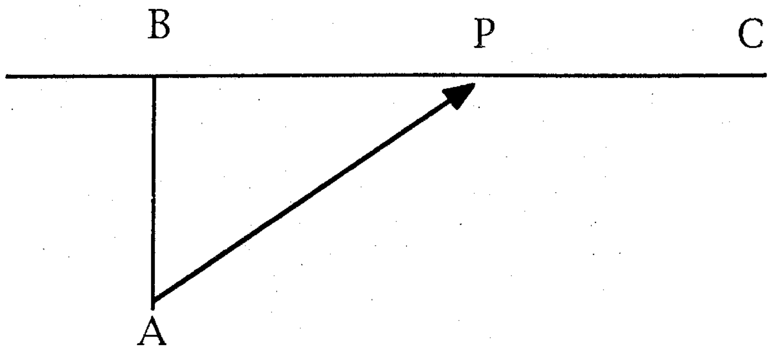
(c) Given $m = \frac{m0}{\sqrt{1 - \frac{v2}{c2}}}$ find $\frac{dm}{dt}$ when $v = \frac{1}{2}c$ and $\frac{dv}{dt} = 0.01c$.

(d) Given $m = \frac{m0}{\sqrt{1 - \frac{v2}{c2}}}$ and $\frac{dm}{dt} = \frac{dm}{dv} \cdot \frac{dv}{dt}$, find $\frac{dm}{dt}$ when $v = \frac{1}{2}c$ and $\frac{dv}{dt} = 0.01c$.

Item 2

(a) If the edge of a contracting cube is decreasing at a rate of 2 centimetres per minute, at what rate is the volume contracting when the volume of the cube is 64 cubic centimetres?

Item 3



(a) The diagram shows a straight road BC running due East. A four wheel drive is in open country at A, 3 km due South of B. It must reach C, 9 km due East of B, as quickly as possible. The driver can travel at 80 kph in open country and 100 kph on the road. Assuming the car proceeds through open country to some point P, and then along the road, what is the distance of P from B so that the journey to C takes the shortest time possible?

Item 4

(a) Find the area of the largest rectangle with its upper vertices on the curve $y = 12 - x$ 2 and lower base on the x-axis.

Teaching Sequence

The basis for the teaching material was Swan (1989) and followed the approach of Barnes (1992). Initially, rates of change were investigated using graphs of physical situations. This led to the secant being seen as representing an average rate of change, the tangent an instantaneous rate of change. Derivative was then developed from the latter. Limits were only considered informally.

Procedure

Four data collections occurred - before, during, immediately after and then six weeks after the teaching sequence. The forty students were arranged into four approximately parallel groups of ten, based on their performance in a previous mathematics exam. The students were unaware of the groupings. Four tests were constructed. Each test included four questions - a different part from each item. Hence, each part of each item occurred on one and only test and each test had one part (a), one part (b) ... These tests were administered in a cyclic fashion to each of the groups over the four data collections so that students did different items each time, but data was still available for the same pool of items in all collections.

In addition, the same sixteen students (four of about equal performance from each group) were interviewed each time within three days of the written data collection. These interviews served to clarify and expand on written responses so that the student's understanding could be better identified.

RESULTS

The number correct at each data collection

Item	Collection 1	2	3	4	Total
1	2	3	8	7	20
2	8	13	26	22	69
3	4	6	14	12	36
4	10	11	16	15	52
Total	24	33	64	56	177

Recalling that scores are out of forty, results in collection 1 show that students in general could not correctly respond to any item. The improvement in the number of correct responses in collection 3 and 4 is substantial, but performance still only exceed 50% in Item 2. The large number who improved suggests the teaching was a positive factor. The equally large number who did not points to inhibiting factors, many of which were able to be identified by a detailed analysis of the responses.

Rates of Change - Items 1 and 2.

In collection 1 only 4/40 students observed that rate of change meant a derivative was required. As well, only 4/20 showed any recognition of the need to use the chain rule in 1(c) and 2(c) where the required derivatives were cued symbolically, but the actual chain rule was not stated. The pattern changed in the collections 3 and 4 with virtually all students at least attempting to translate rates into derivatives, and 34/40 correctly stating the chain rule in 1(c) and 2(c). However, in the last two collections, problems associated with students' **concept of a variable** arose so that, even though derivatives and chain rules were introduced, the number of correct solutions was still lower than might have been expected.

In collections 3 and 4, 36/40 students were able to correctly symbolise derivatives in Item 2, but not in the more complex Item 1 (7/40). In Item 1, 7/8 students who successfully symbolised derivatives gave fully correct responses; whereas for Item 2, only 22/36 of the responses which correctly symbolised the derivatives were actually correct.

Students' under developed concept of a variable manifested itself differently in the two items. Analysis showed that the predominant error in Item 1 was that students focused on the visible symbols, looking for something to fit known manipulation rules, and as a result immediately substituted $\frac{1}{2}c$ for v. Such an attitude was named a "manipulation focus". In Item 2, the main cause for not correctly finishing the item when it had been correctly set up was that students saw $V = x\ 3$ and $V = 64$ as separate cases and either gave two answers or were confused and stopped.

Maximisation - Items 3 and 4.

There were few changes across the four data collections. The main obstacle was that students generally could not model the situation given in part (a) or (b) using algebra (14/80 and 7/80 for Items 3 and 4 respectively). The lower number for the easier Item 4 at first appears odd. The reason was a form of the manipulation focus - named the "x, y syndrome" - where 56/120 students seized on the equation $y = 12 - x\ 2$ and found $\frac{dy}{dx}$ with no thought for modelling the rectangle.

In both Items 3 and 4, all responses which showed correct modelling to part (a) were fully correct (8/8). Admittedly the numbers were small, which of itself may be significant. Correct modelling in part (b) also coincided with a high success rate, but not at the 100% level (10/14).

DISCUSSION

What has been described as the manipulation focus is an indicator of an abstract-apart concept of a variable. Students exhibiting the manipulation focus show they can apply manipulation rules, but have no sense of deciding when such rules are appropriate. Their use of the rules is superficial being based solely on what the symbols look like. The symbols are seen apart from any meaning they might have. Immediately substituting $\frac{1}{2}c$ for v and the confusion between $V = x\ 3$ and $V = 64$ indicates how a particular value for the expression is seen apart from the general expression itself. The x, y syndrome shows how the rule for maximising is equated with the visible symbols used when the rule is first learnt rather than with what they represent. As such, the rule is apart from the situations it generalises.

A majority of students who identified a rate as a derivative, correctly symbolised \f(dV,dt) and $\frac{dx}{dt}$ in Item 2 and correctly wrote down the chain rule, nevertheless made manipulation focus errors. This suggests that these three relationships are superficial operations which can be achieved with thinking at the abstract-apart end of the continuum. On the other hand, being able to correctly symbolise rates of change to the appropriate derivative in structurally complex items (like Item 1) was only achieved by a few students who did not exhibit the manipulation focus. Symbolising in structurally complex situations would seem to indicate meaningful relationships are being used. Such a suggestion makes sense because in more complex situations, there is more than one variable to choose as the dependent and/or the independent variable. Some sense has to be attached to the letters involved; superficial cues alone do not suffice. For example, relating given variables in Items 3 and 4 requires meaningful relationships to be formed because the items are reasonably structurally complex. In contrast, manipulation focus students had no difficulty modelling the volume of the cube by $V = x\ 3$ in the less complex Item 2.

Since few students were able to define appropriate variables in 3(a) and 4(a), and that those who did were always completely correct, there is a strong suggestion that defining and relating variables involves forming meaningful relationships at a higher level than does symbolising, and relating already defined variables. Symbolising in complex situations requires putting appropriate variables to the rates which are already cued; defining variables in a modelling situation indicates the solver is making choices with some plan in mind and the plan has not been laid out in the cues. In this way, the evidence suggests that defining variables indicates relationships at a higher level of generalisation than the pieces of knowledge they connect. For example, in 3(a), information about speed and distance must be pulled together to form a relationship which leads to minimum time. Alone, the link between speed, distance and time is no more general than the cued information. The introduction of Pythagoras' Theorem is also no more general that the cue of the triangle leading to it. However, choosing an appropriate distance to define as the independent variable so that the speed-distance-time relationship and Pythagoras' Theorem can be employed suggests those relationships themselves have been transcended. The executive control required indicates that the solver must reflect on a number of relationships and that seeing how to combine them indicates a relationship at a higher level of abstractness.

IMPLICATIONS

In general, the results present a fairly pessimistic view of where average students are mathematically when they leave high school. Severe doubts have been raised about their ability to learn calculus because of deficiencies in their concept of a variable. As it stands, calculus teaching in school appears to be useless, with students being unable to form even superficial relationships soon after. Tertiary calculus seems to develop the superficial relationships, but not the meaningful ones and as such is worse than useless. Clearly, you cannot successfully teach a tertiary calculus course by teaching calculus only.

There would seem to be a need for a radical change in the way we all teach mathematics. The plot - that mathematics consists of generalisations - seems to have been lost (or never found?). As long as the emphasis is on reproducing procedures in antiseptic situations which depend on key cues that always look the same, the number of students who really understand and appreciate mathematics will continue to be limited to a very elite group. In particular, algebra teaching needs to focus on abstract-general notions of a variable by investigating situations where letters have more than a symbolic context.

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