PROCESSES INVOLVED IN MATHEMATICAL PROBLEM SOLVING IN YEAR 12 CALCULUS

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Mathematical problem solving is a complex activity that necessitates very thoughtful consideration about which are the best ways to teach problemsolving. The purpose of this study is to uncover information about and gaina greater insight into the processes involved in solving problems in year 12 (V.C.E.) Calculus. A qualitative methodology (thinking aloud) has been employed in the study. The verbal protocols of six students are currently being analysed for regularities in overt behaviour and especially the utilisation of potential heuristics in their solutions. The paper will present the research questions, the research method and the plan for the categorisation of the verbal protocols.

Mathematical problem solving is an extremely complex activity that necessitates very thoughtful consideration about what are the best ways to teach problem solving. The process of solving mathematical problems is very important in the mental and conceptual development of the children. Lester (1980) claimed that the ability to solve problems is:

"the ultimate aim of learning mathematics at every level" (p.287).

That problem solving is considered important, can be confirmed by the fact that it has been the subject of research since the turn of the century.

Before proceeding further, two key terms will be defined in this paper, problem and problem solving. A careful examination of many research papers reveals the fact that there is no agreement among researchers as to what is a "problem" and even more disagreement is evident when trying to determine what problem solving is. This is mainly due to the fact that mathematical problem solving appears to a certain extent to be so complex and subtle, as to defy an all-encompassing definition. Lester (1980) writes that:

"A problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution". (p.287)

Sweller and Low (1992), have proposed their own conceptualisation of what a problem is:

"No task can be classed as either an exercise or a problem simply by referring to its structureand components. Alone, the structure of a task cannot reveal its "problem" status. Its status is revealed fully by the novice-expert distinction." (p.84)

In an attempt to define what "universal" problems are, Davis (1992), writes that they are:

"Those problems that are known not to have been solved at a particular time." (p.1-183).

Problems of that kind are therefore problems to every person to whom they are posed. As a result of this short discussion I shall define a mathematical problem as a task posed to an individual or group, who will attempt to decipher the task and obtain a mathematically acceptable solution by not initially having access to a method which completely determines the solution. The task being a problem or not for a particular individual or group depending and being a function of the mathematical knowledge [general and task specific], the executive control mechanisms, the memory capacity, the automation of appropriate skills, the mathematical ability, the utilisation of potential heuristics and the mathematical maturity and creativity of the given individual or group. Problem solving can therefore be defined as the set of actions taken to perform a task, assuming that there exists a desire on the part of the individual or group to perform the task.

HEURISTICS AND OTHER PROBLEM SOLVING STATEGIES:

A useful categorisation of dimensions of mathematical behaviour [incorporating heuristics] has been proposed by Schoenfeld (1985b):

"**COGNITIVE RESOURCES:** The mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand.

HEURISTICS: Strategies and techniques for making progress on unfamiliar or non-standard problems; "rules of thumb" for effective problem-solving.

CONTROL: Global decisions regarding the selection and implementation of resources and strategies.

BELIEF SYSTEMS: One's "mathematical world view," the set of (not necessarily conscious) determinants of an individual's behavior." (p.364)

The importance of teaching problem solving heuristics in school mathematics, as a means of improving problem solving performance, is one of the aims of the study to be described in this paper. Heuristical methods for solving problems include plausible but uncertain actions of a general yet natural character.

Some researchers (Lucas,1974; Kantowski,1977) of mathematical problem solving have endeavoured to teach explicitly task-specific heuristics or general heuristics, in attempting to improve the student's problem solving competence via instruction. The hypothesis they tested was that teaching students problem solving heuristics would improve their problem solving performance. The findings of these research studies suggest that systematic instruction in specific problem solving skills enhances the problem solving ability of the students. The investigators were able to observe positive effects on students being taught to use heuristics. It is disappointing however that few of these studies offer testable evidence that the students learned or were able to transfer the taught heuristics to isomorphic or non-isomorphic problems.

In search for a comprehensive framework of heuristic instruction, Schoenfeld (1985b) identified three major issues regarding heuristic strategies. The first issue is that one of the primary reasons for the lack of success in problem solving instruction through heuristics is attributable to the fact that the strategies were very much unspecified. A second issue is that the description of different ways of implementing the strategies must be done in sufficient detail so that our students can learn to implement them. Schoenfeld argues that the complexity of the implementation process has been greatly underestimated. The fact that in most research studies, investigators did not pay sufficient attention to "fleshing out" such strategies in detail, could partially explain their failure in heuristic instruction. The third issue is related to the detailed specification of strategies:

"If each heuristic label represents a class of a dozen or so more finely defined strategies, then any collection of ten powerful techniques becomes a collection of over 100 specific strategies." (p. 370)

It is evident from the foregoing discussion that teaching heuristics is not as simple as some investigators might have thought a few years ago. Teaching explicitly the strategies along with the underlying substrategies and taking under consideration the student's belief systems and metacognitive behaviour does indeed seem to produce the desirable results.

A different approach has been advocated by Sweller and his colleagues. Owen and Sweller (1989), question the emphasis placed by curriculum developers and mathematics education researchers, on the use of heuristics in problem solving, by arguing that effective problem solving performance is not the result of the use of superior heuristics but the result of students having acquired the appropriate schemata and automation of domain specific skills. A critical comment on Sweller's argument that schema acquisition and automation of domain specific skills are the most effective tools of skilled problem solving performance, has recently been offered by Davis (1992). He claims that good problem solving and the automation of domain specific skills are by no means equivalent, one reason being that such automation is, in many cases, necessary but not sufficient to solve a given problem.

I would argue that a Grand Unified Theory Of Mathematical Problem Solving (GUTOMPS), incorporating the heuristic approach based on a refined Polya model and the approach advocated by Sweller and his colleagues, are two of its constituent components. This is what is actually needed by the mathematics teachers who as practitioners are utilising elements of both theories on a day to day basis.

THE STUDY

The purpose of this investigation is to uncover information about and gain a greater insight into the processes involved in solving problems in year 12 (V.C.E.) Calculus. One of the main hypotheses of this study is that heuristics are techniques which enhance problem solving performance. It is therefore a logical inference that incorporation of heuristics into problem solving strategies is a valuable asset for the problem solver. A number of challenging questions follow: Is it possible to teach heuristics to Year 12 Calculus students? Can a teacher device a set of heuristics and then produce conditions so that students will be enabled to embed these plausible techniques into their problem solving schemata? Can V.C.E. students be guided to adopt more effective strategies than those they are accustomed to apply to problems? How does instruction on heuristics effect performance in terms of time, accuracy of solutions and elegance of solutions?

RESEARCH QUESTIONS

The following hypotheses are proposed by the investigator in the light of the previous discussion:
(a) Implementation of an explicit, carefully planned instruction prógram on heuristics based on Polya's four stage model can provide a vehicle for shifts in strategy on mathematical problems [in Calculus] among a group of year 12 students.
(b) These shifts in problem solving strategy can be observed by the application of a diagnostic instrument deviced by Schoenfeld (1985) and modified slightly by the investigator.
(c) Heuristic-oriented instruction can be incorporated into teaching methods within the framework of the newly designed V.C.E. Mathematical Methods and Specialist Mathematics courses to be implemented in 1994 in Victoria.
(d) Heuristic-oriented instruction will enhance problem solving performance in terms of time, frequency of errors, score, and amount of difficulty.

RESEARCH METHOD

A qualitative methodology (thinking aloud) was employed in this study. The verbal and written protocols of six (three male and three female) Year 12 mathematics students are currently being analysed in detail for regularities in observable sequences of behaviours. Analysis will be made of the various stages of the development of problem solving in elementary Calculus. The instruction phase comprised a major part of the study, "Heuristic teaching" and how it effects problem solving is being considered. The main study was conducted in three phases during the third term of 1990. It included a pretest, the instruction in calculus via heuristics and a posttest. Protocols were recorded during the solution of pretest problems to determine processes used by the students and to enable the investigator analyse possible traces of any (pre-existing) heuristic techniques used by the students, before the instruction phase was initiated. During the instruction phase, the content of Calculus was introduced. During the posttest phase students solved task problems similar to those in the pretest, and protocols were recorded in order to trace the use of heuristics and investigate any improvement in problem solving during its evolution.

PLAN OF ANALYSIS

Categorisation of the interview data

A variation of Schoenfeld's framework for the macroscopic analysis of problem solving protocols (Schoenfeld, 1985) will be used to categorise the transcribed protocols of the students. The written protocols will be parsed into episodes of problem solving behaviour. The episodes and the associated questions [relevant to that type of episode] in each one of them will be matched to the heuristics used in the study. A full analysis of a protocol is obtained by anwering the questions associated with each episode. The episodes used in this study are listed below:

- PRE-READING ACTIVITY
- READING
- ANALYSIS
- EXPLORATION

- UTILIZATION OF NEW INFORMATION AND LOCAL ASSESSMENTS
- PLANNING-IMPLEMENTATION
- VERIFICATION (LOOKING BACK)
- TRANSITION BETWEEN EPISODES

A sample of the questions associated with one of the episodes follows:

NEW INFORMATION AND LOCAL ASSESSMENTS (NILA)

New information points include any previously unnoticed piece of information and they also include the mention of potentially valuable **heuristics** (new processes, new approaches). Local assessment is an evaluation of the current state of the solution at a **microscopic** level.

NILA (1) Does the problem solver assess the current state of his knowledge? (Is it appropriate to do so?)

NILA (2) Does the problem solver assess the relevancy or utility of the new information? (Is it appropriate at this stage?)

NILA (3) What are the consequences for the solution process of these assessments or the absence of them?

PRELIMINARY FINDINGS

The preliminary findings of the study reveal that the students devoted more time in reading the problem questions and planning during the posttest rather than in the pretest. The total time devoted to solving each problem was reduced for most students in the posttest. Finally the students appeared to be more consistent during the posttest in terms of monitoring their progress, by attempting to make local assessments during the solution process.

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